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Abstract

It is a well-known result of Tutte, A homotopy theorem for matroids, I, II, Trans. Amer. Math. Soc. 88 (1958) 144–174. that $U_{2,4}$ is the only non-binary matroid $M$ such that, for every element $e$, both $M\setminus e$ and $M/e$ are binary. Oxley generalized this result by characterizing the non-binary matroids $M$ such that, for every element $e$ of $M$, the deletion $M\setminus e$ or the contraction $M/e$ is binary. We characterize those non-binary matroids $M$ such that, for all elements $e$ and $f$, at least two of $M\setminus e$; $M\setminus e/f$; $M/e\setminus f$; and $M/e,f$ are binary. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper presents a characterization of a class of non-binary matroids that are, in a sense, close to being binary. Oxley [6] characterized those non-binary matroids $M$ such that, for every element, at least half of the minors that arise by removing that element from $M$ are binary. In this paper, we extend that result to characterize those non-binary matroids $M$ such that, for every two elements, at least half of the minors that arise by removing those two elements from $M$ are binary.

The notation and terminology used here will follow Oxley [7]. In particular, if a matroid $M$ has a circuit-hyperplane $X$, that is, a set that is both a circuit and a hyperplane of $M$, then there is another matroid $M'$ on $E(M)$ whose set of bases is $\{X\} \cup \mathcal{B}(M)$. This matroid is a relaxation of $M$ obtained by relaxing $X$. If $M$ has several circuit-hyperplanes, this operation may be repeated. If $M''$ is obtained from $M$ by relaxing two circuit-hyperplanes, then $M''$ is a double relaxation of $M$. Fig. 1 shows the Fano matroid, $F_7$; the non-Fano matroid, $F_7^-$, which, up to isomorphism, is

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the unique relaxation of $F_7$; and $F_7^-$, the unique relaxation of $F_7^-$, and the only double relaxation of $F_7$.

The main theorem of the paper will be stated following the next result from Oxley [6], which it generalizes. The matroids $P_6$, $P_7$, and $J$, which appear in the main theorem, are shown in Fig. 2. The corank of a matroid is the rank of the dual matroid.

**Theorem 1.1.** The following two statements are equivalent for a matroid $M$.

(i) $M$ is non-binary, 3-connected, and, for every element $e$, $M\setminus e$ or $M/e$ is binary.

(ii) (a) $M$ is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n\geq 4$; or

(b) both the rank and corank of $M$ exceed two and $M$ can be obtained from a 3-connected binary matroid by relaxing a circuit-hyperplane.

**Theorem 1.2.** The following two statements are equivalent for a matroid $M$.

(i) $M$ is non-binary, 3-connected, and, for every $\{e,f\} \subseteq E(M)$, at least two of $M\setminus e, f$; $M\setminus e/f$; $M/e\setminus f$; and $M/e, f$ are binary.

(ii) (a) $M$ is isomorphic to $U_{3,6}$, $P_6$, $P_7$, $P_7^*$, or $J$, or to $U_{2,n}$ or $U_{n-2,n}$ for some $n\geq 4$; or

(b) both the rank and corank of $M$ exceed two and $M$ can be obtained from a 3-connected binary matroid by relaxing one or two circuit-hyperplanes.

Theorem 1.2 will be proved in Section 3, while Section 4 extends the theorem to a characterization of all non-binary matroids $M$ in which, for all $\{e,f\} \subseteq E(M)$, at least two of $M\setminus e, f$; $M\setminus e/f$; $M/e\setminus f$; and $M/e, f$ are binary. Section 2 contains several preliminaries that will be needed in the proofs of the main results.
2. Preliminaries

The following theorem from Oxley [6] generalizes Theorem 1.1 to arbitrary matroids.

**Theorem 2.1.** The following two statements are equivalent for a matroid $M$.

(i) $M$ is non-binary and, for every element $e$, $M\setminus e$ or $M/e$ is binary.

(ii) (a) Both the rank and corank of $M$ exceed two and $M$ can be obtained from a connected binary matroid by relaxing a circuit-hyperplane; or
(b) $M$ is isomorphic to a parallel extension of $U_{2,n}$ for some $n \geq 5$; or
(c) $M$ is isomorphic to a series extension of $U_{n-2,n}$ for some $n \geq 5$; or
(d) $M$ can be obtained from $U_{2,4}$ by series extension of a subset $S$ of $E(U_{2,4})$ and parallel extension of a disjoint subset $T$ of $E(U_{2,4})$ where $S$ or $T$ may be empty.

The next lemma, due to Kahn [2], lists some useful properties of relaxation.

**Lemma 2.2.** Let $X$ be a circuit-hyperplane of the matroid $M$ and let $M'$ be the matroid obtained by relaxing $X$.

(i) If $e \in E(M) - X$, then $M/e = M'/e$ and, provided $M$ does not have $e$ as a coloop, $M'\setminus e$ is obtained from $M\setminus e$ by relaxing the circuit-hyperplane $X$ of the latter.

(ii) If $f \in X$, then $M\setminus f = M'\setminus f$ and, provided $M$ does not have $f$ as a loop, $M'/f$ is obtained from $M/f$ by relaxing the circuit-hyperplane $X - f$ of the latter.

The next two lemmas enable one to determine when a matroid is a relaxation or a double relaxation of another matroid. The fundamental circuit of the element $e$ with respect to the basis $B$ is the unique circuit contained in $B \cup e$ and is denoted by $C(e,B)$.

**Lemma 2.3.** A matroid $M$ is obtained from another matroid by relaxing the circuit-hyperplane $B$ if and only if $B$ is a basis of $M$ such that $C(e,B) = B \cup e$ for every $e$ in $E(M) - B$, and neither $B$ nor $E(M) - B$ is empty.

**Lemma 2.4** (Mills [4]). Suppose $M$, $M_1$, and $M_2$ are rank-$r$ matroids on $E$ such that $M$ can be obtained from each of $M_1$ and $M_2$ by relaxing the circuit-hyperplanes $X$ and $Y$, respectively. Then there is a matroid $N$ such that the relaxation of $Y$ in $N$ yields $M_1$ and the relaxation of $X$ in $N$ yields $M_2$ if and only if $|X \cap Y| < r - 1$.

