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On Minors Avoiding Elements in Matroids

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Let $\mathcal{F}$ be a collection of 3-connected matroids, none a proper minor of another, such that if M is a 3-connected matroid having a proper $\mathcal{F}$-minor and e is an element of M, then M has an $\mathcal{F}$-minor avoiding e. This paper proves that there are precisely two collections $\mathcal{F}$ with this property: $\{U_{2,4}\}$ and $\{U_{2,4}, M(K_4)\}$. Several extensions of this result and some similar results for 2-connected matroids are also established.

1. INTRODUCTION

The problem of relating the minors of a matroid to particular elements of the matroid has received considerable research attention recently (see, for example, [1–3, 5–10, 12–16, 18-21]). This paper addresses a new aspect of this problem. Let $\mathcal{F}$ be a collection of matroids. A matroid M' is an *$\mathcal{F}$-minor* of the matroid M if M' is a minor of M that is isomorphic to some member of $\mathcal{F}$. If X is a set of elements of M, then M' *uses* X if $X \subseteq E(M')$. Let k and l be integers with $k \geqslant 2$ and $l \geqslant 1$. Then $\mathcal{F}$ is *(k, l)-rounded* [3] if every member of $\mathcal{F}$ is k-connected [22] having at least four elements and the following condition holds:

(1.1) *If M is a k-connected matroid having an $\mathcal{F}$-minor and X is a subset of $E(M)$ with at most l elements, then M has an $\mathcal{F}$-minor using X.*

Bixby [2] and Seymour [18], respectively, proved that $\{U_{2,4}\}$ is $(2,1)$- and $(3,2)$-rounded. Moreover, numerous other collections of matroids have been shown to possess roundedness properties. Let $\mathcal{W}_r$ denote the r-spoked wheel graph and $\mathcal{W}^r$ the rank-r whirl [23, pp. 80–81]. Then $\mathcal{W}^2 \cong U_{2,4}$. Wheels and whirls feature prominently in several rounded collections. For instance, $\{\mathcal{W}^2, \mathcal{W}^3\}$ is $(3,3)$-rounded [8], $\{\mathcal{W}^2, M(\mathcal{W}_3)\}$ and $\{\mathcal{W}^2, M(\mathcal{W}_4)\}$ are $(3,2)$-rounded [19, 14], and $\{\mathcal{W}^2, M(\mathcal{W}_2)\}$ is $(2,2)$-rounded.

The basic concern with roundedness is in being able to force elements of matroids into minors. The main concern of this paper will be in getting minors that avoid elements. The following definition mimics the definition of a (k, l)-rounded set given above. Let k and m be integers with $k \geqslant 2$ and $m \geqslant 0$, and let $\mathcal{F}$ be a collection of k-connected matroids each having at least four elements. Then $\mathcal{F}$ is *(k, m)-avoiding* if the following condition holds:

(1.2) *If M is a k-connected matroid having a proper $\mathcal{F}$-minor, then there are at most m elements of M that are in every $\mathcal{F}$-minor of M.*

This paper investigates the properties of (k, m)-avoiding sets of matroids when $k \in \{2, 3\}$. In particular, for such k and all m in $\{0, 1, \ldots, k-1\}$, we shall determine all (k, m)-avoiding sets of matroids. The matroid terminology used here will follow Oxley [11]. A three-element circuit of a matroid M will be called a *triangle* and a three-element cocircuit a *triad*. The property that M has no circuit and cocircuit with exactly one common element will be referred to as *orthogonality*. If M_1 and M_2 are matroids on the sets S and $S \cup e$, where $e \notin S$, then M_2 is an *extension* of M_1 if $M_2 \backslash e = M_1$, and M_2 is a *coextension* of M_1 if $M_2/e = M_1$.

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In Section 2, we shall examine the relationship between rounded and avoiding sets. Moreover, we shall describe a quick test for deciding when a set of matroids is (k, m)-avoiding. Section 3 concentrates on (k, m)-avoiding sets for $k = 2$, while Section 4 examines such sets when $k = 3$.

2. Rounded and Avoiding Sets

We shall focus here on (k, l)-rounded and (k, m)-avoiding sets when k is 2 or 3. The study of such sets when $k = 2$ will use the following result of Brylawski [4] and Seymour [16].

