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A class of non-binary matroids with many binary minors

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Abstract

It is a well-known result of Tutte, A homotopy theorem for matroids, I, II, Trans. Amer. Math. Soc. 88 (1958) 144–174. that $U_{2,4}$ is the only non-binary matroid M such that, for every element e, both $M\backslash e$ and M/e are binary. Oxley generalized this result by characterizing the non-binary matroids M such that, for every element e of M, the deletion $M\backslash e$ or the contraction M/e is binary. We characterize those non-binary matroids M such that, for all elements e and f, at least two of $M\backslash e, f$; $M\backslash e/f$; $M/e\backslash f$; and $M/e, f$ are binary. © 1999 Elsevier Science B.V. All rights reserved.

MSC: 05B35

1. Introduction

This paper presents a characterization of a class of non-binary matroids that are, in a sense, close to being binary. Oxley [6] characterized those non-binary matroids M such that, for every element, at least half of the minors that arise by removing that element from M are binary. In this paper, we extend that result to characterize those non-binary matroids M such that, for every *two* elements, at least half of the minors that arise by removing those two elements from M are binary.

The notation and terminology used here will follow Oxley [7]. In particular, if a matroid M has a *circuit-hyperplane* X, that is, a set that is both a circuit and a hyperplane of M, then there is another matroid M' on $E(M)$ whose set of bases is $\{X\} \cup \mathscr{B}(M)$. This matroid is a *relaxation* of M obtained by *relaxing* X. If M has several circuit-hyperplanes, this operation may be repeated. If M'' is obtained from M by relaxing two circuit-hyperplanes, then M'' is a *double relaxation* of M. Fig. 1 shows the Fano matroid, F_7; the non-Fano matroid, F_7^-, which, up to isomorphism, is

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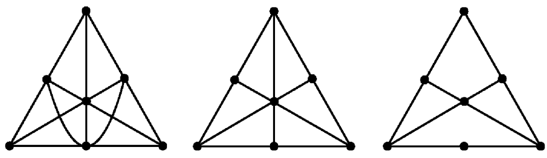
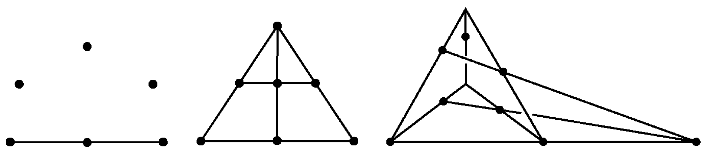


(a) (b) (c)

Fig. 1. (a) F_7. (b) F_7^-. (c) $F_7^=$.



(a) (b) (c)

Fig. 2. (a) P_6. (b) P_7. (c) J.

the unique relaxation of F_7; and $F_7^=$, the unique relaxation of F_7^-, and the only double relaxation of F_7.

The main theorem of the paper will be stated following the next result from Oxley [6], which it generalizes. The matroids P_6, P_7, and J, which appear in the main theorem, are shown in Fig. 2. The *corank* of a matroid is the rank of the dual matroid.

Theorem 1.1. *The following two statements are equivalent for a matroid M.*
(i) *M is non-binary,* 3*-connected, and, for every element e, $M\backslash e$ or M/e is binary.*
(ii) (a) *M is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n\geqslant 4$; or*
(b) *both the rank and corank of M exceed two and M can be obtained from a* 3*-connected binary matroid by relaxing a circuit-hyperplane.*

Theorem 1.2. *The following two statements are equivalent for a matroid M.*
(i) *M is non-binary,* 3*-connected, and, for every $\{e,f\}\subseteq E(M)$, at least two of $M\backslash e,f$; $M\backslash e/f$; $M/e\backslash f$; and $M/e,f$ are binary.*
(ii) (a) *M is isomorphic to $U_{3,6}$, P_6, P_7, P_7^*, or J, or to $U_{2,n}$ or $U_{n-2,n}$ for some $n\geqslant 4$; or*
(b) *both the rank and corank of M exceed two and M can be obtained from a* 3*-connected binary matroid by relaxing one or two circuit-hyperplanes.*

Theorem 1.2 will be proved in Section 3, while Section 4 extends the theorem to a characterization of all non-binary matroids M in which, for all $\{e,f\}\subseteq E(M)$, at least two of $M\backslash e,f$; $M\backslash e/f$; $M/e\backslash f$; and $M/e,f$ are binary. Section 2 contains several preliminaries that will be needed in the proofs of the main results.

2. Preliminaries

The following theorem from Oxley [6] generalizes Theorem 1.1 to arbitrary matroids.

Theorem 2.1. *The following two statements are equivalent for a matroid M.*
(i) *M is non-binary and, for every element e, $M\backslash e$ or M/e is binary.*
(ii) (a) *Both the rank and corank of M exceed two and M can be obtained from a connected binary matroid by relaxing a circuit-hyperplane*; *or*
(b) *M is isomorphic to a parallel extension of $U_{2,n}$ for some $n \geqslant 5$*; *or*
(c) *M is isomorphic to a series extension of $U_{n-2,n}$ for some $n \geqslant 5$*; *or*
(d) *M can be obtained from $U_{2,4}$ by series extension of a subset S of $E(U_{2,4})$ and parallel extension of a disjoint subset T of $E(U_{2,4})$ where S or T may be empty.*

The next lemma, due to Kahn [2], lists some useful properties of relaxation.

Lemma 2.2. *Let X be a circuit-hyperplane of the matroid M and let M' be the matroid obtained by relaxing X.*
(i) *If $e \in E(M) - X$, then $M/e = M'/e$ and, provided M does not have e as a coloop, $M'\backslash e$ is obtained from $M\backslash e$ by relaxing the circuit-hyperplane X of the latter.*
(ii) *If $f \in X$, then $M\backslash f = M'\backslash f$ and, provided M does not have f as a loop, M'/f is obtained from M/f by relaxing the circuit-hyperplane $X - f$ of the latter.*

The next two lemmas enable one to determine when a matroid is a relaxation or a double relaxation of another matroid. The *fundamental circuit of the element e with respect to the basis B* is the unique circuit contained in $B \cup e$ and is denoted by $C(e,B)$.

