6-1-2002

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Recommended Citation

Oxley, J., Semple, C., Vertigan, D., & Whittle, G. (2002). Infinite antichains of matroids with characteristic set {p}. *Discrete Mathematics, 242* (1-3), 175-185. https://doi.org/10.1016/S0012-365X(00)00466-0

Discrete Mathematics 242 (2002) 175–185

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Infinite antichains of matroids with characteristic set $\{p\}$

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Received 19 September 1999; revised 26 September 2000; accepted 9 October 2000

Abstract

For each prime p, we construct an infinite antichain of matroids in which each matroid has characteristic set $\{p\}$. For $p=2$, each of the matroids in our antichain is an excluded minor for the class of matroids representable over the rationals. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 05B35

Keywords: Matroids; Characteristic sets; Infinite antichains

1. Introduction

The *characteristic set* of a matroid M is the set consisting of the characteristic of every field over which M is representable. Rado [9] showed that a matroid cannot have characteristic set $\{0\}$. However, for every prime p, it is known [4,7] that a matroid can have characteristic set $\{p\}$.

For each prime p, Reid [10] conjectured that every matroid that has characteristic set $\{p\}$ and is an excluded minor for $\mathbb{Q}$-representability has at most $2p+2$ elements. Gordon [6] disproved this conjecture, for all p, by exhibiting such matroids which have up to $4p-4$ elements. Furthermore, he showed that, for each p, there are at least 2^{p-2} matroids that have characteristic set $\{p\}$ and are excluded minors for $\mathbb{Q}$-representability. Recall that a set of matroids is an *antichain* if no member of the set is isomorphic to a minor of another member in the set. In this paper, we prove the following result.

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PII: S0012-365X(00)00466-0

Theorem 1.1. *For each prime p, there is an infinite antichain of matroids each member of which has characteristic set $\{p\}$.*

The proof of Theorem 1.1 is constructive in that, for each prime p, we define an infinite antichain of matroids in which each matroid has characteristic set $\{p\}$. For the special case of $p = 2$, every matroid in our constructed antichain has the additional property of being an excluded minor for $\mathbb{Q}$-representability. Thus, the following theorem extends Gordon's result when $p = 2$.

Theorem 1.2. *There is an infinite antichain of matroids each member of which has characteristic set $\{2\}$ and is an excluded minor for $\mathbb{Q}$-representability.*

We conjecture that the analogue of Theorem 1.2 holds for all other prime characteristics.

Conjecture 1.3. *For each prime p, there is an infinite antichain of matroids each member of which has characteristic set $\{p\}$ and is an excluded minor for $\mathbb{Q}$-representability.*

The notation and terminology of this paper will follow [8]. In particular, we denote the characteristic set of a matroid M by $\mathscr{K}(M)$. We will assume that the reader is familiar with the basics of matroid representation theory as discussed, for example, in Chapter 6 of [8].

The paper is organized as follows. In the next section, we describe a canonical triple of perfect matchings of the complete graph K_{4n}. These matchings are fundamental in the construction of each of the antichains that give us Theorem 1.1. In Section 3, we prove Theorem 1.2, and thereby prove Theorem 1.1 for $p = 2$. Section 4 contains the proof of Theorem 1.1 for $p \geqslant 3$.

2. Three perfect matchings of K_{4n}

Let n be a positive integer and consider the complete graph K_{4n}. Label the vertices of K_{4n} as $b_1, b_2, \ldots, b_{4n}$. We shall distinguish three disjoint perfect matchings H_1, H_2, and H_3 of K_{4n}, where

$$\begin{aligned}
H_1 &= \{\{b_1, b_2\}, \{b_3, b_4\}, \ldots, \{b_{4n-1}, b_{4n}\}\},\\
H_2 &= \{\{b_2, b_3\}, \{b_4, b_5\}, \ldots, \{b_{4n}, b_1\}\},\\
H_3 &= \{\{b_1, b_{2n+1}\}\} \cup \{\{b_2, b_{4n}\}, \{b_3, b_{4n-1}\}, \{b_4, b_{4n-2}\}, \ldots, \{b_{2n}, b_{2n+2}\}\}.
\end{aligned}$$

Observe that the union of every distinct pair of such matchings induces a Hamiltonian cycle of K_{4n}. These perfect matchings play an important role in the proof of Theorem 1.1.