THEOREM 2.1. *Let N be a 2-connected minor of a 2-connected matroid M and suppose that $e \in E(M) - E(N)$. Then at least one of $M \backslash e$ and M/e is 2-connected and has N as a minor.*

The next theorem contains a quick test for (k, l)-roundedness when k is 2 or (k, l) is $(3, 1)$ or $(3, 2)$. For $k = 2$, the test is straightforward to derive from the last theorem. For $(k, l) = (3, 1)$, it follows from the test for $(k, l) = (2, 1)$ and from the fact that a $(2, 1)$-rounded collection of matroids is $(3, 1)$-rounded iff all its members are 3-connected. For $(k, l) = (3, 2)$, the theorem was proved by Seymour [19].

THEOREM 2.2. *Suppose that $k = 2$ and $l \geqslant 1$, or that (k, l) is $(3, 1)$ or $(3, 2)$. Let $\mathscr{F}$ be a collection of k-connected matroids each having at least four elements. Then $\mathscr{F}$ is (k, l)-rounded iff the following condition holds: if M is a k-connected matroid that is an extension or coextension of a member of $\mathscr{F}$, and X is a subset of $E(M)$ having at most l elements, then M has an $\mathscr{F}$-minor using X.*

For larger values of l, Bixby and Coullard [3] have given similar results for testing whether a collection of matroids is $(3, l)$-rounded, although these rely on looking at 3-connected matroids M that have $\mathscr{F}$-minors with one, two or three fewer elements than M.

We shall show next that when k is 2 or 3, there are tests for (k, m)-avoidance that are similar to the last result. When $k = 3$, the proof uses the following version of Seymour's splitter theorem [17, (7.3)].

THEOREM 2.3. *Let N be a 3-connected proper minor of a 3-connected matroid M such that $|E(N)| \geqslant 4$. Suppose that if N is a wheel, then M has no larger wheel as a minor, while if N is a whirl, N has no larger whirl as a minor. Then M has a 3-connected minor M' and an element e such that $M' \backslash e$ or M'/e is isomorphic to N.*

THEOREM 2.4. *Suppose that $k \in \{2, 3\}$, m is a non-negative integer, and $\mathscr{F}$ is a collection of k-connected matroids each having at least four elements. Then $\mathscr{F}$ is (k, m)-avoiding iff the following condition holds: if M is a k-connected matroid that is an extension or coextension of a member of $\mathscr{F}$, then there are at most m elements that are in every $\mathscr{F}$-minor of M.*

PROOF. Evidently is suffices to prove that the specified condition guarantees that $\mathscr{F}$ is (k, m)-avoiding. Thus assume that this condition holds. If $k = 2$, then the result follows easily by Theorem 2.1. Now suppose that $k = 3$ and let J be a 3-connected matroid having an $\mathscr{F}$-minor N such that $|E(J) - E(N)| \geqslant 2$. Then, by Theorem 2.3, either (i) N is a wheel or whirl and J has a larger wheel or whirl as a minor; or (ii) J has a 3-connected minor J' having an N-minor N' such that $|E(J') - E(N')| = 1$. In the latter case, there are at most m elements that are in every $\mathcal{F}$-minor of J', so there are at most m elements of J that are in every $\mathcal{F}$-minor of J.

Now suppose that (i) occurs and let J' be a wheel or whirl that is a minor of J and has a proper N-minor. Then, by the symmetry of wheels and whirls, no element of J' is in every N-minor of J'. Hence, for all $m \geq 0$, there are at most m elements of J that are in every $\mathcal{F}$-minor of J. □

Let $\mathcal{F}$ be a collection of k-connected matroids each having at least four elements. Clearly, if $\mathcal{F}$ is (k, m)-avoiding and $m' \geq m$, then $\mathcal{F}$ is (k, m')-avoiding. Moreover, $\mathcal{F}$ is (k, m)-avoiding iff $\mathcal{F}^*$ is (k, m)-avoiding, where $\mathcal{F}^* = \{M^*: M \in \mathcal{F}\}$. Now suppose that M_1 and M_2 are members of $\mathcal{F}$ such that M_1 is a proper minor of M_2. Then $\mathcal{F}$ is (k, m)-avoiding iff $\mathcal{F} - \{M_2\}$ is (k, m)-avoiding. In view of the last observation, we shall frequently restrict attention to (k, m)-avoiding sets in which no member is a proper minor of another.

