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Guided modes and resonant transmission in periodic structures

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GUIDED MODES AND RESONANT TRANSMISSION
IN PERIODIC STRUCTURES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

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by

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Table of Contents

Acknowledgments	ii
List of Figures	v
Abstract	viii
Chapter 1: Introduction	1
1.1 Periodic Slabs	3
1.2 Periodic Pillars	6
1.3 Summary of Dissertation	9
Chapter 2: Wave Scattering and Guided Modes in Periodic Structures	11
2.1 The Wave Equation and Helmholtz Equation	11
2.2 Periodic Structures and Pseudo-periodic Solutions	12
2.3 Plane-wave Scattering by Periodic Slabs	14
2.3.1 Radiation Condition	14
2.3.2 Plane-wave Scattering Problems and Guided Modes	15
2.3.3 Existence of Solutions of Scattering Problems	19
2.4 Guided Modes	22
2.5 Existence and Nonexistence of Guided Modes	25
2.5.1 Existence	25
2.5.2 Nonexistence	27
2.6 Transmission Anomalies	27
Chapter 3: Total Transmission Resonance in Periodic Slabs	30
3.1 Complex Extension	30
3.2 Scattering and Guided Modes with Complex Extension	33
3.3 Analyticity	36
3.4 Main Theorem: Total Resonant Transmission and Reflection for Symmetric Slabs	42
3.4.1 The Reduced Scattering Matrix	42
3.4.2 Resonant Transmission	47
3.5 Nongeneric Resonant Transmission	51
3.5.1 Total Background Reflection and Transmission	51
3.5.2 Multiple Anomalies	57
3.6 Transmission Graphs	65
Chapter 4: Guided Modes in Periodic Pillars	71
4.1 Bessel Functions	71
4.2 Media Structure and Scattering Problem	74

4.2.1	Pillar Structure and Radiation Condition	74
4.2.2	Scattering Problems	76
4.3	Guided Modes	79
4.4	Existence and Nonexistence of Guided Modes	81
4.4.1	Existence	81
4.4.2	Nonexistence	85
Chapter 5: Open Problems and Future Work		92
References		94
Vita		97

List of Figures

1.1	Numerical computation of the percentage of energy $ T ^2$ transmitted across a penetrable waveguide of period 2π as a function of the frequency of the incident plane wave. Here, the wavenumber in the x -direction (Fig. 1.2) is $\kappa = 0.02$ and one period consists of a single circle of radius $\pi/2$ with $\epsilon = 10$ and an ambient medium with $\epsilon = 1$; $\mu = 1$ throughout. The structure supports guided modes at $(\kappa, \omega) = (0, 0.5039\dots)$ and $(\kappa, \omega) = (0, 0.7452\dots)$, both contained within the region \mathcal{D} of one propagating diffractive order (Fig 3.1). Theorem 26 guarantees that the transmission attains minimal and maximal values of 0% and 100% at each of the sharp anomalies near the guided-mode frequencies. This is Figure (6) in [23].	3
1.2	An example of a two-dimensional periodic slab. One period truncated to the rectangle $[-\pi, \pi] \times [-L, L]$ is denoted by Ω	5
1.3	A pillar periodic in z and finite in x and y	7
2.1	Slab structure periodic in x and finite in z ; u^{inc} is transmitted to u^{trans} and reflected to u^{refl}	13
3.1	The diamond \mathcal{D} of one propagating diffractive order within the first Brillouin zone.	31
3.2	$ T ^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. The generic conditions $\frac{\partial \ell}{\partial \omega}, \frac{\partial a}{\partial \omega}, \frac{\partial b}{\partial \omega} \neq 0$ are satisfied at the bound-state pair (κ_0, ω_0) . In (3.15,3.16), $\ell_1 = 0$ so that there is no linear detuning of the anomaly with $\tilde{\kappa}$. Left: $0 < t_2 = 1 < r_2 = 2$ so that the peak is to the left of the dip and both are to the left of ω_0 . Right: $r_2 = -2 < 0 < t_2 = 1$. In both graphs, $r_0 = 0.6, t_0 = 0.8$. The transmission is symmetric in $\tilde{\kappa}$, and the curve without an anomaly is the transmission graph for $\tilde{\kappa} = 0$	66
3.3	$ T ^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.003, \pm 0.006, \pm 0.009$. The generic conditions $\frac{\partial \ell}{\partial \omega}, \frac{\partial a}{\partial \omega}, \frac{\partial b}{\partial \omega} \neq 0$ are satisfied at the bound-state pair (κ_0, ω_0) . In (3.15,3.16), $\ell_1 = 0.9 \neq 0$, so the anomaly is detuned from $\omega = \omega_0$ ($\tilde{\omega} = 0$) in a linear manner in $\tilde{\kappa}$. The coefficients $r_2 = 2$ and $t_2 = 1$ of $\tilde{\kappa}^2$ are distinct real numbers, and $(r_0, t_0) = (0.6, 0.8)$	67

