1-1-2002

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Recommended Citation
Geelen, J., Oxley, J., Vertigan, D., & Whittle, G. (2002). Totally free expansions of matroids. *Journal of Combinatorial Theory. Series B, 84* (1), 130-179. https://doi.org/10.1006/jctb.2001.2068

Journal of Combinatorial Theory, Series B **84**, 130–179 (2002)
doi:10.1006/jctb.2001.2068, available online at http://www.idealibrary.com on IDEAL®

Totally Free Expansions of Matroids

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Received March 9, 1999; published online September 20, 2001

The aim of this paper is to give insight into the behaviour of inequivalent representations of 3-connected matroids. An element x of a matroid M is fixed if there is no extension M' of M by an element x' such that $\{x, x'\}$ is independent and M' is unaltered by swapping the labels on x and x'. When x is fixed, a representation of $M \backslash x$ extends in at most one way to a representation of M. A 3-connected matroid N is totally free if neither N nor its dual has a fixed element whose deletion is a series extension of a 3-connected matroid. The significance of such matroids derives from the theorem, established here, that the number of inequivalent representations of a 3-connected matroid M over a finite field $\mathbf{F}$ is bounded above by the maximum, over all totally free minors N of M, of the number of inequivalent $\mathbf{F}$-representations of N. It is proved that, within a class of matroids that is closed under minors and duality, the totally free matroids can be found by an inductive search. Such a search is employed to show that, for all $r \geqslant 4$, there are unique and easily described rank-r quaternary and quinternary matroids, the first being the free spike. Finally, Seymour's Splitter Theorem is extended by showing that the sequence of 3-connected matroids from a matroid M to a minor N, whose existence is guaranteed by the theorem, may be chosen so that all deletions and contractions of fixed and cofixed elements occur in the initial segment of the sequence.

0095-8956/01 $35.00

1. INTRODUCTION

It is by now a truism to say that the presence of inequivalent representations of matroids over fields is the major barrier to progress in matroid representation theory. Strong results giving characterizations of classes of representable matroids certainly do exist [1, 8, 9, 21–23, 25, 26]. But, in all cases, the class either has a unique representation property, as is the case for binary matroids and ternary matroids over $GF(3)$, or the precise way in which inequivalent representations arise is understood, as is the case for representations of ternary matroids over fields other than $GF(3)$, and quaternary matroids over $GF(4)$. What is needed for progress are techniques that would enable one to characterize the way inequivalent representations arise for more general classes of representable matroids. It is with this problem in mind that the research for this paper was undertaken. What follows is a very relaxed discussion intended to give an intuitive feel for some of the results in this paper.

It has long been noticed that certain elements of a matroid have "freedom," while others are "fixed." Indeed, this notion has been formally studied by Cheung and Crapo [4] and by Duke [6, 7]. An element x of a matroid is *fixed* if the only way to extend the matroid by an element x' so that x and x' are in the same cyclic flats is to put x' in parallel with x. If x is fixed, then any representation of M is determined uniquely by the representation of $M \backslash x$. It is the existence of elements with freedom that gives rise to the potential for inequivalent representations. Consider a minor-closed class $\mathscr{M}$ of matroids. If we knew the matroids in $\mathscr{M}$ whose combined elements have, in some sense, maximum freedom, then we would have insight into the behaviour of inequivalent representations of all members of $\mathscr{M}$. This leads to the definition of a "totally free matroid." For the moment, we can think of a totally free matroid as one for which all elements have freedom (although, of course, the formal definition has to take into account certain technicalities). It turns out that the number of inequivalent representations of a 3-connected matroid M is bounded above by the number of inequivalent representations of a totally free minor. The task, then, is to find all totally free matroids in $\mathscr{M}$. There seems no a priori reason to believe that such totally free matroids do not occur sporadically, but it follows from the main result of this paper that this is not the case. All totally free matroids in $\mathscr{M}$ can be found by an elementary inductive search. Of course, there may well be an infinite number of totally free matroids in $\mathscr{M}$, but, for a natural class, it may be possible to neatly characterize the set of all totally free matroids in that class.

We now illustrate the above ideas on the class of quaternary matroids. Kahn [11] has shown that a 3-connected quaternary matroid is uniquely representable over $GF(4)$. In striking contrast to this is the fact that over any other finite field **F** of non-prime order, no bound can be placed on the number of inequivalent **F**-representations of a 3-connected quaternary matroid. How does this arise? We show in this paper that, for $k \geqslant 4$, there is exactly one totally free rank-k quaternary matroid, namely the "free spike" of rank k. From this it follows that the number of inequivalent representations of a 3-connected quaternary matroid is bounded by the number of inequivalent representations of its largest free-spike minor. This analysis makes it clear that free spikes will play a fundamental role in future work on the structure of subclasses of quaternary matroids.

A "totally free expansion" of a matroid N is, very loosely, a matroid that is totally free relative to N. The core theorems of this paper are proved for this more general concept and, using it, we are able to obtain a substantial strengthening of Seymour's Splitter Theorem [22]. In broad terms, this strengthening asserts that, given a 3-connected minor N of a 3-connected matroid M, we can build from N to M via a chain of 3-connected matroids with the property that elements that are added with freedom are added in the initial segment of the chain.

The techniques of this paper are not particularly difficult. Primarily one analyzes precisely what it means for an element e of a matroid M to have freedom in M and what it means for e to have freedom in M^*. This analysis leads to a number of surprisingly simple lemmas describing the structure that arises, and the main theorems follow without difficulty.

Notation and terminology in this paper follow Oxley [13] with two small exceptions. We denote the simple and cosimple matroid canonically associated with a matroid M by $\text{si}(M)$ and $\text{co}(M)$, respectively. The property that a circuit and a cocircuit of a matroid cannot have exactly one common element is called *orthogonality*.

2. OVERVIEW

Now, we give a more precise description of the results of this paper. Elements x and x' of a matroid M are *clones* if the map that interchanges x and x' and acts as the identity on $E(M)-\{x, x'\}$ is an automorphism of M. In other words, x and x' are clones if they are indistinguishable up to labelling. Thus, an element z of M is *fixed* in M if there is no single-element extension of M by z' in which z and z' are independent clones. Clones provide a very convenient way of capturing the notion of freedom. An element has freedom in the sense described in the introduction if and only if it is not fixed. We also need to consider the dual concept: the element z is *cofixed* in M if there is no single-element coextension of M by z' in which z and z' are coindependent clones. As noted earlier, freedom of an element leads to the possibility of inequivalent representations. We now consider this connection. Assume that M is representable over a field $\mathbf{F}$. Although a given $\mathbf{F}$-representation of $M \backslash x$ may or may not extend to a representation of M, some $\mathbf{F}$-representation must. We will say that a representation of $M \backslash x$ *extends uniquely* if it does extend to a representation of M, and the choice of vector representing x is unique up to scalar multiples. As is well known, representations of a matroid are canonically in one-to-one correspondence with representations of the dual. A representation of M/x *coextends uniquely* if the canonically associated representation of $M^* \backslash x$ extends uniquely to a representation of M^*. We give the very easy proof of the next proposition because it illustrates the usefulness of the notion of clones.

PROPOSITION 2.1. *Let M be representable over the field $\mathbf{F}$. If x is fixed in M, then a representation of $M \backslash x$ that extends to a representation of M does so uniquely. If x is cofixed in M, then a representation of M/x that coextends to a representation of M does so uniquely.*

Proof. Assume that an $\mathbf{F}$-representation A of $M \backslash x$ extends to $\mathbf{F}$-representations $[A \mid \mathbf{x}]$ and $[A \mid \mathbf{x}']$ of M. Evidently $\mathbf{x}$ and $\mathbf{x}'$ are clones in $M[A \mid \mathbf{x}, \mathbf{x}']$. But x is fixed in M, so x cannot be independently cloned. Hence $\{\mathbf{x}, \mathbf{x}'\}$ is a parallel pair, that is, $\mathbf{x}$ and $\mathbf{x}'$ are scalar multiples. ∎

The last result illustrates the fact that the notions of fixed and cofixed elements identify underlying matroid structure that guarantees unique extensions and coextensions of representations. It is of some interest to consider a possible converse to Proposition 2.1. First note that, at times, an element that is not fixed may have a unique extension property because of the size of the field. For example, a representation of $U_{2,4}$ extends uniquely to a representation of $U_{2,5}$ over $GF(4)$, even though $U_{2,5}$ has no fixed elements. But over any larger field, a representation of $U_{2,4}$ does not extend uniquely to a representation of $U_{2,5}$. This is essentially the reason why there is a unique representation theorem for quaternary matroids represented over $GF(4)$, but not for quaternary matroids represented over larger fields. Our initial intuition was that if an element x of an $\mathbf{F}$-representable matroid M is not fixed in M, then, over a sufficiently large extension field of $\mathbf{F}$, there is a representation of $M \backslash x$ that extends to a representation of M, but does not do so uniquely. However, we cannot see how to prove this assertion. Indeed, we conjecture that the converse of Proposition 2.1 does not hold.

Universal stabilizers. Having a guarantee that extensions and coextensions of representations are unique is of considerable value in arguments in matroid representation theory, and theorems that provide such a guarantee for particular situations play a vital role in recent work (see, for example, [8, 25]). General techniques for developing such theorems are badly needed. In a sense, this paper is the third in a series seeking to develop such techniques, the others being [10, 27]. The motivation for this paper was that a promising idea, developed in [10], did not turn out to be quite as fruitful as we had hoped. In that paper the notion of a "universal stabilizer" for a *well-closed class* of matroids was introduced. The latter is a class of matroids that is closed under isomorphism, minors, and duality. There are a number of equivalent characterizations of universal stabilizers. For example, a 3-connected matroid N in a well-closed class $\mathcal{N}$ is a *universal stabilizer* for $\mathcal{N}$ if the following holds for all 3-connected matroids M in $\mathcal{N}$ and all x in $E(M)$: if $M\backslash x$ is 3-connected and has an N-minor, then x is fixed in M, and if M/x is 3-connected and has an N-minor, then x is cofixed in M. It is shown in [10] that, with some natural conditions on the class $\mathcal{N}$, the task of showing that N is a universal stabilizer for $\mathcal{N}$ is an elementary finite check.

A universal stabilizer for a class is a valuable object. By the Splitter Theorem, we see that a representation for a matroid in the class can be built from a representation of the universal stabilizer via a sequence of fixed extensions and cofixed coextensions. Thus a representation of the matroid can be built uniquely from an appropriate representation of the universal stabilizer. The hope was that, for natural classes, one could identify reasonable sets of universal stabilizers. Such is indeed the case for ternary matroids: $U_{2,3}$ is a universal stabilizer for the class of ternary matroids with no $U_{2,4}$-minor and $U_{2,4}$ is a universal stabilizer for the class of all ternary matroids. Also, universal stabilizers have recently proved a very useful tool in the characterizations of [19]. Unfortunately, an example in [10] shows, it seems, that, for classes beyond subclasses of binary and ternary matroids, it is often too much to ask for a reasonable set of universal stabilizers. The theory of totally free expansions was developed to overcome the problems opened up by the existence of this example.

Totally free expansions. For a matroid M, we denote the simple and cosimple matroids canonically associated with M by $\mathrm{si}(M)$ and $\mathrm{co}(M)$, respectively. Let N be a 3-connected matroid with at least four elements, and let M be a matroid with an N-minor. Then M is a *totally free expansion* of N if M is 3-connected and the following holds for all $x \in E(M)$: if $\mathrm{co}(M\backslash x)$ is 3-connected with an N-minor, then x is not fixed in M, and if $\mathrm{si}(M/x)$ is 3-connected with an N-minor, then x is not cofixed in M. The next result, which will be obtained as an immediate corollary of Theorem 7.1, is the key theorem of this paper.

THEOREM 2.2. *Let M be a totally free expansion of a 3-connected matroid N where $M \neq N$, and $|E(N)| \geqslant 4$. Then at least one of the following holds.*

(i) *There is an element d of* $E(M)$ *such that* $M \backslash d$ *is a totally free expansion of* N.

(ii) *There is an element c of* $E(M)$ *such that* M/c *is a totally free expansion of* N.

(iii) *There are elements c and d of* $E(M)$ *such that* $M/c \backslash d$ *is a totally free expansion of* N.

A 3-connected matroid M is *totally free* if it has at least four elements and, for all x in $E(M)$, if $\mathrm{co}(M \backslash x)$ is 3-connected, then x is not fixed in M, while if $\mathrm{si}(M/x)$ is 3-connected, then x is not cofixed in M. We prove, in Corollary 8.6, that a matroid is totally free if and only if it is a totally free expansion of $U_{2,4}$. It follows that Theorem 2.2 can be specialized to totally free matroids. In fact, this specialization can be strengthened somewhat given the particular structure of totally free matroids, and this strengthening is provided in Corollary 8.13. But, in both cases, the crucial point is that totally free expansions and totally free matroids in a minor-closed class can be found by an elementary inductive search—they do not occur sporadically.

Strengthening the Splitter Theorem. One consequence of Theorem 2.2 is the following strengthening of Seymour's Splitter Theorem [22].

THEOREM 2.3. *Let M be a 3-connected matroid with a 3-connected matroid N as a minor, where* $|E(N)| \geqslant 4$. *Assume that if N is a wheel, then N is the largest wheel minor of M, and if N is a whirl, then N is the largest whirl minor of M. Then, for some non-negative integers n and m, there is a sequence*

$$M_0, M_1, \ldots, M_n, M_{n+1}, \ldots, M_{n+m}$$

of 3-connected matroids and a sequence $e_0, e_1, \ldots, e_{m+n-1}$ *of elements of M such that the following hold.*

(i) $M_0 = M$ *and* $M_{n+m} \cong N$.

(ii) *For all i in* $\{0, 1, \ldots, m+n-1\}$, *either* $M_{i+1} = M_i \backslash e_i$ *or* $M_{i+1} = M_i / e_i$.

(iii) *For all i in* $\{0, 1, \ldots, n-1\}$, *if* $M_{i+1} = M_i \backslash e_i$ *then* e_i *is fixed in* M_i, *and if* $M_{i+1} = M_i / e_i$, *then* e_i *is cofixed in* M_i.

(iv) M_n *is a totally free expansion of N, and, for all j in* $\{1, 2, \ldots, m-1\}$, *if* M_{n+j} *is not a totally free expansion of N, then both* M_{n+j-1} *and* M_{n+j+1} *are totally free expansions of N.*

Put in somewhat plainer language, Theorem 2.3 says that, starting from M, one can delete and contract fixed and cofixed elements until a totally free expansion of N is obtained. From then on, one can perform one- or two-element steps to give a sequence of totally free expansions that eventually arrive at N. Alternatively, from a bottom-up point-of-view, we can build M from a minor isomorphic to N via a chain of 3-connected minors having the property that elements with freedom or cofreedom are added in an initial segment of the chain. Theorem 2.3 will be proved in Section 9.

Bounding inequivalent representations. While the strengthening of the Splitter Theorem given here is attractive and potentially useful, from the point of view of the applications given here, the following result is vital.

THEOREM 2.4. *Let* $\mathbf{F}$ *be a finite field and* M *be a 3-connected* $\mathbf{F}$*-representable matroid. If* M *has no totally free minors, then it is uniquely* $\mathbf{F}$*-representable. Otherwise the number of inequivalent* $\mathbf{F}$*-representations of* M *is bounded above by the maximum, over all totally free minors "*N *of* M*, of the number" of inequivalent* $\mathbf{F}$*-representations of* N.

Theorem 2.4, combined with the fact that totally free matroids in a class can be found by an inductive search, provide tools that enable us to examine the behaviour of inequivalent representations for 3-connected members of well-closed classes. The proof of Theorem 2.4 will be given in Section 10.

Totally free quaternary and quinternary matroids. In Sections 11 and 12, we apply the theory to the classes of matroids representable over $GF(4)$ and $GF(5)$ and identify all totally free quaternary and quinternary matroids. It is of interest to note that, apart from matroids of small rank, the totally free matroids that arise are precisely the matroids used in [20] to prove that, for a field $\mathbf{F}$ with at least seven elements, no bound can be placed on the number of inequivalent $\mathbf{F}$-representations of 3-connected matroids. We now outline the results of Sections 11 and 12.

For an integer $k \geqslant 3$, a rank-k *spike* with *tip* p is a rank-k matroid with ground set $\{p, a_1, b_1, a_2, b_2, ..., a_k, b_k\}$ such that

(i) $\{p, a_i, b_i\}$ is a triangle for all i in $\{1, 2, ..., k\}$ and
(ii) $r(\bigcup_{j \in J} \{a_j, b_j\}) = |J| + 1$ for every proper subset J of $\{1, 2, ..., k\}$.

Each pair $\{a_i, b_i\}$ is a *leg* of the spike. The non-spanning circuits of a rank-k spike include the abovementioned triangles containing p together with all sets of the form $\{a_i, b_i, a_j, b_j\}$ for distinct i and j in $\{1, 2, ..., k\}$. All other non-spanning circuits have the form $\{z_1, z_2, ..., z_k\}$ where $z_i \in \{a_i, b_i\}$. If all such sets $\{z_1, z_2, ..., z_k\}$ are independent, then the spike obtained is called the *free* rank-k spike with tip p, and is denoted by Φ_k^+. The *tipless* free rank-k spike Φ_k is the matroid $\Phi_k^+ \backslash p$.

It is shown in [20] that free spikes are representable over all finite fields of non-prime order. Moreover, for every non-prime finite field **F** other than $GF(4)$, the free spike Φ_k has at least 2^{k-1} inequivalent **F**-representations. Hence, for each such field **F**, there is no constant bound on the number of inequivalent **F**-representations of a quaternary 3-connected **F**-representable matroid. Note that Φ_k is a self-dual matroid.

The following are the main results of Section 11.

