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NOTE

A NOTE ON NEGAMI'S POLYNOMIAL INVARIANTS FOR GRAPHS

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Negami has introduced two polynomials for graphs and proved a number of properties of them. In this note, it is shown that these polynomials are intimately related to the well-known Tutte polynomial. This fact is used, together with a result of Brylawski, to answer a question of Negami.

The matroid and graph terminology used here will follow Welsh [10] and Bondy and Murty [1] with the following differences. If M is a matroid, then $E(M)$ and r will denote its ground set and its rank function, respectively. If e is an element of M, then $M\backslash e$ and M/e will denote the deletion and contraction of e from M. The same notation will be used when deleting or contracting an edge from a graph G; the complement of G will be denoted $\bar{G}$.

The Tutte polynomial of a matroid M is defined by

$$T(M;x,y)=\sum_{A\subseteq E(M)}(x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}.$$

This polynomial was originally defined for graphs by Tutte [9], but Crapo [5] showed that the definition could be extended to matroids. Indeed, if G is a graph, its Tutte polynomial $T(G;x,y)$ is precisely $T(M(G);x,y)$ where $M(G)$ is the cycle matroid of G.

Negami [6] introduced two three-variable polynomials for graphs. These polynomials $f(G;t,x,y)$ and $f^*(G;t,x,y)$, or briefly $f(G)$ and $f^*(G)$, are defined recursively as follows:

$$f(\bar{K}_n)=t^n \quad \text{for all} \quad n\geqslant 1; \tag{1.1}$$

$$f(G)=yf(G\backslash e)+xf(G/e) \quad \text{for all} \quad e \text{ in } E(G); \tag{1.2}$$

and

$$f^*(\bar{K}_n)=t^n \quad \text{for all} \quad n\geqslant 1; \tag{2.1}$$

$$f^*(G)=xf^*(G\backslash e)+yf^*(G/e) \quad \text{for all edges } e \text{ that are not loops or cut edges;} \tag{2.2}$$

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$$f^*(G) = (x + ty)f^*(G\backslash e) \text{ if } e \text{ is a loop of } G; \tag{2.3}$$

$$f^*(G) = (x + y)f^*(G/e) \text{ if } e \text{ is a cut edge of } G. \tag{2.4}$$

Negami showed that one could determine the Tutte polynomial $T(G; x, y)$ of a graph G from the polynomial $f(G; t, x, y)$.

Theorem 1 [6, Theorem 2.11(ii)]. *Let G have $\omega(G)$ connected components. Then*

$$f(G; (x-1)(y-1), 1, y-1) = (y-1)^{|V(G)|}(x-1)^{\omega(G)}T(G; x, y).$$

In fact, as we now show, one can also determine both $f(G; t, x, y)$ and $f^*(G; t, x, y)$ from $T(G; x, y)$. This follows essentially by recognizing the functions $t^{-\omega(G)}f(G; t, x, y)$ and $t^{-\omega(G)}f^*(G; t, x, y)$ as generalized Tutte–Grothendieck invariants and then using the characterization of such invariants [8, Theorem 2.1]. However, the proof we give will rely only on properties of the Tutte polynomial.

Theorem 2.

$$f(G; t, x, y) = t^{\omega(G)}y^{r(M^*(G))}x^{r(M(G))}T\left(M(G); \frac{x+ty}{x}, \frac{x+y}{y}\right)$$

$$= \left(\frac{ty}{x}\right)^{\omega(G)}\left(\frac{x}{y}\right)^{|V(G)|}y^{|E(G)|}T\left(G; 1+\frac{ty}{x}, 1+\frac{x}{y}\right);$$

$$f^*(G; t, x, y) = t^{\omega(G)}x^{r(M^*(G))}y^{r(M(G))}T\left(M(G); \frac{x+y}{y}, \frac{x+ty}{x}\right).$$

Proof. Define

$$g(G; t, x, y) = t^{\omega(G)}y^{r(M^*(G))}x^{r(M(G))}T\left(M(G); \frac{x+ty}{x}, \frac{x+y}{y}\right).$$

Since the Tutte polynomial of the empty matroid is 1, $g(\bar{K}_n; t, x, y) = t^n$ for all $n \geq 1$. Moreover, by considering separately the cases when e is a loop, a coloop, and neither a loop nor a coloop, it is straightforward to show that, for all e in $E(G)$,

$$g(G; t, x, y) = yg(G\backslash e; t, x, y) + xg(G/e; t, x, y).$$

Thus $g(G; t, x, y)$ satisfies both (1.1) and (1.2). Since these uniquely determine $f(G; t, x, y)$ [6, Theorem 1.1], we conclude that

$$f(G; t, x, y) = g(G; t, x, y).$$

Similarly, if

$$g^*(G; t, x, y) = t^{\omega(G)}x^{r(M^*(G))}y^{r(M(G))}T\left(M(G); \frac{x+y}{y}, \frac{x+ty}{x}\right),$$

then it is routine to show that g^* satisfies (2.1)–(2.4). Since these conditions uniquely define f^* [6, Theorem 3.2], we conclude that $f^* = g^*$. □

Using the last result and the well-developed theory of Tutte–Grothendieck invariants for matroids [2, 4, 5, 8], one can quickly derive many of Negami's results. We now indicate briefly some examples of such derivations. The edge-connectivity of a connected graph G is the size of a smallest cocircuit in $M(G)$, or equivalently, the size of a smallest circuit in $M^*(G)$. Using [2, Proposition 7.5], one can immediately determine the size of a matroid's smallest circuit from its Tutte polynomial. Negami's Theorem 2.3 comes from combining this with Theorem 2 above.

Theorems 2.8 and 2.9 of [6] express the chromatic and flow polynomials of a graph G in terms of the polynomial f. To deduce these results from Theorem 2, we use the formulas for the chromatic and flow polynomials in terms of the Tutte polynomial of G. These formulas can be found in [9] or, more accessibly, in [10, Section 4] and [11, Theorem 1]. Theorems 3.1 and 3.3 of [6], which relate f and f^*, can be derived from Theorem 2 by using the fact that $T(M; x, y) = T(M^*; y, x)$ for any matroid M [5, Proposition 7]. Similarly, Theorem 5.1 of [6] follows from combining Theorem 2 with Whitney's 2-isomorphism theorem [12, 7].

We conclude this note by observing that Brylawski [3, Corollary 4.7] has proved the existence of pairs of graphs of arbitrarily high connectivity that have the same Tutte polynomial but are not isomorphic. On combining this fact with Theorem 2, one deduces that $f(G; t, x, y)$ is not a complete invariant for 6-connected graphs, thereby answering a question of Negami [6, p. 622].

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