12-6-2005

On pancyclic representable matroids

Brian Beavers
Louisiana State University

James Oxley
Louisiana State University

Available online at www.sciencedirect.com
SCIENCE DIRECT®
Discrete Mathematics 305 (2005) 337–343
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

On pancyclic representable matroids

Brian Beavers, James Oxley
Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana, 70803-4918

Received 16 September 2004; received in revised form 7 October 2005; accepted 11 October 2005
Available online 18 November 2005

Abstract

Bondy proved that an n-vertex simple Hamiltonian graph with at least $n^2/4$ edges has cycles of every length unless it is isomorphic to $K_{n/2,n/2}$. This paper considers finding circuits of every size in $GF(q)$-representable matroids with large numbers of elements. A consequence of the main result is that a rank-r simple binary matroid with at least 2^{r-1} elements either has circuits of all sizes or is isomorphic to $AG(r-1, 2)$.

Keywords: Pancyclic; Matroid; Circuit; $GF(q)$-representable

1. Introduction

A simple graph G with vertex set $V(G)$ is *pancyclic* if it contains cycles of all lengths l, for $3 \leqslant l \leqslant |V(G)|$. Bondy [1] proved the following:

Theorem 1.1. *Let G be a simple Hamiltonian graph with $|V(G)| = n$. If $|E(G)| \geqslant n^2/4$, then G is pancyclic unless G is isomorphic to $K_{n/2,n/2}$.*

The exceptional graph $K_{n/2,n/2}$ is special in that it has many edges and many even cycles, but no odd cycles. A similar role is played in binary matroids by affine geometries, which also have many elements and many even circuits, but no odd circuits. It is natural to ask whether Bondy's theorem has an analog for binary or even for $GF(q)$-representable matroids. Toward this end, we define a simple rank-r matroid M to be *Hamiltonian* if it has

E-mail addresses: beavers@math.lsu.edu (B. Beavers), oxley@math.lsu.edu (J. Oxley).

0012-365X/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2005.10.008

a circuit of size $r+1$ and to be *pancyclic* if it has circuits of all sizes s, for $3 \leqslant s \leqslant r+1$. We will prove the following:

Theorem 1.2. *Let M be a simple rank-r binary matroid. If $|E(M)| \geqslant 2^{r-1}$, then M is pancyclic unless M is isomorphic to the binary affine geometry $AG(r-1, 2)$.*

Note that if we add the condition that M is Hamiltonian, then M must be pancyclic unless it is an affine geometry of even rank. The main result of the paper is a theorem on the existence of circuits of every size in matroids with no $U_{2,q+2}$-minor. This will imply the above result for binary matroids and the following result for $GF(q)$-representable matroids.

Theorem 1.3. *Let M be a simple rank-r $GF(q)$-representable matroid.*

(i) *If $|E(M)| \geqslant (q^{r-1}-1)/(q-1)+q+1$, then, for all s in $\{3, 4, \ldots, r+1\}$ and all e in $E(M)$, there is an s-circuit containing e.*
(ii) *If $|E(M)| \geqslant (q^{r-1}-1)/(q-1)+2$, then, for all s in $\{3, 4, \ldots, r+1\}$ and all but at most one e in $E(M)$, there is an s-circuit containing e.*
(iii) *If $|E(M)| = (q^{r-1}-1)/(q-1)+1$, then M is pancyclic unless M is isomorphic to one of the following matroids*:

  (a) $U_{3,q+2}$ *for q a power of* 2,
  (b) $PG(r-2, q) \oplus U_{1,1}$ *if $r \geqslant 3$, or*
  (c) $AG(r-1, 2)$.

Matroid terminology used here follows Oxley [8] with the following exceptions: the simple matroid associated with a matroid M is denoted by $\mathrm{si}(M)$; and if x and y are elements of a simple matroid M, then xy denotes the line of M spanned by $\{x, y\}$.

2. Main theorem

The next theorem is the main result of the paper. Note that $S(3, 6, 22)$ is the rank-4 paving matroid of the unique Steiner system $S(3, 6, 22)$. The blocks of the Steiner system are the hyperplanes of the matroid.