Note that, in the construction of each of the antichains in this paper, the role of K_{4n} ($n \geqslant 1$) could be replaced by K_{2m} ($m \geqslant 2$). However, doing this requires separating the cases when m is even and when m is odd.

3. Proof of Theorem 1.2

This section is organized as follows. We describe an infinite set of matroids, show that each of the matroids in this set has characteristic set $\{2\}$, and then show that each is an excluded minor for the class of matroids representable over the rationals. It will follow from the last of these proofs that the matroids in the set form an infinite antichain.

Let n be a positive integer, and consider a geometric representation of $U_{3,4n}$. Label the elements of $U_{3,4n}$ by $b_1, b_2, \ldots, b_{4n}$ and recall the matchings H_1, H_2, and H_3 from the last section. For each i in $\{1,2,3\}$, view the elements of H_i as 2-point lines of $U_{3,4n}$ and place a point a_i on the intersection of all these lines so that a_1, a_2, and a_3 are collinear. In the resulting configuration, no two distinct lines have more than one common point. Thus, this configuration is a geometric representation for a rank-3 matroid, which we denote by M_n. In particular, M_1 is isomorphic to the Fano matroid. We shall show that $\{M_n\colon n \geqslant 1\}$ is an infinite antichain of matroids, each of which has characteristic set $\{2\}$ and is an excluded minor for $\mathbb{Q}$-representability.

In constructing a representation for M_n, we shall use the matrix B_n which equals

$$\begin{array}{cccccccccc}
b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & \cdots & b_{2n-1} & b_{2n} \\
\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 1 \\ 1 \\ 0 \end{array} & \begin{array}{c} 1 \\ 1 \\ \alpha_1 \end{array} & \begin{array}{c} 1 \\ \alpha_2 \\ \alpha_1 \end{array} & \begin{array}{c} 1 \\ \alpha_2 \\ \alpha_3 \end{array} & \begin{array}{c} 1 \\ \alpha_4 \\ \alpha_3 \end{array} & \begin{array}{c} 1 \\ \alpha_4 \\ \alpha_5 \end{array} & \begin{array}{c} \\ \cdots \\ \end{array} & \begin{array}{c} 1 \\ \alpha_{2n-4} \\ \alpha_{2n-3} \end{array} & \begin{array}{c} 1 \\ \alpha_{2n-2} \\ \alpha_{2n-3} \end{array} \quad \cdots
\end{array}$$

$$\begin{array}{cccccccc}
b_{2n+1} & b_{2n+2} & \cdots & b_{4n-4} & b_{4n-3} & b_{4n-2} & b_{4n-1} & b_{4n} \\
\begin{array}{c} 1 \\ \alpha_{2n-2} \\ \alpha_{2n-2} \end{array} & \begin{array}{c} 1 \\ \alpha_{2n-3} \\ \alpha_{2n-2} \end{array} & \begin{array}{c} \\ \cdots \\ \end{array} & \begin{array}{c} 1 \\ \alpha_3 \\ \alpha_4 \end{array} & \begin{array}{c} 1 \\ \alpha_3 \\ \alpha_2 \end{array} & \begin{array}{c} 1 \\ \alpha_1 \\ \alpha_2 \end{array} & \begin{array}{c} 1 \\ \alpha_1 \\ 1 \end{array} & \left.\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right]
\end{array}.$$

Lemma 3.1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_{2n-2}$ be algebraically independent transcendentals over some field $\mathbb{F}$. Then the matroid $M[B_n]$ is isomorphic to $U_{3,4n}$.*

Proof. Let D be a 3×3 submatrix of B_n whose columns are indexed by a subset of $\{b_1, b_2, \ldots, b_{4n}\}$. We show that the columns of D are linearly independent. It is easily checked that no two columns are scalar multiples of each other. If the columns of D are indexed by $\{b_1, b_2, b_{4n}\}$, then they are certainly linearly independent. Thus D has a column containing a transcendental. If, for some i in $\{1, 2, \ldots, 2n-2\}$, there is a transcendental α_i that appears in exactly one column of D, then, since the members of $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n-2}\}$ are algebraically independent, the columns of D must be linearly independent. Thus, we may assume that every transcendental appearing in a column of D occurs at least twice in D. If D has a transcendental occurring three or more times, then, by our last assumption, this transcendental must be α_{2n-2} and the columns of D are indexed by $\{b_{2n}, b_{2n+1}, b_{2n+2}\}$. Since these columns are certainly linearly independent, we may now assume that every transcendental α_i in D occurs exactly twice. Hence either

(i) det D has a unique term equal to α_i^2, or
(ii) both occurrences of α_i are in the same row or column of D.