Lemma 2.3. *A matroid M is obtained from another matroid by relaxing the circuit-hyperplane B if and only if B is a basis of M such that $C(e,B) = B \cup e$ for every e in $E(M) - B$, and neither B nor $E(M) - B$ is empty.*

Lemma 2.4 (Mills [4]). *Suppose M, M_1, and M_2 are rank-r matroids on E such that M can be obtained from each of M_1 and M_2 by relaxing the circuit-hyperplanes X and Y, respectively. Then there is a matroid N such that the relaxation of Y in N yields M_1 and the relaxation of X in N yields M_2 if and only if $|X \cap Y| < r - 1$.*

Geometric representations of the matroids appearing in the statement of the next lemma can be found in the appendix of Oxley [7].

Lemma 2.5. *Let M be a 3-connected non-binary matroid having rank and corank at least three. Then M has a minor isomorphic to one of $U_{3,6}$, P_6, Q_6, or $\mathscr{W}^3$.*

The next two lemmas are structural results that relate $U_{2,4}$-minors to particular elements of a non-binary matroid. If $T \subseteq E(M)$, we say that M *uses* T.

Lemma 2.6 (Bixby [1]). *If M is a connected non-binary matroid containing an element e, then M has a $U_{2,4}$-minor using e.*

Lemma 2.7 (Seymour [8]). *If M is a 3-connected non-binary matroid containing distinct elements e and f, then M has a $U_{2,4}$-minor using $\{e,f\}$.*

The next lemma is another generalization of Tutte's excluded-minor characterization of binary matroids.

Lemma 2.8 (Oxley [5]). *Let M be a non-binary matroid such that, for some element e, both $M\backslash e$ and M/e are binary. Then M is obtained from a 4-point line having ground set $\{e,e_1,e_2,e_3\}$ by a sequence of at most three 2-sums where the basepoints of these 2-sums are e_1, e_2, and e_3, the other part of each 2-sum is binary, and each of e_1, e_2, and e_3 is the basepoint of at most one of these 2-sums.*

An immediate consequence of Lemma 2.8 is the following:

Corollary 2.9. *If M is 3-connected, non-binary and, for some element e, both $M\backslash e$ and M/e are binary, then $M \cong U_{2,4}$.*

The next lemma notes some basic properties of a non-binary matroid in which, for every two elements, at least half of the minors that arise by removing both elements are binary.

Lemma 2.10. *Let M be a non-binary matroid so that, for every $\{e,f\} \subseteq E(M)$, at least two of $M\backslash e,f$; $M\backslash e/f$; $M/e\backslash f$; and $M/e,f$ are binary.*
(i) *If H is a hyperplane of M and $|E(M)-H|>1$, then $M|H$ is binary.*
(ii) *If M is the 2-sum, with basepoint p, of M_1 and M_2, and M_1 is a connected binary matroid, then M_1 is isomorphic to $U_{1,n}$ or $U_{n-1,n}$ for some $n \geqslant 3$.*

Proof. We shall only prove (ii), omitting the straightforward proof of (i). Let p be the basepoint of the 2-sum. Assume M_1 is not $U_{1,n}$ or $U_{n-1,n}$. Then, as M_1 has no $U_{2,4}$-minor, it is not uniform. Hence, it has a circuit $\tilde{C}$ with $|\tilde{C}| \leqslant r(M_1)$. Clearly, $\mathrm{cl}_{M_1}(\tilde{C}) \neq E(M_1)$. The closure of any circuit is a union of circuits. Moreover, any closure is an intersection of hyperplanes, so the complement of any closure is a union of cocircuits. Hence, there exists a circuit $C \subseteq \mathrm{cl}_{M_1}(\tilde{C})$ and a cocircuit $C^* \subseteq E(M_1) - \mathrm{cl}_{M_1}(\tilde{C})$, such that $p \in C \cup C^*$. By duality, we may assume that $p \in C$. Let H be the hyperplane $E(M)-C^*$ of M. Since $p \in C$, this hyperplane has the non-binary matroid M_2 as a minor. Thus H is non-binary and, as M_1 is connected, $|E(M)-H|=|C^*|>1$; a contradiction to (i). □

3. The three-connected case

In this section, we prove Theorem 1.2, the main result of the paper. Recall that a *line* of a matroid is a rank-2 flat. We call a line *long* if it contains at least three points.

Proof. We omit the straightforward argument showing that if (ii) holds, then so does (i). Now assume that (i) holds. If $r(M)=2$ or $r^*(M)=2$, then, as M is 3-connected, it is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n\geqslant 4$. Thus we may assume that both the rank and corank of M exceed two. The next lemma determines the possibilities for a matroid M satisfying (i) if its rank or corank is three.

Lemma 3.1. *If $r(M)=3$ or $r^*(M)=3$, then M is isomorphic to one of $U_{3,6}$, P_6, Q_6, $\mathcal{W}^3$, P_7, P_7^*, F_7^-, $(F_7^-)^*$, $F_7^=$, and $(F_7^=)^*$.*

Proof. By duality, we may assume that $r(M)=3$, otherwise replace M by M^* in the argument that follows. Let M be a counterexample to the lemma having the least number of elements. By Lemma 2.5, $|E(M)|>6$. Now assume that $|E(M)|=7$. By Lemma 2.10(i), the matroid M has no $U_{2,4}$-restriction. Thus all long lines of M have exactly three points. If there are 7, 6, or 5 such lines, then an easy combinatorial argument shows that M has a geometric representation as in Fig. 1 or Fig. 2(b). Hence M is isomorphic to one of F_7, F_7^-, $F_7^=$, and P_7. Since F_7 is binary but M is not, we deduce that M satisfies the conclusion of the lemma if it has more than 4 long lines. Thus we may assume that M has at most 4 long lines.

