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ON TERNARY TRANSVERSAL MATROIDS

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The purpose of this paper is to answer a question of Ingleton by characterizing the class of ternary transversal matroids.

1. Introduction

Ingleton [8, p. 123] raised the question of characterizing the class of transversal matroids that are representable over some particular field F. He noted that when $F = \mathrm{GF}(2)$, this problem had already been solved by de Sousa and Welsh [6] who showed that the class of binary transversal matroids coincides with the class of graphic transversal matroids. The latter class had earlier been characterized by Bondy [2] and Las Vergnas [10]. Let C_k^2 be the graph that is obtained from a k-edge cycle by adding a new edge in parallel to each existing edge. On combining the above-mentioned results with Crapo and Rota's Scum Theorem [15, p. 324] and Tutte's well-known characterization of binary matroids [15, p. 167], we get that a matroid is binary and transversal if and only if it has no series minor isomorphic to $U_{2,4}$, $M(K_4)$ or $M(C_k^2)$ for any $k \geq 3$. The purpose of this paper is to give a similar excluded-series-minor characterization of the class of ternary transversal matroids and thereby answer Ingleton's question when $F = \mathrm{GF}(3)$.

The matroid terminology used here will in general follow Welsh [15]. If S is a set, then $S = X_1 \dot\cup X_2 \dot\cup \cdots \dot\cup X_m$ indicates that S is the disjoint union of $X_1, X_2, \ldots, X_m$. The ground set and rank of the matroid M will be denoted by $E(M)$ and $\mathrm{rk}\, M$ respectively. If $T \subseteq E(M)$, then the rank of T will be written as $\mathrm{rk}\, T$ and we shall denote the deletion of T from M by $M \backslash T$ or $M|(E(M) - T)$. The contraction of T from M will be denoted M/T. We shall sometimes write $N \sqsubseteq M$ to indicate that N is a restriction of M having the same rank as M. Flats of M of ranks one and two will be called *points* and *lines*. A line is *non-trivial* if it contains at least three points.

If M_1 and M_2 are matroids on the sets S and $S \dot\cup e$, then M_2 is an *extension* of M_1 if $M_2 \backslash e = M_1$, and M_2 is a *lift* of M_1 if M_2^* is an extension of M_1^*. We call M_2 a *non-trivial extension* of M_1 if e is neither a loop nor a coloop of M_2 and e is not in

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a 2-circuit of M_2. Likewise, M_2 is a *non-trivial lift* of M_1 if M_2^* is a non-trivial extension of M_1^*.

If N_1 is a matroid and $T \subseteq E(N_1)$, then N_1 is a *series extension* of N_1/T if, in N_1, every element of T is in series with an element not in T. The matroid N_2 is a *series minor* of N_1 if, for some subset U of $E(N_1)$, $N_1\backslash U$ is a series extension of N_2. If N_2^* is a series minor of N_1^*, we call N_2 a *parallel minor* of N_1.

We shall assume familiarity with the operations of series and parallel connection of matroids; a detailed discussion of these operations and their properties can be found in [3]. Let $M_1, M_2, \ldots, M_k$ be matroids such that $E(M_i) \cap E(M_j) = \{p\}$ for all pairs, $\{i, j\}$, of distinct elements of $\{1, 2, \ldots, k\}$. The parallel connection of $M_1, M_2, \ldots, M_k$ with respect to the basepoint p will be denoted $P((M_1, p), (M_2, p), \ldots, (M_k, p))$ or just $P(M_1, M_2, \ldots, M_k)$. The series connection of $M_1, M_2, \ldots, M_k$ with respect to p will be denoted $S((M_1, p), (M_2, p), \ldots, (M_k, p))$ or $S(M_1, M_2, \ldots, M_k)$. If each of $M_1, M_2, \ldots, M_m$ is isomorphic to $U_{2,4}$ and each of $M_{m+1}, M_{m+2}, \ldots, M_{m+n}$ is isomorphic to $U_{2,3}$, we shall abbreviate $P(M_1, M_2, \ldots, M_{m+n})$ to $P(mU_{2,4}, nU_{2,3})$. Similarly, if each of $M_1, M_2, \ldots, M_k$ is isomorphic to $U_{2,3}$, we shall abbreviate $P(M_1, M_2, \ldots, M_k)$ to $P(kU_{2,3})$.

A matroid M is *3-connected* if it is connected and $E(M)$ cannot be partitioned into subsets X and Y each having at least two elements such that $\operatorname{rk} X + \operatorname{rk} Y - \operatorname{rk} M = 1$. The following fundamental link between 3-connection and parallel connection was proved by Seymour [14, (2.6)].