Geometric representations of the matroids appearing in the statement of the next lemma can be found in the appendix of Oxley [7].

**Lemma 2.5.** Let $M$ be a 3-connected non-binary matroid having rank and corank at least three. Then $M$ has a minor isomorphic to one of $U_{3,6}$, $P_6$, $Q_6$, or $\mathcal{I}^3$.
The next two lemmas are structural results that relate $U_{2,4}$-minors to particular elements of a non-binary matroid. If $T \subseteq E(M)$, we say that $M$ uses $T$.

**Lemma 2.6 (Bixby [1]).** If $M$ is a connected non-binary matroid containing an element $e$, then $M$ has a $U_{2,4}$-minor using $e$.

**Lemma 2.7 (Seymour [8]).** If $M$ is a $3$-connected non-binary matroid containing distinct elements $e$ and $f$, then $M$ has a $U_{2,4}$-minor using $\{e, f\}$.

The next lemma is another generalization of Tutte’s excluded-minor characterization of binary matroids.

**Lemma 2.8 (Oxley [5]).** Let $M$ be a non-binary matroid such that $M \setminus e$ and $M/e$ are binary. Then $M$ is obtained from a $4$-point line having ground set $\{e, e_1, e_2, e_3\}$ by a sequence of at most three $2$-sums where the basepoints of these $2$-sums are $e_1$, $e_2$, and $e_3$, the other part of each $2$-sum is binary, and each of $e_1$, $e_2$, and $e_3$ is the basepoint of at most one of these $2$-sums.

An immediate consequence of Lemma 2.8 is the following:

**Corollary 2.9.** If $M$ is $3$-connected, non-binary and, for some element $e$, both $M \setminus e$ and $M/e$ are binary, then $M = U_{2,4}$.

The next lemma notes some basic properties of a non-binary matroid in which, for every two elements, at least half of the minors that arise by removing both elements are binary.

**Lemma 2.10.** Let $M$ be a non-binary matroid so that, for every $f \in E(M) - H$, at least two of $M \setminus e,f$; $M \setminus e,f$; $M \setminus e,f$; and $M/e,f$ are binary.

(i) If $H$ is a hyperplane of $M$ and $|E(M) - H| > 1$, then $M\setminus H$ is binary.

(ii) If $M$ is the $2$-sum, with basepoint $p$, of $M_1$ and $M_2$, and $M_1$ is a connected binary matroid, then $M_1$ is isomorphic to $U_{1,n}$ or $U_{n-1,n}$ for some $n \geq 3$.

**Proof.** We shall only prove (ii), omitting the straightforward proof of (i). Let $p$ be the basepoint of the $2$-sum. Assume $M_1$ is not $U_{1,n}$ or $U_{n-1,n}$. Then, as $M_1$ has no $U_{2,4}$-minor, it is not uniform. Hence, it has a circuit $\tilde{C}$ with $|\tilde{C}| \leq r(M_1)$. Clearly, $cl_{M_1}(\tilde{C}) \neq E(M_1)$. The closure of any circuit is a union of circuits. Moreover, any closure is an intersection of hyperplanes, so the complement of any closure is a union of cocircuits. Hence, there exists a circuit $C \subseteq cl_{M_1}(\tilde{C})$ and a cocircuit $C^* \subseteq E(M_1) - cl_{M_1}(\tilde{C})$, such that $p \in C \cup C^*$. By duality, we may assume that $p \in C$. Let $H$ be the hyperplane $E(M) - C^*$ of $M$. Since $p \in C$, this hyperplane has the non-binary matroid $M_2$ as a minor. Thus $H$ is non-binary and, as $M_1$ is connected, $|E(M) - H| = |C^*| > 1$; a contradiction to (i). □
3. The three-connected case

In this section, we prove Theorem 1.2, the main result of the paper. Recall that a line of a matroid is a rank-2 flat. We call a line long if it contains at least three points.

Proof. We omit the straightforward argument showing that if (ii) holds, then so does (i). Now assume that (i) holds. If \( r(M) = 2 \) or \( r^*(M) = 2 \), then, as \( M \) is 3-connected, it is isomorphic to \( U_{2,n} \) or \( U_{n-2,n} \) for some \( n \geq 4 \). Thus we may assume that both the rank and corank of \( M \) exceed two. The next lemma determines the possibilities for a matroid \( M \) satisfying (i) if its rank or corank is three.

Lemma 3.1. If \( r(M) = 3 \) or \( r^*(M) = 3 \), then \( M \) is isomorphic to one of \( U_{3,6} \), \( P_6 \), \( Q_6 \), \( \#^3 \), \( P_7 \), \( P_7^* \), \( F_7^- \), \( F_7^- \), and \( (F_7^-)^* \).

Proof. By duality, we may assume that \( r(M) = 3 \), otherwise replace \( M \) by \( M^* \) in the argument that follows. Let \( M \) be a counterexample to the lemma having the least number of elements. By Lemma 2.5, \( |E(M)| > 6 \). Now assume that \( |E(M)| = 7 \). By Lemma 2.10(i), the matroid \( M \) has no \( U_{2,4} \)-restriction. Thus all long lines of \( M \) have exactly three points. If there are 7, 6, or 5 such lines, then an easy combinatorial argument shows that \( M \) has a geometric representation as in Fig. 1 or Fig. 2(b).

Hence \( M \) is isomorphic to one of \( F_7 \), \( F_7^- \), \( F_7^- \), and \( P_7 \). Since \( F_7 \) is binary but \( M \) is not, we deduce that \( M \) satisfies the conclusion of the lemma if it has more than 4 long lines. Thus we may assume that \( M \) has at most 4 long lines.