THEOREM 2.5. *A quaternary matroid M is totally free if and only if M is isomorphic to one of $U_{2,4}$, $U_{2,5}$, $U_{3,5}$, or Φ_r for some $r \geqslant 3$.*

COROLLARY 2.6. *Let k be a fixed integer exceeding two and* **F** *be a finite field. Then the class of* 3*-connected quaternary matroids with no Φ_k-minor has a bounded number of inequivalent* **F***-representations.*

Next we describe another class of matroids, which, like the free spikes, play a key role in [20]. Suppose $r \geqslant 3$. Begin with the rank-r whirl having spokes, in cyclic order, $s_1, s_2, \ldots, s_r$. Let the non-trivial lines of this whirl be $\{s_1, a_1, s_2\}$, $\{s_2, a_2, s_3\}$, ..., $\{s_r, a_r, s_1\}$. For each i in $\{1, 2, \ldots, r\}$, freely add a new point b_i on the line $\{s_i, a_i, s_{i+1}\}$. Call the resulting matroid the *rank-r jointed swirl*. Let Ψ_r and Ψ_r^+ denote the matroids that are obtained from the rank-r jointed swirl by deleting, respectively, all or all but one of the elements $s_1, s_2, \ldots, s_r$. We shall call Ψ_r the *rank-r swirl*. Note that $\Psi_3 \cong U_{3,6} \cong \Phi_3$. It is shown in [20] that if q is a prime power that exceeds five and is not of the form 2^p where $2^p - 1$ is prime, then the jointed rank-r swirl has at least 2^r inequivalent $GF(q)$-representations. It is not difficult to extend this to establish that Ψ_r also has at least 2^r inequivalent $GF(q)$-representations. Note that Ψ_k is self-dual.

The main results of Section 12 are as follows.

THEOREM 2.7. *A quinternary matroid M is totally free if and only if M is isomorphic to one of $U_{2,4}$, $U_{2,5}$, $U_{2,6}$, $U_{3,5}$, $U_{4,6}$, P_6, or Ψ_r for some $r \geqslant 3$.*

COROLLARY 2.8. *Let k be a fixed integer exceeding two and* **F** *be a finite field. Then the class of* 3*-connected quinternary matroids with no Ψ_k-minor has a bounded number of inequivalent* **R***-representations.*

Corollaries 2.6 and 2.8 establish that, among quaternary and quinternary matroids, free spikes and swirls are the sole obstructions to Kahn's conjecture [11] that there is a constant $n(q)$ such that the number of inequivalent $GF(q)$-representations of a 3-connected $GF(q)$-representable matroid is at most $n(q)$. We know of no other obstructions to the conjecture in general. Indeed, we conjecture the following.

Conjecture 2.9. Let k be an integer exceeding two. Then, for all prime powers q, there is a constant $n(q, k)$ such that every 3-connected $GF(q)$-representable matroid with no minor isomorphic to either Φ_k or Ψ_k has at most $n(q, k)$ inequivalent $GF(q)$-representations.

3. SOME 3-CONNECTIVITY PRELIMINARIES

We assume that the reader is familiar with the theory of connectivity of matroids as set forth in Oxley [13]. Some standard 3-connectivity results play a vital role in this paper and, for convenience, we restate them here. The first is a result of Tutte [24] (see also [13, Lemma 8.4.9]).

LEMMA 3.1 (Tutte's Triangle Lemma). *Let M be a 3-connected matroid with at least four elements and suppose that $\{e, f, g\}$ is a triad of M such that neither M/e nor M/f is 3-connected. Then M has a triangle that contains e and exactly one of f and g.*

The next is a theorem of Bixby [2] (see also [13, Proposition 8.4.6]).

LEMMA 3.2. *Let x be an element of a 3-connected matroid M. Then either $\mathrm{co}(M\backslash x)$ or $\mathrm{si}(M/x)$ is 3-connected.*

The Splitter Theorem [22] plays a vital role in many arguments in matroid structure theory. There are several ways to state this theorem. The version that Theorem 2.3 generalizes is as follows.

THEOREM 3.3. *Let M and N be 3-connected matroids such that N is a minor of M with at least four elements, and if N is a wheel, then M has no larger wheel as a minor, while if N is a whirl, then M has no larger whirl as a minor. Then there is a sequence $M_0, M_1, \ldots, M_n$ of 3-connected matroids such that $M_0 \cong N$, for all i in $\{0, 1, \ldots, n-1\}$, M_i is a single-element deletion or a single-element contraction of M_{i+1}, and $M_n = M$.*

The following consequence of the Splitter Theorem will be used often throughout this paper.

COROLLARY 3.4. *Let M and N be 3-connected matroids such that N is a minor of M. If $M \neq N$, and M is not a wheel or a whirl, then there is an element x of $E(M)$ such that either $M\backslash x$ or M/x is 3-connected with an N-minor.*

The next two lemmas are Lemmas 3.8 and 3.6, respectively, of [27].

LEMMA 3.5. *Let C^* be a rank-3 cocircuit of a 3-connected matroid M. If $x \in C^*$ and x has the property that $\mathrm{cl}_M(C^*) - x$ contains a triangle of M/x, then $\mathrm{si}(M/x)$ is 3-connected.*

LEMMA 3.6. *Let M be a 3-connected matroid, and let x and p be elements of $E(M)$ with the property that $\mathrm{si}(M/x)$ and $\mathrm{si}(M/x, p)$ are 3-connected, but $\mathrm{si}(M/p)$ is not 3-connected. Then $r(M) \geqslant 4$ and x is in a rank-3 cocircuit C^* such that $p \in \mathrm{cl}(C^*) \cap (E(M) - C^*)$.*

LEMMA 3.7. *Let $\{e, f, g\}$ be a triad of a 3-connected matroid M. If neither $\mathrm{si}(M/e)$ nor $\mathrm{si}(M/f)$ is 3-connected, then M has a triangle using e and f. Moreover, M/g is 3-connected unless $M \cong M(K_4)$.*

Proof. The hypotheses of the lemma imply that $r(M) > 2$. We shall first consider the case when M has a triangle T containing g. Then $|T \cap \{e, f, g\}| \geqslant 2$. Moreover, $T \neq \{e, f, g\}$ (see, for example, [13, Proposition 8.1.7]). Thus, without loss of generality, we may assume that $T \cap \{e, f, g\} = \{e, g\}$. Since $\mathrm{si}(M/f)$ is not 3-connected, $\mathrm{co}(M \backslash f)$ is 3-connected. But the last matroid has a parallel pair, and therefore is isomorphic to $U_{1,2}$ or $U_{1,3}$. Thus $\mathrm{co}(M \backslash f)$ has corank one or two. Hence M has corank two or three. Consider M^*. If it has rank two, then one easily checks that $\mathrm{co}(M^* \backslash e)$ is 3-connected; a contradiction. Thus we may assume that $r(M^*) = 3$. Hence $|E(M^*)| \geqslant 6$. Now e, f and g are collinear in M^* and $\{e, g\}$ is contained in the triad T of M^*. Hence $E(M^*) - T$ is a line L of M^* containing f. As $M^* \backslash f$ is not 3-connected, it is a union of two lines. One of these lines must contain two elements of T and can contain at most one element of L. The other line contains the third element of T and hence at most one element of L. Hence, as $|E(M^*)| \geqslant 6$, we deduce that $|E(M^*)| = 6$ and $|L| = 3$. As neither $\mathrm{co}(M^* \backslash e)$ nor $\mathrm{co}(M^* \backslash f)$ is 3-connected, it follows without difficulty that $M^* \cong M(K_4)$ and $\{e, f\}$ is contained in a triangle of M^*. Thus the lemma holds when M has a triangle containing g.

We may now assume that M has no triangle containing g. Since neither M/e nor M/f is 3-connected, Lemma 3.1 implies that M has a triangle, say $\{e, f, z\}$, using e and f. Then $\{e, f, z\}$ is a triangle of M/g. By Lemma 3.2, $\mathrm{si}(M/g)$ is 3-connected. But g is in no triangles, so M/g has no parallel pairs. Hence M/g is 3-connected. ∎

The next lemma will be used frequently. We omit the straightforward proof.

LEMMA 3.8. *Let e be an element of a 3-connected matroid M and suppose that $|E(M)| \geqslant 4$.*

(i) *If* $f \in E(\mathrm{si}(M/e))$, *then* $\mathrm{si}(\mathrm{si}(M/e)/f) = \mathrm{si}(M/e, f)$.
(ii) *If* $f \in E(\mathrm{co}(M\backslash e))$, *then* $\mathrm{co}(\mathrm{co}(M\backslash e)\backslash f) = \mathrm{co}(M\backslash e, f)$.
(iii) *If* $\mathrm{si}(M\backslash e/f)$ *is* 3*-connected, then* $\mathrm{si}(M/f)$ *is* 3*-connected.*
(iv) *If* $\mathrm{co}(M/e\backslash f)$ *if* 3*-connected, then* $\mathrm{co}(M\backslash f)$ *is* 3*-connected.*

Finally, we note another elementary fact.

LEMMA 3.9. *If N is a* 3*-connected matroid with at least four elements, and M is a matroid with an N-minor, then both* $\mathrm{si}(M)$ *and* $\mathrm{co}(M)$ *have N-minors.*

4. CLONES AND FIXED ELEMENTS

The material in this section mainly revises material from [10]. The idea of measuring the relative freedom of elements in matroids is introduced in Cheung and Crapo [4] and Duke [7].

As noted in Section 2, elements x and x' of a matroid M are *clones* if interchanging x and x' is an automorphism of M. Thus clones are elements of a matroid that are indistinguishable up to labelling. If $\{x, x'\}$ is a pair of loops, a pair of coloops, a parallel pair, or a series pair, then x and x' are clones. It is also immediate that x and x' are clones in M if and only if they are clones in M^*.

Let x be an element of a matroid M. The matroid M' is obtained by *cloning x with x'* if M' is a single-element extension of M by x', and x and x' are clones in M'. Dually, we have that M' is obtained by *cocloning x with x'* if M' is a single-element coextension of M by x', and x and x' are clones in M'.

It is always possible to clone x with x': if x is a loop, just add x' as a loop, while if x is not a loop, then add x' in parallel to x. However, it is not always possible to clone x with x' so that $\{x, x'\}$ is independent. In the case that x cannot be cloned with x' so that x and x' are independent, we say that x is *fixed* in M. Dually, x is *cofixed* in M if M has no coextension by x' such that x and x' are coindependent clones in this coextension. In other words, x is cofixed in M if and only if x is fixed in M^*.

If x is not fixed, then there is a matroid M' obtained by cloning x with x' such that $\{x, x'\}$ is independent in M'. We say that M' is obtained by *independently cloning* x with x'. Dually, we refer to $(M')^*$ as being obtained from M^* by *coindependently cocloning* x with x'. Note that knowing that M' is obtained by independently cloning x with x' does not, in general, determine M' up to isomorphism. For example, if $x \in E(U_{3,4})$, then one can obtain both $U_{3,5}$ and $U_{2,4} \oplus_2 U_{2,3}$ by independently cloning x.

Fixed elements can also be characterized in terms of modular cuts. Recall that flats F_1 and F_2 of a matroid form a *modular pair* if $r(F_1 \cup F_2)+r(F_1 \cap F_2)=r(F_1)+r(F_2)$. A *modular cut* in a matroid M is a collection $\mathscr{F}$ of flats of M with the following properties: if F_1 and F_2 are a modular pair of flats in $\mathscr{F}$, then $F_1 \cap F_2$ is in $\mathscr{F}$; and if $F \in \mathscr{F}$, then every flat of M that contains F is also in $\mathscr{F}$. It is known that modular cuts are in one-to-one correspondence with single-element extensions of M; see [5].

Cheung and Crapo [4] have defined the notion of the *degree* of a modular cut and Duke [7] has defined the notion of the *freedom* of an element in a matroid. It is shown in [7, Theorem 3.3] that a modular cut $\mathscr{F}$ has degree k if and only if the freedom of the element of extension in the single-element extension defined by $\mathscr{F}$ is k. Moreover, it follows easily from results in [7] that an element e is fixed in M if and only if it has freedom at most 1, or equivalently, if and only if the modular cut of $M \backslash e$ associated with the extension M has degree at most 1.

A flat of a matroid is *cyclic* if it is a union of circuits. When ordered by inclusion, the collection of modular cuts of a matroid forms a lattice. It follows that, given a set $\mathscr{F}$ of flats of a matroid, there is a unique minimal modular cut containing that set of flats. This is the modular cut $\langle \mathscr{F} \rangle$ *generated* by $\mathscr{F}$. The following proposition is Duke [7, Corollary 3.5].

PROPOSITION 4.1. *Let e be an element of a matroid M. Then e is fixed in M if and only if* $\text{cl}(\{e\})$ *is in the modular cut generated by the cyclic flats of M containing e.*

Next we give some elementary equivalent conditions for x and x' to be clones in a matroid M.

PROPOSITION 4.2. *Let x and x' be elements of a matroid M. Then the following are equivalent.*

(i) *x and x' are clones in M.*

(ii) *Replacing x by x' and fixing every other element is an isomorphism from $M \backslash x'$ to $M \backslash x$.*

(iii) *$M/x \backslash x' = M/x' \backslash x$ and $r(\{x\}) = r(\{x'\})$.*

The next proposition is a straightforward consequence of the definitions. It is a useful way of showing that an element is not fixed in a minor.

PROPOSITION 4.3. *Assume that x and x' are clones in a matroid M', and let $M = M' \backslash x'$. If X and Y are disjoint subsets of $E(M') - \{x, x'\}$, then x and x' are clones in $M' \backslash X/Y$. Moreover, if $\{x, x'\}$ is independent in $M' \backslash X/Y$, then x is not fixed in $M \backslash X/Y$, and if $\{x, x'\}$ is coindependent in $M' \backslash X/Y$, then x is not cofixed in $M \backslash X/Y$.*

The following is an easy consequence of Proposition 4.3.

COROLLARY 4.4. *Let x be an element of a matroid M, and X be a subset of $E(M)-x$.*

(i) *If x is not fixed in M, then x is not fixed in $M\backslash X$.*

(ii) *If x is not cofixed in M, then x is not cofixed in M/X.*

A point p of a matroid M is *freely placed* on a flat F if $p \in F$, and $\mathrm{cl}_M(C) \supseteq F$ for every circuit C of M containing p. The next proposition is a special case of [7, Proposition 3.1].

PROPOSITION 4.5 (Duke). *If p is fixed in M, and F is a flat of M of rank greater than one, then p is not freely placed on F.*

The following corollary of Proposition 4.5 will prove useful in this paper.

COROLLARY 4.6. *Let M be a matroid, a be an element of $E(M)$ that is not a loop or a coloop, and b be an element of $E(M)-\mathrm{cl}(\{a\})$. If a is fixed in M, then there is a cyclic flat of M that contains a but not b.*

The next proposition enables us to deduce that an element is fixed or cofixed in M from the fact that it is fixed or cofixed in certain minors.

PROPOSITION 4.7. *Let x be an element of a matroid M.*

(i) *If M has an element a such that x is fixed in $M\backslash a$, then x is fixed in M.*

(ii) *If M has an element a such that x is cofixed in M/a, then x is cofixed in M.*

(iii) *If M has distinct elements a and b such that $\{a, b, x\}$ is independent in M, and x is fixed in both M/a and M/b, then x is fixed in M.*

(iv) *If M has distinct elements a and b such that $\{a, b, x\}$ is coindependent and x is cofixed in both $M\backslash a$ and $M\backslash b$, then x is cofixed in M.*

Evidently, if x and x' are independent clones in M, then x is not fixed in $M\backslash x'$. The next proposition [10, Proposition 4.9] extends this observation.

PROPOSITION 4.8. *If x and x' are independent clones in M, then x is fixed in neither M nor $M\backslash x'$. Dually, if x and x' are coindependent clones in M, then x is cofixed in neither M nor M/x'.*

It follows that if x and x' are independent, coindependent clones, then x is neither fixed nor cofixed in M. However, it is quite possible for x to be fixed in M/x' and for x to be cofixed in $M\backslash x'$. To see this, consider the rank-r free spike Φ_r where $r \geqslant 4$. For any leg $\{a_i, b_i\}$ of Φ_r, the elements a_i and b_i are independent, coindependent clones. Moreover, it is easily checked that a_i is fixed in M/b_i and cofixed in $M\backslash b_i$. The situation that arises in this example is fundamental, and is the focus of much of the argument in the proof of Theorem 7.1, our main theorem.

PROPOSITION 4.9. *Elements x and y of a matroid M are clones if and only if the set of cyclic flats containing x is equal to the set of cyclic flats containing y.*

Proof. It is evident that if x and y are clones, then a cyclic flat contains x if and only if it contains y. Consider the converse. Assume that a cyclic flat contains x if and only if it contains y. Now let $I \subseteq E(M)-\{x, y\}$ and suppose that $I \cup x$ is independent but that $I \cup y$ is not. Then $I \cup y$ contains a circuit C containing y, and $\mathrm{cl}(C)$ is a cyclic flat contained in $\mathrm{cl}(I)$. But $x \notin \mathrm{cl}(I)$. Thus $\mathrm{cl}(C)$ is a cyclic flat that contains y but not x. We conclude that $I \cup x$ is independent if and only if $I \cup y$ is independent, and it follows easily that x and y are clones. ∎

LEMMA 4.10. *Let $\{a, a'\}$ be a pair of clones of a matroid M, and let x be in $E(M)-\{a, a'\}$.*

(i) *If $\{a, a'\}$ is independent, and x is not fixed in M but is fixed in M/a, then $\{a, a', x\}$ is a triangle of M.*

(ii) *If $\{a, a'\}$ is coindependent, and x is not cofixed in M, but is cofixed in $M\backslash a$, then $\{a, a', x\}$ is a triad of M.*

Proof. Consider part (i). Since a and a' are clones, x is fixed in M/a'. It is now an immediate consequence of Proposition 4.7(iii) that if $\{a, a', x\}$ is independent then x is fixed in M. Hence $\{a, a', x\}$ is dependent. Now x is not a loop in M since it is not fixed in M. Moreover, as a and a' are independent clones, x is not parallel to a or a'. Hence $\{a, a', x\}$ is a triangle of M, that is, (i) holds. It follows by duality that (ii) holds. ∎

5. THE FREE QUASI-ORDER ON MATROID ELEMENTS

Let x and y be elements of a matroid M. Then x is *freer than y in M* if every cyclic flat of M that contains x also contains y. If the matroid M is clear from the context, then we will sometimes say simply that x is freer than y. If x is freer than y but y is not freer than x, then x is *strictly freer* than y.

