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On the minor-minimal 3-connected matroids having a fixed minor

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Abstract

Let N be a minor of a 3-connected matroid M such that no proper 3-connected minor of M has N as a minor. This paper proves a bound on $|E(M) - E(N)|$ that is sharp when N is connected.

MSC 1991: 05B35

1. Introduction

Let N be a minor of a 3-connected matroid M. Suppose that one wants to remove elements from M to maintain both 3-connectedness and the presence of N as a minor. If this cannot be done, what can be said about $|E(M) - E(N)|$? In particular, must this difference be bounded? If N is 3-connected, then clearly the difference is 0. This paper proves a bound on the difference that is sharp when N is connected.

For a matroid N, let $\lambda_1(N)$ denote the number of connected components of N. Now N can be constructed from a collection $\Lambda_2(N)$ of 3-connected matroids by using the operations of direct sum and 2-sum. It follows from results of Cunningham and Edmonds [5] that $\Lambda_2(N)$ is unique up to isomorphism. Let $\lambda_2(N)$ be the number of matroids in $\Lambda_2(N)$. The following is the main result of the paper.

Theorem 1.1. *Let N be a non-empty matroid and M be a minor-minimal* 3*-connected matroid having N as a minor. Then*

$$|E(M)| - |E(N)| \leq 22(\lambda_1(N) - 1) + 5(\lambda_2(N) - 1).$$

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An immediate consequence of the theorem is that if M and N satisfy the hypotheses, then

$$|E(M)| - |E(N)| \leq \alpha(\lambda_1(N) - 1) + \beta(\lambda_2(N) - 1)$$

for all $\alpha \geq 22$ and $\beta \geq 5$. We shall give examples to show that this theorem is sharp when N is connected, so the value of β cannot be reduced below 5. However, we believe that the theorem still holds when the value of α is reduced to 1.

Conjecture 1.2. *Let N be a non-empty matroid and M be a minor-minimal 3-connected matroid having N as a minor. Then*

$$|E(M)| - |E(N)| \leq \lambda_1(N) + 5\lambda_2(N) - 6.$$

The modification of the original problem that insists that M is a minor-minimal 3-connected matroid having N as a restriction was solved by the authors first in the case when N spans M [8] and then in general in joint work with Reid [10]. Another variant of the original problem that requires N to be 3-connected and different from M was solved by Truemper [14] when $|E(N)| \geq 4$ and by Bixby and Coullard [3] when $|E(N)| \leq 3$. That result is as follows.

Theorem 1.3. *Let N be a 3-connected matroid. If M is a minor-minimal 3-connected matroid having N as a proper minor, then $|E(M)| - |E(N)| \leq 3$.*

A third variant of the original problem that requires only that M be a minor-minimal 3-connected matroid having a minor isomorphic to N will be considered at the end of Section 4.

The terminology used in this paper will follow Oxley [11] except that the simplification and cosimplification of a matroid M will be denoted by $\mathrm{si}(M)$ and $\mathrm{co}(M)$, respectively. For a positive integer k, a partition $\{X, Y\}$ of the ground set of a matroid M is a *k-separation* of M if $\min\{|X|, |Y|\} \geq k$ and $r(X) + r(Y) - r(M) \leq k - 1$. When equality holds in the latter inequality, the k-separation $\{X, Y\}$ is *exact*. A matroid is *connected* if it has no 1-separations, and is 3-*connected* if it has no 1- or 2-separations.

The property that a circuit and a cocircuit of a matroid cannot have exactly one common element will be referred to as *orthogonality*. A basic structure in the study of 3-connected matroids consists of an interlocking chain of triangles and triads. Let $T_1, T_2, \ldots, T_k$ be a non-empty sequence of sets each of which is a triangle or a triad of a matroid M such that, for all i in $\{1, 2, \ldots, k-1\}$,

(i) $|T_i \cap T_{i+1}| = 2$;
(ii) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \cdots \cup T_i)$ is empty; and
(iii) in $\{T_i, T_{i+1}\}$, exactly one set is a triangle and exactly one set is a triad.

We call the sequence $T_1, T_2, \ldots, T_k$ a *fan* of M. When this occurs, it is straightforward to show that M has $k+2$ distinct elements $x_1, x_2, \ldots, x_{k+2}$ such that $T_i = \{x_i, x_{i+1}, x_{i+2}\}$ for all i in $\{1, 2, \ldots, k\}$. This terminology differs from that in [12] where the term "chain" is used for what has just been defined as a fan, and where "fan" is used for a maximal chain.

Suppose that the intersection of the ground sets of the matroids M and $M(K_4)$ is Δ and that Δ is a triangle in both matroids. The *generalized parallel connection [4] of* $M(K_4)$ *and* M *across* Δ is the matroid $P_\Delta(M(K_4), M)$ whose ground set is the union of the ground sets of the two matroids and whose flats are those subsets X of the ground set for which $X \cap E(M(K_4))$ is a flat of $M(K_4)$ and $X \cap E(M)$ is a flat of M. If the elements of Δ are deleted from $P_\Delta(M(K_4), M)$, we obtain the matroid that we get by performing a $\Delta - Y$-exchange on M across Δ [1].

This paper is structured as follows. In the next section, we review the results of Cunningham and Edmonds [5] on decomposing a connected matroid into 3-connected pieces. Section 3 proves some technical lemmas that will be used in the proof of the main result. In particular, a result of Seymour [13] is used to show that the destruction of a particular exact 2-separation of the matroid N requires the addition of at most five new elements. In Section 4, the main result is proved in the case that N is connected and it is shown that the theorem is sharp in this case even when M is only required to contain a minor isomorphic to N rather than N itself. Section 5 uses the result for the connected case to obtain a general bound on $|E(M)|$ in terms of $|E(N)|$. This bound tends to be weaker than the bound in the main theorem, which is proved in the last section.

2. Tree decomposition

In this section, we review the results of Cunningham and Edmonds that will be used in the proof of the main result. Let M be a connected matroid. A *tree decomposition* of M is a tree T with edges labelled $e_1, e_2, \dots, e_{k-1}$ and vertices labelled by matroids $M_1, M_2, \dots, M_k$ such that

(i) each M_i is 3-connected having at least four elements or is a circuit or a cocircuit;
(ii) $E(M_1) \cup E(M_2) \cup \cdots \cup E(M_k) = E(M) \cup \{e_1, e_2, \dots, e_{k-1}\}$;
(iii) if the edge e_i joins the vertices M_{j_1} and M_{j_2}, then $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$;
(iv) if no edge joins the vertices M_{j_1} and M_{j_2}, then $E(M_{j_1}) \cap E(M_{j_2})$ is empty;
(v) M is the matroid that labels the single vertex of the tree $T/e_1, e_2, \dots, e_{k-1}$ at the conclusion of the following process: contract the edges $e_1, e_2, \dots, e_{k-1}$ of T one by one in order; when e_i is contracted, its ends are identified and the vertex formed by this identification is labelled by the 2-sum of the matroids that previously labelled the ends of e_i.

Cunningham and Edmonds [5] proved the following result.

Theorem 2.1. *Every connected matroid M has a tree decomposition $T(M)$ in which no two adjacent vertices are both labelled by circuits or are both labelled by cocircuits. Furthermore, the tree $T(M)$ is unique to within relabelling of its edges.*

We shall call $T(M)$ the *canonical tree decomposition of* M and we let $\Lambda_2^u(M)$ be the set of matroids that label vertices of $T(M)$. If a vertex M' of $T(M)$ corresponds to a circuit or a cocircuit with n elements for some $n \geq 4$, then M' has a tree decomposition $T^3(M')$ in which each vertex is labelled by a 3-element circuit when M' is a circuit and by a 3-element cocircuit when M' is a cocircuit. It follows that $T^3(M')$ has $n - 2$ vertices and, indeed, every $(n - 2)$-vertex tree can be labelled so that it is such a tree decomposition of M'.

Now replace the vertex of $T(M)$ labelled by M' by one of the choices for $T^3(M')$. Specifically, delete the vertex of $T(M)$ labelled by M'; take the disjoint union of the resulting graph $T(M)-M'$ with $T^3(M')$; for each edge x of $T(M)$ that joins M' to M'_x, say, add an edge labelled x to $(T(M)-M')\cup T^3(M')$ joining M'_x to the vertex of $T^3(M')$ that is labelled by a matroid having x as an element. Repeat the above process for each vertex of $T(M)$ that is labelled by a circuit or cocircuit with at least four elements. Let the resulting graph be $T^3(M)$. It is not difficult to see that $T^3(M)$ is a tree decomposition of M in which every vertex is labelled by a 3-connected matroid. We call $T^3(M)$ a 3-*c-tree decomposition* of M. Evidently, unlike $T(M)$, the tree $T^3(M)$ is not uniquely determined by M. We let $\Lambda_2(T^3(M))$ be the set of matroids that label vertices of $T^3(M)$. The construction of $T^3(M)$ ensures that the matroid M determines the distribution of isomorphism types of matroids in $\Lambda_2(T^3(M))$ together with the isomorphism type of the matroid M_e that contains e and, if $|E(M_e)| \geq 4$, the isomorphism types of the matroids that share elements with M_e. We shall write $\Lambda_2(M)$ for $\Lambda_2(T^3(M))$ and let $\lambda_2(M)$ be the number of members of $\Lambda_2(M)$. If M has components $M_1, M_2, \ldots, M_t$, we define $\lambda_2(M)$ to be $\sum_{i=1}^{t} \lambda_2(M_i)$. Also we let $\lambda_1(M)$ be t, the number of components of M. Note that this use of $\lambda_2(M)$ differs from that in some earlier work of the authors where $\lambda_2(M)$ did not count the copies of $U_{1,3}$ that arose in the decomposition [8–10].

Let M be a connected matroid and T be a tree decomposition of M. A connected subgraph H of T *induces* a subset X of $E(M)$ if X is the union, over all vertices M_j of H, of $E(M_j)\cap E(M)$. Each edge e of T determines a partition of $E(M)$ into the subsets X_{e1} and X_{e2} that are induced by the components of $T-e$. We shall say that the edge e *displays* the partition $\{X_{e1}, X_{e2}\}$ of $E(M)$ and *displays* the sets X_{e1} and X_{e2}. Now let M' be a vertex of T that is a circuit or a cocircuit. We say that M' *displays* a partition $\{X, Y\}$ of $E(M)$ if every subset of $E(M)$ that is induced by a component of $T-M'$ lies entirely in either X or Y.

The next result of Cunningham and Edmonds [5] does not have an easily accessible proof so we include a proof here.

Lemma 2.2. *Let M be a connected matroid and $\{X_1, X_2\}$ be a partition of $E(M)$ such that $|X_1|, |X_2| \geq 2$. Then the following statements are equivalent.*

(i) *$\{X_1, X_2\}$ is a 2-separation of M;*
(ii) *M has a 3-c-tree decomposition having an edge that displays $\{X_1, X_2\}$; and*
(iii) *$T(M)$ has an edge or a vertex that displays $\{X_1, X_2\}$ where, in the latter case, the vertex is labelled by a circuit or a cocircuit.*

Proof. We show first that (i) implies (ii). Suppose that $\{X_1, X_2\}$ is a 2-separation of M. Then M can be written as the 2-sum, with basepoint b, of two matroids M_1 and M_2 having ground sets $X_1 \cup b$ and $X_2 \cup b$, respectively. We can construct a 3-c-tree decomposition $T^3(M)$ for M inductively as follows. Begin with the two-vertex tree T_1 in which the vertices are labelled by M_1 and M_2 and the edge is labelled by e. Assume that T_k has been constructed for some $k \geq 1$. If every matroid labelling a vertex of T_k is 3-connected, let $T_k = T^3(M)$; otherwise choose a matroid M' that labels a vertex of T_k and is not 3-connected, and let $\{X'_1, X'_2\}$ be a 2-separation of M'. Write M' as the 2-sum of two matroids M'_1 and M'_2 with ground sets $X'_1 \cup b'$ and $X'_2 \cup b'$, respectively; form T_{k+1} by splitting the vertex M' of T_k into two vertices M'_1 and M'_2 joined by the edge b' where each edge e of T_k that meets M' meets the member of $\{M'_1, M'_2\}$ that contains e. Evidently, $T^3(M)$ is a 3-c-tree decomposition of M and the edge b displays $\{X_1, X_2\}$. Thus (i) implies (ii).

Next we show that (ii) implies (iii). Suppose that $T^3(M)$ is a 3-c-tree decomposition of M and the edge b displays $\{X_1, X_2\}$. To obtain $T(M)$ from $T^3(M)$, we look for two adjacent vertices of the latter that are both labelled by circuits or are both labelled by cocircuits. When we find two such vertices, we contract the edge joining them and label the composite vertex resulting by the 2-sum of the two original labels on the ends of the edge. We continue this process until we obtain a tree having no two adjacent vertices both labelled by circuits or both labelled by cocircuits. The uniqueness of $T(M)$ implies that the resulting tree is, indeed, $T(M)$.