In the first case, det D is clearly non-zero. Therefore, we may assume that (i) fails and (ii) holds for every transcendental α_i occurring in D. If D has a column with two copies of the same transcendental, then this column must be b_{2n+1} and, since (i) fails for each i, the other two columns of D must be in $\{b_1, b_2, b_{4n}\}$. It follows that, in this case, det $D \neq 0$. Thus, we may assume that no column of D contains two copies of the same transcendental. Hence, by (ii), either b_3 or b_{4n-1} is a column of D, or each column of D contains two distinct transcendentals. In each case, we easily obtain a contradiction by using (ii) and the structure of B_n. We conclude that $M[B_n] \cong U_{3,4n}$. □

Let A' be the 3×3 matrix

$$\begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \end{array}.$$

Lemma 3.2. *Let p be a prime and let $\alpha_1, \alpha_2, \ldots, \alpha_{2n-2}$ be algebraically independent transcendentals over* GF(p). *Then, over* GF(p)($\alpha_1, \alpha_2, \ldots, \alpha_{2n-2}$), *the matrix $[A'|B_n]$ represents M_n when $p = 2$ and represents M_n' when $p > 2$, where M_n' is obtained from M_n by relaxing the circuit-hyperplane $\{b_1, b_{2n+1}, a_3\}$.*

Proof. By Lemma 3.1, $M[B_n] \cong U_{3,4n}$. The remaining details of the proof are straightforward and are omitted. □

Lemma 3.3. *Let n be a positive integer. Then $\mathscr{K}(M_n) = \{2\}$.*

Proof. By Lemma 3.2, $\mathscr{K}(M_n)$ contains 2. To complete the proof, let $\mathbb{F}$ be a field and D_n be an $\mathbb{F}$-representation of M_n. We shall show that $\mathbb{F}$ has characteristic two. First observe that $M_n|\{a_1, a_2, a_3, b_1, b_2, b_{4n}\}$ is isomorphic to $M(K_4)$. Since $M(K_4)$ is binary, it is uniquely representable over $\mathbb{F}$ [5]. Therefore, we may assume that the columns of D_n corresponding to the elements a_1, a_2, a_3, b_1, b_2, and b_{4n} are identical to their counterparts in $[A'|B_n]$. By successively using the circuits $\{a_2, b_2, b_3\}, \{a_1, b_3, b_4\}, \{a_2, b_4, b_5\}, \ldots, \{a_2, b_{4n-2}, b_{4n-1}\}, \{a_1, b_{4n-1}, b_{4n}\}$ and then the circuits $\{a_3, b_3, b_{4n-1}\}, \{a_3, b_4, b_{4n-2}\}, \ldots, \{a_3, b_{2n}, b_{2n+2}\}$, we deduce that there are elements $d_1, d_2, \ldots, d_{2n-2}$ of $\mathbb{F}$ such that D_n can be obtained from B_n by replacing α_i by d_i for all i in $\{1, 2, \ldots, 2n-2\}$. Since $\{b_1, b_{2n+1}, a_3\}$ is a 3-circuit of M_n, it now follows that $\mathbb{F}$ must have characteristic two. This completes the proof of Lemma 3.3. □

The next lemma follows easily from the symmetry of M_n.

Lemma 3.4. *For each i in $\{1,2,\dots,4n\}$, there is an automorphism of M_n that maps b_i to b_j for some j in $\{1,3n+1,3n+2,\dots,4n\}$.*

Theorem 1.2 will follow by combining Lemma 3.3 with the next result.

Lemma 3.5. *For all n, the matroid M_n is an excluded minor for the class of matroids representable over $\mathbb{Q}$.*

Proof. Every single-element contraction of M_n has rank two and so is representable over $\mathbb{Q}$. The proof of Lemma 3.5 will be completed by showing that every single-element deletion of M_n is representable over $\mathbb{Q}$. Let $B=\{b_1,b_2,\dots,b_{4n}\}$.

There are two cases to consider depending upon whether we are (i) deleting some a_i from M_n, or (ii) deleting some b_j from M_n. We give geometric arguments in both cases.