- 3.4 $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Left: Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} = 0$, and $\frac{\partial b}{\partial \omega} \neq 0$ at (κ_0, ω_0) . In (3.21), $0 < r_1^{(1)} = 0.2 < t_1 = \ell_1 = 2 < r_1^{(2)} = 4$, $(r_2^{(1)}, r_2^{(2)}, t_2) = (7, 7, 0.1)$, and $r_0 = 0.6$. Right: $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} \neq 0$, and $\frac{\partial b}{\partial \omega} = 0$ at (κ_0, ω_0) and $0 < t_1^{(1)} < r_1 = \ell_1 < t_1^{(2)}$ 67
- 3.5 $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} = 0$, and $\frac{\partial b}{\partial \omega} \neq 0$ at (κ_0, ω_0) . In (3.21), $0 < r_1^{(1)} = 0.6 < r_1^{(2)} = 1.2 < \ell_1 = 3$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, 1, 1, 0.2)$ 68
- 3.6 $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Left: Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} = 0$, and $\frac{\partial b}{\partial \omega} \neq 0$ at (κ_0, ω_0) . In (3.21), $r_1^{(1)} = -0.04 < t_1 = \ell_1 = 0 < r_1^{(2)} = 0.06$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, -1, 1, 1)$. Right: Magnification of the graphs for $\tilde{\kappa} = \pm 0.01$, bringing into view the frequencies of total transmission. 68
- 3.7 $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} \neq 0$, and $\frac{\partial b}{\partial \omega} = 0$ at (κ_0, ω_0) and $0 < t_1^{(1)} < r_1 = \ell_1 < t_1^{(2)}$. In (3.21), $t_1^{(1)} = -0.04 < r_1 = \ell_1 = 0 < t_1^{(2)} = 0.06$; $(t_0, t_2^{(1)}, t_2^{(2)}, t_2) = (0.6, -1, 1, 1)$ 68
- 3.8 $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} = 0$, and $\frac{\partial b}{\partial \omega} \neq 0$ at (κ_0, ω_0) . In (3.21), $t_1 = \ell_1 = 0 < r_1^{(1)} = 0.6 < r_1^{(2)} = 0.8$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, -4, 5, 6)$ 69
- 3.9 $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. $\frac{\partial \ell}{\partial \omega} \frac{\partial a}{\partial \omega} \neq 0$, $\frac{\partial b}{\partial \omega} = 0$ at (κ_0, ω_0) . $(r_1^{(1)}, r_1^{(2)}, \ell_1) = (2i, -2i, 0)$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, 2i, 4i, 3)$ 69
- 3.10 $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. $\frac{\partial \ell}{\partial \omega} \frac{\partial a}{\partial \omega} \neq 0$, $\frac{\partial b}{\partial \omega} = 0$ at (κ_0, ω_0) . $(r_1^{(1)}, r_1^{(2)}, \ell_1) = (0.5i, -0.5i, 2)$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, i, i, 1)$ 70
- 3.11 Left: $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.003, \pm 0.006, \pm 0.009$. The partial derivatives of ℓ , a , and b all vanish at (κ_0, ω_0) , whereas their second derivatives are nonzero. In (3.25), $(\ell_1^{(1)}, \ell_1^{(2)}) = (0.7, 0.8)$, $(r_0, t_0) = (0.6, 0.8)$, $r_2^{(1)} = 2 < t_2^{(1)} = 8$ and $t_2^{(2)} = 4 < r_2^{(2)} = 5$. Right: $\tilde{\kappa} = 0.003$ 70

3.12 Left: $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.003, \pm 0.006, \pm 0.009$. The partial derivatives of ℓ , a , and b all vanish at (κ_0, ω_0) , whereas their second derivatives are nonzero. In (3.25), $(\ell_1^{(1)}, \ell_1^{(2)}) = (0.7, 0.8)$, $(r_0, t_0) = (0.6, 0.8)$, $r_2^{(1)} = 2 < t_2^{(1)} = 8$ and $r_2^{(2)} = 4 < t_2^{(2)} = 6$. Right: $\tilde{\kappa} = 0.003$ 70

Abstract

We analyze resonant scattering phenomena of scalar fields in periodic slab and pillar structures that are related to the interaction between guided modes of the structure and plane waves emanating from the exterior. The mechanism for the resonance is the nonrobust nature of the guided modes with respect to perturbations of the wavenumber, which reflects the fact that the frequency of the mode is embedded in the continuous spectrum of the pseudo-periodic Helmholtz equation. We extend previous complex perturbation analysis of transmission anomalies to structures whose coefficients are only required to be measurable and bounded from above and below, and we establish sufficient conditions involving structural symmetry that guarantee that the transmission coefficient reach 0% and 100% at nearby frequencies close to those of the guided modes. Our analysis demonstrates a few more patterns of anomalies in nongeneric cases, including anomalies of two peaks and one dip on the transmission graph with total background transmission, anomalies of one peak and two dips with total background reflection, and multiple anomalies, and we also prove sufficient conditions for these transmission coefficients to reach 0% and 100%. For pillar structures, we establish a fundamental framework using Bessel functions for the analysis of guided modes, and prove the existence and nonexistence in structures in analogy to results for slabs. We provide a new existence result of nontrivial embedded guided modes, which are stable with respect to the wavenumber and nonrobust under perturbations of the structural geometry, in periodic pillars with smaller periodic cells.

Chapter 1

Introduction

Guided modes in periodic structures are very important in composite material designs. They are electromagnetic or acoustic waves that are trapped within certain periodic materials, and this special feature makes many applications possible such as photonic crystal waveguides and light filters discussed in [11].