Now $\{X_1, X_2\}$ is displayed by the edge b in $T(M)$ unless, in $T^3(M)$, both ends of b label circuits or both ends label cocircuits. Consider the exceptional case, assuming, without loss of generality, that both ends of b label circuits. Then b is an edge of a maximal subtree T^c of $T^3(M)$ all of whose vertices are labelled by circuits. In forming $T(M)$, we contract T^c to a single vertex, which we may assume is one of the ends of b, say v. For $T = T^3(M)$, every subset of $E(M)$ induced by a component of $T - v$ lies entirely in X_1 or X_2. This remains true whenever we contract an edge incident with v. Thus, it follows that v displays $\{X_1, X_2\}$ in $T(M)$ and v is labelled by a circuit. Hence (ii) implies (iii).

The proof that (iii) implies (i) is similar to the above and is omitted. □

Let M be a connected matroid. Evidently, $T(M^*)$ and $T^3(M^*)$ can be obtained from $T(M)$ and $T^3(M)$, respectively, by replacing each matroid labelling a vertex of the latter by the dual matroid. Now suppose that $e \in E(M)$ and M/e is connected and non-empty. It is useful to describe the relationship between $T(M)$ and $T(M/e)$. By duality, this also determines the relationship between $T(M)$ and $T(M\backslash e)$ when $M\backslash e$ is connected. Let M' be the matroid labelling a vertex of $T(M)$ such that $e \in E(M')$. To find $T(M/e)$, we proceed as follows:

(i) construct $T(M'/e)$;

(ii) take the disjoint union of $T(M'/e)$ and $T(M) - M'$;

(iii) if b is an edge of $T(M)$ that joins M' to K, add an edge labelled by b joining the vertex K of $T(M) - M'$ to the vertex of $T(M'/e)$ that contains b;

(iv) if a newly added such edge b joins two circuits or two cocircuits, then contract the edge b and relabel the composite vertex by the 2-sum, with basepoint b, of the two matroids that had labelled the ends of b;

(v) if $|E(M'/e)| = 2$ and $T(M)$ has more than one vertex, then contract an edge b joining M'/e with H, say, and relabel the composite vertex by H', the matroid that is obtained from H by relabelling the element b by the unique element b' of $E(M'/e) - b$; finally, if K' is a cocircuit and b' joins K' to another cocircuit K'', then contract b' and label the composite vertex by the 2-sum of the two matroids that had labelled the endpoints of b'.

3. Preliminary lemmas

In this section, we prove some technical lemmas that will be used in the proof of the main result. In particular, we show in Lemma 3.3 that the destruction of a particular exact 2-separation of N requires the addition of at most five new elements. This fact will be crucial in the proof of the main theorem in the case that N is connected.

Let A and B be disjoint subsets of the ground set of a matroid M. Then $k_M(A, B) = \min\{r(X)+r(Y)-r(M)\}$, where the minimum is taken over all partitions $\{X, Y\}$ of $E(M)$ with $X \supseteq A$ and $Y \supseteq B$. This function, which was used by Seymour [13], is closely related to a function $k(M; X, Y)$ introduced by Tutte [15]. Indeed, $k_M(A, B) = k(M; A, B)-1$, so one can easily deduce properties of one function from properties of the other. The following lemma summarizes some useful properties of $k_M(A, B)$.

Lemma 3.1. *Let A and B be disjoint subsets of the ground set of a matroid M. Then*

(i) $k_M(A, B) = k_{M^*}(A, B)$;
(ii) *if N is a minor of M such that $A \cup B \subseteq E(N)$, then $k_N(A, B) \le k_M(A, B)$; and*
(iii) *if $e \in E(M) - (A \cup B)$, then*

$$\max\{k_{M\backslash e}(A, B), k_{M/e}(A, B)\} = k_M(A, B)$$

and

$$\min\{k_{M\backslash e}(A, B), k_{M/e}(A, B)\} \ge k_M(A, B) - 1.$$

Seymour [13] carefully analyses the structure of a matroid M having a minor N such that $\{A, B\}$ is a partition of $E(N)$ and M is minor-minimal having N as a minor and satisfying $k_M(A, B) > k_N(A, B)$. We shall only use this result in the case that $\{A, B\}$ is a 2-separation of N, so we state it only in this case.

Lemma 3.2. *Let N be a matroid and $\{A, B\}$ be an exact 2-separation of N. Suppose that M is a minor-minimal matroid that has N as a minor and satisfies $k_M(A, B) > 1$. Then the following hold.*

(i) *There are unique subsets P and Q of $E(M)$ such that $N = M\backslash P/Q$.*
(ii) *Let M_z be $M\backslash z$ when $z \in P$ and be M/z when $z \in Q$. Then M_z has just one 2-separation $\{X_z, Y_z\}$ such that $A \subseteq X_z$ and $B \subseteq Y_z$, and this 2-separation is exact.*
(iii) *The elements of $P \cup Q$ can be labelled as $z_1, z_2, \dots, z_n$ so that $X_{z_i} = A \cup \{z_1, z_2, \dots, z_{i-1}\}$ and $Y_{z_i} = B \cup \{z_{i+1}, z_{i+2}, \dots, z_n\}$ for all i.*
(iv) *The elements $z_1, z_2, \dots, z_n$ are alternately members of P and Q.*
(v) *M has no circuit C such that $C \subseteq P \cup Q$ and $|C - Q| \le 1$, and M has no cocircuit C^* such that $C^* \subseteq P \cup Q$ and $|C^* - P| \le 1$.*
(vi) *For all $i > 1$, if $z_i \in P$, there is a circuit C of M such that $\{z_{i-1}, z_i\} \subseteq C$ and $C - \{z_{i-1}, z_i\} \subseteq (Q \cap \{z_j : j > i\}) \cup B$. If $z_i \in Q$, there is a cocircuit C^* of M such that $\{z_{i-1}, z_i\} \subseteq C^*$ and $C^* - \{z_{i-1}, z_i\} \subseteq (P \cap \{z_j : j > i\}) \cup A$. Moreover, the corresponding result holds for all $i < n$ with A and B interchanged.*

Lemma 3.3. *Let $\{A, B\}$ be an exact 2-separation of a matroid N and let M be a minor-minimal matroid such that N is a minor of M and $k_M(A, B) > 1$. Then*

$$|E(M)| - |E(N)| \le 5. \tag{1}$$

Moreover, if N is connected and $M\backslash e$ or M/e is disconnected for some e in $E(M) \cup E(N)$, then $|E(M) - E(N)| = 1$. In particular, if $M\backslash e$ is disconnected, then the vertex of $T(M)$ that is labelled by a matroid containing e is a triangle, this vertex has exactly two neighbours in $T(M)$ both of which are labelled by cocircuits, and all four of the sets that are displayed by edges of $T(M)$ incident with this triangle must meet both A and B.

Proof. We shall use the notation of the last lemma. In particular, $E(M)$ has unique subsets P and Q such that $N = M\backslash P/Q$.

We shall prove (1) by contradiction. Let $n = |P| + |Q| = |E(M)| - |E(N)|$ and assume that $n \geq 6$. By taking the dual if necessary, we may assume that $z_1 \in P$. Thus $z_6 \in Q$ by Lemma 3.2(iv). Hence N is a minor of $M\backslash z_1/z_6$. Now, as $\{X_{z_6}, Y_{z_6}\}$ is an exact 2-separation of M/z_6, this matroid is a 2-sum of two matroids N_2 and M_2 where $E(M_2) = Y_{z_6} \cup b$. Since $\{X_{z_1}, Y_{z_1}\}$ is an exact 2-separation of $M\backslash z_1$, it is not difficult to check that $\{X_{z_1}, \{z_2, z_3, z_4, z_5, b\}\}$ is an exact 2-separation of N_2. Thus N_2 is the 2-sum of two matroids M_1 and H where $E(M_1) = X_{z_1} \cup a$ and $E(H) = \{a, z_2, z_3, z_4, z_5, b\}$. We conclude that $M\backslash z_1/z_6$ is obtained by taking the 2-sum of M_1, H, and M_2. Moreover, $M\backslash z_1$ is the 2-sum of M_1 and a matroid N_1 for which $N_1/z_6 = H \oplus_2 M_2$. Next we observe that H is connected, otherwise if a and b are in the same component of H, then P and Q are not unique, while if a and b are in different components of H, then $k_N(A, B) = 0 \neq 1$. Moreover, since P and Q are unique and $k_N(A, B) = 1$, the matroid $H\backslash\{z_3, z_5\}/\{z_2, z_4\}$ is connected and is uniquely determined as a minor of H. Thus $H\backslash\{z_3, z_5\}/\{z_2, z_4\}$ is a circuit on $\{a, b\}$ and hence is also a cocircuit on $\{a, b\}$. The fact that $H\backslash\{z_3, z_5\}/\{z_2, z_4\}$ is uniquely determined as a minor of H implies that $\{a, b, z_2, z_4\}$ is a circuit of H and that $\{a, b, z_3, z_5\}$ is a cocircuit of H. It follows that $\{z_2, z_4\}$ is a line of H. Moreover, $r(H) = r(H^*) = 3$.

By Lemma 3.2(vi), M has a circuit C containing z_4 and z_5 such that $C - \{z_4, z_5\} \subseteq (Q \cap \{z_j : j > 5\}) \cup B$. Let C' be a circuit of M/z_6 such that $C' \subseteq C$ and $C' \cap \{z_4, z_5\} \neq \emptyset$. If $C' \cap B = \emptyset$, then $\{z_4, z_5\} \cup (Q \cap \{z_j : j > 5\})$ contains a circuit of M; a contradiction to Lemma 3.2(v). Thus $C' \cap B \neq \emptyset$, so $b \cup (C' \cap \{z_4, z_5\})$ is a circuit of H. As $C - z_6$ is a union of circuits of M/z_6, we have two possibilities:

(a) both $\{b, z_4\}$ and $\{b, z_5\}$ are circuits of H; or

(b) $\{b, z_4, z_5\}$ is a circuit of H.

Since b is not a loop of H/z_4, it follows that z_4 cannot be parallel to b. Thus $\{b, z_4, z_5\}$ is a circuit of H. Since $\{a, b, z_2, z_4\}$ and $\{b, z_5, z_4\}$ are circuits of H, it follows that H has a circuit containing a and contained in $\{a, b, z_2, z_5\}$. This circuit does not contain b and so is a subset of $\{a, z_2, z_5\}$. Now $\{a, z_2\}$ is not a circuit. Moreover, $\{a, z_5\}$ is not a circuit otherwise $\{a, b, z_4\}$ is a circuit of H. Thus $\{a, z_2, z_5\}$ is a circuit of H. Hence $H\backslash z_3$ is the parallel connection of the two 3-point lines $\{a, z_2, z_5\}$ and $\{b, z_4, z_5\}$. Now, in H/z_5, the element z_3 must either be a loop or be parallel to a or b otherwise $H\backslash\{z_3, z_5\}/\{z_2, z_4\}$ is not uniquely obtainable as a minor of H. Thus z_3 is on the line of H spanned by $\{a, z_5\}$ or the line of H spanned by $\{b, z_5\}$. Since neither $\{a, b, z_2, z_3\}$ nor $\{a, b, z_4, z_3\}$ is a circuit of H, we deduce that z_3 is parallel to one of a, b, and z_5 in H. In the second and third cases, $\{X_{z_3}, Y_{z_3}\}$ is an exact 2-separation of $M\backslash z_3$ and z_3 is spanned by Y_{z_3}, so $\{X_{z_3}, Y_{z_3} \cup z_3\}$ is an exact 2-separation of M; a contradiction. We conclude that $\{z_3, a\}$ is a circuit of H.

Now recall that $M\backslash z_1 = M_1 \oplus_2 N_1$, where $N_1/z_6 = H \oplus_2 M_2$ and $E(M_1) = X_{z_1} \cup a$. Since X_{z_3}, and hence X_{z_1}, does not span z_3 in M, we deduce that $\{a, z_3\}$ is not a circuit of N_1. Therefore $\{a, z_3, z_6\}$ is a circuit of N_1. Hence M has a circuit D that contains $\{z_3, z_6\}$ and is contained in $X_{z_1} \cup \{z_3, z_6\}$. Moreover, by Lemma 3.2(vi), M has a cocircuit D^* that contains $\{z_5, z_6\}$ and is contained in $Y_{z_6} \cup \{z_5, z_6\}$. Thus, $|D \cap D^*| = 1$; a contradiction. We conclude that (1) holds.