To prove (i), again recall the three distinguished perfect matchings of K_{4n} defined in Section 2. Since every distinct pair of such matchings induces a Hamiltonian cycle of K_{4n}, it follows that the matroids $M_n\backslash a_1$, $M_n\backslash a_2$, and $M_n\backslash a_3$ are isomorphic. Thus it suffices to show that $M_n\backslash a_3$ is representable over $\mathbb{Q}$. We do this by finding, for all n, a set T_n of points of the projective plane $\mathrm{PG}(2,\mathbb{Q})$ such that $M_n\backslash a_3$ is isomorphic to $\mathrm{PG}(2,\mathbb{Q})|T_n$.

Suppose that we can find a set $\{(x_j,y_j)\colon j\in\{1,2,\dots,4n\}\}$ of points of the affine plane $\mathrm{AG}(2,\mathbb{Q})$, where b_j is identified with the point (x_j,y_j), so that no three distinct points in this set are collinear and, for each i in $\{1,2\}$, the elements of H_i are lines of a single parallel class. Let S_n be the subset $\{(1,x_j,y_j)\colon j\in\{1,2,\dots,4n\}\}$ of the point set of $\mathrm{PG}(2,\mathbb{Q})$. Let T_n be obtained from S_n by adding, for each i in $\{1,2\}$, the point of $\mathrm{PG}(2,\mathbb{Q})$ that is the common point of intersection of all the lines in the parallel class induced by H_i. Clearly $M_n\backslash a_3\cong \mathrm{PG}(2,\mathbb{Q})|T_n$.

We now define a set $\{(x_j,y_j)\colon j\in\{1,2,\dots,4n\}\}$ of points of $\mathrm{AG}(2,\mathbb{Q})$ that satisfies the initial assumption of the last paragraph. For all j in $\{1,2,\dots,4n\}$, let $b_j=(x_j,y_j)$. For all k in $\{1,2,\dots,2n-1\}$, let $(x_{2k-1},y_{2k-1})=(k^2-k,(k-1)^2)$ and $(x_{2k},y_{2k})=(k^2-k,k^2)$, and let $(x_{4n-1},y_{4n-1})=((2n)^2-2n,(2n-1)^2)$ and $(x_{4n},y_{4n})=((2n)^2-2n,0)$. Then the elements of each of H_1 and H_2 are lines of a single parallel class containing the lines $x=0$ and $y=0$, respectively.

To complete the proof of (i), we should like to show that no three distinct points of B are collinear. To avoid a tedious case analysis here, we can argue as follows. Clearly no horizontal or vertical line contains more than two elements of B. If b_1 or b_2 is collinear with two elements of $B-\{b_1,b_2\}$, then, for some small positive rational number ε_1, where $\varepsilon_1<\frac{1}{10}$, say, we can add $(\varepsilon_1,0)$ to the coordinates for b_1 and b_2 so that there are no longer any lines involving b_1 or b_2 and any two members of $B-\{b_1,b_2\}$. This move maintains the fact that the lines containing $\{b_2,b_3\}$ and $\{b_1,b_{4n}\}$ are horizontal, and the line containing $\{b_1,b_2\}$ is vertical. Next consider $\{b_3,b_4\}$. There is a positive rational number $\varepsilon_2<\frac{1}{10}$ so that we can add $(\varepsilon_2,0)$ to the coordinates for b_3 and b_4 so that there are no lines involving b_3 or b_4 and any two members of $B-\{b_3,b_4\}$.

Moreover, all horizontal or vertical lines containing b_3 or b_4 remain intact. Repeating this process for all the pairs $\{b_5,b_6\},\{b_7,b_8\},\dots,\{b_{4n-1},b_{4n}\}$ ensures that $M_n\backslash a_3$ is $\mathbb{Q}$-representable. This completes the proof of (i).

To prove (ii), it follows from Lemma 3.4 that it suffices to show that $M_n\backslash b_t$ is $\mathbb{Q}$-representable for all t in $\{1,3n+1,3n+2,\dots,4n\}$. For each such t, by using a similar argument to that given for (i), we shall prove (ii) by defining a set of $4n-1$ points of $\mathrm{AG}(2,\mathbb{Q})$, in which each point is identified with exactly one element of $B-b_t$, so that no three distinct points are collinear and, for each i in $\{1,2,3\}$, the elements of the set obtained from H_i by deleting the element containing b_t are lines of a single parallel class.