Next we indicate a relationship between (k, l)-rounded sets and (k, m)-avoiding sets. Here, and throughout the rest of this paper, we shall assume that $k \in \{2, 3\}$.

PROPOSITION 2.5. *Suppose that $\mathcal{F}$ is $(k, n-m)$-rounded, where $n = \max\{|E(M)|: M \in \mathcal{F}\}$. Assume that $\mathcal{F}$ does not contain two members M_1 and M_2 such that M_1 is isomorphic to an extension or coextension of M_2. Then $\mathcal{F}$ is (k, m)-avoiding.*

PROOF. Let M be a k-connected matroid having an $\mathcal{F}$-minor N such that $|E(M) - E(N)| = 1$. Then $M \notin \mathcal{F}$. Suppose that X is a subset of $E(M)$ that is contained in every $\mathcal{F}$-minor of M, and assume that $|X| \geq m + 1$. As $|E(M)| \leq n + 1$, it follows that $|E(M) - X| \leq n - m$. Since $\mathcal{F}$ is $(k, n-m)$-rounded, M has an $\mathcal{F}$-minor N_1 that contains $E(M) - X$. Now $N_1 \neq M$, so N_1 avoids some element of X; a contradiction. Therefore $|X| \leq m$, and so, by Theorem 2.4, $\mathcal{F}$ is (k, m)-avoiding. □

On applying the last result to the various rounded collections noted in the introduction, we deduce that $\{\mathcal{W}^2\}$ is both $(2, 3)$- and $(3, 2)$-avoiding; $\{\mathcal{W}^2, \mathcal{W}^3\}$ is $(3, 3)$-avoiding; $\{\mathcal{W}^2, M(\mathcal{W}_3)\}$ is $(3, 4)$-avoiding; $\{\mathcal{W}^2, M(\mathcal{W}_4)\}$ is $(3, 6)$-avoiding; and $\{\mathcal{W}^2, M(\mathcal{W}_2)\}$ is $(2, 2)$-avoiding. We shall see in Section 4 that stronger results than these can be established for $\{\mathcal{W}^2\}$ and $\{\mathcal{W}^2, M(\mathcal{W}_3)\}$. Moreover, the above observation concerning $\{\mathcal{W}^2, \mathcal{W}^3\}$ can be immediately improved: since $\{\mathcal{W}^2\}$ is $(3, 2)$-avoiding, so too is $\{\mathcal{W}^2, \mathcal{W}^3\}$.

For reference, we note a number of other examples of rounded sets that are also avoiding sets. Seymour [16] showed that $\{U_{2,5}, U_{3,5}, F_7, F_7^*\}$ and $\{U_{2,4}, F_7, F_7^*\}$, the collections of minor-minimal non-ternary and minor-minimal non-regular matroids, are $(2, 1)$-rounded. Hence these collections are both $(2, 6)$-avoiding. Moreover, as all the matroids in these sets are 3-connected, they are also $(3, 1)$-rounded and hence are $(3, 6)$-avoiding. Let P_6 and Q_6 be the matroids for which geometric representations are shown in Figure 1. Then all of $\{\mathcal{W}^3, P_6, Q_6, U_{3,6}\}$, $\{\mathcal{W}^3, P_6, Q_6\}$ and $\{U_{2,6}, U_{4,6}, \mathcal{W}^3, P_6, Q_6, U_{3,6}\}$ are $(3, 2)$-rounded [8, 10, 13]. Hence all are $(3, 4)$-

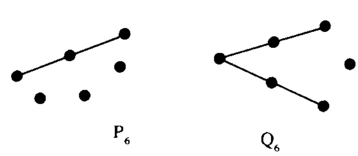

FIGURE 1.

avoiding. Furthermore, $\{M(\mathcal{W}_3), \mathcal{W}^3, P_6, Q_6, U_{3,6}\}$ is $(3,3)$-rounded [8] and is therefore $(3,3)$-avoiding.

