8-1-2005

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Recommended Citation

Hall, R., Oxley, J., & Semple, C. (2005). The structure of equivalent 3-separations in a 3-connected matroid. *Advances in Applied Mathematics, 35* (2), 123-181. https://doi.org/10.1016/j.aam.2005.01.003

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Advances in Applied Mathematics 35 (2005) 123–181
ADVANCES IN Applied Mathematics
www.elsevier.com/locate/yaama

The structure of equivalent 3-separations in a 3-connected matroid

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Received 20 July 2004; accepted 31 January 2005

Available online 25 April 2005

Abstract

Let M be a matroid. When M is 2-connected, Cunningham and Edmonds gave a tree decomposition of M that displays all of its 2-separations. This result was extended by Oxley, Semple, and Whittle, who showed that, when M is 3-connected, there is a corresponding tree decomposition that displays all non-trivial 3-separations of M up to a certain natural equivalence. This equivalence is based on the notion of the full closure $\mathrm{fcl}(Y)$ of a set Y in M, which is obtained by beginning with Y and alternately applying the closure operators of M and M^* until no new elements can be added. Two 3-separations (Y_1, Y_2) and (Z_1, Z_2) are equivalent if $\{\mathrm{fcl}(Y_1), \mathrm{fcl}(Y_2)\} = \{\mathrm{fcl}(Z_1), \mathrm{fcl}(Z_2)\}$. The purpose of this paper is to identify all the structures in M that lead to two 3-separations being equivalent and to describe the precise role these structures have in determining this equivalence.

1. Introduction

A matroid M is 2-connected if it has no 1-separations, where a k-separation is a set X such that $|X|, |E(M) - X| \geqslant k$ and $\mathrm{r}(X) + \mathrm{r}(E(M) - X) < \mathrm{r}(M) + k$. If M has no 1- or

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0196-8858/$ – see front matter
doi:10.1016/j.aam.2005.01.003

2-separations, then it is 3-connected. Many problems in matroid theory are easily reduced to the study of 3-connected matroids, for matroids that are not 3-connected have decompositions using analogues of the graph operations of 1-sum and 2-sum [1]. For some time, it was believed that this paradigm applied to representable matroids so that, in particular, Rota's Conjecture [8] that every finite field has a finite set of excluded minors could be attacked without the need to go beyond 3-connected matroids. Indeed, Kahn [2] conjectured that, for all q, there is an integer $\mu(q)$ such that every 3-connected $GF(q)$-representable matroid has at most $\mu(q)$ inequivalent $GF(q)$ representations. This conjecture was shown to be false for all $q \geqslant 7$ by Oxley, Vertigan, and Whittle [7] using two families of counterexamples of 3-connected matroids, each of which had numerous 3-separations. The abundance of 3-separations in these examples led to the hope that, by imposing some control on the behaviour of 3-separations, one may be able to recover a version of Kahn's conjecture. This motivated a study of the structure of the 3-separations in a 3-connected matroid. Such a study was initiated by Oxley, Semple, and Whittle [6] and this paper is a continuation of that study.

The initial paper in this study [6] has fulfilled the promise that initiated it, with Geelen, Gerards, and Whittle [3] having used the ideas of that paper to find an appropriate constraint on the 3-separations of a 3-connected matroid to allow its number of inequivalent $GF(q)$-representations to be bounded. Rather than terminating this study, this result has provided impetus for its continuation. Indeed, there are numerous matroid structural results, for example, Tutte's Wheels and Whirls Theorem [10] and Seymour's Splitter Theorem [9], where the situation for 3-connected matroids is well understood, but little has been developed beyond that.

In [6], a tree decomposition was obtained that described all non-trivial 3-separations of a 3-connected matroid up to the equivalence based on full closure that was defined in the abstract. The introduction of this equivalence is an essential tool in proving the main result in [6], but this equivalence ignores some of the finer structure of the matroid. The goal of this paper is to make a detailed examination of this equivalence and to explain what substructures in the matroid result in two 3-separations being equivalent.

2. Outline

This paper is long and the proofs of the main results have many technicalities. Because of this, we include here an outline of the paper which not only describes the content of various sections of the paper but, more importantly, includes statements of the main results together with the required definitions.

For a matroid M on a set E, the *connectivity function* λ of M is defined, for all subsets Z of E, by $\lambda(Z) = \mathrm{r}(Z) + \mathrm{r}(E - Z) - \mathrm{r}(M)$. The set Z or the partition $(Z, E - Z)$ is *k-separating* if $\lambda(Z) < k$. Thus the partition $(Z, E - Z)$ is a *k-separation* if it is k-separating and $|Z|, |E - Z| \geqslant k$; and M is *n-connected* if it has no $(n - j)$-separations for all j with $1 \leqslant j \leqslant n - 1$. A k-separating set Z, or k-separating partition $(Z, E - Z)$, or k-separation $(Z, E - Z)$ is *exact* if $\lambda(Z) = k - 1$.

The following is a well-known elementary property of matroids.

Lemma 2.1. *Let e be an element of a matroid M, and Y and Z be disjoint sets whose union is $E(M) - \{e\}$. Then $e \in \mathrm{cl}(Y)$ if and only if $e \notin \mathrm{cl}^*(Z)$.*

Our primary concern throughout this paper will be with exactly 3-separating partitions in 3-connected matroids. In particular, from now on, all matroids considered will be 3-connected unless otherwise specified.

Let (Y, Z) be a 3-separating partition of a 3-connected matroid M with $|Y|, |Z| \geqslant 2$. The *guts* of (Y, Z) is $\mathrm{cl}(Y) \cap \mathrm{cl}(Z)$ and, dually, the *coguts* of (Y, Z) is $\mathrm{cl}^*(Y) \cap \mathrm{cl}^*(Z)$. As M is 3-connected, it is straightforward to show that the guts of (Y, Z) is the set of elements e of E such that $e \in \mathrm{cl}(Y - e) \cap \mathrm{cl}(Z - e)$. Dually, the coguts of (Y, Z) is the set of elements e such that $e \in \mathrm{cl}^*(Y - e) \cap \mathrm{cl}^*(Z - e)$.

The proof of the next result is elementary. On combining this result with Lemma 2.1, we get that, for a partition (Y, e, Z) of the ground set of a 3-connected matroid M, the partitions $(Y, e \cup Z)$ and $(Y \cup e, Z)$ are both exactly 3-separating if and only if e is in the guts or coguts of $(Y, e \cup Z)$. Intuitively, the elements that can be moved from one side of a 3-separation to the other while maintaining a 3-separation are exactly the guts and coguts elements of the 3-separation.

Lemma 2.2. *Let Z be an exactly 3-separating set in a 3-connected matroid M and suppose $e \in E(M) - Z$. Then $Z \cup \{e\}$ is 3-separating if and only if $e \in \mathrm{cl}(Z)$ or $e \in \mathrm{cl}^*(Z)$.*

For a set Y in a matroid M, if Y equals its closure in both M and M^*, we say that Y is *fully closed* in M. The *full closure* of Y, denoted $\mathrm{fcl}(Y)$, is the intersection of all fully closed sets containing Y. One way to obtain the full closure of Y is to take $\mathrm{cl}(Y)$, and then $\mathrm{cl}^*(\mathrm{cl}(Y))$ and so on until neither the closure nor coclosure operator adds any new elements.

The full closure operator enables one to define a natural equivalence on exactly 3-separating partitions as follows. Let M be a matroid, and let Z and Y be exactly 3-separating sets of M. We say that Z is *equivalent* to Y if $\mathrm{fcl}(Z) = \mathrm{fcl}(Y)$. For two exactly 3-separating partitions (Z_1, Z_2) and (Y_1, Y_2) of M, we say (Z_1, Z_2) and (Y_1, Y_2) are *equivalent* if, for some ordering of Y_1 and Y_2, we have Z_1 is equivalent to Y_1, and Z_2 is equivalent to Y_2. If either $\mathrm{fcl}(Z_1)$ or $\mathrm{fcl}(Z_2)$ is $E(M)$, then (Z_1, Z_2) is called *sequential*. As noted in the introduction, Oxley, Semple, and Whittle [6] showed that every 3-connected matroid M has a tree decomposition that displays, up to this equivalence, all non-sequential 3-separations of M. In this paper, we examine, for a non-sequential 3-separation (Y, Z), the members of its equivalence class and we describe the structural relationships between the members of this class. We believe that the techniques of this paper can be extended to include the case when (Y, Z) is sequential but, because the paper is already long, we do not attempt this analysis here.

Let M be a matroid with ground set E and let $\mathcal{K}$ be an equivalence class of non-sequential 3-separations of M. Suppose that $(A_1, B_1) \in \mathcal{K}$. Then $(A_1 - \mathrm{fcl}(B_1), \mathrm{fcl}(B_1))$ and $(\mathrm{fcl}(A_1), B_1 - \mathrm{fcl}(A_1))$ are also 3-separations in $\mathcal{K}$. Let $A = A_1 - \mathrm{fcl}(B_1)$ and

$B = B_1 - \mathrm{fcl}(A_1)$. If $X = E - (A \cup B)$, then there is an ordering $\vec{X}$ of X, say $\vec{X} = (x_1, x_2, \ldots, x_n)$, such that, for some j in $\{0, 1, \ldots, n\}$,

$$(A_1, B_1) = \bigl(A \cup \{x_1, x_2, \ldots, x_j\}, \{x_{j+1}, x_{j+2}, \ldots, x_n\} \cup B\bigr)$$

and, for all i in $\{0, 1, \ldots, n\}$,

$$\bigl(A \cup \{x_1, x_2, \ldots, x_i\}, \{x_{i+1}, x_{i+2}, \ldots, x_n\} \cup B\bigr) \text{ is exactly 3-separating.} \tag{2.1}$$

We define an *exact* (A, B) 3-*sequence* or, more briefly, an (A, B) 3-*sequence* or a 3-*sequence* to be an ordered partition $(A, x_1, x_2, \ldots, x_n, B)$ of the ground set of a matroid M such that $|A|, |B| \geqslant 2$ and (2.1) holds. If (A_2, B_2) is an arbitrary member of the equivalence class $\mathcal{K}$ above, then, without loss of generality, we may assume that $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$. Thus

$$A_2 - \mathrm{fcl}(B_2) = E(M) - \mathrm{fcl}(B_2) = E(M) - \mathrm{fcl}(B_1) = A_1 - \mathrm{fcl}(B_1) = A$$

and, by symmetry,

$$B_2 - \mathrm{fcl}(A_2) = B_1 - \mathrm{fcl}(A_1) = B.$$

Hence there is an ordering $\vec{X'}$ of X, say $\vec{X'} = (x'_1, x'_2, \ldots, x'_n)$ such that $(A, \vec{X'}, B)$ is a 3-sequence and $(A_2, B_2) = (A \cup \{x'_1, x'_2, \ldots, x'_k\}, \{x'_{k+1}, x'_{k+2}, \ldots, x'_n\} \cup B)$ for some k in $\{0, 1, \ldots, n\}$. Thus, to consider the structural relationships amongst the members of an equivalence class of non-sequential 3-separations, it suffices to consider, for an exact 3-sequence $(A, \vec{X}, B)$,

(i) which reorderings of X result in an exact 3-sequence of M, and
(ii) what type of substructures of M result in these reorderings.

If A and B are disjoint subsets of the ground set of a matroid M, an (A, B) 3-*sequence* is a 3-sequence of M of the form $(A, x_1, x_2, \ldots, x_n, B)$. Given such a 3-sequence, we call each x_i a *guts* or *coguts* element depending on whether x_i is in the closure or coclosure of $A \cup \{x_1, x_2, \ldots, x_{i-1}\}$. Lemmas 2.1 and 2.2 imply that no element is both a guts and a coguts element. Moreover, it will be shown in Lemma 5.2 that if x_i is a guts element of some (A, B) 3-sequence, then it is a guts element of all (A, B) 3-sequences. This means that we can designate every element of X in an (A, B) 3-sequence as either a *guts element* or a *coguts element*. We shall refer to this designation as the *type* of the element.

For the rest of the paper, whenever we discuss a 3-sequence (A, X, B), it will be implicit that there is an ordering $\vec{X}$ and a 3-connected matroid M with ground set $A \cup X \cup B$ such that $(A, \vec{X}, B)$ is a 3-sequence in M. Similarly, when we refer to $(A, Z_1, Z_2, \ldots, Z_n, B)$ as a 3-sequence, we shall mean that there are orderings $\vec{Z_1}, \vec{Z_2}, \ldots, \vec{Z_n}$ of $Z_1, Z_2, \ldots, Z_n$ such that $(A, \vec{Z_1}, \vec{Z_2}, \ldots, \vec{Z_n}, B)$ is a 3-sequence. Let (A, X, B) be a 3-sequence. If x and y are elements of X and there are 3-sequences $(A, \vec{X_1}, B)$ and $(A, \vec{X_2}, B)$ such that x precedes y in $\vec{X_1}$, and y precedes x in $\vec{X_2}$, then we say that x is in the *jump-set* J_y of y, and y is in J_x. A crucial ingredient in our discussion of (A, B) 3-sequences will be an analysis of the jump-sets of elements.

Next we introduce the matroid structures that arise within X. Some of these are familiar, while others are less well known. The reader will observe that, while these definitions will often refer to the sets A and B, they are independent of a specific ordering of X.

Let $(A, \overrightarrow{X}, B)$ be a 3-sequence in a matroid M. Then $(A, \overrightarrow{X}, B)$ is also a 3-sequence in M^*. Let S be a subset of X with $|S| \geqslant 4$. Then S is a *segment* if each 3-element subset of S is a triangle; and S is a *cosegment* if each 3-element subset of S is a triad. We call S a *fan* if there is an ordering $(s_1, s_2, \ldots, s_n)$ of the elements of S such that

(i) for all $i \in \{1, 2, \ldots, n-2\}$, the triple $\{s_i, s_{i+1}, s_{i+2}\}$ is either a triangle or a triad, and
(ii) if $i \in \{1, 2, \ldots, n-2\}$ and $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, then $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad, while if $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, then $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

This ordering $(s_1, s_2, \ldots, s_n)$ is called a *fan ordering* of S.

The above definitions impose the requirements that $|S| \geqslant 4$. This is non-standard. We now extend these definitions to include the cases when S is either a 3-element or a 2-element subset of X. Usually a triangle can be viewed as both a segment and a fan, while a triad can be viewed as both a cosegment and a fan. Although we could follow this convention here, in the context of an (A, B) 3-sequence, there is a natural dichotomy within the class of triangles between those that are best viewed as segments and those that behave like small fans. It will be shown in Lemmas 4.1 and 4.6 that a triangle must consist of either three guts elements or two guts elements and a coguts element. In the first case, the triangle will be viewed as a segment, in the second case as a fan. Similarly, a triad consisting of three coguts elements will be viewed as a cosegment, while a triad with exactly two coguts elements will be viewed as a fan. For $|S| = 2$, if there is an (A, B) 3-sequence in which the elements of S are consecutive, then S is a *degenerate segment* if it consists of two guts elements, a *degenerate cosegment* if it consists of two coguts elements, and a *degenerate fan* if it consists of a guts and a coguts element that are not in each other's jump-sets.

Non-degenerate segments and cosegments can be characterised in terms of the jump-sets of their elements. The next result is proved later as Corollary 6.5.

Theorem 2.3. *Let (A, X, B) be a 3-sequence. Let Y be a subset of X having at least three elements. Then Y is a segment or a cosegment if and only if every member y of Y is in the jump-sets of all of the members of $Y - \{y\}$.*

As we shall show in Lemma 6.1 and Theorem 6.9, if Y is a segment, a cosegment, or a fan in a 3-sequence $(A, \overrightarrow{X}, B)$, then there is an ordering $\overrightarrow{X_1}$ of X in which the elements of Y are consecutive and $(A, \overrightarrow{X_1}, B)$ is a 3-sequence. Theorem 6.9 also shows that the elements of a fan Y always occur in the same relative order in every (A, B) 3-sequence. Hence each fan has a *first* and a *last element*, a *left* and a *right end*.

A *maximal segment* in a 3-sequence (A, X, B) is a segment Y in X so that there is no segment Z that properly contains Y. When x is a guts element in X that is not contained in any segment with two or more elements, it will be convenient to view $\{x\}$ as a maximal segment. We call this a *degenerate maximal segment* noting that it is strictly neither a segment nor a degenerate segment. *Maximal cosegments* are defined dually.

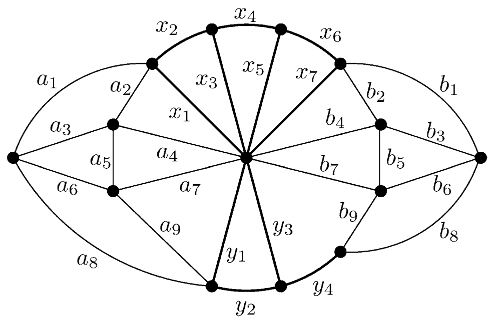

Fig. 1. A graph whose cycle matroid contains the clock $\{x_1, x_2, \ldots, x_7, y_1, y_2, y_3, y_4\}$.

The main result of this paper is that once four special substructures are eliminated from a 3-sequence (A, X, B), the set X can be partitioned into maximal segments and maximal cosegments. Moreover, there is a canonical ordering on these maximal segments and maximal cosegments that induces an ordering $\overrightarrow{X_0}$ on X such that $(A, \overrightarrow{X}, B)$ is a 3-sequence if and only if $\overrightarrow{X}$ is obtained from $\overrightarrow{X_0}$ by arbitrarily permuting the elements within each maximal segment and each maximal cosegment or, at each interface between a maximal segment and a maximal cosegment, by interchanging a guts element and a coguts element.

Before giving a more formal statement of this theorem, we identify the substructures that must be eliminated to get the theorem. An example of the first of these, a clock, is shown in Fig. 1. There $A = \{a_1, a_2, \ldots, a_9\}$ and $B = \{b_1, b_2, \ldots, b_9\}$. In the cycle matroid of the graph G shown, $(A, x_1, x_2, \ldots, x_7, y_1, y_2, y_3, y_4, B)$ is a 3-sequence, as is $(A, y_1, y_2, y_3, y_4, x_1, x_2, \ldots, x_7, B)$.

Let $(A, \overrightarrow{X}, B)$ be a 3-sequence in a matroid M. Let F_1 and F_2 be disjoint fans contained in X such that each is maximal with this property. We call $F_1 \cup F_2$ a *clock* if there is a partition (A', B') of $E(M) - (F_1 \cup F_2)$ such that A' and B' contain A and B, respectively, and each of A', $A' \cup F_1$, $A' \cup F_2$, and $A' \cup F_1 \cup F_2$ is 3-separating. Observe that we allow F_1 or F_2 to be a degenerate fan consisting of one guts and one coguts element. The clock $F_1 \cup F_2$ is *proper* if both $|F_1|$ and $|F_2|$ exceed two. It is *semidegenerate* if one of $|F_1|$ and $|F_2|$ is two and the other exceeds two. It is *degenerate* if both $|F_1|$ and $|F_2|$ are two.

The following characterisation of clocks in terms of jump-sets will follow from Theorem 7.2.

Theorem 2.4. *Let (A, X, B) be a 3-sequence. Then X contains a clock if and only if X contains elements y and z of different types such that $|J_y \cap J_z| \geqslant 2$. Indeed, when such elements y and z exist, the corresponding clock has as its fans $J_y \cap J_z$ and $J_u \cap J_v$ where u and v are arbitrary distinct members of $J_y \cap J_z$.*

In a 3-sequence (A, X, B), if Y is a clock, a maximal segment, or a maximal cosegment, then the elements of Y can be made consecutive in some (A, B) 3-sequence. Moreover, when this is done, the sets L_Y and R_Y of elements of X that occur to the left and right of Y are uniquely determined. Thus such a set Y forms a barrier in the 3-sequence in that no element e of $X - Y$ has all of Y in its jump-set.

There is a natural ordering on the set of non-degenerate clocks in X. Since a clock is the union of two fans, the clock contains the first and last elements of these fans. All other clock elements are called *internal*. A degenerate clock is *even* if its first elements are of the same type, and is *odd* otherwise. The ordering on non-degenerate clocks referred to above can be extended to include odd degenerate clocks. However, even degenerate clocks can interlock in a structure we call a non-degenerate crocodile. Such a structure is built from a maximal segment and a maximal cosegment with the property that at least two elements in the segment have distinct cosegment elements in their jump-sets.

Let $(A, \vec{X}, B)$ be a 3-sequence in a matroid M. Let S and S^* be a maximal segment and a maximal cosegment in X such that $|S \cup S^*| \geqslant 5$. We call $S \cup S^*$ a *crocodile* if, for some (C, D) in $\{(A, B), (B, A)\}$, there is a partition (C', D') of $E(M) - (S \cup S^*)$ such that

(i) C' and D' contain C and D, respectively;
(ii) each of C', $C' \cup S$, and $C' \cup S \cup S^*$ is 3-separating; and
(iii) for some $k \geqslant 2$, there are k-element subsets $S_w = \{s_1, s_2, \dots, s_k\}$ and $S^*_w = \{s^*_1, s^*_2, \dots, s^*_k\}$ of S and S^* such that
  (a) $C' \cup (S - \{s_i\}) \cup \{s^*_i\}$ is 3-separating for all i in $\{1, 2, \dots, k\}$; and
  (b) if $s \in S - S_w$, there is no s^* in S^* such that $C' \cup (S - \{s\}) \cup \{s^*\}$ is 3-separating.

If $(C, D) = (A, B)$, we call $S \cup S^*$ a *segment-first crocodile*; otherwise $S \cup S^*$ is a *cosegment-first crocodile*. The crocodile $S \cup S^*$ is *degenerate* if $|S| = 2$ or $|S^*| = 2$. We shall show in Lemma 8.2 that when $S \cup S^*$ is a non-degenerate crocodile in X, for all distinct i and j in $\{1, 2, \dots, k\}$, there is an even degenerate clock with fans (s_i, s^*_j) and (s_j, s^*_i).

While segments, cosegments, and fans are well-known substructures of matroids, crocodiles are less so. We show next that the matroids that give rise to crocodiles have appeared previously in the literature [5]. For each $k \geqslant 3$, take a basis $\{b_1, b_2, \dots, b_k\}$ of $PG(k-1, \mathbb{R})$ and a line L that is freely placed relative to this basis. By modularity, for each i, the hyperplane of $PG(k-1, \mathbb{R})$ that is spanned by $\{b_1, b_2, \dots, b_k\} - \{b_i\}$ meets L. Let a_i be the point of intersection. We shall denote by Θ_k the restriction of $PG(k-1, \mathbb{R})$ to $\{b_1, b_2, \dots, b_k, a_1, a_2, \dots, a_k\}$. The reader can easily check that Θ_3 is isomorphic to $M(K_4)$. We extend the definition above to include the case $k = 2$. In that case, we begin with two independent points b_1 and b_2 and we add a_1 in parallel with b_2, and a_2 in parallel with b_1. Alternatively, for all $k \geqslant 2$, we can define Θ_k to be the matroid with ground set $\{b_1, b_2, \dots, b_k, a_1, a_2, \dots, a_k\}$ whose circuits consist of all 3-element subsets of $\{a_1, a_2, \dots, a_k\}$; all sets of the form $(\{b_1, b_2, \dots, b_k\} - \{b_i\}) \cup \{a_i\}$, where $i \in \{1, 2, \dots, k\}$; and all sets of the form $(\{b_1, b_2, \dots, b_k\} - \{b_l\}) \cup \{a_g, a_h\}$, where l, g, and h are distinct elements of $\{1, 2, \dots, k\}$ [5, Lemma 2.2]. As noted in [5, Lemma 2.1], for all $k \geqslant 2$, the matroid Θ_k is isomorphic to its dual under the map that interchanges a_i and b_i for all i.

Describing a 3-sequence that contains a crocodile is now straightforward. Begin with Θ_k for some $k \geqslant 2$ and, for some $m \geqslant 2$, add elements $\{z_1, z_2, \dots, z_m, y_1, y_2, \dots, y_n\}$ freely on the line spanned by $\{a_1, a_2, \dots, a_k\}$ to form the matroid M_1. Because these elements were freely added on this line, every 3-element subset of $\{b_1, b_2, \dots, b_k\}$ is a cocircuit of M_1. Then, for some $p \geqslant 2$, add $\{u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_q\}$ freely on the line of M^*_1 that

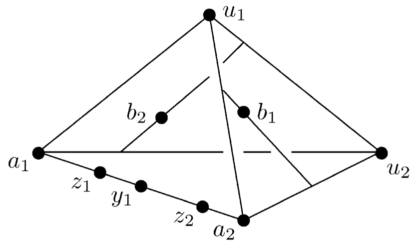

Fig. 2. A geometric representation of a matroid that contains the crocodile $\{y_1, a_1, a_2, b_1, b_2\}$.

is spanned by $\{b_1, b_2, \dots, b_k\}$. It is not difficult to check that if $A = \{z_1, z_2, \dots, z_m\}$ and $B = \{u_1, u_2, \dots, u_p\}$, then, provided $n + q \geqslant 1$,

$$(A, y_1, y_2, \dots, y_n, a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k, v_1, v_2, \dots, v_q, B)$$

is a 3-sequence having $S \cup S^*$ as a crocodile where $S = \{y_1, y_2, \dots, y_n, a_1, a_2, \dots, a_k\}$ and $S^* = \{b_1, b_2, \dots, b_k, v_1, v_2, \dots, v_q\}$. Moreover, $S_w = \{a_1, a_2, \dots, a_k\}$ and $S_w^* = \{b_1, b_2, \dots, b_k\}$ where the swap partner of a_i is b_i. An example with $k = m = p = 2$ and $n - 1 = q = 0$ is shown in Fig. 2.

In Theorem 8.3, we show that every occurrence of a crocodile in a 3-sequence corresponds to the presence of a minor isomorphic to Θ_k.

Let C be a clock whose fans $F_1(C)$ and $F_2(C)$ are non-degenerate. If z is an internal element of C and $z \in F_2(C)$, then the jump-set of z equals $F_1(C)$. This phenomenon of an element being able to jump a non-trivial sequence of guts and coguts elements also occurs in a more general context. This leads us to define a p-flan, a generalisation of the idea of a fan.

Let $(A, \overrightarrow{X}, B)$ be a 3-sequence and z be a guts element of X. For a subset F of $X - \{z\}$, we call $F \cup \{z\}$ a *pointed flan* or *p-flan* if there is an ordered partition $(\{z\}, F_1, F_2, \dots, F_m)$ of $F \cup \{z\}$ with $m \geqslant 3$ such that the following hold:

- (i) for all $i \in \{1, 2, \dots, m\}$, either F_i consists of a single coguts element or $F_i \cup \{z\}$ is a maximal segment;
- (ii) if $i \in \{1, 2, \dots, m - 1\}$, then F_i contains a coguts element if and only if F_{i+1} does not;
- (iii) if $i \in \{1, 2, \dots, m - 2\}$ and F_i is a singleton coguts set, then $F_i \cup F_{i+1} \cup F_{i+2}$ is a cocircuit; and
- (iv) if $i \in \{1, 2, \dots, m - 2\}$ and F_i is a set of guts elements, then $F_i \cup F_{i+1} \cup F_{i+2} \cup \{z\}$ has rank three.

We call z the *tip* of the p-flan $F \cup \{z\}$. Dually, $F \cup \{z\}$ is a *p-coflan* of M with cotip z if it is a p-flan of M with tip z. Note that, in the definition of a p-flan, if F_i contains exactly one guts element, then $F_i \cup \{z\}$ is a degenerate segment. Indeed, in the example of a clock in the previous paragraph, if z is a guts element, then $(z, \{x_1\}, \{x_2\}, \dots, \{x_k\})$ is a p-flan where $(x_1, x_2, \dots, x_k)$ is the fan ordering on $F_1(C)$.

A p-flan $(\{z\}, F_1, F_2, \dots, F_m)$ in a 3-sequence (A, X, B) is *maximal* if there is no p-flan $F \cup \{z\}$ such that F properly contains $F_1 \cup F_2 \cup \dots \cup F_m$.

The next result, which is proved later as Corollary 9.6, shows that maximal p-flans coincide with the jump-sets of guts elements that contain either two coguts elements or two guts elements that are not in a common segment.

Theorem 2.5. *Let $(z, F_1, F_2, \dots, F_m)$ be a maximal p-flan in an exact 3-sequence $(A, \overrightarrow{X}, B)$. Then $F_1 \cup F_2 \cup \cdots \cup F_m = J_z$.*

The main results of the paper appear in Section 10. There we describe three decomposition results for a 3-sequence $(A, \overrightarrow{X}, B)$. The first of these, Theorem 10.2, begins with a non-degenerate or odd degenerate clock C in X and breaks $(A, \overrightarrow{X}, B)$ into two 3-sequences $(A', \overrightarrow{Y}, B)$ and $(A, \overrightarrow{Z}, B')$. Recalling that L_C is the set of elements of $X - C$ that occur to the left of C in an (A, B) 3-sequence having the elements of C consecutive, we have that A' is the union of A with L_C and all of C except the last elements of its fans, and B' is defined symmetrically. The key point about this decomposition is that every pair of orderings $\overrightarrow{Y_1}$ and $\overrightarrow{Z_1}$ of Y and Z such that $(A', \overrightarrow{Y_1}, B)$ and $(A, \overrightarrow{Z_1}, B')$ are 3-sequences can be combined to produce a 3-sequence $(A, \overrightarrow{X_1}, B)$, and every (A, B) 3-sequence arises in this way.

The second decomposition result, Theorem 10.3, has the same flavour as the first. It breaks up a 3-sequence having an even degenerate clock. This decomposition theorem is applicable to 3-sequences with non-degenerate crocodiles. It also applies to degenerate crocodiles that occur within 3-sequences with no non-degenerate clocks. In view of the first decomposition theorem, the imposition of the last restriction is quite natural.

The final decomposition result, Theorem 10.7, treats 3-sequences having no clocks. In such a 3-sequence (A, X, B), when the elements of a p-flan $z \cup F$ are consecutive, it is shown in Lemma 10.4, that the sets L_F and R_F of elements of X occurring to the left and right of $z \cup F$ in X are determined. This enables us to obtain the third decomposition theorem, which has a similar format to the first two.