For each $t\neq 1$, we define such a set $\{(x_j,y_j)\colon j\in\{1,2,3,\dots,t-1,t+1,\dots,4n\}\}$ of points of $\mathrm{AG}(2,\mathbb{Q})$ as follows, where $b_j=(x_j,y_j)$ for all j. Set $b_2=(0,0)$. The remaining points are recursively obtained from b_2:

$$(x_j,y_j)=\begin{cases}(x_{j-1}+j-2,y_{j-1}) & \text{if } 3\leqslant j\leqslant 2n+1 \text{ and } j \text{ is odd},\\ (x_{j-1},y_{j-1}+j-2) & \text{if } 4\leqslant j\leqslant 2n+1 \text{ and } j \text{ is even},\\ (x_{j-1}+4n-j+1,y_{j-1}) & \text{if } 2n+2\leqslant j\leqslant t-1 \text{ and } j \text{ is odd},\\ (x_{j-1},y_{j-1}+4n-j+1) & \text{if } 2n+2\leqslant j\leqslant t-1 \text{ and } j \text{ is even},\\ (0,y_{2n+1}-x_{2n+1}) & \text{if } j=1,\\ (y_{2n+1}-x_{2n+1},y_{2n+1}-x_{2n+1}) & \text{if } j=4n\neq t,\\ (x_{j+1},y_{j+1}+j-4n) & \text{if } t+1\leqslant j\leqslant 4n-1 \text{ and } j \text{ is odd},\\ (x_{j+1}+j-4n,y_{j+1}) & \text{if } t+1\leqslant j\leqslant 4n-1 \text{ and } j \text{ is even}.\end{cases}$$

When $t=1$, we use the first four lines of the above to define (x_j,y_j), replacing the condition $2n+2\leqslant j\leqslant t-1$ in the third and fourth lines by the condition $2n+2\leqslant j$. It is straightforward to check that, for each i in $\{1,2,3\}$, the members of the set obtained from H_i by deleting the element containing b_t are lines of a single parallel class. In particular, these parallel classes contain the lines $x=0, y=0$, and $y=x$, respectively. For $n=4$ and $t=13$, Fig. 1 displays the points (x_j,y_j) in $\mathrm{AG}(2,\mathbb{Q})$.

We need to show that no three distinct points of $B-b_t$ are collinear. To avoid a long case analysis, we shall use a modification of the argument given in case (i) whereby we perturb some of the points slightly to destroy any unwanted lines. An additional difficulty that arises here is that these perturbations must be done so as to maintain three rather than just two parallel classes. We first treat the case when $t=1$. Suppose b_2,b_3,b_{4n-1}, or b_{4n} is collinear with two other members of B. Then, for some positive rational number $\varepsilon_1<\frac{1}{10}$, we can add $(0,\varepsilon_1)$ to each of b_2 and b_3, and add $(-\varepsilon_1,0)$ to each of b_{4n-1} and b_{4n} so as to destroy all lines that contain three elements of B including at least one element of $\{b_2,b_3,b_{4n-1},b_{4n}\}$. This perturbation maintains the parallel classes associated with H_1,H_2, and H_3. We continue this process dealing successively with unwanted lines involving a member of one of $\{b_4,b_5,b_{4n-3},b_{4n-2}\},\{b_6,b_7,b_{4n-5},b_{4n-4}\},\dots,\{b_{2n-2},b_{2n-1},b_{2n+1},b_{2n+2}\}$. At the conclusion of this process, the only remaining unwanted lines must involve all of b_{2n},b_{2n+1}, and b_{2n+2}. Since these three points are not collinear, all unwanted lines have been eliminated.