To show that the hypothesis of Proposition 2.5 cannot be weakened by omitting the second sentence, consider the following example. The only 6-element 3-connected matroids having $U_{2,5}$ as a minor are $U_{2,6}$, $U_{3,6}$, P_6 and Q_6. Using this, together with the fact that $\{\mathcal{W}^3, P_6, Q_6\}$ is $(3,2)$-rounded, we deduce that $\{U_{2,5}, \mathcal{W}^3, P_6, Q_6\}$ is $(3,2)$-rounded. If Proposition 2.5 holds with the second sentence omitted, then $\{U_{2,5}, \mathcal{W}^3, P_6, Q_6\}$ is $(3,4)$-avoiding. But this is true iff $\{U_{2,5}, \mathcal{W}^3\}$ is $(3,4)$-avoiding. To see that the latter does not hold, it suffices to note that five of the elements of Q_6 are in every minor of it that is isomorphic to a member of $\{U_{2,5}, \mathcal{W}^3\}$.

At the end of Section 4, we shall show that the converse of Proposition 2.5 does not hold, although a partial converse does.

3. On $(2, m)$-avoiding Sets

The purpose of this section is to prove the following result which characterizes all $(2,0)$- and $(2,1)$-avoiding sets of matroids.

Theorem 3.1. *Let $\mathcal{F}$ be a collection of 2-connected matroids that is closed under isomorphism and suppose that each member of $\mathcal{F}$ has at least four elements. Then the following statements are equivalent*:

(i) *$\mathcal{F}$ is $(2, 0)$-avoiding*;

(ii) *$\mathcal{F}$ is $(2, 1)$-avoiding*;

(iii) *for some $n \geq 4$, $\mathcal{F}$ contains all 2-connected matroids having exactly n elements, and $\mathcal{F}$ has no members with fewer than n elements.*

The proof of this theorem will use the following well-known result of Tutte [22], which can be deduced from Theorem 2.1 by taking N to be $U_{0,0}$.

Lemma 3.2. *Let e be an element of a 2-connected matroid M. Then $M\backslash e$ or M/e is 2-connected.*

To see that (iii) of (3.1) is not equivalent to $\mathcal{F}$ being $(2,2)$-avoiding, recall from the preceding section that $\{\mathcal{W}^2, M(\mathcal{W}_2)\}$ is $(2,2)$-avoiding, whereas this set clearly does not contain all 2-connected 4-element matroids. The following is an immediate consequence of Theorem 3.1.

Corollary 3.3. *Let $\mathcal{F}$ be a collection of 2-connected matroids that is closed under isomorphism. Suppose that each member of $\mathcal{F}$ has at least four elements and that no member of $\mathcal{F}$ is a proper minor of another. Let m be in $\{0, 1\}$. Then $\mathcal{F}$ is $(2, m)$-avoiding iff, for some $n \geq 4$, $\mathcal{F}$ consists of all 2-connected n-element matroids.*

Proof of Theorem 3.1. Evidently (i) implies (ii). Now suppose that (iii) holds and let M be a 2-connected matroid having exactly $n+1$ elements. If $e \in E(M)$, then, by Lemma 3.2, $M\backslash e$ or M/e is 2-connected. Hence e is not in every $\mathcal{F}$-minor of M. It follows by Theorem 2.4 that (i) holds.

It remains to show that (ii) implies (iii). Thus assume that (ii) holds. Now, from among the members of $\mathcal{F}$ with the least number of elements, choose one, say N, in which the largest parallel class if of maximum size. Let P be a maximum-sized parallel class in N, and let $N+e$ be formed by adding e to N so that it is parallel with all the elements of P. Suppose that $r(N) \geq 2$. Then N has at least two elements, say f and g, that are not in P. Now all of $(N+e)\backslash f$, $(N+e)/f$, $(N+e)\backslash g$ and $(N+e)/g$ have parallel classes containing $P \cup e$. Thus none of these matroids is in $\mathcal{F}$. Hence f and g are in every $\mathcal{F}$-minor of $N+e$. This contradiction implies that $r(N)=1$. Hence $N \cong U_{1,n}$ for some $n \geq 4$, and every member of $\mathcal{F}$ has at least n elements.