A related phenomenon is that of transmission anomalies. For a periodic slab that is bounded in one direction, in some very special settings, the ratio of energy transmitted through the slab can vary dramatically upon a small perturbation of the structural geometry, the frequency ω , or the wavenumber κ . Transmission anomalies are studied in various literature, and applications are suggested such as polarization control, filtering, switching, surface plasmon resonance sensing, and surface-enhanced scattering [8][7]. It is clear that the characterization and prediction of transmission anomalies will continue to contribute to the manufacturing of many devices based on periodic structures.

In this work, we try to understand guided modes and transmission anomalies mathematically. The resonant transmission anomalies can be explained by the dissolving of the frequency of a guided mode into the continuous spectrum, by Fabry-Perot resonance, or by Wood's anomalies near cutoff frequencies of the associated Bloch diffraction, and different models have been developed to describe them [16, 3, 15, 17]. We are concerned with settings of transmission anomalies for which material parameters and the frequency and wavenumber pair (κ, ω) are close to those of guided modes. The transmission anomaly can be understood as caused by the interaction between the incoming waves and the nonrobust embedded guided

mode. More particularly, we consider the resonant transmission appearing when the wavenumber κ is perturbed from a nonrobust guided mode wavenumber κ_0 .

We consider slabs that are finite in one direction and periodic in one or two other directions, as well as pillars periodic in one direction and finite in the other directions perpendicular to them. In lossless isotropic structures, a time-harmonic acoustic wave or electromagnetic wave satisfies the Helmholtz equation

$$\nabla \cdot \frac{1}{\mu} \nabla u + \epsilon \omega^2 u = 0,$$

where μ, ϵ are material parameters. A periodic structure is given by the periodicity of the parameters μ, ϵ . If the Helmholtz equation has a nontrivial solution without source from the exterior of the periodic structure, the solution is a **guided mode**.

Suggested from the above discussions, there are two important problems that bring our interest. One is to design periodic materials that support guided modes robust or nonrobust with respect to perturbations, and to prove the existence and nonexistence theoretically. Another one is to describe transmission anomalies through periodic slabs.

In this dissertation, we answer the first problem for periodic pillars by establishing a systematic framework for the analysis of guided modes using Bessel functions and proving a few existence and nonexistence results similar to those for periodic slabs in [1]. Guided mode analysis for periodic pillars has not yet been discussed adequately, and this work sets up the foundation for future research. We also observe that the embedded guided mode in section 5.2 of [1] is in fact a trivial one, and provide a new nontrivial design. To answer the problem of characterizing transmission anomalies through slabs, we base our work on the framework of [25] and [26]. We establish conditions under which the anomaly is optimized, that is, under which the transmission rate reaches 100% and 0% at nearby frequencies close to that of

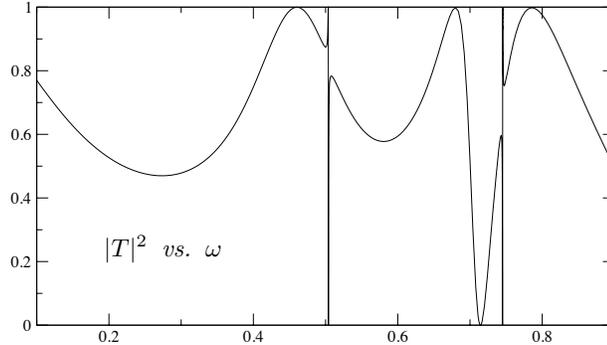


FIGURE 1.1. Numerical computation of the percentage of energy $|T|^2$ transmitted across a penetrable waveguide of period 2π as a function of the frequency of the incident plane wave. Here, the wavenumber in the x -direction (Fig. 1.2) is $\kappa = 0.02$ and one period consists of a single circle of radius $\pi/2$ with $\epsilon = 10$ and an ambient medium with $\epsilon = 1$; $\mu = 1$ throughout. The structure supports guided modes at $(\kappa, \omega) = (0, 0.5039\dots)$ and $(\kappa, \omega) = (0, 0.7452\dots)$, both contained within the region \mathcal{D} of one propagating diffractive order (Fig 3.1). Theorem 26 guarantees that the transmission attains minimal and maximal values of 0% and 100% at each of the sharp anomalies near the guided-mode frequencies. This is Figure (6) in [23].

the guided mode. Our analysis shows more generic forms of anomalies besides a single dip-peak graph, including total background transmission or reflection and multiple anomalies.

1.1 Periodic Slabs

We consider slabs finite in z , periodic in x and invariant in y . (or more generally, periodic in both x and y .) The existence and nonexistence results of guided modes have been studied theoretically in [1][27] and in many special geometries numerically in a lot of literature. The frequencies of some guided modes are in the point spectrum of the associated Helmholtz operator in one period of slab, and are well understood. (See section 4.4 of [1] for some examples of guided modes of this type.) The frequency and wavenumber (κ, ω) of these guided modes lie on a real component of the dispersion relation. Under the perturbation of the wavenumber, a guided mode with a nearby frequency persists, and thus is considered robust under such perturbation.

The more interesting case is when the frequency lies in the essential spectrum of the Helmholtz operator for the wavenumber κ . Ideally the amplitude of a mode is decaying exponentially away from the periodic medium. If the wavenumber κ is perturbed, an embedded guided mode will typically vanish and hence is nonrobust. If we consider the scattering of an incoming wave from one side of the slab, the transmission coefficient that determines the ratio of energy flux transmitted can have sharp spikes as functions of ω and κ in the neighborhood of (κ_0, ω_0) of a guided mode. Numerical experiments show that these spikes occur as a function of frequency ω where κ is perturbed. In many applications, the transmission rate can even reach 100% and 0%. See numerical computations of the transmission by an infinite array of rods in Figure 1.1 from [23].