If x and y are distinct elements of $E(M)$, we call $\{x,y\}$ *good* if x is on at most one long line missing y, and y is on at most one long line missing x. If there are no long lines, then every 2-subset of $E(M)$ is good. Moreover, as there are at most four long lines, every long line contains a 2-subset that is good. Thus $E(M)$ certainly has a good 2-subset. Choose such a subset $\{x,y\}$ for which $M\backslash x,y$ has as few long lines as possible. Then $M\backslash x/y$ and $M/x\backslash y$ are both 5-element rank-2 matroids that have at most one non-trivial parallel class. Since every such parallel class contains at most two elements, $M\backslash x/y$ and $M/x\backslash y$ are non-binary. Thus $M\backslash x,y$ is binary, and it follows that its geometric representation consists of two intersecting 3-point lines. Then, up to an interchange of x and y, the geometric representation of M is as in Fig. 3 where solid lines exist and dashed lines may or may not exist. However, in each of these configurations, $\{x',y'\}$ is a good subset that contradicts the choice of $\{x,y\}$. Thus the lemma holds if $|E(M)|=7$.

We may now assume that $|E(M)|\neq 7$. Then Lemma 2.5 implies that M has a proper minor isomorphic to one of $U_{3,6}$, P_6, Q_6, and $\mathcal{W}^3$. Arbitrarily choose such a minor N. From the Splitter theorem and the fact that M and N have rank 3, it follows that there is an element e of M such that $M\backslash e$ is 3-connected and has an N-minor. Since M is 3-connected and satisfies Theorem 1.2(i), the matroid $M\backslash e$ has rank 3 and satisfies Theorem 1.2(i). Hence, by the minimality of M, we may assume that $M\backslash e$ is isomorphic to one of $U_{3,6}$, P_6, Q_6, $\mathcal{W}^3$, P_7, F_7^-, and $F_7^=$. Thus, as $|E(M)|\neq 7$,

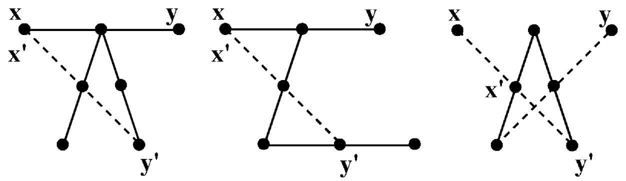


Fig. 3.

the matroid $M\backslash e$ is isomorphic to one of P_7, F_7^-, and $F_7^=$. However, none of these matroids has a $U_{3,6}$- or P_6-minor, so N cannot be isomorphic to $U_{3,6}$ or P_6. Since N was chosen arbitrarily among $U_{3,6}$, P_6, Q_6, and $\mathcal{W}^3$, it follows that M has no $U_{3,6}$- or P_6-minor. Finally, by Lemma 2.10(i), M has no $U_{2,4}$-restriction. Thus we conclude that M is an 8-element matroid with a P_7-, F_7^--, or $F_7^=$-restriction, but no $U_{2,4}$-, $U_{3,6}$- or P_6-restriction. From this, it is not hard to verify that M does not satisfy Theorem 1.2(i); a contradiction. □

We now assume that $r(M) > 3$ and $r^*(M) > 3$. Let $C = C(M) = \{e \in E(M)$: M/e is binary$\}$, $D = D(M) = \{e \in E(M)$: $M\backslash e$ is binary$\}$, and $Z = Z(M) = \{e \in E(M)$: $M\backslash e$ and M/e are non-binary$\}$. The next result, due to Lemos [3], gives valuable information regarding the cardinality of the set $Z(M)$.

Theorem 3.2. *Let H be a 3-connected non-binary matroid. Then $Z(H)$ is empty or $|Z(H)| \geqslant 3$.*

Now if $Z = \emptyset$, then it follows from Theorem 1.1 that (ii) holds. Thus we may assume that $|Z| \geqslant 3$. Moreover, if $e \in C \cap D$, then M is non-binary while $M\backslash e$ and M/e are binary. Thus Corollary 2.9 implies that $M \cong U_{2,4}$. As this contradicts the assumption that $r(M) > 3$, we conclude that $C \cap D = \emptyset$. Therefore, the sets C, D, and Z partition $E(M)$.

We denote by G or $G(M)$ the simple graph that has Z as its vertex set and has uv as an edge if and only if $M\backslash u, v$ is binary. We shall prove that G is either the triangular prism or a complete bipartite graph with equicardinal color classes U_1 and U_{-1}. In the latter case, it will turn out that M is obtained by relaxing the circuit-hyperplanes $U_1 \cup D$ and $U_{-1} \cup D$ of a 3-connected binary matroid. If G is the triangular prism, M will turn out to be isomorphic to J. It follows from the next lemma that $G(M) = G(M^*)$.

Lemma 3.3. *Let e be an element of Z.*

(i) *If f is an element of $E(M) - e$, then exactly one of the matroids $M\backslash e, f$ and $M\backslash e/f$ is binary, while exactly one of $M/e\backslash f$ and $M/e, f$ is binary.*

(ii) *If f is an element of Z, then either*
   (a) *$M\backslash e, f$ and $M/e, f$ are binary while $M\backslash e/f$ and $M/e\backslash f$ are non-binary*; *or*
   (b) *$M\backslash e/f$ and $M/e\backslash f$ are binary while $M\backslash e, f$ and $M/e, f$ are non-binary.*