Theorem 2.1. *Let M be a simple rank-r matroid with no $U_{2,q+2}$-minor, for some integer q greater than one.*

(i) *If $|E(M)| \geqslant (q^{r-1}-1)/(q-1)+q+1$, then, for all s in $\{3, 4, \ldots, r+1\}$ and all e in $E(M)$, there is an s-circuit containing e.*
(ii) *If $|E(M)| \geqslant (q^{r-1}-1)/(q-1)+2$, then, for all s in $\{3, 4, \ldots, r+1\}$ and all but at most one e in $E(M)$, there is an s-circuit containing e.*
(iii) *If $|E(M)| = (q^{r-1}-1)/(q-1)+1$, then M is pancyclic unless M is isomorphic to one of the following matroids*:

  (a) $U_{3,q+2}$,
  (b) $U_{2,q+1} \oplus U_{1,1}$,

(c) $N_q \oplus U_{1,1}$, *where* N_q *is projective plane of order* q,
(d) $PG(r-2,q) \oplus U_{1,1}$ *if* $r > 4$,
(e) $AG(r-1,2)$, *or*
(f) $S(3,6,22)$.

The proof of the theorem uses the following results. The first and second are due to Kung [5] and Murty [7], respectively. The third is a straightforward consequence of the second, while the fourth and fifth use standard techniques. The sixth follows from results of Doyen and Hubaut [3] (see Welsh [9, pp. 214–215]) and Lam et al. [6].

Theorem 2.2. *Let q be an integer exceeding one. If M is a rank-r matroid with no $U_{2,q+2}$-minor, then $|E(M)| \leqslant (q^r - 1)/(q-1)$. For $r \geqslant 4$, equality holds in this bound if and only if $M \cong PG(r-1,q)$. When $r = 3$, equality holds if and only if M is a projective plane of order q.*

Lemma 2.3. *Let C_1 and C_2 be circuits of a matroid M with $C_2 = \{e,f,g\}$ and $C_1 \cap C_2 = \{g\}$. If $(C_1 - g) \cup e$ is independent in M, then $(C_1 \cup C_2) - g$ is a circuit.*

Lemma 2.4. *Let $\{e,f,g\}$ be a circuit of M, and let C_g be a circuit of* $\mathrm{si}(M/e)$ *containing g. Then either $C_g \cup e$ or $(C_g - g) \cup \{e,f\}$ is a circuit of M.*

Proof. As C_g is a circuit of M/e, either $C_g \cup e$ or C_g is a circuit of M. We may assume the latter. Noting that $r_M((C_g - g) \cup e) = r_{M/e}(C_g - g) + r_M(e) = |C_g - g| + 1 = |(C_g - g) \cup e|$, we have that $(C_g - g) \cup e$ is independent. By Lemma 2.3, $(C_g - g) \cup \{e,f\} \in \mathscr{C}(M)$. □

Lemma 2.5. *Let M be a simple rank-r matroid having no $U_{2,q+2}$-minor where $q \geqslant 2$. If $|E(M)| \geqslant (q^{r-1} - 1)/(q-1) + a$ where $a \geqslant 1$ and $e \in E(M)$, then $|E(\mathrm{si}(M/e))| \geqslant (q^{r-2} - 1)/(q-1) + \lceil \frac{a}{q} \rceil$.*

Proof. As every line of M through e has at most q other points, $|E(\mathrm{si}(M/e))| \geqslant \lceil \frac{1}{q}((q^{r-1} - 1)/(q-1) + a - 1) \rceil = \left\lceil \frac{1}{q}((q^{r-2} + q^{r-3} + \cdots + q + 1) + a - 1) \right\rceil = \lceil q^{r-3} + q^{r-4} + \cdots + q + 1 + \frac{a}{q} \rceil = (q^{r-2} - 1)/(q-1) + \lceil \frac{a}{q} \rceil$. □

Lemma 2.6. *Let M be a simple rank-r matroid having no $U_{2,q+2}$-minor where $q \geqslant 2$. Suppose $|E(M)| = (q^{r-1} - 1)/(q-1) + a$ and $|E(\mathrm{si}(M/e))| = (q^{r-2} - 1)/(q-1) + b$. If M/e has exactly c elements in trivial parallel classes, then $c \leqslant b + (b-a)/(q-1)$. Moreover, if equality holds, then each nontrivial parallel class of M/e has exactly q elements.*

Proof. The following inequalities are equivalent:

$$|E(M)| \leqslant q(|E(\mathrm{si}(M/e))| - c) + c + 1,$$

$$\frac{q^{r-1} - 1}{q-1} + a \leqslant q\left(\frac{q^{r-2} - 1}{q-1} + b - c\right) + c + 1,$$

$$q^{r-2}+q^{r-3}+\cdots+q+1+a\leqslant(q^{r-2}+q^{r-3}+\cdots+q)+q(b-c)+c+1,$$

$$a\leqslant qb-qc+c,$$

$$(q-1)c\leqslant qb-a,$$

$$(q-1)c\leqslant(q-1)b+b-a,$$

$$c\leqslant b+\frac{b-a}{q-1}.$$

If equality holds in the last line, then equality must hold in the first line, and so every nontrivial parallel class of M/e has exactly q members. □