We now prove the rest of the lemma. Suppose that N is connected and that $M\backslash e$ or M/e is disconnected for some e in $P \cup Q$. Without loss of generality, we may suppose that $M\backslash e$ is disconnected. As N is connected, M is connected and $e \notin P$, so $e \in Q$. Since $M\backslash e$ is disconnected, the member of $\Lambda_2^u(M)$ containing e is a circuit C. If $E(M) = C$, then $k_M(A, B) = 1$; a contradiction. Thus $E(M)$ properly contains C.

Now M/e has N as a minor. Thus, by the choice of M, there is a 2-separation $\{X, Y\}$ of M/e such that $A \subseteq X$ and $B \subseteq Y$. Moreover, by Lemma 2.2, $\{X, Y\}$ is displayed either by an edge or a vertex of $T(M/e)$. From the last section, there are three possibilities for the way in which $T(M)$ is obtained from $T(M/e)$:

(a) a single vertex of $T(M/e)$ that was labelled by a circuit D of $T(M/e)$ has its label changed to $D \cup e$, a circuit that labels a vertex of $T(M)$;
(b) an edge of $T(M/e)$ is subdivided with the newly inserted vertex being labelled by a triangle containing e; and
(c) a single vertex w of $T(M/e)$ that was labelled by a cocircuit C^* is replaced by three vertices w_0, w_1, and w_2, where w_1 and w_2 are the only two neighbours of w_0; every neighbour of w in $T(M/e)$ is a neighbour of exactly one of w_1 and w_2; the vertices w_1, w_0, and w_2 are labelled by, respectively, a cocircuit C_1^*, a triangle C with ground set $\{x_1, e, x_2\}$, and a cocircuit C_2^*; the edges w_0w_1 and w_0w_2 are labelled by x_1 and x_2, respectively; and $E(C^*) = (E(C_1^*) \cup E(C_2^*)) - \{x_1, x_2\}$.

Since $k_M(A, B) > 1$, there is no 2-separation $\{X', Y'\}$ of M such that $X \subseteq X'$ and $Y \subseteq Y'$. It follows that $\{X, Y\}$ is displayed by a vertex w of $T(M/e)$ where $T(M)$ is obtained from $T(M/e)$ as in (c) above. Therefore the assertion in the last sentence of the lemma holds. In $T(M)$, for each i in $\{1, 2\}$, let $x_i, b_{i1}, b_{i2}, \ldots, b_{im_i}$ be the edges incident with w_i. Now, one by one, contract the edges of $T(M)$ that are not incident with w_1 or w_2 and, after each contraction, label the composite vertex of the result by the 2-sum of the two matroids that had labelled the endpoints of the edge. Then M is the 2-sum of the matroids $C_1^*, C_2^*, C, M_{11}, M_{12}, \ldots, M_{1m_1}, M_{21}, M_{22}, \ldots, M_{2m_2}$ where $E(M_{ij}) - E(M) = \{b_{ij}\}$.

The 2-separation $\{X, Y\}$ has the property that each $E(M_{ij}) - b_{ij}$ is contained in X or Y. Thus each $E(M_{ij}) - b_{ij}$ meets exactly one of A and B. Label b_{ij} by A or B according to which of these two sets is met by $E(M_{ij})$. Similarly, label each element of $C_i^* \cap (A \cup B)$ by A or B according to which of A and B the element belongs. Since $k_M(A, B) > 1$, it follows that, for each i in $\{1, 2\}$, the set C_i^* must have at least one element labelled A and at least one element labelled B. Thus all four of the sets that are displayed by an edge of $T(M)$ incident with w_0 meet both A and B.

Assume that $|E(M) - E(N)| > 1$ and let f be an element of $E(M) - (E(N) \cup e)$. Since N is connected, if $f \in P$, then $M\backslash f$ is connected, while if $f \in Q$, then M/f is connected. Suppose first that $f \in C_1^* - \cup_{j=1}^{m_1} b_{1j}$. Then M/f is disconnected, so $f \notin Q$. Hence $f \in P$ and $M\backslash f$ is connected. Moreover, $C_1^* - f$ must contain x_1 along with an

A-element and a B-element. It follows that $\{X - f, Y - f\}$ is a 1-separation of $M/e/f$, so $k_{M/e/f}(A, B) = 0$. Hence, by Lemma 3.1(iii), $k_{M/f}(A, B) \leq 1 < k_M(A, B)$ and so $k_{M\backslash f}(A, B) = k_M(A, B)$. Thus $M\backslash f$ contradicts the choice of M. We conclude that $f \notin C_1^* - \cup_{j=1}^{m_1} b_{1j}$ and hence, by symmetry, $f \notin C_1^* \cup C_2^*$. Now let M' be $M\backslash f$ if $f \in P$ and be M/f if $f \in Q$. Then the construction of $T(M')$ from $T(M)$ is described in the last section. It follows from that description that $T(M')$ has no edge and no vertex that displays a 2-separation $\{X', Y'\}$ such that $A \subseteq X'$ and $B \subseteq Y'$. Hence $k_{M'}(A, B) > 1$ and so M' contradicts the choice of M. We conclude that $|E(M) - E(N)| = 1$. □

Lemma 3.4. *Let $\{A, B\}$ be a 2-separation of a connected matroid N. Let M be a minor-minimal matroid that has N as a minor and satisfies $k_M(A, B) > 1$. Then either*

(i) $\lambda_2(M) < \lambda_2(N)$*; or*
(ii) *there is a matroid H labelling a vertex of $T(M)$ that has exactly two neighbours such that either H is a triangle and its two neighbours are cocircuits, or H is a triad and its two neighbours are circuits; each of the 2-separations that is displayed by an edge of $T(M)$ meeting H has both its parts meeting both A and B; and the sets $E(H) - E(N)$ and $E(M) - E(N)$ are equal and contain a single element.*

Proof. Observe that M is connected. Moreover, since $k_N(A, B) = 1$, it follows that $M \neq N$ and M is neither a circuit nor a cocircuit. By Lemma 3.3, there are unique sets X and Y such that $N = M\backslash X/Y$. For each H in $\Lambda_2(M)$, we define

$$H^- = H\backslash(X \cap E(H))/(Y \cap E(H)). \quad \square$$

Next we show the following:

3.4.1. *There is a matroid H in $\Lambda_2(M)$ such that $|E(H^-)| \leq 2$.*

Proof. Suppose that $|E(H^-)| \geq 3$ for all H in $\Lambda_2(M)$. Now consider how to construct a 3-c-tree decomposition $T^3(N)$ for N from a 3-c-tree decomposition $T^3(M)$ for M. By assumption, for each matroid H labelling a vertex of $T^3(M)$, the matroid H^- has at least three elements. Thus each vertex of $T^3(H^-)$ is labelled by a matroid with at least three elements. We construct $T^3(N)$ from $T^3(M)$ by replacing each vertex H of the latter by the tree $T^3(H^-)$ where an edge b of $T^3(M)$ that meets H corresponds to an edge of $T^3(N)$ that meets the vertex of $T^3(H^-)$ that is labelled by a matroid using b. We deduce that

$$\lambda_2(N) = \sum_{H\in\Lambda_2(M)} \lambda_2(H^-) \geq \sum_{H\in\Lambda_2(M)} \lambda_2(H) = \sum_{H\in\Lambda_2(M)} 1 = \lambda_2(M). \tag{2}$$

We may assume that we have equality throughout (2), otherwise the result follows. Therefore,

$$\lambda_2(H^-) = 1, \qquad \text{for every } H \text{ in } \Lambda_2(M). \tag{3}$$

Now, by Lemma 2.2, we can construct a 3-c-tree decomposition $T^3(N)$ for N having an edge that displays the 2-separation $\{X_1, X_2\}$. By (3), $T^3(M)$ can be obtained from $T^3(N)$ just by relabelling each vertex H^- of the latter by the corresponding matroid H. Thus $T^3(M)$ has an edge that displays a 2-separation $\{X_1', X_2'\}$ where $X_1' \supseteq X_1$ and $X_2' \supseteq X_2$. Therefore $k_M(X_1, X_2) = 1$; a contradiction. Thus (3.4.1) holds. □

3.4.2. *If $|E(H^-)| \leq 2$ for some H in $\Lambda_2(M)$, then $|E(H^-)| = 2$, the matroid H is a triangle or a triad, and $|E(M) - E(N)| = 1$.*

Proof. If $E(H^-) = \emptyset$, then H is a component of M. This is a contradiction to the uniqueness of X and Y because N is a minor of both $M\backslash f$ and M/f, when $f \in E(H)$. Thus $E(H^-) \neq \emptyset$. Now suppose that $|E(H^-)| = 1$. Then H^- is a loop or a coloop. As H^- is uniquely determined as a minor of H, it follows that H is a circuit or a cocircuit. Thus, as $|E(H)| \geq 3$, if $f \in E(H) \cap (X \cup Y)$, then $H\backslash f$ or H/f is disconnected. Therefore $M\backslash f$ or M/f is disconnected. Hence, by Lemma 3.3, $|E(M) - E(N)| = 1$. But $|E(M) - E(N)| \geq |E(H) - E(H^-)| \geq 2$; a contradiction. Hence we may assume that $|E(H^-)| = 2$. Observe that H^- is connected because N is connected. Hence H^- is isomorphic to $U_{1,2}$. Now take g in $E(H) \cap (X \cup Y)$. By switching to the dual if necessary, we may assume that $g \in Y$. Then $H\backslash g$ is disconnected because H^- is uniquely determined as a minor of H and so $H\backslash g$ does not have a circuit that contains $E(H^-)$. Thus, as H is 3-connected having at least three elements but $H\backslash g$ is disconnected, it follows that $H \cong U_{2,3}$. Since $H\backslash g$ is disconnected, $M\backslash g$ is disconnected. Thus, by Lemma 3.3, $|E(M) - E(N)| = 1$. □

Now, by (3.4.1), there is a member H of $\Lambda_2(M)$ such that $|E(H^-)| \leq 2$. By (3.4.2), $|E(H^-)| = 2$ and $|E(M) - E(N)| = 1$, so H is unique. Let $E(M) - E(N) = \{g\}$. Then $g \in E(H) - E(H^-)$. By switching to the dual if necessary, we have that H is a triangle and $N = M/g$. Moreover, it follows by the last part of Lemma 3.3 that H labels a vertex of $T(M)$ that has exactly two neighbours both of which are labelled by cocircuits. Furthermore, both of the edges incident with H in $T(M)$ display two sets and, by the last part of Lemma 3.3, all four of these sets meet both A and B. Thus the lemma is proved.

Lemma 3.5. *Let N be a simple connected matroid having at least four elements. Suppose that M is a minor-minimal* 3*-connected matroid having N as a minor. If $e \in E(M) - E(N)$ and N is a minor of both $M\backslash e$ and M/e, then e belongs to a triad T_e^* of M such that $T_e^* - e \subseteq E(N)$. Moreover, if $N = M\backslash X/Y$ and $X' \subseteq X$, then every component H of $M/Y\backslash X'$ that does not meet $E(N)$ is a coloop.*

Proof. Since N is a simple minor of M/e, we may choose the elements of $\mathrm{si}(M/e)$ so that it has N as a minor. By the choice of N, it follows that $\mathrm{si}(M/e)$ is not 3-connected. Thus, by a result of Bixby [2], $\mathrm{co}(M\backslash e)$ is 3-connected and each series class of $M\backslash e$ has at most two elements. The choice of M implies that the elements of $\mathrm{co}(M\backslash e)$ cannot be chosen so that N is a minor of it.

Assume that no non-trivial series class of $M\backslash e$ is contained in $E(N)$. Then, to obtain N from M, we must delete or contract an element from every such series class. If $\{a, b\}$ is such a series class where $a \notin E(N)$, then either N is a minor of $M\backslash e\backslash a$, or N is a minor of $M\backslash e/a$. In the former case, since b is a coloop of $M\backslash e\backslash a$ but not of N, it follows that $b \notin E(N)$, so N is a minor of $M\backslash e\backslash a\backslash b$, which equals $M\backslash e\backslash b/a$. Thus, in both cases, N is a minor of $M\backslash e/a$. Therefore the 3-connected matroid $\mathrm{co}(M\backslash e)$ has N as a minor; a contradiction. We conclude that $M\backslash e$ has a non-trivial series class contained in $E(N)$, so M has a triad T_e^* containing e such that $T_e^* - e \subseteq E(N)$.

To prove the second part of the lemma, we argue by contradiction. Suppose that H is not a coloop. If $f \in E(H)$, then, as $E(H)$ does not meet $E(N)$, both $M\backslash f$ and M/f have

N as a minor. Hence T_f^* exists and is a triad of M/Y. Thus T_f^* is a union of cocircuits of $M/Y\backslash X'$. But $T_f^* \cap E(H) = \{f\}$, so H is a coloop; a contradiction. □

The hypotheses of the next lemma are satisfied, for example, when M is a rank-4 wheel and N is the restriction to its rim. In that case, $|E(M)| = 8$, and the lemma shows that this equation holds in general.