Proof. We prove statement (i) first. Suppose that $f \in E(M) - e$. In addition, assume that $(M\backslash e)\backslash f$ and $(M\backslash e)/f$ are binary matroids. Now, as $e \in Z$, the matroid $M\backslash e$ is non-binary. Since M satisfies Theorem 1.2(i), at least two of $(M\backslash e)\backslash u,v$; $(M\backslash e)\backslash u/v$; $(M\backslash e)/u\backslash v$; and $(M\backslash e)/u,v$ are binary for every $\{u,v\} \subseteq E(M) - e$. Moreover, as M is a 3-connected matroid with at least four elements, $M\backslash e$ is connected. It follows from Lemmas 2.8 and 2.10(ii) that $M\backslash e$ is obtained from a 4-point line having ground set $\{f, e_1, e_2, e_3\}$ by series extending a subset S of $\{e_1, e_2, e_3\}$ and parallel extending a disjoint subset T of $\{e_1, e_2, e_3\}$ where S or T may be empty. But, as M is 3-connected, $M\backslash e$ cannot have a non-trivial parallel class. Thus $M\backslash e$ can be obtained from $U_{2,4}$ by series extending up to 3 elements. However, as e is not a coloop of M and $r^*(M\backslash e)=2$, it follows that $r^*(M)=3$, contrary to the assumption that $r^*(M) > 3$. We conclude that, for every element e of Z and every element f of $E(M)-e$, at least one of $M\backslash e,f$ and $M\backslash e/f$ is non-binary. Dually, at least one of $M/e,f$ and $M/e\backslash f$ is non-binary. Since at least two of the four minors of M obtained by removing e and f must be binary, exactly one of $M\backslash e,f$ and $M\backslash e/f$ is binary, and exactly one of $M/e\backslash f$ and $M/e,f$ is binary. Hence (i) holds. Now, if f is also an element of Z, then, by symmetry, exactly one of $M\backslash f,e$ and $M\backslash f/e$ is binary, and exactly one of $M/f\backslash e$ and $M/f,e$ is binary. Thus (ii) holds. □

For each element e of Z, we define $X_M(e)$ or $X(e)$ to be the set of neighbors of e in G and we define $Y_M(e)$ or $Y(e)$ to be the set of non-neighbors of e in G. Then $X(e) \cap Y(e) = \emptyset$ and $X(e) \cup Y(e) = Z - e$. Thus, for each e in Z, the sets C, D, $X(e)$, and $Y(e) \cup e$ partition $E(M)$. Moreover, Lemma 3.3(ii) implies that

$$X(e) = \{x \in Z - e\colon\ M\backslash e,x \text{ and } M/e,x \text{ are binary while } M\backslash e/x \text{ and } M/e\backslash x \text{ are non-binary}\}$$

and

$$Y(e) = \{y \in Z - e\colon\ M\backslash e/y \text{ and } M/e\backslash y \text{ are binary while } M\backslash e,y \text{ and } M/e,y \text{ are non-binary}\}.$$

Clearly, $X_M(e) = X_{M^*}(e)$ and $Y_M(e) = Y_{M^*}(e)$, while $D(M^*) = C(M)$.

Lemma 3.4. *Suppose that $e \in Z$. Then $M\backslash e$ is obtained from a connected binary matroid by relaxing the circuit-hyperplane $X(e) \cup D$ while M/e is obtained from a connected binary matroid by relaxing the circuit-hyperplane $Y(e) \cup D$.*

Proof. For every element f of $E(M) - e$, we have, by Lemma 3.3, that either $(M\backslash e)\backslash f$ or $(M\backslash e)/f$ is binary. Thus Theorem 2.1 implies that one of (a)–(d) of Theorem 2.1(ii) holds with $M\backslash e$ replacing M. But each of (ii)(b)–(d) contradicts the assumption that both $r(M)$ and $r^*(M)$ exceed 3. Hence $M\backslash e$ is obtained from a binary matroid $N_d(e)$ by relaxing a circuit-hyperplane H.

We now show that $H = X(e) \cup D$. Assume $x \in X(e) \cup D$. Then $M\backslash e,x$ is binary. Now, if $x \notin H$, then Lemma 2.2(i) implies that $M\backslash e/x$ equals the binary matroid

$N_d(e)/x$. Since this contradicts Lemma 3.3, we conclude that $x \in H$, and $X(e) \cup D \subseteq H$. Moreover, Lemma 2.2(ii) implies that, for every h in H, the matroid $M \backslash e, h$ equals the binary matroid $N_d(e) \backslash h$. It follows that $h \in X(e) \cup D$, and thus $H \subseteq X(e) \cup D$. Therefore, $X(e) \cup D = H$ and $M \backslash e$ is obtained from a connected binary matroid by relaxing the circuit-hyperplane $X(e) \cup D$.

It follows from duality and the above argument that $M^* \backslash e$ is obtained from a binary matroid by relaxing the circuit-hyperplane $X(e) \cup D(M^*) = X(e) \cup C(M)$. As the complement of $X(e) \cup C(M)$ in $E(M) - e$ is $Y(e) \cup D(M)$, the matroid $(M^* \backslash e)^* = M/e$ is obtained from a binary matroid by relaxing $Y(e) \cup D(M)$ and the lemma holds. □

We now list some consequences of Lemma 3.4. First, as $X(e) \cup D$ and $Y(e) \cup D$ are bases of $M \backslash e$ and M/e, respectively, it follows that, for every e in Z.

3.5. *$X(e) \cup D$ and $(Y(e) \cup e) \cup D$ are bases of M.*

Hence, setting $n = r(M) - |D|$, we see that G is an n-regular graph on $2n$ vertices. Moreover, $n \geqslant 2$ since $|Z| \geqslant 3$.

Let e be an element of Z. Now, as M/e is obtained by relaxing the circuit-hyperplane $Y(e) \cup D$, it follows from Lemma 2.3 that $C_{M/e}(f, Y(e) \cup D) = Y(e) \cup D \cup f$ for every f in $E - (Y(e) \cup e \cup D)$. Thus we deduce that,

3.6. *For every f in $E - (Y(e) \cup e \cup D)$, either $Y(e) \cup D \cup f$ or $Y(e) \cup D \cup \{e, f\}$ is a circuit of M.*

Similarly, as $C_{M \backslash e}(f, X(e) \cup D) = X(e) \cup D \cup f$ for every f in $E - (X(e) \cup D \cup e)$, it follows that, for each e in Z,

3.7. *$C_M(f, X(e) \cup D) = X(e) \cup D \cup f$, for every f* in $E - (X(e) \cup D \cup e)$.

The next lemma concerns circuits of M contained in $Z - e$. This result will enable us to obtain useful information about the triangles in G.

