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James G. Oxley
Louisiana State University

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A Characterization of Certain Excluded-Minor Classes of Matroids

JAMES G. OXLEY

A result of Walton and the author establishes that every 3-connected matroid of rank and corank at least three has one of five six-element rank-3 self-dual matroids as a minor. This paper characterizes two classes of matroids that arise when one excludes as minors three of these five matroids. One of these results extends the author's characterization of the ternary matroids with no $M(K_4)$ -minor, while the other extends Tutte's excluded-minor characterization of binary matroids.

1. INTRODUCTION

Many important matroid results characterize the matroids with a certain basic property by a list of excluded minors [1, 9, 12, 13]. In this paper, we consider the problem of constructively characterizing various classes of matroids that are specified by a small list of excluded minors. In previous work on this problem, motivated by Tutte's wheels and whirls theorem [14], we have excluded low-rank wheels and whirls [4, 6, 7]. Here the choice of matroids to be excluded is motivated by the following theorem. Euclidean representations for the matroids P_6 and Q_6 appearing in the theorem are shown in Figure 1. We also denote by \mathcal{W}^3 and $M(\mathcal{W}_3)$ the rank-3 whirl and the cycle matroid of the three-spoked wheel, respectively.

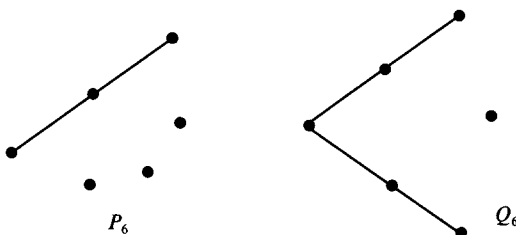


FIGURE 1.

(1.1) THEOREM [3, Theorem 2.5; 15, Lemma 3.3.7]. *Let M be a 3-connected matroid having rank and corank exceeding 2. Then M has a minor isomorphic to one of $M(\mathcal{W}_3)$, \mathcal{W}^3 , Q_6 , P_6 or $U_{3,6}$.*

Each of the five matroids listed in this theorem is 3-connected, has six elements and is isomorphic to its dual. Moreover, each of the last four matroids in the list can be obtained from its predecessor by *relaxing* a circuit-hyperplane. This operation creates a new matroid M' from an existing matroid M by taking as bases of M' all bases of M , together with a set X that is both a circuit and a hyperplane of M .

Theorem 1.1 raises the question as to what can be said about the structure of a class of matroids that is obtained by excluding as minors certain of the five matroids listed in the theorem. As all matroids that are not 3-connected can be built up from 3-connected ones by using the operations of direct sum and 2-sum [10, (2.10)], the structure of such a class is determined by its 3-connected members. If $EX(M(\mathcal{W}_3), \mathcal{W}^3, P_6, Q_6, U_{3,6})$ denotes the class of matroids obtained by excluding as minors all five of the listed matroids, a consequence of Theorem 1.1 is that the 3-connected members of this class are $U_{0,1}$, $U_{1,1}$ and $U_{1,2}$, together

with $U_{2,n}$ and $U_{n-2,n}$ for all $n \geq 3$. Moreover, Walton [15] showed that the 3-connected members of $EX(M(\mathcal{W}_3), \mathcal{W}^3, P_6, Q_6) - EX(M(\mathcal{W}_3), \mathcal{W}^3, P_6, Q_6, U_{3,6})$ are precisely the uniform matroids of rank and corank at least 3. The following result is a restatement of Theorem 3.1 of [5].

(1.2) THEOREM. *Let M be a 3-connected matroid having rank and corank at least 3. Then M is in $EX(\mathcal{W}^3, P_6, Q_6, U_{3,6})$ iff M is binary.*

The main results of this paper characterize the classes $EX(M(\mathcal{W}_3), P_6, Q_6)$ and $EX(\mathcal{W}^3, P_6, Q_6)$ by specifying their 3-connected members. Elsewhere [8], we use the latter characterization to determine the structure of $EX(\mathcal{W}^3, P_6)$. From that result and those of this paper, we can specify the structure of all the classes of matroids that are obtained by excluding four of $M(\mathcal{W}_3), \mathcal{W}^3, P_6, Q_6$ and $U_{3,6}$ except for $EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$. It is not clear why some of these classes are easier to characterize than others.