Theorem 1.1. *A connected matroid M is not 3-connected if and only if there are matroids M_1 and M_2 each of which has at least three elements and each of which is isomorphic to a series minor of M such that $M = P((M_1, p), (M_2, p))\backslash p$, where p is not a loop or a coloop of M_1 or M_2.* □

We shall need to use the construction of a series minor of M isomorphic to M_1. This proceeds as follows. Let C be a circuit of M meeting both $E(M_1) - p$ and $E(M_2) - p$. Choose an element z of C. Now delete $E(M_2) - p - C$ from M. In the resulting restriction of M, the elements of $C - E(M_1)$ are all in series. If we now contract $C - z - E(M_1)$ we obtain a series minor M_1' of M. Moreover, the bijection from $E(M_1)$ to $E(M_1')$ that fixes every element of $E(M_1) - \{p\}$ and maps p to z is an isomorphism between M_1 and M_1'.

Two fundamental classes of 3-connected matroids are the whirls and the cycle matroids of wheels. Suppose that $r \geq 3$. The *wheel $\mathcal{W}_r$ of order r* is a graph having $r + 1$ vertices, r of which lie on a cycle (the *rim*); the remaining vertex is joined by a single edge (a *spoke*) to each of the other vertices. The *whirl $\mathcal{W}^r$* of order r is the matroid on $E(\mathcal{W}_r)$ that has as its circuits all cycles of $\mathcal{W}_r$ other than the rim as well as all sets of edges formed by adding a single spoke to the set of edges of the rim. The terms 'rim' and 'spoke' will be used in the obvious way in $\mathcal{W}^r$ as well as in $M(\mathcal{W}_r)$. It will also be convenient here to view the matroid $U_{2,4}$ as the whirl $\mathcal{W}^2$.

If $\mathscr{A} = (A_1, A_2, \ldots, A_n)$ is a family of subsets of a finite set S, the transversal matroid M induced on S by the family $\mathscr{A}$ will be denoted by $M[\mathscr{A}]$. We call $\mathscr{A}$ a *presentation* for M. It is well known that the class $\mathscr{T}$ of transversal matroids is not closed under minors since it is not closed under contraction (see, for example, [15, p. 105]). However, $\mathscr{T}$ is closed under series minors [2, Lemmas 1.3–1.4]. Moreover, every series extension of a transversal matroid is transversal [2, Lemma 1.5]. A *gammoid* is a matroid that is isomorphic to a transversal matroid or a contraction of a transversal matroid. The class of gammoids is closed under both minors and duality [9].

In Section 2 we present various elementary results concerning transversal matroids that will be needed later in the paper. Sections 3 and 4 contain the main results of the paper. In Section 3 we specify which ternary gammoids are transversal, while in Section 4 we describe the class of ternary gammoids in terms of excluded series minors. The combination of these results is an excluded-series-minor description for the class of ternary transversal matroids.

The next two theorems will be of fundamental importance in this paper. The Fano matroid is denoted F_7, while $U_{k,n}$ denotes the uniform matroid of rank k on an n-element set.

Theorem 1.2 (Bixby [1, Corollary 7.6.1]). *A matroid is ternary if and only if it has no series minor isomorphic to any of F_7, F_7^*, $U_{2,5}$ or $U_{k-2,k}$ for $k \geq 5$.* □

The matroid P_7 that appears in the next theorem is the 7-element

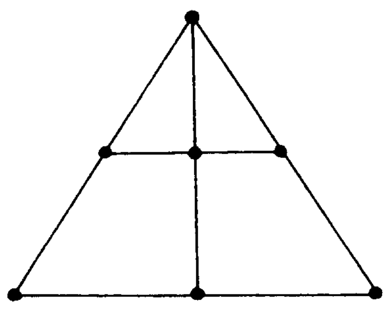

Fig. 1.

rank-3 matroid for which a Euclidean representation is shown in Fig. 1.

Theorem 1.3 [13, (4.1), (4.3)]. *A matroid M is a ternary gammoid if and only if it has no minor isomorphic to any of the matroids $U_{2,5}$, $U_{3,5}$, $M(K_4)$, P_7 or P_7^*. Moreover, M is a 3-connected ternary gammoid having at least 4 elements if and only if $M \cong \mathscr{W}^r$ for some $r \geq 2$.* □

2. Transversal matroids and their presentations

In this section we note some properties of transversal matroids that will be needed later in the paper. We begin with some observations concerning the presentations of transversal matroids. It is well-known that if $\mathscr{A} = (A_1, A_2, \ldots, A_n)$ is a presentation of a rank-r transversal matroid M and $\mathscr{A}' = (A_{i_1}, A_{i_2}, \ldots, A_{i_r})$ has a transversal, then $M = M[\mathscr{A}']$ (see, for example, [15, p. 244]). Using this, it follows easily that

Lemma 2.1. *If M is a rank-r transversal matroid having no coloops, then every presentation of M has exactly r non-empty sets.* □

If $\mathscr{A} = (A_1, A_2, \ldots, A_r)$ is a presentation for the transversal matroid M and e is an element of M that occurs in exactly one of $A_1, A_2, \ldots, A_r$, then we shall say that e *occurs exactly once in the presentation* $\mathscr{A}$.