Lemma 3.8. *Suppose that e is an element of Z and C_1 is a circuit of M contained in $Z - e$.*
(i) *If C_1 contains neither $X(e)$ nor $Y(e)$, then C_1 contains at least two elements of each of $X(e)$ and $Y(e)$.*
(ii) *If $D \neq \emptyset$, then C_1 contains at least two elements of each of $X(e)$ and $Y(e)$.*

Proof. We prove (i) and (ii) simultaneously. Since $X(e)$ and $Y(e)$ are independent in M, every circuit of $M|(Z - e)$ contains elements of both $X(e)$ and $Y(e)$. Suppose $C_1 \cap Y(e) = \{y\}$. Then, by (3.7), the set $X(e) \cup D \cup y$ is a circuit of M. Moreover, if $D \neq \emptyset$ or C_1 does not contain $X(e)$, then C_1 is properly contained in $X(e) \cup D \cup y$; a contradiction. Thus we conclude that $|C_1 \cap Y(e)| \geqslant 2$. Similarly if $C_1 \cap X(e) = \{x\}$, then,

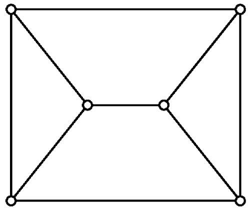


Fig. 4.

whenever $D \neq \emptyset$ or C_1 does not contain $Y(e)$, statement (3.6) leads to the contradiction that C_1 is properly contained in a circuit. Thus $|C_1 \cap X(e)| \geqslant 2$. □

The next two lemmas describe properties of the triangles of G.

Lemma 3.9. *If uvw is a triangle of G and $x, y \in Y(w)$, then $\{u, v, x, y\}$ is a circuit and a cocircuit in M.*

Proof. As $G(M) = G(M^*)$, it suffices to prove that $\{u, v, x, y\}$ is a circuit in M. It follows from (3.7) that $X(w) \cup D \cup x$ and $X(w) \cup D \cup y$ are circuits in M. Hence $(X(w) - \{u, v\}) \cup D \cup x$ and $(X(w) - \{u, v\}) \cup D \cup y$ are circuits in the binary matroid $M/u, v$. Therefore, their symmetric difference $\{x, y\}$ is dependent in $M/u, v$. Consequently, $\{u, v, x, y\}$ is dependent in M. If $\{u, v, x, y\}$ is not a circuit of M, then it properly contains a circuit C_1. Then Lemma 3.8 implies that $D = \emptyset$ and C_1 contains either $X(w)$ or $Y(w)$. Since $|Y(w)| \geqslant 2$, we have $|X(w)| \geqslant 3$. Now, as $|C_1 \cap X(w)| \leqslant 2$, we deduce that C_1 contains $Y(w)$ and thus $|Y(w)| \leqslant 2$. Therefore $n = |X(w)| \leqslant 3$. But this contradicts the fact that $|D| + n = r(M) \geqslant 4$ and we conclude that $\{u, v, x, y\}$ is a circuit of M. □

Lemma 3.10. *No two distinct triangles of G share an edge.*

Proof. Let uvw and vwz be two different triangles in G. Since $|Y(w)| = n - 1 = |X(w)| - 1 \geqslant |\{u, v, z\}| - 1 = 2$, there are at least two vertices of G that are not adjacent to w. Let x and y be two such non-neighbors of w. By Lemma 3.9, both $\{u, v, x, y\}$ and $\{v, z, x, y\}$ are circuits of M. Hence, by circuit-elimination, $\{u, v, z, x\}$ is dependent in M. Now (3.7) implies that $X(w) \cup D \cup x$ is a circuit of M. Since $\{u, v, z, x\}$ is a subset of $X(w) \cup D \cup x$, it follows that $D = \emptyset$ and $X(w) = \{u, v, z\}$. Thus $|D| = 0$ and $n = 3$, contradicting the fact that $|D| + n = r(M) \geqslant 4$. □

The next lemma determines the possibilities for the graph G. The triangular prism is shown in Fig. 4. Recall that G is an n-regular graph on $2n$ vertices.

Lemma 3.11. *G is either isomorphic to the complete bipartite graph $K_{n,n}$ for some $n \geqslant 2$ or G is the triangular prism.*

Proof. First assume that G has no triangles and fix an edge uv of G. Then $X(u)$ and $X(v)$ are disjoint stable sets in G. Now, as both sets have n elements, $X(u)$ and $X(v)$ partition Z. Thus, for each y in $X(u)$, we have that $X(y) \subseteq X(v)$. Hence, as y has degree n, it follows that $X(y) = X(v)$ for each y in $X(u)$. Moreover, by the symmetry between u and v, we have $X(y) = X(u)$ for each y in $X(v)$. Thus G is isomorphic to $K_{n,n}$.

Now assume that uvw is a triangle of G. By Lemma 3.10, no two triangles in G share an edge. Thus the sets $X(u) - \{v,w\}$, $X(v) - \{u,w\}$, and $X(w) - \{u,v\}$ must be pairwise disjoint subsets of $Z - \{u,v,w\}$. Since each of these three sets has $n-2$ elements, we have $3 + 3(n-2) \leqslant 2n$. Hence $n \leqslant 3$. Furthermore, as no 2-regular graph on four vertices has a triangle, $n = 3$. Therefore each of $X(u) - \{v,w\}$, $X(v) - \{u,w\}$, and $X(w) - \{u,v\}$ is a singleton. Let these sets be $\{u'\}$, $\{v'\}$, and $\{w'\}$, respectively. Now, as $|Z| = 2n = 6$, we see that $Z = \{u,v,w,u',v',w'\}$. Moreover, G contains the edges $uv, vw, uw, uu', vv', ww'$, and possibly $u'v', v'w'$, and $u'w'$. Since G is 3-regular, each of $u'v', v'w'$, and $u'w'$ must be an edge of G. Thus G is the triangular prism. □

We now divide the proof of Theorem 1.2 into two cases:

Case A: $G \cong K_{n,n}$ for some $n \geqslant 2$; and

Case B: G is the triangular prism.