The matroid terminology used here will, in general, follow Welsh [16]. The ground set, rank and corank of the matroid M will be denoted by $E(M)$, $\text{rk } M$ and $\text{cork } M$, respectively. If $T \subseteq E(M)$, the deletion and contraction of T from M will be denoted by $M \setminus T$ and M/T . A *trivial* circuit of M is a circuit of M having exactly $\text{rk } M + 1$ elements. The basic results on matroid connectivity that are used here but are not explicitly stated may be found in the introduction of [4].

The characterization of $EX(M(\mathcal{W}_3), P_6, Q_6)$ that we shall prove here extends the following theorem, the main result of [4]. The matroids $S(5, 6, 12)$ and J appearing in this theorem are defined as follows: $S(5, 6, 12)$ is the rank-6 matroid on the set of 12 elements of the Steiner system $S(5, 6, 12)$, the hyperplanes of which are the blocks of the Steiner system; J is the eight-element rank-4 matroid for which a Euclidean representation is shown in Figure 2.

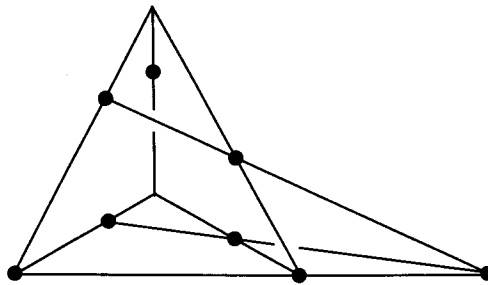


FIGURE 2.

(1.3) THEOREM. [4, Theorem 2.1]. *A matroid M is a 3-connected member of $EX(U_{2,5}, U_{3,5}, M(\mathcal{W}_3))$ iff*

- (i) $M \simeq W^r$ for some $r \geq 2$;
- (ii) $M \simeq J$; or
- (iii) M is isomorphic to a 3-connected minor of $S(5, 6, 12)$.

A complete list of the 3-connected minors of $S(5, 6, 12)$ is given in [4].

The next two theorems, the main results of this paper, will be proved in Section 2. In Section 3 some consequences of these theorems will be noted.

(1.4) THEOREM. *A matroid M is a 3-connected member of $EX(M(\mathcal{W}_3), P_6, Q_6)$ iff*

- (i) $M \simeq U_{r,n}$, where both r and $n - r$ exceed one; or
- (ii) M is a 3-connected member of $EX(M(\mathcal{W}_3), U_{2,5}, U_{3,5})$.

- (1.5) THEOREM. *A matroid M is a 3-connected member of $EX(\mathcal{W}^3, P_6, Q_6)$ iff*
- (i) *$M \simeq U_{r,n}$, where both r and $n - r$ exceed one; or*
 - (ii) *M is binary and 3-connected.*

The proofs of these theorems will use the following result, a straightforward combination of Lemmas 3.2 and 3.3 of [5].

(1.6) THEOREM. *The following statements are equivalent for a 3-connected matroid M having rank and corank at least 3:*

- (i) *M has a $U_{2,5}$ -minor;*
- (ii) *M has a $U_{3,5}$ -minor;*
- (iii) *M has a minor isomorphic to one of P_6, Q_6 or $U_{3,6}$.*

An interesting consequence of this last result comes from combining it with the next theorem. F_7 denotes the Fano matroid.

(1.7) THEOREM [1, 9]. *A matroid is ternary iff it has no minor isomorphic to $U_{2,5}, U_{3,5}, F_7$ or F_7^* .*

(1.8) COROLLARY. *A 3-connected matroid having rank at least 3 is ternary iff it has no minor isomorphic to $U_{3,5}, F_7$ or F_7^* .*

2. THE PROOFS

In this section we shall prove Theorems 1.4 and 1.5. The core of both proofs is the following result.