In view of the three decomposition theorems, it is natural to exclude clocks, p-flans, and p-coflans from the 3-sequences we are considering. In Lemma 10.8, we show that, in such a 3-sequence, every guts element is in a unique maximal segment and every coguts element is in a unique maximal cosegment. Moreover, for such a 3-sequence, we have the following theorem, the main result of the paper. A slightly more explicit statement of this result appears later as Theorem 10.13.

Theorem 2.6. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence that contains no clocks, no p-flans, and no p-coflans and suppose that $|X| \geqslant 3$. Let $T_1, T_2, \dots, T_n$ be the collection of maximal segments and maximal cosegments in X. Then there is a unique ordering on these sets such that $(A, T_1, T_2, \dots, T_n, B)$ is a 3-sequence. Moreover, every (A, B) 3-sequence can be obtained from this one by the following two steps*:

(i) *arbitrarily reorder the elements of each T_i; and*
(ii) *look among these reorderings at when the last element of T_i is in the jump-set of the first element of T_{i+1}. These swap pairs are disjoint and, for each i and $i+1$ in $\{1, 2, \dots, n\}$, there is at most one such pair. Pick some subset of these swap pairs and swap each element with its partner.*

To conclude this section, we now briefly describe the organisation of the paper. In Section 3, we present some basic matroid preliminaries. Section 4 describes some elementary properties of 3-sequences, while Section 5 considers how elements can be moved around in 3-sequences. Section 6 treats segments, cosegments, and fans in 3-sequences, while Sections 7, 8, and 9 examine the properties of, respectively, clocks, crocodiles, and pointed flans. The main results of the paper, the decomposition theorems described above, appear in Section 10. The last section of the paper gives an algorithm to determine the jump-set of every element x of X in a 3-sequence (A, X, B).

3. Preliminaries

In this section, we present some matroid results, which we shall need in the paper but which have not yet appeared. Any otherwise unexplained matroid terminology will follow [4]. In a matroid M, if e is an element of M and $A \subseteq E(M)$, we shall write $e \in \mathrm{cl}^{(*)}(A)$ to indicate that $e \in \mathrm{cl}(A)$ or $e \in \mathrm{cl}^*(A)$.

The connectivity functions of a matroid M and its dual M^* are equal. Moreover, the connectivity function of M is submodular, that is, $\lambda(X) + \lambda(Y) \geqslant \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E$. One useful consequence of this is the following:

Lemma 3.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geqslant 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geqslant 2$, then $X \cap Y$ is 3-separating.*

The following well-known result, whose proof is straightforward, helps to explain why segments, cosegments, and fans are so omnipresent in this paper.

Lemma 3.2. *Let S be a set in a 3-connected matroid. If S has an ordering $(s_1, s_2, \dots, s_n)$ such that, for all $i \in \{1, 2, \dots, n-2\}$, the triple $\{s_i, s_{i+1}, s_{i+2}\}$ is 3-separating, then the 3-sets $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \dots, \{s_{n-2}, s_{n-1}, s_n\}$ are all triangles, all triads, or are alternately triangles and triads.*

If X and Y are subsets of the ground set of a matroid M, the *local connectivity* $\sqcap(X, Y)$ between X and Y is defined by

$$\sqcap(X, Y) = \mathrm{r}(X) + \mathrm{r}(Y) - \mathrm{r}(X \cup Y).$$

In particular, if (X, Y) is a partition of $E(M)$, then $\sqcap(X, Y) = \lambda(X)$. If M is a representable matroid and we view it as a restriction of a projective geometry P, then the modularity of P means that $\sqcap(X, Y)$ is the rank of the intersection of the closures, in P, of X and Y. The next lemma is just a restatement of Lemma 8.2.10 of [4].

Lemma 3.3. *Let X_1, X_2, Y_1, and Y_2 be subsets of the ground set of a matroid M. If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then $\sqcap(X_1, X_2) \leqslant \sqcap(Y_1, Y_2)$.*

By the definition of $\sqcap$, we have that $\mathrm{r}(\mathrm{cl}(X) \cap \mathrm{cl}(Y)) \leqslant \sqcap(X, Y)$. The next lemma is an immediate consequence of this inequality.

Lemma 3.4. *Let M be a 3-connected matroid with at least four elements, let X and Y be subsets of $E(M)$, and let $Z = \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. If $\sqcap(X, Y) = 2$, then $\mathrm{r}(Z) \leqslant 2$; if $\sqcap(X, Y) = 1$, then $|Z| \leqslant 1$; and if $\sqcap(X, Y) = 0$, then $Z = \emptyset$.*

4. Properties of 3-sequences

In this section, we begin our analysis of 3-sequences by noting some of their elementary properties.

Lemma 4.1. *Let $(A, x_1, x_2, \ldots, x_n, B)$ be a 3-sequence of a 3-connected matroid, and let x_i be an element of the sequence. Then either*

(i) $x_i \in \mathrm{cl}(A \cup \{x_1, \ldots, x_{i-1}\}) \cap \mathrm{cl}(\{x_{i+1}, \ldots, x_n\} \cup B)$ *or*
(ii) $x_i \in \mathrm{cl}^*(A \cup \{x_1, \ldots, x_{i-1}\}) \cap \mathrm{cl}^*(\{x_{i+1}, \ldots, x_n\} \cup B)$,

but not both.

Proof. Since $(A, x_1, \ldots, x_n, B)$ is a 3-sequence,

$$\mathrm{r}\big(A \cup \{x_1, \ldots, x_{i-1}\}\big) + \mathrm{r}\big(\{x_i, \ldots, x_n\} \cup B\big) = \mathrm{r}(M) + 2$$

and

$$\mathrm{r}\big(A \cup \{x_1, \ldots, x_i\}\big) + \mathrm{r}\big(\{x_{i+1}, \ldots, x_n\} \cup B\big) = \mathrm{r}(M) + 2.$$

Therefore, if $x_i \in \mathrm{cl}(A \cup \{x_1, \ldots, x_{i-1}\})$, then $x_i \in \mathrm{cl}(\{x_{i+1}, \ldots, x_n\} \cup B)$, while if $x_i \notin \mathrm{cl}(A \cup \{x_1, \ldots, x_{i-1}\})$, then $x_i \notin \mathrm{cl}(\{x_{i+1}, \ldots, x_n\} \cup B)$. The result now readily follows by Lemma 2.1. $\square$

Given a 3-sequence $(A, x_1, x_2, \ldots, x_n, B)$ of a 3-connected matroid, we say that x_i is a *guts element of* $\overrightarrow{X}$ if $x_i \in \mathrm{cl}(A \cup \{x_1, \ldots, x_{i-1}\}) \cap \mathrm{cl}(\{x_{i+1}, \ldots, x_n\} \cup B)$, and we say that x_i is a *coguts element of* $\overrightarrow{X}$ if $x_i \in \mathrm{cl}^*(A \cup \{x_1, \ldots, x_{i-1}\}) \cap \mathrm{cl}^*(\{x_{i+1}, \ldots, x_n\} \cup B)$.

An immediate consequence of Lemmas 4.1 and 2.1 is the following.

Corollary 4.2. *Let $(A, x_1, x_2, \ldots, x_n, B)$ be a 3-sequence of a 3-connected matroid. If $i \in \{1, 2, \ldots, n\}$, then x_i is either a guts element of $\overrightarrow{X}$ or a coguts element of $\overrightarrow{X}$, but it cannot be both.*

Let M be a matroid with ground set E. A partition (X, Y) of E is *displayed* by a sequence $(A, x_1, x_2, \ldots, x_n, B)$ if $\{X, Y\} = \{A \cup x_1 \cup \cdots \cup x_i, x_{i+1} \cup \cdots \cup x_n \cup B\}$ for some i in $\{0, 1, \ldots, n\}$.

Lemma 4.3. *Let* $(A, x_1, x_2, \dots, x_n, B)$ *be a* 3*-sequence of a matroid* M *and let* $i < j$*. Suppose that either* $x_j \in \mathrm{cl}(A \cup \{x_1, \dots, x_i\})$ *or* $x_j \in \mathrm{cl}^*(A \cup \{x_1, \dots, x_i\})$*. Then*

$$(A, x_1, \dots, x_i, x_j, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n, B)$$

is also a 3*-sequence of* M*.*

Proof. For all k in $\{i, i+1, \dots, j-1\}$, the set $A \cup \{x_1, x_2, \dots, x_k\}$ is 3-separating so, by Lemma 2.2, $A \cup \{x_1, x_2, \dots, x_k\} \cup x_j$ is also 3-separating. The lemma follows. □

For an ordered set $\overrightarrow{X} = (x_1, x_2, \dots, x_n)$ and an ordinary set Y, we denote the ordered set obtained from $\overrightarrow{X}$ by deleting the elements of Y in X by $\overrightarrow{X} - Y$; and we denote the ordered set that consists of the members of $X \cap Y$ by $\overrightarrow{X} \cap Y$. For example, if $Y = \{x_2, x_3, x_5\}$, then $\overrightarrow{X} - Y = (x_1, x_4, x_6, \dots, x_n)$ and $\overrightarrow{X} \cap Y = (x_2, x_3, x_5)$.

Lemma 4.4. *Let* $(A, \overrightarrow{X}, B)$ *be a* 3*-sequence. Let* $\overrightarrow{X} = (x_1, x_2, \dots, x_n)$*. If* $A \cup \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$ *is* 3*-separating for some* $i_1, i_2, \dots, i_k$ *with* $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n$*, then* $(A, x_{i_1}, x_{i_2}, \dots, x_{i_k}, \overrightarrow{X} - \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}, B)$ *is a* 3*-sequence.*

Proof. For each j in $\{1, 2, \dots, n\}$, both of the sets $A \cup \{x_1, x_2, \dots, x_j\}$ and $A \cup \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$ are 3-separating, and each avoids B. Thus, by Lemma 3.1, their intersection is 3-separating. Hence $A \cup \{x_{i_1}, x_{i_2}, \dots, x_{i_s}\}$ is 3-separating for all s in $\{0, 1, \dots, k\}$. Moreover, by Lemma 4.1, $x_t \in \mathrm{cl}^{(*)}(A \cup \{x_1, x_2, \dots, x_{t-1}\})$ for all t in $\{1, 2, \dots, n\}$. Hence if $t \notin \{i_1, i_2, \dots, i_k\}$, then $x_t \in \mathrm{cl}^{(*)}(A \cup \{x_1, x_2, \dots, x_{t-1}\} \cup \{x_{i_1}, \dots, x_{i_k}\})$. Therefore $(A, x_{i_1}, x_{i_2}, \dots, x_{i_k}, \overrightarrow{X} - \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}, B)$ is indeed a 3-sequence. □

The following is a straightforward consequence of the last lemma.

Corollary 4.5. *Let* $(A, x_1, x_2, \dots, x_n, B)$ *and* $(A, y_1, y_2, \dots, y_n, B)$ *be* 3*-sequences. Let* $\overrightarrow{Y} = (y_1, y_2, \dots, y_n)$*. Then, for all* k*, both*

$$\bigl(A, x_1, x_2, \dots, x_k, \overrightarrow{Y} - \{x_1, x_2, \dots, x_k\}, B\bigr)$$

and

$$\bigl(A, \overrightarrow{Y} - \{x_{n-k+1}, x_{n-k+2}, \dots, x_n\}, x_{n-k+1}, x_{n-k+2}, \dots, x_n, B\bigr)$$

are 3*-sequences.*

Proof. By symmetry, it suffices to prove that the second-last sequence is a 3-sequence. It follows immediately from the last lemma that $(A, \overrightarrow{Y} \cap \{x_1, x_2, \dots, x_k\}, \overrightarrow{Y} - \{x_1, x_2, \dots, x_k\}, B)$ is a 3-sequence because $A \cup \{x_1, x_2, \dots, x_k\}$ is 3-separating and $(A, y_1, y_2, \dots, y_n, B)$ is a 3-sequence. To see that $(A, x_1, x_2, \dots, x_k, \overrightarrow{Y} - \{x_1, x_2, \dots, x_k\}, B)$ is a 3-sequence, we note that $(A, x_1, x_2, \dots, x_n, B)$ is a 3-sequence, and hence $A \cup \{x_1, x_2, \dots, x_i\}$ is 3-separating for all i in $\{0, 1, \dots, k\}$. □

Lemma 4.6. *Let S and S' be (A, B) 3-sequences in a matroid M. Let $x \in E(M) - (A \cup B)$. If x is a guts element of S, then x is a guts element of S'. Furthermore, if x is a coguts element of S, then x is a coguts element of S'.*

Proof. Without loss of generality, we may assume that $S = (A, \overrightarrow{X_1}, x, \overrightarrow{X_2}, B)$ and $S' = (A, \overrightarrow{Y_1}, x, \overrightarrow{Y_2}, B)$, where the disjoint unions of X_1 and X_2, and Y_1 and Y_2 are $E(M) - (A \cup \{x\} \cup B)$. By duality, we may assume that x is a guts element of S. By Corollary 4.5, S' can be modified to produce the 3-sequence $(A, \overrightarrow{X_1}, \overrightarrow{Y_1} - X_1, x, \overrightarrow{Y_2} - X_1, B)$. Since x is a guts element of S, we have that $x \in \mathrm{cl}(A \cup X_1)$ and so $x \in \mathrm{cl}(A \cup X_1 \cup Y_1)$. Therefore, by Lemma 4.1, $x \in \mathrm{cl}(B \cup (Y_2 - X_1))$, which in turn implies that $x \in \mathrm{cl}(B \cup Y_2)$. Thus x is a guts element of S'. □

The last lemma means that in a 3-sequence (A, X, B), we can designate every element as either a *guts* or a *coguts element*. We shall refer to this designation as the *type* of the element.

5. Jumping and sorting

Lemma 4.3 illustrates the idea that an element x_i in a 3-sequence $(A, x_1, x_2, \dots, x_n, B)$ can jump over other elements in the sequence and be inserted somewhere else in the ordered set $(x_1, x_2, \dots, x_n)$, so that the resulting sequence is also a 3-sequence. The next series of lemmas, which culminates with Theorem 5.5, describes exactly when this can be done. This section also begins the discussion of the structural implications of an element being able to jump over other elements.

Let $(A, \overrightarrow{X}, B)$ be a 3-sequence and let $x \in X$. Recall from Section 2 that the jump-set, J_x, of x is the set of elements z of X such that there is an (A, B) 3-sequence with z appearing to the left of x and an (A, B) 3-sequence with z appearing to the right of x. The *left-set*, L_x, of x and the *right-set*, R_x, of x are the sets of elements of X that appear to the left and right, respectively, of x in every (A, B) 3-sequence.

The next lemma says that, for every element x, there is an (A, B) 3-sequence in which the elements of J_x are consecutive.

Lemma 5.1. *Let $(A, \overrightarrow{X_1}, x, \overrightarrow{X_2}, B)$ and $(A, \overrightarrow{Y_1}, x, \overrightarrow{Y_2}, B)$ be (A, B) 3-sequences of a matroid M in which the cardinalities of X_1 and Y_2 are minimised. Then $X_1 = L_x$ and $Y_2 = R_x$ and both $(A, \overrightarrow{X_1}, \overrightarrow{J_x}, x, \overrightarrow{Y_2}, B)$ and $(A, \overrightarrow{X_1}, x, \overrightarrow{J_x}, \overrightarrow{Y_2}, B)$ are 3-sequences of M. Moreover, if $\overrightarrow{J_x} = (y_1, y_2, \dots, y_k)$, then $(A, \overrightarrow{L_x}, y_1, \dots, y_i, x, y_{i+1}, \dots, y_k, \overrightarrow{R_x}, B)$ is a 3-sequence for all i in $\{1, 2, \dots, k-1\}$.*

Proof. Let $S_1 = (A, \overrightarrow{X_1}, x, \overrightarrow{X_2}, B)$ and $S_2 = (A, \overrightarrow{Y_1}, x, \overrightarrow{Y_2}, B)$. Applying Corollary 4.5 to S_1 and S_2, we deduce that $S_3 = (A, \overrightarrow{X_1}, \overrightarrow{Y_1} - X_1, x, \overrightarrow{Y_2} - X_1, B)$ is a 3-sequence of M. Consider this 3-sequence. We first observe that $X_1 \cap Y_2 = \emptyset$; for otherwise the number of elements to the right of x in S_3 is strictly less than $|Y_2|$, contradicting the minimality of $|Y_2|$. Thus $S_3 = (A, \overrightarrow{X_1}, \overrightarrow{Y_1} - X_1, x, \overrightarrow{Y_2}, B)$. Now, by duality, we may assume that x is a guts element of S_3. Therefore, by Lemma 4.6, x is a guts element of S_1 and S_2, so

$x \in \mathrm{cl}(A \cup X_1)$ and $x \in \mathrm{cl}(Y_2 \cup B)$. By Lemma 4.3, this implies that x may be moved in S_3 to any position that lies between the last member of X_1 and the first member of Y_2. Thus every element of $Y_1 - X_1$ is an element of the jump-set J_x of x. We show next that $J_x = Y_1 - X_1$.

Suppose that there is an element z of Y_2 that is also an element of J_x. Then there is an (A, B) 3-sequence in which z is to the left of x and another (A, B) 3-sequence of M in which z is to the right of x. Now z is to the right of x in S_2 as $z \in Y_2$. Let $S_4 = (A, \overrightarrow{Z_1}, x, \overrightarrow{Z_2}, B)$ be some 3-sequence of M in which z is to the left of x. Then, by applying Corollary 4.5 to S_2 and S_4, we get that $(A, \overrightarrow{Z_1}, \overrightarrow{Y_1 - Z_1}, x, \overrightarrow{Y_2 - Z_1}, B)$ is a 3-sequence of M. But then, as $z \in Z_1 \cap Y_2$, the 3-sequence $(A, \overrightarrow{Z_1}, \overrightarrow{Y_1 - Z_1}, x, \overrightarrow{Y_2 - Z_1}, B)$ has the property that the number of elements of X to the right of x is strictly less that $|Y_2|$, contradicting the minimality of $|Y_2|$. Hence, no element of Y_2 is an element of J_x. A similar argument shows that no element of X_1 is also an element of J_x.

Having established that $J_x = Y_1 - X_1$, we now conclude that $S_3 = (A, \overrightarrow{X_1}, \overrightarrow{J_x}, x, \overrightarrow{Y_2}, B)$, where $X_1 = L_x$ and $Y_2 = R_x$. Since x can be moved to any position between the last member of X_1 and the first member of Y_2 in S_3 maintaining a 3-sequence, the remainder of the lemma follows. □

The following are two useful consequences of Lemma 5.1.

Corollary 5.2. *Let $(A, \overrightarrow{X}, B)$ be an (A, B) 3-sequence S, and suppose $x \in X$. If x is a guts element of S, then $x \in \mathrm{cl}(A \cup L_x) \cap \mathrm{cl}(R_x \cup B)$. Dually, if x is a coguts element of S, then $x \in \mathrm{cl}^*(A \cup L_x) \cap \mathrm{cl}^*(R_x \cup B)$.*

Corollary 5.3. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence. Let x and y be elements of X. If $x \in J_y$, then there is an (A, B) 3-sequence in which x and y are consecutive.*

The next lemma says that, whenever we have a 3-sequence $(A, \overrightarrow{X}, B)$ in which an element $x \in X$ is adjacent to elements of its jump-set, x can jump over these elements and the resulting sequence is also an (A, B) 3-sequence.

Lemma 5.4. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence of a matroid M. Let $x \in X$. If x immediately follows an element x' in $\overrightarrow{X}$ and $x' \in J_x$, then interchanging x' and x results in an (A, B) 3-sequence of M.*

Proof. Let $(A, \overrightarrow{X}, B) = (A, \overrightarrow{L}, x', x, \overrightarrow{R}, B)$. Then $L_x \subseteq L$ so, by Corollary 5.2, $x \in \mathrm{cl}(A \cup L)$ or $x \in \mathrm{cl}^*(A \cup L)$. Thus, by Lemma 4.3, $(A, \overrightarrow{L}, x, x', \overrightarrow{R}, B)$ is a 3-sequence. □

Suppose $(A, \overrightarrow{X}, B)$ is an (A, B) 3-sequence of a matroid M. Lemma 5.4 enables us to determine exactly which orderings of X give an (A, B) 3-sequence of M provided, for each $x \in X$, we know L_x. This is the content of the next theorem.

Theorem 5.5. *Let M be a matroid, and suppose that there exists an (A, B) 3-sequence of M. Let $X = E(M) - (A \cup B)$ and let $\overrightarrow{X} = (x_1, x_2, \dots, x_n)$ be an ordering of the elements of X. Then $(A, \overrightarrow{X}, B)$ is an (A, B) 3-sequence of M if and only if $L_{x_i} \subseteq \{x_1, x_2, \dots, x_{i-1}\}$ for each $i \in \{1, 2, \dots, n\}$.*

Proof. If $(A, x_1, x_2, \dots, x_n, B)$ is a 3-sequence of M, then, by definition, $L_{x_i} \subseteq \{x_1, x_2, \dots, x_{i-1}\}$ for each $i \in \{1, 2, \dots, n\}$.

For the converse, suppose that $(x_1, x_2, \dots, x_n)$ is an ordering of the elements of X such that $L_{x_i} \subseteq \{x_1, x_2, \dots, x_{i-1}\}$ for each i. We shall show that, with this ordering, $(A, \overrightarrow{X}, B)$ is a 3-sequence of M. We begin by showing that $R_{x_i} \subseteq \{x_{i+1}, x_{i+2}, \dots, x_n\}$ for each i. If not, then $x_j \in R_{x_i}$ for some i and j with $j < i$. But then $x_i \in L_{x_j}$, yet $x_i \notin \{x_1, x_2, \dots, x_{j-1}\}$; a contradiction.

By our initial assumptions, there is an (A, B) 3-sequence of M. Let $S_0 = (A, y_1, y_2, \dots, y_n, B)$ be such a sequence. Then there exists some $j \in \{1, 2, \dots, n\}$ such that $y_j = x_1$. Since $L_{x_1} = \emptyset$ and $R_{x_1} \subseteq \{y_{j+1}, y_{j+2}, \dots, y_n\}$, we see that $\{y_1, y_2, \dots, y_{j-1}\} \subseteq J_{x_1} = J_{y_j}$. Therefore, by repeated application of Lemma 5.4, $S_1 = (A, y_j, y_1, y_2, \dots, y_{j-1}, y_{j+1}, \dots, y_n, B)$ is an (A, B) 3-sequence of M. Moreover, the sequence $(A, \overrightarrow{X}, B)$ and S_1 agree in the first two coordinates.

Set $S_1 = (A, z_1, z_2, \dots, z_n, B)$ by replacing $y_j, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n$ with $z_1, z_2, \dots, z_n$, respectively. Note that $z_1 = x_1$. Now there is some $j \in \{2, 3, \dots, n\}$ such that $z_j = x_2$. As $L_{x_2} \subseteq \{x_1\} = \{z_1\}$ and $R_{x_2} = R_{z_j} \subseteq \{z_{j+1}, z_{j+2}, \dots, z_n\}$, we deduce that $\{z_2, z_3, \dots, z_{j-1}\} \subseteq J_{x_2} = J_{z_j}$. Therefore, by repeated application of Lemma 5.4, $S_2 = (A, z_1, z_j, z_2, \dots, z_{j-1}, z_{j+1}, \dots, z_n, B)$ is an (A, B) 3-sequence of M that agrees with $(A, \overrightarrow{X}, B)$ in the first three coordinates. By considering S_2 and the sequence $(A, \overrightarrow{X}, B)$, and repeating this process, we eventually obtain the sequence $(A, \overrightarrow{X}, B)$. This shows that $(A, \overrightarrow{X}, B)$ is indeed an (A, B) 3-sequence of M. □

Lemma 5.6. *Let (A, X, B) be a 3-sequence. If y and z are elements of X such that $y \in L_z$, then there is a 3-sequence in which the elements of $J_y \cap J_z$ are consecutive. Moreover, if there is a 3-sequence in which y and z are consecutive, then $(A, L_y, J_y - J_z, y, J_y \cap J_z, z, J_z - J_y, R_z, B)$ is a 3-sequence.*

Proof. Since $y \in L_z$, it follows that $L_y \subseteq L_z$, so $L_y \cap (J_z \cup R_z) = \emptyset$ and, similarly, $R_z \cap (L_y \cup J_y) = \emptyset$. By Lemma 5.1, all of L_y, J_y, R_y, L_z, J_z, and R_z have orderings such that $(A, \overrightarrow{L_y}, y, \overrightarrow{J_y}, \overrightarrow{R_y}, B)$ and $(A, \overrightarrow{L_z}, \overrightarrow{J_z}, z, \overrightarrow{R_z}, B)$ are 3-sequences. By Corollary 4.5, there is a 3-sequence that begins like the first sequence above and ends like the second, namely $(A, \overrightarrow{L_y}, y, \overrightarrow{J_y}, \overrightarrow{L_z} \cap R_y, \overrightarrow{J_z} - J_y, z, \overrightarrow{R_z}, B)$. By Corollary 4.5 again, there is a 3-sequence that begins like the second sequence and ends like the last 3-sequence, namely $(A, \overrightarrow{L_z}, \overrightarrow{J_y} - L_z, \overrightarrow{J_z} - J_y, z, \overrightarrow{R_z}, B)$. But $J_y \cap R_z = \emptyset$, so $J_y - L_z = J_y \cap J_z$, and hence this sequence is $(A, \overrightarrow{L_z}, \overrightarrow{J_y} \cap J_z, \overrightarrow{J_z} - J_y, z, \overrightarrow{R_z}, B)$. Evidently, in this sequence, the elements of $J_y \cap J_z$ occur consecutively.

Now assume there is a 3-sequence in which y and z are consecutive. Then $L_z \cap R_y = \emptyset$. By Corollary 4.5 again, there is a 3-sequence that ends like the last 3-sequence in the previous paragraph and begins like the first 3-sequence, namely $(A, \overrightarrow{L_y}, y, \overrightarrow{J_y} \cap L_z, \overrightarrow{J_y} \cap J_z, \overrightarrow{J_z} - J_y, z, \overrightarrow{R_z}, B)$. Now, since $J_y \cap R_z = \emptyset$, we have that $\overrightarrow{J_y} \cap L_z = \overrightarrow{J_y} - J_z$. Furthermore, by repeated application of Lemma 5.4, y can be moved to the right of the elements of $J_y - J_z$, while z can be moved to the left of $J_z - J_y$. Hence, as required, $(A, \overrightarrow{L_y}, \overrightarrow{J_y} - J_z, y, \overrightarrow{J_y} \cap J_z, z, \overrightarrow{J_z} - J_y, \overrightarrow{R_z}, B)$ is a 3-sequence. □

Lemma 5.7. *Let (A, X, B) be a 3-sequence. If y and z are elements of X such that $y \in J_z$, then $(A, L_y \cup L_z, \{y, z\} \cup (J_y \cap J_z), R_y \cup R_z, B)$ is a 3-sequence.*

Proof. Observe that $z \in J_y$, so y and z are symmetric under the hypotheses. Suppose $x \in L_y \cap R_z$ for some $x \in X$. Then, as $y \in J_z$, there is an (A, B) 3-sequence in which y occurs to the left of z. But x must occur to the left of y in this sequence, putting x to the left of z; a contradiction. Thus $L_y \cap R_z$ is empty and, by symmetry, so is $L_z \cap R_y$. By Lemma 5.1, both $(A, \overrightarrow{L_y}, y, \overrightarrow{J_y}, \overrightarrow{R_y}, B)$ and $(A, \overrightarrow{L_z}, \overrightarrow{J_z}, z, \overrightarrow{R_z}, B)$ are 3-sequences. By Lemma 4.5, there is an (A, B) 3-sequence that begins $A, \overrightarrow{L_y}$ and finishes as in the second sequence. In this sequence, all the elements of $L_y \cup L_z$ are used before any other elements of X. By symmetry, there is an (A, B) 3-sequence in which all the elements of $R_y \cup R_z$ occur after all the other elements of X. Then, since $L_y \cap R_z$ and $L_z \cap R_y$ are empty, Lemma 4.5 implies that the last two sequences can be used to produce an (A, B) 3-sequence that uses all the elements of $L_y \cup L_z$ as the first elements of X and uses all the elements of $R_y \cup R_z$ as the last elements of X. This sequence must be $(A, L_y \cup L_z, \{y, z\} \cup (J_y \cap J_z), R_y \cup R_z, B)$. □

Lemma 5.8. *Let $(A, x_1, x_2, \ldots, x_n, B)$ be a 3-sequence. If x_i and x_{i+1} are of the same type, then*

$$(A, x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n, B)$$

is also a 3-sequence.

Proof. By duality, we may suppose that x_i and x_{i+1} are both guts elements. Then, as $x_i \in \mathrm{cl}(A \cup \{x_1, \ldots, x_{i-1}\})$ and $x_{i+1} \in \mathrm{cl}(A \cup \{x_1, \ldots, x_i\})$, it follows that $x_{i+1} \in \mathrm{cl}(A \cup \{x_1, \ldots, x_{i-1}\})$. By Lemma 4.3, this now implies that

$$(A, x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n, B)$$

is a 3-sequence. □

Given an (A, B) 3-sequence, the next result characterises when two adjacent elements, one guts and one coguts can be interchanged.

Lemma 5.9. *Let $(A, x_1, x_2, \ldots, x_n, B)$ be a 3-sequence. Suppose that x_i is a guts element and x_{i+1} is a coguts element. Let $A' = A \cup \{x_1, \ldots, x_{i-1}\}$ and $B' = \{x_{i+2}, \ldots, x_n\} \cup B$. Then*

$$(A, x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n, B)$$

is a 3-sequence if and only if $x_i \in \mathrm{cl}(A') \cap \mathrm{cl}(B')$ and $x_{i+1} \in \mathrm{cl}^*(A') \cap \mathrm{cl}^*(B')$.