This transmission anomaly is studied in [25][26]. One can represent the field by boundary integrals for piecewise structures, and numerical implementation of the boundary integral equations shows the spike of transmission coefficient as a function of frequency ω [25]. In [26], they consider the case for which the frequency and wavenumber pair (κ_0, ω_0) lies in a real diamond domain admitting one single energy-carrying propagating harmonic. One example of such is an antisymmetric guided mode in a slab that is symmetric about an axis perpendicular to it, established in [1]. The pair (κ, ω) is allowed to be in the complex domain, and the generalized guided mode problem can be thought of as an eigenvalue problem in operator form such that the eigenvalue ℓ is equal to 0 at complex pair (κ, ω) of a guided mode. The nonrobust nature of the embedded guided mode implies that (κ_0, ω_0) is the only real pair in a complex neighborhood of it that satisfies the dispersion relation $\ell(\kappa, \omega) = 0$ for guided modes. Within this framework, one can analytically connect the scattering problem to the guided modes, and perturbation analysis of the field and eigenvalue ℓ can be done to obtain an asymptotic for-

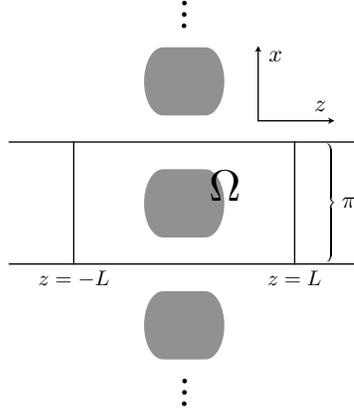


FIGURE 1.2. An example of a two-dimensional periodic slab. One period truncated to the rectangle $[-\pi, \pi] \times [-L, L]$ is denoted by Ω .

mula for the transmission coefficient in terms of perturbations for two-dimensional structures [26, 21, 23]. More details on the proof and transmission graphs using the formula are discussed in [26][23].

My work develops this idea and aims to find when the transmission can reach 100% and 0%, as shown in Figure 1.1. We consider a slab that is not only symmetric in x , but also symmetric with respect to an axis parallel to the slab, as shown in Figure 1.2. The unitary scattering matrix possesses special symmetric properties due to the symmetry of the slab structure. Our main result in Chapter 3 is a proof for slabs under generic assumptions that ensure the transmission magnitude of 100% and 0%. Specifically, if a two-dimensional lossless periodic slab is symmetric about an axis parallel to the slab, and if the slab supports an embedded guided mode nonrobust in κ at a real pair of frequency and wavenumber (κ_0, ω_0) , then total transmission and reflection is necessarily attained for pairs (κ, ω) close to those of the guided mode. The frequencies that admit total transmission and reflection are real-analytic functions of the wavenumber in the real (κ, ω) plane that intersect tangentially at (κ_0, ω_0) . In the proof of this result, the special symmetries of the scattering matrix give more information on the transmission and reflection than

the analysis in [26][23], and is just what we need to show the real analyticity of the total transmission and total reflection curves.

In our complex perturbation analysis of transmission anomalies, we extend to structures whose coefficients are only required to be measurable and bounded from below and above. In stead of using boundary integral representations for piecewise constant slabs, we utilize only the analyticity of the solution of the eigenvalue problem for which the operator has the form of the identity map plus an analytic compact operator.

We also show that a more intricate patterns of transmission anomalies can be excited by the perturbation of wavenumber if we relax our generic assumptions to nongeneric cases. One type of anomaly possesses two peaks and one dip on the transmission graph, which corresponds to total background transmission; similarly one peak and two dips corresponds to total background reflection. We give conditions for the total background transmission case for which the transmission coefficient is nearly 100% and reaches 100% at two frequencies, but drops to 0% at one frequency, and similarly for the total background reflection case. We also analyze a case of multiple anomalies, for which the transmission coefficient reaches 100% and 0% twice in a narrow range of the frequency ω .

1.2 Periodic Pillars

In the remaining part of this dissertation, we consider a structure illustrated by Figure 1.3, periodic in one direction and bounded in the other two directions. The field is governed by the Helmholtz equation in which the material is homogeneous in the exterior.

We have not found adequate foundational work on the scattering problem and guided modes for periodic pillars in the literature, so we present here a system-

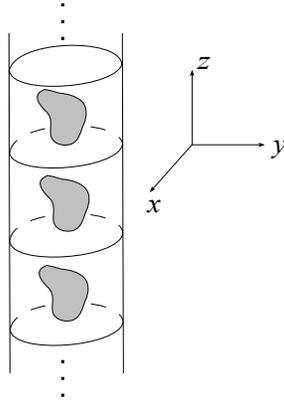


FIGURE 1.3. A pillar periodic in z and finite in x and y .