Lemma 2.7. *Let M be a simple matroid with rank $r\geqslant 4$. If M has no triangles and if every single-element contraction of M is a projective space, then $M\cong S(3,6,22)$.*

Proof. By Doyen and Hubaut [3], if $r>4$, then $M\cong PG(r-1,q)$ or $M\cong AG(r-1,q)$; and if $r=4$, then (i) $M\cong PG(3,q)$, (ii) $M\cong AG(3,q)$, (iii) every single-element contraction of M is a projective plane of order 4, or (iv) every single-element contraction of M is a projective plane of order 10. Because M has no triangles, M/e is simple for all $e\in E(M)$. Thus M cannot be isomorphic to $PG(r-1,q)$ or $AG(r-1,q)$ for any $r\geqslant 4$. By Lam et al. [6], there are no projective planes of order 10, and, by Doyen and Hubaut [3], $S(3,6,22)$ is the unique matroid all of whose single-element contractions are projective planes of order 4. □

Proof of Theorem 2.1. We argue by induction on r to prove all three parts simultaneously. The result is easily checked if $r=2$. Assume $r=3$. If $|E(M)|=q+2$, then either $M\cong U_{3,q+2}$, or M has a nontrivial line and at least one other point not on this line. If there is exactly one point not on the line, then $M\cong U_{2,q+1}\oplus U_{1,1}$. If there are at least two points not on the line, then there is a 4-circuit containing these two points. Thus M has a 3-circuit and a 4-circuit and (iii) holds.

Now let $|E(M)|\geqslant q+3$. Suppose $e\in E(M)$ and $|\mathrm{si}(M/e)|>2$. Then $3\leqslant|\mathrm{si}(M/e)|\leqslant q+1$ and there is at least one 2-circuit $\{f,g\}$ in M/e. As $\mathrm{si}(M/e)$ is a nontrivial line, it has a triangle C through g. Since $\{e,f,g\}$ is a triangle of M, Lemma 2.4 implies that $C\cup e$ or $(C-g)\cup\{e,f\}$ is a 4-circuit of M containing e. Thus if $|\mathrm{si}(M/e)|>2$, then e is in both a 3-circuit and a 4-circuit of M. We deduce that (i) and (ii) hold unless M has an element e such that $|\mathrm{si}(M/e)|=2$. Consider the exceptional case. Then $|E(M)|<2q+2$ and M consists of two lines meeting in e. Thus M has 3- and 4-circuits through every point except e. Hence, in the exceptional case, (ii) holds and (i) holds vacuously since $|E(M)|<(q^{3-1}-1)/(q-1)+q+1$. We conclude that the theorem holds when $r=3$.

Assume the theorem holds for $r<k$ and let $r=k>3$. First, we consider (i). Suppose that $|E(M)|\geqslant(q^{r-1}-1)/(q-1)+q+1$ and let $e\in E(M)$. Then M has at least two nontrivial lines through e since $|E(\mathrm{si}(M/e))|\leqslant(q^{r-1}-1)/(q-1)$. By Lemma 2.5, $|E(\mathrm{si}(M/e))|\geqslant(q^{r-2}-1)/(q-1)+2$. Then, by the induction hypothesis, every element but at most one of $\mathrm{si}(M/e)$ is in circuits of all sizes from 3 to k. By choosing a triangle containing e and an element of $\mathrm{si}(M/e)$ that is in circuits of all sizes from 3 to k, we apply

Lemma 2.4 to get circuits in M of all sizes from 4 to $k+1$ through e. Since e is also in a triangle, (i) holds.

Next we consider (ii). Assume $|E(M)|=(q^{r-1}-1)/(q-1)+a$ with $2\leqslant a\leqslant q$ and let $e\in E(M)$. Then, as $a\geqslant 2$, it follows that e is in a triangle of M. Moreover, $|E(\mathrm{si}(M/e))|\geqslant(q^{r-2}-1)/(q-1)+1$ by Lemma 2.5. If $|E(\mathrm{si}(M/e))|\geqslant(q^{r-2}-1)/(q-1)+q+1$, then every element of $\mathrm{si}(M/e)$ is in circuits of every size from 3 to k. Choose an element g of $\mathrm{si}(M/e)$ that is in a triangle of M with e. By Lemma 2.4, the triangle containing both e and g and the circuits of every size from 3 to k containing g yield circuits of M containing e of all sizes from 3 to $k+1$.