Proof. Let $S = (A, x_1, x_2, \ldots, x_n, B)$ and $S' = (A, x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n, B)$. If $x_{i+1} \in \mathrm{cl}^*(B')$, then, by Lemma 4.3, S' is a 3-sequence.

For the converse, suppose that S' is a 3-sequence. Since x_i is a guts element and x_{i+1} is a coguts element of S, it follows that $x_i \in \mathrm{cl}(A')$ and $x_{i+1} \in \mathrm{cl}^*(B')$. Moreover, as both S and S' are (A, B) 3-sequences, Lemma 4.6 implies that x_i is a guts element and x_{i+1} is a coguts element of S'. Therefore $x_i \in \mathrm{cl}(B')$ and $x_{i+1} \in \mathrm{cl}^*(A')$. □

The next result states that if $(A, \overrightarrow{X}, B)$ is a 3-sequence, then no guts element can have two adjacent coguts elements in its jump-set. This will be useful when discussing the structure of large jump-sets.

Lemma 5.10. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence of a matroid M. Let $y \in X$, and suppose that x and x' are adjacent elements in $\overrightarrow{X}$ with x preceding x'.*

(i) *If y is a guts element, but x and x' are both coguts elements of this 3-sequence, then J_y does not contain both x and x'.*
(ii) *Dually, if y is a coguts element, but x and x' are both guts elements of this 3-sequence, then J_y does not contain both x and x'.*

Proof. By duality, it suffices to prove that (i) holds. Suppose that y is a guts element, but x and x' are both elements of J_y. Since x and x' are adjacent in $(A, \overrightarrow{X}, B)$, it follows by Lemma 5.1 that, for some fixed ordering of the subsets X_1 and X_2 of X, each of $(A, \overrightarrow{X_1}, y, x, x', \overrightarrow{X_2}, B)$, $(A, \overrightarrow{X_1}, x, y, x', \overrightarrow{X_2}, B)$, and $(A, \overrightarrow{X_1}, x, x', y, \overrightarrow{X_2}, B)$ are 3-sequences of M. Let $A' = A \cup X_1$ and $B' = B \cup X_2$. Using Lemma 5.9 and considering the first and third of these sequences, we deduce that $y \in \mathrm{cl}(A') \cap \mathrm{cl}(B')$. Similarly, we also deduce that $x \in \mathrm{cl}^*(A') \cap \mathrm{cl}^*(B' \cup \{x'\})$, and $x' \in \mathrm{cl}^*(A' \cup \{x\}) \cap \mathrm{cl}^*(B')$. It now follows by Lemma 2.1 that $\mathrm{r}(A' \cup \{y, x, x'\}) = \mathrm{r}(A' \cup \{y, x\}) + 1$, $\mathrm{r}(A' \cup \{y, x\}) = \mathrm{r}(A' \cup \{y\}) + 1$, and $\mathrm{r}(A' \cup \{y\}) = \mathrm{r}(A')$. Thus $\mathrm{r}(A' \cup \{y, x, x'\}) = \mathrm{r}(A') + 2$ and so, as $(A' \cup \{y, x, x'\}, B')$ is an 3-separation of M, $\mathrm{r}(A' \cup \{y, x, x'\}) + \mathrm{r}(B') - \mathrm{r}(M) = 2$. Since $\mathrm{r}(A' \cup \{y, x, x'\}) = \mathrm{r}(A') + 2$, this implies that $\mathrm{r}(A') + \mathrm{r}(B') = \mathrm{r}(M)$. As $y \in \mathrm{cl}(A') \cap \mathrm{cl}(B')$, it follows by submodularity that $\mathrm{r}(A') + \mathrm{r}(B') \geqslant \mathrm{r}(A' \cup B') + 1$ and so $\mathrm{r}(A' \cup B') \leqslant \mathrm{r}(M) - 1$. But this means that, as $y \in \mathrm{cl}(A' \cup B')$, the set $\{x, x'\}$ is 2-separating, contradicting the fact that M is 3-connected. We conclude that J_y does not contain both x and x'. □

The following corollary is an immediate consequence of Lemma 5.10.

Corollary 5.11. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence and suppose that $y \in X$. If y is a guts element of X and J_y contains two coguts elements x and x', then x and x' are not adjacent in any (A, B) 3-sequence. Dually, if y is a coguts element of this sequence and J_y contains two guts elements x and x', then x and x' are not adjacent in any (A, B) 3-sequence.*

6. Segments, cosegments, and fans

In this section, we explain why segments, cosegments, and fans occur in 3-sequences and we analyse how such substructures behave in a 3-sequence (A, X, B). The reader may recall from Section 2 that the 3-element segments in X coincide with all triangles of X

containing only guts elements, while a 2-element subset of X is a segment if and only if it consists of two guts elements that can be made consecutive in some (A, B) 3-sequence. The second part of the next result helps to explain the reason for these definitions.

Lemma 6.1. *Let $(A, \vec{X}, B)$ be a 3-sequence and let Y be a subset of X with at least three elements such that every 3-element subset of Y is a triangle.*

(i) *Let $\vec{X} \cap Y = (y_1, y_2, \dots, y_k)$. Then there is an (A, B) 3-sequence in which the elements of Y are consecutive and preserve the ordering $(y_1, y_2, \dots, y_k)$.*
(ii) *If $|Y| \geqslant 4$, then each element of Y is a guts element of $(A, \vec{X}, B)$.*

Proof. Let $S_0 = (A, \vec{X}, B)$. Since every 3-element subset of Y is a triangle, $y_i \in \mathrm{cl}(\{y_{i+1}, y_{i+2}\})$ for all $i \in \{1, 2, \dots, k-2\}$. Therefore, by Lemma 4.3, the sequence S_1 obtained from S_0 by moving y_{k-2} to the right so that it immediately precedes y_{k-1} is also an (A, B) 3-sequence. So too is the sequence S_2 that is obtained from S_1 by moving y_{k-3} to the right so that it immediately precedes y_{k-2}. By repeating this process, we eventually obtain an (A, B) 3-sequence S_{k-2} in which $y_1, y_2, \dots, y_{k-1}$ is a consecutive subsequence having y_k to its right. A final application of Lemma 4.3 allows y_k to be moved to the left so that it immediately succeeds y_{k-1}, and (i) follows.

To prove (ii), let $y \in Y$. Then, in $(A, \vec{X}, B)$, there are two elements y' and y'' of Y such that, in $\vec{X}$, both occur to the right of y or both occur to the left of y. Since $y \in \mathrm{cl}(\{y', y''\})$, it now follows by Lemma 4.1 that y is a guts element of $(A, \vec{X}, B)$. $\square$

In this section, we establish some structural properties for segments that are contained in an (A, B) 3-sequence. All of the results in this section are written for segments. However, each of them can be dualised to obtain the corresponding results for cosegments by simply interchanging "segment" with "cosegment", "guts" with "coguts", and "closure" with "coclosure".

Corollary 6.2 is an immediate consequence of Lemma 5.8.

Corollary 6.2. *Let $(A, \vec{X}, B)$ be a 3-sequence in a matroid M, and suppose that Y is a segment of M whose elements are consecutive in $\vec{X}$. Then replacing the subsequence of $(A, \vec{X}, B)$ consisting of the elements of Y with any other ordering of Y results in an (A, B) 3-sequence of M.*

Lemma 6.3. *Let $(A, \vec{X}, B)$ be a 3-sequence. For any consecutive subsequence $\vec{Y}$ of $\vec{X}$ consisting entirely of guts elements, if $|Y| \geqslant 2$, then Y is a segment of M.*

Proof. Let $\vec{X} = (x_1, x_2, \dots, x_n)$ and suppose all of $x_i, x_{i+1}, \dots, x_{i+j}$ are guts elements for some $j \geqslant 1$. Then $x_k \in \mathrm{cl}(A \cup \{x_1, x_2, \dots, x_{k-1}\})$ for all $k \in \{i, i+1, \dots, i+j\}$. Using a simple induction argument, this implies that, for all such k,

$$x_k \in \mathrm{cl}\big(A \cup \{x_1, x_2, \dots, x_{i-1}\}\big).$$

Since $(A, \overrightarrow{X}, B)$ is a 3-sequence,

$$\mathrm{r}\big(A \cup \{x_1, x_2, \dots, x_{i-1}\}\big) + \mathrm{r}\big(\{x_i, x_{i+1}, \dots, x_n\} \cup B\big) = \mathrm{r}(M) + 2.$$

Therefore, as $\mathrm{r}(A \cup \{x_1, x_2, \dots, x_{i-1}\}) = \mathrm{r}(A \cup \{x_1, x_2, \dots, x_{i+j}\})$, we have $\mathrm{r}(A \cup \{x_1, x_2, \dots, x_{i+j}\}) + \mathrm{r}(\{x_i, x_{i+1}, \dots, x_n\} \cup B) = \mathrm{r}(M) + 2$. By submodularity, this implies that $\mathrm{r}(\{x_i, x_{i+1}, \dots, x_{i+j}\}) \leqslant 2$, and so $\{x_i, x_{i+1}, \dots, x_{i+j}\}$ is a segment. □

Lemma 6.4. *If, in a* 3*-sequence* (A, X, B)*, each of* x*,* y*, and* z *is in* X *and is in the jump-set of each of the others, then* $\{x, y, z\}$ *is a segment or a cosegment.*

Proof. Without loss of generality, we may assume that there is a 3-sequence $(A, L_x, \overrightarrow{J'_x}, x, y, \overrightarrow{J''_x}, z, \overrightarrow{J'''_x}, R_x, B)$ where $J_x = J'_x \cup y \cup J''_x \cup z \cup J'''_x$. We argue by induction on $|J''_x|$ to establish the result. If $|J''_x| = 0$, then, without loss of generality, by Theorem 5.4, we can arrange x, y, and z consecutively in the sequence so that the first two are guts elements. Since the last element can jump over the first two, it follows, by the dual of Lemma 5.10, that all three elements are guts elements. Then Lemma 6.3 implies that $\{x, y, z\}$ is a triangle.

Now assume the result holds for $|J''_x| = n$ and let $|J''_x| = n + 1$, say $\overrightarrow{J''_x} = (a_1, a_2, \dots, a_{n+1})$. By the induction assumption, $a_1 \notin J_y$ and $a_{n+1} \notin J_z$. Because y and z are in each other's jump-sets, there is a 3-sequence S_{yz} in which y and z occur consecutively. Because $a_{n+1} \notin J_z$, we must have that a_{n+1} occurs to the left of y in S_{yz}. Thus $a_{n+1} \in J_y$. Similarly, as $a_1 \notin J_y$, we must have that a_1 occurs to the right of z in S_{yz}. Hence $a_1 \in J_z$. This means that there is a 3-sequence $S_{a_1 z}$ in which a_1 and z occur consecutively. As $a_{n+1} \notin J_z$, we must have that a_{n+1} occurs to the left of a_1 in $S_{a_1 z}$. Hence $a_{n+1} \in J_{a_1}$. Now $(A, L_x, \overrightarrow{J'_x}, y, x, a_1, a_2, \dots, a_{n+1}, z, \overrightarrow{J'''_x}, R_x, B)$ is a 3-sequence and each of x, a_1, and a_{n+1} is in the jump-set of the other. Thus, by the induction assumption, $\{x, a_1, a_{n+1}\}$ is a triangle or a triad. Therefore, by Lemma 4.3, $(A, L_x, \overrightarrow{J'_x}, x, y, a_1, a_{n+1}, a_2, \dots, a_n, z, \overrightarrow{J'''_x}, R_x, B)$ is a 3-sequence. As a_{n+1} is in all of J_{a_1}, J_y, and J_x, we get that $(A, L_x, \overrightarrow{J'_x}, a_{n+1}, x, y, a_1, a_2, \dots, a_n, z, \overrightarrow{J'''_x}, R_x, B)$ is a 3-sequence. Then the induction assumption implies that $\{x, y, z\}$ is a triangle of guts elements or a triad of coguts elements, that is, a segment or a cosegment. □

The next result characterises non-degenerate segments and cosegments in terms of the jump-sets of their elements.

Corollary 6.5. *Let* (A, X, B) *be a* 3*-sequence. Let* Y *be a subset of* X *having at least three elements. Then* Y *is a segment or a cosegment if and only if every member of* Y *is in the jump-sets of all members of* $Y - \{y\}$*.*

Proof. If Y is a segment or a cosegment, then, by Lemma 6.1, there is an (A, B) 3-sequence in which the elements of Y are consecutive. Moreover, Corollary 6.2 implies that if $y \in Y$, then y is in the jump-sets of every member of $Y - \{y\}$. The converse is a straightforward consequence of Lemma 6.4. □

In the characterisation of non-degenerate segments and cosegments just given, the hypothesis did not require that all elements of Y are of the same type. By adding this hypothesis, we can extend the last result to characterise degenerate segments and cosegments where we recall, for example, that two guts elements form a segment if there is a 3-sequence in which they are consecutive. The proof is a straightforward combination of Lemmas 5.1 and 5.8.

Corollary 6.6. *Let $(A, \vec{X}, B)$ be a 3-sequence. Let x and y be elements of X of the same type. Then $\{x, y\}$ is a segment or a cosegment if and only if $x \in J_y$.*

The next result is our first example of a number of results about a set of elements in an (A, B) 3-sequence forming a barrier over which no element can jump. Its proof is an easy combination of Corollary 5.3 and Lemma 6.4.

Corollary 6.7. *Let $(A, \vec{X}, B)$ be a 3-sequence. Let x and y be elements of X of the different types. If x and y are in each other's jump-sets, then there is an (A, B) 3-sequence in which x and y are consecutive. Moreover, there are sets L and R such that, for every (A, B) 3-sequence of the form $(A, Z_1, \{x, y\}, Z_2, B)$, the sets Z_1 and Z_2 equal L and R, respectively.*

In a 3-sequence $(A, \vec{X}, B)$, a segment S is *maximal* if there is no segment T that is contained in X and properly contains S. The next result gives several useful properties of a maximal segment S showing, for example, that, in any (A, B) 3-sequence, most of the members of S are consecutive. Moreover, whenever all the members of S are consecutive in an (A, B) 3-sequence, the sets of elements occurring to the left and right of S are uniquely determined. Indeed, S forms a barrier in every (A, B) 3-sequence with no element being able to jump over all of S.

Theorem 6.8. *Let S be a maximal segment in a 3-sequence $(A, \vec{X}, B)$ and suppose that $(s_1, s_2, \ldots, s_k)$ is the ordering induced on S by $\vec{X}$. Then the following hold.*

(i) *No element of $X - S$ is in the jump-sets of two distinct elements of S.*
(ii) *There are subsets L_S and R_S of $X - S$ such that, in every (A, B) 3-sequence of the form (A, L, S, R, B), the sets L and R equal L_S and R_S, respectively.*
(iii) *The elements $s_2, s_3, \ldots, s_{k-1}$ are consecutive.*
(iv) *If $k \geqslant 3$ and s_1 is not adjacent to s_2 in $\vec{X}$, then every element of $\vec{X} - S$ lying between s_1 and s_2 is in $J_{s_1} \cap L_S$.*
(v) *If $k \geqslant 3$ and s_k is not adjacent to s_{k-1} in $\vec{X}$, then every element of $\vec{X} - S$ lying between s_k and s_{k-1} is in $J_{s_k} \cap R_S$.*

Proof. Suppose that $z \in X - S$ and z is in the jump-sets of s_i and s_j for some distinct i and j. As $\{s_i, s_j\} \subseteq S$, by Corollary 6.5 or 6.6, s_i and s_j jump each other. Hence s_i, s_j, and z are mutually jumping so, by Lemma 6.4, $\{s_i, s_j, z\}$ is contained in a segment. Hence $S \cup \{z\}$ is a segment, contradicting the maximality of S. Thus (i) holds.

Part (ii) follows from (i) by noting that if (A, L, S, R, B) and (A, L', S, R', B) are 3-sequences and $z \in L \cap R'$, then z jumps every element of S. Hence $L \cap R' = \emptyset$ and, similarly, $R \cap L' = \emptyset$.

Part (iii) is trivial if $k \leqslant 3$ so assume that $k \geqslant 4$. Let z be an element of $X - S$ that lies between s_2 and s_{k-1} in $\overrightarrow{X}$. By Lemma 6.1, there is an (A, B) 3-sequence in which $s_1, s_2, \ldots, s_k$ occurs as a consecutive subsequence. The element z must occur to the left or right of this subsequence. In both cases, z is in the jump-sets of two elements of S, contradicting (i).

Parts (iv) and (v) follow by similar arguments to the above. □

The final result of this section, which summarises some basic properties of fans when they occur in (A, B) 3-sequences, justifies the definition of a degenerate fan. In the next section, we shall discuss a special substructure in a 3-sequence called a clock. As we shall see, every clock is a union of two fans. If $s_1, s_2, \ldots, s_k$ is a fan ordering of a fan F, then $s_k, s_{k-1}, \ldots, s_1$ is also a fan ordering of F. Moreover, when $k \geqslant 5$, this reversal is the only other fan ordering of F. When $k = 4$, there are two other fan orderings, namely s_1, s_3, s_2, s_4 and its reversal.

Theorem 6.9. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence and F be a subset of X such that F is a fan. Let $x_1, x_2, \ldots, x_k$ be the ordering of F induced by $\overrightarrow{X}$. Then the following hold.*

(i) *There is an (A, B) 3-sequence having $x_1, x_2, \ldots, x_k$ as a consecutive subsequence.*
(ii) *The elements $x_1, x_2, \ldots, x_k$ are alternately guts and coguts elements.*
(iii) *The ordering $x_1, x_2, \ldots, x_k$ is a fan ordering of F.*
(iv) *In all (A, B) 3-sequences, the elements of F occur in the order $x_1, x_2, \ldots, x_k$.*

Proof. If $k = 2$, then the definition of a degenerate fan ensures that all four parts hold. Now suppose that $k = 3$. By duality, we may assume that $\{x_1, x_2, x_3\}$ is a triangle. Since $x_3 \in \mathrm{cl}(\{x_1, x_2\})$ and $x_1 \in \mathrm{cl}(\{x_2, x_3\})$, we deduce that x_1 and x_3 are guts elements. By our definition of a three-element fan, it follows that x_2 is a coguts element. Moreover, by Lemma 4.5, we can move x_1 and x_3 in $\overrightarrow{X}$ to get an (A, B) 3-sequence having x_1, x_2, x_3 as a consecutive subsequence. To see that (iv) holds when $k = 3$, note that, if not, then there is an (A, B) 3-sequence in which the fan elements occur in the order x_3, x_2, x_1. In that case, x_1, x_2, and x_3 are mutually jumping so, by Lemma 6.4, all these elements are of the same type; a contradiction. We conclude that (i)–(iv) hold for $k = 2$ and $k = 3$.

Now assume that (i)–(iv) hold for $k < n$ and let $k = n \geqslant 4$. If x_k is in both a triangle and a triad of F, then $x_k \in \mathrm{cl}(\{x_1, x_2, \ldots, x_{k-1}\})$ and $x_k \in \mathrm{cl}^*(\{x_1, x_2, \ldots, x_{k-1}\})$. Thus x_k is both a guts and a coguts element of $\overrightarrow{X}$; a contradiction to Corollary 4.2. Hence, by duality, we may assume that x_k is in a triangle but no triad of F. It follows that $F - \{x_k\}$ is a fan. Likewise, $F - \{x_1\}$ is a fan. By the induction assumption, (i)–(iv) hold for both $F - \{x_1\}$ and $F - \{x_k\}$. It is straightforward to deduce that (ii) and (iv) hold for F. Moreover, as there is an (A, B) 3-sequence having $x_1, x_2, \ldots, x_{k-1}$ as a consecutive subsequence and (iv) holds for F, the element x_k occurs to the right of this subsequence. By Lemma 4.5, since $x_k \in \mathrm{cl}(\{x_1, x_2, \ldots, x_{k-1}\})$, it follows that (i) holds for F. Finally, since $x_2, x_3, \ldots, x_k$ is a fan ordering of $F - \{x_1\}$, and x_k is in a triangle but no triad of F, we deduce that

$\{x_{k-2}, x_{k-1}, x_k\}$ is a triangle. Combining this with the fact that $x_1, x_2, \ldots, x_{k-1}$ is a fan ordering of $F - \{x_k\}$ gives that $x_1, x_2, \ldots, x_k$ is a fan ordering of F. □

7. Clocks

In this section, we investigate what happens when two elements y and z have jump-sets that meet in two or more elements. One case when we know this occurs is when y and z have the same type and belong to a common segment or cosegment with four or more elements. The other case, which will be the focus of this section arises, for example, when y and z have different types.

Lemma 7.1. *Let (A, X, B) be a 3-sequence in a matroid M and let y and z be elements of X that can be made consecutive in some (A, B) 3-sequence with y to the left of z. Suppose that $J_y \cap J_z = \{x_1, x_2, \ldots, x_n\}$ for some $n \geqslant 2$. Then one of the following holds*:

(i) $\{y, z, x_1, x_2, \ldots, x_n\}$ *is a segment or a cosegment*; *or*
(ii) (a) $y \in L_z$;
  (b) *the elements of $J_y \cap J_z$ occur in the same order, say $x_1, x_2, \ldots, x_n$, in every (A, B) 3-sequence*;
  (c) $(x_1, x_2, \ldots, x_n)$ *is a fan whose elements are alternately guts and coguts elements*;
  (d) *exactly one of y and z is a guts element*; *and*
  (e) $J_{x_i} \cap J_{x_{i+s}} = J_{x_j} \cap J_{x_{j+t}}$ *for all non-zero s and t such that $\{i, i+s, j, j+t\} \subseteq \{1, 2, \ldots, n\}$.*

Proof. First suppose that $y \in J_z$. Then, for each i in $\{1, 2, \ldots, n\}$, the elements y, z, and x_i are all in each other's jump-sets. Thus, by Lemma 6.4, $\{y, z, x_i\}$ is either a segment or a cosegment. It follows that $\{y, z, x_1, x_2, \ldots, x_n\}$ is a segment or a cosegment.

Now assume that $y \in L_z$. By Lemma 5.6, there is an (A, B) 3-sequence in which $(y, J_y \cap J_z, z)$ is a consecutive subsequence. Let this subsequence be $(y, x_1, x_2, \ldots, x_n, z)$. Then there are (A, B) 3-sequences in which $(y, z, x_1, x_2, \ldots, x_n)$ and $(x_1, x_2, \ldots, x_n, y, z)$ occur as consecutive subsequences. It follows by Lemma 5.8 that exactly one of y and z is a guts element. Moreover, for some disjoint sets A' and B' that contain A and B, respectively and satisfy $E(M) - (A' \cup B') = \{y, z, x_1, x_2, \ldots, x_n\}$, both $A' \cup \{x_1, x_2, \ldots, x_i\}$ and $\{x_j, x_{j+1}, \ldots, x_n\} \cup B'$ are 3-separating for all i and j in $\{1, 2, \ldots, n\}$. Hence, for all i in $\{1, 2, \ldots, n-2\}$, by Lemma 3.1, $\{x_i, x_{i+1}, x_{i+2}\}$ is 3-separating. By Lemma 3.2, the 3-sets $\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \ldots, \{x_{n-2}, x_{n-1}, x_n\}$ are all triangles, all triads, or are alternately triangles and triads. Since one of y and z is a guts element and the other is a coguts element, and both are in the jump-set of each x_i, it follows by Lemma 5.10 that the elements $x_1, x_2, \ldots, x_n$ are alternately guts and coguts elements. Hence if $n \geqslant 3$, then $\{x_1, x_2, \ldots, x_n\}$ is neither a segment nor a cosegment; so it is a fan. It follows, by Theorem 6.9, that, when $n \geqslant 3$, the elements $x_1, x_2, \ldots, x_n$ occur in the same order in all (A, B) 3-sequences. Now let $n = 2$. Then, since y jumps both x_1 and x_2 and these are of different types, they are not in each other's jump-sets, so x_1, x_2 is a degenerate fan. Thus (c) and (b) hold for $n = 2$.

To prove (ii)(e), first note that we may assume that $n \geqslant 3$. Next we show that, for all $s \geqslant 1$ such that $i, i+s \in \{1, 2, \dots, n\}$,

$$J_{x_i} \cap J_{x_{i+1}} = J_{x_i} \cap J_{x_{i+s}}. \tag{7.1}$$

Clearly, we may assume that $s \geqslant 2$. By (ii)(b), $x_1, x_2, \dots, x_n$ always occur in that order in all (A, B) 3-sequences. Thus if $a \in J_{x_i} \cap J_{x_{i+s}}$, then there are (A, B) 3-sequences in which a is to the left of x_i and hence of x_{i+1}, and to the right of x_{i+s} and hence of x_{i+1}. Thus $a \in J_{x_{i+1}}$ and $J_{x_i} \cap J_{x_{i+1}} \supseteq J_{x_i} \cap J_{x_{i+s}}$. To establish the reverse inclusion, suppose that $a \in J_{x_i} \cap J_{x_{i+1}}$. Then there is certainly an (A, B) 3-sequence in which a is to the left of x_i and hence of x_{i+s}. Moreover, there is an (A, B) 3-sequence in which a is to the right of x_{i+1}. Because $x_i \in \mathrm{cl}^{(*)}(\{x_{i+1}, x_{i+2}\})$, we can use Lemma 4.3 to move x_i in this sequence so that it immediately precedes x_{i+1}. Similarly, we can move $x_{i+2}, x_{i+3}, \dots, x_{i+s}$ one at a time so that these elements occur consecutively. As a is to the right of x_{i+s} in the last sequence, we deduce that $a \in J_{x_{i+s}}$ and (7.1) follows.

By repeatedly applying (7.1), we deduce that (ii)(e) holds. □

Theorem 7.2. *Let (A, X, B) be a 3-sequence. Let y and z be elements of X such that $J_y \cap J_z = \{x_1, x_2, \dots, x_n\}$ for some $n \geqslant 2$ and $\{y, z\} \cup (J_y \cap J_z)$ contains both a guts and a coguts element. Let $J_{x_1} \cap J_{x_2} = \{y_1, y_2, \dots, y_m\}$. Then*

(i) *X contains a clock with fans $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_m\}$.*
(ii) *$\{y, z\} \subseteq \{y_1, y_2, \dots, y_m\}$.*
(iii) *$J_{x_i} \cap J_{x_{i+s}} = \{y_1, y_2, \dots, y_m\}$ for all i and s such that $1 \leqslant i < i+s \leqslant n$; these elements have the same ordering, say $(y_1, y_2, \dots, y_m)$, in every (A, B) 3-sequence; and this ordering is a fan ordering.*
(iv) *$J_{y_j} \cap J_{y_{j+t}} = \{x_1, x_2, \dots, x_n\}$ for all j and t such that $1 \leqslant j < j+t \leqslant m$; these elements have the same ordering, say $(x_1, x_2, \dots, x_n)$, in every (A, B) 3-sequence; and this ordering is a fan ordering.*
(v) *There are (A, B) 3-sequences in which each of $(y_1, y_2, \dots, y_m, x_1, x_2, \dots, x_n)$ and $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$ occurs as a consecutive subsequence.*

Proof. By Lemmas 5.6 and 5.7, there is an (A, B) 3-sequence in which the elements of $J_y \cap J_z$ are consecutive with the ordering $(x_1, x_2, \dots, x_n)$, say. Then y and z are both in $J_{x_1} \cap J_{x_2}$ and Lemma 7.1 implies that either

(a) the elements of $\{x_1, x_2\} \cup (J_{x_1} \cap J_{x_2})$ form a segment or a cosegment; or
(b) the elements of $J_{x_1} \cap J_{x_2}$ form a fan whose elements occur in the same order in every (A, B) 3-sequence and are alternately guts and coguts elements; exactly one of x_1 and x_2 is a guts element; and $x_1 \in L_{x_2}$.

In the first case, there is an (A, B) 3-sequence in which y and z are consecutive. Then, since y and z are of the same type, it follows by Lemma 7.1, that the elements of $\{y, z\} \cup (J_y \cap J_z)$ are all of the same type, contradicting the hypothesis.

We conclude that (b) holds. Now let $(y_1, y_2, \dots, y_m)$ be the ordering on the elements of $J_{x_1} \cap J_{x_2}$ in some (A, B) 3-sequence. Then, by Lemma 7.1(ii), $J_{x_1} \cap J_{x_2} = J_{x_i} \cap J_{x_{i+s}}$ for all i and s with $1 \leqslant i < i + s \leqslant n$, and these elements occur in the order $(y_1, y_2, \dots, y_m)$ in every (A, B) 3-sequence.

Let $y = y_j$ and $z = y_{j+t}$ for some $t \geqslant 1$. Since y_j and y_{j+1} occur consecutively in some (A, B) 3-sequence, and exactly one is a guts element, by Lemma 7.1(ii) again, $J_{y_j} \cap J_{y_{j+1}} = J_y \cap J_z = J_{y_k} \cap J_{y_{k+t}} = \{x_1, x_2, \dots, x_n\}$ for all k and t such that $1 \leqslant k < k + t \leqslant m$. Moreover, $\{x_1, x_2, \dots, x_n\}$ is a fan and, since the elements of this fan occur in the order $(x_1, x_2, \dots, x_n)$ in some (A, B) 3-sequence, they occur in this order in every such sequence.

By Lemma 7.1, $y_1, x_1, x_2, \dots, x_n, y_2$ occur consecutively in that order in some (A, B) 3-sequence. Hence so do $y_1, y_2, x_1, x_2, \dots, x_n$ and $x_1, x_2, \dots, x_n, y_1, y_2$. Because $(y_1, y_2, \dots, y_m)$ is a fan, we know that $y_{i+2} \in \mathrm{cl}^{(*)}(\{y_i, y_{i+1}\})$ provided $1 \leqslant i \leqslant n - 2$. Thus we can move the elements $y_3, y_4, \dots, y_m$ one at a time so that both $(y_1, y_2, \dots, y_m, x_1, x_2, \dots, x_n)$ and $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$ occur as consecutive subsequences of (A, B) 3-sequences. We conclude that (v) holds and also that $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_m\}$ are fans F_x and F_y such that for some set A' containing A but avoiding $B \cup F_x \cup F_y$, all of A', $A' \cup F_x$, $A' \cup F_y$, and $A' \cup F_x \cup F_y$ are 3-separating.