atic mathematical framework. My work uses standard variational techniques in [4, 6, 12], and the analysis is similar the theory established in [1] for slabs. Bessel functions are naturally introduced in cylindrical coordinates to characterize the Fourier harmonics. The general solution of the Helmholtz equation in the exterior domain with constant parameters is expanded as an infinite superposition of Fourier harmonics:

$$u(r, \theta, z) = \sum_{m, \ell=-\infty}^{\infty} [A_{\ell} H_{\ell}^1(\eta_m r) + B_{\ell} H_{\ell}^2(\eta_m r)] e^{i\ell\theta} e^{i(m+\kappa)z},$$

where $\eta_m^2 = \epsilon_0 \mu_0 \omega^2 - (m + \kappa)^2$ and $H_{\ell}^1 = J_{\ell} + iY_{\ell}$ and $H_{\ell}^2 = J_{\ell} - iY_{\ell}$ are Hankel functions. The Hankel functions' monotonicity and phase change orientation make it possible to separate between outgoing and incoming harmonics and impose appropriate radiating boundary conditions through a Dirichlet-to-Neumann map. Upon this, the solvability of the plane wave scattering problem, the characterization of the guided-mode frequencies, and existence results analogous to those of slabs can all be built.

There are a few nonexistence and existence results we establish in this dissertation.

One is nonexistence. In [27], by introducing an augmented medium structure, Shipman and Volkov give a proof of the nonexistence of guided modes in piecewise inverse photonic slabs, i.e., piecewise structures that have higher wave speed in the pillar than in the exterior. In their study, the proof of the nonexistence is contingent on a restriction on the width of the slabs. Similar to their analysis, we show here the nonexistence of guided modes in inverse pillars such as a periodic array of bubbles in glass. Certain restrictions on the geometry of the structures are needed in our proofs of the nonexistence, and whether the restrictions can be removed remains an open problem.

There have been limited results on the existence of embedded guided modes, even for periodic slabs. One example of an embedded guided mode is an antisymmetric guided mode in periodic structures symmetric in the direction of waveguide, as discussed earlier. This guided mode typically only exists at an isolated real pair (κ_0, ω_0) .

A non-isolated but artificially constructed example of embedded guided modes is provided by Bonnet-Bendhia and Starling in section 5.2 of [1]. If a structure of period p has smaller periodic cells, say of period $q < p$, then on the subspace F consisting of all the functions with the smaller period q , the infimum of the essential spectrum of the Helmholtz operator is strictly larger than that of the space of functions with period p . One can find eigenfunctions with their frequencies lying below the cutoff frequency for the restriction to F , which are non-embedded and easy to find, but these are embedded in the essential spectrum on the full period function space. However, these guided modes are simply non-embedded guided modes for a smaller periodic structure. In other words, for a given periodic structure, a larger period is artificially chosen so that the frequency of a non-

embedded guided mode is embedded in the artificial essential spectrum, and this is therefore a trivial example. An essentially nontrivial example is strongly desired.

My main achievement in Chapter 4 is a proof of nontrivial embedded guided modes in periodic pillars that are robust under perturbations of κ . In our construction, the period of the mode is genuinely larger than that of the structure. Our proof relies on choosing the material parameters so that the Helmholtz operator is invariant on a subspace where the propagating harmonics automatically vanish. This solution does not depend upon the exact choice of the wavenumber and so is a guided mode robust in κ . On the other hand, the existence is based on special properties of the structure, and is thus nonrobust with respect to the perturbation of material geometries.

1.3 Summary of Dissertation

The structure of this dissertation is as follows.

In chapter 2, we give a brief introduction to the scattering problems and guided modes, as well as the transmission anomaly phenomena. The focus is put on periodic slabs and we provide some standard tools used in the analysis of periodic structures.

In chapter 3, for a slab that is symmetric with respect to an axis parallel to it, we present the proof of the existence of total transmission and reflection associated with a nonrobust guided mode. We also discuss the cases that the slab admits a single anomaly and multiple anomalies. The diagrams of our results are shown the last section of this chapter, based on the approximation formula given in [26, 23].

In chapter 4, we study the scattering problem and guided modes for pillar structures that are finite in two directions. Bessel functions are used to do the analysis systematically. We provide a proof for the existence of embedded guided modes,

and nonexistence of guided modes for some special geometries in the last section of this chapter.

In the chapter 5, we point out some restrictions of our work and pose challenges for future work. We also provide some new open problems on the nature of transmission resonances.

Chapter 2

Wave Scattering and Guided Modes in Periodic Structures

In this chapter, we give a brief introduction to plane-wave scattering problems and guided modes for time-harmonic wave equations, as well as previous analysis on transmission anomalies.

We first introduce the wave equation and the Helmholtz equation, and explain the periodic structures and the solutions in these structures. The plane wave scattering problem is presented in section 2.3. We give existence and nonexistence results for guided modes in periodic slabs. The proofs can be found in [1, 27]. The nonrobust guided modes shown in [27] typically vanish as the wavenumber κ is perturbed from 0, and the transmission coefficients can reach a magnitude of 100% and 0%. The transmission anomaly for piecewise periodic slabs is studied in [25, 26], and an asymptotic formula of the transmission coefficient as a function of the perturbations of κ, ω is obtained.

In this chapter, except in the last section, we assume that the wave frequency and wavenumber (κ, ω) are real. The wave frequency and wavenumber can be extended to the complex domain, which we will use to prove our main result.