It is an immediate consequence of the definition that the relation on the elements of M defined above is transitive and reflexive. Hence it is a quasi-order. We call this quasi-order the *free quasi-order* on the elements of M. This quasi-order is introduced and studied by Duke [7, 6]. As with all quasi-orders, one can easily obtain an equivalence relation $\cong$ on $E(M)$ by defining $x \cong y$ if and only if x is freer than y and y is freer than x. The next proposition is an immediate consequence of Proposition 4.9.

PROPOSITION 5.1. *Let x and y be elements of a matroid M. Then $x \cong y$ if and only if x and y are clones in M.*

We call the equivalence classes of the above relation *clonal classes* of M. It is well known, and easily seen, that F is a cyclic flat of M if and only if $E(M) - F$ is a cyclic flat of M^*. The following is an elementary consequence of this observation.

PROPOSITION 5.2. *Let x and y be elements of a matroid M. Then x is freer than y in M if and only if y is freer than x in M^*.*

PROPOSITION 5.3. *The following are equivalent for elements x and y of a matroid M.*

(i) *x is freer than y.*

(ii) *$r(A \cup x) \geqslant r(A \cup y)$ for all $A \subseteq E(M) - \{x, y\}$.*

(iii) *For all $A \subseteq E - \{x, y\}$, if $x \in \mathrm{cl}(A)$, then $y \in \mathrm{cl}(A)$.*

PROPOSITION 5.4. *Let x and y be elements of a matroid M.*

(i) *If x is fixed in M/y, but not in M, then x is freer than y.*

(ii) *If x is cofixed in $M \backslash y$, but not in M, then y is freer than x.*

Proof. Assume that x is fixed in M/y but not in M. Let M' be a matroid obtained by independently cloning x with x'. Assume that $\{x, x', y\}$ is independent. Then x and x' are independent clones in M/y, so that x is not fixed in M/y. Hence $\{x, x', y\}$ is a triangle. Let F be a cyclic flat of M containing x. Then, since x and x' are clones, $\mathrm{cl}_{M'}(F)$ contains x', and therefore y. But $\mathrm{cl}_{M'}(F) = F \cup x'$. Hence $y \in F$. It follows that x is freer than y. ∎

The straightforward proof of the next proposition is omitted.

PROPOSITION 5.5. *Let x and y be distinct elements of a matroid M such that x is freer than y in M.*

(i) *If y is not fixed in M, then x is not fixed in M.*

(ii) *If x is not cofixed in M, then y is not cofixed in M.*

(iii) *If x is strictly freer than y, and y is not a loop, then x is not fixed in M.*

(iv) *If x is strictly freer than y, and x is not a coloop, then y is not cofixed in M.*

PROPOSITION 5.6. *Let x and y be elements of a matroid M, and V and W be disjoint subsets of $E(M)-\{x, y\}$. If x is freer than y in M, then x is freer than y in $M\backslash V/W$.*

Proof. Suppose z is an element of $E-\{x, y\}$. Let F be a cyclic flat of $M\backslash z$ containing x. Then either F or $F \cup z$ is a cyclic flat of M containing x. Hence either F or $F \cup z$ contains y. In either case, F contains y. Thus x is freer than y in $M\backslash z$.

Consider M/z. By Proposition 5.2, y is freer than x in M^* and, by the above argument, y is freer than x in $M^*\backslash x$. Again, by Proposition 5.2, x is freer than y in $(M^*\backslash x)^*$, that is, in M/x. The proposition now follows by an elementary induction. ∎

Recall that if M and M' are matroids on a common ground set, then M' is a rank-preserving weak-map image of M if M and M' have the same rank, and every independent set of M' is also independent in M. If M' is a rank-preserving weak-map image of M, then, following [17], we say that M is *freer* than M'. We observe that this definition differs from that in [13] by requiring M and M' to have the same rank.

PROPOSITION 5.7. *Let x and y be elements of a matroid M where x is freer than y.*

(i) *Let M' be the matroid obtained from $M\backslash y$ by relabelling x as y. If x is not a coloop of M, then M' is freer than $M\backslash x$.*

(ii) *Let M'' be the matroid obtained from M/y by relabelling x as y. If x is not a loop of M, then M/x is freer than M''.*

Proof. We begin by proving (i). Let C be a circuit of M'. If C does not contain y, then C is also a circuit of $M\backslash x$. Assume that $y \in C$. Then $\mathrm{cl}_M((C-y) \cup x)$ is a cyclic flat of M containing x. Hence $\mathrm{cl}_M((C-y) \cup x)$ contains y. But $C-y$ is a basis for this flat. Hence $(C-y) \cup y$ contains a circuit, so $(C-y) \cup y$ is dependent in M. We conclude that every dependent set of M' is also dependent in $M\backslash x$. Moreover, as x is freer than y, and x is not a coloop of M, the ranks of M' and $M\backslash x$ are equal. Hence M' is freer than $M\backslash x$, that is, (i) holds. Part (ii) follows from (i) by using duality together with Proposition 5.2 and the fact that M_2 is freer than M_1 if and only if M_2^* is freer than M_1^* (see, for example, [13, Corollary 7.3.13]). ∎

Let k be a positive integer. Then it is well known that if M is k-connected and M' is freer than M, then M' is also k-connected. The following corollary is an immediate consequence of this fact and Proposition 5.7.

COROLLARY 5.8. *Let x and y be elements of a matroid M, where x is freer than y.*

(i) *If $M \backslash x$ is 3-connected and x is not a coloop of M, then $M \backslash y$ is* 3*-connected.*

(ii) *If M / y is* 3*-connected and y is not a loop of M, then M / x is* 3*-connected.*

LEMMA 5.9. *Let c and d be elements of a matroid M, and assume that d is fixed in M/c but not in M. Then either c and d are clones, or c is fixed in M.*

Proof. By Proposition 5.4(i), d is freer than c. Assume that c is not fixed in M. Let M' be a matroid obtained by independently cloning c with c'. We show next that d is not fixed in M'. Since c is not fixed in M', it suffices to show that d is freer than c in M'. Let F be a cyclic flat of M' containing d. If $c' \notin F$, then F is a cyclic flat of M containing d and, since d is freer than c in M, it follows that $c \in F$. If $c' \in F$, then, as c and c' are clones in M', we again obtain that $c \in F$. We conclude that every cyclic flat of M' containing d also contains c. Hence d is not fixed in M'. But, by Corollary 4.4, d is fixed in M'/c. Thus, by Lemma 4.10(i), $\{c, c', d\}$ is a triangle of M'. Hence c is freer than d in M' and so, by Proposition 5.6, c is freer than d in M. Thus c and d are clones in M. ∎

6. LEMMAS ON TRIANGLES AND TRIADS

In this paper, just as with many papers in matroid structure theory, much of the argument focuses on the behaviour of triads and triangles. The lemmas of this section examine triads and triangles in the context of the notions defined in the two previous sections.

LEMMA 6.1. *Let a, b, and c be elements of a matroid M.*

(i) *If $\{a, b, c\}$ is a triangle of M and neither a nor b is fixed in M, then a and b are clones.*

(ii) *If $\{a, b, c\}$ is a triad of M and neither a nor b is cofixed in M, then a and b are clones.*

Proof. Consider part (i). Let M' be obtained by independently cloning a by a'. As $\{a, b, c\}$ is a triangle, so is $\{a', b, c\}$. Hence a, a', b, and c are collinear. If F is a cyclic flat of M containing a, then $F \cup a'$ is a cyclic flat of M' containing a'. Hence $F \cup a'$ contains b, that is, F contains b. We conclude that a is freer than b. By the symmetry of the hypothesis, we deduce that b is also freer than a. Thus a and b are clones. Part (ii) follows by duality. ▮

LEMMA 6.2. *Let a, b, and c be elements of the connected matroid M.*

(i) *If $\{a, b, c\}$ is a coindependent triangle and a is not fixed in M, then neither b nor c is cofixed in M.*

(ii) *If $\{a, b, c\}$ is an independent triad and a is not cofixed in M, then neither b nor c is fixed in M.*

Proof. By duality, it suffices to prove part (i). Independently clone a by a' to obtain a matroid M'. Then $\{a', b, c\}$ is a triangle of M', so M' has a line L containing $\{a, a', b, c\}$. Since a and a' are clones, every cyclic flat of M' that contains a also contains a', and so contains L. Thus a is freer than b in M' and hence also in M. Now either a and b are clones in M, or a is strictly freer than b. In the first case, since $\{a, b, c\}$ is coindependent we see that a and b are coindependent clones and b is not cofixed. The same conclusion holds in the second case by Proposition 5.5(iv). ▮

LEMMA 6.3. *Let a, b, and c be elements of a* 3*-connected matroid M for which* $|E(M)| \geqslant 4$.

(i) *If $\{a, b, c\}$ is a triangle, a is not fixed, and no triad of M contains both a and b, then $M \backslash b$ is* 3*-connected.*

(ii) *If $\{a, b, c\}$ is a triad, a is not cofixed, and no triangle of M contains both a and b, then M / b is* 3*-connected.*

Proof. By duality, it suffices to prove (i). Assume that $\{a, b, c\}$ is a triangle and that a is not fixed. We show that if $M \backslash b$ is not 3-connected, then there is a triad of M using a and b. Let M' be a matroid obtained by independently cloning a with a'. Evidently $M' | \{a, a', b, c\} \cong U_{2,4}$. Let $\{A, Z\}$ be a 2-separation of $M \backslash b$, where $a \in A$. It is easily seen that we lose no generality in assuming that A is closed. If $c \in A$, then $r_M(A \cup b) = r_M(A)$, and $\{A \cup b, Z\}$ is a 2-separation of M, contradicting the fact that M is 3-connected. Hence $c \in Z$. Consider $M' \backslash b$. In this matroid, $\{a, a', c\}$ is a triangle. Suppose that $a' \in \mathrm{cl}_{M' \backslash b}(A)$. Since A is a flat of $M \backslash b$, it follows that $\mathrm{cl}_{M' \backslash b}(A) = A \cup a'$. Since $\{a, a'\}$ spans $\{a, a', c\}$, we deduce that $c \in A \cup a'$, that is, $c \in A$. From this contradiction, we deduce that $a' \notin \mathrm{cl}_{M' \backslash b}(A)$. Thus a' is a coloop of $M' | (A \cup a')$. Since a and a' are clones, a is a coloop of $M' \mid (A \cup a')$, and hence of $M \mid A$. From this, it is easily deduced that if $|A-a|>1$, then $\{A-a, Z \cup a\}$ is also a 2-separation of $M \backslash b$. But now $\{a, c\} \subseteq Z \cup a$ and arguing as before, we deduce that M is not 3-connected. Hence $|A-a|=1$, say $A-a=t$. In this case, $r(Z)=r(M)-1$. Hence $\{a, b, t\}$ contains a cocircuit of M. It now follows from the fact that M is 3-connected having at least four elements that $\{a, b, t\}$ is a triad, as required. ▮

The next lemma gives a useful necessary and sufficient condition for an element in a triangle to be fixed.

Lemma 6.4. *Let $\{a, b, c\}$ be a triangle in a matroid M. Then a is fixed in M if and only if M has a circuit whose closure meets $\{a, b, c\}$ in $\{a\}$.*

Proof. If M has a circuit C whose closure meets $\{a, b, c\}$ in $\{a\}$, then $\{\mathrm{cl}(\{a, b, c\}), \mathrm{cl}(C)\}$ is a modular pair whose intersection is $\mathrm{cl}(\{a\})$. Thus a is fixed in M. Conversely, if a is fixed in M, then, by Proposition 4.5, a is not freely placed on $\mathrm{cl}(\{a, b, c\})$. Thus M has a circuit C containing a such that $\mathrm{cl}(C) \nsupseteq \mathrm{cl}(\{a, b, c\})$. Thus $|\mathrm{cl}(C) \cap \mathrm{cl}(\{a, b, c\})| \leqslant 1$. Hence $\mathrm{cl}(C) \cap \mathrm{cl}(\{a, b, c\}) = \{a\}$. ▮

Lemma 6.5. *Let $\{X, Y\}$ be a 3-separation of a 3-connected matroid M. Suppose that $\{a, b\} \subseteq \mathrm{cl}(X) \cap \mathrm{cl}(Y)$ and a is strictly freer than b. Then b is fixed in M.*

Proof. Since a is strictly freer than b, there is a cyclic flat F containing b but not a. Now clone b by b' to obtain the matroid M'. Then $F' = \mathrm{cl}_{M'}(F)$ contains b and b' but not a. Also $b' \in \mathrm{cl}_{M'}(X)$ and $b' \in \mathrm{cl}_{M'}(Y)$. Set $Z = \mathrm{cl}_{M'}(X) \cap \mathrm{cl}_{M'}(Y)$. Then $\{b, b'\} \subseteq Z$. As X, Y is a 3-separation, $r(Z) \leqslant 2$. Moreover, $a \in Z$, but $a \notin F'$. Hence $r(F' \cap Z) = 1$. But $\{b, b'\} \subseteq (F' \cap Z)$. Hence $\{b, b'\}$ is a parallel pair so that b is fixed in M. ▮

7. PROOF OF THE MAIN THEOREM

Let N be a 3-connected matroid with at least four elements, and let M be a 3-connected matroid with an N-minor. Recall from the introduction that M is a *totally free expansion of* N if the following holds for all x in $E(M)$: if $M \backslash x$ has an N-minor and $\mathrm{co}(M \backslash x)$ is 3-connected, then x is not fixed in M, and if M/x has an N-minor and $\mathrm{si}(M/x)$ is 3-connected, then x is not cofixed in M. In this section, we prove our main theorem, the following result.

THEOREM 7.1. *Let N be a 3-connected matroid with at least four elements and M be a totally free expansion of N such that $M \neq N$. Then either*

(i) *M has an element d such that $M \backslash d$ is a totally free expansion of N; or*

(ii) *M has an element c such that M/c is a totally free expansion of N; or*

(iii) *neither* (i) *nor* (ii) *holds and M has an element c such that M/c is* 3-*connected with an N-minor. Moreover, for, every such element c, there is an element d of $E(M)$ such that $M \backslash d$ is* 3-*connected with an N-minor, d is fixed in M/c, and $M \backslash d/c$ is a totally free expansion of N.*

Note that Theorem 7.1 is a strengthening of Theorem 2.2. It follows from the Splitter Theorem that, unless M is a wheel or a whirl, M has an element c such that $M \backslash c$ or M/c is 3-connected with an N-minor. Ideally we would like such a matroid to be a totally free expansion of N, but this is not always the case. Hence the necessity for part (iii) of Theorem 7.1. In the lemmas that follow, we examine the structure that arises. The first lemma follows immediately from the fact that the definition of a totally free expansion is self dual.

LEMMA 7.2. *M is a totally free expansion of N if and only if M^* is a totally free expansion of N^*.*

Let N be a 3-connected minor of a 3-connected matroid M, where $|E(N)| \geqslant 4$. Let c be an element of M. Then M is an *almost totally free expansion of N relative to c* if either $M \backslash c$ or M/c is 3-connected with an N-minor and, for all y in $E(M)-c$, if $\mathrm{si}(M/y)$ is 3-connected with an N-minor, then y is not cofixed in M, while if $\mathrm{co}(M \backslash y)$ is 3-connected with an N-minor, then y is not fixed in M.

LEMMA 7.3. *Let C^* be a cocircuit of a matroid M for which $|C^*| \geqslant 3$ and C^* has an element c such that no element of C^*-c is cofixed. Then there is a subset S of C^*-c such that C^*-c-S contains a pair of clones in $M \backslash S$.*

Proof. We shall argue by induction on $|C^*|$. If C^* is a triad, then the result follows from Lemma 6.1(ii) by taking S to be the empty set.

Now assume the lemma holds for $|C^*| < k$ and let $|C^*| = k > 3$. Choose a in C^*-c. If C^*-c-a contains an element b that is cofixed in $M \backslash a$, then, by the dual of Lemma 5.9, a and b are clones in M. Hence the result follows by taking S to be the empty set. Thus we may assume that no element of C^*-c-a is cofixed in $M \backslash a$. Then, by the induction assumption, there is a subset S' of C^*-c-a such that $C^*-c-a-S'$ contains a pair of clones of $M \backslash a \backslash S$. The result now follows by taking s to be $S' \cup a$. We conclude, by induction, that the lemma holds. ▮

Lemma 7.4. *Let M be an almost totally free expansion of N relative to c such that M/c is 3-connected with an N-minor. If x is an element of M such that* $\mathrm{si}(M/c, x)$ *is 3-connected, and M/c, x has an N-minor, then x is not cofixed in M/c.*

Proof. If $\mathrm{si}(M/x)$ is 3-connected, then it follows from the definition of an almost totally free expansion that x is not cofixed in M, and thus, by Corollary 4.4(ii), x is not cofixed in M/c. Therefore we may assume that $\mathrm{si}(M/x)$ is not 3-connected. By Lemma 3.6, M has a rank-3 cocircuit C^* containing c such that $x \in \mathrm{cl}(C^*) \cap (E(M) - C^*)$. We now prove:

7.4.1. (i) If $y \in C^*$, then both $M/x, y$ and M/y have N-minors.