If \( x \) and \( y \) are distinct elements of \( E(M) \), we call \( \{x, y\} \) good if \( x \) is on at most one long line missing \( y \), and \( y \) is on at most one long line missing \( x \). If there are no long lines, then every 2-subset of \( E(M) \) is good. Moreover, as there are at most four long lines, every long line contains a 2-subset that is good. Thus \( E(M) \) certainly has a good 2-subset. Choose such a subset \( \{x, y\} \) for which \( M \setminus x, y \) has as few long lines as possible. Then \( M \setminus x / y \) and \( M / x \setminus y \) are both 5-element rank-2 matroids that have at most one non-trivial parallel class. Since every such parallel class contains at most two elements, \( M \setminus x / y \) and \( M / x \setminus y \) are non-binary. Thus \( M \setminus x, y \) is binary, and it follows that its geometric representation consists of two intersecting 3-point lines. Then, up to an interchange of \( x \) and \( y \), the geometric representation of \( M \) is as in Fig. 3 where solid lines exist and dashed lines may or may not exist. However, in each of these configurations, \( \{x', y'\} \) is a good subset that contradicts the choice of \( \{x, y\} \). Thus the lemma holds if \( |E(M)| = 7 \).

We may now assume that \( |E(M)| \neq 7 \). Then Lemma 2.5 implies that \( M \) has a proper minor isomorphic to one of \( U_{3,6} \), \( P_6 \), \( Q_6 \), and \( \#^3 \). Arbitrarily choose such a minor \( N \). From the Splitter theorem and the fact that \( M \) and \( N \) have rank 3, it follows that there is an element \( e \) of \( M \) such that \( M \setminus e \) is 3-connected and has an \( N \)-minor. Since \( M \) is 3-connected and satisfies Theorem 1.2(i), the matroid \( M \setminus e \) has rank 3 and satisfies Theorem 1.2(i). Hence, by the minimality of \( M \), we may assume that \( M \setminus e \) is isomorphic to one of \( U_{3,6} \), \( P_6 \), \( Q_6 \), \( \#^3 \), \( P_7 \), \( F_7^- \), and \( F_7^- \). Thus, as \( |E(M)| \neq 7 \),
the matroid $M \setminus e$ is isomorphic to one of $P_7$, $F_7^-$, and $F_7^-$. However, none of these matroids has a $U_{3,6}$- or $P_6$-minor, so $N$ cannot be isomorphic to $U_{3,6}$ or $P_6$. Since $N$ was chosen arbitrarily among $U_{3,6}$, $P_6$, $Q_6$, and $W^3$, it follows that $M$ has no $U_{3,6}$- or $P_6$-minor. Finally, by Lemma 2.10(i), $M$ has no $U_{2,4}$-restriction. Thus we conclude that $M$ is an 8-element matroid with a $P_7$, $F_7^-$, or $F_7^-$-restriction, but no $U_{2,4}$-, $U_{3,6}$-, or $P_6$-restriction. From this, it is not hard to verify that $M$ does not satisfy Theorem 1.2(i); a contradiction. □

We now assume that $r(M) > 3$ and $r^*(M) > 3$. Let $C = C(M) = \{e \in E(M): M/e$ is binary$\}$, $D = D(M) = \{e \in E(M): M \setminus e$ is binary$\}$, and $Z = Z(M) = \{e \in E(M): M \setminus e$ and $M/e$ are non-binary$\}$. The next result, due to Lemos [3], gives valuable information regarding the cardinality of the set $Z(M)$.

**Theorem 3.2.** Let $H$ be a 3-connected non-binary matroid. Then $Z(H)$ is empty or $|Z(H)| \geq 3$.

Now if $Z = \emptyset$, then it follows from Theorem 1.1 that (ii) holds. Thus we may assume that $|Z| \geq 3$. Moreover, if $e \in C \cap D$, then $M$ is non-binary while $M \setminus e$ and $M/e$ are binary. Thus Corollary 2.9 implies that $M \cong U_{2,4}$. As this contradicts the assumption that $r(M) > 3$, we conclude that $C \cap D = \emptyset$. Therefore, the sets $C$, $D$, and $Z$ partition $E(M)$.

We denote by $G$ or $G(M)$ the simple graph that has $Z$ as its vertex set and has $uv$ as an edge if and only if $M \setminus u, v$ is binary. We shall prove that $G$ is either the triangular prism or a complete bipartite graph with equicardinal color classes $U_1$ and $U_{-1}$. In the latter case, it will turn out that $M$ is obtained by relaxing the circuit-hyperplanes $U_1 \cup D$ and $U_{-1} \cup D$ of a 3-connected binary matroid. If $G$ is the triangular prism, $M$ will turn out to be isomorphic to $J$. It follows from the next lemma that $G(M) = G(M^*)$.

**Lemma 3.3.** Let $e$ be an element of $Z$.

(i) If $f$ is an element of $E(M) - e$, then exactly one of the matroids $M \setminus e, f$ and $M \setminus e \setminus f$ is binary, while exactly one of $M/e \setminus f$ and $M/e, f$ is binary.

(ii) If $f$ is an element of $Z$, then either

(a) $M \setminus e, f$ and $M/e, f$ are binary while $M \setminus e \setminus f$ and $M/e \setminus f$ are non-binary; or

(b) $M \setminus e \setminus f$ and $M/e \setminus f$ are binary while $M \setminus e, f$ and $M/e, f$ are non-binary.
Theorem 2.1(ii) holds with exactly one of \( r(M) = 3 \), contrary to the assumption that \( r(M) > 3 \). We conclude that, for every element \( e \) of \( Z \) and every element \( f \) of \( E(M) - e \), at least one of \( M \setminus e \), \( f \) and \( M \setminus e \setminus f \) is non-binary. Dually, at least one of \( M/e, f \) and \( M/e \setminus f \) is non-binary. Since at least two of the four minors of \( M \) obtained by removing \( e \) and \( f \) must be binary, exactly one of \( M \setminus f \) and \( M \setminus f \setminus e \) is binary, and exactly one of \( M/f \setminus e \) and \( M/f, e \) is binary. Hence (i) holds. Now, if \( f \) is also an element of \( Z \), then, by symmetry, exactly one of \( M \setminus f \setminus e \) and \( M \setminus f, e \) is binary. Thus (ii) holds.  