(2.1) THEOREM. *Let M be a 3-connected matroid having a $U_{2,5}$ -minor. Then either M is uniform or M has a minor isomorphic to P_6 or Q_6 .*

PROOF. Assume that M is not uniform. Then it follows that $\text{rk } M \geq 3$. Hence, by Theorem 1.6, we may also suppose that M has a $U_{3,6}$ -minor. Thus, by the splitter theorem [10, (7.3)], there is a sequence $M_0, M_1, M_2, \dots, M_n$ of 3-connected matroids such that $M_0 \simeq U_{3,6}, M_n = M$ and, for all i in $\{1, 2, \dots, n\}, M_{i-1}$ is a single-element deletion or contraction of M_i . Let $j = \min\{i: M_i \text{ is not uniform}\}$ and $M_j = N$. By duality, we may assume that $N \setminus e \simeq U_{k,n}$, where both k and $n - k$ exceed two. Evidently N has at least one non-trivial circuit and every such circuit contains e .

Suppose first that N has exactly one non-trivial circuit C . Contract $|C| - 3$ elements from C , keeping e . The resulting matroid N' has a unique non-trivial circuit; this circuit has three elements and contains e . Now contract $k - |C|$ elements and delete $n - k - 2$ elements from N' , keeping the three remaining elements of C . The resulting matroid has six elements, rank 3 and a unique non-trivial circuit, this having three elements. Therefore this matroid is isomorphic to P_6 .

We may now assume that e is in at least two non-trivial circuits of N . Let C be such a circuit. Contract $|C| - 3$ elements from it, keeping e . Let the resulting matroid be N_1 and the resulting circuit be C_1 . Then $N_1 \setminus e \simeq U_{s,t}$, where both s and $t - s$ exceed 2. If N_1 has C_1 as its only non-trivial circuit then, from above, N_1 has a P_6 -minor. Thus we may assume that N_1 has a non-trivial circuit C_2 that is different from C_1 and has minimum cardinality among such circuits. Evidently $e \in C_2$.

As $e \in C_1 \cap C_2$, by circuit elimination, N_1 has a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) - \{e\}$. Now, C_3 does not contain e and so is a trivial circuit. But $|C_1| = 3$, so $|C_3| \leq |C_2| + 1$.

Therefore $|C_3| = |C_2| + 1$ and so $|C_1 \cap C_2| = 1$. Hence $C_1 \cap C_2 = \{e\}$. Moreover, $\text{rk } N_1 = |C_2|$.

Let $C_2 = \{e, b_1, b_2, \dots, b_p\}$ and contract b_3, b_4, \dots, b_p from N_1 . The resulting matroid N_2 has $\{e, b_1, b_2\}$ as a circuit and has rank 3. Let $C_1 = \{e, a_1, a_2\}$. If C_1 is not a circuit of N_2 , then some proper subset D_1 of C_1 is a circuit of N_2 . Thus $D_1 \cup \{b_3, b_4, \dots, b_p\}$ contains a circuit D_2 of N_1 . Now $|D_2| \leq |C_2| - 1$, so D_2 is a non-trivial circuit. But this contradicts the choice of C_2 . Hence C_1 is indeed a circuit of N_2 .

Consider $N_2 \setminus e$. It is uniform of rank 3 and corank at least 2. Thus $N_2 \setminus e$ has a $U_{3,5}$ -minor N_3 the ground set of which contains $\{a_1, a_2, b_1, b_2\}$. We conclude that the restriction of N_2 to $E(N_3) \cup e$ is isomorphic to Q_6 . This completes the proof of the theorem. \square

PROOF OF THEOREM 1.4. It is straightforward to check that if (i) or (ii) holds, then M is a 3-connected member of $EX(M(\mathcal{W}_3), P_6, Q_6)$. Now suppose that M is a 3-connected member of $EX(M(\mathcal{W}_3), P_6, Q_6)$. If M is ternary then, by Theorem 1.7, $M \in EX(M(\mathcal{W}_3), U_{2,5}, U_{3,5})$. Thus we may assume that M is non-ternary. Hence, using Theorem 1.7 again and the fact that M has no $M(\mathcal{W}_3)$ -minor, we deduce that M has a minor isomorphic to $U_{2,5}$ or $U_{3,5}$. The theorem now follows by applying Theorem 2.1 and its dual.