(ii) No triangle of $M|(C^* \cup x)$ meets $\{x, c\}$.

(iii) If $y \in C^* - c$, then $\mathrm{si}(M/y)$ is 3-connected.

(iv) If $y \in C^* - c$, then y is not cofixed in M.

Proof. Evidently C^* is a rank-2 cocircuit of M/x. It is now easily checked that if $y \in C^*$, then $\mathrm{si}(M/x, y) \cong \mathrm{si}(M/x, c)$. But $\mathrm{si}(M/x, c)$ has an N-minor, so $\mathrm{si}(M/x, y)$ has an N-minor. It follows immediately that M/y has an N-minor, so part (i) holds.

Suppose that (ii) fails. As M/c is 3-connected, $M|(C^* \cup x)$ certainly has no triangles containing c. Thus we may assume that $M|(C^* \cup x)$ has a triangle $\{x, z, z'\}$. Denote $E(M) - C^*$ by H. If $x \notin \mathrm{cl}(H - x)$, then $\{H - x, C^* \cup x\}$ is a 2-separation of M, contradicting the fact that M is 3-connected. Hence $x \in \mathrm{cl}(H - x)$. Thus $H - x$ contains a cyclic flat F containing x. Evidently $r(F \cup \{x, z, z'\}) = r(F) + 1 = r(F) + r(\{x, z, z'\}) - 1$. Thus $\{F, \mathrm{cl}(\{x, z, z'\})\}$ is a modular pair of flats of M. But $F \cap \mathrm{cl}(\{x, z, z'\}) = \{x\}$. We can now deduce, by Proposition 4.1, that x is fixed in M. By (i), $M/x, z$ has an N-minor. Now z' is a loop of $M/x, z$ so $M/x, z \backslash z'$ has an N-minor. But $\{x, z'\}$ is a parallel pair of M/z. Hence $M/x, z \backslash z' = M/z, z' \backslash x$. Therefore $M \backslash x$ has an N-minor. Since $\mathrm{si}(M/x)$ is not 3-connected, Lemma 3.2 implies that $\mathrm{co}(M \backslash x)$ is 3-connected. Hence $\mathrm{co}(M \backslash x)$ is 3-connected with an N-minor. As $x \neq c$, it follows from the definition of an almost totally free expansion that x is not fixed in M. We conclude from this contradiction that part (ii) holds.

Let y and y' be distinct elements of $C^* - c$. Now $r(C^*) = 3$ and, by (ii), $M|(C^* \cup x)$ has no triangles meeting $\{x, c\}$. It follows that $\{x, y', c\}$ is a triangle of M/y. Hence, by Lemma 3.5, $\mathrm{si}(M/y)$ is 3-connected, and part (iii) holds. Part (iv) follows from parts (i) and (iii) and the definition of an almost totally free expansion. ∎

By Lemma 7.3, there is a subset S of $C^* - c$ such that $C^* - c - S$ contains a pair $\{w, z\}$ of clones of $M \backslash S$. By (7.4.1), $\{c, w, z\}$ is not a triangle. Hence $\{w, z\}$ is a pair of independent clones of $M \backslash S/c$.

Consider M/c. All points of S are on the line of M/c spanned by $\{x, w, z\}$. None of these points is parallel to either w or z. It is now straightforward to argue that, for a subset D of $E(M/c)-\{w, z\}$, the set $D \cup w$ is a circuit of M/c if and only if $D \cup z$ is a circuit of M/c. Thus w and z are clones of M/c. We conclude that neither w nor z is fixed in M/c. Hence, by Lemma 6.2(i), x is not cofixed in M/c. ∎

Dualizing Lemma 7.4, we immediately obtain the following result.

COROLLARY 7.5. *Let M be an almost totally free expansion of N relative to d such that $M\backslash d$ is 3-connected with an N-minor. If x is an element of $M\backslash d$ such that $\text{co}(M\backslash d, x)$ is 3-connected, and $M\backslash d, x$ has an N-minor, then x is not fixed in $M\backslash d$.*

LEMMA 7.6. *Let M be an almost totally free expansion of N relative to d, and assume that $M\backslash d$ is 3-connected with an N-minor.*

(i) *If d is fixed in M, then $M\backslash d$ is a totally free expansion of N.*

(ii) *If M is a totally free expansion of N, but $M\backslash d$ is not a totally free expansion of N, then there, is an element z of $E(M)-d$ such that $\text{si}(M\backslash d/z)$ is 3-connected with an N-minor and z is cofixed in $M\backslash d$. Moreover, every such element also has the property that M/z is 3-connected with an N-minor.*

Proof. Assume that $M\backslash d$ is not a totally free expansion of N. Suppose that $y \in E(M)-d$ is such that $\text{co}(M\backslash d, y)$ is 3-connected with an N-minor. Then, by Corollary 7.5, y is not fixed in $M\backslash d$. Therefore there is an element z of $E(M)-d$ such that $\text{si}(M\backslash d/z)$ is 3-connected with an N-minor and z is cofixed in $M\backslash d$. By Lemma 3.8(ii), $\text{si}(M/z)$ is 3-connected with an N-minor. Hence z is not cofixed in M. By Proposition 5.4(ii), d is freer than z.

Assume that d is fixed in M. Then, since d is freer than z, either d and z are parallel or z is a loop of M, contradicting the fact that M is 3-connected with more than four elements. Hence part (i) holds, and we may assume that d is not fixed in M.

Consider part (ii). Assume that M is a totally free expansion of N. We have already established the first assertion of part (ii). It remains to prove that, for every element z satisfying this first assertion, M/z is 3-connected. Assume that, for some such element, M/z is not 3-connected. As $\text{si}(M/z)$ is 3-connected, it follows that there is a triangle T of M using z. Suppose first that $d \in T$, say $T = \{d, d', z\}$. Evidently $M\backslash d/z \cong M\backslash d'/z$. Hence $M\backslash d'$ has an N-minor. Since $M\backslash d$ is 3-connected, no triad of M uses d. Hence, by Lemma 6.3(i), $M\backslash d'$ is 3-connected. Thus, by the definition of a totally free expansion, d' is not fixed in M. Hence, by Lemma 6.1(i), d and

d' are clones. It now follows by Lemma 4.10(ii) that $\{z, d, d'\}$ is a triad of M. The only 3-connected matroid in which a triad is also a triangle is $U_{2,4}$. But $|E(N)| \geqslant 4$, and N is a proper minor of M, so $M \not\cong U_{2,4}$. It follows that z is not in a triangle of M containing d.

We may now assume that $T = \{a, b, z\}$ where $d \notin \{a, b\}$. Next we prove that no triad of M contains $\{a, b\}$. Suppose that $\{a, b, t\}$ is a triad T^* of M. Since $M \backslash d$ is 3-connected, $t \neq d$. As M is 3-connected but $M \not\cong U_{2,4}$, we deduce that $|E(M) - T^*| \geqslant 3$. Thus $\{T^*, E(M) - T^*\}$ is a 3-separation of M. But $z \in \text{cl}(T^*)$, so if $r(E(M) - T^*) \geqslant 3$, then $\{T^*, E(M) - (T^* \cup z)\}$ is a vertical 2-separation of M/z, contradicting the fact that $\text{si}(M/z)$ is 3-connected. Hence $E(M) - T^*$ is a line. Moreover, $E(M) - T^*$ contains d and z, and has at least three points. Thus M has a triangle containing $\{d, z\}$, contradicting the fact that M has no such triangles. We conclude that no triad of M contains $\{a, b\}$.

Since $\{a, b\}$ is a parallel pair of M/z, and $\text{si}(M/z)$ has an N-minor, $M \backslash a$ and $M \backslash b$ both have N-minors. Assume that a is not fixed in M. No triad of M contains $\{a, b\}$, so, by Lemma 6.3(i), $M \backslash b$ is 3-connected. By the definition of a totally free expansion, b is not fixed in M. Hence, by Lemma 6.1(i), a and b are clones. Thus a and b are also clones in $M \backslash d$ and are therefore not fixed in $M \backslash d$. It now follows, by applying Lemma 6.2(i) to $M \backslash d$, that z is not cofixed in $M \backslash d$. This contradiction implies that both a and b are fixed in M. Since both $M \backslash a$ and $M \backslash b$ have N-minors, we deduce that neither $\text{co}(M \backslash a)$ nor $\text{co}(M \backslash b)$ is 3-connected. In this case, we deduce from the dual of Lemma 3.7 that there is a triad of M containing $\{a, b\}$, contradicting the fact that no such triad exists. We conclude that M/z is 3-connected, as required. ∎

Lemma 7.7. *Let M be a totally free expansion of N such that, for all x in $E(M)$, neither $M \backslash x$ nor M/x is a totally free expansion of N. Assume that an element c of M has the property that M/c is 3-connected and has an N-minor. Then there is an element d of $E(M) - c$ with the property that $M \backslash d$ is 3-connected, d is fixed in M/c, and $M/c \backslash d$ is a totally free expansion of N.*

Proof. Let S_{del} consist of those elements whose deletion from M is 3-connected with an N-minor, and let $V_{\text{del}} = \{(x, \text{del}) : x \in S_{\text{del}}\}$. Let S_{con} consist of those elements whose contraction from M is 3-connected with an N-minor, and let $V_{\text{con}} = \{(x, \text{con}) : x \in S_{\text{con}}\}$. We now construct a directed bipartite graph with vertex set $V_{\text{con}} \cup V_{\text{del}}$ as follows. There is a directed edge from (z, del) to (z', con) if and only if $\text{si}(M \backslash z/z')$ is 3-connected with an N-minor and z' is cofixed in $M \backslash z$, and there is a directed edge from (z', con) to (z, del) if and only if $\text{co}(M/z' \backslash z)$ is 3-connected with an N-minor and z is fixed in M/z'.

Suppose that $z \in S_{\text{del}}$. By assumption, $M \backslash z$ is not a totally free expansion of N. Moreover, by Lemma 7.6(ii), there is an element z' of M such that $\text{si}(M \backslash z/z')$ is 3-connected with an N-minor and z' is cofixed in $M \backslash z$. Furthermore, M/z' is 3-connected with an N-minor. Hence $z' \in S_{\text{con}}$. We conclude that each vertex of V_{del} has out degree at least one and, by duality, so too does each vertex of V_{con}. This shows that each vertex of the bipartite graph has outdegree at least one. Assume that some vertex, say (d, del), of V_{del} has indegree greater than one. Then there are elements c_1 and c_2 of S_{con} such that d is fixed in both M/c_1 and M/c_2. Since d is not fixed in M, it follows from Lemma 4.7(iii) that $\{d, c_1, c_2\}$ is a triangle, contradicting the fact that M/c_1 is 3-connected. Thus each vertex of V_{del} has indegree at most one. By a dual argument, the same is true for each vertex in V_{con}. It now follows by elementary graph theory that each vertex has indegree and outdegree exactly one.

Choose $c \in S_{\text{con}}$. By the above, there is a unique element d of S_{del} such that $\text{co}(M/c \backslash d)$ is 3-connected with an N-minor and d is fixed in M/c. By Lemma 7.4, if x is an element of $E(M) - c$ such that $\text{si}(M/c, x)$ is 3-connected with an N-minor, then x is not cofixed in $M/c, x$. Also, $M \backslash d$ is 3-connected, so $M \backslash d$ has no series pairs. Hence $M \backslash d/c$ has no series pairs. Therefore $\text{co}(M/c \backslash d) = M/c \backslash d$. But we know $\text{co}(M/c \backslash d)$ is 3-connected. Hence $M/c \backslash d$ is 3-connected. It follows that M/c is an almost totally free expansion of N relative to d. Thus, by Lemma 7.6(i), $M/c \backslash d$ is a totally free expansion of N. ∎

Proof of Theorem 7.1. Suppose first that M is a wheel or whirl. Since $|E(M)| \geqslant 5$, it follows that $r(M) \geqslant 3$. One readily checks that if $\text{co}(M \backslash x)$ is 3-connected, then x is fixed in M, and it follows by duality that if $\text{si}(M/x)$ is 3-connected, then x is cofixed in M. Moreover, it is easily seen that, since N is a proper minor of M, there is an element x such that either $\text{co}(M \backslash x)$ or $\text{si}(M/x)$ is 3-connected with an N-minor. We conclude that M is not a totally free expansion of N; a contradiction.

We may now assume that M is not a wheel or a whirl. By the Splitter Theorem, the set of elements whose deletion or contraction from M is 3-connected with an N-minor is non-empty. If, for some such element x, either $M \backslash x$ or M/x is a totally free expansion of N, then (i) or (ii) holds. Thus we may assume that neither (i) nor (ii) holds. Then there is an element x of M such that either M/x or $M \backslash x$ is 3-connected having an N-minor. In the second case, by the dual of Lemma 7.7, M^* has an element y such that $M^* \backslash y$ is 3-connected having an N^*-minor. Hence M/y is 3-connected having an N-minor. Thus, in both the first and second cases, M has an element c such that M/c is 3-connected having an N-minor. It now follows by Lemma 7.7 that (iii) holds. ∎

8. TOTALLY FREE MATROIDS

Recall from the introduction that a 3-connected matroid M is *totally free* if it has at least four elements and, for all x in $E(M)$, if $\mathrm{co}(M\backslash x)$ is 3-connected, then x is not fixed in M, while if $\mathrm{si}(M/x)$ is 3-connected, then x is not cofixed in M. The main purpose of this section is to present a strengthening of Theorem 7.1 for totally free matroids. Theorem 7.1 can also be strengthened for totally free expansions of a matroid N as long as a reasonably natural condition is placed on the matroid N, and we begin by showing this. Recall that a matroid N' is *strictly freer* than N if N is a rank-preserving weak-map image of N', and $N \neq N'$.

THEOREM 8.1. *Let $\mathcal{N}$ be a minor-closed class of matroids that is closed under isomorphism, and let N be a 3-connected matroid in $\mathcal{N}$ with $|E(N)| \geqslant 4$ such that no matroid in $\mathcal{N}$ is strictly freer than N. Let M be a matroid in $\mathcal{N}$ that is a totally free expansion of N. Then at least one of the following holds.*

(i) *There is an element x of $E(M)$ for which either $M\backslash x$ or M/x is a totally free expansion of N.*

(ii) *The set of elements x for which either $M\backslash x$ or M/x is 3-connected with an N-minor can be partitioned into 2-element subsets with the property that if $\{a, a'\}$ is a block in this partition, then $\{a, a'\}$ is a clonal class of M and $M\backslash a/a'$ is a totally free expansion of N.*

Before proving Theorem 8.1, we note two lemmas. The first holds for any totally free expansion of a 3-connected matroid. When, as in the next proof, we refer to a *clonal triple* or a *clonal pair*, we mean a subset of size three or two, respectively, of a clonal class.

LEMMA 8.2. *Let N be a 3-connected matroid with at least four elements. Let M be a totally free expansion of N, and assume that the element a of M belongs to a clonal class of size at least three.*

(i) *If $M\backslash a$ is 3-connected with an N-minor, then $M\backslash a$ is a totally free expansion of N.*

(ii) *If M/a is 3-connected with an N-minor, then M/a is a totally free expansion of N.*

Proof. By duality, it suffices to prove (i). Assume that $M\backslash a$ is 3-connected with an N-minor, but that $M\backslash a$ is not a totally free expansion of N. Then, by Lemma 7.6(ii), there is an element z of $E(M)-a$ such that z is cofixed in $M\backslash a$. Now M has elements b and c such that $\{a, b, c\}$ is a clonal triple, and hence b and c are clones in $M\backslash a$. Moreover, since $M\backslash a$ is

3-connected with at least four elements, $\{b, c\}$ is coindependent in $M\backslash a$. Thus neither b nor c is cofixed in $M\backslash a$. Hence $z \notin \{b, c\}$.

Assume next that a is freer than z. Since a and b are clones, b is also freer than z. By Proposition 5.6, b is freer than z in $M\backslash a$. But then, by Lemma 5.5(ii), z is not cofixed in $M\backslash a$. It follows from this contradiction that we may assume that a is not freer than z. Then, by Proposition 5.4(ii), z is not cofixed in $M\backslash a$. This contradiction completes the proof of the lemma. ∎

Proof of Theorem 8.1. Assume that there is no element x such that either $M\backslash x$ or M/x is a totally free expansion of N. Suppose that the set of elements x of M for which $M\backslash x$ or M/x is 3-connected with an N-minor is non-empty. By duality, we may assume that M has an element c for which M/c is 3-connected with an N-minor. By Theorem 7.1, there is an element d of $E(M)-c$ such that $M\backslash d$ is 3-connected with an N-minor, d is fixed in M/c, and $M/c\backslash d$ is a totally free expansion of N. As M is a totally free expansion of N and $M\backslash d$ is 3-connected having an N-minor, it follows that d is not fixed in M. By Proposition 5.4(i), d is freer than c. Let M' be the matroid obtained from $M\backslash c$ by relabelling d as c. By Proposition 5.7(i), M' is freer than $M\backslash d$. Since N is a minor of $M\backslash d$, there is an independent set I and a coindependent set J in $M\backslash d$ such that $(M\backslash d)/I\backslash J \cong N$. One readily checks that $M'/I\backslash J$ is freer than $(M\backslash d)/I\backslash J$, that is, $M'/I\backslash J$ is freer than N. But $\mathcal{N}$ is closed under isomorphism and minors, so $M'/I\backslash J \in \mathcal{N}$. It now follows by the hypothesis of the theorem that $M'/I\backslash J$ is not strictly freer than N, so $M'/I\backslash J \cong N$. Hence M' has an N-minor. Moreover, since $M\backslash d$ is 3-connected and d is freer than c in M, it follows by Corollary 5.8 that $M\backslash c$ is 3-connected. Since $M\backslash c$ also has an N-minor, the definition of a totally free expansion implies that c is not fixed in M. We conclude, by Lemma 5.9, that c and d are clones. It now follows from Lemma 8.2 that $\{c, d\}$ is a clonal class. ∎

We now turn our attention to totally free matroids. We start by showing that being a totally free matroid is equivalent to being a totally free expansion of $U_{2,4}$. This will follow from the next three lemmas. The straightforward proof of the first of these is given in [10].