Finally, we note that if there is an element x_0 of X such that $x_0, x_1, \dots, x_n$ is a fan, then, as in the last paragraph, both (x_0, F_x, F_y) and (F_y, x_0, F_x) occur as consecutive subsequences of (A, B) 3-sequences. Hence $x_0 \in J_{y_1} \cap J_{y_2}$; a contradiction. □

The next result shows that the fans in a clock are uniquely determined by the clock.

Corollary 7.3. *Let (A, X, B) be an (A, B) 3-sequence and let Z and Z' be clocks contained in X with fans F_1 and F_2, and F_1' and F_2', respectively. If $Z' \supseteq Z$, then $Z' = Z$, and $\{F_1', F_2'\} = \{F_1, F_2\}$.*

Proof. Let $F_1' = \{x_1, x_2, \dots, x_n\}$ and $F_2' = \{y_1, y_2, \dots, y_m\}$. Then $J_{x_1} \cap J_{x_2} \supseteq F_2'$ and $J_{y_1} \cap J_{y_2} \supseteq F_1'$. By Theorem 7.2, $J_{x_1} \cap J_{x_2}$ and $J_{y_1} \cap J_{y_2}$ are disjoint fans whose union is a clock. Since F_2' and F_1' are maximal disjoint fans whose union is a clock, we conclude that $J_{x_1} \cap J_{x_2} = F_2'$ and $J_{y_1} \cap J_{y_2} = F_1'$. By Theorem 7.2 again, we may assume that both $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_m)$ maintain these orderings in every (A, B) 3-sequence. Hence $J_{x_i} \cap F_1' = \emptyset$ for all i and $J_{y_j} \cap F_2' = \emptyset$ for all i.

Now consider Z. Its elements are contained in $F_1' \cup F_2'$, and every element of F_1 is in the jump-set of every element of F_2. Without loss of generality, we may assume that F_1 contains some x_i. Then no element of F_1' is in F_2. Hence $F_2 \subseteq F_2'$, so no element of F_2' is in F_1. Thus $F_1 \subseteq F_1'$. We conclude, since F_1 and F_2 are maximal disjoint fans contained in X whose union is a clock, that $F_1 = F_1'$ and $F_2 = F_2'$. Thus $Z' = Z$. □

The next result describes how the elements of a clock can be permuted within the clock in an (A, B) 3-sequence.

Theorem 7.4. *Let (A, X, B) be a 3-sequence with a clock $F_x \cup F_y$, where F_x and F_y are the fans of this clock with orderings $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_m)$, respectively, in this sequence. Let $(A, L, \overrightarrow{(F_x \cup F_y)_1}, R, B)$ be a 3-sequence in which the clock elements are consecutive. Then a sequence of the form $(A, L, \overrightarrow{(F_x \cup F_y)_2}, R, B)$ is a 3-sequence if and only if the subsequence of the elements of F_x has the ordering $(x_1, x_2, \ldots, x_n)$ and the subsequence of the elements of F_y has the ordering $(y_1, y_2, \ldots, y_m)$.*

Proof. By Theorem 7.2, the elements of F_x and F_y occur in the order $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_m)$, respectively, in $(A, L, \overrightarrow{(F_x \cup F_y)_1}, R, B)$, and these orderings are fan orderings for F_x and F_y. Let $A' = A \cup L$ and $B' = R \cup B$. By the definition of a clock, $x_1 \in \mathrm{cl}^{(*)}(A')$ and $x_2 \in \mathrm{cl}^{(*)}(A' \cup x_1)$; and $y_1 \in \mathrm{cl}^{(*)}(A')$ and $y_2 \in \mathrm{cl}^{(*)}(A' \cup y_1)$. Thus, there is an (A, B) 3-sequence that begins A, L, x_1, x_2. Since $(x_1, x_2, \ldots, x_n)$ is a fan ordering for F_x, using Lemma 4.3, we can move $x_3, x_4, \ldots, x_n$ one at a time in this sequence until the sequence becomes $(A, L, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m, R, B)$. Call this sequence S_1. By symmetry, $(A, L, y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_n, R, B)$ is also an (A, B) 3-sequence.

It remains to show that every ordering of $F_x \cup F_y$ in which the elements of F_x and F_y occur in the order $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_m)$, respectively, gives an (A, B) 3-sequences. To see this, we observe, from S_1 and S_2, that $\{y_1, y_2, \ldots, y_m\} \subseteq J_{x_i}$ for each $i \in \{1, 2, \ldots, n\}$, and $\{x_1, x_2, \ldots, x_n\} \subseteq J_{y_j}$ for each $j \in \{1, 2, \ldots, m\}$. The result follows by repeatedly applying Lemma 5.4. □

Let (A, X, B) be a 3-sequence and $F_x \cup F_y$ be a clock contained in X where F_x and F_y are the fans of this clock. By Theorem 7.2, the ordering of each of F_x and F_y is the same in every (A, B) 3-sequence. This means that each of F_x and F_y has a first and last element in X. These elements are called the *ends* of the clock. All other elements of the clock are called *internal clock elements*.

The next result, a consequence of Theorem 7.2, tells us that no element can jump over a clock, so the clock forms a barrier between the elements to its left and those to its right.

Corollary 7.5. *Let (A, X, B) be a 3-sequence and $F_x \cup F_y$ be a clock contained in X where F_x and F_y, the fans of this clock, have orderings $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_m)$, respectively, in (A, X, B). Suppose $z \in X - (F_x \cup F_y)$. Then both $J_z \cap F_x$ and $J_z \cap F_y$ have at most one element.*

Proof. Suppose that J_z contains x_i and x_j for some distinct i and j. Then $z \in J_{x_i} \cap J_{x_j}$. But this contradicts the fact that $J_{x_i} \cap J_{x_j} = F_y$. □

As an immediate consequence of this, we get that when the elements of a clock are consecutive in a 3-sequence, the sets of elements occurring to the left and to the right of the clock are uniquely determined.

Corollary 7.6. *Let (A, X, B) be an (A, B) 3-sequence and let Z be a clock contained in X. If (A, A_1, Z, B_1, B) and (A, A_2, Z, B_2, B) are 3-sequences, then $A_1 = A_2$, and $B_1 = B_2$.*

We now show that only the four ends of a clock can have jump-set members that are not in the clock, so no internal elements of a clock can interact with any non-clock elements. In particular, when a clock occurs in a 3-sequence, all of its internal elements must be consecutive except for the possible insertion of clock ends.

Lemma 7.7. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence and $F_x \cup F_y$ be a clock contained in X where F_x and F_y, the fans of this clock, have orderings $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_m)$, respectively, in (A, X, B).*

- (i) *If z is a clock element whose jump-set contains some element s of $X - (F_x \cup F_y)$, then $z \in \{x_1, x_n, y_1, y_m\}$. In particular, no element outside of the clock jumps an internal clock element.*
- (ii) *There is a subset Z of the clock $F_x \cup F_y$ that contains all of the internal clock elements such that the elements of Z are consecutive in X.*
- (iii) *If x_2 occurs to the left of y_{m-1}, then every element of $\overrightarrow{X}$ occurring between x_2 and y_{m-1} is in $F_x \cup F_y$.*

Proof. To prove (i), we may suppose, by symmetry, that $z = x_i$ for some i that is strictly between 1 and n. Let S_1 and S_2 be (A, B) 3-sequences in which s is, respectively, to the left and to the right of x_i. Then, in S_1, both x_i and x_{i+1} lie to the right of s, while, in S_2, both x_{i-1} and x_i lie to the left of s. Hence, in S_1, by using Lemma 4.3, all elements of F_x can be moved to the right of s to obtain another 3-sequence. Similarly, in S_2, all elements of F_x can be moved to the left of s to obtain another 3-sequence. Hence $s \in J_{x_j}$ for all $j \in \{1, 2, \ldots, n\}$. But then Theorem 7.2 implies that $s \in F_y$; a contradiction.

Suppose that (ii) fails. Then there is an element s of $X - (F_x \cup F_y)$ such that s has internal clock elements to both its left and its right. Without loss of generality, we may assume that x_i occurs to the left of s for some $i \geqslant 2$. If x_{n-1} occurs to the right of s, then x_{n-1} can be moved so that $x_1, x_2, \ldots, x_{n-1}$ are consecutive and to the left of s, contrary to (i). Hence we may assume that $\{x_1, x_2, \ldots, x_{n-1}\} \subseteq L_s$. Similarly, the result follows by (i) unless $\{y_2, y_3, \ldots, y_m\} \subseteq R_s$. Now there is an (A, B) 3-sequence in which $y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_n$ is a consecutive subsequence. But s cannot occur to the left or right of this subsequence without contradicting (i).

For (iii), we note that, by Corollary 7.6 and Theorem 7.4, there is a 3-sequence of the form $(A, A_1, F_x, F_y, B_1, B)$. Now suppose that $(A, \overrightarrow{X}, B)$ has the form $(A, Z_1, x_2, Z_2, y_{m-1}, Z_3, B)$. If $z \in Z_2 - (F_x \cup F_y)$, then $z \in A_1$ or $z \in B_1$. In the first case, z jumps x_1 and x_2 and (ii) gives the contradiction that $z \in F_y$. In the second case, z jumps both y_m and y_{m-1} so $z \in F_x$; a contradiction. We conclude that $Z_2 \subseteq F_x \cup F_y$. □

The following is another characterisation of clocks.

Theorem 7.8. *Let (A, X, B) be a 3-sequence. A subset Z of X is a clock with fans F and F' if and only if Z is a maximal subset of X that can be partitioned into sets F and F' each having at least two elements such that every element of F is in the jump-set of every element of F' and, in every (A, B) 3-sequence, the elements of F occur in the same order.*

Proof. Assume that Z is a maximal subset of X that can be partitioned into sets F and F', each having at least two elements such that every element of F is in the jump-set of every element of F' and, in every (A, B) 3-sequence, the elements of F occur in the same order. Let F and F' have elements $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ and assume that the elements of F occur in this order in every (A, B) 3-sequence. Then $J_{y_1} \cap J_{y_2} \supseteq F$.

Suppose that $\{y_1, y_2\} \cup F$ does not contain both a guts and a coguts element. Then, by Theorem 7.1, this set is a segment or a cosegment. Hence, by Lemma 6.1, its elements can be made consecutive in some (A, B) 3-sequence and hence can be permuted arbitrarily within this consecutive subsequence. This contradicts the fact that the elements of F occur in the same order in every (A, B) 3-sequence. We conclude that $\{y_1, y_2\} \cup F$ contains both a guts and a coguts element. Thus, we can apply Theorem 7.2 to get that X contains a clock, the fans of which are $J_{y_1} \cap J_{y_2}$ and $J_{x_1} \cap J_{x_2}$. If $J_{y_1} \cap J_{y_2} \neq F$ or $J_{x_1} \cap J_{x_2} \neq F'$, then the maximality of Z is contradicted. We conclude that Z is a clock with fans F and F'.

Conversely, if Z_0 is a clock with fans F_x and F_y, then $|F_x|, |F_y| \geqslant 2$; every element of F_x is in the jump-set of every element of F_y; and the elements of F_x occur in the same order in every (A, B) 3-sequence. Thus Z satisfies the specified conditions except that we are not guaranteed that it is a maximal such set. Assume that $Z_0 \subseteq Z$ where Z is a maximal subset of X that can be partitioned into sets F and F' satisfying the specified conditions. Then, by the first part, Z is a clock with fans F and F'. By Corollary 7.3, $Z_0 = Z$ and $\{F_x, F_y\} = \{F_1, F_2\}$. □

Finally, we specify how different clocks can overlap.

Lemma 7.9. *Let (A, X, B) be a 3-sequence and let Z be a clock contained in X.*

- (i) *The clock Z has at most two common elements with any other clock Z' contained in X.*
- (ii) *If z is an internal element of Z, then Z is the unique clock containing z that is contained in X.*
- (iii) *If z is an end of Z and Z is non-degenerate, then X contains at most one other non-degenerate clock containing z. Moreover, when there is a second non-degenerate clock containing z, that clock has z as an end.*
- (iv) *If the distinct clocks Z and Z' have two common elements, then*
  - (a) *these elements are in different fans in each of Z and Z'; and*
  - (b) *these elements are ends of their respective clocks.*

Proof. Let the fans of Z be F_x and F_y and suppose $z \in F_x$. Let Z' be another clock containing z and suppose its fans are F'_x and F'_y where $z \in F'_x$.

Assume that a fan F of Z meets both F'_x and F'_y with x' and y', respectively, being in these intersections. Since x' and y' are both in F, they occur in the same order in every (A, B) 3-sequence. But, since $x' \in F'_x$ and $y' \in F'_y$, we have $x' \in J_{y'}$; a contradiction. We conclude that each of F_x and F_y meets at most one of F'_x and F'_y. Likewise, each of F'_x and F'_y meets at most one of F_x and F_y.

If a fan F of Z meets a fan F' of Z' in at least two elements, say f_1 and f_2, then $J_{f_1} \cap J_{f_2}$ is one of the fans of each of Z and Z'. Moreover, if g_1 and g_2 are in $J_{f_1} \cap J_{f_2}$, then the other fan of each of Z and Z' is $J_{g_1} \cap J_{g_2}$. This implies that $Z = Z'$; a contradiction.

On combining the last two paragraphs and using the fact that $z \in F_x \cap F_{x'}$, we deduce that $Z \cap Z' = \{z\}$, or $Z \cap Z' = \{z, z'\}$, where $z' \in F_y \cap F'_y$. Hence (i) and (iv)(a) hold.

Now suppose that z is an internal element of Z. By Lemma 7.7(i), the jump-set of z is F_y. But $z \in F'_x$, so its jump-set contains F'_y. Hence $F'_y \subseteq F_y$; a contradiction. We conclude that (ii) holds.

Next suppose that z is an end of Z, say the left end of F_x. Suppose that Z and Z' are non-degenerate. Then, by (ii), z is not an internal element of Z'. Thus z is an end of F'_x. We shall show that z must be the right end of F'_x. Part (iii) follows immediately from this. Assume that z is the left end of F'_x.

Suppose that $|F'_x| \geqslant 3$. Then there is an (A, B) 3-sequence in which $F_y, z, F_x - \{z\}$ occurs as a consecutive subsequence. Since z is the left end of F'_x, this subsequence has $F'_x - \{z\}$ to its right. Hence, we can move z in this 3-sequence so that it immediately precedes the first element of $F'_x - \{z\}$. Thus z can jump $F_x - \{z\}$; a contradiction. We conclude that $|F'_x| = 2$. By symmetry, $|F_x| = 2$.

As Z and Z' are both non-degenerate, F_y and F'_y have internal elements y and y', say. By (ii), these elements are distinct. Let $F'_x = \{z, x'_1\}$ and $F_x = \{z, x_1\}$. Then we have (A, B) 3-sequences having as consecutive subsequences, (z, x'_1, F'_y) and (z, x_1, F_y). As x_1 and x'_1 are in R_z, the element x_1 occurs to the right of F'_y in the first 3-sequence while x'_1 occurs to the right of F_y in the second 3-sequence. Thus the first 3-sequence has as a subsequence one of (I) (z, x'_1, F'_y, x_1, y); (II) (z, x'_1, F'_y, y, x_1); or (III) (y, z, x'_1, F'_y, x_1). The second 3-sequence has as a subsequence one of (a) (z, x_1, F_y, x'_1, y'); (b) (z, x_1, F_y, y', x'_1); or (c) (y', z, x_1, F_y, x'_1). If (I) or (II) occurs, then neither (a) nor (b) does because, by Lemma 7.7(ii), $y' \notin J_y$. If (I) or (II) occurs, then (c) does not, otherwise x'_1 is in the jump-sets of both y and the element of F_y to the right of y. Thus, by Lemma 7.7(ii), $x'_1 \in F_x$; a contradiction. We deduce that (III) occurs and, by symmetry, so does (c). But this implies that $y' \in J_y$; a contradiction. Hence (iii) holds.

Finally, suppose that $Z \cap Z' = \{z, z'\}$. Then it follows by (ii) and (iv)(a), that z is an end of both F_x and F'_x, while z' is an end of both F_y and F'_y. Hence (iv)(b) holds. □

We know from above that the elements of a clock Z form a barrier between the elements to their left and those to their right. Suppose that e_1 lies to the left of the clock but can jump over the clock's two left ends x_1 and y_1. Suppose also that e_2 lies to the right of the clock and can jump its two right ends. Then, provided Z is non-degenerate, there is at least one internal clock element lying between e_1 and e_2, so e_1 and e_2 cannot interact with each other. Now suppose that Z is degenerate. Then, after the jumps described above, e_1 and e_2 are adjacent and may be able to jump each other. This means that while a degenerate clock still forms a barrier preventing elements from jumping over it completely, for some non-degenerate clocks, the elements on the left and right may still interact with each other within this barrier.

Let us consider further which degenerate clocks allow the kind of interaction described in the previous paragraph. For x_1, y_1 and e_1 to all be able to jump each other, by duality, we may assume, by Lemma 6.4, that $\{x_1, y_1, e_1\}$ is a segment. As Z is degenerate, x_2 and y_2 are coguts elements and so, as x_2, y_2 and e_2 can jump each other, $\{x_2, y_2, e_2\}$ is a cosegment. Thus the situation we are looking at is where we have a segment and a cosegment lying side by side, and there is a pairing between some segment and cosegment elements, such that a segment element can jump over its cosegment partner. In the next section, we shall examine this situation in more detail. What we also see with the example above is that $(x_1, x_2) \cup (y_1, y_2)$, $(x_1, e_2) \cup (e_1, y_2)$ and $(y_1, e_2) \cup (e_1, x_2)$ are all degenerate clocks (with fan orderings as shown). Thus, although proper clocks and semi-degenerate clocks may only lie side by side in an (A, B) 3-sequence, the above example raises the possibility that degenerate clocks can lie on top of each other. We showed in Fig. 2 that this situation does actually arise and leads to what we call a crocodile. We observe, however, that, for this to occur, the degenerate clocks must all have their first elements of the same type. We recall from Section 2 that such degenerate clocks are called even.

8. Crocodiles

This section analyses the properties of crocodiles in a 3-sequence (A, X, B). Recall from Section 2 that the definition of a crocodile involves S and S^*, a maximal segment and maximal cosegment contained in X, and subsets $\{s_1, s_2, \dots, s_k\}$ and $\{s_1^*, s_2^*, \dots, s_k^*\}$ of S and S^*, respectively. We shall show that, when a crocodile arises in a 3-sequence, it corresponds to the presence of a minor isomorphic to the matroid Θ_k where we recall from Section 2 that, for example, $\Theta_3 \cong M(K_4)$.

Lemma 8.1. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence in a matroid M, let $S \cup S^*$ be a segment-first crocodile contained in X, and let (C', D') be the associated partition of $E(M) - (S \cup S^*)$. Then*

(i) *there is an (A, B) 3-sequence in which the elements of $S \cup S^*$ are consecutive with those in S preceding those in S^*;*
(ii) *an element of X whose jump-set contains two elements of S is in S, and an element of X whose jump-set contains two elements of S^* is in S^*;*
(iii) *if $(A, A_1, S \cup S^*, B_1, B)$ is a 3-sequence, then $A_1 = C' - A$ and $B_1 = D' - B$;*
(iv) *for each i in $\{1, 2, \dots, k\}$, there is an (A, B) 3-sequence in which s_i^* lies to the left of s_i, specifically, $(A, C' - A, S - \{s_i\}, s_i^*, s_i, S^* - \{s_i^*\}, D' - B, B)$;*
(v) *if s is an element of S and s^* is an element of S^* such that s occurs to the right of s^* in $\overrightarrow{X}$, then $(s, s^*) = (s_i, s_i^*)$ for some i in $\{1, 2, \dots, k\}$;*
(vi) *in every 3-sequence in which the elements of $S \cup S^*$ are consecutive, the first $|S| - 1$ elements of $S \cup S^*$ are in S and the last $|S^*| - 1$ elements of $S \cup S^*$ are in S^*;*
(vii) $\mathrm{r}(C' \cup S) = \mathrm{r}(C')$ *and* $\mathrm{r}^*(D' \cup S^*) = \mathrm{r}^*(D')$; *and*
(viii) *for each distinct i and j in $\{1, 2, \dots, k\}$, there is a clock in X whose fans are $J_{s_i} \cap J_{s_j^*}$ and $J_{s_j} \cap J_{s_i^*}$ and these fans meet $S \cup S^*$ in $\{s_j, s_i^*\}$ and $\{s_i, s_j^*\}$, respectively.*

Proof. Let $C' - A = A_0$ and $D' - B = B_0$. Since each of $A \cup A_0$, $A \cup A_0 \cup S$, and $A \cup A_0 \cup S \cup S^*$ is 3-separating, by Lemma 4.4, there are 3-sequences $(A, A_0, \overrightarrow{X} - A_0, B)$, $(A, A_0 \cup S, \overrightarrow{X} - (A_0 \cup S), B)$, and $(A, A_0 \cup S \cup S^*, \overrightarrow{X} - (A_0 \cup S \cup S^*), B)$. By Corollary 4.5, there is a 3-sequence that begins A, A_0 as in the first of these 3-sequences and then finishes as in the second. This 3-sequence is $(A, A_0, S, \overrightarrow{X} - (A_0 \cup S), B)$. Applying Corollary 4.5 again, we get that there is a 3-sequence that begins A, A_0, S and then finishes as in the third 3-sequence above. This 3-sequence is (A, A_0, S, S^*, B_0, B). Hence (i) holds. The fact that $\mathrm{r}(C' \cup S) = \mathrm{r}(C')$ is immediate from this because S is a segment and so consists entirely of guts elements. By duality, $\mathrm{r}^*(D' \cup S^*) = \mathrm{r}^*(D')$. Thus (vii) holds.

For (ii), suppose that $z \in X - S$ and the jump-set of z contains elements s and t of S. By (i), s and t can jump each other. Thus, z, s, and t are mutually jumping, so, by Lemma 6.4, $\{z, s, t\}$ is a triangle of guts elements. Hence $S \cup \{z\}$ is a segment, contradicting the maximality of S. This establishes the first part of (ii), and the second part follows by duality.

To prove (iii), let $(A, A_1, S \cup S^*, B_1, B)$ be a 3-sequence. If $A_1 \neq A_0$, then there is an element z that precedes $S \cup S^*$ in one of $(A, A_1, S \cup S^*, B_1, B)$ and $(A, A_0, S \cup S^*, B_0, B)$ and follows $S \cup S^*$ in the other. Thus z is in the jump-set of every element of S, contradicting (ii).

For (iv), note that, by (i), (iii), and the fact that S is a segment and S^* a cosegment, we get that there is an (A, B) 3-sequence of the form $(A, A_0, S - \{s_i\}, s_i, s_i^*, S^* - \{s_i^*\}, B_0, B)$. Because $A \cup A_0 \cup (S - \{s_i\}) \cup \{s_i^*\}$ is 3-separating, we can interchange s_i and s_i^* in the last 3-sequence to obtain another (A, B) 3-sequence with the desired property.

For (v), we note first that if $s = s_i$, then $s^* = s_i^*$, otherwise s, s^*, and s_i^* are mutually jumping; a contradiction. The converse of the last assertion follows similarly. We may now assume that $s \notin S_w$ and $s^* \notin S_w^*$. As in the last paragraph, we have an (A, B) 3-sequence of the form $(A, A_0, S - \{s\}, s, s^*, S^* - \{s^*\}, B_0, B)$. By (iii), $A_0 = C' - A$. By hypothesis, we also have such a sequence of the form $(A, \overrightarrow{W_1}, s^*, W_2, s, W_3, B)$. Thus we have such a sequence $(A, \overrightarrow{X_1}, B)$ where $\overrightarrow{X_1}$ begins $A_0, S - \{s\}$ and ends as in $(A, \overrightarrow{W_1}, s^*, W_2, s, W_3, B)$. If $W_1 \subseteq A_0 \cup (S - \{s\})$, then the element of $\overrightarrow{X_1}$ following $S - \{s\}$ is s^*. We deduce that $A \cup A_0 \cup (S - \{s\}) \cup \{s^*\}$ is 3-separating and so (iii)(b) in the definition of a crocodile is contradicted. Thus there is an element in $W_1 - (A_0 \cup (S - \{s\}))$. Let w be the left-most element of $\overrightarrow{W_1}$ in this set. By (ii), $w \notin S^* - \{s^*\}$, otherwise w, s^*, and s are mutually jumping. Hence $w \in B_0$ and w is the first element following $S - \{s\}$ in $\overrightarrow{X_1}$. Thus w is in the jump-sets of every element of S^*. This contradiction to (ii) establishes (v).

For (vi), by duality, it suffices to observe, using (i), that if an element s^* of S^* has two elements, s and t, of S to its right, then s^*, s, and t are mutually jumping; a contradiction to (ii).

For (viii), we note that, by (iv), $J_{s_i} \cap J_{s_j^*}$ contains $\{s_j, s_i^*\}$. Then, by Theorem 7.2, there is a clock whose fans are $J_{s_i} \cap J_{s_j^*}$ and $J_{s_j} \cap J_{s_i^*}$. Since the elements of each of these fans occur in the same order in every (A, B) 3-sequence, while there are (A, B) 3-sequences in which every possible permutation of each of S and S^* occurs, we deduce that $J_{s_i} \cap J_{s_j^*}$ and $J_{s_j} \cap J_{s_i^*}$ meet $S \cup S^*$ in $\{s_j, s_i^*\}$ and $\{s_i, s_j^*\}$, respectively. $\square$

The last lemma does not use the fact that a crocodile must have at least five elements. The reason we have added this requirement is revealed in (viii), namely to prevent a crocodile from being contained in a clock.

Parts (iv) and (v) of the last lemma establish that the pairing of the elements s_i and s_i^* in a crocodile is uniquely determined. We call s_i and s_i^* *swap partners* in the crocodile and call the pair (s_i, s_i^*) a *swap pair*.

Lemma 8.2. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence and $S \cup S^*$ be a segment-first non-degenerate crocodile contained in X. Let the orderings of S and S^* in $\overrightarrow{X}$ be $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$. Then*

- (i) *the elements of $(S - \{x_1\}) \cup (S^* - \{y_n\})$ are consecutive in $\overrightarrow{X}$;*
- (ii) *if $z \in X - (S \cup S^*)$, then z is in the jump-set of at most one element of $S \cup S^*$;*
- (iii) *for all distinct i and j in $\{1, 2, \ldots, k\}$, there is an even degenerate clock in X with fans (s_i, s_j^*) and (s_j, s_i^*).*

Proof. By Lemma 8.1(i), we have a 3-sequence $(A, \overrightarrow{A_0}, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, \overrightarrow{B_0}, B)$. By Lemma 5.11, $x_2, x_3, \ldots, x_{m-1}$ and $y_2, y_3, \ldots, y_{n-1}$ are consecutive subsequences of $\overrightarrow{X}$. If $y_2, y_3, \ldots, y_{n-1}$ precedes $x_2, x_3, \ldots, x_{m-1}$ in $\overrightarrow{X}$, then y_1 is in the jump-set of both x_{m-1} and x_m contradicting Lemma 8.1(ii). Thus $(A, \overrightarrow{X}, B)$ has the form

$$\left(A, A_1', x_1, A_1'', x_2, \ldots, x_{m-1}, \overrightarrow{Z_1}, y_2, \ldots, y_{n-1}, B_1'', y_n, B_1', B\right)$$

and $y_1 \in Z_1$. Moreover, $x_m \in Z_1$ otherwise x_m is in the jump-sets of y_1 and y_2, contradicting Lemma 8.1(ii). Similarly, there is no element z in $B_0 \cap (A_1' \cup A_1'')$, otherwise z is in the jump-sets of both y_1 and y_2. Thus $B_0 \cap (A_1' \cup A_1'') = \emptyset$. If $z \in A_0 \cap (B_1' \cup B_1'' \cup Z_1)$, then z is in the jump-set of both x_1 and x_2. Thus $A_0 \cap (B_1' \cup B_1'' \cup Z_1) = \emptyset$, so $A_1' \cup A_1'' = A_0$. Now $B_0 \cap Z_1$ is empty since an element in this set is in the jump-sets of y_2 and y_3. Hence Z_1 is $\{x_m, y_1\}$ and we conclude that the elements of $(S - \{x_1\}) \cup (S^* - \{y_n\})$ are consecutive in $\overrightarrow{X}$.

To prove (ii), let $z \in X - (S \cup S^*)$ and suppose that z is in the jump-set of two elements of $S \cup S^*$. Then, by Lemma 8.1(ii), one of these elements is in S while the other is in S^*. We may assume without loss of generality that $z \in A_0$ so that z is to the left of all members of $S \cup S^*$ in $(A, \overrightarrow{A_0}, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, B_0, B)$. Let $(A, \overrightarrow{Z}, B)$ be a 3-sequence in which z is to the right of y_i, where $z \in J_{y_i}$. Then, by Lemma 6.4, in $\overrightarrow{Z}$, at most one member of S is right of y_i and at most one member of S is left of z, and hence left of y_i. Thus $|S| \leqslant 2$, contradicting the fact that $S \cup S^*$ is non-degenerate. Thus (ii) holds.

To prove (iii), let $F_1 = J_{s_j} \cap J_{s_i^*}$ and $F_2 = J_{s_i} \cap J_{s_j^*}$. Then, by (ii), neither F_1 nor F_2 contains any member of $X - (S \cup S^*)$. It follows easily that $F_1 = (s_i, s_j^*)$ and $F_2 = (s_j, s_i^*)$, and then Theorem 7.8 implies that $F_1 \cup F_2$ is an even degenerate clock. □

Next we briefly consider the behaviour of degenerate crocodiles. Let $S \cup S^*$ be a crocodile in which $|T| = 2$ for some $T \in \{S, S^*\}$. Let $T = \{t_1, t_2\}$ and define t_i^* to be s_i^* or s_i depending on whether t_i is s_i or s_i^*, respectively. Consider the clock K whose fans are $J_{t_1} \cap J_{t_2^*}$ and $J_{t_2} \cap J_{t_1^*}$. This clock may be degenerate, in which case $S \cup S^*$ behaves like a non-degenerate crocodile. When K is non-degenerate, $S \cup S^* \cup K$ is a (degenerate) crocodile attached to a non-degenerate clock. Alternatively, we can view $S \cup S^* \cup K$ as a non-degenerate clock for which both first elements or both last elements belong to the same non-degenerate segment or cosegment. This situation can be analysed more thoroughly and an earlier version of this paper included this analysis. However, the decomposition results in Section 10 circumvent the need for this analysis.