Suppose that c elements of M/e are in trivial parallel classes. Assume that $|E(\mathrm{si}(M/e))|=(q^{r-2}-1)/(q-1)+b$ with $2\leqslant b\leqslant q$. Then, by Lemma 2.6, $c\leqslant b+(b-a)/(q-1)$. Since $b\geqslant 2$ and $a\leqslant q$, we assert that $c\leqslant q$. To see this, suppose that $c\geqslant q+1$. Then $b+(b-a)/(q-1)\geqslant q+1$, and so $(q-1)b+b-a\geqslant(q+1)(q-1)$. Thus $qb-a\geqslant q^2-1$, and hence we obtain the contradiction that $-1\geqslant 1-a\geqslant q^2-qb=q(q-b)\geqslant 0$. We conclude that $c\leqslant q$. Let U be the set of elements of M/e that are in trivial parallel classes. By the induction hypothesis, all but at most one element, say p, of $\mathrm{si}(M/e)$ is in circuits of all sizes from 3 to k in $\mathrm{si}(M/e)$. Assume p is not in a trivial parallel class of M/e. Adjoin to U all points on the line ep of M. Thus U has at most $2q+1$ elements. As $|E(M)|=(q^{r-1}-1)/(q-1)+a$ and $r\geqslant 4$, it follows that $|E(M)|\geqslant q^2+q+1+a$. Thus $|E(M)-U|\geqslant(q^2+q+1+a)-(2q+1)=q^2-q+a>0$. Hence M has at least q^2-q+a elements that are in nontrivial parallel classes of M/e and avoid U. Take g to be one such element that is also in $\mathrm{si}(M/e)$. As g is not p, there are circuits of all sizes from 3 to k containing g in $\mathrm{si}(M/e)$, and $\{e, g\}$ is contained in a triangle of M. Thus, by Lemma 2.4, M has circuits of all sizes from 3 to $k+1$ containing e.

Now assume $|E(\mathrm{si}(M/e))|=(q^{r-2}-1)/(q-1)+1$. By Lemma 2.6, $c\leqslant 1+(1-a)/(q-1)<1$. Then every element of M/e is in a nontrivial parallel class. Moreover, by the induction hypothesis, $\mathrm{si}(M/e)$ has circuits of all sizes from 3 to k unless $\mathrm{si}(M/e)$ is one of the exceptions (a)–(f). By Lemma 2.4, we deduce that M has circuits containing e of all sizes from 3 to $k+1$ unless $\mathrm{si}(M/e)$ is one of (a)–(f). Now part (ii) holds unless there are at least two elements f and g of M such that each of $\mathrm{si}(M/f)$ and $\mathrm{si}(M/g)$ is one of (a)–(f). We may assume that $g\in\mathrm{si}(M/f)$. Because every element of M/g is in a nontrivial parallel class, g is in a triangle with every other element of $\mathrm{si}(M/f)$. This is not possible in any of (a)–(f), so (ii) holds.

Finally, we consider (iii). Assume that $|E(M)|=(q^{r-1}-1)/(q-1)+1$. Suppose first that M has no triangles. Then, for all e in $E(M)$, we have $|E(\mathrm{si}(M/e))|=(q^{r-1}-1)/(q-1)$, and so, by Theorem 2.2, every single-element contraction of M is a projective space. By Lemma 2.7, $M\cong S(3,6,22)$.

We may now assume that M has a triangle and that this triangle contains e. If $|E(\mathrm{si}(M/e))|\geqslant(q^{r-2}-1)/(q-1)+q+1$, then every element of $\mathrm{si}(M/e)$ is in circuits of all sizes from 3 to k. So M has circuits of all sizes from 3 to $k+1$ by Lemma 2.4.

If $|E(\mathrm{si}(M/e))|=(q^{r-2}-1)/(q-1)+b$ for $2\leqslant b\leqslant q$, then all but at most one element, say p, of $\mathrm{si}(M/e)$ is in circuits of all sizes from 3 to k. By Lemma 2.6, $c\leqslant b+(b-1)/(q-1)\leqslant b+1\leqslant q+1$. Let U be the set consisting of those elements of M/e that are in trivial parallel classes. Assume p is in a nontrivial parallel class and adjoin to U all points on the line ep. Thus $|U|\leqslant 2q+2$. Since $r\geqslant 4$, we have $|E(M)|\geqslant q^2+q+2$. Hence $|E(M)-U|\geqslant(q^2+q+2)-(2q+2)=q^2-q>0$. So we may choose g from $E(M)-U$

in si(M/e) such that $\{e, g\}$ is in a triangle of M and e, g, and p are not collinear. Then, since si(M/e) has circuits of all sizes from 3 to k containing g, Lemma 2.4 implies that M has circuits of all sizes from 3 to $k+1$ containing e.