Lemma 3.6. *Let N be a 4-element circuit. Suppose that M is a minor-minimal 3-connected matroid having N as a minor. If there is a non-spanning circuit C of M such that $E(N) \subseteq C$, then $|E(M)| = 8$.*

Proof. Suppose that the lemma is not true and let (M, N) be a counterexample for which $|E(M)|$ is minimal. Let

$$U = \{e \in E(M) - E(N)\text{: } N \text{ is a minor for both } M\backslash e \text{ and } M/e\}.$$

Evidently, $E(M) - \mathrm{cl}(C) \subseteq U$. Let $\mathcal{T}$ be the set of triangles of M such that $T - \mathrm{cl}(C)$ is non-empty. Since $\mathrm{cl}(C) \neq E(M)$, there is a cocircuit D^* of M avoiding $\mathrm{cl}(C)$. Moreover, for all e in D^*, the matroid M/e has N as a minor. Thus M/e is not 3-connected. Therefore, by a result of Lemos [6], M has at least two triangles meeting D^*. Thus $|\mathcal{T}| \geq 2$.

Suppose that $T \in \mathcal{T}$. As $T - \mathrm{cl}(C) \subseteq U$, it follows by Lemma 3.5 that each element e of $T - \mathrm{cl}(C)$ is in a triad T_e^* such that $T_e^* - e \subseteq E(N)$. Hence, as $|T - \mathrm{cl}(C)| \geq 2$, the matroid M has two different triads T_{1T}^* and T_{2T}^* such that $T_{1T}^* \cap (T - \mathrm{cl}(C))$ and $T_{2T}^* \cap (T - \mathrm{cl}(C))$ are distinct single-element sets and $|T_{iT}^* \cap E(N)| = 2$ for each i. Hence T_{1T}^*, T, T_{2T}^* is a fan whose rim R_T is contained in $E(N)$. In particular,

$$|T \cap E(N)| = 1.$$

Now choose T' to be a member of $\mathcal{T} - \{T\}$. Since N does not contain a triangle, $R_T \neq R_{T'}$. Thus $|R_T \cup R_{T'}| \geq 4$. Since $R_T \cup R_{T'} \subseteq E(N)$ and $|E(N)| = 4$, we deduce that $R_T \cup R_{T'} = E(N)$. Moreover, each element of $E(N)$ is in one of $T_{1T}^*, T_{2T}^*, T_{1T'}^*$, or $T_{2T'}^*$. Thus $E(N)$ is contained in a series class S of $M \mid \mathrm{cl}(C)$.

We show next that $\mathrm{cl}(C) = C$. Suppose that $e \in \mathrm{cl}(C) - C$. Then M has a circuit C' such that $e \in C' \subseteq C \cup e$. Moreover, by circuit elimination and orthogonality, we may choose C' so that $C' \cap S = \emptyset$. Thus $C' \subseteq E(M) - E(N)$, so e is a loop of $M/(C - E(N))$. This contradicts the last part of Lemma 3.5. Therefore e does not exist and we conclude that $\mathrm{cl}(C) = C$.

Clearly either

(i) $T \cap T' \neq \emptyset$, or
(ii) $T \cap T' = \emptyset$.

Consider (i). Without loss of generality, we may suppose that $T' \cap T_{2T}^* \neq \emptyset$ and that $T_{2T}^* = T_{1T'}^*$. As N does not contain a triangle, it follows that $T_{1T}^*, T, T_{2T}^*, T', T_{2T'}^*$ is a fan of M. The rim of this fan is $R_T \cup R_{T'}$, which equals $E(N)$.

Now suppose that $f \in U - (T \cup T')$. Then, by Lemma 3.5, f belongs to a triad T_f^* of M such that $T_f^* - f \subseteq E(N)$. But orthogonality implies that $T_f^* - f$ avoids $(T \cup T') \cap E(N)$. Thus $T_f^* \cap E(N) = E(N) - (T \cup T')$. Assume that $|U - (T \cup T')| \geq 2$ and let f and g be distinct elements of $U - (T \cup T')$. Then $T_f^* \cap E(N) = T_g^* \cap E(N)$.

Hence $\{f, g\} \cup (E(N) - (T \cup T'))$ is a 4-point line in M^*, so $M^* \backslash f$ is 3-connected. Therefore M/f is 3-connected and, since $f \in U$, the matroid M/f has N as a minor. Thus the choice of M is contradicted. Hence $|U - (T \cup T')| \leq 1$.

Consider $M \mid (C \cup T \cup T')$. It has $C - (T \cup T')$ as a non-trivial series class. Thus $C \cup T \cup T' \neq E(M)$. Since $C \cup U = E(M)$, we deduce that $|U - (T \cup T')| \geq 1$. Hence $|U - (T \cup T')| = 1$. Take e in $U - (T \cup T')$. Then $M \mid (C \cup T \cup T') = M \backslash e$ and $C - (T \cup T')$ is a series class of this matroid. Thus $e \cup (C - (T \cup T'))$ is a line of M^*. Now $M^* \backslash e$ is not 3-connected otherwise M/e contradicts the choice of M. Thus $|C - (T \cup T')| = 2$ and we conclude that $|E(M)| = 8$. This completes the proof in case (i).

Now assume that (ii) holds. By orthogonality, we must have that $R_T - T = R_{T'} - T'$. We show next that $E(M) - C$ is a cocircuit of M. Assume the contrary. Then, as $\mathrm{cl}(C) = C$ and $|T - \mathrm{cl}(C)| = 2$, it follows by cocircuit elimination that M has a cocircuit D^* that is contained in $E(M) - C$ and avoids T. By applying Lemos's result [6] again, we get that D^* meets two triangles of M, one of which, say T'', must be different from T'. Hence $T'' \in \mathcal{T}$ so $|T'' \cap E(N)| = 1$ and $|T'' - C| = 2$. As $T'' - C$ meets D^*, it follows by orthogonality that $T'' - C \subseteq D^*$ so $T \cap (T'' - \mathrm{cl}(C)) = \emptyset$. Therefore, by the orthogonality of T'' with each of T^*_{1T} and T^*_{2T}, we deduce that $T'' \cap R_T = \emptyset$, otherwise $T'' = R_T$. Since $|E(N)| = 4$, it follows that the unique element of $E(N) - R_T$ is in both T' and T''. As $E(N)$ does not contain a triangle, it follows by comparing the fans containing T' and T'' that $T' = T''$; a contradiction. We conclude that $E(M) - C$ is a cocircuit of M. We call this cocircuit C^*.

Next we show that $|C^*| = 4$. By cocircuit elimination, $(T^*_{1T} \cup T^*_{2T}) - (T \cap E(N))$ contains a cocircuit D^*_T of M which, by orthogonality with both T and C, must equal $(T^*_{1T} \cup T^*_{2T}) - (T \cap E(N))$. Take $e \in R_T - T$. Then $e \in D^*_T \cup D^*_{T'}$, so $(D^*_T \cup D^*_{T'}) - e$ contains a cocircuit of M. This cocircuit contains at most one element of C and must therefore equal C^*. Hence $C^* = (T \cup T') - C$, so $|C^*| = 4$.

Let $M' = M/(C - E(N))$. Then M' is connected by Lemma 3.5. We show next that M' is 3-connected. Evidently, $E(N)$ is a circuit and C^* is a cocircuit of M'. As each of T^*_{1T}, T^*_{2T}, $T^*_{1T'}$, and $T^*_{2T'}$ is contained in $E(M')$, each of these triads of M is also a triad of M'. Moreover, each of these triads contains a single element of C^* and these elements are distinct. Thus M' has no 2-circuit meeting C^*. Furthermore, since $E(N)$ is a circuit of M', there is no 2-circuit of M' contained in $E(N)$. Thus M' is simple. But M' is also cosimple since M is cosimple. Hence M' has no trivial 2-separations. Let $\{X, Y\}$ be a non-trivial 2-separation of M'. We may assume that $|X \cap T| \geq 2$ and that X is closed in both M' and $(M')^*$. Thus X contains T and hence it contains T^*_{1T} and T^*_{2T}. Therefore X contains $E(N)$, so $|X| \geq 6$ and $|Y| \leq 2$; a contradiction. We conclude that M' is 3-connected. Since M' has N as a minor, it follows that $M' = M$, so $|E(M)| = |E(M')| = 8$. □

4. The connected case

In this section, we prove the main result in the case that N is connected. We also show that the bound in this case is sharp. In particular, we prove the following result.

Theorem 4.1. *Let N be a non-empty connected matroid. If M is a minor-minimal 3-connected matroid having N as a minor, then*

$$|E(M)| - |E(N)| \leq 5(\lambda_2(N) - 1).$$

Proof. Suppose the theorem fails and choose a counterexample (M, N) which is minimal with respect to the lexicographic order on $(|E(M)|, -|E(N)|)$. Observe that N is not 3-connected. In particular, $\lambda_2(N) \geq 2$. Thus

$$|E(N)| \geq 4. \tag{4}$$

Moreover, since (M, N) is a counterexample to the theorem, it follows that $|E(M)| > |E(N)| + 5(\lambda_2(N) - 1)$. Therefore

$$|E(M)| \geq 10. \tag{5}$$

4.1.1. *Let $\{X_1, X_2\}$ be a 2-separation of N and let N' be a minor of M that is minor-minimal having N as a minor and satisfying $k_{N'}(X_1, X_2) > 1$. Then $T(N')$ is a 3-vertex path with central vertex H such that*

(i) $|E(H) \cap E(M)| = |E(N') - E(N)| = 1$;
(ii) *both neighbours of H meet both X_1 and X_2; and*
(iii) *either*

  (a) *H is a triangle whose two neighbours in $T(N')$ are cocircuits, and N is a cocircuit; or*
  (b) *H is a triad whose two neighbours in $T(N')$ are circuits, and N is a circuit.*

Proof. Observe that N' is connected. Moreover, since $k_N(X_1, X_2) = 1$, it follows that $N' \neq N$. By Lemma 3.3,

$$|E(N')| - |E(N)| \leq 5. \tag{6}$$

By the choice of (M, N), the theorem holds for the pair (M, N'). Hence

$$|E(M)| - |E(N')| \leq 5(\lambda_2(N') - 1).$$

Substituting from (6) into the last inequality, we obtain

$$|E(M)| - |E(N)| \leq 5\lambda_2(N').$$

Since the theorem fails for the pair (M, N), we have that

$$\lambda_2(N') \geq \lambda_2(N). \tag{7}$$

Thus $\lambda_2(N') \geq 2$ and, as N is connected, each member of $\Lambda_2(N')$ has at least three elements. Moreover, by Lemma 3.4 and by switching to the dual if necessary, we may assume that there is a vertex H of $T(N')$ that is labelled by a triad and has exactly two neighbours, each of which labels a circuit such that both X_1 and X_2 meet all four of the sets displayed by the edges incident with H. Since the 2-separation $\{X_1, X_2\}$ was arbitrary, we deduce that every 2-separation of N is displayed by a vertex but not by an edge of $T(N)$. Thus $T(N)$ has no edges, so N is a circuit or a cocircuit and the lemma follows. □

We now know that N is a circuit or a cocircuit. Let $N = M\backslash X/Y$. Next we establish the following:

4.1.2. *The sets X and Y are not unique.*

Proof. Suppose that X and Y are unique. Let $\{X_1, X_2\}$ be a 2-separation of N. Then, by (4.1.1), M has a minor N_1' having an element e_1 such that $N_1'\backslash e_1 = N$. Moreover, $T(N_1')$ is a 3-vertex path in which the central vertex is labelled by a triad containing e_1 and the other two vertices are labelled by circuits C_{11} and C_{12} where each C_{1i} meets each X_j. Hence $|C_{1i} \cap E(N)| \geq 2$ for each i. Now consider the 2-separation $\{C_{11} \cap E(N), C_{12} \cap E(N)\}$ of N. Again, M has a minor N_2' having an element e_2 such that $N_2'\backslash e_2 = N$. Moreover, $T(N_2')$ is a 3-vertex path in which the central vertex is labelled by a triad containing e_2 and the other two vertices are labelled by circuits C_{21} and C_{22} each of which meets both $C_{11} \cap E(N)$ and $C_{12} \cap E(N)$. By the uniqueness of X and Y, both e_1 and e_2 are in X. Let $M' = M\backslash(X - \{e_1, e_2\})/Y$. Then $E(N)$ is a circuit of M', and M' has corank 3. Now $M'\backslash e_2 = N_1'$ and $M'\backslash e_1 = N_2'$, and it is straightforward to check that the dual of M' is a matroid in which $\{e_1\}$, $\{e_2\}$, and $\{e_1, e_2\}$ are flats and for which the simplification is isomorphic to $M(K_4)$. Since $M'\backslash\{e_1, e_2\} \cong N$, it is easily checked that $\lambda_2(M') = \lambda_2(N) - 1$. Now $|E(M')| > |E(N)|$, so (M, M') is not a counterexample to the theorem. Hence

$$\begin{aligned}|E(M)| - (|E(N)| + 2) &= |E(M)| - |E(M')| \\ &\leq 5(\lambda_2(M') - 1) \leq 5(\lambda_2(N) - 1) - 5.\end{aligned}$$

Therefore $|E(M)| - |E(N)| \leq 5(\lambda_2(N) - 1)$; a contradiction. Thus (4.1.2) holds. □

Let

$$U = \{e \in E(M) - E(N): N \text{ is a minor of both } M\backslash e \text{ and } M/e\}.$$

By (4.1.2), $U \neq \emptyset$. Choose $e \in U$. Since $|E(N)| \geq 4$ and M/e has N as a minor, $\text{si}(M/e)$ has N as a minor so $\text{si}(M/e)$ is not 3-connected. Therefore, by a result of Bixby [2], $\text{co}(M\backslash e)$ is 3-connected and every non-trivial series class of $M\backslash e$ has exactly two elements. Since $M\backslash e$ is not 3-connected, the set $\{T_1^*, T_2^*, \ldots, T_n^*\}$ of triads of M containing e is non-empty. Moreover, $T_1^* - e, T_2^* - e, \ldots,$ and $T_n^* - e$ are pairwise disjoint.