2.1 The Wave Equation and Helmholtz Equation

We consider a physical structure that is three-dimensional but invariant in the y -direction. In this structure, the Maxwell system of electromagnetics is y -independent and has two polarizations that are simplified to the Helmholtz equation for the out-of-plane components of the E field and the H field. We consider harmonic fields with positive angular frequency ω . Given a frequency ω , plane waves and guided

modes are characterized by their propagation constant κ in the direction parallel to the slab. We take the y component E_y of a harmonic E -polarized field with propagation constant κ to be of the following pseudoperiodic form

$$E_y(x, z, t) = u(x, z)e^{i(\kappa x - \omega t)},$$

$$u(x + 2\pi n, z) = u(x, z) \text{ for } n \in \mathbb{Z}.$$

E_y satisfies the wave equation

$$\epsilon \frac{\partial^2}{\partial t^2} E_y(x, y, z; t) = \nabla \cdot \frac{1}{\mu} \nabla E_y(x, y, z; t) \quad (2.1)$$

We are looking for time-harmonic waves $E_y(x, y, z, t) = \tilde{u}(x, y, z)e^{-i\omega t}$. The spatial factor of the wave satisfies the Helmholtz equation

$$\nabla \cdot \frac{1}{\mu} \nabla \tilde{u} + \epsilon \omega^2 \tilde{u} = 0. \quad (2.2)$$

2.2 Periodic Structures and Pseudo-periodic Solutions

We consider periodic slab structures that are finite in the z -direction, periodic in the x -direction and invariant in the y -direction.

The periodic slab is defined by the material parameters $\epsilon(x, z)$ and $\mu(x, z)$ for $x, z \in \mathbb{R}$. We take these parameters to be bounded from below and above by positive numbers:

$$\begin{aligned} \epsilon(x + 2\pi n, z) &= \epsilon(x, z), \quad \mu(x + 2\pi n, z) = \mu(x, z), \quad \text{for } n \in \mathbb{Z}, \\ \epsilon(x, z) &= \epsilon_0, \quad \mu(x, z) = \mu_0, \quad \text{for } |z| \geq L, \\ 0 < \epsilon_- &< \epsilon(x, z) < \epsilon_+, \quad 0 < \mu_- < \mu(x, z) < \mu_+. \end{aligned} \quad (2.3)$$

The field \tilde{u} satisfies the *pseudo-periodic*, or *quasi-periodic* or κ -*periodic* conditions:

$$\tilde{u}(x, z; \kappa) = e^{i\kappa \cdot x} u(x, z), \quad (2.4)$$

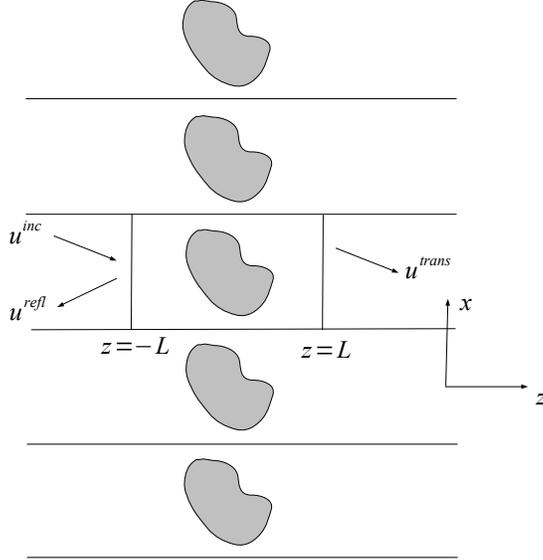


FIGURE 2.1. Slab structure periodic in x and finite in z ; u^{inc} is transmitted to u^{trans} and reflected to u^{refl} .

where $u(x, z)$ is 2π -periodic in x . We call such solutions *Bloch waves* and the number κ *Bloch wavenumber*. This can be understood as follows. The periodicity of the slab implies that the values of the same incident field viewed at the point $(x + 2\pi, z)$ and at (x, z) have only a phase change. Thus, the field at $(x + 2\pi, z)$ can be seen as the field at (x, z) multiplied by the phase change factor $e^{2\pi\kappa i}$.

The field \tilde{u} has a *Bloch factor* $e^{i\kappa x}$, for which we have $e^{i\kappa(x+2\pi)} = e^{i\kappa x} e^{2\pi\kappa i}$ and hence we can regard that the κ -periodicity is caused by the Bloch factor. It is noticed that $e^{2\pi(\kappa+m)i} = e^{2\pi\kappa i}, \forall m \in \mathbb{Z}$. This means the wave number κ and $\kappa + m$ have the same effect on the Bloch factor. Thus, we can reduce the wavenumber κ by an integer and deal with the cases that κ lies in the *first Brillouin zone* $B = [-1/2, 1/2)$.

The periodic factor u satisfies the following modified Helmholtz equation

$$(\nabla + i\boldsymbol{\kappa})\mu^{-1}(\nabla + i\boldsymbol{\kappa})u(x, z) + \epsilon\omega^2 u(x, z) = 0, \quad (2.5)$$

where $\boldsymbol{\kappa} = (\kappa, 0)^T$.