For each element \( e \) of \( Z \), we define \( X_M(e) \) or \( X(e) \) to be the set of neighbors of \( e \) in \( G \) and we define \( Y_M(e) \) or \( Y(e) \) to be the set of non-neighbors of \( e \) in \( G \). Then \( X(e) \cap Y(e) = \emptyset \) and \( X(e) \cup Y(e) = Z - e \). Thus, for each \( e \) in \( Z \), the sets \( C, D, X(e), \) and \( Y(e) \cup e \) partition \( E(M) \). Moreover, Lemma 3.3(ii) implies that

\[
X(e) = \{ x \in Z - e : M \setminus e, x \text{ and } M/e, x \text{ are binary while } M \setminus e \setminus x \text{ and } M/e \setminus x \text{ are non-binary} \}
\]

and

\[
Y(e) = \{ y \in Z - e : M \setminus e, y \text{ and } M/e, y \text{ are binary while } M \setminus e, y \text{ and } M/e, y \text{ are non-binary} \}.
\]

Clearly, \( X_M(e) = X_M^*(e) \) and \( Y_M(e) = Y_M^*(e) \), while \( D(M^*) = C(M) \).

Lemma 3.4. Suppose that \( e \in Z \). Then \( M \setminus e \) is obtained from a connected binary matroid by relaxing the circuit-hyperplane \( X(e) \cup D \) while \( M/e \) is obtained from a connected binary matroid by relaxing the circuit-hyperplane \( Y(e) \cup D \).

Proof. For every element \( f \) of \( E(M) - e \), we have, by Lemma 3.3, that either \( (M \setminus e) \setminus f \) or \( (M \setminus e) \setminus f \) is binary. Thus Theorem 2.1 implies that one of (a)–(d) of Theorem 2.1(ii) holds with \( M \setminus e \) replacing \( M \). But each of (ii)(b)–(d) contradicts the assumption that both \( r(M) \) and \( r^*(M) \) exceed 3. Hence \( M \setminus e \) is obtained from a binary matroid \( N_d(e) \) by relaxing a circuit-hyperplane \( H \).

We now show that \( H = X(e) \cup D \). Assume \( x \in X(e) \cup D \). Then \( M \setminus e, x \) is binary. Now, if \( x \not\in H \), then Lemma 2.2(i) implies that \( M \setminus e/x \) equals the binary matroid...
$N_{D}(e)/x$. Since this contradicts Lemma 3.3, we conclude that $x \in H$, and $X(e) \cup D \subseteq H$. Moreover, Lemma 2.2(ii) implies that, for every $h \in H$, the matroid $M \setminus e, h$ equals the binary matroid $N_{D}(e)/h$. It follows that $h \in X(e) \cup D$, and thus $H \subseteq X(e) \cup D$. Therefore, $X(e) \cup D = H$ and $M \setminus e$ is obtained from a connected binary matroid by relaxing the circuit-hyperplane $X(e) \cup D$.

It follows from duality and the above argument that $M^* \setminus e$ is obtained from a binary matroid by relaxing the circuit-hyperplane $X(e) \cup D(M^*) = X(e) \cup C(M)$. As the complement of $X(e) \cup C(M)$ in $E(M) - e$ is $Y(e) \cup D(M)$, the matroid $(M^* \setminus e)^* = M/e$ is obtained from a binary matroid by relaxing $Y(e) \cup D(M)$ and the lemma holds.

We now list some consequences of Lemma 3.4. First, as $X(e) \cup D$ and $Y(e) \cup D$ are bases of $M \setminus e$ and $M/e$, respectively, it follows that, for every $e$ in $Z$.

3.5. $X(e) \cup D$ and $(Y(e) \cup e) \cup D$ are bases of $M$.

Hence, setting $n = r(M) - |D|$, we see that $G$ is an $r$-regular graph on $2n$ vertices. Moreover, $n \geq 2$ since $|Z| \geq 3$.

Let $e$ be an element of $Z$. Now, as $M/e$ is obtained by relaxing the circuit-hyperplane $Y(e) \cup D$, it follows from Lemma 2.3 that $C_{M/e}(f, Y(e) \cup D) = Y(e) \cup D \cup f$ for every $f$ in $E - (Y(e) \cup e \cup D)$. Thus we deduce that,

3.6. For every $f$ in $E - (Y(e) \cup e \cup D)$, either $Y(e) \cup D \cup f$ or $Y(e) \cup D \cup \{e, f\}$ is a circuit of $M$.

Similarly, as $C_{M \setminus e}(f, X(e) \cup D) = X(e) \cup D \cup f$ for every $f$ in $E - (X(e) \cup D \cup e)$, it follows that, for each $e$ in $Z$,

3.7. $C_{M}(f, X(e) \cup D) = X(e) \cup D \cup f$, for every $f$ in $E - (X(e) \cup D \cup e)$.

The next lemma concerns circuits of $M$ contained in $Z - e$. This result will enable us to obtain useful information about the triangles in $G$.

Lemma 3.8. Suppose that $e$ is an element of $Z$ and $C_1$ is a circuit of $M$ contained in $Z - e$.

(i) If $C_1$ contains neither $X(e)$ nor $Y(e)$, then $C_1$ contains at least two elements of each of $X(e)$ and $Y(e)$.

(ii) If $D \neq \emptyset$, then $C_1$ contains at least two elements of each of $X(e)$ and $Y(e)$.

Proof. We prove (i) and (ii) simultaneously. Since $X(e)$ and $Y(e)$ are independent in $M$, every circuit of $M/(Z - e)$ contains elements of both $X(e)$ and $Y(e)$. Suppose $C_1 \cap Y(e) = \{y\}$. Then, by (3.7), the set $X(e) \cup D \cup y$ is a circuit of $M$. Moreover, if $D \neq \emptyset$ or $C_1$ does not contain $X(e)$, then $C_1$ is properly contained in $X(e) \cup D \cup y$; a contradiction. Thus we conclude that $|C_1 \cap Y(e)| \geq 2$. Similarly if $C_1 \cap X(e) = \{x\}$, then,
whenever $D \neq \emptyset$ or $C_1$ does not contain $Y(e)$, statement (3.6) leads to the contradiction that $C_1$ is properly contained in a circuit. Thus $|C_1 \cap X(e)| \geq 2$. □

The next two lemmas describe properties of the triangles of $G$.