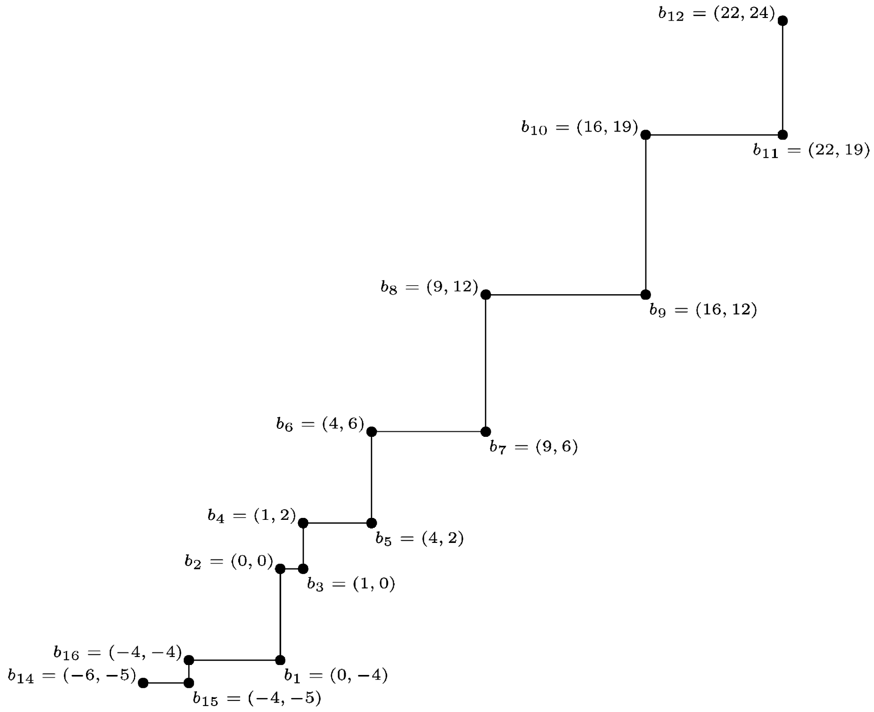

Fig. 1. The points (x_j, y_j) in the affine plane $\mathrm{AG}(2, \mathbb{Q})$ before perturbation, where $n = 4$ and $t = 13$.

Now, suppose that $t \neq 1$ and follow the procedure just described until dealing with the 4-set $\{b_{2i}, b_{2i+1}, b_{4n-(2i-1)}, b_{4n-(2i-2)}\}$ that contains b_t. In that case, for a suitably chosen small rational number ε_i, move b_{2i} and b_{2i+1} by $(0, \varepsilon_i)$ and the member of $\{b_{4n-(2i-1)}, b_{4n-(2i-2)}\} - \{b_t\}$ by $(-\varepsilon_i, 0)$. Then continue dealing with the remaining 4-sets as before. When this process concludes, the only remaining unwanted lines must involve three of $b_1, b_{2n}, b_{2n+1}, b_{2n+2}$. Since none of these four points has been moved, it is easily checked that no such line exists and the lemma follows. □

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 for all primes $p \geqslant 3$, and thereby complete the proof of Theorem 1.1.

This proof will use matroids that are defined using the operation of generalized parallel connection [1]. Let N_1 and N_2 be matroids such that $N_1|T = N_2|T$, where $T = E(N_1) \cap E(N_2)$. Let $N_1|T = N$ and suppose that T is a modular flat of N_1. The *generalized parallel connection* $P_N(N_1, N_2)$ of N_1 and N_2 across N is the matroid on $E(N_1) \cup E(N_2)$ whose flats are those subsets X of $E(N_1) \cup E(N_2)$ such that $X \cap E(N_1)$ is a flat of N_1, and $X \cap E(N_2)$ is a flat of N_2.

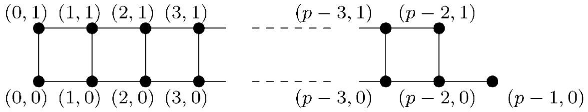

Fig. 2. The points (x_k, y_k) in the affine plane AG$(2, p)$.

For all positive integers n, recall the construction and labelling of M_n from the last section. Let M_n'' denote the matroid that can be obtained from M_n by relaxing the circuit-hyperplane $\{b_1, b_{2n+1}, a_3\}$ and then placing a point a_4 on the intersection of the lines $\{a_1, a_2, a_3\}$ and $\{b_1, b_{2n+1}\}$. Thus, $M_n''\backslash a_4$ is the matroid M_n' defined in Lemma 3.2.

Lemma 4.1. *$\{M_n''\colon n \geqslant 1\}$ is an infinite antichain.*

Proof. Suppose that M_j'' is isomorphic to a minor of M_k'' for some $j < k$. As both M_j'' and M_k'' have rank three, there is a map $\phi : E(M_j'') \to E(M_k'')$ under which M_j'' is isomorphic to some restriction of M_k''. Because each of M_j'' and M_k'' has a unique 4-point line, namely $\{a_1, a_2, a_3, a_4\}$, this set must be fixed by ϕ.