The next lemma will be important in the proof of the main result of the next section.

Lemma 2.2. *Suppose that N_1 and N_2 are connected matroids such that $E(N_1) \cap E(N_2) = \{p\}$ and $|E(N_1)|$, $|E(N_2)| \geqslant 2$. Then $P((N_1, p), (N_2, p))$ is transversal if and only if N_1 and N_2 are both transversal having presentations in which p occurs exactly once.*

Proof. Suppose that $P((N_1, p), (N_2, p))$ is transversal and let N_1 have rank r_1 and N_2 have rank r_2. Then the parallel connection has rank $r_1 + r_2 - 1$. Let $\mathscr{A} = (A_1, A_2, \ldots, A_{r_1+r_2-1})$ be a presentation for this parallel connection. Then, for i in $\{1, 2\}$, N_i is transversal having $\mathscr{A}_i$ as a presentation where $\mathscr{A}_i = (A_1 \cap E(N_i), A_2 \cap E(N_i), \ldots, A_{r_1+r_2-1} \cap E(N_i))$. As N_i has no coloops, Lemma 2.1 implies that $\mathscr{A}_i$ has precisely r_i non-empty sets. We may therefore assume that the non-empty sets in $\mathscr{A}_1$ are $A_1 \cap E(N_1), A_2 \cap E(N_1), \ldots, A_{r_1} \cap E(N_1)$, and the non-empty sets in $\mathscr{A}_2$ are $A_{r_1} \cap E(N_2)$, $A_{r_1+1} \cap E(N_2), \ldots, A_{r_1+r_2-1} \cap E(N_2)$. Since $E(N_1) \cap E(N_2) = \{p\}$, it follows that $p \in A_{r_1}$ but p is in no other member of the family $\mathscr{A}$.

To prove the converse, we note that, as neither N_1 nor N_2 has any coloops, neither has a presentation using more sets than its rank. Let $(X_1, X_2, \ldots, X_{r_1} \dot{\cup} p)$ be a presentation for N_1 that uses p exactly once, and $(Y_1 \dot{\cup} p, Y_2, \ldots, Y_{r_2})$ be a presentation for N_2 that uses p exactly once. It is straightforward to check that $P((N_1, p), (N_2, p))$ is transversal having $(X_1, X_2, \ldots, X_{r_1} \dot{\cup} Y_1 \dot{\cup} p, Y_2, \ldots, Y_{r_2})$ as a presentation. We omit the details. □

One further tool that we shall require in the proofs of our main theorems is Brylawski's affine representation for transversal matroids [4]. If e is an element of the matroid M and F is a flat of $M \backslash e$, then e is *freely added* to F if $\operatorname{rk}(F \cup e) = \operatorname{rk} F$ and every circuit of M that contains e contains F in its closure. If $F = E(M \backslash e)$, we shall also say that e is *free* in M.

Let E^{n-1} denote $(n-1)$-dimensional Euclidean space. A matroid M on a set S is *free-simplicial* with *spanning simplex* B if there is an affinely independent subset $B=\{x_1, x_2, \ldots, x_n\}$ of E^{n-1} such that every element e of $S-B$ is freely added to some flat of $E^{n-1}|(B \cup (S-e))$ that is spanned by a subset of B. We observe that each element x_i of B may or may not be in S. The members of B are called the *vertices* of the spanning simplex. Evidently, two elements x and y are parallel in M if and only if, for some element x_i of B, $\{x, y\}$ is contained in the closure of $\{x_i\}$ in E^{n-1}. The following result of Brylawski [4, Corollary 3.1] means that transversal matroids can be treated geometrically.

Theorem 2.3. *A matroid is transversal if and only if it is free-simplicial.* □

3. Which ternary gammoids are transversal?

In this section we specify which ternary gammoids are transversal in terms of excluded series minors. Since we already have an excluded-minor characterization of ternary gammoids (Theorem 1.3), the combination of these two theorems gives one characterization of the class of ternary transversal matroids. In the next section we give a second such characterization that is entirely in terms of excluded series minors.

Before stating the main theorem of this section we note, for comparison, the corresponding result for binary gammoids. This result is easily obtained by combining the characterizations of binary transversal matroids [2, 10, 6] and binary gammoids [3, 4, 7].