Consider Case A. Let U_1 and U_{-1} be the color classes of G. For each i in $\{1,-1\}$, if $v \in U_i$, then $X(v) = U_{-i}$ and $Y(v) \cup \{v\} = U_i$.

Lemma 3.12. *Suppose $i \in \{1,-1\}$. Then $U_i \cup D$ is a basis of M and, for each element f of $E - (U_i \cup D)$, the fundamental circuit $C_M(f, U_i \cup D) = U_i \cup D \cup f$.*

Proof. Suppose $i \in \{1,-1\}$ and f is an element of $E - (U_i \cup D)$. Let e be an element of $U_{-i} - \{f\}$. Then $X(e) = U_i$ and, by (3.5), the set $U_i \cup D = X(e) \cup D$ is a basis of M. Furthermore, $f \in E - (U_i \cup D \cup e)$ and (3.7) implies that $C_M(f, U_i \cup D) = U_i \cup D \cup f$. □

On combining the last lemma with Lemma 2.3, we deduce that, for each i in $\{1,-1\}$, there is a matroid N_i such that M is obtained from N_i by relaxing the circuit-hyperplane $U_i \cup D$. Moreover, as $|(U_1 \cup D) \cap (U_{-1} \cup D)| = |D| = r(M) - n < r(M) - 1$, it follows from Lemma 2.4 that there is a matroid N with circuit-hyperplanes $U_1 \cup D$ and $U_{-1} \cup D$ such that relaxing $U_i \cup D$ yields N_{-i} for each i in $\{1,-1\}$ and such that relaxing both circuit-hyperplanes yields M. In other words, M is a double relaxation of N. The next two lemmas establish that N is 3-connected and binary. It will then follow that, if $G(M) \cong K_{n,n}$ for some $n \geqslant 2$, the matroid M satisfies Theorem 1.2(ii)(b) and the theorem holds.

Lemma 3.13. *N is 3-connected.*

Proof. Suppose that N is not 3-connected. Then there is a partition (S,T) of $E(N)$ such that for some k in $\{1,2\}$, we have $|S|,|T|\geqslant k$, and

3.14. $r_N(S)+r_N(T)-r(N)=k-1.$

Now $E(N)=E(M)$ and all subsets of $E(M)$ except $U_1\cup D$ and $U_{-1}\cup D$ have the same rank in N as they do in M. Since M is 3-connected, we deduce that S or T equals $U_1\cup D$ or $U_{-1}\cup D$. Suppose D and C are empty and $\{S,T\}=\{U_1,U_{-1}\}$. Since $r_N(U_i\cup D)=r(N)-1$ for each i in $\{1,-1\}$, it follows from (3.14) that $r(N)=k+1$. However, $r(N)=r(M)\geqslant 4$ and $k+1\leqslant 3$; a contradiction. We may now assume that $D\cup C$ is non-empty. In addition, suppose that $T=U_1\cup D$ and $S=E-(U_1\cup D)=U_{-1}\cup C$. It follows from (3.14) that $r_N(U_{-1}\cup C)=k$. Moreover, as $D\cup C\neq\emptyset$, the set $U_{-1}\cup C$ has the same rank in both M and N. In particular, $r_M(U_{-1}\cup C)=k$ for some k in $\{1,2\}$. Now $|U_{-1}\cup C|\geqslant 4$ since $U_1\cup D$ is a basis of M and $r^*(M)\geqslant 4$. Thus, as M is 3-connected, $k\neq 1$. Hence $k=2$. Therefore, $U_{-1}\cup C$ is a cobasis of M contained in a line L that has at least 4 points. Since $r(M)\geqslant 4$, we deduce that M has a non-binary hyperplane, contradicting Lemma 2.10(i). We conclude that N is 3-connected. □

Lemma 3.15. *N is binary.*

Proof. Suppose N is non-binary and $e\in Z(M)$. Now $\{X(e),Y(e)\cup e\}=\{U_1,U_{-1}\}$. Suppose $e\in U_i$. Then $X(e)=U_{-i}$ and Lemma 3.4 implies that $M\backslash e$ is obtained from a binary matroid $N_d(e)$ by relaxing the circuit-hyperplane $U_{-i}\cup D$. Since M is obtained from N_{-i} by relaxing the same circuit-hyperplane and $e\in U_i$, Lemma 2.2(i) implies that $M\backslash e$ is obtained from $N_{-i}\backslash e$ by relaxing $U_{-i}\cup D$. Thus $\mathscr{B}(N_{-i}\backslash e)=\mathscr{B}(N_d(e))$ and we have $N_{-i}\backslash e=N_d(e)$. Now, as $e\in U_i\cup D$ and N_{-i} is obtained from N by relaxing $U_i\cup D$, it follows from Lemma 2.2(ii) that $N\backslash e=N_{-i}\backslash e$. Thus $N\backslash e=N_{-i}\backslash e=N_d(e)$. In particular, $N\backslash e$ is binary since $N_d(e)$ is binary.

Dually, Lemma 3.4 implies that $M^*\backslash e$ is obtained from a binary matroid by relaxing the circuit-hyperplane $U_{-i}\cup C$. In addition, M^* is obtained from N^*_{-i} by relaxing $U_i\cup C$ while N^*_{-i} is obtained from N^* by relaxing $U_{-i}\cup C$. It follows from the argument above that $(N^*\backslash e)$ is binary. Thus $N/e=(N^*\backslash e)^*$ is binary. Since both $N\backslash e$ and N/e are binary, Corollary 2.9 implies that $N\cong U_{2,4}$, contrary to the fact that $r(N)=r(M)\geqslant 4$. We conclude that N is binary. □

This completes the proof of Theorem 1.2 for M in Case A, that is, when $G(M)\cong K_{n,n}$. Next we consider Case B, that is, when $G(M)$ is the triangular prism.