2.3 Plane-wave Scattering by Periodic Slabs

2.3.1 Radiation Condition

The periodic solution of equation (2.5) has a Fourier series expansion

$$u(x, z) = \sum_m u_m(z) e^{imx}, \quad (2.6)$$

and the pseudo-periodic field is

$$\tilde{u}(x, z) = u(x, z) e^{i\kappa x} = \sum_m u_m(z) e^{i(m+\kappa)x} \quad (2.7)$$

If $|z| > L$, then $u_m(z) = (u_m^-(z), u_m^+(z))$, for $z < -L$ or $z > L$, are solutions of the ordinary differential equation

$$u_m'' + \eta_m^2 u_m = 0$$

where

$$\eta_m^2 = \epsilon_0 \mu_0 \omega^2 - (m + \kappa)^2. \quad (2.8)$$

The solutions $u_m(z) = c_m^1 u^1(z) + c_m^2 u^2(z)$, called *spatial harmonics*, where $u_m^{1,2}(z)$ are independent solutions of the linear ordinary differential equation, belong to the following three classes:

$$\left\{ \begin{array}{ll} u_m = c_m^1 e^{i\eta_m z} + c_m^2 e^{-i\eta_m z} \in \mathcal{Z}_p & \text{(propagating),} \quad \text{if } \eta_m^2 > 0; \text{ we take } \eta_m = |\eta_m|; \\ u_m = c_m^1 e^{i\eta_m z} + c_m^2 e^{-i\eta_m z} \in \mathcal{Z}_e & \text{(evanescent),} \quad \text{if } \eta_m^2 < 0; \text{ we take } \eta_m = |\eta_m| i; \\ u_m = c_m^1 + c_m^2 z \in \mathcal{Z}_l & \text{(linear),} \quad \text{if } \eta_m^2 = 0. \end{array} \right. \quad (2.9)$$

The classes \mathcal{Z}_p is finite, \mathcal{Z}_l is generically empty but has at most one harmonic, and the class \mathcal{Z}_e is infinite. As long as η_m^2 are nonzero for all integers m , the general solution of this equation (2.5) admits a Fourier expansion on each side of the slab

$$u(x, z) = \begin{cases} \sum_{m=-\infty}^{\infty} (A_m^+ e^{i\eta_m z} + B_m^+ e^{-i\eta_m z}) e^{imx}, & \text{for } z > L, \\ \sum_{m=-\infty}^{\infty} (A_m^- e^{i\eta_m z} + B_m^- e^{-i\eta_m z}) e^{imx}, & \text{for } z < -L. \end{cases} \quad (2.10)$$

The spatial harmonics $e^{i\eta_m z}, e^{-i\eta_m z}$ for $m \in \mathcal{Z}_p$ represent right-going or left-going traveling waves, whose angles α_m are

$$\alpha_m = \arcsin \frac{\kappa + m}{\omega \sqrt{\epsilon_0 \mu_0}}. \quad (2.11)$$

The harmonics $e^{\pm i\eta_m z} = e^{\mp |\eta_m| z}$ for $m \in \mathcal{Z}_e$ represent exponentially decaying harmonics for $\pm z > L$, while $e^{\mp i\eta_m z} = e^{\pm |\eta_m| z}$ represent exponentially growing harmonics for $\pm z > L$. The linear orders $\eta_m = 0$ correspond to “grazing incidence”, and will not play a role in the present study.

Dropping the exponentially growing harmonics, a function u is said to be *radiating* or *outgoing* if in the form (2.10) the coefficients $B_m^+ = A_m^- = 0$. We introduce the following *radiation condition*:

Condition 1 (Radiation). A complex field u defined on \mathbb{R}^2 satisfies the *radiation condition* if there exist complex coefficients $\{c_m^\pm\}$ such that

$$u(x, z) = \sum_{m \in \mathcal{Z}} c_m^\pm e^{\pm i\eta_m z} e^{imx} \quad \text{for } \pm z > L.$$

2.3.2 Plane-wave Scattering Problems and Guided Modes

An incident wave with frequency and wavenumber (κ, ω) is a linear superposition of the propagating Fourier harmonics

$$u^{inc}(x, z) = \sum_{m \in \mathcal{Z}_p} (A_m e^{i\eta_m z} + B_m e^{-i\eta_m z}) e^{imx}. \quad (2.12)$$

Problem 2 (Plane-wave scattering). Given $\omega > 0$ and $\kappa \in B$, find a function u on \mathbb{R}^2 that is 2π -periodic in x with Bloch wavenumber κ and satisfying the modified Helmholtz equation $(\nabla + i\boldsymbol{\kappa}) \cdot \mu^{-1}(\nabla + i\boldsymbol{\kappa})u(x, z) + \epsilon\omega^2 u(x, z) = 0$, and such that

$$u(x, z) = u^{inc}(x, z) + u^{sc}(x, z)$$

in which $u^{inc}(x, z)$ is an incident wave (2.12) and u^{sc} satisfies the radiation condition 1.

The scattering problem 2 can be formulated using standard variational techniques in the truncated domain of one period

$$\Omega = \{(x, z) \in \mathbb{R}^2 : -\pi < x < \pi, |z| < L\}. \quad (2.13)$$

Let $\Gamma_{\pm} = \{(x, z) \in \mathbb{R}^2 : -\pi < x < \pi, z = \pm L\}$ and $\Gamma = \Gamma_- \cup \Gamma_+$. We make use of the Dirichlet-to-Neumann map $T = T(\kappa, \omega)$ on the right and left boundaries Γ_{\pm} to characterize outgoing fields. It is a bounded linear operator from $H^{\frac{1}{2}}(\Gamma)$ to $H^{-\frac{1}{2}}(\Gamma)$ defined as follows. For any $f \in H^{\frac{1}{2}}(\Gamma)$, let $\hat{f}_m = (\hat{f}_m^+, \hat{f}_m^-)$ be the Fourier coefficients of f , that is, $f(\pm L, x) = \sum_m \hat{f}_m^{\pm} e^{imx}$. Then

$$\begin{aligned} T : H^{\frac{1}{2}}(\Gamma) &\rightarrow H^{-\frac{1}{2}}(\Gamma), \\ \widehat{(Tf)}_m &= -i\eta_m \hat{f}_m. \end{aligned} \quad (2.14)$$