LEMMA 8.3 [10, Lemma 5.6]. *Let x be an element of a connected binary matroid M with at least two elements.*

(i) *If $M\backslash x$ is connected, then x is fixed in M.*
(ii) *If M/x is connected, then x is cofixed in M.*

LEMMA 8.4. *Let a and a' be clones of a 3-connected matroid M such that $r(M) \geqslant 3$. Then M/a is non-binary. Moreover, if $r^*(M) \geqslant 3$, then $M/a\backslash a'$ is non-binary.*

Proof. Since M is 3-connected, $M\backslash a$ is connected. But a is not fixed in M, so, by Lemma 8.3(i), M is non-binary. If $r^*(M) = 2$, then $M \cong U_{n,n+2}$ for some $n \geqslant 3$, and the result clearly holds. Assume that M has corank at least three. In this case, it follows from [13, Corollary 11.2.19] that M has a minor isomorphic to one of $\mathcal{W}^3$, P_6, Q_6, and $U_{3,6}$, where $\mathcal{W}^3$ denotes the rank-3 whirl, and the matroids P_6 and Q_6 are as defined in the appendix of [13]. But it is shown in [14] that this set of four matroids is 2-rounded. This means that M has a minor M' that uses both a and a' and is isomorphic to one of $\mathcal{W}^3$, P_6, Q_6, and $U_{3,6}$. By Proposition 4.3, a and a' are clones in M'. But $\mathcal{W}^3$ has no pairs of clones, so $M' \not\cong \mathcal{W}^3$. An easy check shows that a matroid obtained by contracting one member of a clonal pair and deleting the other is non-binary in each of the other three possibilities for M', that is, $M'/a\backslash a'$ is non-binary. We immediately deduce that $M/a\backslash a'$ is non-binary. ∎

LEMMA 8.5. *Let M be a totally free expansion of $U_{2,4}$ of rank at least three. If $z \in E(M)$, then M/z is non-binary.*

Proof. It is a straightforward consequence of the Splitter Theorem that there is an element a of $E(M)$ such that $\mathrm{si}(M/a)$ is 3-connected and non-binary. If $z = a$, then M/z is certainly non-binary. Hence we may assume that $z \neq a$.

Suppose that no triangle of M contains $\{z, a\}$. Then z is not in a 2-circuit of M/a. It now follows easily from the fact that $\mathrm{si}(M/a)$ is 3-connected that $M/a, z$ is connected. By the definition of a totally free expansion, a is not cofixed in M, so, by Corollary 4.4(ii), a is not cofixed in M/z. By Lemma 8.3(ii), if M/z were binary, then, given that $M/a, z$ is connected, a would be cofixed in M/z. We deduce that M/z is non-binary.

It remains to consider the case when M has a triangle containing $\{z, a\}$. Let $\{a, y, y'\}$ be such a triangle, where $z \in \{y, y'\}$. Suppose that M has a triad T^* containing $\{y, y'\}$. Since $M \not\cong U_{2,4}$, it follows that $a \notin T^*$. If $r(M) \geqslant 4$, then one easily deduces, since $a \in \mathrm{cl}(T^*)$, that $\mathrm{si}(M/a)$ is not 3-connected. From this contradiction, we conclude that $r(M) = 3$. In that case, $\mathrm{si}(M/a) \cong U_{2,3}$; a contradiction.

We may now assume that M has no triad that contains $\{y, y'\}$. Since both y and y' are in a non-trivial parallel class of the non-binary matroid M/a, both $M/a\backslash y$ and $M/a\backslash y'$ are non-binary. Hence both $M\backslash y$ and $M\backslash y'$ are non-binary. Suppose that neither $M\backslash y$ nor $M\backslash y'$ is 3-connected. Then, by Lemma 3.1 and the fact that $\{y, y'\}$ is in no triad, it follows that M has a triad containing $\{a, y\}$ and a different triad containing $\{a, y'\}$. We deduce that a is cofixed in M; a contradiction. We conclude that either $M\backslash y$ or $M\backslash y'$ is 3-connected. Without loss of generality, assume that $M\backslash y$ is 3-connected. Then, by the definition of a totally free expansion, y is not fixed in M. But then, since no triad of M uses both y and y', it follows from Lemma 6.3 that $M\backslash y'$ is 3-connected. Again, by the definition of a totally free expansion we deduce that y' is not fixed in M. Hence, by Lemma 6.1(i), y and y' are clones. But now, by Lemma 8.4, both M/y and M/y' are non-binary. Since $z \in \{y, y'\}$, we conclude that M/z is non-binary. ∎

COROLLARY 8.6. *A matroid M is totally free if and only if it is a totally free expansion of $U_{2,4}$.*

Proof. Suppose that M is totally free. It is an immediate consequence of Lemmas 3.2 and 8.3 that M is not binary, that is, M has a $U_{2,4}$-minor. It now follows immediately that M is a totally free expansion of $U_{2,4}$. Conversely, assume that M is a totally free expansion of $U_{2,4}$. If $r(M) = 2$, then evidently $M \cong U_{2,n}$ for some $n \geqslant 4$, and it is easily verified that M is totally free. Dually, M is totally free if $r^*(M) = 2$. Assume that M has rank and corank greater than two. Choose x in $E(M)$. Suppose $\mathrm{si}(M/x)$ is 3-connected. By Lemma 8.5, M/x has a $U_{2,4}$-minor. Thus, since M is a totally free expansion of $U_{2,4}$, the element x is not cofixed in M. Dually, if $\mathrm{co}(M\backslash x)$ is 3-connected, then x is not fixed in M. ∎

Since the class of all matroids is minor-closed and no matroid is freer than $U_{2,4}$, Theorem 8.1 holds when the matroid M is a totally free expansion of $U_{2,4}$, that is, when M is a totally free matroid. However, in the special case of totally free matroids, Theorem 8.1 can be strengthened somewhat. This strengthening, Corollary 8.13, will require several more preliminaries.

LEMMA 8.7. *If $\{a, b, c\}$ is a triad of a totally free matroid M with at least five elements, then no triangle of M meets $\{a, b, c\}$.*

Proof. Let $H = E(M) - \{a, b, c\}$. Assume that M has a triangle meeting $\{a, b, c\}$. Then, as $M \not\cong U_{2,4}$, this triangle is not $\{a, b, c\}$. Thus, without loss of generality, we may assume that it is $\{a, b, z\}$ for some z in H. If z is a coloop of $M|H$, then $\{\{a, b, c, z\}, E(M) - \{a, b, c, z\}\}$ is a 2-separation of M; a contradiction. Thus z is not a coloop of $M|H$. Therefore, by Lemma 6.4, z is fixed in M. As M is totally free, it follows that $\mathrm{co}(M\backslash z)$ is not 3-connected. Thus, by Lemma 3.2, $\mathrm{si}(M/z)$ is 3-connected. But $\mathrm{si}(M/z)$ has a 2-cocircuit and so is isomorphic to $U_{2,3}$. On combining this with the fact that $\mathrm{co}(M\backslash z)$ is not 3-connected, it is not difficult to show that M is isomorphic to a rank-3 wheel or whirl. But neither of these matroids is totally free; a contradiction. ▮

LEMMA 8.8. *If $\{a, b, c\}$ is a triad or a triangle of a totally free matroid M, then $\{a, b, c\}$ is a clonal triple.*

Proof. The lemma certainly holds if $|E(M)| = 4$ since, in that case, $M \cong U_{2,4}$. Thus suppose that $|E(M)| \geqslant 5$. Assume $\{a, b, c\}$ is a triad. By Lemma 8.7 and the dual of Lemma 3.1, there is an element z of $\{a, b, c\}$ such that M/z is 3-connected. Without loss of generality, assume that M/a is 3-connected. By the definition of a totally free matroid, a is not cofixed in M. It now follows from Lemmas 6.3 and 8.7 that M/b and M/c are 3-connected. Hence neither b nor c is cofixed, that is, no member of $\{a, b, c\}$ is cofixed. We conclude from Lemma 6.1(ii) that $\{a, b, c\}$ is a clonal triple. ▮

PROPOSITION 8.9. *Let x be an element of a totally free matroid M. Then either $M\backslash x$ or M/x is 3-connected.*

Proof. By Lemma 3.2, either $\text{co}(M\backslash x)$ or $\text{si}(M/x)$ is 3-connected. By duality, we may assume that $\text{si}(M/x)$ is 3-connected. If M/x is not 3-connected, then x belongs to a triangle, T, and, by Lemma 8.8, T is a clonal triple. By the dual of Lemma 8.7, no member of T belongs to a triad, so, by Tutte's Triangle Lemma (3.1), T has a member z such that $M\backslash z$ is 3-connected. But z and x are clones, so $M\backslash x$ is 3-connected. ▮

LEMMA 8.10. *Let M be a totally free matroid and a and b be elements of M such that $M/a\backslash b$ is 3-connected and b is fixed in M/a. Then $\{a, b\}$ is a clonal class of M.*

Proof. Suppose first that b is fixed in M. Then $M\backslash b$ is not 3-connected. But $M\backslash b/a$ is 3-connected. Hence M has a triad T^* containing $\{a, b\}$. Thus, by Lemma 8.8, T^* is a clonal triple of M containing b, so b is not fixed in M; a contradiction.

We may now assume that b is not fixed in M. Then, by Lemma 5.9, either

(i) $\{a, b\}$ is a clonal pair in M; or

(ii) a is fixed in M.

Suppose that (ii) holds. Then $M\backslash a$ is not 3-connected. Let $\{X, Y\}$ be a 2-separation of $M\backslash a$ in which $b \in X$. Now assume that b is not a coloop of $M|X$. Then $M|X$ has a cyclic flat containing b. As $a \notin \text{cl}_M(X)$, it follows that a is not in this cyclic flat. Thus b is not freer than a in M. However, b is fixed in M/a but not in M, so, by Proposition 5.4, b is freer than a in M;

a contradiction. We conclude that b is a coloop of $M|X$. Now both $|X|$ and $|Y|$ exceed two, otherwise a is in a triad of M, so, by Lemma 8.8, a is not fixed in M. It follows that $\{X, Y \cup a\}$ is a 3-separation of M and hence of M^*. As $a \notin \mathrm{cl}_M(Y)$, we deduce that $a \in \mathrm{cl}_{M^*}(X)$, so $a \in \mathrm{cl}_{M^*}(x) \cap \mathrm{cl}_{M^*}(Y \cup a)$. Moreover, as $b \notin \mathrm{cl}_M(X-b)$, we deduce that $b \in \mathrm{cl}_{M^*}(Y \cup a)$, so $b \in \mathrm{cl}_{M^*}(X) \cap \mathrm{cl}_{M^*}(Y \cup a)$.

We now know that $\{a, b\} \subseteq \mathrm{cl}_{M^*}(X) \cap \mathrm{cl}_{M^*}(Y \cup a)$. Moreover, as a is fixed in M but b is not, b is strictly freer than a in M, so a is strictly freer than b in M^*. Then, by Lemma 6.5, b is fixed in M^*, that is, b is cofixed in M. Thus M/b is not 3-connected, so, by Corollary 5.8(ii), M/a is not 3-connected. But $M/a\backslash b$ is 3-connected, so $\{a, b\}$ is contained in a triangle T of M. Thus, by Lemma 8.8, T is a clonal triple of M containing b, so b is not fixed in M/a; a contradiction. We conclude that a is not fixed in M, that is, (ii) fails. Hence (i) holds, that is, $\{a, b\}$ is a clonal pair in M. Moreover, if the clonal class of M containing a has size at least three, then b is not fixed in M/a. It follows that b is the unique clone of a in M. ▮

LEMMA 8.11. *Let M be a totally free matroid and a be an element of M such that M/a is 3-connected.*

(i) *If* $\mathrm{co}(M/a\backslash x)$ *is 3-connected but x is fixed in M/a, then $M/a\backslash x$ is 3-connected and $\{a, x\}$ is a clonal class of M.*

(ii) *If* $\mathrm{si}(M/a/x)$ *is 3-connected, then x is not cofixed in M/a.*

Proof. Assume that $\mathrm{co}(M/a\backslash x)$ is 3-connected and x is fixed in M/a. If $M/a\backslash x$ is 3-connected, then, by Lemma 8.10, $\{a, x\}$ is a clonal class of M. Thus we may assume that $M/a\backslash x$ is not 3-connected. Since $\mathrm{co}(M/a\backslash x)$ is 3-connected, it follows that x is in a triad $\{x, s, t\}$, say, of M/a. Then $\{x, s, t\}$ is a triad of M. As M is totally free, by Lemma 8.8, $\{x, s, t\}$ is a clonal triple of M and hence of M/a. Thus x is not fixed in M/a; a contradiction. We conclude that (i) holds.

Now suppose that $\mathrm{si}(M/a/x)$ is 3-connected but x is cofixed in M/a. Then, by Proposition 4.7, x is cofixed in M. But M is totally free, so $\mathrm{si}(M/x)$ is not 3-connected. Then, by Lemma 3.6, M has a rank-3 cocircuit C^* containing a such that $x \in \mathrm{cl}(C^*) \cap (E(M)-C^*)$. In M/a, the elements of $(C^*-a) \cup x$ are collinear. Thus, if $\{j, k\} \subseteq C^*-a$, then $\{x, j, k\}$ is a triangle of M/a and x is cofixed in M/a. Then, by Lemma 6.2, j is fixed in M/a. Hence every element of C^*-a is fixed in M/a.

Suppose that $|C^*|=3$. Then, by Lemma 8.8, C^* is a clonal triple of M. Hence C^*-a is a clonal pair in M/a, that is, no element of C^*-a is fixed in M/a; a contradiction. We may now assume that $|C^*| \geqslant 4$. Then $|(C^*-a) \cup x| \geqslant 4$. Since the elements of $(C^*-a) \cup x$ are collinear in M/a, it follows that if $j \in C^*-a$, then $M/a\backslash j$ is 3-connected. Since j is also fixed in M/a, Lemma 8.10 implies that $\{j, a\}$ is a clonal class of M. But this is a contradiction since the last assertion must hold for every j in C^*-a and $|C^*-a| \geqslant 3$. We conclude that (ii) holds. ∎

THEOREM 8.12. *Let M be a totally free matroid with $|E(M)| \geqslant 5$. If a is an element of M such that either $M\backslash a$ is 3-connected but not totally free, or M/a is 3-connected but not totally free, then*

(i) *a has a unique clone a' in M;*
(ii) *both $M\backslash a/a'$ and $M/a\backslash a'$ are totally free; and*
(iii) *both $M\backslash a$ and M/a are 3-connected.*

Proof. The hypotheses imply that both $r(M)$ and $r^*(M)$ exceed two. By duality, we may assume that M/a is 3-connected but not totally free. Then M/a has an element x such that either

(i) $\mathrm{co}(M/a\backslash x)$ is 3-connected but x is fixed in M/a; or
(ii) $\mathrm{si}(M/a/x)$ is 3-connected but x is cofixed in M/a.

By Lemma 8.11, (ii) does not hold and, since (i) must hold, $\{a, x\}$ is a clonal class of M, and $M/a\backslash x$ is 3-connected.

We show next that M/a is an almost totally free expansion of $U_{2,4}$ relative to x. Certainly $M/a\backslash x$ is 3-connected and, by Lemma 8.4, $M/a\backslash x$ is non-binary. Now suppose that $y \in E(M/a)-x$. Then, by Lemma 8.11, if $\mathrm{si}(M/a/y)$ is 3-connected, then y is not cofixed in M/a; and if $\mathrm{co}(M/a\backslash y)$ is 3-connected, then y is not fixed in M/a since $y \neq x$. Thus M/a is indeed an almost totally free expansion of $U_{2,4}$ relative to x. Moreover, $M/a\backslash x$ is 3-connected with a $U_{2,4}$-minor, and x is fixed in M/a. Hence, by Lemma 7.6, $M/a\backslash x$ is a totally free expansion of $U_{2,4}$. Thus, by Corollary 8.6, $M/a\backslash x$ is totally free. Since x and a are clones, $M/x\backslash a$ is also totally free.

Finally, we note that, since $\{a, x\}$ is a clonal class, Lemma 8.8 implies that $\{a, x\}$ is not in a triangle or a triad of M. Thus, as $M\backslash a/x$ is 3-connected, so too are $M\backslash a$ and M/x. ∎

COROLLARY 8.13. *Let M be a totally free matroid such that $|E(M)| \geqslant 5$ and, for all x in $E(M)$, neither $M\backslash x$ nor M/x is totally free. Then the ground set of M is the union of 2-element clonal classes. Moreover, if $a \in E(M)$, then $M\backslash a$ and M/a are both 3-connected, and if a' is the unique clone of a in M, then $M\backslash a/a'$ is totally free.*

Proof. Choose a in $E(M)$. By Proposition 8.9, $M\backslash a$ or M/a is 3-connected. The result now follows immediately from Theorem 8.12. ∎

We conclude this section with a result of a somewhat different nature from earlier ones. It shows that, while it is possible to remove elements or pairs of elements from a totally free matroid and remain totally free, if elements are removed in the wrong way, we may soon be far from being totally free. It also shows that, in general, totally free matroids may be highly complicated objects.

THEOREM 8.14. *Let N be a matroid. Then there is a totally free matroid M with an N-minor such that* $|E(M)-E(N)|=4$.