To conclude this section, we prove that, whenever a crocodile occurs in a 3-sequence, the associated matroid has a minor isomorphic to Θ_k. The matroid $\mathcal{W}^3$, which also appears in the next result, is the rank-3 whirl.

Theorem 8.3. *Let (A, A_0, S, S^*, B_0, B) be a 3-sequence in which $S \cup S^*$ is a crocodile with S as its segment and S^* as its cosegment. Then*

$$M\big\backslash\big[A \cup A_0 \cup \big(S - \{s_1, s_2, \dots, s_k\}\big)\big]\big/\big[B \cup B_0 \cup \big(S^* - \big\{s_1^*, s_2^*, \dots, s_k^*\big\}\big)\big] \cong \Theta_k.$$

Moreover, if $k = 2$ and $S - \{s_1, s_2\}$ and $S^* - \{s_1^*, s_2^*\}$ contain elements s_0 and s_0^*, then

$$M\big\backslash\big[A \cup A_0 \cup \big(S - \{s_0, s_1, s_2\}\big)\big]\big/\big[B \cup B_0 \cup \big(S^* - \big\{s_0^*, s_1^*, s_2^*\big\}\big)\big] \cong \mathcal{W}^3.$$

Proof. Since $S \cup S^*$ is a crocodile, we may assume that $|S^*| \geqslant 3$. If $|S| = 2$, then extend M by an element s_0 that is freely placed on $\mathrm{cl}(S)$. In the resulting matroid M', we have that $(A, A_0, s_0, S, S^*, B_0, B)$ is a 3-sequence and $\{s_0\} \cup S \cup S^*$ is a crocodile with $\{s_0\} \cup S$ as its segment and S^* as its cosegment. In this case, let $S' = \{s_0\} \cup S$. If $|S| \geqslant 3$, then we let $(M', S') = (M, S)$.

We show next that $M'\backslash(A \cup A_0)$ is 3-connected. Suppose $M'\backslash(A \cup A_0)$ has a j-separation (U, V) for some j in $\{1, 2\}$. Then,

$$\mathrm{r}_{M'\backslash(A\cup A_0)}(U) + \mathrm{r}_{M'\backslash(A\cup A_0)}(V) - \mathrm{r}\big(M'\backslash(A \cup A_0)\big) \leqslant j - 1. \tag{8.1}$$

Without loss of generality, $|U \cap S'| \geqslant 2$. Thus $\mathrm{r}(U \cup S') = \mathrm{r}(U)$. Moreover, we recall that $\mathrm{r}(A \cup A_0 \cup S') = \mathrm{r}(A \cup A_0)$ by Lemma 8.1(v). Therefore,

$$\begin{aligned}
\mathrm{r}(A \cup A_0 \cup U) &= \mathrm{r}\big(A \cup A_0 \cup U \cup S'\big)\\
&\leqslant \mathrm{r}\big(A \cup A_0 \cup S'\big) + \mathrm{r}(U) - \mathrm{r}\big(U \cap S'\big)\\
&\leqslant \mathrm{r}(A \cup A_0) + \mathrm{r}(U) - 2\\
&= \mathrm{r}(U) + \big[\mathrm{r}(A \cup A_0) - 2\big]\\
&= \mathrm{r}(U) + \big[\mathrm{r}\big(M'\big) - \mathrm{r}\big(M'\backslash(A \cup A_0)\big)\big]
\end{aligned}$$

where the last step follows since $(A \cup A_0, E(M') - (A \cup A_0))$ is a 3-separation of M'. Thus

$$\mathrm{r}(A \cup A_0 \cup U) - \mathrm{r}\big(M'\big) \leqslant \mathrm{r}(U) - \mathrm{r}\big(M'\backslash(A \cup A_0)\big).$$

Substituting into (8.1) establishes that M' is not 3-connected; a contradiction. We conclude that $M'\backslash(A \cup A_0)$ is 3-connected. By applying the above argument to $[M'\backslash(A \cup A_0)]^*$, we obtain that $M'\backslash(A \cup A_0)/(B \cup B_0)$ is 3-connected. This matroid has S' as a segment and

S^* as a cosegment. Moreover, for all i in $\{1, 2, \ldots, k\}$, as $A \cup A_0 \cup (S' - \{s_i\}) \cup \{s_i^*\}$, is 3-separating in M', it follows without difficulty that $(S' - \{s_i\}) \cup \{s_i^*\}$ is 3-separating in $M' \backslash (A \cup A_0)/(B \cup B_0)$.

Now suppose that $k \geqslant 3$. Then we can delete $S' - \{s_1, s_2, \ldots, s_k\}$ from $M' \backslash (A \cup A_0)/(B \cup B_0)$ to get a 3-connected matroid. Finally, contracting $S^* - \{s_1^*, s_2^*, \ldots, s_k^*\}$ leaves a 3-connected matroid N having $\{s_1, s_2, \ldots, s_k\}$ as a segment, $\{s_1^*, s_2^*, \ldots, s_k^*\}$ as a cosegment, and $(\{s_1, s_2, \ldots, s_k\} - \{s_i\}) \cup \{s_i^*\}$ as a 3-separating set for all i. One easily checks that N has rank k having $\{s_1^*, s_2^*, \ldots, s_k^*\}$ as a basis. Evidently, every 3-element subset of $\{s_1, s_2, \ldots, s_k\}$ is a circuit and every 3-element subset of $\{s_1^*, s_2^*, \ldots, s_k^*\}$ is a cocircuit. The latter implies that a non-spanning circuit C of N that meets $\{s_1^*, s_2^*, \ldots, s_k^*\}$ must contain at least $k - 1$ elements of this set. As the whole set is a basis of N, it follows that C contains exactly $k - 1$ of the elements of $\{s_1^*, s_2^*, \ldots, s_k^*\}$, say all but s_i^*. As $(\{s_1^*, s_2^*, \ldots, s_k^*\} - \{s_i^*\}) \cup \{s_i\}$ is 3-separating in N, the last set must have rank $\mathrm{r}(N) - 1$. Hence it is a circuit and also a hyperplane. We deduce that this circuit is C, and conclude that $N \cong \Theta_k$, that is,

$$\Theta_k \cong M \Big\backslash \big[A \cup A_0 \cup \big(S - \{s_1, s_2, \ldots, s_k\}\big)\big] \big/ \big[B \cup B_0 \cup \big(S^* - \big\{s_1^*, s_2^*, \ldots, s_k^*\big\}\big)\big].$$

Now suppose that $k = 2$. Then $M' \backslash [A \cup A_0 \cup (S - \{s_0, s_1, s_2\})]/[B \cup B_0 \cup (S^* - \{s_0^*, s_1^*, s_2^*\})]$ is a 3-connected matroid having $\{s_0, s_1, s_2\}$ as a segment and $\{s_0^*, s_1^*, s_2^*\}$ as a cosegment. Moreover, $\{s_0, s_2, s_1^*\}$ and $\{s_0, s_1, s_2^*\}$ are 3-separating, so their complements are 3-point lines. We conclude that this matroid is isomorphic to $\mathcal{W}^3$. To see this, observe that the only other possible 3-circuit is $\{s_0, s_1^*, s_2^*\}$ but it is not a circuit, for if $|S| = 2$, then s_0 was freely added to $\mathrm{cl}(S)$, while if $|S| \geqslant 3$, then $s_0 \notin S_w$.

Finally, we note that contracting s_0^* and deleting s_0 from the last matroid gives Θ_2. $\square$

9. Pointed flans

This section discusses how pointed flans arise in 3-sequences and, when they arise, how they behave. We observe that, although the definition of a p-flan in Section 2 imposes an ordering on the sets $F_1, F_2, \ldots, F_m$, the definition does not explicitly link this ordering to any (A, B) 3-sequence. One of the tasks of this section will be to prove that, when a p-flan occurs in a 3-sequence $(A, \overrightarrow{X}, B)$, the ordering induced on $F_1 \cup F_2 \cup \cdots \cup F_m$ has one of the forms $(F_1, F_2, \ldots, F_m)$ or $(F_m, F_{m-1}, \ldots, F_1)$, and this form is the same for every other 3-sequence $(A, \overrightarrow{X_1}, B)$.

The other main task of this section is to show that p-flans are associated with elements that have non-trivial jump-sets. Let z be a guts element in an (A, B) 3-sequence. We define z to be *wild* if either J_z contains at least two coguts elements, or J_z contains at least two guts elements that are not in a common segment. By dualising this definition, we get the definition of a *wild coguts element*.

The next lemma is an immediate consequence of Lemma 6.4.

Lemma 9.1. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence of a matroid M and let y be an element of this sequence. Suppose that J_y contains both a guts element x and a coguts element x'. Then x is not in the jump-set of x', and x' is not in the jump-set of x.*

The proof that a wild guts element gives rise to a p-flan will use the following result.

Lemma 9.2. *Let M be a matroid with a 3-sequence $(A, \overrightarrow{A_1}, \overrightarrow{Z_1}, \overrightarrow{Z_2}, \overrightarrow{Z_3}, \overrightarrow{B_1}, B)$.*

(i) *If Z_1 and Z_3 are non-empty sets of guts elements and Z_2 is a singleton coguts set, then $Z_1 \cup Z_2 \cup Z_3$ has rank two or three.*
(ii) *If Z_1 and Z_3 are singleton coguts sets and Z_2 is a non-empty set of guts elements, then $\mathrm{r}(A \cup A_1) + \mathrm{r}(B \cup B_1) = \mathrm{r}(M)$.*

Proof. Let $A' = A \cup A_1$ and $B' = B \cup B_1$. Assume that Z_1 and Z_3 are non-empty sets of guts elements and Z_2 is a singleton coguts set. Since $(A' \cup Z_1, B' \cup Z_2 \cup Z_3)$ is a 3-separation of M and Z_2 is a singleton coguts set,

$$\begin{aligned}\mathrm{r}(M) + 2 &= \mathrm{r}\big(A' \cup Z_1\big) + \mathrm{r}\big(B' \cup Z_2 \cup Z_3\big) \\ &= \mathrm{r}\big(A' \cup Z_1 \cup Z_2\big) - 1 + \mathrm{r}\big(B' \cup Z_2 \cup Z_3\big) \\ &= \mathrm{r}\big(A' \cup Z_1 \cup Z_2 \cup Z_3\big) + \mathrm{r}\big(B' \cup Z_1 \cup Z_2 \cup Z_3\big) - 1 \\ &\geqslant \mathrm{r}(M) + \mathrm{r}(Z_1 \cup Z_2 \cup Z_3) - 1 \quad \text{by submodularity.}\end{aligned}$$

Thus $\mathrm{r}(Z_1 \cup Z_2 \cup Z_3) \leqslant 3$ so $\mathrm{r}(Z_1 \cup Z_2 \cup Z_3) \in \{2, 3\}$ and (i) holds.

For the proof of (ii), let Z_1 and Z_3 be singleton coguts sets, and Z_2 be a non-empty set of guts elements. Then $\mathrm{r}(A' \cup Z_1) = \mathrm{r}(A') + 1$, $\mathrm{r}(A' \cup Z_1 \cup Z_2) = \mathrm{r}(A') + 1$, and $\mathrm{r}(A' \cup Z_1 \cup Z_2 \cup Z_3) = \mathrm{r}(A') + 2$. Therefore, as B' is 3-separating, $\mathrm{r}(A' \cup Z_1 \cup Z_2 \cup Z_3) + \mathrm{r}(B') = \mathrm{r}(M) + 2$, and so $\mathrm{r}(A') + \mathrm{r}(B') = \mathrm{r}(M)$. □

Theorem 9.3. *Let z be a guts element of a 3-sequence $(A, \overrightarrow{X}, B)$. Suppose that J_z contains two guts elements that are not in a common segment or J_z contains two coguts elements. Then $J_z \cup \{z\}$ is a p-flan of M with tip z.*

Proof. By Lemma 5.1, there is an (A, B) 3-sequence $(A, \overrightarrow{X_1}, B)$ in which the elements of J_z are consecutive. Let $\overrightarrow{J_z}$ be the ordering imposed on J_z by $\overrightarrow{X_1}$. This ordering induces an ordered partition $(F_1, F_2, \dots, F_n)$ on J_z where $\overrightarrow{J_z} = \overrightarrow{F_1}, \overrightarrow{F_2}, \dots, \overrightarrow{F_n}$, the elements of each F_i are all of the same type and this type is different from the type of the elements of F_{i+1}. By Lemma 6.4, if F_i consists of coguts elements, then $|F_i| = 1$. Since J_z contains two guts elements that are not in a common segment or J_z contains two coguts elements, it follows that $n \geqslant 3$.

We show first that if $F_i \cup z$ is a segment, then it is maximal. If not, then there is an element x such that $F_i \cup z \cup x$ is a segment. By Lemma 5.4, there are 3-sequences having $z, \overrightarrow{F_i}$ and $\overrightarrow{F_i}, z$ as consecutive subsequences. Since $x \in \mathrm{cl}(F_i \cup z)$, it follows by Lemma 4.3 that there is a 3-sequence $(A, \overrightarrow{X_2}, B)$ in which the elements of $F_i \cup x \cup z$ are consecutive.

Hence $x \in J_z$. Without loss of generality, we may assume that $x \in F_j$ for some $j > i$. By Lemma 9.1, the coguts element c in F_{i+1} cannot jump any element of F_i. Thus, in $\overrightarrow{X_2}$, the element x occurs to the left of c. Hence x jumps c, contradicting Lemma 9.1. Thus $F_i \cup z$ is indeed a maximal segment.

Suppose F_i is a set of guts elements. Since $F_i \cup z$ is a maximal segment, it follows by Lemma 9.2 that $F_i \cup F_{i+1} \cup F_{i+2} \cup \{z\}$ has rank three.

Now suppose that F_i is a singleton coguts set. Let A' be the union of A and the elements to the left of F_i in $\overrightarrow{X_1}$, and let B' be the union of B and the elements of X to the right of F_{i+2} in $\overrightarrow{X_1}$. We show that $F_i \cup F_{i+1} \cup F_{i+2}$ is a cocircuit by showing that $\mathrm{r}(A' \cup B') = \mathrm{r}(M) - 1$ and that no member of $F_i \cup F_{i+1} \cup F_{i+2}$ is in $\mathrm{cl}(A' \cup B')$.

By Lemma 9.2, $\mathrm{r}(A') + \mathrm{r}(B') = \mathrm{r}(M)$. Furthermore, as z is a guts element, it follows by Lemmas 4.1 and 5.1 that $z \in \mathrm{cl}(A')$ and $z \in \mathrm{cl}(B')$. Thus $\mathrm{cl}(A') \cap \mathrm{cl}(B')$ is non-empty and so, by submodularity, $\mathrm{r}(A') + \mathrm{r}(B') \geqslant \mathrm{r}(A' \cup B') + 1$. Therefore $\mathrm{r}(A' \cup B') \leqslant \mathrm{r}(M) - 1$. Now

$$\begin{aligned}
\mathrm{r}(M) &= \mathrm{r}\big(A' \cup B' \cup F_i \cup F_{i+1}\big) \\
&= \mathrm{r}\big(A' \cup B' \cup F_i\big) \quad \text{as } F_{i+1} \subseteq \mathrm{cl}\big(A' \cup F_i\big), \\
&\leqslant \mathrm{r}\big(A' \cup B'\big) + 1 \quad \text{as } |F_i| = 1, \\
&\leqslant \mathrm{r}(M).
\end{aligned}$$

We deduce that equality holds throughout the last chain of inequalities, so $\mathrm{r}(A' \cup B') = \mathrm{r}(M) - 1$ and $\mathrm{r}(A' \cup B' \cup F_i) = \mathrm{r}(M)$. By symmetry, $\mathrm{r}(A' \cup B' \cup F_{i+2}) = \mathrm{r}(M)$. Hence $A' \cup B'$ is a hyperplane of M unless $\mathrm{cl}(A' \cup B')$ contains some element x of F_{i+1}. In the exceptional case, since $z \in \mathrm{cl}(A')$ and $\{x, z\}$ spans F_{i+1}, we deduce that $\mathrm{cl}(A' \cup B') \supseteq F_{i+1}$. Hence $F_i \cup F_{i+2}$ is a 2-element cocircuit of M. This contradiction implies that $A' \cup B'$ is a hyperplane, so $F_i \cup F_{i+1} \cup F_{i+2}$ is a cocircuit. □

Lemma 9.4. *Let (A, X, B) be a 3-sequence and $\{z\}, F_1, F_2, \dots, F_m$ be a collection of disjoint subsets of X. If $(z, F_1, F_2, \dots, F_m)$ is a p-flan, then there is an (A, B) 3-sequence that has as a consecutive subsequence either $z, F_1, F_2, \dots, F_m$ or $z, F_m, F_{m-1}, \dots, F_1$.*

Proof. We argue by induction on m. Let $m = 3$. Suppose first that F_1 is a singleton coguts element. As $F_2 \cup z$ is a maximal segment, there is an (A, B) 3-sequence $(A, \overrightarrow{X_1}, B)$ in which the elements of $F_2 \cup z$ are consecutive. Now if both the elements in $F_1 \cup F_3$ occur to the left of $F_2 \cup z$, then when we read from the left, the last element of F_2 in X_1 must be a coguts element because $F_1 \cup F_2 \cup F_3$ is a cocircuit. But this contradicts the fact that every element of F_2 is a guts element. It follows that one element of $F_1 \cup F_3$ occurs to the left of $F_2 \cup z$ while the other element occurs to the right of $F_2 \cup z$. Then, again since $F_1 \cup F_2 \cup F_3$ is a cocircuit, we can move the elements of F_1 and F_3 so that either (z, F_1, F_2, F_3) or (z, F_3, F_2, F_1) occurs as a consecutive subsequence.

Now suppose that F_1 is a set of guts elements. There is an (A, B) 3-sequence in which the elements of $F_1 \cup z$ are consecutive. In that sequence, the elements of F_3 all lie to the left or all to the right of $F_1 \cup z$, otherwise we can move an element of F_3 so that it is consecutive with $F_1 \cup z$ contradicting the fact that $F_1 \cup z$ is a maximal segment. We shall assume that the elements of F_3 all lie to the right of $F_1 \cup z$. If e is the first element of F_3, then we can move the other elements of F_3 so that they all immediately follow e. Now choose $(A, \overrightarrow{X_1}, B)$ to be an (A, B) 3-sequence in which both the sets $F_1 \cup z$ and F_3 are consecutive. Suppose $\overrightarrow{X_1}$ has $(F_1 \cup z, T, F_3)$ as a consecutive subsequence. Because F_2 is in the closure of $F_1 \cup z \cup F_3$, we deduce that $F_2 \subseteq T$. If $|T| = 1$, then the required result holds. Thus we may assume that $|T| > 1$. Because $F_1 \cup z$ is a maximal segment, it follows that the first element of T is a coguts element.

Assume that $|F_3| > 1$. Then we can move z so that it is consecutive with the elements of F_3. Thus $T \subseteq J_z$ and, as $F_3 \cup z$ is a maximal segment, the last element of T is a coguts element. Since $T \subseteq J_z$, it follows that T does not contain two consecutive coguts elements. If we consider all the elements between F_2 and the closest coguts element in T, including both the coguts elements, then we have a cocircuit by Lemma 9.2 and the proof of Lemma 9.3. But this cocircuit has a unique common element with a circuit that contains F_2 and is contained in $F_1 \cup F_2 \cup F_3$; a contradiction.

We may now assume that $|F_3| = 1$ and, by symmetry, we may also assume that $|F_1| = 1$. For each i in $\{1, 2, 3\}$, let $F_i = \{f_i\}$. Let $(A, \overrightarrow{X_2}, B)$ be an (A, B) 3-sequence in which f_1 and z are consecutive such that the number of elements occurring between f_1 and f_3 is minimised. Without loss of generality, we may assume that $\overrightarrow{X_2}$ has, as a consecutive subsequence, $(z, f_1, \overrightarrow{Z_2}, f_3)$. There is also an (A, B) 3-sequence in which f_3 and z are consecutive. Choose such a 3-sequence in which the number of elements occurring between f_1 and f_3 is minimised. Now f_1 and f_3 are not mutually jumping, otherwise f_1, f_3, and z are mutually jumping; a contradiction. Hence we may assume that $\overrightarrow{X_3}$ has, as a consecutive subsequence, $(f_1, \overrightarrow{Z_3}, f_3, z)$. Since $\{z, f_1, f_2, f_3\}$ has rank 3 and is spanned by any of its 3-element subsets containing z, we deduce that $f_2 \in \mathrm{cl}(\{z, f_1, f_3\})$. But f_2 is a coguts element. Hence $f_2 \in Z_2 \cap Z_3$. Moreover, since $f_3 \in \mathrm{cl}(\{z, f_1, f_2\})$, the choice of $\overrightarrow{X_2}$ implies that f_2 is the last element of $\overrightarrow{Z_2}$. By symmetry, f_2 is the first element of $\overrightarrow{Z_3}$.

Suppose that $(Z_2 - f_2) \cap (Z_3 - f_2)$ contains some element e. Then e, f_2, and z are mutually jumping; a contradiction. Therefore $(Z_2 - f_2) \cap (Z_3 - f_2) = \emptyset$. Now if $Z_2 - f_2$ is empty, then the result holds. Thus we may consider the last element z_2 of $\overrightarrow{Z_2} - f_2$. In $\overrightarrow{X_3}$, either z_2 precedes f_1, or z_2 succeeds z. In the former case, z_2, f_1, and z are mutually jumping; a contradiction. Hence z_2 succeeds z in $\overrightarrow{X_3}$, so f_2 jumps z_2. Hence, in $\overrightarrow{X_2}$, we can interchange z_2 and f_2 maintaining an (A, B) 3-sequence. As before, since $f_3 \in \mathrm{cl}(\{z, f_1, f_2\})$, we can next move f_3 so that it immediately follows f_2. We now have an (A, B) 3-sequence that contradicts the choice of $\overrightarrow{X_2}$ since it has $|Z_2| - 1$ elements between f_1 and f_3. We conclude that the result holds if $m = 3$.

Now assume the result holds for $m < n$ and let $m = n > 3$. Then there is an (A, B) 3-sequence in which $z, F_1, F_2, \dots, F_{m-1}$ or $z, F_{m-1}, F_{m-2}, \dots, F_1$ occurs as a consecutive subsequence. If F_m is a singleton coguts element, then because $F_{m-2} \cup F_{m-1} \cup F_m$ is a cocircuit, we deduce that F_m must follow F_{m-1} in the first case, and precede it in the second. Moreover, it can be moved to immediately follow F_{m-1} in the first case, or to immediately precede F_{m-1} in the second. Thus the required result holds if F_m is a singleton coguts element. Hence we may assume that F_m is a set of guts elements.

By the induction assumption again, there is an (A, B) 3-sequence in which $z, F_2, F_3, \dots, F_m$ or $z, F_m, F_{m-1}, \dots, F_2$ occurs as a consecutive subsequence. Arguing as in the last paragraph, it follows that we may assume that F_1 is a set of guts elements. We conclude that m is odd, so $m \geqslant 5$.

If there are (A, B) 3-sequences in which both $z, F_1, F_2, \ldots, F_{m-1}$ and $z, F_m, F_{m-1}, \ldots, F_2$ occur as consecutive subsequences, then the elements of $F_2 \cup F_3 \cup F_4$ are mutually jumping; a contradiction. By symmetry, we may assume that there are 3-sequences $(A, \overrightarrow{X_1}, B)$ and $(A, \overrightarrow{X_2}, B)$ in which $(z, F_1, F_2, \ldots, F_{m-1})$ and $(z, F_2, F_3, \ldots, F_m)$, respectively, occur as consecutive subsequences. If, in $\overrightarrow{X_1}$, the elements of F_m occur to the left of z, then F_m and F_{m-2} are mutually jumping. Thus the elements of $F_{m-2} \cup F_m \cup z$ are mutually jumping, contradicting the fact that $F_m \cup z$ is a maximal segment. We deduce that, in $\overrightarrow{X_1}$, the elements of F_m occur to the right of F_{m-1}. Since both $z \cup F_{m-2} \cup F_{m-1}$ and $z \cup F_{m-2} \cup F_{m-1} \cup F_m$ have rank 3, it follows that $F_m \subseteq \mathrm{cl}(z \cup F_{m-2} \cup F_{m-1})$. Thus we can move the elements of F_m in $\overrightarrow{X_1}$ so that they immediately follow those of F_{m-1}. Hence we have an (A, B) 3-sequence in which $(z, F_1, F_2, \ldots, F_m)$ occurs as a consecutive subsequence. The lemma follows immediately by induction. □

Lemma 9.5. *Let $(z, F_1, F_2, \ldots, F_m)$ be a p-flan in a 3-sequence (A, X, B). Then $F_1 \cup F_2 \cup \cdots \cup F_m \subseteq J_z$. Moreover, either*

(i) *in every (A, B) 3-sequence, the subsequence consisting of the elements of $F_1 \cup F_2 \cup \cdots \cup F_m$ is $(F_1, F_2, \ldots, F_m)$; or*
(ii) *in every (A, B) 3-sequence, the subsequence consisting of the elements of $F_1 \cup F_2 \cup \cdots \cup F_m$ is $(F_m, F_{m-1}, \ldots, F_1)$.*

Proof. Without loss of generality, by Lemma 9.4, we may assume that there is an (A, B) 3-sequence in which $(z, F_1, F_2, \ldots, F_m)$ is a consecutive subsequence.

If F_k is a set of guts elements for some k in $\{1, 2, \ldots, m\}$, then, because $F_k \cup z$ is a maximal segment, its elements can be made consecutive in some (A, B) 3-sequence. Hence every guts element of $F_1 \cup F_2 \cup \cdots \cup F_m$ is in J_z. Now suppose that x and y are guts elements in F_i and F_j, respectively, where $i < j$. If there is an (A, B) 3-sequence in which y precedes x, then x, y, and z are mutually jumping. Thus $\{x, y, z\}$ is contained in a maximal segment and this will be the unique maximal segment containing $\{x, z\}$. But $\{x, z\}$ is contained in the maximal segment $F_i \cup z$ and this segment does not contain y; a contradiction. We conclude that if F_i and F_j are sets of guts elements with $i < j$, then, in every (A, B) 3-sequence, every element of F_i precedes every element of F_j.

Now let F_i be a singleton coguts set with $i \leqslant m - 2$. Because $F_i \cup F_{i+1} \cup F_{i+2}$ is a cocircuit, in every (A, B) 3-sequence, the first and last elements $F_i \cup F_{i+1} \cup F_{i+2}$ must be coguts elements. Thus the subsequence consisting of the elements of $F_i \cup F_{i+1} \cup F_{i+2}$ is F_i, F_{i+1}, F_{i+2} or F_{i+2}, F_{i+1}, F_i. But the latter does not occur since it implies that the elements of $F_i \cup F_{i+1} \cup F_{i+2}$ are mutually jumping. We deduce that (i) holds unless F_m is a set of guts elements. By symmetry, (i) holds unless F_1 is a set of guts elements.

To show that $F_1 \cup F_2 \cup \cdots \cup F_m \subseteq J_z$, assume first that F_m is a set of guts elements. Then there is an (A, B) 3-sequence $(A, \overrightarrow{X_1}, B)$ in which the elements of $F_m \cup z$ are consecutive. From above, we know that every guts element of $F_1 \cup F_2 \cup \cdots \cup F_{m-1}$ occurs to the left of F_m. Suppose c is a coguts element of $F_1 \cup F_2 \cup \cdots \cup F_{m-1}$. Either c is in J_z, or c occurs to the right of $F_m \cup z$ in $\overrightarrow{X_1}$. In the latter case, because c is in the span of the set of guts elements in $F_1 \cup F_2 \cup \cdots \cup F_m \cup z$, we get a contradiction. We conclude that if F_m is a set of guts elements, then J_z contains $F_1 \cup F_2 \cup \cdots \cup F_m$. By symmetry, the same conclusion holds if F_1 is a set of guts elements. We deduce that $F_1 \cup F_2 \cup \cdots \cup F_m \subseteq J_z$ unless both F_1 and F_m are singleton coguts sets.

Consider the exceptional case. Then, from above, (i) holds since F_1 is not a set of guts elements. Now the elements of $F_2 \cup z$ can be made consecutive in some 3-sequence $(A, \overrightarrow{X_2}, B)$. By (i), the subsequence consisting of the elements of $F_1 \cup F_2 \cup F_3$ is F_1, F_2, F_3. We may assume that z is the first element of $F_2 \cup z$ in $\overrightarrow{X_2}$. Then, since $F_1 \cup F_2 \cup F_3$ is a cocircuit, we obtain another (A, B) 3-sequence from $(A, \overrightarrow{X_2}, B)$ by moving the unique element of F_1 so that it immediately follows z. Thus $F_1 \subseteq J_z$. By symmetry, $F_m \subseteq J_z$. If $m = 3$, then, as $F_2 \cup z$ is a maximal segment, $F_2 \subseteq J_z$, so $F_1 \cup F_2 \cup \cdots \cup F_m \subseteq J_z$. If $m \geqslant 4$, then $(z, F_1, F_2, \ldots, F_{m-1})$ is a p-flan whose last set is a set of guts elements. Hence, from the previous paragraph, J_z contains $F_1 \cup F_2 \cup \cdots \cup F_{m-1}$. We conclude that, in general, $F_1 \cup F_2 \cup \cdots \cup F_m \subseteq J_z$. It follows that if $x \in F_i$ and $y \in F_j$ for some distinct i and j, then x, y, and z are not mutually jumping. Since there is an (A, B) 3-sequence in which $z, F_1, F_2, \ldots, F_m$ is a consecutive subsequence, we deduce that (i) holds. □

In view of the last result, from now on, when we refer to a p-flan $(\{z\}, F_1, F_2, \ldots, F_m)$ in a 3-sequence $(A, \overrightarrow{X}, B)$, we shall assume that the ordering on the subsequence of $\overrightarrow{X}$ consisting of the elements of $F_1 \cup F_2 \cup \cdots \cup F_m$ is $(F_1, F_2, \ldots, F_m)$.