**Lemma 3.9.** If $uvw$ is a triangle of $G$ and $x, y \in Y(w)$, then $\{u, v, x, y\}$ is a circuit and a cocircuit in $M$.

**Proof.** As $G(M) = G(M^*)$, it suffices to prove that $\{u, v, x, y\}$ is a circuit in $M$. It follows from (3.7) that $X(w) \cup D \cup x$ and $X(w) \cup D \cup y$ are circuits in $M$. Hence $(X(w) - \{u, v\}) \cup D \cup x$ and $(X(w) - \{u, v\}) \cup D \cup y$ are circuits in the binary matroid $M/u, v$. Therefore, their symmetric difference $\{x, y\}$ is dependent in $M/u, v$. Consequently, $\{u, v, x, y\}$ is dependent in $M$. If $\{u, v, x, y\}$ is not a circuit of $M$, then it properly contains a circuit $C_1$. Then Lemma 3.8 implies that $D = \emptyset$ and $C_1$ contains either $X(w)$ or $Y(w)$. Since $|Y(w)| \geq 2$, we have $|X(w)| \geq 3$. Now, as $|C_1 \cap X(w)| \leq 2$, we deduce that $C_1$ contains $Y(w)$ and thus $|Y(w)| \leq 2$. Therefore $n = |X(w)| \leq 3$. But this contradicts the fact that $|D| + n = r(M) \geq 4$ and we conclude that $\{u, v, x, y\}$ is a circuit of $M$. □

**Lemma 3.10.** No two distinct triangles of $G$ share an edge.

**Proof.** Let $uvw$ and $vwz$ be two different triangles in $G$. Since $|Y(w)| = n - 1 = |X(w)| - 1 \geq |\{u, v, x, z\}| - 1 = 2$, there are at least two vertices of $G$ that are not adjacent to $w$. Let $x$ and $y$ be two such non-neighbors of $w$. By Lemma 3.9, both $\{u, v, x, y\}$ and $\{v, z, x, y\}$ are circuits of $M$. Hence, by circuit-elimination, $\{u, v, z, x\}$ is a circuit of $M$. Now (3.7) implies that $X(w) \cup D \cup x$ is a circuit of $M$. Since $\{u, v, z, x\}$ is a subset of $X(w) \cup D \cup x$, it follows that $D = \emptyset$ and $X(w) = \{u, v, z\}$. Thus $|D| = 0$ and $n = 3$, contradicting the fact that $|D| + n = r(M) \geq 4$. □

The next lemma determines the possibilities for the graph $G$. The triangular prism is shown in Fig. 4. Recall that $G$ is an $n$-regular graph on $2n$ vertices.

**Lemma 3.11.** $G$ is either isomorphic to the complete bipartite graph $K_{n,n}$ for some $n \geq 2$ or $G$ is the triangular prism.
Proof. First assume that $G$ has no triangles and fix an edge $uv$ of $G$. Then $X(u)$ and $X(v)$ are disjoint stable sets in $G$. Now, as both sets have $n$ elements, $X(u)$ and $X(v)$ partition $Z$. Thus, for each $y$ in $X(u)$, we have that $X(y) \subseteq X(v)$. Hence, as $y$ has degree $n$, it follows that $X(y) = X(v)$ for each $y$ in $X(u)$. Moreover, by the symmetry between $u$ and $v$, we have $X(y) = X(u)$ for each $y$ in $X(v)$. Thus $G$ is isomorphic to $K_{n,n}$.

Now assume that $uvw$ is a triangle of $G$. By Lemma 3.10, no two triangles in $G$ share an edge. Thus the sets $X(u) - \{v,w\}$, $X(v) - \{u,w\}$, and $X(w) - \{u,v\}$ must be pairwise disjoint subsets of $Z - \{u,v,w\}$. Since each of these three sets has $n - 2$ elements, we have $3 + 3(n - 2) \leq 2n$. Hence $n \leq 3$. Furthermore, as no 2-regular graph on four vertices has a triangle, $n = 3$. Therefore each of $X(u) - \{v,w\}$, $X(v) - \{u,w\}$, and $X(w) - \{u,v\}$ is a singleton. Let these sets be $\{u'\}$, $\{v'\}$, and $\{w'\}$, respectively. Now, as $|Z| = 2n = 6$, we see that $Z = \{u,v,w,u',v',w'\}$. Moreover, $G$ contains the edges $uv, vw, uw, uu', vv', ww'$, and possibly $u'v', v'w'$, and $u'w'$. Since $G$ is 3-regular, each of $u'v', v'w'$, and $u'w'$ must be an edge of $G$. Thus $G$ is the triangular prism. □

We now divide the proof of Theorem 1.2 into two cases:

Case A: $G \cong K_{n,n}$ for some $n \geq 2$; and

Case B: $G$ is the triangular prism.

Consider Case A. Let $U_i$ and $U_{-i}$ be the color classes of $G$. For each $i$ in $\{1,-1\}$, if $v \in U_i$, then $X(v) = U_{-i}$ and $Y(v) \cup \{v\} = U_i$.

Lemma 3.12. Suppose $i \in \{1,-1\}$. Then $U_i \cup D$ is a basis of $M$ and, for each element $f$ of $E - (U_i \cup D)$, the fundamental circuit $C_M(f, U_i \cup D) = U_i \cup D \cup f$.