Now suppose that $\mathcal{F}$ does not contain all 2-connected n-element matroids. Let M be a 2-connected n-element matroid that is not in $\mathcal{F}$ so that, among all such matroids, M has its largest parallel class of maximum size. Let Q be a maximum-sized parallel class of M. Let $M+e$ be formed from M by adding e in parallel with all the elements of Q. Choose f in Q. Then neither $(M+e)/e$ nor $(M+e)/f$ is in $\mathcal{F}$ since each has a loop. Moreover, neither $(M+e)\backslash e$ nor $(M+e)\backslash f$ is in $\mathcal{F}$ since each is isomorphic to M. Therefore every $\mathcal{F}$-minor of $M+e$ contains $\{e, f\}$. But, since $\mathcal{F}$ is (2, 1)-avoiding, it follows that $M+e$ has no $\mathcal{F}$-minor. As $U_{1,n} \in \mathcal{F}$, $M \not\cong U_{1,n}$, so $r(M) \geq 2$ and M has an element g that is not in Q. Thus, by Lemma 3.2, for some M_1 in $\{(M+e)\backslash g, (M+e)/g\}$, M_1 is 2-connected. Clearly M_1 has a parallel class containing $Q \cup e$. It follows, by the choice of M, that $M_1 \in \mathcal{F}$. This contradiction to the fact that $M+e$ has no $\mathcal{F}$-minor completes the proof that (ii) implies (iii). □

4. On $(3, m)$-avoiding Sets

The main result of this section is the following analogue of Theorem 3.1 for $(3, m)$-avoiding sets of matroids.

Theorem 4.1. *Let $\mathcal{F}$ be a collection of 3-connected matroids that is closed under isomorphism. Suppose that each member of $\mathcal{F}$ has at least four elements and that no member of $\mathcal{F}$ is a proper minor of another. Then the following statements are equivalent*:
(i) *$\mathcal{F}$ is (3, 0)-avoiding*;
(ii) *$\mathcal{F}$ is (3, 1)-avoiding*;
(iii) *$\mathcal{F}$ is (3, 2)-avoiding*;
(iv) *$\mathcal{F}$ is $\{\mathcal{W}^2\}$ or $\{\mathcal{W}^2, M(\mathcal{W}_3)\}$.*

The proof of this theorem will use the next two results, which are stronger than is needed just to prove the theorem but are of some independent interest.

Theorem 4.2. *Let $\mathcal{F}$ be a (3, 2)-avoiding set of matroids that is closed under isomorphism. Then $\mathcal{W}^2 \in \mathcal{F}$.*

Proof. From among the members of $\mathcal{F}$ with the least number of elements, choose one, say N, in which the maximum cardinality of a line is as large as possible. Let L be a line of N of maximum cardinality. Now form $N+e$ by placing the element e freely on the line L. Suppose that $r(N) > 2$. Then we can choose three distinct elements x, y and z from $E(N) - L$. If $f \in \{x, y, z\}$, then neither $(N+e)\backslash f$ nor $(N+e)/f$ is a member of $\mathcal{F}$ since each has a line containing $L \cup e$. Thus $\{x, y, z\}$ is contained in every $\mathcal{F}$-minor of $N+e$. This contradiction to the fact that $\mathcal{F}$ is (3, 2)-avoiding implies thast $r(N)=2$. Thus $N \cong U_{2,n}$ for some $n \geq 4$.

If $n=5$, then $U_{2,5} \in \mathcal{F}$. But Q_6 has a $U_{2,5}$-minor and there are four elements that are in every 3-connected minor of Q_6 with at least five elements. Since $\mathcal{F}$ is (3, 2)-avoiding, it follows that $n \neq 5$.

Now suppose that $n \geq 6$ and let Q_n denote the rank-3 n-element matroid for which a geometric representation is shown in Figure 2. Let M be the matroid obtained from Q_n by adding the element e freely on the line L. Then $M/n \in \mathcal{F}$. Now, for all x in

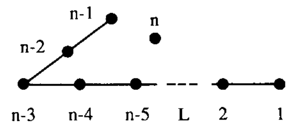

FIGURE 2.