Corollary 9.6. *Let $(z, F_1, F_2, \ldots, F_m)$ be a maximal p-flan in a 3-sequence $(A, \overrightarrow{X}, B)$. Then $F_1 \cup F_2 \cup \cdots \cup F_m = J_z$.*

Proof. By Lemma 9.5, $F_1 \cup F_2 \cup \cdots \cup F_m \subseteq J_z$. As $m \geqslant 3$, by Theorem 9.3, $J_z \cup \{z\}$ is a p-flan with tip z. The result follows immediately. □

The next lemma uses the local connectivity function to recognise a p-flan in an (A, B) 3-sequence of a special form. The result is proved in more generality than is needed here as it will be used again at the end of the next section when 3-sequences of this form will arise naturally.

Lemma 9.7. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence. Suppose that $\overrightarrow{X} = (\overrightarrow{A_1}, \overrightarrow{T_1}, \overrightarrow{T_2}, \ldots, \overrightarrow{T_{2k+1}}, \overrightarrow{B_1})$ where each of $T_1, T_3, \ldots, T_{2k+1}$ contains a single coguts element and each of $T_2, T_4, \ldots, T_{2k}$ is a non-empty set of guts elements. Then $\sqcap(A \cup A_1, B \cup B_1) = 1$ if and only if, for all j in $\{0, 1, \ldots, k-1\}$, the set $T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}$ contains a unique cocircuit and this cocircuit contains $T_{1+2j} \cup T_{3+2j}$. Moreover, if $\sqcap(A \cup A_1, B \cup B_1) = 1$ and, for some j in $\{0, 1, \ldots, k-1\}$, the set $T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}$ is not a cocircuit, then*

(i) *T_{2+2j} contains an element z such that $(T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}) - \{z\}$ is a cocircuit;*
(ii) *$T_{1+2i} \cup T_{2+2i} \cup T_{3+2i}$ is a cocircuit for all i in $\{0, 1, \ldots, k-1\} - \{j\}$; and*
(iii) *$(z, T_1, T_2, \ldots, T_{1+2j}, T_{2+2j} - \{z\}, T_{3+2j}, \ldots, T_{2k+1})$ is a p-flan with tip z.*

Proof. Let $A' = A \cup A_1$ and $B' = B \cup B_1$. Suppose first that, for all j in $\{0, 1, \dots, k-1\}$, the set $T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}$ contains a cocircuit containing $T_{1+2j} \cup T_{3+2j}$. Then

$$\mathrm{r}\big(A' \cup B'\big) \leqslant \mathrm{r}(M) - k. \tag{9.1}$$

Now, because each of $T_1, T_3, \dots, T_{2k+1}$ contains a single coguts element while each of $T_2, T_4, \dots, T_{2k}$ is a non-empty set of guts elements, we have, for all i in $\{0, 1, \dots, k\}$ and all j in $\{1, 2\}$, that

$$\mathrm{r}\big(A' \cup T_1 \cup T_2 \cup \dots \cup T_{2i+j}\big) + \mathrm{r}\big(B'\big) = \mathrm{r}\big(A'\big) + (i+1) + \mathrm{r}\big(B'\big). \tag{9.2}$$

Thus, as $(A' \cup T_1 \cup T_2 \cup \dots \cup T_{2k+1}, B')$ is a 3-separation of M,

$$\mathrm{r}\big(A'\big) + \mathrm{r}\big(B'\big) = \mathrm{r}(M) - (k-1). \tag{9.3}$$

On combining (9.1) and (9.3), we get that

$$\sqcap\big(A', B'\big) = \mathrm{r}\big(A'\big) + \mathrm{r}\big(B'\big) - \mathrm{r}\big(A' \cup B'\big) \geqslant \big[\mathrm{r}(M) - (k-1)\big] - \big[\mathrm{r}(M) - k\big] = 1.$$

But, by Lemma 3.3, (9.2), and (9.3) and using the fact that $T_{2k-1} \cup T_{2k} \cup T_{2k+1}$ is a cocircuit, we have

$$\begin{aligned}
\sqcap\big(A', B'\big) &\leqslant \sqcap\big(A' \cup T_1 \cup T_2 \cup \dots \cup T_{2k-2}, B'\big)\\
&= \mathrm{r}\big(A' \cup T_1 \cup T_2 \cup \dots \cup T_{2k-2}\big) + \mathrm{r}\big(B'\big)\\
&\quad - \mathrm{r}\big(A' \cup T_1 \cup T_2 \cup \dots \cup T_{2k-2} \cup B'\big)\\
&= \mathrm{r}\big(A'\big) + (k-1) + \mathrm{r}\big(B'\big) - \big(\mathrm{r}(M) - 1\big)\\
&= 1.
\end{aligned}$$

We conclude that $\sqcap(A', B') = 1$.

Conversely, suppose that $\sqcap(A', B') = 1$. Then, by Lemma 3.3, for all j in $\{0, 1, \dots, k-1\}$

$$\begin{aligned}
1 &\leqslant \sqcap\big(A' \cup T_1 \cup T_2 \cup \dots \cup T_{2j}, T_{2j+4} \cup \dots \cup T_{2k+1} \cup B'\big)\\
&= \mathrm{r}\big(A' \cup T_1 \cup T_2 \cup \dots \cup T_{2j}\big) + \mathrm{r}\big(T_{2j+4} \cup \dots \cup T_{2k+1} \cup B'\big)\\
&\quad - \mathrm{r}\big(E(M) - (T_{1+2j} \cup T_{2+2j} \cup T_{3+2j})\big)\\
&= \mathrm{r}(M) - \mathrm{r}\big(E(M) - (T_{1+2j} \cup T_{2+2j} \cup T_{3+2j})\big) \quad \text{by Lemma 9.2(ii).}
\end{aligned}$$

Thus $T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}$ contains a cocircuit of M. Since both $E(M) - (T_{1+2j} \cup T_{2+2j})$ and $E(M) - (T_{2+2j} \cup T_{3+2j})$ span T_{2+2j} and hence span M, we deduce that each cocircuit contained in $T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}$ contains $T_{1+2j} \cup T_{3+2j}$. By cocircuit elimination, it follows that $T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}$ contains a unique cocircuit.

Now suppose that, for some j in $\{0, 1, \ldots, k-1\}$, the set $T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}$ is not a cocircuit. Since T_{2+2j} is a single guts element or a segment, the cocircuit C^* contained in $T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}$ avoids at most one element of T_{2+2j}. Thus $C^* = (T_{1+2j} \cup T_{2+2j} \cup T_{3+2j}) - \{z\}$ for some z in T_{2j+2}.

Clearly $\mathrm{r}(A \cup A_1 \cup T_1 \cup \cdots \cup T_{1+2j} \cup \{z\}) = \mathrm{r}(A \cup A_1 \cup T_1 \cup \cdots \cup T_{2j}) + 1$. If $\mathrm{r}(A \cup A_1 \cup T_1 \cup \cdots \cup T_{2j} \cup \{z\}) = \mathrm{r}(A \cup A_1 \cup T_1 \cup \cdots \cup T_{2j}) + 1$, then $A \cup A_1 \cup T_1 \cup \cdots \cup T_{2j} \cup \{z\}$ spans T_{1+2j}; a contradiction. Thus $\mathrm{r}(A \cup A_1 \cup T_1 \cup \cdots \cup T_{2j} \cup \{z\}) = \mathrm{r}(A \cup A_1 \cup T_1 \cup \cdots \cup T_{2j})$. Hence, in $\overrightarrow{X}$, we can remove z from T_{2+2j} and add it to T_{2j}. This new sequence obeys the same hypotheses as the original sequence. In particular, the value of $\sqcap(A', B')$ remains at 1. Using the equivalence established in the first part of the proof and applying it to the new sequence with z moved, we get that $T_{2j-1} \cup (T_{2j} \cup \{z\}) \cup T_{2j+1}$ contains a unique cocircuit D^*. But applying the equivalence to the initial sequence before z was moved, we have that $T_{2j-1} \cup T_{2j} \cup T_{2j+1}$ contains a unique cocircuit. This cocircuit must be D^* and it avoids at most one element of the segment $T_{2j} \cup \{z\}$. Hence z is the unique element of $T_{2j} \cup \{z\}$ avoided by D^*. We conclude that $T_{2j-1} \cup T_{2j} \cup T_{2j+1}$ is a cocircuit. Moreover, z jumps $T_{2j} \cup T_{2j+1}$. By repeating this argument, we deduce that (ii) holds and z jumps every element of $T_1 \cup T_2 \cup \cdots \cup T_{2k+1}$ except itself. It follows that each of $T_2 \cup \{z\}, T_4 \cup \{z\}, \ldots, T_{2k} \cup \{z\}$ is a maximal segment for if there is an element y that can be added to one of these sets to get a larger segment, say to T_{2t}, then y jumps both z and a coguts element c in T_{2t-1} or T_{2t+1}. Thus y, z, and c are mutually jumping; a contradiction. We conclude that $(z, T_1, T_2, \ldots, T_{1+2j}, T_{2+2j} - \{z\}, T_{3+2j}, \ldots, T_{2k+1})$ is a p-flan with tip z. □

10. Decompositions

In earlier sections, we have identified various structures in a matroid M that allow for the elements of X to be permuted in an 3-sequence $(A, \overrightarrow{X}, B)$ to give another such sequence $(A, \overrightarrow{X_1}, B)$. In this section, we shall complete the task of describing all possible such permutations $\overrightarrow{X_1}$. Because of the complexity of the behaviour of such permutations, we shall adopt a strategy of breaking up the sequence $\overrightarrow{X}$ when certain previously discussed structures occur within X. The first such break-up will occur when we have a clock C. We know that C is the union of two fans $F_1(C)$ and $F_2(C)$ each of which has a fan ordering meaning that, in every (A, B) 3-sequence, all its elements occur in the same order. For each i in $\{1, 2\}$, let f_i be the first element of $F_i(C)$ and let l_i be its last element. Let I_C be the set of internal elements of C, that is, $I_C = C - \{f_1, f_2, l_1, l_2\}$. By Corollary 7.6, when the elements of C are consecutive in an (A, B) 3-sequence, the sets L_C and R_C of elements that occur to the left and right of C are uniquely determined.

Now suppose that C is non-degenerate. Let $A' = A \cup L_C \cup (C - \{l_1, l_2\})$ and $B' = (C - \{f_1, f_2\}) \cup R_C \cup B$. We break $(A, \overrightarrow{X}, B)$ into $(A', \overrightarrow{R_C \cup \{l_1, l_2\}}, B)$ and $(A, \overrightarrow{L_C \cup \{f_1, f_2\}}, B')$ where the orderings $\overrightarrow{L_C \cup \{f_1, f_2\}}$ and $\overrightarrow{R_C \cup \{l_1, l_2\}}$ are induced by $\overrightarrow{X}$. Both $(A', \overrightarrow{R_C \cup \{l_1, l_2\}}, B)$ and $(A, \overrightarrow{L_C \cup \{f_1, f_2\}}, B')$ can be shown to be 3-sequences. More significantly, if we take orderings $\overrightarrow{[R_C \cup \{l_1, l_2\}]_1}$ and $\overrightarrow{[L_C \cup \{f_1, f_2\}]_1}$ of these sets such that both $(A', \overrightarrow{[R_C \cup \{l_1, l_2\}]_1}, B)$ and $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1}, B')$ are 3-sequences, then $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1}, \overrightarrow{I_C}, \overrightarrow{[R_C \cup \{l_1, l_2\}]_1}, B)$ is a 3-sequence where $\overrightarrow{I_C}$ is any ordering of the internal elements of C that respects the fan orderings in C. Furthermore, we can obtain every (A, B) 3-sequence in this way by allowing as a final step, the movement of clock ends over adjacent elements in their jump-sets.

In this section, we shall describe three break-up results. The first treats non-degenerate or odd degenerate clocks; the second treats even degenerate clocks; and the last treats p-flans in the absence of clocks. By applying these three results and the dual of the third, we will be left with a 3-sequence in which there are no clocks, no p-flans, and no p-coflans. Such 3-sequences are treated in the last theorem of the section. There it is shown that such a 3-sequence breaks up into a collection of segments and cosegments. Moreover, the only allowable movements of elements within this 3-sequence are arbitrary shufflings of the elements within each segment and cosegment along with swaps of pairs of elements at the interfaces between segments and cosegments.

Before stating our first decomposition theorem, we consider how the elements of a degenerate clock can sit in an (A, B) 3-sequence. Recall that the type of an element is its designation as a guts or a coguts element.

Lemma 10.1. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence and C be a degenerate clock in X. Suppose $\overrightarrow{X} = \overrightarrow{Z_1}, f_i, \overrightarrow{Z_2}, x_u, \overrightarrow{Z_3}, x_v, \overrightarrow{Z_4}, l_p, \overrightarrow{Z_5}$, where $\{f_i, x_u, x_v, l_p\} = \{f_1, f_2, l_1, l_2\}$.*

(i) *If f_i and x_u are of different types, then $Z_3 = \emptyset$. In particular, if C is odd, then $Z_3 = \emptyset$.*
(ii) *If f_i and x_u are of the same type, then $\{f_i, x_u\} = \{f_1, f_2\}$ and C is even. Moreover, either Z_3 is empty; or there is a crocodile for which the segment S is $\mathrm{cl}(\{f_1, f_2\})$ or $\mathrm{cl}(\{l_1, l_2\}$ and the cosegment S^* is $\mathrm{cl}^*(\{l_1, l_2\})$ or $\mathrm{cl}^*(\{f_1, f_2\}$. In the latter case, $Z_3 \subseteq S \cup S^*$.*
(iii) *$Z_1 \cup Z_2 \subseteq L_C$ and $Z_4 \cup Z_5 \subseteq R_C$.*

Proof. To prove (i) and (ii), suppose that Z_3 is non-empty and consider e in Z_3. Since there is an (A, B) 3-sequence in which f_i, x_u, x_v, l_p is a consecutive subsequence, e jumps both f_i and x_u or both x_v and l_p. If $\{f_i, x_u\} = F_i(C)$, then Corollary 7.5 implies that $e \in C$; a contradiction. Thus, we may assume that $\{f_i, x_u\} = \{f_1, f_2\}$. Then either f_1, f_2, and e are mutually jumping or l_1, l_2, and e are mutually jumping. Hence f_1 and x_u are of the same type and (i) holds. Moreover, C is even and, since the elements of C can be made consecutive in some 3-sequence, it follows that either

(a) f_1 and f_2 are guts elements, in which case, $\mathrm{cl}(\{f_1, f_2\})$ is a maximal segment S and $\mathrm{cl}^*(\{l_1, l_2\})$ is a maximal cosegment S^*; or
(b) f_1 and f_2 are coguts elements, in which case, $\mathrm{cl}^*(\{f_1, f_2\})$ is a maximal cosegment S^* and $\mathrm{cl}(\{l_1, l_2\})$ is a maximal segment S.

In both cases, $S \cup S^*$ is a (possibly degenerate) crocodile and $Z_3 \subseteq S \cup S^*$.

Part (iii) is an immediate consequence of Corollary 7.5. □

The next result describes how the presence of a clock C enables us to break a 3-sequence (A, X, B). We have assumed here that C is a non-degenerate or odd degenerate clock. The break-up associated with an even degenerate clock will be treated in the subsequent result.

Theorem 10.2. *Let C be a non-degenerate or odd degenerate clock in a 3-sequence $(A, \overrightarrow{X}, B)$. Let $\overrightarrow{R_C \cup \{l_1, l_2\}}$ and $\overrightarrow{L_C \cup \{f_1, f_2\}}$ be the orderings induced on these sets by the ordering $\overrightarrow{X}$. Let $A' = A \cup L_C \cup (C - \{l_1, l_2\})$ and $B' = B \cup R_C \cup (C - \{f_1, f_2\})$. Then*

(i) $(A', \overrightarrow{R_C \cup \{l_1, l_2\}}, B)$ *and* $(A, \overrightarrow{L_C \cup \{f_1, f_2\}}, B')$ *are* 3*-sequences.*
(ii) *For some orderings $\overrightarrow{[R_C \cup \{l_1, l_2\}]_1}$ and $\overrightarrow{[L_C \cup \{f_1, f_2\}]_1}$ of $R_C \cup \{l_1, l_2\}$ and $L_C \cup \{f_1, f_2\}$, let $(A', \overrightarrow{[R_C \cup \{l_1, l_2\}]_1}, B)$ and $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1}, B')$ be* 3*-sequences. Let I'_C be obtained from I_C by adjoining the last element of $\overrightarrow{[L_C \cup \{f_1, f_2\}]_1}$ if this last element is in $\{f_1, f_2\}$, and adjoining the first element of $\overrightarrow{[R_C \cup \{l_1, l_2\}]_1}$ if this first element is in $\{l_1, l_2\}$. Let $\overrightarrow{I'_C}$ be an ordering of I'_C that respects the fan orderings on $F_1(C)$ and $F_2(C)$. Then $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1} - I'_C, \overrightarrow{I'_C}, \overrightarrow{[R_C \cup \{l_1, l_2\}]_1} - I'_C, B)$ is an (A, B)* 3*-sequence. Moreover, every (A, B)* 3*-sequence arises in this way from some* 3*-sequences $(A', \overrightarrow{[R_C \cup \{l_1, l_2\}]_1}, B)$ and $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1}, B')$.*

Proof. By Theorem 7.2, there is a 3-sequence of the form $(A, \overrightarrow{L_C}, \overrightarrow{C}, \overrightarrow{R_C}, B)$. Moreover, by Theorem 7.4, we may choose $\overrightarrow{C}$ so that its first two elements are f_1 and f_2 and its last two elements are l_1 and l_2. Thus $(A \cup L_C \cup (C - \{l_1, l_2\}, R_C \cup \{l_1, l_2\} \cup B)$ and $(A \cup L_C \cup \{f_1, f_2\}, R_C \cup (C - \{f_1, f_2\}) \cup B)$ are exact 3-separations of M.

We want to show that $(A \cup Y, Z \cup B')$ is a 3-separation for any Y and Z such that $\overrightarrow{L_C \cup \{f_1, f_2\}} = \overrightarrow{Y}, \overrightarrow{Z}$. Certainly $(A \cup Y, Z \cup B')$ is a 3-sequence if $\overrightarrow{Y}$ coincides with a consecutive initial subsequence of $\overrightarrow{X}$. Assume that C is non-degenerate. By Lemma 7.7(ii), in $\overrightarrow{X}$, there is a consecutive subsequence $\overrightarrow{X'}$ that contains I_C and has all its elements in C. By Lemma 7.7(iii), neither f_1 nor f_2 occurs to the right of $\overrightarrow{X'}$. If f_1 or f_2 occurs in X', then we can move these elements so that they maintain their relative order and occur at the left end of $\overrightarrow{X'}$. Likewise, if l_1 or l_2 occurs in X', then we can move these elements so that they maintain their relative order and occur at the right end of $\overrightarrow{X'}$. By Theorem 7.4, these moves do not alter the fact that we have an (A, B) 3-sequence. Thus we have an (A, B) 3-sequence in which the elements of I_C are consecutive, so we may assume that $\overrightarrow{X}$ has the form $\overrightarrow{Y_1}, f_i, \overrightarrow{Y_2}, f_j, \overrightarrow{Y_3}, \overrightarrow{I_C}, \overrightarrow{Z_3}, l_p, \overrightarrow{Z_2}, l_q, \overrightarrow{Z_1}$, where $\{i, j\} = \{1, 2\} = \{p, q\}$.

By Theorem 7.2, if an element e of X has two elements of one of the fans of C in its jump-set, then $e \in C$. It follows that $Y_1 \cup Y_2 \subseteq L_C$ and $Z_2 \cup Z_1 \subseteq R_C$. Moreover, as C is non-degenerate, $Y_3 \subseteq L_C$ and $Z_3 \subseteq R_C$. Hence, if C is non-degenerate, then $Y_1 \cup Y_2 \cup Y_3 \cup \{f_1, f_2\} = L_C \cup \{f_1, f_2\}$ and so $(A, \overrightarrow{L_C \cup \{f_1, f_2\}}, B')$ is a 3-sequence.

Now assume that C is an odd degenerate clock and let $\overrightarrow{X} = \overrightarrow{Z_1}, f_i, \overrightarrow{Z_2}, x_u, \overrightarrow{Z_3}, x_v, \overrightarrow{Z_4}, l_p, \overrightarrow{Z_5}$, where $\{f_i, x_u, x_v, l_p\} = \{f_1, f_2, l_1, l_2\}$. By Lemma 10.1, $Z_3 = \emptyset$ and L_C and R_C are $Z_1 \cup Z_2$ and $Z_4 \cup Z_5$, respectively. Evidently, $\{x_u, x_v\} = \{l_j, f_k\}$, where $\{j, k\} = \{1, 2\}$, so we can interchange x_u and x_v if necessary to get that $x_u = f_j$. Thus (i) holds if C is an odd degenerate clock.

To prove (ii), assume that C is a non-degenerate or an odd degenerate clock, and let $(A', \overrightarrow{[R_C \cup \{l_1, l_2\}]_1}, B)$ and $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1}, B')$ be 3-sequences. Take $\overrightarrow{I_C}$ to be an ordering of I_C that respects the fan orderings on $F_1(C)$ and $F_2(C)$. We show first that $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1}, \overrightarrow{I_C}, \overrightarrow{[R_C \cup \{l_1, l_2\}]_1} B)$ is a 3-sequence. To do this, it suffices to show that $(A \cup L_C \cup \{f_1, f_2\} \cup I_{C,1}, I_{C,2} \cup \{l_1, l_2\} \cup R_C \cup B)$ is a 3-sequence for every $I_{C,1}$ and $I_{C,2}$ such that $\overrightarrow{I_C} = \overrightarrow{I_{C,1}}, \overrightarrow{I_{C,2}}$. But $f_1, f_2, \overrightarrow{I_{C,1}}, \overrightarrow{I_{C,2}}, l_1, l_2$ respects the fan orderings on $F_1(C)$ and $F_2(C)$, and so, by Theorem 7.4, there is a 3-sequence of the form

$(A, L_C, f_1, f_2, \overrightarrow{I_{C,1}}, \overrightarrow{I_{C,2}}, l_1, l_2, R_C, B)$. Hence $(A \cup L_C \cup \{f_1, f_2\} \cup I_{C,1}, I_{C,2} \cup \{l_1, l_2\} \cup R_C \cup B)$ is indeed a 3-sequence.

Let I'_C be obtained from I_C as follows: if the last element of $\overrightarrow{[L_C \cup \{f_1, f_2\}]_1}$ is in $\{f_1, f_2\}$, then adjoin this last element to I_C; and if the first element of $\overrightarrow{[R_C \cup \{l_1, l_2\}]_1}$ is in $\{l_1, l_2\}$, then adjoin this element to I_C. Let $\overrightarrow{I'_C}$ be an ordering of I'_C that respects the fan orderings on $F_1(C)$ and $F_2(C)$. Then $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1} - I'_C, \overrightarrow{I'_C}, \overrightarrow{[R_C \cup \{l_1, l_2\}]_1} - I'_C, B)$ is an (A, B) 3-sequence because it can be obtained from the (A, B) 3-sequence $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1}, \overrightarrow{I_C}, \overrightarrow{[R_C \cup \{l_1, l_2\}]_1}, B)$ by allowing some f_i or some l_j to jump elements in C from their jump-sets.

To complete the proof, we need to show that every permutation $\overrightarrow{X_1}$ of X such that $(A, \overrightarrow{X_1}, B)$ is a 3-sequence arises as described. By Lemma 7.7, the elements of I_C are consecutive in $\overrightarrow{X_1}$ except for the possible insertion of clock ends. When C is non-degenerate, let $\overrightarrow{I''_C}$ be the longest consecutive subsequence of $\overrightarrow{X_1}$ that contains I_C and is contained in C. When C is an odd degenerate clock, by Lemma 10.1, the second and third elements of C are consecutive in $\overrightarrow{X_1}$. In this case, we let $\overrightarrow{I''_C}$ be the longest consecutive subsequence of $\overrightarrow{X_1}$ that is contained in C and contains these second and third elements. Let $\overrightarrow{X_1} = \overrightarrow{L}, \overrightarrow{I''_C}, \overrightarrow{R}$. Because there is a 3-sequence of the form (A, L_C, C, R_C, B), we deduce, using the fact that either $I_C \neq \emptyset$ or Lemma 10.1 applies, that $R \cap L_C = \emptyset$ and $L \cap R_C = \emptyset$. Thus $L - C = L_C$ and $R - C = R_C$. If both f_1 and f_2 are in I''_C, then the first element of $\overrightarrow{I''_C}$ is f_1 or f_2. In this case, we remove that element from $\overrightarrow{I''_C}$. Likewise, if both l_1 and l_2 are in I''_C, then the last element of $\overrightarrow{I''_C}$ is l_1 or l_2. In this case, we remove this element from $\overrightarrow{I''_C}$. Let $\overrightarrow{I'_C}$ be the result of these two potential removals. Now if f_i is in $\overrightarrow{I'_C}$, we can move it so that it is the first element of $\overrightarrow{I'_C}$ and this move maintains a 3-sequence. Similarly, if l_j is in $\overrightarrow{I'_C}$, we can move it so that it is the last element of $\overrightarrow{I'_C}$ and this move maintains a 3-sequence. These reorderings maintain the orderings $\overrightarrow{[L_C \cup \{f_1, f_2\}]_1}$ and $\overrightarrow{[R_C \cup \{l_1, l_2\}]_1}$ induced on these sets by the ordering $\overrightarrow{X_1}$. Moreover, the ordering we now have on X is $\overrightarrow{[L_C \cup \{f_1, f_2\}]_1}, \overrightarrow{I_C}, \overrightarrow{[R_C \cup \{l_1, l_2\}]_1}$, where $\overrightarrow{I_C}$ is the ordering induced on this set by the ordering $\overrightarrow{X_1}$. We conclude that $\overrightarrow{X_1}$ can be obtained by following the procedure described beginning with some 3-sequences $(A', \overrightarrow{[R_C \cup \{l_1, l_2\}]_1}, B)$ and $(A, \overrightarrow{[L_C \cup \{f_1, f_2\}]_1}, B')$. □

Next we consider breaking up an (A, B) 3-sequence when it has an even degenerate clock C. If g_1 and g_2 are the guts elements of C, and c_1 and c_2 are its coguts elements, then $\mathrm{cl}(\{g_1, g_2\})$ and $\mathrm{cl}^*(\{c_1, c_2\})$ are, respectively, a segment S and a cosegment S^*. If $|S \cup S^*| \geqslant 5$, then it is easily seen that $S \cup S^*$ is a crocodile. Thus the next result includes a decomposition result for a non-degenerate crocodile. To enable the result to be applied to degenerate crocodiles, one may, for example, add the assumption that no non-degenerate clocks are present. In view of Theorem 10.2, this is a natural assumption.

Theorem 10.3. *Let C be an even degenerate clock in a 3-sequence $(A, \overrightarrow{X}, B)$. Assume that the first elements of the fans $F_1(C)$ and $F_2(C)$ are both guts elements. Let S be the segment that is the closure of these first elements and let S^* be the cosegment that is the coclosure of the last elements of $F_1(C)$ and $F_2(C)$. Let $\overrightarrow{R_C \cup S^*}$ and $\overrightarrow{L_C \cup S}$ be the orderings induced on these sets by the ordering $\overrightarrow{X}$. Let $A' = A \cup L_C \cup S$ and $B' = B \cup R_C \cup S^*$. Then*

(i) $(A', \overrightarrow{R_C \cup S^*}, B)$ *and* $(A, \overrightarrow{L_C \cup S}, B')$ *are* 3*-sequences.*
(ii) *For some orderings* $\overrightarrow{[R_C \cup S^*]_1}$ *and* $\overrightarrow{[L_C \cup S]_1}$ *of* $R_C \cup S^*$ *and* $L_C \cup S$*, respectively, let* $(A', \overrightarrow{[R_C \cup S^*]_1}, B)$ *and* $(A, \overrightarrow{[L_C \cup S]_1}, B')$ *be* 3*-sequences. Then* $(A, \overrightarrow{[L_C \cup S]_1}, \overrightarrow{[R_C \cup S^*]_1}, B)$ *is an* (A, B) 3*-sequence; and the last element of* $\overrightarrow{[L_C \cup S]_1}$ *is in* S *while the first element of* $\overrightarrow{[R_C \cup S^*]_1}$ *is in* S^**. If these elements are in each other's jump-sets, then they can be interchanged to give another* (A, B) 3*-sequence. Moreover, every* (A, B) 3*-sequence arises in this way from some* 3*-sequences* $(A', \overrightarrow{[R_C \cup S^*]_1}, B)$ *and* $(A, \overrightarrow{[L_C \cup S]_1}, B')$*.*

Proof. First we note the following easy consequence of the fact that there is a 3-sequence in which the elements of C are consecutive.