Next, we prove that

4.1.3. *$T_i^* - e \subseteq E(N)$ for all i in $\{1, 2, \ldots, n\}$.*

Proof. Suppose that $f \in T_i^* - (e \cup E(N))$ for some i in $\{1, 2, \ldots, n\}$, say $i = 1$. Then $n \geq 2$, otherwise $M\backslash e/f$ is 3-connected having N as a minor, and the choice of M is contradicted. The choice of M also implies that M/f is not 3-connected. By applying the dual of Tutte's triangle lemma [16] (or [11, Lemma 8.4.9]) to T_1^*, we get that, for each x in $\{e, f\}$, there is a triangle T_x of M such that $x \in T_x$ and $|T_x \cap T_1^*| = 2$. Suppose that $T_e \neq T_f$. Then T_f, T_1^*, T_e, T_2^* is a fan in M, so $\text{si}(M/f) \cong M/f\backslash(T_1^* - \{e, f\})$ and thus $\text{si}(M/f)$ is 3-connected and its elements can be chosen so that it has N as a minor. This contradiction to the choice of M implies that $T_e = T_f$. In this case, T_1^*, T_e, T_2^* is a fan in M and $n = 2$. Now switch attention to M^*. Let g be the unique element of $T_e - \{e, f\}$ and, for each i in $\{1, 2\}$, let e_i be the unique element of $T_i^* - T_e$. Since $\text{co}(M^*\backslash e)$ is not

3-connected, it follows, by a result of Akkari and Oxley [1], that M^* has a triangle that meets $\{e, f, g\}$ in $\{f, g\}$. Let the third element of this triangle be e_3. Then Akkari and Oxley's result implies that $\{e_1, e_2, e_3\}$ is a triangle Δ of M^* and $M^* = P_\Delta(M(K_4), M^*\backslash T_e)$ where the triangles of $M(K_4)$ other than Δ are T_1^*, T_2^*, and $\{f, g, e_3\}$, and $M^*\backslash T_e$ is 3-connected. Now, by (5), $|E(M)| \geq 10$. Hence, for all i, the matroid $\mathrm{si}(M^*/e_i)$ is not 3-connected. Thus $\mathrm{co}(M^*\backslash e_i)$ is 3-connected for all i and, as no e_i can be in a triad of M^*, it follows that $M^*\backslash e_i$ is 3-connected for all i. Now N^* is a cocircuit and it is a minor of M^*/e. Thus, if $e_1 \notin E(N)$, then $M^*/e\backslash e_1$ has N^* as a minor. Hence $M^*\backslash e_1$ has N^* as a minor, a contradiction to the choice of M^*. Therefore $e_1 \in E(N)$ and, by symmetry, $e_2 \in E(N)$. Moreover, $M^*\backslash e/f$ has N^* as a minor and therefore so does M^*/f. Thus, if $e_3 \notin E(N)$, then N is a minor of $M^*/f\backslash e_3$ and hence of $M^*\backslash e_3$. Therefore $e_3 \in E(N)$.

The matroid $M^*\backslash T_e$ is 3-connected and so, by the choice of (M, N), does not have N^* as a minor. Thus $g \in E(N)$. Clearly $M^*\backslash T_e$ has $N^*\backslash g$ as a minor, and $N^*\backslash g$ is connected since N^* is a cocircuit. Evidently, $|E(N^*\backslash g)| = |E(N)| - 1$ and $\lambda_2(N^*\backslash g) = \lambda_2(N) - 1$.

We now distinguish two cases:

(i) $|E(N^*\backslash g)| \geq 4$;
(ii) $|E(N^*\backslash g)| = 3$.

In case (i), choose M' to be a 3-connected minor of $M^*\backslash T_e$ that is minor-minimal having $N^*\backslash g$ as a minor. By the choice of (M, N), the theorem holds for $(M', N^*\backslash g)$ and so

$$|E(M')| - |E(N^*\backslash g)| \leq 5(\lambda_2(N^*\backslash g) - 1). \tag{8}$$

In case (ii), choose M' to be a 3-connected minor of $M^*\backslash T_e$ that is minor-minimal having $N^*\backslash g$ as a proper minor. Then, by Theorem 1.3,

$$|E(M')| - |E(N^*\backslash g)| \leq 3. \tag{9}$$

Now $M^* = P_\Delta(M(K_4), M^*\backslash T_e)$. Let $M' = (M^*\backslash T_e)\backslash X'/Y'$. The choice of M' ensures that M' has Δ as a triangle. Thus $M^*\backslash X'/Y' = P_\Delta(M(K_4), M^*\backslash T_e\backslash X'/Y')$. As M' is 3-connected, so is $M^*\backslash X'/Y'$. Moreover, as $N^*\backslash g$ is a minor of M', it follows that N^* is a minor of $M^*\backslash X'/Y'$. We deduce, by the choice of M^*, that $X' = Y' = \emptyset$. Hence $M' = M^*\backslash T_e$. Therefore, in case (i), by (8),

$$|E(M^*\backslash T_e)| - |E(N^*\backslash g)| \leq 5(\lambda_2(N^*\backslash g) - 1).$$

Thus $(|E(M)| - 3) - (|E(N)| - 1) \leq 5((\lambda_2(N) - 1) - 1)$, so $|E(M)| - |E(N)| \leq 5(\lambda_2(N) - 1) - 3$; a contradiction. In case (ii), by (9), $(|E(M)| - 3) - (|E(N)| - 1) \leq 3$, so $|E(M)| - |E(N)| \leq 5$. But $|E(N)| = 4$, so $\lambda_2(N) = 2$, and so $|E(M)| - |E(N)| \leq 5(\lambda_2(N) - 1)$. This contradiction completes the proof of (4.1.3). $\square$

Now let $T_i^* = \{e, a_i, b_i\}$ for all i. We define $M' = M\backslash e/\{a_1, a_2, \ldots, a_n\}$ and $N' = N/\{a_1, a_2, \ldots, a_n\}$. Then $M' \cong \mathrm{co}(M\backslash e)$, so M' is 3-connected.

We show next that

4.1.4. $|E(N')| \leq 3$.

Proof. Assume that $|E(N')| \geq 4$. Let M'' be a 3-connected minor of M' that is minor-minimal having N' as a minor. By the choice of (M, N), we have that

$$|E(M'')| - |E(N')| \leq 5(\lambda_2(N') - 1). \tag{10}$$

Suppose that $M'' = M' \backslash X'/Y'$. Then $M'' = M\backslash(X' \cup e)/Y'/\{a_1, a_2, \ldots, a_n\}$. The matroid $M\backslash(X' \cup e)/Y'$ can be obtained from the 3-connected matroid M'' by adding a_i in series with b_i for all i. Thus $\lambda_2(M\backslash(X' \cup e)/Y') = n+1$. Now $E(M\backslash(X' \cup e)/Y') \supseteq E(N)$ so, in the lexicographic order,

$$(|E(M)|, -|E(M\backslash(X' \cup e)/Y')|) \leq (|E(M)|, -|E(N)|).$$

But, by (4.1.1), in every lexicographically minimal counterexample (M, N) to the theorem, the second coordinate is a circuit or a cocircuit. As $M\backslash(X' \cup e)/Y'$ is not a circuit or cocircuit, it follows that the theorem holds for $(M, M\backslash(X' \cup e)/Y')$. Hence

$$|E(M)| - |E(M\backslash(X' \cup e)/Y')| \leq 5(\lambda_2(M\backslash(X' \cup e)/Y') - 1) = 5n.$$

We also have that $|E(M\backslash(X' \cup e)/Y')| = |E(M'')| + n$ and so $|E(M)| - |E(M'')| \leq 6n$. Adding the last inequality to (10), we get that $|E(M)| - |E(N')| \leq 6n + 5(\lambda_2(N') - 1)$. As $|E(N')| = |E(N)| - n$, we obtain $|E(M)| - |E(N)| \leq 5(\lambda_2(N') + n - 1)$. But $\lambda_2(N) = \lambda_2(N') + n$ and so we obtain a contradiction. We conclude that (4.1.4) holds. □

Now N' is a circuit and, since $|E(N)| \geq 4$, the construction of N' implies that $|E(N')| \geq 2$. Then, by (4.1.4), $|E(N')| \in \{2, 3\}$. If $M' = N'$, then, as $\lambda_2(N) \geq 2$, we have

$$|E(M)| - |E(N)| = 1 \leq 5(\lambda_2(N) - 1);$$

a contradiction. Thus $M' \neq N'$. Next we define a matroid M''. If $|E(N')| = 3$, then, by Theorem 1.3, M' has a 3-connected minor M'' such that $|E(M'')| - |E(N')| \leq 3$ and M'' is minor-minimal having N' as a proper minor. If $|E(N')| = 2$, then M' is a loopless extension of N', so M' has a minor M'' isomorphic to $U_{1,3}$ such that $E(N') \subseteq E(M'')$.

Suppose that $M'' = M' \backslash X'/Y'$ where Y' is chosen so that $|Y'|$ is maximal. Then $M'' = [M\backslash(X' \cup e)/Y']/\{a_1, a_2, \ldots, a_n\}$. Let $M''' = M\backslash(X' \cup e)/Y'$. It is obtained from the 3-connected matroid M'' by adding a_i in series with b_i for all i. Thus, the only 2-separations of M''' are $\{\{a_i, b_i\}, E(M''') - \{a_i, b_i\}\}$ for all i.

We show next that

4.1.5. $n = 1$.

Proof. Assume that $n \geq 2$. We show first that e is not a coloop in $M\backslash X'/Y'$. Assume the contrary. Then M has a cocircuit C^* such that $e \in C^* \subseteq X' \cup e$. Take $f \in C^* - e$. Then $\{e, f\}$ is a union of cocircuits of $M\backslash(X' - f)/Y'$. Thus f is a coloop of $M\backslash[(X' - f) \cup e]/Y'$ so $M\backslash(X' \cup e)/Y' = M\backslash[(X' - f) \cup e]/(Y' \cup f)$. Therefore $M' \backslash X'/Y' = M\backslash(X' - f)/(Y' \cup f)$; a contradiction to the choice of Y'. Hence e is not a coloop in $M\backslash X'/Y'$. As $\{e, a_i, b_i\}$ is a cocircuit of M for each i, it is a union of cocircuits of $M\backslash X'/Y'$. As e is not a coloop of $M\backslash X'/Y'$, this matroid is connected. Moreover, by orthogonality, since $n \geq 2$, there is no 2-circuit in $M\backslash X'/Y'$ containing e.