$\{n-2, n-1\}$, neither $M\backslash x$ nor M/x is in $\mathscr{F}$ since neither is 3-connected. But $\mathscr{F}$ is $(3,2)$-avoiding. Thus $n-2$ and $n-1$ are the two elements that are in every $\mathscr{F}$-minor of M. Consequently, one of $M\backslash e$ and M/e has an $\mathscr{F}$-minor and so belongs to $\mathscr{F}$. As M/e is not simple, it is not in $\mathscr{F}$. Hence $M\backslash e \in \mathscr{F}$; that is, $Q_n \in \mathscr{F}$.

Let M' be the 3-connected, rank-4, $(n+1)$-element matroid for which a geometric representation is shown in Figure 3. Clearly $M'/e \simeq Q_n$, so M' has an $\mathscr{F}$-minor. But each of $n-3$, $n-2$ and $n-1$ is in both a triangle and a triad of M'. Since every member of $\mathscr{F}$ has at least n elements, it follows that every $\mathscr{F}$-minor of M' contains $\{n-3, n-2, n-1\}$. This contradiction to the fact that $\mathscr{F}$ is $(3,2)$-avoiding implies that $n<6$. Since $n \neq 5$, we conclude that $n=4$; that is, $\mathscr{W}^2 \in \mathscr{F}$. □

THEOREM 4.3. *Let $\mathscr{F}$ be a $(3, m)$-avoiding set of matroids that is closed under isomorphism and suppose that no member of $\mathscr{F}$ is a proper minor of another. If $\mathscr{W}^2 \in \mathscr{F}$, then either $\mathscr{F}$ is $\{\mathscr{W}^2\}$ or $\{\mathscr{W}^2, M(\mathscr{W}_3)\}$, or $m \geq 6$.*

The proof of this theorem is very similar to the proof of [12, Theorem 1.4]. In particular, we shall use the notion of a chain of triangles and triads in a matroid. Let $T_1, T_2, \ldots, T_t$ be a non-empty sequence of sets, each of which is a triangle or a triad of a matroid M such that, for all i in $\{1, 2, \ldots, t-1\}$,
(i) exactly one of T_i and T_{i+1} is a triangle,
(ii) $|T_i \cap T_{i+1}| = 2$, and
(iii) $(T_{i+1} - T_i) \cap (T_i \cup T_2 \cup \cdots T_i)$ is empty.
Then $T_1, T_2, \ldots, T_t$ is called a *chain* of M of *length t*. Evidently $T_1, T_2, \ldots, T_t$ is a chain of M iff it is a chain of M^*.

PROOF OF THEOREM 4.3. Suppose that $m \leq 5$, let $\mathscr{F}' = \mathscr{F} - \{\mathscr{W}^2\}$, and suppose that $\mathscr{F}'$ is non-empty. Then, by Tutte's excluded-minor characterization of binary matroids (see, for example, [23, p. 167]), every member of $\mathscr{F}'$ is binary. Now let N be a smallest 3-connected member of $\mathscr{F}'$. Then, by Theorem 2.3, it follows that N has a 3-connected wheel as a minor. Thus $|E(N)| \geq 6$. Moreover, if equality holds here, then $N \simeq M(\mathscr{W}_3)$. But in that case, since Theorem 2.3 implies that every 3-connected matroid with at least four elements has a $\mathscr{W}^2$-or $M(\mathscr{W}_3)$-minor, we deduce that

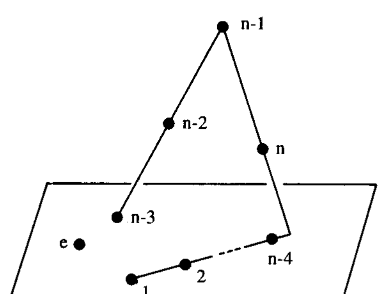

FIGURE 3.

$\mathcal{F} = \{\mathcal{W}^2, M(\mathcal{W}_3)\}$. Hence we may assume that $|E(N)| \geq 7$.