Theorem 3.1. *A binary gammoid is transversal if and only if it has no series minor isomorphic to* $[P(kU_{2,3})\backslash p]^*$ *for any* $k \geq 3$. □

We observe here that $[P(kU_{2,3})\backslash p]^* \cong M(C_k^2)$. In the next theorem, $\mathcal{W}_+^3$ will denote the matroid that is obtained from $\mathcal{W}^3$ by adding an element in parallel to one of the rim elements.

Theorem 3.2. *A ternary gammoid is transversal if and only if it has no series minor isomorphic to* $\mathcal{W}_+^3$ *or* $[P(mU_{2,4}, nU_{2,3})\backslash p]^*$ *for any non-negative integers* m *and* n *with* $m+n \geq 3$.

Proof. Let M be a ternary gammoid and suppose that M is transversal. Then every series minor of M is transversal. Now it is easy to see that $\mathcal{W}_+^3$ is not free-simplicial. Hence, by Theorem 2.3, $\mathcal{W}_+^3$ is not transversal.

We continue the proof of Theorem 3.2 with a series of lemmas.

Lemma 3.3. *If $m+n \geq 3$, then $[P(mU_{2,4}, nU_{2,3})\backslash p]^*$ is not transversal.*

Proof. Assume that $[P(mU_{2,4}, nU_{2,3})\backslash p]^*$ is transversal. Then it is a free-simplicial matroid whose spanning simplex has $2m+n-1$ vertices. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let the ith copy of $U_{2,4}$ and the jth copy of $U_{2,3}$ in $P(mU_{2,4}, nU_{2,3})$ have ground sets $\{a_i, b_i, c_i, p\}$ and $\{s_j, t_j, p\}$ respectively. Now both $\{a_i, b_i, c_i\}$ and $\{s_j, t_j\}$ are cocircuits of $P(mU_{2,4}, nU_{2,3})\backslash p$. Hence both are circuits of $[P(mU_{2,4}, nU_{2,3})\backslash p]^*$. Thus, $\{s_j, t_j\}$ is contained in the closure of a vertex, say v_j, of the spanning simplex. Moreover, all of a_i, b_i and c_i lie on an edge, say u_iw_i, of the spanning simplex.

Since $m+n \geq 3$, no cocircuit of $P(mU_{2,4}, nU_{2,3})\backslash p$ meets exactly two of the sets $\{a_1, b_1, c_1\}, \ldots, \{a_m, b_m, c_m\}, \{s_1, t_1\}, \ldots, \{s_n, t_n\}$. It follows that the vertices $v_1, v_2, \ldots, v_n, u_1, w_1, u_2, w_2, \ldots, u_m, w_m$ are all distinct. Therefore the spanning simplex has at least $2m+n$ vertices; a contradiction. □

As M is transversal, but $\mathcal{W}^3_+$ and $[P(mU_{2,4}, nU_{2,3})\backslash p]^*$ $(m+n \geq 3)$ are not, we conclude that M has no series minor isomorphic to one of these matroids. To prove the converse, suppose that M is a ternary gammoid having no series minor isomorphic to $\mathcal{W}^3_+$ or $[P(mU_{2,4}, nU_{2,3})\backslash p]^*$ for any m and n with $m+n \geq 3$. We shall argue by induction on $|E(M)|$ to show that M is transversal. If $|E(M)| \leq 3$, this is certainly true. Assume it true for $|E(M)| < n$ and let $|E(M)| = n \geq 4$. Since the direct sum of transversal matroids is transversal, we may assume that M is connected.

Lemma 3.4. *If M is 3-connected, it is transversal.*

Proof. If M is 3-connected, it is a ternary 3-connected gammoid having at least four elements. By Theorem 1.3, M is a whirl. But all whirls are transversal (see, for example, [15, p. 241]). Therefore M is transversal. □

By the last lemma, we may assume that M is connected but not 3-connected.