Now, as G has six vertices, $n=3$ and $r(M)=|D|+3$. Next we show that $|D|=|C|=1$. Suppose $\{d_1,d_2\}\subseteq D$. As M is non-binary and 3-connected, Lemma 2.7 implies that there is a $U_{2,4}$-minor of M using d_1 and d_2. Let $M\backslash S/T$ be such a minor where $|T|=r(M)-2$. Then $C\cap T=\emptyset$. Since $r(M)=|D|+3$, we have $|T|=|D|+1$. Moreover, neither d_1 nor d_2 is an element of T, since both are used in the $U_{2,4}$-minor. Thus T contains at least three elements of Z. Assume that $\{u,v,w\}\subseteq Z\cap T$. Every collection of

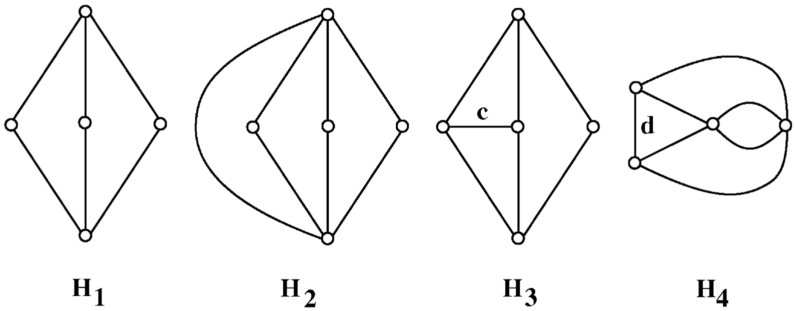


Fig. 5.

three vertices of G contains a pair of adjacent vertices. Thus we may assume that uv is an edge of G. Then $M\backslash u,v$ is binary, and it follows from Lemma 3.3(ii) that $M/u,v$ is also binary. However, as $M\backslash S/T \cong U_{2,4}$, the matroid $M/u,v$ is non-binary. As a result of this contradiction, we conclude that $|D| \leqslant 1$. However, if $D = \emptyset$, then $r(M) = 3$; a contradiction. Thus $|D| = 1$. By duality, $|C| = 1$, and it follows that $|E(M)| = 8$. We set $D = \{d\}$ and $C = \{c\}$.

Lemma 3.16. *M is ternary.*

Proof. Suppose M is non-ternary. Then, by duality, we may assume that M has a $U_{2,5}$- or an F_7-minor. First suppose that M has an F_7-minor. Then, as M has corank 4 and F_7 is binary, $M/c \cong F_7$. Let uv be an edge of G that is in no triangles. Then $X(u)$ and $X(v)$ are disjoint. By (3.7), the sets $X(u) \cup \{d,c\}$ and $X(v) \cup \{d,c\}$ are circuits of M. Hence $X(u) \cup \{d\}$ and $X(v) \cup \{d\}$ are 4-circuits in M/c that share exactly one element. However, F_7 does not have such a pair of 4-circuits. Thus M has no F_7-minor and we may now assume that M has a $U_{2,5}$-minor. Hence $M/x,y\backslash z \cong U_{2,5}$ for certain x, y, and z. One of x and y is an element of Z as neither of them is in C and $|D| = 1$. Thus we may assume that $x \in Z$. For each element w of $E - \{x,y,z\}$, we have that $M/x\backslash w$ is non-binary since $M/x\backslash w/y\backslash z \cong U_{2,4}$. Thus $E - \{x,y,z\} \subseteq X(x)$ and we get the contradiction $5 = |E - \{x,y,z\}| \leqslant |X(x) \cup C| = 4$. □

Let $Z = \{u,v,w,u',v',w'\}$ and suppose that uvw and $u'v'w'$ are the triangles in $G(M)$ while uu', vv', and ww' are the edges that are contained in no triangles. The graphs H_1, H_2, H_3, and H_4, which appear in the next few lemmas, are shown in Fig. 5.

Lemma 3.17. *$M\backslash d$ is graphic, and $M\backslash d,c \cong M(H_1)$. Moreover, the series classes of H_1 correspond to the pairs $\{u,u'\},\{v,v'\}$, and $\{w,w'\}$.*

Proof. The matroid $M\backslash d$ is ternary and binary and has fewer elements than $M^*(K_5)$ and $M^*(K_{3,3})$. Hence $M\backslash d$ is graphic. Then $M\backslash d,c$ is the cycle matroid of a connected graph H. By Lemma 3.9, each 4-cycle in the triangular prism G is a 4-circuit in M. It is now easy to verify that H is isomorphic to either K_4 or the graph H_1. However, as M is 3-connected, $r(M\backslash d, c)=4$ implying that H has five vertices. Therefore $M\backslash d, c \cong M(H_1)$. The statement about the three series classes now follows immediately. □

Lemma 3.18. *$M\backslash d$ is isomorphic to one of $M(H_2)$ and $M(H_3)$ with c as indicated in Fig.* 5.

Proof. This follows from Lemma 3.17 and the fact that $M\backslash d$ has no parallel elements, loops, or coloops. □

Lemma 3.19. *$M\backslash d$ or $M^*\backslash c$ is isomorphic to $M(H_2)$.*

Proof. If not, then, by Lemma 3.18 and duality, $M\backslash d$ and $M^*\backslash c$ are both isomorphic to $M(H_3)$. Hence M/c is isomorphic to $M(H_4)$, with d as indicated in Fig. 5. Therefore, $M\backslash d/c$ is isomorphic to the cycle matroids of both H_3/c and $H_4\backslash d$. However, $M(H_3/c) \not\cong M(H_4\backslash d)$; a contradiction. □

By duality, we may now assume that $M\backslash d \cong M(H_2)$. Then Lemma 3.17 implies that the set $\{c, u, v, w\}$ is a basis of $M\backslash d$, and hence also of M. Therefore, by using row and column scaling, we may assume that M has a ternary representation of the following form:

$$\begin{array}{cccccccc} c & u & v & w & u' & v' & w' & d \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & \alpha \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & \beta \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & \gamma \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & \delta \end{bmatrix}.$$

Now, as M has no series elements, each of β, γ, and δ is non-zero. Since (3.7) implies that $X(u) \cup d = \{u', v, w, d\}$ is a basis of M, we have that $\alpha \neq \beta$. Moreover, as $\{u', v, w, d\}$ is also a basis of the binary matroid $M\backslash v', w'$, we conclude that $\alpha \neq -\beta$. Hence $\alpha = 0$.