Proof. Let $N_0=N$ and, for each i in $\{1,2\}$, let N_i be obtained from N_{i-1} by freely extending by the element d_i. Then, for each j in $\{1,2\}$, let N_{j+2} be obtained from N_{j+1} by freely coextending by the element c_j. Finally, let $M=N_4$. Certainly M has an N-minor and has exactly four more elements than N. It is straightforward to check, by, say, comparing collections of bases [13, Exercise 7.2.5], that the operation of free extension by an element e commutes with the operation of free extension by an element f and also with the operation of free coextension by an element g. It follows that c_1 and c_2 are clones in M, as are d_1 and d_2. Thus none of c_1, c_2, d_1 and d_2 is fixed or cofixed in M.

We shall show next that no element of $E(M)-\{c_1, c_2, d_1, d_2\}$ is fixed or cofixed in M. By duality, it suffices to show that no such element is fixed. To do this, we first observe that, since a free extension of a matroid has no coloops while a free extension has no loops, M has no loops or coloops. Now suppose that $x \in E(M)-c_2$ and x is fixed in M. The construction of M guarantees that $\{d_2\}$ is a flat of M. Then, by Corollary 4.6, M has a cyclic flat F containing x but not c_2. Thus $E(M)-F$ is a cyclic flat of M^* containing c_2 but not x. But c_2 is free in M^* so the unique cyclic flat of M^* containing c_2 is $E(M)$. This contradiction implies that x is not fixed in M. Hence no element of $E(M)-c_2$ is fixed in M.

We now know that no element of $E(M)$ is fixed or cofixed in M. Hence M is totally free provided it is 3-connected. But a free extension by p' of a loopless matroid M' is certainly connected since the union of p' with a basis of M' is a circuit of the free extension that meets every component of M'. Thus M is certainly connected. Moreover, since M can be obtained from another matroid by two free coextensions, M is simple. Suppose that M has a 2-separation $\{X, Y\}$. Then $r(X)+r(Y)=r(M)+1$. Without loss of generality, we may assume that $d_1 \in X$. If $r(X-d_1)=r(X)$, then, since d_1 is free in M, it follows that $r(X)=r(M)$. Hence $r(Y)=1$; a contradiction since $|Y| \geqslant 2$. We deduce that $r(X-d_1)=r(X)-1$, so $r(X-d_1)+r(Y)=r(M \backslash d_1)$. Therefore $M \backslash d_1$ is disconnected. But $M \backslash d_1$ is a free extension of a loopless matroid and so, from above, is connected. This contradiction completes the proof that M is 3-connected. ∎

9. AN EXTENSION OF THE SPLITTER THEOREM

In this section, we shall prove Theorem 2.3. This theorem will be deduced as a consequence of the next theorem. The proof of the latter will use some ideas from [15, 18]. Tutte [24] called an element e of a 3-connected matroid N *essential* if neither $N\backslash e$ nor N/e is 3-connected. A *chain of triangles and triads* [18] in a 3-connected matroid M is a non-empty sequence of sets $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, ..., \{a_k, a_{k+1}, a_{k+2}\}$, each link $\{a_i, a_{i+1}, a_{i+2}\}$ of which is a triangle or a triad such that no two consecutive links are triangles, no two consecutive links are triads, and the elements a_1, a_2, ..., a_{k+2} are distinct. When M is not a wheel or whirl, a maximal chain in M is called a *fan*. If a fan has at least two links, then the fan contains exactly two non-essential elements, namely the two elements that are in only one link of the fan. These elements are the *ends* of the fan.

THEOREM 9.1. *Let M be a 3-connected matroid and N be a 3-connected minor of M having at least four elements. If M is not a wheel or a whirl, and M is not a totally free expansion of N, then there is an element x of $E(M)$ such that either $M\backslash x$ is 3-connected with an N-minor and x is fixed in M, or M/x is 3-connected with an N-minor and x is cofixed in M.*

Proof. Since M is not a totally free expansion of N, there is an element x such that either $\text{co}(M\backslash x)$ or $\text{si}(M/x)$ is 3-connected with an N-minor, and x is, respectively, either fixed or cofixed in M. By duality, we may assume that $\text{co}(M\backslash x)$ is 3-connected with an N-minor and that x is fixed in M.

The desired result holds if $M\backslash x$ is 3-connected. Thus we may assume that $M\backslash x$ is not 3-connected. As $\text{co}(M\backslash x)$ is 3-connected, x is in a triad $\{x, y, z\}$ of M. Since $\{y, z\}$ is a series pair of $M\backslash x$, and $\text{co}(M\backslash x)$ has an N-minor, it follows that both $M\backslash x/y$ and $M\backslash x/z$ have N-minors.

We now show that the theorem holds if either M/y or M/z is 3-connected. Suppose that M/y is 3-connected. It has an N-minor. Thus the theorem holds unless y is not cofixed in M. In the exceptional case, by Lemma 6.2(ii), neither x nor z is fixed in M. This is a contradiction to the assumption that x is fixed. Hence y is cofixed in M. The same argument shows that the theorem holds if M/z is 3-connected.

We may now assume that neither M/y nor M/z is 3-connected. Then, by the dual of Lemma 3.1, M has a triangle containing y and exactly one of x and z. If there is a triangle containing y and z, then $\text{co}(M\backslash x)$ has a parallel pair. Therefore, since $|E(\text{co}(M\backslash x))| \geqslant |E(N)| \geqslant 4$, we obtain the contradiction that $\text{co}(M\backslash x)$ is not 3-connected. Thus it follows, without loss of generality, that M has a triangle containing $\{x, y\}$. The existence of this triangle shows that M/x is not 3-connected, and, by assumption, M/z

is not 3-connected. Thus, again by the dual of Lemma 3.1, M has a triangle containing z and exactly one of x and y. But, as noted above, M does not have a triangle containing $\{y, z\}$. Hence M has a triangle containing $\{x, z\}$. Thus M has a chain of triangles and triads that contains x and has length at least three. It now follows from Oxley and Wu [18] that x is in a unique fan of M unless x is in exactly three 5-element fans. But the exceptional case does not arise because M has no triangle containing $\{y, z\}$. We conclude that x is in a unique fan of M. Hence x is in exactly two triangles of M and exactly one triad, $\{x, y, z\}$. The fan of M containing x is of one of three types.

For the first type, the fan consists of a chain

$$\{x_1, y_1, x_2\}, \{y_1, x_2, y_2\}, ..., \{x_{k-1}, y_{k-1}, x_k\}$$

of triangles and triads where $\{x_1, y_1, x_2\}$ and $\{x_{k-1}, y_{k-1}, x_k\}$ are triangles, neither x_1 nor x_k is essential, and $x \in \{x_2, x_3, ..., x_{k-1}\}$. Since x is in a unique triad of M, it follows, by [18], that $\mathrm{co}(M \backslash x) \cong M \backslash x_2 / y_2$. But

$$M \backslash x_2 / y_1 \cong M / y_1 \backslash x_1 = M \backslash x_1 / y_1.$$

Since x_1 is non-essential, $M \backslash x_1$ is 3-connected. Moreover, $M \backslash x_1$ has an N-minor.

We shall show next that x_1 is fixed in M. Assume it is not. By [18], M is the generalized parallel connection across a triangle T of a k-spoked wheel and a minor M_1 of M, where

$$M_1 = M \backslash x_2, x_3, ..., x_{k-1} / y_2, y_3, ..., y_{k-1}$$

with y_1 renamed as z, and $T = \{x_1, x_k, z\}$. Moreover, either M_1 is 3-connected, or z is in a unique 2-circuit $\{z, h\}$ of M_1, and $M_1 \backslash z$ is 3-connected. In either case, we deduce that $M_1 \backslash z$ is connected having at least two elements. It follows from this that there is a circuit C of $M_1 \backslash z$ containing x_1. As

$$M_1 \backslash z = M \backslash y_1, y_2, ..., y_{k-1}, x_2, x_3, ..., x_{k-1},$$

$\mathrm{cl}_M(C)$ is a cyclic flat of M. Moreover, by orthogonality with the triads in the fan, this cyclic flat meets $\{x_1, y_1, x_2\}$ in x_1. We conclude, by Lemma 6.4, that x_1 is fixed in M.

The second type of fan is dual to the first type. In this case, the chain of triangles and triads begins and ends with triads. Since it has at least two triangles, it contains at least seven elements of M. Now $\mathrm{co}(M \backslash x) \cong M \backslash x / y$, so $M \backslash x / y$ has an N-minor. Moreover, $M \backslash x / y \cong \mathrm{si}(M / y)$, so $M^* \backslash y / x \cong \mathrm{co}(M^* \backslash y)$. Thus $\mathrm{co}(M^* \backslash y)$ is 3-connected having an

N^*-minor and we may now argue as in the first case to deduce that M^* has an element y' such that $M^*\backslash y'$ is 3-connected with an N^*-minor and y' is fixed in M^*. Thus M/y' is 3-connected with an N-minor and y' is cofixed in M.

If x is in a fan of the third type, this fan consists of a chain

$$\{x_1, y_2, x_2\}, \{y_1, x_2, y_2\}, \ldots, \{y_{k-1}, x_k, y_k\}$$

of triangles and triads where the first member is a triangle and the last is a triad. Moreover, $k \geqslant 3$. As with a fan of the first type, M is the generalized parallel connection, across the triangle $\{x_1, x_k, z\}$, of a k-spoked wheel and the matroid M_1 where M_1 is $M\backslash x_2, x_3, \ldots, x_{k-1}/y_2, y_3, \ldots, y_{k-1}$ with y_1 renamed as z. We may now argue as in the first case to deduce that x_1 is fixed in M. ▮

Proof of Theorem 2.3. Define $M = M_0$. Assume that $M_0, M_1, \ldots, M_i$ have been defined with each having an N-minor. If M_i is a wheel or a whirl, then it follows easily from the hypotheses that $M_i = N$. We may now assume that M_i is not a wheel or a whirl. Either

(i) M_i is a totally free expansion of N; or

(ii) M_i is not a totally free expansion of N.

In case (ii), by Theorem 9.1, M_i has an element e_i such that either

(a) $M_i\backslash e_i$ is 3-connected with an N-minor and e_i is fixed in M_i; or

(b) M_i/e_i is 3-connected with an N-minor and e_i is cofixed in M_i.

If (a) occurs, let $M_{i+1} = M_i\backslash e_i$; otherwise let $M_{i+1} = M_i/e_i$.

In case (i), if M_i has an element e_i such that $M_i\backslash e_i$ is a totally free expansion of N, then let $M_{i+1} = M_i\backslash e_i$. If M_i has no element e such $M_i\backslash e$ that is a totally free expansion of N, but M_i does have an element e_i such that M_i/e_i is a totally free expansion of N, then let $M_{i+1} = M_i/e_i$. Finally, if M_i has no element e for which $M_i\backslash e$ or M_i/e is a totally free expansion of N, then, by Theorem 7.1, M_i has elements e_i and $e_{i=1}$ such that $M_i\backslash e_i/e_{i+1}$ is a totally free expansion of N, and $M_i\backslash e_i$ is 3-connected. In that case, we let $M_{i+1} = M_i\backslash e_i$ and $M_{i+2} = M_{i+1}/e_{i+1}$. ▮

The next result is an extension of Theorem 2.3 for totally free matroids. The proof, a straightforward combination of Proposition 8.9 and Theorem 8.12, is omitted.

THEOREM 9.2. *Let M be a totally free matroid. Then there is a sequence*

$$M_0, M_1, \ldots, M_n$$

of 3-connected matroids and a sequence $e_0, e_1, \ldots, e_{n-1}$ of elements of M such that the following hold.

(i) *$M_0 = M$ and $M_n \cong U_{2,4}$.*

(ii) *For all i in $\{0, 1, ..., n-1\}$, either $M_{i+1} = M_i \backslash e_i$ or $M_{i+1} = M_i / e_i$.*

(iii) *For all i in $\{1, 2, ..., n-1\}$, if M_i is not totally free, then both M_{i-1} and M_{i+1} are totally free, $M_{i+1} = M_{i-1} \backslash e_{i-1} / e_i$, and $\{e_{i-1}, e_i\}$ is a clonal class of M_{i-1}.*

10. TOTALLY FREE MATROIDS AND INEQUIVALENT REPRESENTATIONS

The original motivation for studying totally free matroids was to gain insight into inequivalent representations. Theorem 2.4 bounds the number of such representations over a finite field of a 3-connected matroid, and we now prove that result.

Proof of Theorem 2.4. By Corollary 8.6, M has no totally free minors if and only if M is binary. In that case, by results of Brylawski and Lucas [3], M is uniquely **F**-representable. We may now assume that M is non-binary. The second part of the theorem will be proved by induction on $|E(M)|$. If $|E(M)| = 4$, the result certainly holds. Now let $|E(M)| = n > 4$ and assume that the result holds for all non-binary 3-connected matroids with fewer than n elements. We may suppose that M is not totally free, otherwise the result holds. Then M has an element x such that either $\mathrm{co}(M \backslash x)$ is 3-connected and x is fixed in M, or $\mathrm{si}(M/x)$ is 3-connected and x is cofixed in M. We lose no generality in assuming the latter. In that case, by Proposition 2.1, M has no more inequivalent **F**-representations than M/x. Moreover, representations of $\mathrm{si}(M/x)$ are in one-to-one correspondence with representations of M/x. Now either $\mathrm{si}(M/x)$ is binary or it is not. In the first case, $\mathrm{si}(M/x)$, and hence M, is uniquely **F**-representable. In the second case, the induction assumption implies that the number of inequivalent **F**-representations of $\mathrm{si}(M/x)$ is bounded above by the maximum, over all totally free minors N of $\mathrm{si}(M/x)$, of the number of inequivalent **F**-representations of N. Since such minors are also minors of M, the result follows by induction. ▌

An immediate consequence of Theorem 2.4 is the following.

COROLLARY 10.1. *Let* **F** *be a finite field and $\mathscr{M}$ be a minor-closed class of* **F**-*representable matroids. Suppose that, for some positive integer k, every totally free matroid in $\mathscr{M}$ has at most k inequivalent* **F**-*representations. Then every* 3-*connected matroid in $\mathscr{M}$ has at most k inequivalent* **F**-*representations.*

In particular, if $\mathcal{M}$ contains only a finite number of totally free matroids, then, for some integer k', every 3-connected member of $\mathcal{M}$ has at most k' inequivalent **F**-*representations.*

Finally, we note an analogue of Theorem 2.4 for totally free expansions. It follows by a similar argument to that of Theorem 2.4. It is also a more-or-less immediate corollary of Theorem 9.1.

COROLLARY 10.2. *Let* **F** *be a finite field, N be a 3-connected* **F**-*representable matroid with at least four elements, and M be a 3-connected matroid M with an N-minor. Then the number of inequivalent* **F**-*representations of M is bounded above by the maximum, over all minors N' of M that are totally free expansions of N, of the number of inequivalent* **F**-*representations of N'.*

11. TOTALLY FREE QUATERNARY MATROIDS

In this section, we determine all totally free quaternary matroids. Some of the preliminaries here will also deal with totally free quinternary matroids and the determination of all matroids of the latter type will be completed in the next section.

First we shall determine which small quaternary or quinternary matroids are totally free. We shall use two preliminary lemmas in proving that result.

LEMMA 11.1. *The matroid M that is obtained from $U_{3,6}$ by freely adding a point on some line is not quinternary.*

Proof. Let $E(U_{3,6}) = \{1, 2, ..., 6\}$ and assume that M is obtained by freely adding x on the line through 5 and 6. View M as a restriction of $PG(2, 5)$. Let L be the line of this projective space spanned by $\{5, 6\}$. Then each of the six lines of $PG(2, 5)$ that are spanned by two points from $\{1, 2, 3, 4\}$ meets L in one of the three points of $L - \{5, 6, x\}$. Moreover, for each i in $\{1, 2, 3, 4\}$, the three lines through i and each of the members of $\{1, 2, 3, 4\} - i$ meet L in distinct points. Thus we may assume that $\{1, 2, p_2\}$, $\{1, 3, p_3\}$, and $\{1, 4, p_4\}$ are circuits of $PG(2, 5)$ where $L - \{5, 6, x\} = \{p_2, p_3, p_4\}$. Since the three lines through two of 1, 2, and 3 meet L in distinct points, we deduce that $\{2, 3, p_4\}$ is a circuit. Similarly, $\{2, 4, p_3\}$ and $\{3, 4, p_2\}$ are circuits. It follows that $PG(2, 5) \mid \{1, 2, 3, 4, p_2, p_3, p_4\} \cong F_7$; a contradiction. ∎

LEMMA 11.2. *Let M be a quinternary extension of $U_{3,6}$ by the element x. Then x is fixed in M.*

Proof. We shall argue geometrically. We may assume that M is simple otherwise x is certainly fixed. Now $U_{3,7}$ is not $GF(5)$-representable. Hence x lies on at least one 3-point line with two points of $U_{3,6}$. If x lies on at least two such lines in M, then it is certainly fixed. Thus we may assume that x lies on exactly one non-trivial line of M. In that case, by Lemma 11.1, M is not quinternary; a contradiction. ∎

LEMMA 11.3. *Let M be a matroid with at most eight elements and suppose that M is quaternary or quinternary and that M is totally free. Then M is isomorphic to one of $U_{2,4}$, $U_{2,5}$, $U_{2,6}$, $U_{3,5}$, $U_{3,6}$, $U_{4,6}$, P_6, Φ_4, or Ψ_4.*

Proof. If $r(M)=2$, then, since M is quaternary or quinternary but non-binary, M is isomorphic to one of $U_{2,4}$, $U_{2,5}$ or $U_{2,6}$. Now assume that $r(M)=3$. If $r^*(M)=2$, then $M \cong U_{3,5}$. Thus we may assume that $r^*(M)=3$. Then, by [13, Corollary 11.2.9], M has $\mathcal{W}^3$, $U_{3,6}$, P_6, or Q_6 as a minor. The last matroid has a fixed element whose deletion is $U_{3,5}$ and so Q_6 cannot occur as a restriction of M. Suppose that M has $\mathcal{W}^3$ as a restriction. Each spoke x of $\mathcal{W}^3$ is fixed in $\mathcal{W}^3$ and hence in M. Moreover, $\text{co}(\mathcal{W}^3 \backslash x)$ is 3-connected. Thus $\text{co}(M \backslash x)$ is not 3-connected. Since $\mathcal{W}^3$ has three spokes, it is now straightforward to obtain a contradiction. Hence M does not have $\mathcal{W}^3$ as a restriction. If M has $U_{3,6}$ as a restriction, then, by Lemma 11.2, $M \cong U_{3,6}$. Thus, we may assume that M has P_6 as a minor but that $M \not\cong P_6$. Hence M is not quaternary, so M is quinternary. Moreover, M has a single-element extension of P_6 as a restriction. It is not difficult to see that there are exactly three single-element extensions of P_6 in which no element whose deletion is 3-connected is fixed: the matroids consisting of

(i) four points and a 3-point line freely placed in the plane;
(ii) three points and a 4-point line freely placed in the plane; and
(iii) two disjoint 3-point lines and a single point freely placed in the plane.