We show next that $M\backslash X'/Y'$ is 3-connected. If not, it has a 2-separation $\{J \cup e, K\}$ where $e \notin J$ and $|J| \geq 2$. Then $\{J, K\}$ is a 2-separation of $M\backslash(X' \cup e)/Y'$ so J or K is $\{a_i, b_i\}$ for some i. In each case, e is in a circuit of $M\backslash X'/Y'$ that meets some $\{e, a_j, b_j\}$ in a single element. This contradiction to orthogonality implies that $M\backslash X'/Y'$ is indeed 3-connected. Since the last matroid has N as a minor, the choice of M implies that $X' = Y' = \emptyset$. Therefore $M'' = M'$ and so

$$\begin{aligned}|E(M)| - |E(N)| &= (1 + |E(M'')| + n) - (|E(N')| + n)\\ &= |E(M'')| - |E(N')| + 1 \leq 4.\end{aligned}$$

Since $\lambda_2(N) \geq 2$, we deduce that $|E(M)| - |E(N)| \leq 5(\lambda_2(N) - 1)$; a contradiction. We conclude that $n = 1$. □

On combining (4.1.4) and (4.1.5) with the fact that $|E(N)| \geq 4$, we deduce that

$$|E(N)| = 4 \qquad (11)$$

and that M''' has just one 2-separation, which is induced by $\{a_1, b_1\}$. We relabel the cocircuit $\{e, a_1, b_1\}$ by T_e^*. As e was chosen arbitrarily in U, it follows that T_e^* is defined for every element e of U.

4.1.6. *There is a spanning circuit D of M such that $E(M) - D$ is a 3-element subset of U whose elements can be labelled by f, g, and h such that $T_g^* \cap T_h^* = \emptyset$ and T_f^* meets each of T_g^* and T_h^* in exactly one element.*

Proof. Since M''' is neither a circuit nor a cocircuit, the theorem holds for the pair (M, M''') so $|E(M)| - |E(M''')| \leq 5$. Now, by (11), $|E(N)| = 4$, so

$$|E(M''')| - |E(N)| = |E(M'')| - |E(N')| \leq 3.$$

It follows that $|E(M)| - |E(M''')| \geq 3$ because

$$\begin{aligned}5 = 5(\lambda_2(N) - 1) < |E(M)| - |E(N)| &= (|E(M)| - |E(M''')|)\\ &\quad +(|E(M''')| - |E(N)|).\end{aligned}$$

We are now going to apply Lemma 3.2 to the exact 2-separation $\{T_e^* - e, E(M''') - T_e^*\}$ of M'''. Evidently M has M''' as a minor and, as M is 3-connected, $k_M(T_e^* - e, E(M''') - T_e^*) > 1$. Now let M_1 be a minor of M that is minor-minimal having M''' as a minor and satisfying $k_{M_1}(T_e^* - e, E(M''') - T_e^*) > 1$. Assume that $M_1 \neq M$. Then, since M_1 has N as a minor, the choice of M implies that M_1 is not 3-connected. Thus, as $\lambda_2(M''') = 2$, we deduce that $\lambda_2(M_1) \geq \lambda_2(M''')$. Now $T(M''')$ has two vertices, one a triangle and the other isomorphic to M''. But, by Lemma 3.4, $T(M_1)$ has at least three vertices including a triangle or triad H that contains the unique element x of $E(M_1) - E(M''')$. Thus $T(M''')$, which is $T(M_1/x)$ if H is a triangle and is $T(M_1\backslash x)$ if H is a triad, has a vertex corresponding to a circuit or a cocircuit with at least four elements. This contradiction implies that $M_1 = M$.

We now know that M is minor-minimal having M''' as a minor and satisfying $k_M(T_e^* - e, E(M''') - T_e^*) > 1$. Then, by Lemma 3.2, there are unique sets P and Q such that $M''' = M\backslash P/Q$ and the elements of $P \cup Q$ can be labelled $z_1, z_2, \ldots, z_m$ such that these elements are alternately in P and Q. Now, by Lemma 3.2(iii), $X_{z_1} = T_e^* - e$.

Moreover, $\{X_{z_1}, Y_{z_1}\}$ is a 2-separation of the 2-connected matroid M_{z_1}. Thus X_{z_1} is a 2-circuit or a 2-cocircuit of M_{z_1}. As M''' is a minor of M_{z_1} and X_{z_1} is a cocircuit of M''', we deduce that X_{z_1} is a 2-cocircuit of M_{z_1}. Since X_{z_1} is not a 2-cocircuit of M, it follows that $z_1 \in P$ and $X_{z_1} \cup z_1$ is a triad of M. Therefore $e = z_1$, otherwise $M^* \mid (T_e^* \cup z_1) \cong U_{2,4}$, so M/e is 3-connected; a contradiction.

Now $X_{z_2} = T_e^*$. Since $\{X_{z_2}, Y_{z_2}\}$ is a 2-separation of M/z_2 but $\{X_{z_2}, Y_{z_2} \cup z_2\}$ is not a 2-separation of M, we deduce that T_e^* spans z_2 in M. Thus M has a circuit C such that $z_2 \in C \subseteq T_e^* \cup z_2$. If $e \notin C$, then C is a 3-element set that contains a circuit and a cocircuit of $M \backslash e$, so $\text{co}(M \backslash e)$ is not 3-connected; a contradiction. Thus $e \in C$.

Since $e \in U$, it follows that N is a minor of M/e. As $T_e^* - e \subseteq E(N)$ by (4.1.3), and $T_e^* - e$ spans z_2 in M/e, we must delete z_2 from M/e to obtain N. Thus

$$N \text{ is a minor of } M/e\backslash z_2. \tag{12}$$

Since $z_2 \in Q$, we deduce that $z_2 \in U$ and so $T_{z_2}^*$ exists. By orthogonality, $T_{z_2}^* \cap C \neq \{z_2\}$, so $T_{z_2}^* \cap T_e^*$ contains an element of C. Now $T_{z_2}^* - z_2 \neq T_e^* - e$, otherwise $\{T_{z_2}^* \cup T_e^*, E(M) - (T_{z_2}^* \cup T_e^*)\}$ is a 2-separation of M. Thus

$$|T_{z_2}^* \cap T_e^*| = 1 \qquad \text{and} \qquad T_{z_2}^* \cap T_e^* \subseteq C. \tag{13}$$

Hence $(T_{z_2}^* - z_2) \cup (T_e^* - e)$ is a 3-element subset of $E(N)$ that is a union of cocircuits of $M \backslash \{e, z_2\}$. This 3-element set must be contained in a series class S of $M \backslash \{e, z_2\}$, otherwise it is a union of coloops of $M \backslash \{e, z_2\}$ so $M^* \mid (T_e^* \cup T_{z_2}^*) \cong U_{2,5}$ and we obtain the contradiction that M/e is 3-connected. We deduce that $M \backslash \{e, z_2\}$ has a circuit D that contains S and an element of $E(N) - S$. But $|E(N)| = 4$, so $E(N) \subseteq D$. By (5), $|E(M)| \geq 10$. Therefore, by Lemma 3.6, D is a spanning circuit of M.

Next we show that

$$|U| \geq 3. \tag{14}$$

Now N is a minor of $M \backslash e/z_2$, so $N = M \backslash e/z_2 \backslash I^*/I$, where I is independent and I^* is coindependent in $M \backslash e/z_2$. Thus $r(N) = r(M \backslash e/z_2) - |I| = r(M) - 1 - |I|$. But N is a 4-element circuit, so $r(N) = 3$. Hence

$$|I| = r(M) - 4.$$

Suppose that $I^* \cap D = \emptyset$. Then $D - E(N) \subseteq I$. But

$$|D - E(N)| = |D| - |E(N)| = (r(M) + 1) - 4 = r(M) - 3.$$

This contradiction implies that $I^* \cap D \neq \emptyset$. Thus if $f \in I^* \cap D$, then N is a minor of $M \backslash f$. But, since f is in the circuit D and N is also a circuit, it follows that N is a minor of M/f. Thus $f \in U$. Since $f \notin \{e, z_2\}$, we deduce that $|U| \geq 3$.

Choose e' in $U - \{e, z_2\}$ such that, if possible, $e' \notin D$. Next we show that

$$T_{e'}^* \cap C = \emptyset. \tag{15}$$

Suppose not. Since $e' \notin T_e^* \cup z_2$ and $C \subseteq T_e^* \cup z_2$, we have, by orthogonality, that $T_{e'}^* \cap C = T_{e'}^* - e'$. Now $e, z_2 \notin T_{e'}^*$, so $T_{e'}^* - e' = T_{e'}^* \cap C \subseteq C - e \subseteq T_e^* - e$. Thus $M^* \mid (T_e^* \cup T_{e'}^*) \cong U_{2,4}$ and so M/e is 3-connected; a contradiction. Hence (15) holds.

Now, using e' in place of e in the argument above, we deduce that U contains an element z_2' such that $T^*_{z_2'} \cap T^*_{e'} \neq \emptyset$ and M has a circuit C' such that $\{z_2', e'\} \subseteq C' \subseteq T^*_{e'} \cup z_2'$. Thus $|C' \cap E(N)| \leq 2$ and so, as $|D \cap E(N)| = |E(N)| = 4$, we deduce that $C' \nsubseteq D$. Moreover, $C' - \{e', z_2'\} \subseteq E(N) \subseteq D$. Thus $e' \notin D$ or $z_2' \notin D$. By the choice of e', we deduce that $e' \notin D$. Hence $D \cap \{e, z_2, e'\} = \emptyset$. On combining (13) and (15), we deduce that

$$T^*_e \cap T^*_{e'} \cap T^*_{z_2} = \emptyset. \tag{16}$$

Next, we show that

$$T^*_e \cap T^*_{e'} = \emptyset \qquad \text{or} \qquad T^*_{z_2} \cap T^*_{e'} = \emptyset. \tag{17}$$

Assume that (17) is false. We know that $(T^*_e - e) \cup (T^*_{z_2} - z_2)$ is contained in a series class S of $M\backslash\{e, z_2\}$. Now $|(T^*_e - e) \cup (T^*_{z_2} - z_2)| = 3$ and $|(T^*_e - e) \cap (T^*_{z_2} - z_2)| = 1$. Since $T^*_{e'} - e'$ must meet both $T^*_e - e$ and $T^*_{z_2} - z_2$ but, by (16), $T^*_{e'} - e'$ avoids $(T^*_e - e) \cap (T^*_{z_2} - z_2)$, we deduce that $T^*_{e'} - e' \subseteq (T^*_e - e) \cup (T^*_{z_2} - z_2)$. Thus either e' is a coloop of $M\backslash\{e, z_2\}$, or $e' \in S$. In the former case, $\{e', e, z_2\}$ is a triad of M that avoids the spanning circuit D; a contradiction. Hence $e' \in S$. Thus, in $M\backslash\{e, z_2\}\backslash e'$, the elements of $(T^*_e - e) \cup (T^*_{z_2} - z_2)$ are coloops. But these coloops are contained in D, which is a circuit of $M\backslash\{e, z_2\}\backslash e'$; a contradiction. We conclude that (17) holds.

To complete the proof of (4.1.6), we shall show that $E(M) - D \subseteq \{e', e, z_2\}$. Now $D \cap \{e', e, z_2\} = \emptyset$ and D is a spanning circuit of M. Since each of $T^*_{e'} - e'$, $T^*_e - e$, and $T^*_{z_2} - z_2$ is a union of cocircuits of $M\backslash\{e', e, z_2\}$ contained in D, we deduce that each of $T^*_{e'} - e'$, $T^*_e - e$, and $T^*_{z_2} - z_2$ is a cocircuit of $M\backslash\{e', e, z_2\}$. By (17), two of these 2-cocircuits are disjoint and so their union is $E(N)$. Since $|(T^*_e - e) \cap (T^*_{z_2} - z_2)| = 1$, it follows that the third 2-cocircuit meets the other two. Thus $E(N)$ is contained in a series class of $M\backslash\{e', e, z_2\}$. Suppose $f \in E(M) - D - \{e', e, z_2\}$. Then, since D is spanning in M, there is a circuit of $M\backslash\{e', e, z_2\}$ that contains f. Moreover, this circuit may be chosen to avoid $E(N)$ since $E(N)$ is contained in a series class of $M\backslash\{e', e, z_2\}$ contained in D. We deduce that f is a loop of $M\backslash\{e', e, z_2\}/(D - E(N))$. Since $N = M/(D - E(N))\backslash(E(M) - D)$, Lemma 3.5 implies that f is a coloop of $M\backslash\{e', e, z_2\}/(D - E(N))$; a contradiction. We conclude that (4.1.6) holds. □

Let D be a spanning circuit of M whose existence is guaranteed by (4.1.6). Let $E(M) - D = \{f, g, h\}$ where $T^*_h \cap T^*_g = \emptyset$. Let $M' = M/(D - E(N))$. Then $(M')^*$ is a rank-4 matroid in which T^*_f is a triangle to which triangles T^*_g and T^*_h have been attached at different points via parallel connection. It follows that M' is connected, that $T^*_g - T^*_f$ and $T^*_h - T^*_f$ are disjoint 2-element parallel classes of M', and that M' has no other non-trivial parallel classes. Now let Z be a minimal subset of $D - E(M)$ such that $M/(D - (E(N) \cup Z))$ has no non-trivial parallel classes. It follows, since $M/(D - E(N))$ has exactly two non-trivial parallel classes each with exactly two elements, that $|Z| \leq 2$. Let $M'' = M/(D - (E(N) \cup Z))$. Then, for each x in $\{g, h\}$, the parallel class $T^*_x - T^*_f$ of $M/(D - E(N))$ is contained in a triangle T_x of M''.