Lemma 3.5. *Either M is transversal or $M = P(M_1, M_2)\backslash p$, where M_1 and M_2 are isomorphic to series minors of M and each has at least three elements and rank at least two.*

Proof. Assume that M is not transversal. By Theorem 1.1, since M is connected but not 3-connected, $M = P(M_1, M_2)\backslash p$, where $|E(M_1)|, |E(M_2)| \geq 3$. Moreover, M_1 and M_2 are both isomorphic to series minors of M. Now suppose that M_1 and M_2 cannot be chosen so that both have rank exceeding one. Then it is not difficult to show using duality and [14, (5.1)] that M is a parallel extension of a 3-connected matroid N. As M is a ternary gammoid, so is N. If $\operatorname{rk} N = 1$, then M is certainly transversal. If $\operatorname{rk} N = 2$, then $N \cong U_{2,4}$ and M is free-simplicial and hence is transversal unless it has $M(C_3^2)$ as a restriction. But $M(C_3^2) \cong [P(3U_{2,3})\backslash p]^*$ and this is excluded as a series minor of M. Thus if $\operatorname{rk} N = 2$, then M is transversal. If $\operatorname{rk} N = 3$, then $N \cong \mathcal{W}^3$ and M is free-simplicial and hence is transversal unless it has $\mathcal{W}_+^3$ as a restriction. But, by assumption, M has no series minor isomorphic to $\mathcal{W}_+^3$. Hence if $\operatorname{rk} N = 3$, M is transversal. Now suppose that $\operatorname{rk} N = r \geq 4$. Then $N \cong \mathcal{W}^r$ and M is transversal unless it has as a restriction the matroid that is obtained by adding an element in parallel to a rim element of $\mathcal{W}^r$. As the last matroid has $M(C_3^2)$ as a series minor, it cannot occur as a restriction of M. Hence M is transversal. This completes the proof of the lemma. □

By the last lemma, we may assume that $M = P((M_1, p), (M_2, p))\backslash p$ where both M_1 and M_2 have at least three elements and rank at least two. As M_1 and M_2 are isomorphic to proper series minors of M, the induction assumption implies that both are transversal. For the rest of the proof of Theorem 3.2, we shall concentrate on whether M_1 and M_2 have presentations in which p occurs exactly once. If both M_1 and M_2 have such presentations, then by Lemma 2.2, $P(M_1, M_2)$ is transversal, hence so is $P(M_1, M_2)\backslash p$. If one of M_1 and M_2 fails to have such a presentation, then we shall be able to determine the structure of the other and this information will enable us to complete the proof of the theorem.

Since the number of non-empty sets in a presentation of a transversal matroid N can only differ from $\operatorname{rk} N$ if N has coloops, the next two lemmas concern the existence of coloops in certain minors of M. The first two are quite straightforward.

Lemma 3.6. *If $e \in E(M)$ and $M\backslash e$ has a coloop, then M is transversal.*

Proof. If $M\backslash e$ has a coloop, then e is in a 2-cocircuit of M. In that case, M is a series extension of M/e. But, by the induction assumption, M/e is transversal. Hence M is transversal. □

Lemma 3.7. *If $M_1\backslash p$ or $M_2\backslash p$ has a coloop, then M is transversal.*

Proof. It suffices to consider the case when $M_2\backslash p$ has a coloop f. Then $\{p, f\}$ is a cocircuit of M_2, so $M_2\backslash f$ is disconnected. Since $M\backslash f = P(M_1, M_2\backslash f)\backslash p$, it follows that $M\backslash f$ is disconnected (see, for example, [11, (1.13)]). Therefore, by [3, Proposition 4.10], $M = S(M', M'')$ for some matroids M' and M''. Now both M' and M'' are isomorphic to proper series minors of M and so, by the induction assumption, both are transversal. As the union of transversal matroids is transversal (see, for example, [15, p. 243]), the series connection of transversal matroids is transversal. Hence M is transversal. □

Lemma 3.8. *If $e \in E(M_1) - \{p\}$ and $M_1\backslash e, p$ has no coloops, then M is transversal or M_2 has a presentation in which p occurs exactly once.*

Proof. Evidently $M\backslash e = P(M_1\backslash e, M_2)\backslash p$. Moreover, by the induction assumption, $M\backslash e$ is transversal. Let $\mathscr{A}$ be a presentation for $M\backslash e$. By Lemma 3.6, we can assume that $M\backslash e$ has no coloops. Therefore $\mathscr{A}$ has $r_1 + r_2 - 1$ non-empty sets where $r_i = \operatorname{rk} M_i$ for $i = 1, 2$. Now $M_1\backslash e, p$ has no coloops and, by the preceding lemma, we can suppose that neither $M_1\backslash p$ nor $M_2\backslash p$ has any coloops. Therefore when we restrict $\mathscr{A}$ to $E(M_1) - \{e, p\}$ and to $E(M_2) - \{p\}$, we get presentations for $M_1\backslash e, p$ and $M_2\backslash p$ that have r_1 and r_2 non-empty sets respectively. Let these presentations be $(A_1, A_2, \ldots, A_{r_1})$ and $(A'_{r_1}, A_{r_1+1}, \ldots, A_{r_1+r_2-1})$ respectively. Then, without loss of generality, we can assume that $\mathscr{A} = (A_1, A_2, \ldots, A_{r_1} \cup A'_{r_1}, A_{r_1+1}, \ldots, A_{r_1+r_2-1})$.