For each i in $\{j, k\}$, there are two perfect matchings of K_{4i} associated with a_1 and a_2 such that the union of these matchings is a cycle of length $4i$. It follows that M_j'' cannot be isomorphic to a restriction of M_k''. □

The infinite antichain $\{M_n''\colon n \geqslant 1\}$ does not, in itself, prove Theorem 1.1 for, as we shall see, the characteristic set of every M_n'' contains all primes exceeding two. The infinite antichain that will prove the theorem will be obtained by attaching a fixed matroid with characteristic set $\{p\}$ to every member of $\{M_n''\colon n \geqslant 1\}$ to form a set of rank-4 matroids. We shall now describe this construction more formally. Let p be a prime exceeding two. For all k in $\{1, 2, \ldots, 2p-1\}$, let $c_k = (1, x_k, y_k)$ where

$$(x_k, y_k) = \begin{cases} \left(\dfrac{k-1}{2}, 0\right) & \text{if } k \text{ is odd,} \\[2ex] \left(\dfrac{k-2}{2}, 1\right) & \text{if } k \text{ is even.} \end{cases}$$

Now, view the elements of $\{(x_k, y_k)\colon k \in \{1, 2, \ldots, 2p-1\}\}$ as points of AG$(2, p)$, as shown in Fig. 2, and consider the extension of this plane to the projective plane PG$(2, p)$. We shall distinguish a set A consisting of four collinear points a_1, a_2, a_3, and a_4 of PG$(2, p)$, where $a_1 = (0, 1, 0), a_2 = (0, 0, 1), a_3 = (0, 1, -1)$, and $a_4 = (0, 1, 1)$. We can view each of these points as the common point of intersection of all the lines in a parallel class in AG$(2, p)$, these classes containing, respectively, the lines $y = 0, x = 0, y = -x$, and $y = x$. Let N_p be the restriction of PG$(2, p)$ to the set consisting of $c_1, c_2, \ldots, c_{2p-1}$ and all points on the line L spanned by A. Let

$L - A = \{a_5, a_6, \ldots, a_{p+1}\}$. Clearly L is a modular line of N_p. Let M_n''' be obtained from M_n'' by freely placing $(p+1)-4$ points on the line of M_n'' spanned by A, labelling these points by the elements of $L - A$. Then $P_L(N_p, M_n''')$ is well-defined. Let N_p^n be the matroid obtained from $P_L(N_p, M_n''')$ by deleting $L - A$. Clearly, the ground set of N_p^n is the union of $E(N_p \backslash (L-A))$ and $E(M_n'')$.

Lemma 4.2. *For all primes* p *exceeding two, the set* $\{N_p^n\colon n \geqslant 1\}$ *is an infinite antichain of matroids.*

Proof. Suppose that N_p^j is isomorphic to a minor of N_p^k for some $j < k$. Then, since N_p^j and N_p^k have the same rank, there is a map $\phi\colon E(N_p^j) \to E(N_p^k)$ under which N_p^j is isomorphic to some restriction of N_p^k. For each i in $\{j,k\}$, the matroid N_p^i/e is vertically 3-connected if and only if $e \notin A$. Thus ϕ fixes the set A. Let $\{s,t,u,v\} = \{1,2,3,4\}$. Then N_p^i/a_s has $\{a_t, a_u, a_v\}$ as a parallel class and is the parallel connection with basepoint a_t of two rank-2 matroids. Thus, if ϕ does not fix the set $E(N_p \backslash (L-A))$, then it maps this set to a subset of $E(M_k'')$. But the latter cannot occur since $N_p \backslash (L-A)$ has a $(p+1)$-point line different from A but M_k'' has no such line. We deduce that ϕ must fix $E(N_p \backslash (L-A))$. Thus ϕ maps $E(M_j'')$ to a subset of $E(M_k'')$, so M_j'' is isomorphic to a minor of M_k'', a contradiction to Lemma 4.1. □

The next two lemmas will be combined to show that each member of $\{N_p^n\colon n \geqslant 1\}$ has characteristic set $\{p\}$.

Lemma 4.3. $\mathscr{K}(N_p \backslash (L-A)) = \{p\}$.

Proof. Order the elements of $N_p \backslash (L-A)$ as follows: the first eight elements are $c_1, a_1, a_2, c_4, c_2, c_3, a_3, a_4$ and the remaining elements are $c_5, c_6, \ldots, c_{2p-1}$. Suppose D is a matrix representing $N_p \backslash (L-A)$ over some field $\mathbb{F}$. Then, without loss of generality, we may assume that the submatrix of D indexed by its first four columns is

$$\begin{array}{c} \begin{array}{cccc} c_1 & a_1 & a_2 & c_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