We show next that M'' is 3-connected. By Lemma 3.5, since M'' clearly has no coloops, M'' is connected. Moreover, since $M/(D - E(N))$ is cosimple, so is M''; and, as $M/(D - E(N))$ has exactly two 2-circuits neither of which is a 2-circuit of M'', it follows

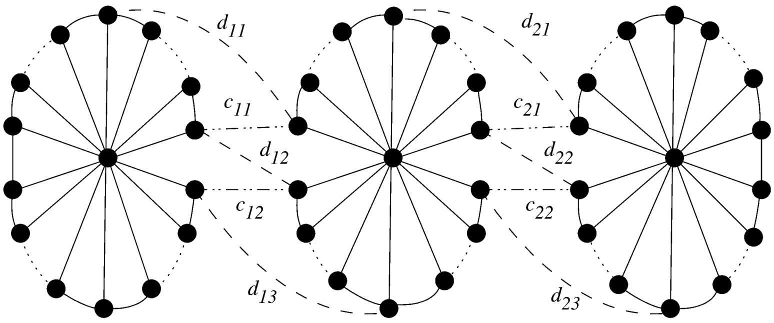

Fig. 1. An extremal example for Theorem 4.1.

that M'' is simple. We deduce that M'' has no non-trivial 2-separations. Let $\{X, Y\}$ be a 2-separation of M''. Then $\min\{|X|, |Y|\} \geq 3$. Now $E(M'') = T_f^* \cup T_g \cup T_h$. Evidently X or Y, say X, meets at least two of T_f^*, T_g, and T_h in at least two elements. By symmetry, we may assume that $|X \cap T_g| \geq 2$. We may also assume that X is closed in both M'' and $(M'')^*$. Thus $X \supseteq T_g$ and, since $|T_g \cap T_g^*| = 2$, it follows that $X \supseteq T_g^*$. Note that $|X \cap T_h| \not\geq 2$, otherwise $X \supseteq T_h \cup T_h^*$ and so $|Y| \leq 1$; a contradiction. Thus $|X \cap T_f^*| \geq 2$, so $X \supseteq T_f^*$. Hence $Y \subseteq T_h$ and, as $|Y| \geq 3$, it follows that $Y = T_h$ and that T_h and T_g are disjoint. Thus $r(M'') = 5$ and so $r(X) = r(M'') + 1 - r(Y) = 4$. Therefore T_h is a triad of M'' and hence of M. Thus $M^* \mid (T_h \cup T_h^*) \cong U_{2,4}$ so M/h is 3-connected; a contradiction. We conclude that M'' is indeed 3-connected.

Since M'' has N as a minor and is a 3-connected minor of M, it follows that $M'' = M$. But $|E(M'')| \leq 9$, whereas, by (5), $|E(M)| \geq 10$. This contradiction completes the proof of Theorem 4.1. $\square$

To conclude this section, we show that, for every integer n exceeding one, there are infinitely many matroids N that attain the bound of Theorem 4.1 such that $\lambda_2(N) = n$. In fact, our examples will show that the bound in Theorem 4.1 cannot be improved if we require only that M has a minor *isomorphic to*, rather than equal to, N. For each i in $\{1, 2, \ldots, n\}$, let G_i be isomorphic to a wheel for which the vertices of the rim are, in cyclic order, $v_{i1}, v_{i2}, \ldots, v_{i(4m+6)}$, where m is large, say $m = 100n$. Let G be formed from the vertex-disjoint union of $G_1, G_2, \ldots, G_n$ by, for all i in $\{1, 2, \ldots, n-1\}$, adding the edges $d_{i1}, d_{i2}, d_{i3}, c_{i1}, c_{i2}$ and deleting the edges $v_{i1}v_{i2}$ and $v_{(i+1)(2m+4)}v_{(i+1)(2m+5)}$, where $d_{i1} = v_{i(m+3)}v_{(i+1)(2m+4)}$, $c_{i1} = v_{i2}v_{(i+1)(2m+4)}$, $d_{i2} = v_{i2}v_{(i+1)(2m+5)}$, $c_{i2} = v_{i1}v_{(i+1)(2m+5)}$, and $d_{i3} = v_{i1}v_{(i+1)(3m+6)}$. Now take $M = M(G)$ and $N = M(H)$ where

$$H = G \backslash \cup_{i=1}^{n-1} \{d_{i1}, d_{i2}, d_{i3}\} / \cup_{i=1}^{n-1} \{c_{i1}, c_{i2}\}.$$

In the case $n = 3$, the graph G is illustrated in Fig. 1. We shall show that M is a minor-minimal 3-connected matroid having a minor isomorphic to N. A cocircuit in a connected matroid whose deletion leaves a connected matroid is called a *vertex cocircuit*. We observe that, in N, the edges meeting the hub of each wheel G_i form a vertex cocircuit with

$4m + 6$ elements. Moreover, for all i in $\{1, 2, \ldots, n - 1\}$, the two vertices that result from identifying the end vertices of c_{i1} and c_{i2} in G induce a 2-separation of N; and every 2-separation of N is of this type. To obtain a minor of M isomorphic to N, we must delete and contract a total of $5(n - 1)$ elements. Assume that $M\backslash D/C \cong N$. We shall show first that $E(M) - E(N) = C \cup D$. If a spoke s of one of the wheels G_i is in C, then $M(G/s)$ has a 2-separation such that, in the corresponding 2-sum, one of the two matroids is a series–parallel network with at least $m - 3$ elements. As $m = 100n$ and $|E(M) - E(N)| = 5(n - 1)$, it is not possible for $M(G/s)$ to have a minor isomorphic to N otherwise N has a disallowed 2-separation. Thus no spoke of any G_i is in C and, similarly, no rim element of any G_i is in D.

If some spoke s of one of the wheels G_i is in D, then $M(G\backslash s)$ has $n - 1$ vertex cocircuits of size $4m + 6$, one vertex cocircuit of size $4m + 5$, and all its remaining vertex cocircuits of size at most 5. But N has exactly n vertex cocircuits of size $4m + 6$. Since $m = 100n$ but $|E(M) - E(N)| = 5(n - 1)$, the structure of N means that the only way for N to obtain the required number of vertex cocircuits of size $4m + 6$ is by contracting a spoke of one of the wheels, which we have already ruled out. We deduce that none of the spokes of any G_i is in D. Hence none of the rim elements of any G_i is in C otherwise, since N is simple, D must contain a spoke adjacent to this rim element. We conclude that $C \cup D = E(M) - E(N)$.

Next we show that $D = \cup_{i=1}^{n-1}\{d_{i1}, d_{i2}, d_{i3}\}$ and $C = \cup_{i=1}^{n-1}\{c_{i1}, c_{i2}\}$. Consider the sequence $d_{i1}, c_{i1}, d_{i2}, c_{i2}, d_{i3}$. The deletion from M of two consecutive elements from this sequence leaves a matroid with a 2-separation one side of which corresponds to a series–parallel network with at least $2m$ elements. It follows that this matroid cannot have a minor isomorphic to N. Thus no two consecutive members of $d_{i1}, c_{i1}, d_{i2}, c_{i2}, d_{i3}$ are in D. Clearly D cannot contain four or more elements of $d_{i1}, c_{i1}, d_{i2}, c_{i2}, d_{i3}$. Thus D contains at most three such elements. From the structure of M, it follows that D is coindependent and C is independent in M. We deduce that $|C| = r(M)-r(N) = 2(n-1)$. Therefore C contains exactly two elements of each set $\{d_{i1}, c_{i1}, d_{i2}, c_{i2}, d_{i3}\}$ otherwise D contains at least four elements of one such set. Because no two consecutive elements of $d_{i1}, c_{i1}, d_{i2}, c_{i2}, d_{i3}$ are in D, we deduce that $\{d_{i1}, d_{i2}, d_{i3}\} \subseteq D$ and $\{c_{i1}, c_{i2}\} \subseteq C$. Thus $D = \cup_{i=1}^{n-1}\{d_{i1}, d_{i2}, d_{i3}\}$ and $C = \cup_{i=1}^{n-1}\{c_{i1}, c_{i2}\}$. Hence the only minor of M isomorphic to N is N itself. The deletion of any of d_{i1}, d_{i2}, and d_{i3} or the contraction of any of c_{i1} and c_{i2} from M produces a matroid that is not 3-connected and has no 3-connected minor having a minor isomorphic to N. Thus N is indeed minor-minimal having a minor isomorphic to N. We conclude that we cannot sharpen the bound in Theorem 4.1 even if we allow N to be replaced by an isomorphic copy.

5. A bound in general

In this section, we combine the main result of the last section with some extremal results for connected matroids to prove a bound on $|E(M)|$ in terms of $|E(N)|$ alone, when M is a minor-minimal 3-connected matroid having N as a minor. We begin by recalling an analogue of the main result for the case when we require only that M is connected [7].

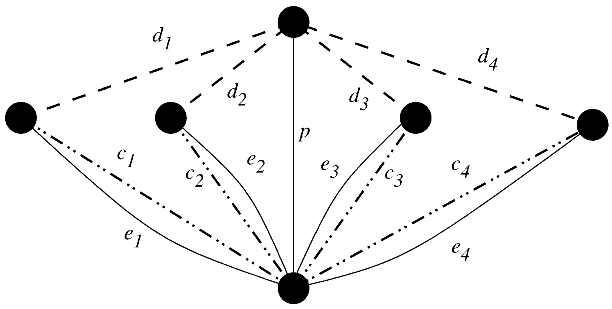

Fig. 2. An extremal example for Lemma 5.2.

Lemma 5.1. *Let N be a non-empty matroid and M be a minor-minimal connected matroid having N as a minor. Then*

$$|E(M) - E(N)| \leq 2\lambda_2(N) - 2.$$

Lemma 5.2. *Let N be a non-empty matroid. If M is a minor-minimal connected matroid having N as a minor, then*

$$|E(M)| \leq 3|E(N)| - 2.$$

Proof. The result follows immediately from Lemma 5.1 because $|E(N)| \geq \lambda_1(N) \geq 1$. $\square$

For all i in $\{1, 2, \ldots, n\}$, let M_i be the cycle matroid of the graph that is obtained from a triangle $\{p, d_i, c_i\}$ by adding an edge e_i in parallel with c_i. Let M be the parallel connection of $M_1, M_2, \ldots, M_n$ across the basepoint p and let $N = M\backslash\{d_1, d_2, \ldots, d_n\}/\{c_1, c_2, \ldots, c_n\}$. When $n = 4$, the matroid M is the cycle matroid of the graph in Fig. 2, where the edges to be deleted are dashed, while those to be contracted are dotted and dashed. Evidently N is the direct sum of n loops, $e_1, e_2, \ldots, e_n$, and one coloop p. Thus $|E(N)| = n + 1$. Moreover, $|E(M)| = 3n + 1 = 3|E(N)| - 2$. Thus M is an extremal example for the last lemma provided M is a minor-minimal connected matroid having N as a minor. But, in order to make e_i but not p a loop in a minor of M, we must delete d_i and contract c_i. Deleting d_i or contracting c_i from M produces a matroid that has a component contained in $\{e_i, c_i\}$ and so is disconnected. Hence M shows that the bound in Lemma 5.2 is sharp.

Lemma 5.3. *If M is a connected matroid such that $|E(M)| \geq 3$, then*

$$\lambda_2(M) \leq |E(M)| - 2.$$

Proof. We prove this result by induction on $|E(M)|$. If $|E(M)| = 3$, then M is isomorphic to $U_{1,3}$ or $U_{2,3}$ and the result follows. Suppose that $|E(M)| \geq 4$. The result also follows when M is 3-connected. Thus we may suppose that M is not 3-connected. Then there are matroids M_1 and M_2 such that $M = M_1 \oplus_2 M_2$. By induction, we have that

$$\lambda_2(M_i) \leq |E(M_i)| - 2,$$

for each i in $\{1, 2\}$. Observe that

$$\lambda_2(M) = \lambda_2(M_1) + \lambda_2(M_2) \leq |E(M_1)| + |E(M_2)| - 4.$$

The result follows because $|E(M)| = |E(M_1)| + |E(M_2)| - 2$. □

It is not difficult to show that the bound in the last lemma holds if and only if every vertex of $T(M)$ is a circuit or a cocircuit. Thus, in each of the last two lemmas, the bounds are sharp. By contrast, the bound in the next theorem seems far from best-possible.