Now consider the transversal matroids N_1 and N_2 that have presentations $(A_1, A_2, \ldots, A_{r_1} \cup p)$ and $(A'_{r_1} \cup p, A_{r_1+1}, \ldots, A_{r_1+r_2-1})$. By the proof of Lemma 2.2, $P((N_1, p), (N_2, p))\backslash p = M[\mathscr{A}]$ and therefore $P((N_1, p), (N_2, p))\backslash p = P(M_1\backslash e, M_2)\backslash p$. If we now use the procedure described in the introduction to construct minors of $P(N_1, N_2)\backslash p$ and $P(M_1\backslash e, M_2)\backslash p$ isomorphic to N_2 and M_2 respectively, then, since $E(N_2) = E(M_2)$, we get that $N_2 = M_2$. Thus M_2 has a presentation in which p occurs exactly once. □

By the last lemma, if $M_1\backslash p$ and $M_2\backslash p$ have elements e_1 and e_2 respectively so that neither $M_1\backslash e_1, p$ nor $M_2\backslash e_2, p$ has any coloops, then M is transversal or both M_2 and M_1 have presentations in which p occurs exactly once. In the latter case, Lemma 2.2 implies that M is transversal. Therefore we may assume that, for all elements e of $E(M_1) - p$, $M_1\backslash e, p$ has a coloop. By Lemma 3.7, we may also assume that $M_1\backslash p$ has no coloops. Hence, if $e \in E(M_1) - \{p\}$, e is in a 2-cocircuit $\{e, e'\}$ of $M_1\backslash p$. If this 2-cocircuit is a cocircuit of M_1, it is also a cocircuit of M. In that case, $M\backslash e$ has a coloop and so, by Lemma 3.6, M is transversal. Thus we may assume that $\{e, e', p\}$ is a cocircuit of M_1. Hence, in M_1^*, every element is in a 3-circuit containing p. Hence, every element of M_1^* lies on a non-trivial line through p. As M_1^* is ternary every such line contains either three or four points. Suppose that there are t such lines.

Lemma 3.9. $\operatorname{rk} M_1^* = t + 1$.

The proof of this lemma will use the following result [13, Lemma 2.2].

Lemma 3.10. *Let N be a rank-3 ternary matroid having a 4-point line as a restriction. If N has at least 7 points and is not the parallel connection of two 4-point lines, then M has an $M(K_4)$-minor.* □

Proof of Lemma 3.9. The lemma is certainly true if t is 1 or 2. Now assume that $t \geq 3$ and consider M_1^*/p. It has t points, no loops, and rank equal to $\operatorname{rk} M_1^* - 1$. Let the points of M_1^*/p be $X_1, X_2, \ldots, X_t$. If $\operatorname{rk}(M_1^*/p) \leq t - 1$, then M_1^*/p has a circuit C that meets each of $X_1, X_2, \ldots, X_t$ in at most one element. Assume, without loss of generality, that this circuit is $\{x_1, x_2, \ldots, x_s\}$, where $x_i \in X_i$ for $1 \leqslant i \leqslant s$. Evidently $s \geqslant 3$. Now let $X = \{x_4, x_5, \ldots, x_s\}$, $Y = \bigcup_{i=4}^{s} (X_i - x_i)$ and $Z = \bigcup_{i=s+1}^{t} X_i$. Then $M_1^*/(p \cup X)\backslash(Y \cup Z)$ is a line having X_1, X_2 and X_3 as its points. Each of these points contains either 2 or 3 elements. Next consider $M_1^*/X\backslash(Y \cup Z)$. This matroid has rank 3, is simple and ternary and has $\{p\} \cup \{x : x \in X_1 \cup X_2 \cup X_3\}$ as its set of points. It therefore has at least 7 points. Since it is not the parallel connection of two 4-point lines, Lemma 3.10 implies that if it has a 4-point line, it has an $M(K_4)$-minor; a contradiction to Theorem 1.3. If it has no 4-point lines, it is a rank-3 7-point 3-connected ternary gammoid; this again contradicts Theorem 1.3. We conclude that $\mathrm{rk}(M_1^*/p) \geqslant t$ and so $\mathrm{rk}\, M_1^* \geqslant t+1$. But a matroid having t lines meeting at a point has rank at most $t+1$ with equality being attained exactly when the matroid is the parallel connection of the lines. Hence $\mathrm{rk}\, M_1^* = t+1$. □

As an immediate consequence of the last part of the preceding proof we have

Corollary 3.11. *For some non-negative integers m_1 and n_1 such that $m_1 + n_1 = t$, $M_1^* \cong P(m_1 U_{2,4}, n_1 U_{2,3})$.* □

Now that we know the structure of M_1, we consider M_2.