Proof. Suppose $i \in \{1,-1\}$ and $f$ is an element of $E - (U_i \cup D)$. Let $e$ be an element of $U_{-i} - \{f\}$. Then $X(e) = U_i$ and, by (3.5), the set $U_i \cup D = X(e) \cup D$ is a basis of $M$. Furthermore, $f \in E - (U_i \cup D \cup e)$ and (3.7) implies that $C_M(f, U_i \cup D) = U_i \cup D \cup f$. □

On combining the last lemma with Lemma 2.3, we deduce that, for each $i$ in $\{1,-1\}$, there is a matroid $N_i$ such that $M$ is obtained from $N_i$ by relaxing the circuit-hyperplane $U_i \cup D$. Moreover, as $|(U_i \cup D) \cap (U_{-i} \cup D)| = |D| = r(M) - n < r(M) - 1$, it follows from Lemma 2.4 that there is a matroid $N$ with circuit-hyperplanes $U_i \cup D$ and $U_{-i} \cup D$ such that relaxing $U_i \cup D$ yields $N_{-i}$ for each $i$ in $\{1,-1\}$ and such that relaxing both circuit-hyperplanes yields $M$. In other words, $M$ is a double relaxation of $N$. The next two lemmas establish that $N$ is 3-connected and binary. It will then follow that, if $G(M) \cong K_{n,n}$ for some $n \geq 2$, the matroid $M$ satisfies Theorem 1.2(ii)(b) and the theorem holds.

Lemma 3.13. $N$ is 3-connected.
Proof. Suppose that \( N \) is not 3-connected. Then there is a partition \((S,T)\) of \( E(N) \) such that for some \( k \) in \( \{1,2\} \), we have \(|S|,|T| \geq k \), and

3.14. \( r_N(S) + r_N(T) - r(N) = k - 1 \).

Now \( E(N) = E(M) \) and all subsets of \( E(M) \) except \( U_1 \cup D \) and \( U_{-1} \cup D \) have the same rank in \( N \) as they do in \( M \). Since \( M \) is 3-connected, we deduce that \( S \) or \( T \) equals \( U_1 \cup D \) or \( U_{-1} \cup D \). Suppose \( D \) and \( C \) are empty and \( \{S,T\} = \{U_1, U_{-1}\} \). Since \( r_N(U_i \cup D) = r(N) - 1 \) for each \( i \) in \( \{1,-1\} \), it follows from (3.14) that \( r(N) = k + 1 \). However, \( r(N) = r(M) \geq 4 \) and \( k + 1 \leq 3 \); a contradiction. We may now assume that \( D \cup C \) is non-empty. In addition, suppose that \( T = U_1 \cup D \) and \( S = E - (U_1 \cup D) = U_{-1} \cup C \). It follows from (3.14) that \( r_N(U_{-1} \cup C) = k \). Moreover, as \( D \cup C \neq \emptyset \), the set \( U_{-1} \cup C \) has the same rank in both \( M \) and \( N \). In particular, \( r_d(U_{-1} \cup C) = k \) for some \( k \) in \( \{1,2\} \). Now \( |U_{-1} \cup C| \geq 4 \) since \( U_1 \cup D \) is a basis of \( M \) and \( r^*(M) \geq 4 \). Thus, as \( M \) is 3-connected, \( k \neq 1 \). Hence \( k = 2 \). Therefore, \( U_{-1} \cup C \) is a cobasis of \( M \) contained in a line \( L \) that has at least 4 points. Since \( r(M) \geq 4 \), we deduce that \( M \) has a non-binary hyperplane, contradicting Lemma 2.10(i). We conclude that \( N \) is 3-connected. \( \square \)

**Lemma 3.15.** \( N \) is binary.

Proof. Suppose \( N \) is non-binary and \( e \in Z(M) \). Now \( \{X(e), Y(e) \cup e\} = \{U_1, U_{-1}\} \). Suppose \( e \in U_1 \). Then \( X(e) = U_{-1} \) and Lemma 3.4 implies that \( M \setminus e \) is obtained from a binary matroid \( N_d(e) \) by relaxing the circuit-hyperplane \( U_{-1} \cup D \). Since \( M \) is obtained from \( N_{-1} \) by relaxing the same circuit-hyperplane and \( e \in U_1 \), Lemma 2.2(i) implies that \( M \setminus e \) is obtained from \( N_{-1} \setminus e \) by relaxing \( U_{-1} \cup D \). Thus \( D(N_{-1} \setminus e) = D(N_d(e)) \) and we have \( N_{-1} \setminus e = N_d(e) \). Now, as \( e \in U_1 \cup D \) and \( N_{-1} \) is obtained from \( N \) by relaxing \( U_1 \cup D \), it follows from Lemma 2.2(ii) that \( N \setminus e = N_{-1} \setminus e \). Thus \( N \setminus e = N \setminus e = N_d(e) \). In particular, \( N \setminus e \) is binary since \( N_d(e) \) is binary.

Dually, Lemma 3.4 implies that \( M^* \setminus e \) is obtained from a binary matroid by relaxing the circuit-hyperplane \( U_{-1} \cup C \). In addition, \( M^* \) is obtained from \( N^*_d \) by relaxing \( U_1 \cup C \) while \( N^*_{d} \) is obtained from \( N^* \) by relaxing \( U_{-1} \cup C \). It follows from the argument above that \( (N^* \setminus e) = (N^* \setminus e)^* \) is binary. Since both \( N \setminus e \) and \( N^* \setminus e \) are binary, Corollary 2.9 implies that \( N \cong U_{2,4} \), contrary to the fact that \( r(N) = r(M) \geq 4 \). We conclude that \( N \) is binary. \( \square \)

This completes the proof of Theorem 1.2 for \( M \) in Case A, that is, when \( G(M) \cong K_{n,n} \). Next we consider Case B, that is, when \( G(M) \) is the triangular prism.

Now, as \( G \) has six vertices, \( n = 3 \) and \( r(M) = |D| + 3 \). Next we show that \( |D| = |C| = 1 \). Suppose \( \{d_1, d_2\} \subseteq D \). As \( M \) is non-binary and 3-connected, Lemma 2.7 implies that there is a \( U_{2,4} \)-minor of \( M \) using \( d_1 \) and \( d_2 \). Let \( M \setminus S/T \) be such a minor where \( |T| = r(M) - 2 \). Then \( C \cap T = \emptyset \). Since \( r(M) = |D| + 3 \), we have \( |T| = |D| + 1 \). Moreover, neither \( d_1 \) nor \( d_2 \) is an element of \( T \), since both are used in the \( U_{2,4} \)-minor. Thus \( T \) contains at least three elements of \( Z \). Assume that \( \{u, v, w\} \subseteq Z \cap T \). Every collection of
three vertices of $G$ contains a pair of adjacent vertices. Thus we may assume that $uv$ is an edge of $G$. Then $M'\setminus u,v$ is binary, and it follows from Lemma 3.3(ii) that $M/uv$ is also binary. However, as $M'\setminus S/T \cong U_{2,4}$, the matroid $M/uv$ is non-binary. As a result of this contradiction, we conclude that $|D| < 1$. However, if $D = \emptyset$, then $r(M) = 3$; a contradiction. Thus $|D| = 1$. By duality, $|C| = 1$, and it follows that $|E(M)| = 8$. We set $D = \{d\}$ and $C = \{c\}$.