10.3.1. $S - \{f_1, f_2\} \subseteq L_C$ *and* $S^* - \{l_1, l_2\} \subseteq R_C$*.*

Next we prove the first part of (ii). Let $(A', \overrightarrow{[R_C \cup S^*]_1}, B)$ and $(A, \overrightarrow{[L_C \cup S]_1}, B')$ be 3-sequences. Consider $(A, \overrightarrow{[L_C \cup S]_1}, \overrightarrow{[R_C \cup S^*]_1}, B)$. Because $A' = A \cup L_C \cup S$, it is clear that $(A \cup L_C \cup S, R_C \cup S^* \cup B)$ is a 3-separation. It follows immediately from this that $(A, \overrightarrow{[L_C \cup S]_1}, \overrightarrow{[R_C \cup S^*]_1}, B)$ is a 3-sequence. Let s be the last element of S in $\overrightarrow{[L_C \cup S]_1}$ and let s^* be the first element of S^* in $\overrightarrow{[R_C \cup S^*]_1}$. Let Z be the set of elements that occur between s and s^* in $(A, \overrightarrow{[L_C \cup S]_1}, \overrightarrow{[R_C \cup S^*]_1}, B)$. By Lemmas 10.1 and 8.1, there is a 3-sequence of the form $(A, L_C - S, S - s, s, s^*, S^* - s^*, R_C - S^*, B)$. It follows that every element of Z jumps all of S or all of S^*. Thus every such element is in S or S^*; a contradiction. We conclude that Z is empty, so s is the last element of $\overrightarrow{[L_C \cup S]_1}$ and s^* is the first element of $\overrightarrow{[R_C \cup S^*]_1}$. Evidently, if these two elements are in each other's jump-sets, then we can interchange them to get another 3-sequence. This proves all of (ii) except for the final assertion.

It remains to prove (i) and to check that every permutation $\overrightarrow{X_1}$ of X such that $(A, \overrightarrow{X_1}, B)$ is a 3-sequence arises as described. Let $\overrightarrow{X_1} = \overrightarrow{Z_1}, f_i, \overrightarrow{Z_2}, x_u, \overrightarrow{Z_3}, x_v, \overrightarrow{Z_4}, l_p, \overrightarrow{Z_5}$, where $\{f_i, x_u, x_v, l_p\} = \{f_1, f_2, l_1, l_2\}$.

Suppose first that $\{f_i, x_u\}$ is a fan of C. Then, by Lemma 10.1, $Z_3 = \emptyset$ and we have $L_C = Z_1 \cup Z_2$ and $R_C = Z_4 \cup Z_5$. Now x_u and x_v are adjacent and are in each other's jump-sets. It follows that if we interchange x_u and x_v in $(A, \overrightarrow{X_1}, B)$, we get $(A, \overrightarrow{L_C \cup S}, \overrightarrow{R_C \cup S^*}, B)$ as a 3-sequence. We deduce that, in this case, (i) and the last part of (ii) hold.

We may now suppose that $\{f_i, x_u\} = \{f_1, f_2\}$. Then, by Lemma 10.1, $Z_3 \subseteq S \cup S^*$. Moreover, by Lemma 8.1, no element in $Z_3 \cap S^*$ has two elements of $Z_3 \cap S$ to its right, and no element of $Z_3 \cap S$ has two elements of $Z_3 \cap S^*$ to its left. Thus $\overrightarrow{Z_3}$ consists of a consecutive sequence of elements of S followed by a consecutive sequence of elements of S^* with possibly the last S-element and the first S^*-element being interchanged. In the latter case, these two elements are in each other's jump-sets and interchanging them gives $(A, \overrightarrow{L_C \cup S}, \overrightarrow{R_C \cup S^*}, B)$ as a 3-sequence. Using this, it is easy to complete the proof that (i) and the last part of (ii) hold. □

We now turn attention to p-flans in (A, B) 3-sequnces with no clocks. We begin with a straightforward lemma.

Lemma 10.4. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence with no non-degenerate clocks. Let $F \cup z$ be a p-flan in X with tip z. If $(A, L, F \cup z, R, B)$ and $(A, L', F \cup z, R', B)$ are 3-sequences, then $L = L'$ and $R = R'$.*

Proof. If L' contains an element e of R, then e jumps $F \cup z$. But, by Lemma 9.5, $F \subseteq J_z$, and so $|J_e \cap J_z| \geqslant 3$. Therefore, by Theorem 7.2, X contains a non-degenerate clock; a contradiction. Thus $L' \cap R = \emptyset$. Similarly, $L \cap R' = \emptyset$. Hence $L = L'$ and $R = R'$. □

In view of the last lemma, for a p-flan $F \cup z$ with tip z in a 3-sequence $(A, \overrightarrow{X}, B)$ with no non-degenerate clocks, we can define L_F and R_F to be the sets of elements that, respectively, precede and succeed $F \cup z$ when the elements of the last set are consecutive.

Lemma 10.5. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence with no clocks. Let $F \cup z$ be a p-flan in X with tip z. Then no element of $X - (F \cup z)$ has two elements of F in its jump-set.*

Proof. Suppose that e is in $X - (F \cup z)$ and e has two elements of F in its jump-set. Let the p-flan $F \cup z$ be $(z, F_1, F_2, \ldots, F_m)$. As the subsequence of $\overrightarrow{X}$ induced by F has the form $\overrightarrow{F_1}, \overrightarrow{F_2}, \ldots, \overrightarrow{F_m}$ and there is an (A, B) 3-sequence $(A, \overrightarrow{X_1}, B)$ in which $(\overrightarrow{F_1}, \overrightarrow{F_2}, \ldots, \overrightarrow{F_m})$ is a consecutive subsequence, we deduce that J_e contains two elements of F_1, or J_e contains two elements of F_m, or J_e contains both a guts and a coguts element of F. But $F \subseteq J_z$ so $|J_z \cap J_e| \geqslant 2$. Since we have no clocks, Theorem 7.2 and the fact that z is a guts element implies that all the elements of $(J_z \cap J_e) \cup \{z, e\}$ are guts elements. We conclude that J_e contains two guts elements of F_1 or two guts elements of F_m. These two elements with e form a segment and so $F_1 \cup z \cup e$ or $F_m \cup z \cup e$ is a segment contradicting the fact that $F_1 \cup z$ and $F_m \cup z$ are maximal segments. □

Lemma 10.6. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence with no clocks. Let $F \cup z$ be a p-flan in X with tip z. In $\overrightarrow{X}$, let f and l be the first and last elements of F. Then the elements of $F - \{f, l\}$ are consecutive except for the possible insertion of z. Moreover, apart from possibly z, the elements to the left and to the right of $F - \{f, l\}$ are, respectively, $L_F \cup f$ and $R_F \cup l$. If $\overrightarrow{X}$ does not have a consecutive subsequence whose elements are $(F - \{f, l\}) \cup z$, then either*

(i) *f immediately precedes $F - \{f, l\}$ and z precedes f; also each element between z and f is in J_z. In particular, if the p-flan is maximal, then z immediately precedes f; or*
(ii) *l immediately follows $F - \{f, l\}$ and z follows l; also each element between l and z is in J_z. In particular, if the p-flan is maximal, then z immediately follows l.*

Proof. Let the p-flan $F \cup z$ be $(z, F_1, F_2, \ldots, F_m)$. Now suppose that $e \in X - (F \cup z)$. If e has two elements of F to its left and two to its right, then because the elements of F can be made consecutive in some (A, B) 3-sequence, it follows that e has two elements of F in its jump-set; a contradiction to Lemma 10.5.

We now know that the elements of $F - \{f, l\}$ are consecutive except for the possible insertion of z. If there is an element g to the left of $F - \{f, l\}$ that is in R_F or there is an element h to the right of $F - \{f, l\}$ that is in L_F, then g or h has two elements of F in its jump-set, contradicting Lemma 10.5. Hence, apart from possibly z, the sets of elements that precede and succeed $F - \{f, l\}$ are $L_F \cup f$ and $R_F \cup l$, respectively.

Assume that the elements of $F - \{f, l\}$ are consecutive and that z does not immediately precede or succeed this subsequence. Without loss of generality, we may assume that z occurs to the left of $F - \{f, l\}$. There are two possibilities for consecutive subsequences of $\overrightarrow{X}$:

(a) $(z, Z_1, f, Z_2, F - \{f, l\}, Z_3, l)$; or
(b) $(f, Z_1, z, Z_2, F - \{f, l\}, Z_3, l)$

In both cases, since there is an (A, B) 3-sequence in which $(z, f, F - \{f, l\}, l)$ occurs as a consecutive subsequence, each element e of Z_2 jumps $F - f$ or jumps $\{f, z\}$. If such an e jumps $F - f$, then e jumps at least two elements including both a guts element and a coguts element, so X contains a clock; a contradiction. Thus each element e of Z_2 jumps f and z. Hence e, f, and z are mutually jumping. As z is a guts element, it follows that $\{e, f, z\}$ is contained in a segment so the maximality of the segment $F_1 \cup z$ is contradicted. Thus $Z_2 = \emptyset$. This eliminates case (b). In case (a), since no element of Z_1 jumps all of F, every element of Z_1 is in J_z. Hence, when $F \cup z$ is a maximal p-flan, every element of J_z is in the p-flan so Z_1 is empty. □

Theorem 10.7. *In a* 3*-sequence* $(A, \overrightarrow{X}, B)$ *with no clocks, let* $F \cup z$ *be a maximal p-flan* $(z, F_1, F_2, \dots, F_m)$. *Let* $\overrightarrow{L_F \cup F_1}$ *and* $\overrightarrow{R_F \cup F_m}$ *be the orderings induced on these sets by the ordering* $\overrightarrow{X}$. *Let* $A' = A \cup L_F \cup (F - F_m) \cup z$ *and* $B' = B \cup R_F \cup (F - F_1) \cup z$. *Then*

(i) $(A', \overrightarrow{R_F \cup F_m}, B)$ *and* $(A, \overrightarrow{L_F \cup F_1}, B')$ *are* 3*-sequences.*
(ii) *Let* $(A', \overrightarrow{[R_F \cup F_m]_1}, B)$ *and* $(A, \overrightarrow{[L_F \cup F_1]_1}, B')$ *be* 3*-sequences and* $\overrightarrow{F_2}, \overrightarrow{F_3}, \dots, \overrightarrow{F_{m-1}}$ *be arbitrary orderings of each of these sets. Then* $(A, \overrightarrow{[L_F \cup F_1]_1}, z, \overrightarrow{F_2}, \overrightarrow{F_3}, \dots, \overrightarrow{F_{m-1}}, \overrightarrow{[R_F \cup F_m]_1}, B)$ *is a* 3*-sequence; so too is every sequence that is obtained from this one by any series of moves each involving* z *jumping an element of* F *to which it is adjacent. Moreover, every* (A, B) 3*-sequence arises in this way from some* 3*-sequences* $(A', \overrightarrow{[R_F \cup F_m]_1}, B)$ *and* $(A, \overrightarrow{[L_F \cup F_1]_1}, B')$.

Proof. In $\overrightarrow{X}$, let f and l be the first and last elements of F. Evidently, $f \in F_1$ and $l \in F_m$. By Lemma 10.6, the elements of $F - \{f, l\}$ are consecutive in $\overrightarrow{X}$ except for the possible insertion of z; and, again apart from possibly z, the elements to the left and to the right of $F - \{f, l\}$ are, respectively, $L_F \cup f$ and $R_F \cup l$. Moreover, either

(a) $\overrightarrow{X}$ has a consecutive subsequence whose elements are $(F - \{f, l\}) \cup z$; or
(b) $\overrightarrow{X}$ has $(z, f, F - \{f, l\})$ or $(F - \{f, l\}, l, z)$ as a consecutive subsequence.

In each case, since the induced order on $F - \{f, l\}$ is $(\overrightarrow{F_1} - f, \overrightarrow{F_2}, \overrightarrow{F_3}, \dots, \overrightarrow{F_{m-1}}, \overrightarrow{F_m} - l)$, we can move z over elements in its jump-set to obtain the 3-sequences $(A, \overrightarrow{L_F \cup F_1}, z, \overrightarrow{F_2}, \overrightarrow{F_3}, \dots, \overrightarrow{F_{m-1}}, \overrightarrow{R_F \cup F_m}, B)$ and $(A, \overrightarrow{L_F \cup F_1}, \overrightarrow{F_2}, \overrightarrow{F_3}, \dots, \overrightarrow{F_{m-1}}, z, \overrightarrow{R_F \cup F_m}, B)$ where $\overrightarrow{L_F \cup F_1}$ and $\overrightarrow{R_F \cup F_m}$ are the orderings induced on these sets by the ordering $\overrightarrow{X}$. It is immediate from this that $(A', \overrightarrow{R_F \cup F_m}, B)$ and $(A, \overrightarrow{L_F \cup F_1}, B')$ are 3-sequences where $A' = A \cup L_F \cup (F - F_m) \cup z$ and $B' = B \cup R_F \cup (F - F_1) \cup z$. Hence (i) holds.

Now let $(A', \overrightarrow{[R_F \cup F_m]_1}, B)$ and $(A, \overrightarrow{[L_F \cup F_1]_1}, B')$ be 3-sequences and $\overrightarrow{F_2}, \overrightarrow{F_3}, \ldots, \overrightarrow{F_{m-1}}$ be arbitrary orderings of each of these sets. Clearly $(A, \overrightarrow{[L_F \cup F_1]_1}, z, \overrightarrow{F_2}, \overrightarrow{F_3}, \ldots, \overrightarrow{F_{m-1}}, \overrightarrow{[R_F \cup F_m]_1}, B)$ is a 3-sequence; and we obtain other 3-sequences by moving z so that it immediately precedes or immediately succeeds any element of $\overrightarrow{F_2}, \overrightarrow{F_3}, \ldots, \overrightarrow{F_{m-1}}$. Finally, if, for some non-negative integers k and j, the last k elements of $\overrightarrow{[L_F \cup F_1]_1}$ and the first j elements of $\overrightarrow{[R_F \cup F_m]_1}$ are in F_1 and F_m, respectively, then we can move z so that it immediately precedes or immediately succeeds any of these elements. The theorem follows. □

Recall that a 2-element segment consists of two guts elements that can be made consecutive in some (A, B) 3-sequence. It is also convenient to extend the definition of a maximal segment to include a single guts element that is not contained in any segment with two or more elements. Dually, we include, as a maximal cosegment, a single coguts element that is not contained in any cosegment with two or more elements.

Lemma 10.8. *Let (A, X, B) be a 3-sequence that contains no p-flans and no p-coflans. Then every guts element is in exactly one maximal segment and every coguts element is in exactly one maximal cosegment.*

Proof. By duality, we may suppose that a guts element e is in maximal segments $S_1 \cup e$ and $S_2 \cup e$ where both S_1 and S_2 have at least one element and neither contains e. Then the jump-set of e contains $S_1 \cup S_2$. By Lemma 5.1, there is an (A, B) 3-sequence in which the elements of $J_e \cup e$ are consecutive. If J_e contains no coguts elements, then the fact that $S_1 \cup e$ is maximal is contradicted. Hence, by Theorem 9.3, there is a p-flan in X with tip e; a contradiction. □

Lemma 10.9. *Let (A, X, B) be a 3-sequence that contains no clocks, no p-flans, and no p-coflans. Let T be a maximal segment or a maximal cosegment in X and suppose that $x \in T$. Then x is in the jump-set of at most one element not in T and, when such an element exists, it is not of the same type as x.*

Proof. Suppose x is in the jump-set of an element y of $X - T$ where x and y are of the same type. Then, by Corollary 5.3, $\{x, y\}$ is contained in a maximal segment which is clearly different from T. This contradicts Lemma 10.8. Thus $J_x - T$ contains no element of the same type as x. If x is in the jump-set of at least two elements of the other type, then, by Theorem 9.3, $J_x \cup x$ is a p-flan or a p-coflan; a contradiction. □

The next result gives a fundamental structural result about (A, B) 3-sequences in which there are no clocks, no p-flans, and no p-coflans. Recall that we are viewing a single guts element g as a maximal segment if there is no segment that properly contains $\{g\}$.

Theorem 10.10. *Let (A, X, B) be a 3-sequence that contains no clocks, no p-flans, and no p-coflans. Let $\{T_1, T_2, \ldots, T_n\}$ be the set of maximal segments and maximal cosegments in X. Then the sets $T_1, T_2, \ldots, T_n$ can be relabelled so that, for any orderings $\overrightarrow{T_1}, \overrightarrow{T_2}, \ldots, \overrightarrow{T_n}$ of these sets, $(A, \overrightarrow{T_1}, \overrightarrow{T_2}, \ldots, \overrightarrow{T_n}, B)$ is a 3-sequence. Moreover, after this relabelling, if $|X| \geqslant 3$, and σ is a permutation of $\{1, 2, \ldots, n\}$ such that there is a 3-sequence of the form $(A, T_{\sigma(1)}, T_{\sigma(2)}, \ldots, T_{\sigma(n)}, B)$, then σ is the identity permutation.*

Proof. We begin with the initial labelling on $T_1, T_2, \dots, T_n$ and assume that $|X| \geqslant 3$. We shall define a relation $<$ on $\{T_1, T_2, \dots, T_n\}$ and show that $<$ is a total order on this set. To do this, we shall require some preliminaries. First recall that if T is a segment or a cosegment in X and $|T| \geqslant 2$, then there is an (A, B) 3-sequence in which the elements of T are consecutive. Moreover, by Theorem 6.8(ii), for all such (A, B) 3-sequences, the sets L_T and R_T of elements that occur to the left and right of T are the same. We follow the convention below that, whenever some T_i has a single element, this element will be denoted by t_i.

10.10.1. *Suppose that $|T_i| \geqslant 2$ and $|T_j| \geqslant 2$ where $i \neq j$. Then either $T_j \subseteq L_{T_i}$ or $T_j \subseteq R_{T_i}$.*

To see this, assume that T_j has an element e in L_{T_i} and an element f in R_{T_i}. Because e and f can be made consecutive in some (A, B) 3-sequence, each element of T_i is in J_e or J_f. If all of T_i is contained in J_e, then, by Corollary 6.5, as e and the elements of T_i are mutually jumping, $T_i \cup e$ is contained in a segment or a cosegment, contradicting the maximality of T_i. Thus T_i contains distinct elements e' and f' that are in the jump-sets of e and f, respectively. Then $J_e \cap J_{f'} \supseteq \{f, e'\}$. Since there are no clocks, we deduce, by Theorem 7.2, that all of e, f, e', and f' are of the same type. Since e and e' are mutually jumping, we have a contradiction to Lemma 10.9. We conclude that 10.10.1 holds.

10.10.2. *Suppose that $|T_i| \geqslant 2$ and $|T_j| \geqslant 2$ where $i \neq j$. Then $T_j \subseteq L_{T_i}$ if and only if $T_i \subseteq R_{T_j}$.*

Suppose that $T_j \subseteq L_{T_i}$. Then, by 10.10.1, $T_i \subseteq L_{T_j}$ or $T_i \subseteq R_{T_j}$. In the first case, every element of T_i is in the jump-set of every element of T_j, contradicting Lemma 10.9. Thus $T_i \subseteq R_{T_j}$ and 10.10.2 follows.

10.10.3. *Suppose that $|T_i| = |T_j| = 1$ where $i \neq j$. If $|X| \geqslant 3$, then either t_i precedes t_j in every (A, B) 3-sequence, or t_j precedes t_i in every (A, B) 3-sequence.*

Suppose that there are (A, B) 3-sequences in which t_i precedes t_j and in which t_j precedes t_i. Then, by Lemma 10.9, t_i and t_j are of different types. Since t_i and t_j are in each other's jump-sets, there is a 3-sequence $(A, \overrightarrow{X_1}, B)$ in which t_i and t_j are consecutive. Since $|X| \geqslant 3$, in either $(A, \overrightarrow{X_1}, B)$ or the 3-sequence obtained from it by interchanging t_i and t_j, we have an (A, B) 3-sequence in which t_i or t_j is adjacent to an element of the same type. This contradicts the maximality of T_i or of T_j, and 10.10.3 follows.

When $|X| \geqslant 3$, we now define the relation $<$ on $\{T_1, T_2, \dots, T_n\}$ as follows. If $i \neq j$, then

$$T_i < T_j$$

provided

(a) $|T_i| \geqslant 2$ and $T_j \subseteq R_{T_i}$;
(b) $|T_j| \geqslant 2$ and $T_i \subseteq L_{T_j}$; or
(c) $|T_i| = |T_j| = 1$ and t_i precedes t_j in every (A, B) 3-sequence.

By 10.10.1–10.10.3, this relation is well defined. Moreover, for all distinct i and j, either $T_i < T_j$ or $T_j < T_i$, but not both. To ensure that this defines a total order on $\{T_1, T_2, \dots, T_n\}$, we need to show that this relation is transitive.

10.10.4. *If i, j, and k are distinct elements of $\{1, 2, \dots, n\}$ and $T_i < T_j$ and $T_j < T_k$, then $T_i < T_k$.*

Assume not. Then $T_k < T_i$. Suppose $|T_j| \geqslant 2$. Then $T_i \subseteq L_{T_j}$ and $T_k \subseteq R_{T_j}$. As $T_k < T_i$, it follows that every element of T_k is in the jump-set of every element of T_i. This contradicts Lemma 10.9 unless $|T_i| = |T_k| = 1$. But, in the exceptional case, 10.10.3 is contradicted.

We may now assume that $|T_j| = 1$. If $|T_i| = 1$, then, as $T_i < T_j$, it follows from the definition that $T_i < T_k$; a contradiction. Hence, we may assume that $|T_i| \geqslant 2$, and, by symmetry, that $|T_k| \geqslant 2$. Since $T_k < T_i$, we have $T_k \subseteq L_{T_i}$. As $T_i < T_j$, we have $T_j \subseteq R_{T_i}$. Thus, there is an (A, B) 3-sequence in which t_j occurs to the right of T_k. But $T_j < T_k$ so $T_j \subseteq L_{T_k}$. Hence every element of T_k is in the jump-set of t_j and Lemma 10.9 is contradicted. We conclude that 10.10.4 holds.

Now relabel $T_1, T_2, \dots, T_n$ so that $T_1 < T_2 < \cdots < T_n$. We show next that, for any orderings $\overrightarrow{T_1}, \overrightarrow{T_2}, \dots, \overrightarrow{T_n}$ of these sets, $(A, \overrightarrow{T_1}, \overrightarrow{T_2}, \dots, \overrightarrow{T_n}, B)$ is a 3-sequence. Again this will be done in several steps.

10.10.5. *If $1 \leqslant i \leqslant n-1$ and $|T_i| \geqslant 2$ or $|T_{i+1}| \geqslant 2$, then $(A \cup T_1 \cup \cdots \cup T_i, T_{i+1} \cup \cdots \cup T_n \cup B)$ is a 3-separation.*

Assume that $|T_i| \geqslant 2$. By the definition of $<$, in a 3-sequence of the form $(A, L_{T_i}, T_i, R_{T_i}, B)$, we have $T_j \subseteq L_{T_i}$ when $1 \leqslant j \leqslant i-1$ and $T_j \subseteq R_{T_i}$ when $i+1 \leqslant j \leqslant n$. From this, 10.10.5 follows when $|T_i| \geqslant 2$. By symmetry, it also follows when $|T_{i+1}| \geqslant 2$.

10.10.6. *If $1 \leqslant i \leqslant n-1$ and $|T_i| = |T_{i+1}| = 1$, then $(A \cup T_1 \cup \cdots \cup T_i, T_{i+1} \cup \cdots \cup T_n \cup B)$ is a 3-separation.*

Take a 3-sequence $(A, \overrightarrow{Z_1}, t_i, \overrightarrow{Z_2}, t_{i+1}, \overrightarrow{Z_3}, B)$ in which the number of elements occurring between t_i and t_{i+1} is a minimum. If $e \in Z_2$, then, without loss of generality, we may assume that $e \in T_j$ for some $j < i$. Evidently $|T_j| \geqslant 2$ and we have a 3-sequence of the form $(A, \overrightarrow{Y_1}, \overrightarrow{T_j}, \overrightarrow{Y_2}, t_i, \overrightarrow{Y_3}, t_{i+1}, \overrightarrow{Y_4}, B)$. By combining the last two sequences using Corollary 4.5, we get the following 3-sequence, where $Y = Y_1 \cup T_j \cup Y_2$:

$$\left(A, \overrightarrow{Y_1}, \overrightarrow{T_j}, \overrightarrow{Y_2}, \overrightarrow{Z_1} - Y, t_i, \overrightarrow{Z_2} - Y, t_{i+1}, \overrightarrow{Z_3} - Y, B\right).$$

As $e \in Z_2 \cap Y$, this last 3-sequence contradicts the choice of $(A, \overrightarrow{Z_1}, t_i, \overrightarrow{Z_2}, t_{i+1}, \overrightarrow{Z_3}, B)$. We deduce that $Z_2 = \emptyset$.

Now if $|T_k| = 1$, then T_k is contained in Z_1 or Z_3 depending on whether $k < i$ or $k > i+1$. Next assume that $|T_k| \geqslant 2$ and $k < i$. No element e of T_k is in Z_3 otherwise e jumps t_i and t_{i+1}, which contradicts Lemma 10.9. Hence $Z_1 = T_1 \cup T_2 \cup \cdots \cup T_{i-1}$ and $Z_3 = T_{i+2} \cup \cdots \cup T_n$ and 10.10.6 follows.

On combining 10.10.5, 10.10.6, and the fact that each of A and B is exactly 3-separating, we immediately get:

10.10.7. *If* $0 \leqslant i \leqslant n$*, then* $(A \cup T_1 \cup \cdots \cup T_i, T_{i+1} \cup \cdots \cup T_n \cup B)$ *is exactly* 3*-separating.*

Finally, to establish that, for any orderings $\overrightarrow{T_1}, \overrightarrow{T_2}, \ldots, \overrightarrow{T_n}$ of these sets, $(A, \overrightarrow{T_1}, \overrightarrow{T_2}, \ldots, \overrightarrow{T_n}, B)$ is a 3-sequence, we show the following.

10.10.8. *Suppose* $|T_i| \geqslant 2$ *for some* $1 \leqslant i \leqslant n$*. If* (T_i', T_i'') *is a partition of* T_i *into non-empty sets, then* $(A \cup T_1 \cup \cdots \cup T_{i-1} \cup T_i', T_i'' \cup T_{i+1} \cup \cdots \cup T_n \cup B)$ *is a* 3*-separation.*

To see this, take a 3-sequence $(A, \overrightarrow{L}, \overrightarrow{T_i}, \overrightarrow{R}, B)$. Then by Corollary 6.2, we get another 3-sequence by permuting the elements of T_i so that $\overrightarrow{T_i}$ has the form T_i', T_i''. Moreover, $L = T_1 \cup \cdots \cup T_{i-1}$ and $R = T_{i+1} \cup \cdots \cup T_i$, so the result follows.

Assume that we have a 3-sequence of the form $(A, T_{\sigma(1)}, T_{\sigma(2)}, \ldots, T_{\sigma(n)}, B)$. If, in this sequence, T_i occurs before T_j where $i > j$, then every element of T_i jumps every element of T_j so we get a contradiction to 10.10.3 or Lemma 10.9. □

Lemma 10.11. *Let* (A, X, B) *be a* 3*-sequence that contains no clocks,no p-flans, and no p-coflans. Let* $\{T_1, T_2, \ldots, T_n\}$ *be the collection of maximal segments and maximal cosegments in* X*, and assume that they are labelled so that there is an* (A, B) 3*-sequence of the form* $(A, T_1, T_2, \ldots, T_n, B)$*.*

- (i) *If* $x \in T_i$ *and* x *has an element* y *in its jump-set such that* $y \notin T_i$*, then* y *is in* T_{i-1} *or* T_{i+1}*.*
- (ii) *If* $2 \leqslant i \leqslant n-1$ *and* T_i *has a single element* t_i*, then* t_i *has no element in its jump-set.*
- (iii) *If* $|T_i| = 2$*, then at most one element of* T_i *has an element of the other type in its jump-set.*
- (iv) *If* $|T_i| \geqslant 3$*, then at most two elements of* T_i *have elements of the other type in their jump-sets. Moreover, when there are two such elements, one has its jump partner in* T_{i-1} *and the other has its jump partner in* T_{i+1}*.*

Proof. Suppose $x \in T_i$ and $y \in T_j$, and x and y are in each other's jump-sets. Assume that $j > i+1$. Then, because there is an (A, B) 3-sequence in which x and y are consecutive, it follows that, for some k with $i+1 \leqslant k < j$, there is an element z of T_k such that z jumps x or y. But this contradicts Lemma 10.9. We conclude that $j \leqslant i+1$ and, similarly, $j \geqslant i-1$. Thus (i) holds.

Now suppose that $T_i = \{t_i\}$ and $2 \leqslant i \leqslant n-1$. If t_i has an element e in its jump-set, then, by (i), we may assume that $e \in T_{i-1}$. Since there is an (A, B) 3-sequence of the form $(A, \overrightarrow{T_1}, \overrightarrow{T_2}, \ldots, \overrightarrow{T_n}, B)$, we may assume that e is the last element of T_{i-1}. Then, interchanging e and t_i shows that $T_{i+1} \cup e$ is a segment or cosegment that contradicts the maximality of T_{i+1}. Hence (ii) holds.

Next suppose that $1 \leqslant i \leqslant n$ and $|T_i| \geqslant 2$. Let $\{s_i, t_i\} \subseteq T_i$ and assume that each of s_i and t_i has an element outside of T_i in its jump-set. If s_i' and t_i' are in the jump-sets of s_i and t_i, respectively, and s_i' and t_i' are both in T_{i-1} or are both in T_{i+1}, then, by Theorem 7.2, we have a clock; a contradiction. Thus we may assume that $s_i' \in T_{i-1}$ and $t_i' \in T_{i+1}$. Part (iv) follows from this. Now let $|T_i| = 2$. As (s_i', s_i, t_i, t_i') occurs as a consecutive subsequence of an (A,B) 3-sequence, it follows that (s_i, s'_i, t'_i, t_i) does too. Thus $J_{s_i} \cap J_{t'_i}$, which equals $\{t_i, s'_i\}$, contains an element of each type, so we have a clock; a contradiction. □

The next lemma is the final preliminary result needed for the proof of the main theorem of the paper.