PROOF OF THEOREM 1.5. Evidently, if M is listed in (i) or (ii), then it is a 3-connected member of $EX(\mathcal{W}^3, P_6, Q_6)$. Now suppose that M is a 3-connected member of $EX(\mathcal{W}^3, P_6, Q_6)$ and assume that M is non-binary. If $\text{rk } M = 2$ or $\text{cork } M = 2$, then M is uniform. Thus we may assume that both the rank and corank of M exceed 2. Then, by Theorem 1.2, M has a minor isomorphic to \mathcal{W}^3, P_6, Q_6 or $U_{3,6}$. The first three are excluded by assumption. Hence M has a $U_{3,6}$ -minor. As M has no P_6 - or Q_6 -minor, it follows by Theorem 2.1 that M is uniform. \square

3. SOME CONSEQUENCES

In this section we note some consequences of the main theorems. Evidently $M(\mathcal{W}_3) \simeq M(K_4)$. Using this, the next result follows easily from Theorems 1.3 and 2.1.

- (3.1) **COROLLARY.** *M is a 3-connected member of $EX(U_{2,5}, M(K_4))$ iff*
- (i) $M \simeq \mathcal{W}^r$ for some $r \geq 2$;
 - (ii) $M \simeq J$;
 - (iii) M is isomorphic to a 3-connected minor of $S(5, 6, 12)$; or
 - (iv) $M \simeq U_{n,n+2}$ for some $n \geq 3$.

Evidently, the dual of the last result gives a characterization of $EX(U_{3,5}, M(K_4))$, where we note that both J and $S(5, 6, 12)$ are isomorphic to their duals [4].

It was proved in [4, Theorem 5.1] that, for a simple rank- r member M of $EX(U_{2,5}, U_{3,5}, M(K_4))$,

$$|E(M)| \leq \begin{cases} 4r - 3 & \text{if } r \text{ is odd,} \\ 4r - 4 & \text{if } r \text{ is even.} \end{cases}$$

A straightforward consequence of the last corollary is that the same bound holds if M is a simple member of $EX(U_{2,5}, M(K_4))$. Indeed, the same proof that was given for $EX(U_{2,5}, U_{3,5}, M(K_4))$ works for the larger class. Moreover, the complete list of matroids in $EX(U_{2,5}, U_{3,5}, M(K_4))$ that attain this bound [4, Theorem 5.1] does not need to be augmented. A general bound on the number of elements in a simple member of $EX(U_{2,q+2}, M(K_4))$ has been derived by Kung [2].

We conclude by noting the following extension of Theorem 1.2 that is an immediate consequence of Theorem 1.5.

(3.2) COROLLARY. *Let M be a non-binary 3-connected matroid having rank and corank at least 3. Then either M is uniform or M has a minor isomorphic to \mathcal{W}^3 , P_6 or Q_6 .*

The last corollary can be strengthened by using the following result, the proof of which is an easy consequence of the main result of [11].

(3.3) LEMMA. *Let M be a 3-connected matroid and $\{x, y\}$ be a subset of $E(M)$. Suppose that M has a minor isomorphic to a member of $\{\mathcal{W}^3, P_6, Q_6\}$. Then M has such a minor whose ground set contains $\{x, y\}$.*

On combining the last two results, we immediately obtain the following result that strengthens Corollary 3.2 as well as Corollary 3.5 of [5].

(3.4) COROLLARY. *Let M be a non-binary 3-connected matroid having rank and corank at least 3 and suppose that $\{x, y\} \subseteq E(M)$. Then either M is uniform or M has a minor isomorphic to one of \mathcal{W}^3 , P_6 or Q_6 whose ground set contains $\{x, y\}$.*

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JAMES G. OXLEY
 Mathematics Department, Louisiana State University,
 Baton Rouge, LA 70803, U.S.A.