Lemma 3.16. $M$ is ternary.

Proof. Suppose $M$ is non-ternary. Then, by duality, we may assume that $M$ has a $U_{2,5}$- or an $F_7$-minor. First suppose that $M$ has an $F_7$-minor. Then, as $M$ has corank 4 and $F_7$ is binary, $M/c \cong F_7$. Let $uv$ be an edge of $G$ that is in no triangles. Then $X(u)$ and $X(v)$ are disjoint. By (3.7), the sets $X(u) \cup \{d,c\}$ and $X(v) \cup \{d,c\}$ are circuits of $M$. Hence $X(u) \cup \{d\}$ and $X(v) \cup \{d\}$ are 4-circuits in $M/c$ that share exactly one element. However, $F_7$ does not have such a pair of 4-circuits. Thus $M$ has no $F_7$-minor and we may now assume that $M$ has a $U_{2,5}$-minor. Hence $M/xy \cong U_{2,5}$ for certain $x$, $y$, and $z$. One of $x$ and $y$ is an element of $Z$ as neither of them is in $C$ and $|D| = 1$. Thus we may assume that $x \in Z$. For each element $w$ of $E - \{x,y,z\}$, we have that $M/xyw$ is non-binary since $M/xyw\setminus z \cong U_{2,4}$. Thus $E - \{x,y,z\} \subseteq X(x)$ and we get the contradiction $5 = |E - \{x,y,z\}| \leq |X(x) \cup C| = 4$. □

Let $Z = \{u,v,w,u',v',w'\}$ and suppose that $uw'$ and $uw'v'$ are the triangles in $G(M)$ while $uw'$, $vv'$, and $ww'$ are the edges that are contained in no triangles. The graphs $H_1$, $H_2$, $H_3$, and $H_4$, which appear in the next few lemmas, are shown in Fig. 5.

Lemma 3.17. $M\setminus d$ is graphic, and $M\setminus d,c \cong M(H_1)$. Moreover, the series classes of $H_1$ correspond to the pairs $\{u,u'\}$, $\{v,v'\}$, and $\{w,w'\}$.

Proof. The matroid $M\setminus d$ is ternary and binary and has fewer elements than $M^*(K_5)$ and $M^*(K_{3,3})$. Hence $M\setminus d$ is graphic. Then $M\setminus d,c$ is the cycle matroid of a connected graph $H$. By Lemma 3.9, each 4-cycle in the triangular prism $G$ is a 4-circuit in $M$. It
is now easy to verify that \( H \) is isomorphic to either \( K_4 \) or the graph \( H_1 \). However, as \( M \) is 3-connected, \( r(M \setminus d, c) = 4 \) implying that \( H \) has five vertices. Therefore \( M \setminus d, c \cong M(H_1) \). The statement about the three series classes now follows immediately.

**Lemma 3.18.** \( M \setminus d \) is isomorphic to one of \( M(H_2) \) and \( M(H_3) \) with \( c \) as indicated in Fig. 5.

**Proof.** This follows from Lemma 3.17 and the fact that \( M \setminus d \) has no parallel elements, loops, or coloops. □

**Lemma 3.19.** \( M \setminus d \) or \( M^* \setminus c \) is isomorphic to \( M(H_5) \).

**Proof.** If not, then, by Lemma 3.18 and duality, \( M \setminus d \) and \( M^* \setminus c \) are both isomorphic to \( M(H_5) \). Hence \( M/c \) is isomorphic to \( M(H_4) \), with \( d \) as indicated in Fig. 5. Therefore, \( M \setminus d/c \) is isomorphic to the cycle matroids of both \( H_3/c \) and \( H_4 \setminus d \). However, \( M(H_3/c) \not\cong M(H_4 \setminus d) \); a contradiction. □

By duality, we may now assume that \( M \setminus d \cong M(H_2) \). Then Lemma 3.17 implies that the set \( \{c, u, v, w\} \) is a basis of \( M \setminus d \), and hence also of \( M \). Therefore, by using row and column scaling, we may assume that \( M \) has a ternary representation of the following form:

\[
\begin{bmatrix}
c & u & v & w & u' & v' & w' & d \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & x \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & \beta \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & \gamma \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & \delta 
\end{bmatrix}
\]

Now, as \( M \) has no series elements, each of \( \beta, \gamma, \) and \( \delta \) is non-zero. Since (3.7) implies that \( X(u) \cup d = \{u', v, w, d\} \) is a basis of \( M \), we have that \( x \neq \beta \). Moreover, as \( \{u', v, w, d\} \) is also a basis of the binary matroid \( M \setminus v', w' \), we conclude that \( x \neq -\beta \). Hence \( x = 0 \).

As \( u \) and \( u' \) are adjacent in \( G(M) \), the matroid \( M/u, u' \) is non-binary. Hence \( \gamma = \delta \).

From the symmetry between \( u \) and \( w \), we also have that \( \beta = \gamma \). In addition, by scaling the last column, we may assume that \( \beta = \gamma = \delta = 1 \). Thus \( M \) has the following ternary representation:

\[
\begin{bmatrix}
c & u & v & w & u' & v' & w' & d \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 
\end{bmatrix}
\]
Since this is the ternary representation of \( J \), it has been shown that if \( G(M) \) is the triangular prism, then \( M \cong J \). This completes the proof in Case B and thereby completes the proof of Theorem 1.2. \( \square \)

4. The general case

In this section, we determine all non-binary matroids \( M \) such that at least half of the minors that arise by removing two elements from \( M \) are binary. Recall that \( C(M) = \{ e \in E(M) : M/e \text{ is binary} \} \), and \( D(M) = \{ e \in E(M) : M/e \text{ is binary} \} \).