Suppose that $|E(N)| = 7$. Then $N \cong F_7$ or F_7^*. Let S_8 denote the matroid that is represented by the following matrix over $GF(2)$:

$$\left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \end{array}\right].$$

Then one easily checks that S_8 has only two elements x for which $S_8\backslash x$ or S_8/x is 3-connected. Therefore there are at least six elements that are in every $\mathcal{F}$-minor of S_8; a contradiction. Hence $|E(N)| \geq 8$.

Now let $r(N) = r$ and let $V(r, 2)$ denote the matroid of the r-dimensional vector space over $GF(2)$. Evidently, we may identify N with the restriction of $V(r, 2)$ to some set S. Suppose that no smallest member of $\mathcal{F}'$ has a triangle. Let a and b be distinct elements of N, and let e be the third element on the line of $V(r, 2)$ that is spanned by $\{a, b\}$. Let $N' = V(r, 2) \mid (S \cup e)$. Since $|E(N')| \geq 9$ and $m \leq 5$, there is an element x of $E(N') - \{a, b, e\}$ such that x is not in every $\mathcal{F}$-minor of N'. Because N' is binary, every $\mathcal{F}$-minor of N' is in $\mathcal{F}'$. Moreover, by the choice of N, no $\mathcal{F}'$-minor of N' has fewer than $|E(N')| - 1$ elements. Thus $N'\backslash x$ or N'/x is in $\mathcal{F}'$. But both of the last two matroids have $\{a, b, e\}$ as a triangle. This contradiction implies that $\mathcal{F}'$ does indeed have a smallest member having a triangle.

From among the smallest members of $\mathcal{F}'$, choose one, say M, in which the longest chain is of maximum length. Let $T_1, T_2, \ldots, T_t$ be such a maximum-length chain in M, and let $r(M) = r_1$. Clearly, we can identify M with the restriction of $V(r_1, 2)$ to some set E. Now a straightforward argument using orthogonality and 3-connectivity gives that $T_1 \cup T_2 \cup \cdots \cup T_t$ has $t + 2$ distinct elements $a_1, a_2, \ldots, a_{t+2}$ such that, for all i in $\{1, 2, \ldots, t\}$, $T_i = \{a_i, a_{i+1}, a_{i+2}\}$. We may assume that T_t is a triad of M, for otherwise we replace M by M^* in the argument that follows.

Let v be the third element of the line of $V(r_1, 2)$ that is spanned by $\{a_{t+1}, a_{t+2}\}$. We shall show that v belongs to E. Assume the contrary and let M' be $V(r_1, 2) \mid (E \cup e)$. Suppose that $\{a_t, a_{t+1}, a_{t+2}\}$ is not a triad of M'. Then $\{a_t, a_{t+1}, a_{t+2}, v\}$ is a cocircuit of M' meeting the triangle $\{a_{t+1}, a_{t+2}, v\}$ in exactly three elements. Since this contradicts the fact that M' is binary, we deduce that $\{a_t, a_{t+1}, a_{t+2}\}$ is indeed a triad of M'. Thus, it follows by orthogonality that, for all j in $\{1, 2, \ldots, t\}$, if T_j is a triad of M, then it is a triad of M'. Thus all of $a_2, a_3, \ldots, a_{t+2}$ are in both a triangle and a triad of M'. Hence every such element f is in every $\mathcal{F}$-minor of M' since neither $M'\backslash f$ nor M'/f is 3-connected.

Suppose that, for some element g of $E(M') - \{a_1, a_2, \ldots, a_{t+2}, v\}$, there is an $\mathcal{F}'$-minor of M' avoiding g. Then, for some M_1 in $\{M'\backslash g, M'/g\}$, M_1 is in $\mathcal{F}'$ and hence is 3-connected. Thus M_1 has no 2-circuits and no 2-cocircuits. Hence $T_1, T_2, \ldots, T_t, \{a_{t+1}, a_{t+2}, v\}$ is a chain of M_1, contrary to the fact that $T_1, T_2, \ldots, T_t$ has maximum length among chains in smallest members of $\mathcal{F}'$. We conclude that, for all elements x of $E(M') - \{a_1, v\}$, there is no $\mathcal{F}'$-minor of M' avoiding x. Thus at least $|E(M)| - 1$ elements of M' are in every $\mathcal{F}$-minor of it. Since $|E(M)| - 1 \geq 7$, this is a contradiction.