The first matroid was shown in Lemma 11.1 to be non-quinternary. To see that the second and third matroids are not quinternary, one can argue similarly: take one of the non-trivial lines L of the matroid. If the matroid is quinternary, all of the lines through two points not on L must meet the closure of L in $PG(2, 5)$ at points not in L. But this is easily seen to be impossible.

We may now assume that $r(M) > 3$. By duality, we may also assume that $r^*(M) > 3$. Thus M is totally free of rank four having eight elements. As there are no 7-element totally free quaternary or quinternary matroids, we deduce, by Corollary 8.13, that the ground set of M is the union of four 2-element clonal classes, $\{a_1, b_1\}$, $\{a_2, b_2\}$, $\{a_3, b_3\}$, $\{a_4, b_4\}$. Moreover, for all i, the matroid $M\backslash a_i/b_i$ is totally free, and $M\backslash a_i$ and M/b_i are 3-connected. Therefore $M\backslash a_i/b_i$ is isomorphic to $U_{3,6}$ or P_6. But $E(M)-\{a_i, b_i\}$ is the union of three clonal pairs in M. Hence $M\backslash a_i/b_i$ has no clonal classes of size three, and so $M\backslash a_i/b_i \not\cong P_6$. Therefore $M\backslash a_i/b_i \cong U_{3,6}$. Since neither $U_{3,7}$ nor the matroid in (i) above is quaternary or quinternary, it follows that M/b_i is isomorphic to one of Ψ_3^+ or Φ_3^+, where, in both cases, a_i corresponds to the point that is on more than one non-trivial line. Observe that, because every single-element deletion and contraction of M is 3-connected, M has no triangles and no triads. Thus no 4-circuit C of M contains exactly one element of $\{a_i, b_i\}$ otherwise $\mathrm{cl}_M(C)$ contains $\{a_i, b_i\}$ and so $E(M)-\mathrm{cl}_M(C)$ is a triad of M.

We now know that every 4-circuit of M has the form $\{a_j, b_j, a_k, b_k\}$ for some $\{j, k\} \subseteq \{1, 2, 3, 4\}$. Construct an auxiliary graph G with vertex set $\{1, 2, 3, 4\}$ such that jk is an edge if and only if $\{a_j, b_j, a_k, b_k\}$ is a circuit of M. Then a vertex i of G has degree 2 or 3 depending on whether M/b_i is isomorphic to Ψ_3^+ or Φ_3^+, respectively. It follows that G is isomorphic to K_4, K_4-e, or a 4-cycle.

Suppose that G has K_4-34 as a subgraph. Then the five edges of this subgraph imply that M has five 4-circuits of the form $\{a_i, b_i, a_j, b_j\}$. These five circuits include three containing each of a_1, b_1, a_2, and b_2. Thus none of these elements is in any more 4-circuits since M/x has at most three 3-circuits for all x. Hence the only other possible 4-circuit of M is $\{a_3, b_3, a_4, b_4\}$. If M does have this 4-circuit, then the set of 4-circuits of M coincides with that of Φ_4. If M does not have this 4-circuit, then its set of 4-circuits coincides with that of the Vámos matroid, V_8. Thus the set of non-spanning circuits of M coincides with that of Φ_4 or V_8, so $M \cong \Phi_4$ or V_8 [16]. As M is representable, it follows that $M \cong \Phi_4$.

We may now assume that G is a 4-cycle. Then M has the same set of 4-circuits as Ψ_4, so $M \cong \Psi_4$. ∎

The converse of the last lemma is also true, but it will be simpler not to prove it yet. We shall require one further preliminary result.

LEMMA 11.4. *If $r \geqslant 3$ and M is a connected quaternary matroid such that $M\backslash x \cong \Phi_r$, then x is fixed in M.*

Proof. If M is not 3-connected, then x is in a non-trivial parallel class and is certainly fixed in M. Thus we may assume that M is 3-connected. We argue by induction on r. The result holds if $r=3$ by [10, Lemma 8.2(vi)]. Now assume that the lemma holds for $r<k$ and let $r=k \geqslant 4$. We may assume that $M\backslash x$ has a leg $\{a, b\}$ such that $\{a, b, x\}$ is independent otherwise x is certainly fixed in M. Then $M\backslash a/b\backslash x \cong \Phi_{r-1}$ and so $M\backslash a/b$ is connected. Thus, by the induction assumption, x is fixed in $M\backslash a/b$,

and hence in M/b. Similarly, x is fixed in M/a. It now follows from Proposition 4.7(iii) that x is fixed in M. ▮

We are now able to prove Theorem 2.5. For convenience, we restate it here.

THEOREM 11.5. *A quaternary matroid M is a totally free if and only if M is isomorphic to one of $U_{2,4}$, $U_{2,5}$, $U_{3,5}$, or Φ_r for some $r \geqslant 3$.*

Proof. Certainly each of the matroids listed is totally free and quaternary. It remains to show that there are no other totally free quaternary matroids. By Lemma 11.3, there are no other such matroids with at most eight elements for Ψ_4 has a P_6-minor and so is not quaternary. Let M be a totally free quaternary matroid. We shall show, by induction on $|E(M)|$ that if $|E(M)| \geqslant 8$, then $M \cong \Phi_r$ for some $r \geqslant 4$. This is true if $|E(M)| = 8$. Assume it true for $|E(M)| < k$ and let $|E(M)| = k \geqslant 8$.

Now suppose that M has an element d such that $M \backslash d$ is totally free. Then, by the induction assumption, $M \backslash d \cong \Phi_r$ for some r. But, by Lemma 11.4, d is fixed in M, contradicting the fact that M is totally free. Thus there is no element d such that $M \backslash d$ is totally free and, by duality, there is no element c such that M/c is totally free. It now follows, by Corollary 8.13, that the ground set of M is the union of 2-element clonal classes, that for every element a of $E(M)$, both $M \backslash a$ and M/a are 3-connected, and that, for every clonal pair $\{a, a'\}$, the matroid $M \backslash a/a'$ is totally free.

Choose a clonal pair $\{c, d\}$ of M. By the induction assumption and the fact that $M \backslash d/c$ is totally free, we deduce that $M \backslash d/c \cong \Phi_r$ for some $r \geqslant 4$. Then $M \backslash d/c$ is represented over $GF(4)$ by the matrix $[I_r \mid D_r]$, where D_r is

$$
\begin{array}{c} \\ b_1 \\ b_2 \\ \vdots \\ b_{r-1} \\ b_r \end{array}
\begin{array}{c}
\begin{array}{ccccc} a_1 & a_2 & \cdots & a_{r-1} & a_r \end{array} \\
\begin{pmatrix}
\omega & 1 & \cdots & 1 & 1 \\
1 & \omega & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & \omega & 1 \\
1 & 1 & \cdots & 1 & \omega
\end{pmatrix},
\end{array}
$$

and $\omega^2 = \omega + 1$.

Elements of $E(M) - \{c, d\}$ that are clones in M remain clones in $M \backslash d/c$. The clonal classes of $M \backslash d/c$ are the pairs $\{a_i, b_i\}$ for all i in

$\{1, 2, \ldots, r\}$. Therefore, these pairs, together with $\{c, d\}$, are the clonal classes of M. Evidently, M is represented by the matrix $[I_{r+1} \mid D'_{r+1}]$ where D'_{r+1} is

$$\begin{array}{c} \\ b_1 \\ b_2 \\ \vdots \\ b_r \\ c \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & \cdots & a_r & d \end{array} \\ \begin{pmatrix} \omega & 1 & \cdots & 1 & y_1 \\ 1 & \omega & \cdots & 1 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & \omega & y_r \\ x_1 & x_2 & \cdots & x_r & z \end{pmatrix} \end{array}.$$

Now $M \backslash b_1 / a_1 \cong \Phi_r$ and has $\{c, d\}$, $\{a_2, b_2\}$, $\{a_3, b_3\}$, ..., $\{a_r, b_r\}$ as its clonal classes. Given this, and the unique representability of 3-connected quaternary matroids [11], we can deduce that $y_2 = y_3 = \cdots = y_r \neq 0$ and that $x_2 = x_3 = \cdots = x_r \neq 0$. By scaling the last row and last column of D'_{r+1} if necessary, we may assume that $x_2 = x_3 = \cdots = x_r = 1$ and that $y_2 = y_3 = \cdots = y_r = 1$. It now follows that $z = \omega$. Finally, we deduce that $x_1 = y_1 = 1$ by considering $M \backslash b_2 / a_2$, which is isomorphic to Φ_r. We conclude that $M \cong \Phi_{r+1}$ and the theorem follows by induction. ∎

Corollary 2.6 follows without difficulty by combining Theorem 11.5 and Corollary 10.1. The last result of this section is a consequence of Corollary 2.6 and the following result.

LEMMA 11.6. *Let p be a prime exceeding 4. If $r \geqslant p-1$, then Φ_r is not $GF(p)$-representable. Moreover, Φ_3 is $GF(p)$-representable but is not binary or ternary.*

Proof. The last statement follows easily from the fact that $\Phi_3 \cong U_{3,6}$. Now assume that $r \geqslant p-1$ and Φ_r is $GF(p)$-representable. Then so too are Φ_{p-1} and Φ^+_{p-1} We may assume that Φ^+_{p-1} is represented by the matrix $[I_{p-1} \mid D_{p-1} \mid \mathbf{1}]$, where $\mathbf{1}$, the column of all ones, corresponds to the tip, the matrix D_k is

$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_k \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_k \end{array} \\ \begin{pmatrix} 1+\alpha_1 & 1 & 1 & \cdots & 1 \\ 1 & 1+\alpha_2 & 1 & \cdots & 1 \\ 1 & 1 & 1+\alpha_3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \end{array},$$

and $\alpha_1, \alpha_2, ..., \alpha_k$ are non-zero members of $GF(p)$. As Wu [28] notes, one can show by induction (or see Mirsky [12, Exercise I.13]) that

$$\det(D_k) = \left(1 + \sum_{i=1}^{k} \alpha_i^{-1}\right) \prod_{i=1}^{k} \alpha_i.$$

Consider the multiset $S = \{\alpha_1^{-1}, \alpha_2^{-1}, ..., \alpha_{p-1}^{-1}\}$. We shall show next that some submultiset of S sums to -1. This is surely known but we prove it here for completeness. It is certainly true if all the members of S are equal. Thus we may assume that all $\alpha_1^{-1} \neq \alpha_2^{-1}$. Now let $\{s_1, s_2, ..., s_m\}$ be the set of distinct sums of submultisets of $\{\alpha_1^{-1}, \alpha_2^{-1}, ..., \alpha_{j-1}^{-1}\}$. We show that, among the submultisets of $\{\alpha_1^{-1}, \alpha_2^{-1}, ..., \alpha_{j-1}^{-1}, \alpha_j^{-1}\}$, there are at least $m+1$ distinct sums. If not, then

$$\{s_1, s_2, ..., s_m\} = \{s_1 + \alpha_j^{-1}, s_2 + \alpha_j^{-1}, ..., s_m + \alpha_j^{-1}\}.$$

Thus there is a permutation σ of $\{1, 2, ..., m\}$ such that $s_i = s_{\sigma(i)} + \alpha_j^{-1}$ for all i. Therefore

$$\sum_{i=1}^{m} (s_i - s_{\sigma(i)}) = m\alpha_j^{-1},$$

that is, $\sum_{i=1}^{m} s_i - \sum_{i=1}^{m} s_{\sigma(i)} = m\alpha_j^{-1}$, so $m\alpha_j^{-1} = 0$; a contradiction. Since $\alpha_1^{-1} \neq \alpha_2^{-1}$, the submultisets of $\{\alpha_1^{-1}, \alpha_2^{-1}\}$ have three distinct sums. We deduce that the submultisets of $\{\alpha_1^{-1}, \alpha_2^{-1}, ..., \alpha_{p-1}^{-1}\}$ have at least, and hence have exactly, p distinct sums. Thus, for some subset T of $\{1, 2, ..., p-1\}$, we have $\sum_{t \in T} \alpha_t^{-1} = -1$. Then Φ_{p-1} has as a circuit the set $\{a_t : t \in T\} \cup \{b_i : i \in \{1, 2, ..., p-1\} - T\}$; a contradiction. ∎

Corollary 11.7. *Let p be a prime number. Then there is an integer k such that every 3-connected $GF(p)$-representable quaternary matroid has at most k inequivalent $GF(p)$-representations.*

12. TOTALLY FREE $GF(5)$-REPRESENTABLE MATROIDS

The purpose of this section is to specify all totally free quinternary matroids. We begin by noting some basic properties of swirls. In the first result, all subscripts should be interpreted modulo r.

Lemma 12.1. *If $r = 3$, then $\Psi_r \cong U_{3,6}$ and Ψ_r has no non-spanning circuits. However, if $r > 3$, then the collection of non-spanning circuits of Ψ_r consists of all sets of the form*

$$\{a_i, b_i, e_{i+1}, e_{i+2}, ..., e_{j-1}, a_j, b_j\},$$

where $e_t \in \{a_t, b_t\}$ *for all* t, *the elements* i *and* j *are distinct members of* $\{1, 2, ..., r\}$, *and* $\{i, i+1, ..., j-1, j\} \neq \{1, 2, ..., r\}$.

Proof. This is straightforward using the observation that, for all i, the matroid $\Psi_r \backslash \{a_i, b_i\}$ can be constructed as follows. Let N_0 be a 3-point line on $\{a_{i+1}, b_{i+1}, s_{i+1}\}$. Then, for all j in $\{1, 2, ..., r-1\}$, let N_j be obtained by taking the 2-sum, across the basepoint s_{i+j}, of N_{j-1} and a 4-point line on $\{s_{i+j}, a_{i+j}, b_{i+j}, s_{i+j+1}\}$. Finally, $\Psi_r \backslash \{a_i, b_i\}$ is the 2-sum, across the basepoint s_{i-1}, of N_{r-1} and a 3-point line on $\{s_{i-1}, a_{i-1}, b_{i-1}\}$. We omit the remaining details. ▮

Swirls have many similar properties to free spikes. The next lemma summarizes some of these basic properties. The proof, which is not difficult, is omitted.

LEMMA 12.2. *Let r be an integer greater than* 2.

(i) $\Psi_r^* = \Psi_r$.
(ii) *If* $r > 3$, *and* $e \in \{a_i, b_i\}$, *then* $\Psi_r / e \cong \Psi_{r-1}^+$.
(iii) $\Psi_r \backslash a_i / b_i = \Psi_r \backslash b_i / a_i \cong \Psi_{r-1}$ *for all* i *in* $\{1, 2, ..., r\}$.
(iv) Ψ_r *and* Ψ_r^+ *are 3-connected.*

LEMMA 12.3. Ψ_r *is quinternary for all* $r \geqslant 3$. *Moreover, if* $r \geqslant 4$, *then every representation of* Ψ_r *over* $GF(5)$ *is equivalent to* $[I_r \mid D]$ *where the columns of this matrix are labelled, in order,* $b_1, b_1, ..., b_r, a_1, b_r, a_1, a_2, ..., a_r$, *and either D or its transpose is*

$$\begin{bmatrix} 3 & 2 & 2 & \cdots & 2 & 2 \\ 4 & 3 & 2 & \cdots & 2 & 2 \\ 4 & 4 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 4 & 4 & 4 & \cdots & 3 & 2 \\ 4 & 4 & 4 & \cdots & 4 & 3 \end{bmatrix}.$$

Proof. Certainly Ψ_3, which is isomorphic to $U_{3,6}$, is quinternary. Now let $r \geqslant 4$ and suppose that $[I_r \mid D]$ is a quinternary representation of Ψ_r where the columns of this matrix are labelled in the order specified above.

By scaling rows and columns, we may certainly assume that D is

$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_r \end{array}\begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & \cdots & a_r \end{array} \\ \begin{pmatrix} z_1 & x_1 & x_1 & x_1 & \cdots & x_1 \\ y_1 & z_2 & d_{23} & d_{24} & \cdots & d_{2r} \\ y_1 & d_{32} & z_3 & d_{34} & \cdots & d_{3r} \\ y_1 & d_{42} & d_{43} & z_4 & \cdots & d_{4r} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1 & d_{r2} & d_{r3} & d_{r4} & \cdots & z_r \end{pmatrix} \end{array},$$

where all the entries of D are non-zero. The circuit $\{a_1, b_1, a_2, b_2\}$ implies that $d_{32} = d_{42} = \cdots = d_{r2}$. Then, from considering $\{a_i, b_i, a_{i+1}, b_{i+1}\}$ for consecutive elements i of $\{2, 3, \ldots, r-1\}$, we deduce that all below-diagonal entries of D, except possibly those in the first column, take a common value. Thus we may assume, in the above notation, that $d_{ij} = y$ for all $i > j$. Similarly, from considering the fact that each of the circuits above is also a cocircuit, we deduce that all above-diagonal entries in D, except possibly those in the first row, take a common value. Thus we may assume that $d_{ij} = x$, say, for all $j > i$. By rescaling the first row and first column and then all the rows, we may assume that D is

$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_r \end{array}\begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & \cdots & a_r \end{array} \\ \begin{pmatrix} z'_1 & 2 & 2 & 2 & \cdots & 2 \\ y' & z'_2 & 2 & 2 & \cdots & 2 \\ y' & y' & z'_3 & 2 & \cdots & 2 \\ y' & y' & y' & z'_4 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y' & y' & y' & y' & \cdots & z'_r \end{pmatrix} \end{array}.$$

Moreover, none of $z'_1, z'_2, \ldots, z'_r$; is in $\{2, y'\}$, and, since $\{a_1, b_1, a_3, b_3\}$ is independent in M, the elements y' and 2 are distinct.