Theorem 5.4. *Let N be a non-empty matroid. If M is a 3-connected minor-minimal matroid having N as a minor, then*

$$|E(M)| \leq 18|E(N)| - 27.$$

Proof. Let N' be a minor-minimal connected minor of M having N as a minor. By Lemmas 5.2 and 5.3,

$$\lambda_2(N') \leq |E(N')| - 2 \leq 3|E(N)| - 4.$$

By Theorem 4.1, we obtain

$$|E(M)| - |E(N')| \leq 5(\lambda_2(N') - 1) \leq 15|E(N)| - 25.$$

The result follows by Lemma 5.2. □

6. The proof of the main result

The main result was proved when N is connected in Section 4. In this section, we complete its proof in general. The main tool in the proof, apart from Theorem 4.1, is the next result.

Lemma 6.1. *Let N be a non-empty matroid and M be a minor-minimal connected matroid having N as a minor. Then*

$$\lambda_2(M) \leq 4(\lambda_1(N) - 1) + \lambda_2(N).$$

Proof. Suppose the theorem fails and choose a counterexample (M, N) which is minimal with respect to the lexicographic order on $(|E(M)|, -|E(N)|)$. Choose a minor N' of M such that N' is minor-minimal having N as a minor and satisfying $\lambda_1(N') < \lambda_1(N)$. By the choice of N', there is just one component H of N' such that H is not a component of N. Let X and Y be disjoint subsets of elements of H such that $N' \backslash X/Y = N$. If $z \in X \cup Y$, and N is a minor of both $N' \backslash z$ and N'/z, then, since $\lambda_1(N' \backslash z) = \lambda_1(N')$ or $\lambda_1(N'/z) = \lambda_1(N')$, the minimality of N' is contradicted. We deduce that the sets X and Y are unique. By taking the dual if necessary, we may assume that $X \neq \emptyset$. Choose an element e of X.

Now the matroid $H \backslash e$ is disconnected otherwise $\lambda_1(N' \backslash e) = \lambda_1(N')$ and $N' \backslash e$ contradicts the choice of N'. Thus the member of $\Lambda_2^u(H)$ containing e is a circuit C with at least three elements. If $(C \cap (X \cup Y)) - e$ is non-empty and f is in this set, then f is a coloop of $N' \backslash e$ contradicting the fact that the sets X and Y are unique. Therefore $C \cap (X \cup Y) = \{e\}$. Let $C - e = \{e_1, e_2, \ldots, e_k\}$ where $C \cap E(N) = \{e_{l+1}, e_{l+2}, \ldots, e_k\}$. Then $k \geq 2$. Each element of $C \cap E(N)$ is a coloop of N. In $T(H)$, the edges incident with the vertex corresponding to C are $e_1, e_2, \ldots, e_l$. One by one, contract the edges of $T(H)$ other than $e_1, e_2, \ldots, e_l$ and relabel the vertex that is obtained by contracting each edge g by the 2-sum of the matroids that previously labelled the ends of g. At the conclusion of this process, let H_i be the matroid different from C that labels an end of e_i. Since the end of e_i other than C is not labelled by a circuit in $T(N)$, the matroid $H_i \backslash e_i$ is connected. Thus the components of $H \backslash e$ are $H_1 \backslash e_1, H_2 \backslash e_2, \ldots, H_l \backslash e_l$ together with $k - l$ coloops on $e_{l+1}, e_{l+2}, \ldots, e_k$.

Each component of $H \backslash e$ must meet $E(N)$ for if there is such a component avoiding $E(N)$, then the sets X and Y are not unique. Moreover, each component of $H \backslash e$ contains the elements of just one component of N, otherwise $N' \backslash e$ has N as a minor but has fewer components than N. Thus

$$\lambda_1(N) = \lambda_1(N') + k - 1. \tag{18}$$

Recall that we were able to assume that $X \neq \emptyset$ by duality. Next we show that $|X| = 1$. Suppose that $f \in X - e$. Without loss of generality, we may assume that $f \in E(H_1) - e_1$. If $H_1 \backslash f$ is connected, then $H \backslash f$ is connected so $\lambda_1(N' \backslash f) = \lambda_1(N')$ and the choice of N' is contradicted. Thus $H_1 \backslash f$ is disconnected. Since the last matroid cannot have e_1 as a coloop because $H_1 \backslash e_1$ is connected, we deduce that $H_1 \backslash f, e_1$ is disconnected. Since $H_1 \backslash e_1$ contains elements from just one component of N, it follows that $H_1 \backslash f, e_1$ has a component avoiding $E(N)$; a contradiction to the fact that the sets X and Y are unique. We conclude that f does not exist, so $|X| = 1$. Since X was assumed to be non-empty by duality, we have actually established the following:

6.1.1. *If $X \neq \emptyset$, then $|X| = 1$.*

An immediate consequence of this is that:

6.1.2. *If $Y \neq \emptyset$, then $|Y| = 1$.*

Next we show that:

6.1.3. *If $X = \{e\}$ and $Y = \{f\}$, then $|C| = 3$ and the element of $\Lambda_2^u(H)$ containing f is a 3-element cocircuit C^*.*

We may assume that $f \in E(H_1) - e_1$. Then H_1/f is disconnected otherwise N'/f contradicts the choice of N'. Moreover, $H_1/f \backslash e_1$ is connected otherwise $H_1/f \backslash e_1$ has a component that avoids $E(N)$ and so the uniqueness of X and Y is contradicted. It follows that the element of $\Lambda_2^u(H)$ containing f is a 3-element cocircuit C^* that also contains e_1. By duality, the element of $\Lambda_2^u(H)$ containing e, namely C, also has three elements. Hence (6.1.3) holds.

By the choice of (M, N), the lemma holds for (M, N'), that is,

$$\lambda_2(M) \le 4(\lambda_1(N') - 1) + \lambda_2(N').$$

Substituting from (18) into this inequality, we obtain

$$\lambda_2(M) \le 4(\lambda_1(N) - 1) + \lambda_2(N') - 4(k - 1). \tag{19}$$

By duality, (6.1.1), and (6.1.2), we have the following two cases:

(i) $|X| = 1$ and $|Y| = 0$; or
(ii) $|X| = 1$ and $|Y| = 1$.

In case (i), we have that

$$\begin{aligned}\lambda_2(N') - \lambda_2(N) &= \lambda_2(H) - \lambda_2(H\backslash X/Y)\\ &= \left(|C| - 2 + \sum_{i=1}^{l}\lambda_2(H_i)\right) - \left(k - l + \sum_{i=1}^{l}\lambda_2(H_i\backslash e_i)\right)\\ &= l - 1 + \sum_{i=1}^{l}(\lambda_2(H_i) - \lambda_2(H_i\backslash e_i)),\end{aligned}$$

where we recall that $|C| = k + 1$. Now, it is not difficult to see that

$$\lambda_2(H_i) - \lambda_2(H_i\backslash e_i) \le 1$$

for all i. Thus

$$\lambda_2(N') - \lambda_2(N) \le 2l - 1 \le 2k - 1.$$

On combining this inequality with (19), we obtain that

$$\lambda_2(M) \le 4(\lambda_1(N) - 1) + \lambda_2(N) - 2k + 3.$$

This is a contradiction to the fact that (M, N) is a counterexample to the lemma, because $k \ge 2$. We conclude that (i) does not hold.

In case (ii), by (6.1.3), $|C| = |C^*| = 3$. Thus $k = 2$. Using the same notation as above, let $C = \{e, e_1, e_2\}$ and $C^* = \{f, e_1, e_1'\}$. First suppose that $\{e_2, e_1'\} \subseteq E(N)$. In this case, e_2 is a coloop and e_1' is a loop of N, and $H\backslash X/Y$ is the direct sum of the loop e_1' and the coloop e_2. In this case, $\lambda_2(N) = \lambda_2(N')$. Substituting this into (19), we obtain the contradiction that (M, N) is not a counterexample to the lemma. We may now suppose that $|\{e_2, e_1'\} \cap E(N)| \le 1$. By taking the dual if necessary, we may assume that $e_1' \notin E(N)$. Now $H_1\backslash e_1$ is connected and $N = N'\backslash e/f$. Thus

$$\lambda_2(H) = 2 + \lambda_2(H_1\backslash e_1) + \sum_{i=2}^{l}\lambda_2(H_i),$$

and

$$\lambda_2(H\backslash X/Y) = 2 - l + \lambda_2(H_1\backslash e_1/f) + \sum_{i=2}^{l}\lambda_2(H_i\backslash e_i).$$

Thus

$$\begin{aligned}\lambda_2(N') - \lambda_2(N) &= \lambda_2(H) - \lambda_2(H\backslash X/Y)\\ &= \lambda_2(H_1\backslash e_1) - \lambda_2(H_1\backslash e_1/f) + l + \sum_{i=2}^{l}(\lambda_2(H_i) - \lambda_2(H_i\backslash e_i)).\end{aligned}$$

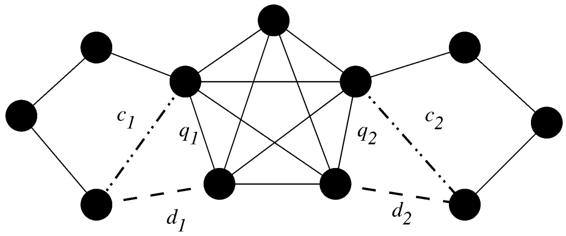

Fig. 3. An extremal example for Lemma 6.1.

Now each component of $H_1\backslash e_1/f$ must meet $E(N)$ otherwise X and Y are not unique. Thus $H_1\backslash e_1/f$ is connected otherwise $N'\backslash e/f$ contradicts the choice of N' since it has fewer components than N and has N as a minor. It follows that

$$\lambda_2(H_1\backslash e_1) - \lambda_2(H_1\backslash e_1/f) \leq 1.$$

Similarly, for each i in $\{2, \ldots, l\}$,

$$\lambda_2(H_i) - \lambda_2(H_i\backslash e_i) \leq 1.$$

Thus

$$\lambda_2(N') - \lambda_2(N) \leq 2l \leq 2k = 4.$$

Substituting this into (19), we obtain

$$\lambda_2(M) \leq 4(\lambda_1(N) - 1) + \lambda_2(N) + 4 - 4(k - 1).$$

Since $k = 2$, we have a contradiction that completes the proof. □

To see that the bound in the last lemma is sharp, consider the following example. For each i in $\{1, 2, \ldots, n\}$, let G_i be a 7-edge graph consisting of a 5-cycle C_i with two chords c_i and q_i, where c_i makes a triangle $\{c_i, p_i, d_i\}$ with two of the edges of C_i, and q_i is parallel to p_i. Let G_0 be a graph that is isomorphic to K_{n+3} and has $p_1, p_2, \ldots, p_n$ as distinct edges. Form M from $M(G_0)$ by attaching $M(G_1), M(G_2), \ldots, M(G_n)$ via 2-sums at $p_1, p_2, \ldots, p_n$, respectively. Let $N = M\backslash\{d_1, d_2, \ldots, d_n\}/\{c_1, c_2, \ldots, c_n\}$. For the case when $n = 2$, one possibility for the matroid M is the cycle matroid of the graph shown in Fig. 3 where the edges to be deleted are dashed, while those to be contracted are dotted and dashed. It is not difficult to check that M is a minor-minimal connected matroid having N as a minor. Moreover, $\lambda_1(N) = \lambda_2(N) = n + 1$, while $\lambda_2(M(G_i)) = 5$ for all $i \geq 1$, so $\lambda_2(M) = 5n + 1$. Hence this example attains equality in the bound in the last lemma.

We are now ready to complete the proof of the main result.

Proof of Theorem 1.1. Let N' be a connected minor of M that is minor-minimal having N as a minor. By Lemma 6.1,

$$\lambda_2(N') \leq 4(\lambda_1(N) - 1) + \lambda_2(N).$$

As M is a minor-minimal 3-connected matroid having N' as a minor, it follows from Theorem 4.1 that

$$|E(M)| - |E(N')| \leq 5(\lambda_2(N') - 1).$$

Hence

$$|E(M)| - |E(N')| \leq 20(\lambda_1(N) - 1) + 5(\lambda_2(N) - 1).$$

By Lemma 5.1, we have that

$$|E(N')| - |E(N)| \leq 2(\lambda_1(N) - 1).$$

By adding the last two inequalities, we obtain the theorem. □

Since the hypotheses of Theorems 1.1 and 5.4 are the same, it is natural to compare their bounds. It is not difficult to show that the bound in the former is sharper than that in the latter provided the average number of elements per component of N is at least 2. In particular, Theorem 1.1 is sharper than Theorem 5.4 if N has no loops and no coloops. However, if, for example, N is the direct sum of n loops and n coloops, then Theorem 5.4 is sharper than Theorem 1.1.

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