If $|E(\text{si}(M/e))| = (q^{r-2}-1)/(q-1)+1$, then by Lemma 2.6, $c \leqslant 1+(1-1)/(q-1)=1$, that is, at most one element of M/e is in a trivial parallel class. Hence M has a 3-circuit. Moreover, we get a 4-circuit in M by taking two elements from each of two nontrivial parallel classes of M/e. If si(M/e) has circuits of all sizes from 3 to k, then M has circuits of all sizes from 3 to $k+1$ by Lemma 2.4. Thus we may assume that si(M/e) is one of the exceptions (a)–(f), and next we consider each of these, noting that we have already shown that M has both 3- and 4-circuits. Suppose first that si(M/e) is $U_{3,q+2}$. Then we use Lemma 2.4 to get a circuit of size 5. Suppose next that si(M/e) is $S(3, 6, 22)$. Then M has 5- and 6-circuits by Lemma 2.4. Next suppose that si(M/e) is the direct sum of a coloop g and a projective space of rank at least two. Either g is the unique element of M/e in a trivial parallel class or not. In the first case, g is also a coloop of M. By Lemma 2.6, each nontrivial parallel class of M/e has q elements. Thus $|E(M\backslash g)| = (q^{r-1}-1)/(q-1)$ and, by Theorem 2.2, $M\backslash g$ is a projective space, and M is (c) or (d). Now suppose g is in a nontrivial parallel class. We now have that M is the parallel connection, with basepoint e, of the line eg and matroid of rank $r-1$, and that the line eg has at least one other point f. We may use circuits of sizes from 3 to $r-1$ of si(M/e) to obtain circuits of M of sizes 4 to r that contain e and avoid all other points on the line eg. Then, we take an r-circuit C of M containing e and apply Lemma 2.3 to get that $(C-e)\cup\{f, g\}$ is an $(r+1)$-circuit of M.

Finally, we consider the case when si(M/e) is a binary affine geometry. Then $q=2$ and so M is binary, as M has no $U_{2,4}$-minor. In M, there is exactly one trivial line through e. We can obtain a binary representation for a single-element extension M' of M as follows. If $AG(r-2, 2)$ is represented by the matrix A, then $\begin{bmatrix} 1 & \mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & A & A \end{bmatrix}$ represents M', where the first column of this matrix corresponds to e, and $\mathbf{0}$ and $\mathbf{1}$ are vectors of all zeros and all ones, respectively, of appropriate size. Since A can be chosen so that its columns are all vectors of $V(r-1, 2)$ with first coordinate 1, it follows that $M'\backslash e \cong AG(r-1, 2)$. Thus M' is the unique simple rank-r binary single-element extension of $AG(r-1, 2)$ and hence M is pancyclic. □

The next two lemmas were proved by Kantor [4] (see Welsh [9, p. 215]) and Bose [2] (see Oxley [8, p. 206]), respectively.

Lemma 2.8. *The matroid $S(3, 6, 22)$ is not representable over any field.*

Lemma 2.9. *The matroid $U_{3,q+2}$ is representable over $GF(q)$ if and only if q is even.*

On combining these lemmas with Theorem 2.1, we immediately obtain Theorems 1.2 and 1.3.

Acknowledgements

The second author's work was partially supported by grants from the National Security Agency.

References

[1] J.A. Bondy, Pancyclic graphs I, J. Combin. Theory Ser. B 11 (1971) 80–84.
[2] R.C. Bose, Mathematical theory of the symmetrical factorial design, Sankyhā 8 (1947) 107–166.
[3] J. Doyen, X. Hubaut, Finitely regular locally projective spaces, Math. Z. 119 (1971) 83–88.
[4] W.M. Kantor, Dimension and embedding theorems for geometric lattices, J. Combin. Theory Ser. A 17 (1974) 173–195.
[5] J.P.S. Kung, Extremal matroid theory, in: N. Robertson, P. Seymour (Eds.), Graph Structure Theory, Contemp. Math. 147 (1993) 21–61.
[6] C.W.H. Lam, L. Thiel, S. Swiercz, The non-existence of finite projective planes of order 10, Canad. J. Math. 41 (1989) 1117–1123.
[7] U.S.R. Murty, Matroids with Sylvester property, Aequationes Math. 4 (1970) 44–50.
[8] J.G. Oxley, Matroid Theory, Oxford University Press, Oxford, 1992.
[9] D.J.A. Welsh, Matroid Theory, Academic Press, London, 1976.