Lemma 10.12. *Let $(A, \vec{X}, B)$ be a 3-sequence that contains no clocks. Suppose that (u, v, w, x) is a consecutive subsequence of $\vec{X}$ such that u and v are of different types and w and x are of different types. Then either u and v cannot jump each other, or w and x cannot jump each other.*

Proof. Assume that u and v can jump each other, and w and x can jump each other. Then, since (v, u, w, x) is also a consecutive subsequence of an (A, B) 3-sequence, we may assume that v and w are of the same type. Then u and x are also of the same type. As (v, u, x, w), and hence (v, x, u, w), occurs as a consecutive subsequence of an (A, B) 3-sequence, we have that $J_u \cap J_w \supseteq \{v, x\}$. Thus, by Theorem 7.2, $\vec{X}$ contains a clock; a contradiction. □

We are now ready to prove Theorem 10.13, which describes the very simple structure of an (A, B) 3-sequence with no clocks, no p-flans, and no p-coflans.

Theorem 10.13. *Let $(A, \vec{X}, B)$ be a 3-sequence that contains no clocks, no p-flans, and no p-coflans and suppose that $|X| \geqslant 3$. Let $T_1, T_2, \ldots, T_n$ be the collection of maximal segments and maximal cosegments in X. Then there is a unique ordering on these sets such that $(A, T_1, T_2, \ldots, T_n, B)$ is a 3-sequence, and every (A, B) 3-sequence can be obtained from this one by the following two steps*:

(i) *arbitrarily reorder the elements of each T_i; and*
(ii) *look among these reorderings at when the last element of T_i is in the jump-set of the first element of T_{i+1}. Such guts-coguts swap pairs are disjoint. Pick some subset of these swap pairs and swap each element with its partner.*

Furthermore, the swaps in (ii) *are subject to the following restrictions*:

(a) *if T_i contains a single element, then this element has empty jump-set*;
(b) *if T_i contains two elements, then at most one of these elements has an element outside of T_i in its jump-set, and the latter such element, which is unique, is in T_{i-1} or T_{i+1}; and*
(c) *if T_i has at least three elements, then T_i has a subset T'_i containing at most two elements such that no element of $T_i - T'_i$ has any elements outside of T_i in its jump-set*; *each element e of T'_i has a unique element $j(e)$ outside of T_i in its jump-set*; *$j(e)$ is in T_{i-1} or T_{i+1}*; *and if T'_i contains e and f, then one of $j(e)$ and $j(f)$ is in T_{i-1} and the other is in T_{i+1}.*

Proof. The fact that there is a unique ordering on the sets $T_1, T_2, \ldots, T_n$ such that $(A, T_1, T_2, \ldots, T_n, B)$ is a 3-sequence was proved in Theorem 10.10. Moreover, it is clear that every sequence obtained from $(A, T_1, T_2, \ldots, T_n, B)$ by (i) and (ii) is a 3-sequence. We now need to show that every 3-sequence $(A, \overrightarrow{X_0}, B)$ can be obtained by the procedure described. Clearly $\overrightarrow{X_0}$ breaks into a collection of maximal consecutive subsequences of elements of the same type. We shall call such subsequences *pieces*.

We should like to show that we can recover a sequence of the form $T_1, T_2, \ldots, T_n$ from $\overrightarrow{X_0}$ by a sequence of swaps of two consecutive elements. More precisely, in $\overrightarrow{X_0}$, consider all pairs (e, f) of elements such that

(a) e and f are consecutive;
(b) e and f are of different types;
(c) e and f are in each other's jump-sets; and
(d) replacing (e, f) by (f, e) will reduce the number of pieces into which $\overrightarrow{X_0}$ is divided.

By Lemma 10.9, each element is in the jump-set of at most one element of the other type. In particular, distinct pairs satisfying (a)–(d) are disjoint. Moreover, by Lemma 10.12, between any two consecutive such pairs, there is at least one element. From these facts, we deduce that the swapping of one pair (e_1, f_1) satisfying (a)–(d) will have no affect on any other pair (e_2, f_2) satisfying (a)–(d). Thus these swaps can be done independently of each other. Let $\overrightarrow{X_1}$ be obtained from $\overrightarrow{X_0}$ by swapping all pairs (e, f) satisfying (a)–(d).

We want to show that $\overrightarrow{X_1}$ has the form $T_1, T_2, \ldots, T_n$. Assume it does not. Let $\overrightarrow{X_1} = (x_1, x_2, \ldots, x_m)$. Take a sequence of the form $(T_1, T_2, \ldots, T_n)$ and order the elements of each T_i to get a sequence $\overrightarrow{X_2} = (x_1, x_2, \ldots, x_{k-1}, y_k, y_{k+1}, \ldots, y_m)$ where $y_k \neq x_k$ and k is as large as possible. Then x_k and y_k are in each other's jump-sets. If they belong to the same T_i, then we can move x_k in $\overrightarrow{X_2}$ so that it immediately follows x_{k-1}, contradicting the choice of $\overrightarrow{X_2}$. Thus x_k and y_k belong to different T_i's. By the definition of $\overrightarrow{X_2}$ and Lemma 10.11, we may assume that $y_k \in T_i$ and $x_k \in T_{i+1}$.

Now, by Lemma 10.9, x_k is in the jump-set of no other element of T_i. It follows that $T_i \subseteq \{x_1, x_2, \ldots, x_{k-1}, y_k\}$. If $k+1 = m$, then, by interchanging x_k and y_k in $\overrightarrow{X_1}$, we reduce the number of pieces into which this sequence is divided, contradicting the choice of $\overrightarrow{X_1}$. Thus $m > k + 1$.

Assume that $y_k \neq x_{k+1}$. Because of $\overrightarrow{X_2}$, we see that y_k jumps both x_k and x_{k+1}. Thus, by Lemma 5.10, x_k and x_{k+1} are of different types. As x_k and y_k are also of different types, we deduce that y_k and x_{k+1} are of the same type. Also y_k jumps x_{k+1}. Therefore, since $y_k \in T_i$, Lemma 10.9 implies that, $x_{k+1} \in T_i$. Then, from considering $\overrightarrow{X_1}$, we deduce that y_k, x_k, and x_{k+1} are mutually jumping; a contradiction. We conclude that $y_k = x_{k+1}$.

If $k = 1$, then $y_k \in T_1$ and, since x_k is the first element of $\overrightarrow{X_1}$, we deduce that x_k jumps every element of T_1. Thus $|T_1| = 1$, so x_{k+2} is of a different type to y_k and hence is of the same type as x_k. Thus, interchanging y_k and x_k in $\overrightarrow{X_1}$ reduces the number of pieces; a contradiction. We may now assume that $k > 1$.

Clearly $x_{k-1} \in T_{i-1}$ or $x_{k-1} \in T_i$. In the first case, x_{k-1} and y_k are of different types, so x_{k-1} and x_k are of the same type. Since they are consecutive in $\overrightarrow{X_1}$, we deduce that $\{x_{k-1}, x_k\} \subseteq T_{i-1}$; a contradiction. Thus $x_{k-1} \in T_i$. We also know that $y_k \in T_i$ and that $x_{k+2} \notin T_i$. It follows, since y_k and x_{k+2} are consecutive in $\overrightarrow{X_1}$, that x_{k+2} and y_k are of different types. Thus x_{k+2} and x_k are of the same type and, interchanging x_k and y_k in $\overrightarrow{X_1}$ reduces its number of pieces; a contradiction.

We conclude that $\overrightarrow{X_1}$ has the form $T_1, T_2, \dots, T_n$. We can now recover $\overrightarrow{X_0}$ by reversing all the swaps that were done to produce $\overrightarrow{X_1}$ from $\overrightarrow{X_0}$, noting that these swaps satisfy (ii). $\square$

The last theorem tells us that once we eliminate clocks, p-flans, and p-coflans, we are left with 3-sequences that are essentially just sequences of maximal segments and maximal cosegments. In order to achieve this result, we needed to view a single coguts element that was not contained in any 2-element cosegment as a maximal cosegment. This may seem unsatisfactory. To remedy this, we now consider sequences that consists of single coguts elements broken by sets of consecutive guts elements. Such a structure arises naturally when we have a p-flan with its elements consecutive. We show in the next result that, when this situation arises, the union of two consecutive coguts elements and the set of guts elements between them is either a cocircuit or a coindependent set. In the case when we have a consecutive sequence of these sets all being cocircuits, the local connectivity function and Lemma 9.7 shows us that we have a structure that can basically be viewed as a p-flan without a tip. The other situation, when the union of two consecutive coguts elements and the set of guts elements between them is coindependent essentially corresponds geometrically to having the segments that immediately precede and succeed this part of the 3-sequence being skew lines.

Corollary 10.14. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence that contains no clocks, no p-flans, and no p-coflans and suppose that $|X| \geqslant 3$. Let $\overrightarrow{X} = (T_1, T_2, \dots, T_n)$ where each T_i is a maximal segment or a maximal cosegment in X. Suppose each maximal cosegment contains a single element. If T_i is a cosegment and $1 \leqslant i \leqslant n-2$, then*

(i) $T_i \cup T_{i+1} \cup T_{i+2}$ *is a cocircuit; or*
(ii) $T_i \cup T_{i+1} \cup T_{i+2}$ *is coindependent.*

Furthermore, if $A' = A \cup T_1 \cup \dots \cup T_{i-1}$ and $B' = T_{i+3} \cup \dots \cup T_n \cup B$, then $\sqcap(A', B')$ is 1 *or* 0 *depending on which of* (i) *and* (ii) *occurs.*

Proof. By Lemma 9.2, $\mathrm{r}(A') + \mathrm{r}(B') = \mathrm{r}(M)$. Thus $\sqcap(A', B') = \mathrm{r}(M) - \mathrm{r}(A' \cup B')$. Since

$$\mathrm{r}\big(A' \cup B'\big) \geqslant \mathrm{r}\big(A' \cup B' \cup T_i\big) - 1 = \mathrm{r}\big(A' \cup B' \cup T_i \cup T_{i+1}\big) - 1 = \mathrm{r}(M) - 1,$$

we deduce that $\sqcap(A', B')$ is 0 or 1 with the first possibility occurring if and only if $T_i \cup T_{i+1} \cup T_{i+2}$ is coindependent. We may now assume that $\sqcap(A', B') = 1$. Then, by Lemma 9.7, $T_i \cup T_{i+1} \cup T_{i+2}$ contains a unique cocircuit and this cocircuit C^* contains $T_i \cup T_{i+2}$. Assume that C^* avoids some element z of T_{i+1}. Then

$$\mathrm{r}\big(A' \cup T_i \cup T_{i+1}\big) = \mathrm{r}\big(A' \cup T_i \cup \{z\}\big) = \mathrm{r}\big(A' \cup T_i\big) = \mathrm{r}\big(A'\big) + 1.$$

If $\mathrm{r}(A' \cup \{z\}) = \mathrm{r}(A')$, then we can move z in $\overrightarrow{X}$ so that it precedes T_i. This contradicts Lemma 10.11. Hence $\mathrm{r}(A' \cup \{z\}) = \mathrm{r}(A') + 1 = \mathrm{r}(A' \cup T_i \cup \{z\})$. Thus $A' \cup \{z\}$ spans T_i, so $\mathrm{cl}(A' \cup B' \cup \{z\}) = E(M)$; a contradiction. We conclude that, when $\sqcap(A', B') = 1$, the set $T_i \cup T_{i+1} \cup T_{i+2}$ is a cocircuit. □

11. An algorithm

Let $(A, \overrightarrow{X}, B)$ be a 3-sequence. The definitions of many of the structures that appear earlier in the paper rely on knowing the jump-sets of each element x of X.

In this section, we describe an algorithm that finds L_x for every x in X. By a symmetric algorithm, one can find each R_x, and hence J_x since $J_x = X - (L_x \cup R_x \cup \{x\})$. We assume that we have a rank oracle, which determines in unit time the rank of any specified set Y. Equivalently, it suffices to have a closure oracle that will determine in unit time whether a specified element is in the closure or the coclosure of a specified set.

The basis of the algorithm to determine L_x is the observation made in Lemma 5.1 that if $(A, \overrightarrow{Y_1}, x, \overrightarrow{Z_1}, B)$ is a 3-sequence in which $|Y_1|$ is minimal, then $Y_1 = L_x$. We shall say that a 3-sequence of the form (A, L_x, x, Z, B) is one in which x *is as far left as possible*. Given a 3-sequence $(A, \overrightarrow{X_1}, B)$ and an element x of X for which we know L_x, we *move* x *as far left as possible* by replacing $(A, \overrightarrow{X_1}, B)$ by the sequence $(A, \overrightarrow{L_x}, x, \overrightarrow{X_1} - L_x, B)$ where the ordering on L_x is that induced by the ordering on $\overrightarrow{X_1}$. It follows by Lemma 5.1 that $A \cup L_x \cup x$ is 3-separating. The fact that $(A, \overrightarrow{L_x}, x, \overrightarrow{X_1} - L_x, B)$ is a 3-sequence is now an immediate consequence of Lemma 4.4.

Left-set Algorithm.

Given a 3-sequence $(A, \overrightarrow{X}, B)$ with $\overrightarrow{X} = (x_1, x_2, \ldots, x_n)$ and the sets $L_{x_1}, L_{x_2}, \ldots, L_{x_{i-1}}$, the algorithm constructs sequence S_i as follows.

Step 0 Let $S^0 = (A, x_1, x_2, \ldots, x_n, B)$ and $L_0 = \emptyset$.

Step j (where $j \geqslant 1$) Start with the sequence S^{j-1} and the set L_{j-1} obtained at step $j-1$.

- (i) Ask whether there is an element of X to the left of x_i in S^{j-1}.
  - If no, then let $S^j = S^{j-1}$ and $L_j = \emptyset$; stop and output $S^j = S_i$.
  - If yes, then go to (ii).
- (ii) Let x_{i-1}^{j-1} be the element of S^{j-1} to the immediate left of x_i in S^{j-1}. Ask whether x_{i-1}^{j-1} jumps x_i.
  - If yes, then interchange x_i and x_{i-1}^{j-1} in S^{j-1} to give S^j, let $L_j = L_{j-1}$, and go to step $j+1$.
  - If no, then move x_{i-1}^{j-1} as far left as possible in S^{j-1} to give S^j, let $L_j = L_{j-1} \cup \{x_{i-1}^{j-1}\}$, and go to (iii).
- (iii) Ask whether x_{i-1}^{j}, the element of S^j to the immediate left of x_i, is in L_j.
  - If yes, then stop and output $S^j = S_i$.
  - If no, then go to step $j+1$.

Theorem 11.1. *The left-set algorithm is well defined and terminates within i steps. If the sequence S_i that it produces is $(A, \overrightarrow{Y_i}, x_i, \overrightarrow{Z_i}, B)$, then S_i is an (A, B) 3-sequence having $Y_i = L_{x_i}$.*

Proof. We begin with the 3-sequence $S^0 = (A, x_1, x_2, \dots, x_n, B)$ and argue by induction on i. If $i = 1$, then the algorithm gives $S_1 = S^0$ in one step and, evidently, $L_{x_1} = \emptyset$, so the result holds for $i = 1$. Assume it holds for $i < m$ and let $i = m \geqslant 2$. The algorithm performs two types of moves on an (A, B) 3-sequence. The first involves interchanging two consecutive jumping elements and, by Lemma 5.4, gives another (A, B) 3-sequence. The second involves moving an element of the left-set of x_i as far left as possible. Initially, the set of elements to the left of x_i is $\{x_1, x_2, \dots, x_{i-1}\}$. Because each element that we want to move as far left as possible is an element of L_{x_i}, it is in $\{x_1, x_2, \dots, x_{i-1}\}$ so, by the induction assumption, we know its left-set and so the move can be performed. Furthermore, as noted prior to the algorithm's statement, this move produces another (A, B) 3-sequence.

To see that the algorithm terminates within i steps, note that it begins with $i - 1$ elements to the left of x_i. Step j takes an element to the left of x_i and either moves it to the right of x_i or adds it to L_j. Since the algorithm will certainly terminate when there are no elements to the left of x_i that do not belong to L_j, it will finish within i steps.

Now suppose that the algorithm terminates at step j outputting S^j. Consider the reason why the algorithm stopped. One possibility is that, in S^j, there is no element of $\{x_1, x_2, \dots, x_n\}$ to the left of x_i. In that case, $Y_i = \emptyset = L_{x_i}$ and the required result holds. The other possible reason for stopping is that x^j_{i-1} is in L_j. Consider when x^j_{i-1} was added to L_j. This occurred because $x^j_{i-1} = x^{j-t}_{i-1}$ for some $t \geqslant 1$ and we found that x^j_{i-1} could not jump x_i, so $x^j_{i-1} \in L_{x_i}$. Then the algorithm moved x^j_{i-1} as far left as possible. After that move, the set of elements to the left of x^j_{i-1} coincides with its left-set, so all such elements are also in L_{x_i}. Subsequent moves in the algorithm considered the elements x^{j-s}_{i-1} with $1 \leqslant s \leqslant t - 1$. These elements either jump x_i if possible, or are moved as far left as possible with the latter occurring because x^{j-s}_{i-1} is in L_{x_i}. If such a move takes an element x_k from the right to the left of x^j_{i-1}, then the move maintains the property that all elements to the left of x^j_{i-1} are in L_{x_i}. Finally, at step j, the elements x^j_{i-1} and x_i are consecutive again. At that stage, the set of elements to the left of x_i coincides with L_{x_i}. Since the algorithm terminates then because $x^j_{i-1} \in L_j$, we deduce, by induction, that the theorem holds. □

Theorem 11.2. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence where $\overrightarrow{X} = (x_1, x_2, \dots, x_n)$. Suppose that, for all i in $\{1, 2, \dots, n\}$, the sets L_{x_i}, J_{x_i}, and R_{x_i} are known along with the type of x_i. Then all segments, cosegments, clocks, crocodiles, p-flans, and p-coflans can be determined.*

Proof. To find all segments, let G be the set of all guts elements in X. If $g \in G$, then $J_g \cap G$ contains every element that is in some segment with g. If $g' \in J_g \cap G$, then the maximal segment containing g and g' is $J_g \cap J_{g'}$. By duality, all cosegments in X can be found.

To find all clocks, take elements y and z in X that are of different types such that $z \in R_y$. If $|J_y \cap J_z| \leqslant 1$, then y and z do not belong to a common clock. If $|J_y \cap J_z| \geqslant 2$, then y and z are both members of a clock, one fan of which is $J_y \cap J_z$, and the other fan of which is $J_u \cap J_v$, where u and v are distinct members of $J_y \cap J_z$.

To find all crocodiles, see if there is a maximal segment Y and a maximal cosegment Z such that, for some $k \geqslant 2$, there are distinct elements $y_1, y_2, \dots, y_k$ of Y and $z_1, z_2, \dots, z_k$ of Z such that $y_i \in J_{z_i}$ for all i.

To find all p-flans, take a guts element y, and see if J_y contains two coguts elements or two guts elements that are not in a common segment. In both cases, $y \cup J_y$ is a maximal p-flan with tip y. Since we know L_z and R_z for all $z \in J_y$, we know the structure of the p-flan. By duality, all p-coflans can be found. □

12. A partial order

Let (A, X, B) be a 3-sequence. For x, y in X, we write $x \leqslant y$ if $x = y$ or $x \in L_y$. It is easily seen that this defines a partial order on X. We call it the *canonical partial order* on X. Recall that a *linear extension* of a partial order P is a linear order Q on the same ground set as P such that if $x \leqslant y$ in P, then $x \leqslant y$ in Q. The following is an immediate consequence of Theorem 5.5.

Corollary 12.1. *Let M be a matroid having an (A, B) 3-sequence and let $X = E(M) - (A \cup B)$. Then $(A, \overrightarrow{X}, B)$ is a 3-sequence if and only if the linear order determined by $\overrightarrow{X}$ is a linear extension of the canonical partial order on X.*

Let (A, X, B) be a 3-sequence and suppose that the sets of guts and coguts elements of X are G_X and C_X, respectively. For each g in G_X and c in C_X, let $M_{gc} = M\backslash(G_X - g)/(C_X - c)$. Jim Geelen (private communication) suggested that the canonical partial order on X provides a compact way of specifying all orderings of X that give rise to 3-sequences and conjectured that this partial order could be determined by considering all of the matroids M_{gc}. The next theorem verifies this conjecture.

Theorem 12.2. *Let (A, X, B) be a 3-sequence. The set $\{M_{gc}\colon g \in G_X, c \in C_X\}$ of matroids determines the set $\{L_x\colon x \in X\}$ of left-sets and hence determines all orderings $\overrightarrow{X}$ of X for which $(A, \overrightarrow{X}, B)$ is a 3-sequence.*

The proof of this will be broken into a sequence of lemmas, some of which are of interest in their own right.

Lemma 12.3. *In a 3-sequence $(A, \overrightarrow{X}, B)$, the sets G_X and C_X of guts elements and coguts elements of X are coindependent and independent, respectively.*

Proof. It suffices to prove that G_X is coindependent. Let $\overrightarrow{X} = (x_1, x_2, \ldots, x_n)$. Assume that G_X contains a cocircuit C^* and let x_i be the first element of this cocircuit in $\overrightarrow{X}$. Then $x_i \in \mathrm{cl}^*(\{x_{i+1}, x_{i+2}, \ldots, x_n\} \cup B)$, so x_i is not a guts element of X; a contradiction. □

Lemma 12.4. *Let (A, X, B) be a 3-sequence, and x and y be distinct elements of X. If $x \in L_y$ and $(A, \overrightarrow{Z_1}, x, \overrightarrow{Z_2}, y, \overrightarrow{Z_3}, B)$ is a 3-sequence in which $|Z_2|$ is minimised, then no element of Z_2 is in $J_x \cup J_y$.*

Proof. Assume that Z_2 contains an element e that is in the jump-set of x. Then there is a 3-sequence $(A, \overrightarrow{W_1}, x, \overrightarrow{W_2}, y, \overrightarrow{W_3}, B)$ with e in W_1. Now it is easily seen that $(A, \overrightarrow{W_1}, \overrightarrow{Z_1} -$

$W_1, x, \overrightarrow{Z_2} - W_1, y, \overrightarrow{Z_3} - W_1, B)$ is a 3-sequence in which the number of elements between x and y is less than $|Z_2|$; a contradiction. □

Lemma 12.5. *Let x and y be guts elements in a 3-sequence (A, X, B). If $x \in L_y$, then there is a coguts element z such that $x \in L_z$ and $z \in L_y$.*

Proof. Take a 3-sequence $(A, \overrightarrow{Z_1}, x, \overrightarrow{Z_2}, y, \overrightarrow{Z_3}, B)$ in which the number of elements occurring between x and y is minimised. Then, by the last lemma, no element between x and y is in J_x or J_y. Since there must certainly be a coguts element between x and y, it follows that, for such an element z, we have $x \in L_z$ and $z \in L_y$. □

Lemma 12.6. *Let (A, X, B) be a 3-sequence, and let g be a guts element and c a coguts element of X. If $A \cup c$ is not 3-separating in M_{gc}, then $g \in L_c$. Conversely, if $g \in L_c$ and there is a 3-sequence $(A, \overrightarrow{X}, B)$ in which g and c are consecutive, then $A \cup c$ is not 3-separating in M_{gc}.*

Proof. Let $(A, \overrightarrow{X}, B)$ be a 3-sequence in which c precedes g and let F^g be the set of coguts elements of $\overrightarrow{X}$ that follow g. Then, by using Lemma 12.3 and the fact that C_X is the set of coguts elements of X, we get

$$\begin{aligned}
&\mathrm{r}_{M_{gc}}(A \cup c) + \mathrm{r}_{M_{gc}}(B \cup g) - \mathrm{r}(M_{gc}) \\
&\quad = \mathrm{r}\big(A \cup C_X\big) + \mathrm{r}\big(B \cup g \cup (C_X - c)\big) - \mathrm{r}(C_X - c) - \mathrm{r}(M) \\
&\quad = \mathrm{r}\big(A \cup \big(C_X - F^g\big)\big) + \big|F^g\big| + \mathrm{r}\big(B \cup F^g \cup g\big) + \big[|C_X - c| - \big|F^g\big|\big] \\
&\qquad - |C_X - c| - \mathrm{r}(M) \\
&\quad = \mathrm{r}\big(A \cup \big(C_X - F^g\big)\big) + \mathrm{r}\big(B \cup F^g \cup g\big) - \mathrm{r}(M) = 2.
\end{aligned}$$

Thus $A \cup c$ is 3-separating in M_{gc}. Hence if $A \cup c$ is not 3-separating, then $g \in L_c$.

Now assume that $g \in L_c$, and that g and c are consecutive in some 3-sequence $(A, \overrightarrow{X}, B)$. Then, arguing as above, we get that $B \cup c$ and hence $A \cup g$ is 3-separating in M_{gc}. Let P^g be the set of coguts elements of X that precede g in $\overrightarrow{X}$. Then, as $A \cup g$ is 3-separating in M_{gc},

$$\begin{aligned}
\mathrm{r}(M) + 2 &= \mathrm{r}\big(A \cup P^g \cup g\big) + \mathrm{r}\big(B \cup \big(C_X - P^g\big)\big) \\
&= \mathrm{r}\big(A \cup P^g\big) + \mathrm{r}\big(B \cup \big(C_X - P^g\big)\big) \\
&= \mathrm{r}\big(A \cup P^g \cup c\big) + \mathrm{r}\big(B \cup \big(C_X - P^g - c\big)\big).
\end{aligned}$$

Now assume that $A \cup c$ is 3-separating in M_{gc}. Then, since g and c are consecutive in $\overrightarrow{X}$ and every element of P^g is a coguts element, it follows that

$$\mathrm{r}(M) + 2 = \mathrm{r}\big(A \cup P^g \cup c\big) + \mathrm{r}\big(B \cup g \cup \big(C_X - P^g - c\big)\big).$$

The last two equations imply that $\mathrm{r}(B \cup g \cup (C_X - P^g - c)) = \mathrm{r}(B \cup (C_X - P^g - c))$. Thus we can interchange g and c in $\overrightarrow{X}$ and retain a 3-sequence, contradicting the fact that $g \in L_c$. □

The matroid M_{gc} need not determine directly whether $g \in L_c$ or $c \in L_g$ since it is possible for one of these to occur and yet for both $A \cup g$ and $A \cup c$ to be 3-separating in M_{gc}. However, by Lemma 12.6, if g and c are consecutive in some 3-sequence, then M_{gc} does tell us whether $g \in L_c$ or $c \in L_g$. Theorem 12.2 will be proved by making repeated use of the last fact. Duality will also be used several times in the next proof without explicit mention.

Proof of Theorem 12.2. If we can determine from M_{xy} that $x \in L_y$, then we write $x \to y$. Now suppose that $g \in L_c$. We argue by induction on the minimum number d of elements separating g from c in a 3-sequence that there are sequences $g_0, g_1, \ldots, g_k$ and $c_0, c_1, \ldots, c_k$ of guts and coguts elements, respectively, such that $g_0 \to c_0 \to g_1 \to \cdots \to g_k \to c_k$ where $g_0 = g$ and $c_k = c$. This is immediate from Lemma 12.6 if $d = 0$. Now assume it true when $d < n$ and let $d = n \geqslant 1$. Take a 3-sequence $(A, \overrightarrow{X}, B)$ in which the number of elements between g and c is n and let the consecutive subsequence of $\overrightarrow{X}$ with ends g and c be $g, C_1, G_1, \ldots, C_m, G_m, c$ where, for each i, the sets C_i and G_i consist entirely of coguts and guts elements, respectively. Then $m \geqslant 1$. By the choice of this sequence, $c_1 \in R_g$ for all c_1 in C_1, so $g \to c_1$ for all such c_1. Moreover, no element of G_1 has all of C_1 in its jump-set otherwise such an element is in J_g, contrary to Lemma 12.4. Hence, for $g_1 \in G_1$, there is a c_1 in C_1 such that $g \to c_1$ and $c_1 \to g_1$. Now $g_1 \notin J_c$, so $g_1 \in L_c$ and we have a 3-sequence in which g_1 and c are separated by fewer than n elements. The required assertion follows immediately by applying the induction assumption.

By the assertion just established, if u and v are of different types, we can certify that $u \in L_v$ by considering all the matroids M_{xy} with $x \in G_X$ and $y \in C_X$. To complete the determination of each L_x, we need to find all the elements in this set that are of the same type as x. If x and y are both guts elements and $y \in L_x$, then, by Lemma 12.5, there is a coguts element z such that $y \in L_z$ and $z \in L_x$. As y and z have different types and z and x have different types, we have already certified that $y \in L_z$ and $z \in L_x$. Combining these facts enables us to certify that $y \in L_x$. □

Acknowledgments

The authors thank Geoff Whittle for suggesting the problem that this paper addresses and for numerous helpful discussions while the research was being done. The second author was supported by the National Security Agency. The third author was supported by a University of Canterbury Research Grant.

References

[1] W.H. Cunningham, J. Edmonds, A combinatorial decomposition theory, Canad. J. Math. 32 (1980) 734–765.

[2] J. Kahn, On the uniqueness of matroid representation over $GF(4)$, Bull. London Math. Soc. 20 (1988) 5–10.
[3] J. Geelen, A.M.H. Gerards, G. Whittle, Inequivalent representations of matroids I: An overview, in preparation.
[4] J.G. Oxley, Matroid Theory, Oxford University Press, 1992.
[5] J. Oxley, C. Semple, D. Vertigan, Generalized $\Delta - Y$ exchange and k-regular matroids, J. Combin. Theory Ser. B 79 (2000) 1–65.
[6] J. Oxley, C. Semple, G. Whittle, The structure of the 3-separations of 3-connected matroids, J. Combin. Theory Ser. B 92 (2004) 257–293.
[7] J. Oxley, D. Vertigan, G. Whittle, On inequivalent representations of matroids, J. Combin. Theory Ser. B 67 (1996) 325–343.
[8] G.-C. Rota, Combinatorial theory, old and new, in: Proc. Internat. Cong. Math., Nice, 1970, Gauthier–Villars, Paris, 1971, pp. 229–233.
[9] P.D. Seymour, Decomposition of regular matroids, J. Combin. Theory Ser. B 28 (1980) 305–359.
[10] W.T. Tutte, Connectivity in matroids, Canad. J. Math. 18 (1966) 1301–1324.