If D' is a matrix obtained from D by deleting any $r-3$ rows and the corresponding $r-3$ columns, then

$$D' = \begin{bmatrix} z'_i & 2 & 2 \\ y' & z'_j & 2 \\ y' & y' & z'_k \end{bmatrix}$$

and $[I_3 \mid D']$ represents $U_{3,6}$. Thus none of $z'_i z'_j$, $z'_i z'_k$ and $z'_j z'_k$ equals $2y'$. Since each of z'_i, z'_j, and z'_k is in $\{1, 2, 3, 4\} - \{2, y'\}$, at least two of z'_i, z'_j, and z'_k are equal. Thus $y' \neq 3$ otherwise $2y' = 1$, and $\{1, 2, 3, 4\} - \{2, y'\} = \{1, 4\}$ in which case one of $z'_i z'_j$, $z'_i z'_k$ and $z'_j z'_k$ is in $\{1^2, 4^2\} = \{1\}$; a contradiction. If $y' = 1$, then z'_i, z'_j, and z'_k are all in $\{3, 4\}$. But $3 \times 4 = 2 = 2y'$, so $z'_i = z'_j = z'_k$ Similarly, if $y' = 4$, then z'_i, z'_j, and z'_k are all in $\{1, 3\}$. But $1 \times 3 = 3 = 2y'$, so $z'_i = z'_j = z'_k$. We conclude that either

(i) $y' = 1$ and $z'_j = z'_2 = \cdots = z'_r \in \{3, 4\}$; or
(ii) $y' = 4$ and $z'_1 = z'_2 = \cdots = z'_r \in \{1, 3\}$.

The matrix D' cannot have zero determinant so, in cases (i) and (ii), the common values of all the z'_i are 4 and 3, respectively. Thus, in case (ii), the matrix D certainly has the desired form. Moreover, in case (i), by multiplying all entries of D by 2, we obtain the transpose of the matrix in (ii).

Now let D be the $r \times r$ matrix in which all entries on the main diagonal equal 3, all entries above the main diagonal equal 2, and all entries below the main diagonal equal 4. To complete the proof of the lemma, it suffices to show that, when their columns are labelled $b_1, b_2, \ldots, b_r, a_1, a_2, \ldots, a_r$, both of the matrices $[I_r \mid D]$ and $[I_r \mid D^T]$ represent Ψ_r over $GF(5)$. The matroids $M[I_r \mid D]$, $M[I_r \mid D^T]$, and Ψ_r all certainly have the same ground set. In the proof of Lemma 12.1, we described $\Psi_r \backslash \{a_i, b_i\}$ constructively. Using this, it is straightforward to check that both $M[I_r \mid D] \backslash \{a_i, b_i\}$ and $M[I_r \mid D^T] \backslash \{a_i, b_i\}$ equal $\Psi_r \backslash \{a_i, b_i\}$. Thus both $M[I_r \mid D]$ and $M[I_r \mid D^T]$ have the same sets of non-spanning circuits as Ψ_r unless one of the former has a non-spanning circuit meeting all of $\{a_1, b_1\}$, $\{a_2, b_2\}$, ..., $\{a_r, b_r\}$. It is not difficult to check that the exceptional case does not arise. We conclude that both $M[I_r \mid D]$ and $M[I_r \mid D^T]$ equal Ψ_r. ▌

Next we prove the analogue of Lemma 11.4.

LEMMA 12.4. *If $r \geqslant 3$ and M is a connected quinternary matroid such that $M \backslash x \cong \Psi_r$, then x is fixed in M.*

Proof. Assume that the lemma fails and let M be a counterexample for which r is minimal. Then, by Lemma 11.2, $r \geqslant 4$. Moreover, since x is not fixed in M, it is in no non-trivial parallel classes. Hence M is 3-connected. Now, for some i in $\{1, 2, \ldots, r\}$, the set $\{x, a_i, b_i\}$ is independent. Since $\Psi_r / a_i \backslash b_i \cong \Psi_{r-1}$, the matroid $M / a_i \backslash b_i$ is a connected quinternary extension of Ψ_{r-1}. The minimality of r implies that x is fixed in $M / a_i \backslash b_i$. We conclude that x is fixed in M / a_i. Similarly, x is fixed in M / b_i. Hence, by Proposition 4.7(iii), x is fixed in M; a contradiction. ▌

Next we prove Theorem 2.7, again restating the result for convenience.

THEOREM 12.5. *A quinternary matroid M is totally free if and only if M is isomorphic to one of $U_{2,4}$, $U_{2,5}$, $U_{2,6}$, $U_{3,5}$, $U_{4,6}$, P_6, or Ψ_r for some $r \geqslant 3$.*

Proof. Certainly all of the matroids listed are totally free and quinternary. It remains to prove the converse. Let M be totally free and quinternary. By Lemma 11.3, if $|E(M)| \leqslant 8$, then M is isomorphic to one of the matroids listed since, as is not difficult to check, Φ_4 is not quinternary. We complete the proof by arguing by induction on $|E(M)|$ that if $|E(M)| \geqslant 8$, then $M \cong \Psi_r$ for some r. This is certainly true if $|E(M)| = 8$. Assume it true if $|E(M)| < n$ and suppose that $|E(M)| = n > 8$. If M has an element d such that $M \backslash d$ is totally free, then, by the induction assumption, $M \backslash d \cong \Psi_r$ for some r. But, by Lemma 12.4, d is fixed in M, a contradiction. Thus M has no element d such that $M \backslash d$ is totally free and, by duality, M has no element c such that M/c is totally free. Corollary 8.13 now implies that the ground set of M is the union of 2-element clonal classes. Moreover, if $\{c, d\}$ is such a class, then $M \backslash d/c$ is totally free. Thus, by the induction assumption, $M \backslash d/c \cong \Psi_r$ for some $r \geqslant 4$. Then letting the clonal classes of M other than $\{c, d\}$ be $\{a_1, b_1\}$, $\{a_2, b_2\}$, ..., $\{a_r, b_r\}$, we may assume, by Lemma 12.3, that M is represented by the matrix $[I_{r+1} \mid D]$ where D is

$$
\begin{array}{c}
\\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{r-1} \\ b_r \\ c
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} a_1 & a_2 & a_3 & \cdots & a_{r-1} & a_r & d \end{array} \\
\begin{pmatrix}
3 & \alpha & \alpha & \cdots & \alpha & \alpha & y_1 \\
\beta & 3 & \alpha & \cdots & \alpha & \alpha & y_2 \\
\beta & \beta & 3 & \cdots & \alpha & \alpha & y_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\beta & \beta & \beta & \cdots & 3 & \alpha & y_{r-1} \\
\beta & \beta & \beta & \cdots & \beta & 3 & y_r \\
x_1 & x_2 & x_3 & \cdots & x_{r-1} & x_r & z
\end{pmatrix}
\end{array},
$$

and (α, β) is $(2, 4)$ or $(4, 2)$.

Suppose first that, for all i in $\{1, 2, ..., r\}$, the set $\{a_i, b_i, a_{i+1}, b_{i+1}\}$ is a circuit of M where, throughout this proof, all subscripts are interpreted modulo r. We shall show that $\{b_1, a_1, a_2, ..., a_{r-1}\}$ spans $E(M) - \{c, d\}$. Certainly $\{b_1, a_1, a_2, ..., a_{r-1}\}$ spans b_2 and hence spans b_3. Continuing with this pattern, we deduce that $\{b_1, a_1, a_2, ..., a_{r-1}\}$ spans $\{b_2, b_3, ..., b_{r-1}\}$. But $\{a_1, b_1, a_r, b_r\}$ and $\{a_{r-1}, b_{r-1}, a_r, b_r\}$ are both circuits of M.

Therefore, by elimination, both $\{a_1, b_1, a_{r-1}, b_{r-1}, a_r\}$ and $\{a_1, b_1, a_{r-1}, b_{r-1}, b_r\}$ contain circuits of M. From considering $M\backslash d/c$, we deduce that neither of these sets contains a 4-circuit, so both are circuits of M. We conclude that $\{b_1, a_1, a_2, ..., a_{r-1}\}$ also spans $\{a_r, b_r\}$. Hence $\{b_1, a_1, a_2, ..., a_{r-1}\}$ does indeed span $E(M)-\{c, d\}$, so $\{c, d\}$ contains a cocircuit of M; a contradiction.

Without loss of generality, we may now assume that $\{a_1, b_1, a_r, b_r, c\}$ is a circuit of M. Hence $\{a_1, b_1, a_r, b_r, d\}$ is also a circuit of M. Thus both $\{a_1, a_r, b_r, c, d\}$ and $\{b_1, a_r, b_r, c, d\}$ contain circuits of M. Since neither $M\backslash a_1/b_1$ nor $M\backslash b_1/a_1$ has a 3-circuit, we deduce that either

(i) $\{a_r, b_r, c, d\}$ is a circuit of M; or

(ii) both $\{b_1, a_r, b_r, c, d\}$ and $\{a_1, a_r, b_r, c, d\}$ are circuits of M.

It follows, by symmetry, that either

(iii) $\{a_1, b_1, c, d\}$ is a circuit of M; or

(iv) both $\{b_r, a_1, b_1, c, d\}$ and $\{a_r, a_1, b_1, c, d\}$ are circuits of M.

By the dual argument to that used for circuits, we obtain that, for some i, both $\{a_i, b_i, a_{i+1}, b_{i+1}, d\}$ and $\{a_i, b_i, a_{i+1}, b_{i+1}, c\}$ are cocircuits of M. Since the first of these sets meets the circuit $\{a_1, b_1, a_r, b_r, d\}$, we deduce that $\{i, i+1\}$ meets $\{1, r\}$. Thus $i \in \{1, r-1, r\}$. Hence, by symmetry, we may assume that one of the following holds:

(v) both $\{a_1, b_1, a_2, b_2, d\}$ and $\{a_1, b_1, a_2, b_2, c\}$ are cocircuits of M;

or

(vi) both $\{a_1, b_1, a_r, b_r, d\}$ and $\{a_1, b_1, a_r, b_r, c\}$ are cocircuits of M.

For all j in $\{1, 2, ..., r\}$, either $\{a_j, b_j, a_{j+1}, b_{j+1}\}$ is a circuit, or both $\{a_j, b_j, a_{j+1}, b_{j+1}, c\}$ and $\{a_j, b_j, a_{j+1}, b_{j+1}, d\}$ are circuits of M. If $\{j, j+1\}$ avoids $\{1, 2, r\}$, then, in both cases (v) and (vi), orthogonality implies that $\{a_j, b_j, a_{j+1}, b_{j+1}\}$ is a circuit of M. Similarly, it follows from (i) and (ii) that $\{a_i, b_i, a_{i+1}, b_{i+1}\}$ is a cocircuit of M for all i in $\{2, 3, ..., r-2\}$.

Now recall the matrix $[I_{r+1} \mid D]$ that represents M. The cocircuits $\{a_i, b_i, a_{i+1}, b_{i+1}\}$ for i in $\{2, 3, ..., r-2\}$ imply that $y_2 = y_3 = \cdots = y_{r-1} = y$ say. Similarly, the circuits $\{a_j, b_j, a_{j+1}, b_{j+1}\}$ for j in $\{3, 4, ..., r-2\}$ imply that $x_3 = x_4 = \cdots = x_{r-1} = x$, say. Moreover, since (i) or (ii) holds, $\{a_r, b_r, c, d\}$ is a circuit of $M/b_1\backslash a_1$ and hence, by Lemma 12.2(i), is a cocircuit of $M/b_1\backslash a_1$. Thus $x_2 = x$. By scaling the last row and the last column of D, we may assume that $x = \beta$ and $y = \alpha$. Thus D is

$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{r-1} \\ b_r \\ c \end{array} \begin{array}{c} \begin{array}{ccccccc} a_1 & a_2 & a_3 & \cdots & a_{r-1} & a_r & d \end{array} \\ \left(\begin{array}{ccccccc} 3 & \alpha & \alpha & \cdots & \alpha & \alpha & y_1 \\ \beta & 3 & \alpha & \cdots & \alpha & \alpha & \alpha \\ \beta & \beta & 3 & \cdots & \alpha & \alpha & \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \beta & \beta & \beta & \cdots & 3 & \alpha & \alpha \\ \beta & \beta & \beta & \cdots & \beta & 3 & y_r \\ x_1 & \beta & \beta & \cdots & \beta & x_r & z \end{array}\right) \end{array}.$$

Assume that (v) occurs. Then, by orthogonality, $\{a_{r-1}, b_{r-1}, a_r, b_r\}$ is a circuit of M, so $x_r = \beta$. Thus $\{a_{r-1}, b_{r-1}, a_r, b_r\}$ is a circuit of $M/b_1 \backslash a_1$. As the last matroid is isomorphic to Ψ_r, it follows that $\{a_{r-1}, b_{r-1}, a_r, b_r\}$ is a cocircuit of $M/b_1 \backslash a_1$, and so $y_r = \alpha$. Now consider $M/b_r \backslash a_r$. It too is isomorphic to Ψ_r. For each i in $\{2, 3, ..., r-2\}$, the set $\{a_i, b_i, a_{i+1}, b_{i+1}\}$ is a circuit and a cocircuit of $M/b_r \backslash a_r$. Moreover, as (iii) or (iv) holds, $\{a_1, b_1, c, d\}$ is a circuit and hence a cocircuit of $M/b_r \backslash a_r$. Thus exactly one of (I) $\{a_2, b_2, c, d\}$ and (II) $\{a_{r-1}, b_{r-1}, c, d\}$ is both a circuit and a cocircuit of $M/b_r \backslash a_r$. In case (I), the fact that $\{a_2, b_2, c, d\}$ is a cocircuit of $M/b_r \backslash a_r$ implies that $x_1 = \beta^2/\alpha$. But then, we deduce from the matrix D that $\{a_1, b_1, a_r, b_r\}$ is a circuit of M; a contradiction. Hence case (II) holds. Then the fact that $\{a_{r-1}, b_{r-1}, c, d\}$ is a circuit of $M/b_r \backslash a_r$ implies that $y_1 = \alpha$. Hence $\{a_1, b_1, a_2, b_2\}$ is a cocircuit of M; a contradiction.

We may now assume that (vi) holds and that (v) does not. Then $\{a_1, b_1, a_2, b_2\}$ is a cocircuit and hence a circuit of $M/b_r \backslash a_r$. It follows that $x_1 = \beta$ and $y_1 = \alpha$. Then the sets of 4-circuits and 4-cocircuits of $M/b_1 \backslash a_1$ coincide and include $\{a_2, b_2, a_3, b_3\}$, $\{a_3, b_3, a_4, b_4\}$, ..., $\{a_{r-2}, b_{r-2}, a_{r-1}, b_{r-1}\}$, and $\{c, d, a_r, b_r\}$. Thus exactly one of $\{a_2, b_2, c, d\}$ and $\{a_{r-1}, b_{r-1}, c, d\}$ is a circuit and a cocircuit of $M/b_1 \backslash a_1$. In the first case, $y_r = \alpha$ and $x_r = \beta$. In the second case, $y_r = \beta$ and $x_r = \alpha$, and so $\{a_1, b_1, a_r, b_r\}$ is a circuit of M; a contradiction. Thus we may assume that the first case holds. Then, since $M/\{b_1, b_2, ..., b_{r-2}\} \backslash \{a_1, a_2, ..., a_{r-2}\} \cong U_{3,6}$, it follows that $z \notin \{\alpha, \beta\}$ so $z \in \{1, 3\}$. The first possibility implies that $\{a_{r-1}, a_r, d\}$ is a circuit of this minor; a contradiction. Hence $z = 3$ and so $M \cong \Psi_{r+1}$. The theorem now follows by induction. ∎

Corollary 2.8 follows without difficulty on combining Theorem 12.5 and Corollary 10.1. Kahn's conjecture for $GF(5)$ was proved in [20, Theorem 4.1]. We conclude by noting that it can be derived from the results above.

COROLLARY 12.6. *A 3-connected matroid has at most six inequivalent representations over GF(5).*

Proof. By Corollary 2.4 and Theorem 12.5, it suffices to show that none of the matroids $U_{2,4}$, $U_{2,5}$, $U_{2,6}$, $U_{3,5}$, $U_{3,6}$, $U_{4,6}$, P_6, or Ψ_r for $r \geqslant 4$ has more than six inequivalent $GF(5)$-representations. By Lemma 12.3, Ψ_r has exactly two such representations. Hence the corollary is reduced to checking that each of the seven matroids listed above has at most six inequivalent $GF(5)$-representations. This straightforward check appears in [20, Lemmas 4.2 and 4.3]. ∎

ACKNOWLEDGMENTS

Geelen was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada, Oxley and Vertigan were partially supported by grants from the National Security Agency, Vertigan was also partially supported by a grant from the Louisiana Education Quality Support Fund and Whittle was partially supported by a grant from the New Zealand Marsden Fund.

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