It now follows that we may assume that $v \in E$. Using this and following the proof of [12, Theorem 1.4], it is straightforward to deduce that $M \cong M(\mathcal{W}_r)$ for some $r \geq 3$. We omit the details of this argument. As $M \not\cong M(\mathcal{W}_3)$, $r \geq 4$. Moreover, if we add an edge e to $\mathcal{W}_r$ joining two non-adjacent vertices of the rim, then we obtain a graph G such that $M(G)$ has at least $2(r - 1)$ elements in every $\mathcal{F}'$-minor. Since at most m elements of

$M(G)$ are in every $\mathscr{F}$-minor of it and $m \leq 5$, we have a contradiction that finishes the proof of Theorem 4.3. □

We remark here that the bound on m in the last theorem cannot be improved for, as noted in Section 2, both $\{\mathscr{W}^2, M(\mathscr{W}_4)\}$ and $\{\mathscr{W}^2, F_7, F_7^*\}$ are $(3, 6)$-avoiding.

Proof of Theorem 4.1. Evidently (i) implies (ii), and (ii) implies (iii). Now suppose that $\mathscr{F}$ satisfies (iii). Then, by Theorem 4.2, $\mathscr{W}^2 \in \mathscr{F}$. Hence, by Theorem 4.3, $\mathscr{F}$ is $\{\mathscr{W}^2\}$ or $\{\mathscr{W}^2, M(\mathscr{W}_3)\}$; that is, (iv) holds. It remains to show that (iv) implies (i); that is, that both $\{\mathscr{W}^2\}$ and $\{\mathscr{W}^2, M(\mathscr{W}_3)\}$ are $(3, 0)$-avoiding. But the only 3-connected, 5-element matroids with a $\mathscr{W}^2$-minor are $U_{2,5}$ and $U_{3,5}$. Hence, by symmetry, $\{\mathscr{W}^2\}$ is $(3, 0)$-avoiding. Similarly, $\{\mathscr{W}^2, M(\mathscr{W}_3)\}$ is $(3, 0)$-avoiding since F_7 and F_7^* are the only 3-connected, binary, 7-element matroids having an $M(\mathscr{W}_3)$-minor. □

To see that (i)–(iv) of Theorem 4.1 are not equivalent to $\mathscr{F}$ being $(3, 3)$-avoiding, recall from Section 2 that $\{M(\mathscr{W}_3), \mathscr{W}^3, P_6, Q_6, U_{3,6}\}$ is $(3, 3)$-avoiding.

We remark there that the fact that $\{\mathscr{W}^2\}$ is $(3, 0)$-avoiding was proved in [8] when it was shown that if M is a non-binary 3-connected matroid other than $U_{2,4}$, then, for all elements e of M, $M\backslash e$ or M/e is non-binary. For non-ternary and non-regular matroids, the corresponding results are restatements of the assertions that $\{U_{2,5}, U_{3,5}, F_7, F_7^*\}$ and $\{\mathscr{W}^2, F_7, F_7^*\}$ are $(3, 6)$-avoiding: if M is non-ternary and 3-connected, then there are at least six elements e such that both $M\backslash e$ and M/e are ternary; and if M is non-regular and 3-connected, then there are at most six elements e such that both $M\backslash e$ and M/e are regular. To see that 'six' cannot be replaced by 'five' in these results, one needs only to consider the matroid S_8.

To conclude the paper, we briefly consider the converse of Proposition 2.5. Since $\{\mathscr{W}^2, M(\mathscr{W}_3)\}$ is $(3, 2)$-avoiding but is not $(3, 4)$-rounded, this converse does not hold. However, it is not difficult to show using Theorem 2.2 that the converse does hold under the added assumption that the members of $\mathscr{F}$ are equicardinal. On combining this partial converse with Proposition 2.5, we obtain the following:

Corollary 4.4. *Let $\mathscr{F}$ be a collection of k-connected matroids each having exactly n elements for some $n \geq 4$. Suppose that $k = 2$ and $n - m \geq 1$, or that $k = 3$ and $n - m \in \{1, 2\}$. Then $\mathscr{F}$ is (k, m)-avoiding iff $\mathscr{F}$ is $(k, n - m)$-rounded.*

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