Lemma 3.12. *If, for all elements e of $E(M_2) - \{p\}$, the matroid $M_2\backslash e, p$ has a coloop, then M is transversal.*

Proof. Assume that M is not transversal. Then, arguing as above, we get that $M_2^* \cong P(m_2 U_{2,4}, n_2 U_{2,3})$ for some non-negative integers m_2 and n_2 with $m_2 + n_2 \geqslant 1$. Now $M^* = P(M_1^*, M_2^*)\backslash p$, hence $M \cong [P((m_1+m_2)U_{2,4}, (n_1+n_2)U_{2,3})\backslash p]^*$. Since the last matroid is excluded as a series minor of M when $m_1 + m_2 + n_1 + n_2 \geqslant 3$, we can assume that $m_1 + m_2 + n_1 + n_2 \leqslant 2$. Now $\mathrm{rk}\, M_i = 2m_i + n_i$ for i in $\{1, 2\}$. Moreover, $\mathrm{rk}\, M_i \geqslant 2$. Thus $m_1 = m_2 = 1$ and $n_1 = n_2 = 0$. Therefore $M \cong [P(U_{2,4}, U_{2,4})\backslash p]^* \cong P(U_{2,4}, U_{2,4})\backslash p$. As the last matroid is free-simplicial and is therefore transversal, M is transversal; a contradiction. □

By the preceding lemma, we can assume that there is an element e of $E(M_2) - \{p\}$ so that $M_2\backslash e, p$ has no coloops. Hence, by Lemma 3.8, M_1 has a presentation in which p occurs exactly once or M is transversal.

The next lemma shows that the first of these cannot occur and thereby completes the proof of Theorem 3.2.

Lemma 3.13. *M_1 has no presentation in which p occurs exactly once.*

To prove this we use the following

Lemma 3.14. *p is free in M_1.*

Proof. By [12, Lemma 2.2], as p is not a coloop of M_1, p is free in M_1 if and only if p is in every dependent flat of M_1^*. But, as $M_1^* \cong P(m_1U_{2,4}, n_1U_{2,3})$, the latter is easily seen to be true. □

Proof of Lemma 3.13. As M_1 has rank $2m_1+n_1$ and has no coloops, every presentation of it has exactly $2m_1+n_1$ non-empty sets. Assume that M_1 has such a presentation $(A_1, A_2, \ldots, A_{2m_1+n_1})$ in which p occurs in A_1 but in no other A_i. Now $M_1^* \cong P(m_1U_{2,4}, n_1U_{2,3})$. Let the ith copy of $U_{2,4}$ have ground set $\{a_i, b_i, c_i, p\}$ $(1 \leq i \leq m_1)$ and the jth copy of $U_{2,3}$ have ground set $\{s_j, t_j, p\}$ $(1 \leq j \leq n_1)$. As p is free in M_1, adding p to any independent set of $M_1\backslash p$ of size $\operatorname{rk} M_1 - 1$ gives a basis for M_1. Since p is in A_1 but no other A_i, every independent set of $M_1\backslash p$ of size $\operatorname{rk} M_1 - 1$ is a transversal of $(A_2, A_3, \ldots, A_{2m_1+n_1})$.

Now $\{p, a_1, a_2, \ldots, a_{m_1}, s_1, s_2, \ldots, s_{n_1}\}$ is a basis of M_1^*, so $\{b_1, c_1, b_2, c_2, \ldots, b_{m_1}, c_{m_1}, t_1, t_2, \ldots, t_{n_1}\}$ is a basis of M_1. Deleting any element from this basis gives a transversal for $(A_2, A_3, \ldots, A_{2m_1+n_1})$. As $\{a_i, b_i, c_i\}$ is a circuit of M_1, $a_i \notin A_1$ $(1 \leq i \leq m_1)$. By symmetry, neither b_i nor c_i is in A_1. Similarly, $\{s_j, t_j\}$ is a circuit of M_1, so $s_j \notin A_1$ $(1 \leq j \leq n_1)$ and, by symmetry, $t_j \notin A_1$. We conclude that $A_1 = \{p\}$. Thus p is a coloop of M_1; a contradiction. □

4. The excluded series minors for ternary gammoids

In this section we characterize the class of ternary gammoids in terms of excluded series minors. Using this and the main result of the last section, we then deduce an excluded-series-minor characterization for the class of ternary transversal matroids.