If we consider the remaining elements of $N_p \backslash (L-A)$ in the order specified, it is not difficult to check that $N_p \backslash (L-A)$ is *sequentially unique* [2], that is, each element lies on the intersection of two lines spanned by points that occur earlier in the sequence. Using this, it follows straightforwardly that, for each element of $N_p \backslash (L-A)$, the corresponding column of D agrees with the coordinates originally assigned to that element of $N_p \backslash (L-A)$. But c_2, a_4, and c_{2p-1} are collinear in $N_p \backslash (L-A)$. Thus $p = 0$ in $\mathbb{F}$. Since $\mathscr{K}(N_p \backslash (L-A))$ clearly contains p, the lemma follows. □

Let A'' be the matrix

$$\begin{array}{c} \begin{array}{cccccc} a_4 & a_5 & a_6 & \cdots & a_p & a_{p+1} \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & & 0 & 0 \\ 1 & 1 & 1 & & 1 & 1 \\ 1 & 2 & 3 & \cdots & p-3 & p-2 \end{array}\right] \end{array},$$

and recall the matrices A' and B_n from the last section.

Lemma 4.4. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_{2n-2}$ *be algebraically independent transcendentals over* GF(p). *Then* $[A'|B_n|A'']$ *represents* M_n''' *over* GF$(p)(\alpha_1, \alpha_2, \ldots, \alpha_{2n-2})$.

Proof. By Lemma 3.2, $[A'|B_n]$ represents M_n', which equals $M_n'''\backslash\{a_4, a_5, \ldots, a_{p+1}\}$. Thus, it suffices to show that, in $M[A'|B_n|A'']$,

(i) $\{b_1, b_{2n+1}, a_4\}$ is a line; and
(ii) each of $a_5, a_6, \ldots, a_{p+1}$ is freely placed on the line spanned by $\{a_1, a_2\}$.

Now (i) is easily checked. To check (ii), suppose that it fails. Then $\{b_i, b_j, a_k\}$ is a circuit of $M[A'|B_n|A'']$ for some distinct i and j in $\{1, 2, \ldots, 4n\}$ and some k in $\{5, 6, \ldots, p+1\}$. Thus, the matrix

$$\begin{array}{c} \begin{array}{ccc} b_i & b_j & a_k \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 0 \\ x & y & 1 \\ u & v & k-3 \end{array}\right] \end{array}$$

has zero determinant. Hence,

$$u - v = (k-3)(x-y). \tag{1}$$

Now $k - 3 \in \mathrm{GF}(p) - \{0, 1, -1\}$. Thus $u = v$ if and only if $x = y$. But b_i and b_j are distinct so $u \neq v$ and $x \neq y$. Moreover, from (1), the number of members of the multiset $\{u, v, x, y\}$ that are transcendentals is 0, 2, or 4. In the first case, it follows that $\{b_i, b_j\} = \{b_2, b_{4n}\}$ and so $k - 3 = p - 1$; a contradiction. In the second and third cases, the structure of B_n implies that $u - v = \pm(x - y)$, so $k - 3 \in \{1, -1\}$. This contradiction completes the proof of the lemma. □

To prove Theorem 1.1, we shall combine the last three lemmas.

Proof of Theorem 1.1. We shall show that, for all primes p exceeding 2, every member of $\{N_p^n \colon n \geqslant 1\}$ has characteristic set $\{p\}$. Since $\{N_p^n \colon n \geqslant 1\}$ is an infinite antichain, the theorem will follow.

We take the representations for N_p and M_n''' described above and adjoin a row of zeros to each so that the new rows become the first and last rows, respectively. This gives representations for N_p and M_n''' over GF$(p)(\alpha_1, \alpha_2, \ldots, \alpha_{2n-2})$ in which L has a common representation. By a result of Brylawski [3, Proposition 7.6.11], it follows that $P_L(N_p, M_n''')$ has a GF$(p)(\alpha_1, \alpha_2, \ldots, \alpha_{2n-2})$-representation. Since N_p^n is a restriction of

$P_L(N_p, M_n''')$, the characteristic set of the former contains $\{p\}$. But N_p^n has $N_p\backslash(L-A)$ as a restriction and the last matroid has characteristic set equal to p. Thus N_p^n also has characteristic set equal to $\{p\}$ and so the theorem holds. $\square$

Acknowledgements

Oxley and Vertigan were partially supported by grants from the National Security Agency; Vertigan was also partially supported by a grant from the Louisiana Education Quality Support Fund; Semple and Whittle were partially supported by the Marsden Fund of New Zealand.

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