**Theorem 4.1.** The following two statements about a matroid \( M \) are equivalent.

(i) \( M \) is non-binary and, for every \( \{ e, f \} \subseteq E(M) \), at least two of \( M \setminus e, f; M \setminus e/f; M/e \setminus f \) and \( M/e, f \) are binary.

(ii) (a) \( M \) is isomorphic to \( U_{2,n} \) or \( U_{n-2,n} \) for some \( n \geq 4 \); or

(b) both the rank and corank of \( M \) exceed two and \( M \) can be obtained from a connected binary matroid by relaxing a circuit-hyperplane; or

(c) both the rank and corank of \( M \) exceed two and \( M \) can be obtained from a connected binary matroid by relaxing two circuit-hyperplanes; or

(d) \( M \) is isomorphic to one of \( U_{3,6}, P_6, P_7, P_7^* \), and \( J \); or

(e) \( M \) is isomorphic to \( U_{2,4} \oplus U_{2,4} \); or

(f) \( M \) is obtained from a matroid \( \tilde{M} \) described in (a) or (b) by the addition of a loop or coloop, or by series extension of a subset \( S \) of \( D(\tilde{M}) \) or parallel extension of a subset \( T \) of \( C(\tilde{M}) \) where \( S \cap T = \emptyset \); or

(g) \( M \) is obtained from a matroid \( \tilde{M} \) described in (a), (b), (c), or (d) by series extension of a subset \( S \) of \( D(\tilde{M}) \) or parallel extension of a subset \( T \) of \( C(\tilde{M}) \) where \( S \cap T = \emptyset \).

**Proof.** We omit the straightforward proof showing that if (ii) holds, then so does (i). Now assume (i) holds. We argue by induction on \( |E(M)| \) to show that (ii) holds. If \( M \) is 3-connected, then the result follows easily from Theorem 1.2. Assume the result is true for all matroids satisfying the hypotheses and having fewer elements than \( M \).

If \( M \) is disconnected, then \( M = M_1 \oplus M_2 \) where \( M_1 \) or \( M_2 \) is non-binary. Suppose \( M_2 \) is non-binary. If \( e, f \in E(M_1) \), then each of \( M \setminus e, f; M \setminus e/f; M/e \setminus f \) has the non-binary matroid \( M_2 \) as a minor; a contradiction. Thus \( M_1 \) consists of an element \( f \) that is either a loop or a coloop. If there is an \( e \) in \( E(M_2) \) such that \( M_2 \setminus e \) and \( M_2/e \) are non-binary, then \( M \setminus e, f; M \setminus e/f; M/e \setminus f \) are non-binary; a contradiction. Thus we may assume that, for every element \( e \) of \( M_2 \), at least one of \( M_2 \setminus e \) and \( M_2/e \) is binary. It follows from Theorem 2.1 that \( M \) satisfies (ii)(f) if \( M \) is disconnected.

Suppose that \( M \) is connected but not 3-connected. Then \( M = M_1 \oplus M_2 \) for some connected matroids \( M_1 \) and \( M_2 \) such that \( E(M_1) \cap E(M_2) = \{ p \} \) and \( |E(M_1)|, |E(M_2)| \geq 3 \). Since \( M \) is connected, we may assume that \( M_1 \) and \( M_2 \) are connected. Suppose \( M_1 \) and \( M_2 \) are both non-binary. Then, by Lemma 2.6, both \( M_1 \) and \( M_2 \) have a \( U_{2,4} \)-minor
using the basepoint \( p \). Thus \( M \) has \( U_{2,4} \oplus 2 U_{2,4} \) as a minor. By duality, \( M \) has a connected but not 3-connected single-element extension \( M_1 \) of \( U_{2,4} \oplus 2 U_{2,4} \) as a minor. It is not difficult to check that each of the three possibilities for \( M_1 \) implies that \( M \) fails to satisfy (i).

We may now assume that \( M = M_1 \oplus M_2 \) and exactly one of \( M_1 \) and \( M_2 \) is non-binary. Suppose \( M_2 \) is non-binary. By Lemma 2.10(ii), we may assume that \( M_2 \cong U_1 \oplus n \) for some \( n \geq 3 \), otherwise we replace \( M \) by \( M' \) in the argument that follows. We may also suppose that \( M_1 \) has no elements parallel with the basepoint \( p \), since any such element may be taken to be in \( M_2 \) rather than \( M_1 \). Hence \( M \) is obtained from \( M_1 \) by replacing \( p \) by \( n \) elements parallel to \( p \). Moreover, \( M_1 / p \) is binary. To see this, suppose \( M_1 / p \) is non-binary and let \( s \) and \( t \) be elements of \( E(M_2) - p \). Then, as \( s \) and \( t \) are parallel in \( M \), we have \( M \setminus s \cup t \cong M' \setminus s \cup t \cong M / s \cup t \). Moreover, \( M / s \cup t \setminus (E(M_2) - \{p, s, t\}) \) equals \( M_1 / p \). If \( M_1 / p \) is non-binary, then at least three of the minors of \( M \) that involve the elimination of \( s \) and \( t \) are non-binary; a contradiction. We conclude that \( M_1 / p \) is binary. Thus \( p \in C(M_1) \).

By the induction assumption, one of (ii)(a)–(g) holds for \( M_1 \). Notice that it is impossible for \( M_1 \) to be isomorphic to \( U_{2,4} \oplus 2 U_{2,4} \) since \( p \in C(M_1) \), yet \( U_{2,4} \oplus 2 U_{2,4} \) has no single-element contraction that is binary. Thus one of (ii)(a)–(g), other than (ii)(e), holds for \( M_1 \). Since \( M \) is obtained from \( M_1 \) by the parallel extension of the element \( p \) of \( C(M_1) \), it is clear that \( M \) satisfies (ii)(f) or (ii)(g). \( \Box \)

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References