Theorem 4.1. *A matroid is a ternary gammoid if and only if it has no series minor isomorphic to $U_{2,5}$, $U_{k-2,k}$ for $k \geq 5$, F_7^*, P_7, or any of the fifteen matroids N for which N^* is a restriction of* PG(2, 3) *having $M(K_4)$ or P_7 as a restriction.*

Since the class of ternary gammoids is closed under duality, we can dualize this theorem to get an excluded-parallel-minor characterization of the class of ternary gammoids.

The determination of the fifteen non-isomorphic matroids N for which $\mathrm{PG}(2,3) \supseteq N^* \supseteq M(K_4)$ or P_7 is straightforward since each such N^* is uniquely determined by its complement in PG(2, 3) [5, p. 94]. In Table 1, we have listed the 15 possibilities for the complement $\mathrm{PG}(2,3) - N^*$ of N^*. When N^* has a common name, this has also been given. The empty matroid has been denoted by $\emptyset$.

Table 1. Possibilities for PG(2, 3) − N^* when PG(2, 3) $\supseteq N^* \supseteq M(K_4)$ or P_7

$\|E(N)\|$	PG(2, 3) − N^*	Common name for N^*
13	$\emptyset$	PG(2, 3)
12	$U_{1,1}$	
11	$U_{2,2}$	
10	$U_{2,3}$	
	$U_{3,3}$	
9	$U_{2,4}$	AG(2, 3)
	$U_{2,3} \oplus U_{1,1}$	
	$U_{3,4}$	
8	$U_{2,4} \oplus U_{1,1}$	
	$P(U_{2,3}, U_{2,3})$	
	$P(U_{2,4}\ U_{2,3})\backslash p$	
7	$P(U_{2,4}, U_{2,3})$	P_7
	$\mathcal{W}^3$	
	$M(K_4)$	Non-Fano
6	Non-Fano	$M(K_4)$

Proof of Theorem 4.1. Let the set of matroids listed in the theorem be Φ. Assume that M is a ternary gammoid. Then, by Theorem 1.2, M has no series minor isomorphic to any of $U_{2,5}$, $U_{k-2,k}$ for $k \geqslant 5$, or F_7^*. Now M^* is also a ternary gammoid. Hence, by Theorem 1.3, M^* has no minor isomorphic to $M(K_4)$, P_7 or P_7^*. It follows that no series minor of M is isomorphic to a member of Φ.

Now suppose that M has no series minor isomorphic to a member of Φ. We shall use Theorems 1.2 and 1.3 to show that M is a ternary gammoid.

Lemma 4.2. *M has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$.*

Proof. M has no series minor isomorphic to any of $U_{2,5}$, $U_{k-2,k}$ $(k \geqslant 5)$, F_7^* or $M(K_4)$. Thus M has no series minor isomorphic to any of $U_{2,5}$, $U_{k-2,k}$ $(k \geqslant 5)$, F_7^* or F_7, and so, by Theorem 1.2, M is ternary. Thus M has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$. □

Lemma 4.3. *M has no $M(K_4)$-minor.*

Proof. Suppose M has an $M(K_4)$-minor. Then M^* also has an $M(K_4)$-minor. Let T be a maximal subset of $E(M^*)$ for which $M^*/T\backslash U \cong M(K_4)$. Then M^*/T is a rank-3 loopless ternary matroid. Thus M^* has as a parallel minor a matroid N^* that is a restriction of PG(2, 3) and has $M(K_4)$ as a restriction. Hence M has a member of Φ as a series minor; a contradiction. □

By a similar argument to that just given we also have that

Lemma 4.4. *M has no P_7^*-minor.* □

It follows from Theorem 1.3 that the next lemma completes the proof that M is a ternary gammoid.

Lemma 4.5. *M has no P_7-minor.*

Proof. Suppose M does have a P_7-minor. Since M does not have a series minor isomorphic to P_7, it must have a non-trivial lift N of P_7 as a minor. But, by Lemma 4.3, M has no $M(K_4)$-minor. Thus, by [13, Lemmas 2.8–2.10], N is an extension of P_7^*. This contradicts the preceding lemma. □

On combining Theorems 3.2 and 4.1 we get the following

Corollary 4.6. *A matroid is ternary and transversal if and only if it has no series minor isomorphic to $U_{2,5}$, $U_{k-2,k}$ for $k \geq 5$, F_7^*, P_7, $\mathcal{W}_+^3$, $[P(mU_{2,4}, nU_{2,3})\backslash p]^*$ for $m+n \geq 3$, or any of the fifteen matroids N for which N^* is a restriction of* PG(2, 3) *having $M(K_4)$ or P_7 as a restriction.* □

We conclude the paper by noting that it is straightforward to modify the last result to get excluded-series-minor characterizations of which ternary matroids are transversal and which transversal matroids are ternary.

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