As u and u' are adjacent in $G(M)$, the matroid $M/u\backslash u'$ is non-binary. Hence $\gamma = \delta$. From the symmetry between u and w, we also have that $\beta = \gamma$. In addition, by scaling the last column, we may assume that $\beta = \gamma = \delta = 1$. Thus M has the following ternary representation:

$$\begin{array}{cccccccc} c & u & v & w & u' & v' & w' & d \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Since this is the ternary representation of J, it has been shown that if $G(M)$ is the triangular prism, then $M \cong J$. This completes the proof in Case B and thereby completes the proof of Theorem 1.2. □

4. The general case

In this section, we determine all non-binary matroids M such that at least half of the minors that arise by removing two elements from M are binary. Recall that $C(M) = \{e \in E(M) : M/e \text{ is binary}\}$, and $D(M) = \{e \in E(M) : M\backslash e \text{ is binary}\}$.

Theorem 4.1. *The following two statements about a matroid M are equivalent.*

(i) *M is non-binary and, for every $\{e, f\} \subseteq E(M)$, at least two of $M\backslash e, f$; $M\backslash e/f$; $M/e\backslash f$; and $M/e, f$ are binary.*

(ii) (a) *M is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n \geqslant 4$; or*
(b) *both the rank and corank of M exceed two and M can be obtained from a connected binary matroid by relaxing a circuit-hyperplane; or*
(c) *both the rank and corank of M exceed two and M can be obtained from a connected binary matroid by relaxing two circuit-hyperplanes; or*
(d) *M is isomorphic to one of $U_{3,6}$, P_6, P_7, P_7^*, and J; or*
(e) *M is isomorphic to $U_{2,4} \oplus_2 U_{2,4}$; or*
(f) *M is obtained from a matroid $\hat{M}$ described in* (a) *or* (b) *by the addition of a loop or coloop, or by series extension of a subset S of $D(\hat{M})$ or parallel extension of a subset T* of *$C(\hat{M})$ where $S \cap T = \emptyset$; or*
(g) *M is obtained from a matroid $\hat{M}$ described in* (a), (b), (c), *or* (d) *by series extension of a subset S of $D(\hat{M})$ or parallel extension of a subset T of $C(\hat{M})$ where $S \cap T = \emptyset$.*

Proof. We omit the straightforward proof showing that if (ii) holds, then so does (i). Now assume (i) holds. We argue by induction on $|E(M)|$ to show that (ii) holds. If M is 3-connected, then the result follows easily from Theorem 1.2. Assume the result is true for all matroids satisfying the hypotheses and having fewer elements than M.

If M is disconnected, then $M = M_1 \oplus M_2$ where M_1 or M_2 is non-binary. Suppose M_2 is non-binary. If $e, f \in E(M_1)$, then each of $M\backslash e, f$; $M\backslash e/f$; and $M/e\backslash f$ has the non-binary matroid M_2 as a minor; a contradiction. Thus M_1 consists of an element f that is either a loop or a coloop. If there is an e in $E(M_2)$ such that $M_2\backslash e$ and M_2/e are non-binary, then $M\backslash e, f$, $M\backslash e/f$, and $M/e\backslash f$ are non-binary; a contradiction. Thus we may assume that, for every element e of M_2, at least one of $M_2\backslash e$ and M_2/e is binary. It follows from Theorem 2.1 that M satisfies (ii)(f) if M is disconnected.

Suppose that M is connected but not 3-connected. Then $M = M_1 \oplus_2 M_2$ for some connected matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \{p\}$ and $|E(M_1)|, |E(M_2)| \geqslant 3$. Since M is connected, we may assume that M_1 and M_2 are connected. Suppose M_1 and M_2 are both non-binary. Then, by Lemma 2.6, both M_1 and M_2 have a $U_{2,4}$-minor using the basepoint p. Thus M has $U_{2,4} \oplus_2 U_{2,4}$ as a minor. By duality, M has a connected but not 3-connected single-element extension M_1 of $U_{2,4} \oplus_2 U_{2,4}$ as a minor. It is not difficult to check that each of the three possibilities for M_1 implies that M fails to satisfy (i).

We may now assume that $M = M_1 \oplus_2 M_2$ and exactly one of M_1 and M_2 is non-binary. Suppose M_2 is non-binary. By Lemma 2.10(ii), we may assume that $M_2 \cong U_{1,n}$ for some $n \geqslant 3$, otherwise we replace M by M^* in the argument that follows. We may also suppose that M_1 has no elements parallel with the basepoint p, since any such element may be taken to be in M_2 rather than M_1. Hence M is obtained from M_1 by replacing p by $n-1$ parallel elements. Moreover, M_1/p is binary. To see this, suppose M_1/p is non-binary and let s and t be elements of $E(M_2) - p$. Then, as s and t are parallel in M, we have $M/s\backslash t \cong M\backslash s/t \cong M/s,t$. Moreover, $M/s,t\backslash(E(M_2) - \{p,s,t\})$ equals M_1/p. If M_1/p is non-binary, then at least three of the minors of M that involve the elimination of s and t are non-binary; a contradiction. We conclude that M_1/p is binary. Thus $p \in C(M_1)$.

By the induction assumption, one of (ii)(a)–(g) holds for M_1. Notice that it is impossible for M_1 to be isomorphic to $U_{2,4} \oplus_2 U_{2,4}$ since $p \in C(M_1)$, yet $U_{2,4} \oplus_2 U_{2,4}$ has no single-element contraction that is binary. Thus one of (ii)(a)–(g), other than (ii)(e), holds for M_1. Since M is obtained from M_1 by the parallel extension of the element p of $C(M_1)$, it is clear that M satisfies (ii)(f) or (ii)(g). □

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