This operator has the property that

$$\partial_n u + Tu = 0 \text{ on } \Gamma \iff u \text{ is outgoing.}$$

The operator T has a nonnegative real part T_r and a nonpositive imaginary part T_i :

$$\begin{aligned} T &= T_r + iT_i, \\ \widehat{(T_r f)}_m &= \begin{cases} -i\eta_m \hat{f}_m & \text{if } m \in \mathcal{Z}_e, \\ 0 & \text{otherwise.} \end{cases} \\ \widehat{(T_i f)}_m &= \begin{cases} -\eta_m \hat{f}_m & \text{if } m \in \mathcal{Z}_p, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.15)$$

In the periodic Sobolev space

$$H_{\text{per}}^1(\Omega) = \{u \in H^1(\Omega) : u(\pi, z) = u(-\pi, z) \text{ for all } z \in (-L, L)\},$$

in which evaluation on the boundaries of Ω is in the sense of the trace map, we also define the following forms in $H_{\text{per}}^1(\Omega)$:

$$\begin{aligned}
p(v) &= \mu_0^{-1} \int_{\Gamma} (\partial_n u^{\text{inc}} + T u^{\text{inc}}) \bar{v}, \\
a(u, v) &= a_{\kappa, \omega}(u, v) = \int_{\Omega} \mu^{-1} (\nabla + i\boldsymbol{\kappa}) u \cdot (\nabla - i\boldsymbol{\kappa}) \bar{v} + \mu_0^{-1} \int_{\Gamma} (T u) v, \\
a_r(u, v) &= \int_{\Omega} \mu^{-1} (\nabla + i\boldsymbol{\kappa}) u \cdot (\nabla - i\boldsymbol{\kappa}) \bar{v} + \mu_0^{-1} \int_{\Gamma} (T_r u) v, \\
a_i(u, v) &= \mu_0^{-1} \int_{\Gamma} (T_i u) v, \\
b(u, v) &= \int_{\Omega} \epsilon u \bar{v}.
\end{aligned}$$

We have $a = a_r + i a_i$.

Problem 3 (Scattering problem, variational form). Given a pair (κ, ω) , find a function $u \in H_{\text{per}}^1(\Omega)$ such that

$$a(u, v) - \omega^2 b(u, v) = p(v), \quad \text{for all } v \in H_{\text{per}}^1(\Omega) \quad (2.16)$$

Theorem 4. The problem 2 and the problem 3 are equivalent.

Proof. We observe that

$$\left[(\nabla + i\boldsymbol{\kappa}) \cdot \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa}) u \right] \bar{v} = \nabla \cdot \left[\left(\frac{1}{\mu} (\nabla + i\boldsymbol{\kappa}) u \right) \bar{v} \right] - \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa}) u \cdot (\nabla - i\boldsymbol{\kappa}) \bar{v}.$$

Integrating it implies

$$\int_{\Omega} \left[(\nabla + i\boldsymbol{\kappa}) \cdot \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa}) u \right] \bar{v} = \int_{\Gamma} \left[\left(\frac{1}{\mu} (\nabla + i\boldsymbol{\kappa}) u \right) \bar{v} \right] \cdot \mathbf{n} - \int_{\Omega} \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa}) u \cdot (\nabla - i\boldsymbol{\kappa}) \bar{v}.$$

We multiply the modified Helmholtz equation by \bar{v} and integrate to obtain

$$\int_{\Gamma} \left[\left(\frac{1}{\mu} (\nabla + i\boldsymbol{\kappa}) u \right) \bar{v} \right] \cdot \mathbf{n} - \int_{\Omega} \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa}) u \cdot (\nabla - i\boldsymbol{\kappa}) \bar{v} + \int_{\Omega} \epsilon \omega^2 u \bar{v} = 0,$$

or

$$\int_{\Gamma} \frac{1}{\mu} \partial_n u \bar{v} - \int_{\Omega} \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa}) u \cdot (\nabla - i\boldsymbol{\kappa}) \bar{v} + \int_{\Omega} \epsilon \omega^2 u \bar{v} = 0.$$

By the radiation condition $\partial_{\mathbf{n}}(u - u^{inc}) = -T(u - u^{inc})$, we have

$$\int_{\Omega} \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa})u \cdot (\nabla - i\boldsymbol{\kappa})\bar{v} + \int_{\Gamma} \frac{1}{\mu} (Tu)\bar{v} - \int_{\Omega} \epsilon\omega^2 u\bar{v} = \int_{\Gamma} \frac{1}{\mu_0} \bar{v} (\partial_{\mathbf{n}}u^{inc} + Tu^{inc})$$

This is the weak form (2.16).

Conversely, if a function $u \in H_{\text{per}}^1(\Omega)$ satisfies (2.16), we can take test functions $v \in C_0^\infty(\Omega)$ to get $(\nabla + i\boldsymbol{\kappa}) \cdot \frac{1}{\mu} (\nabla - i\boldsymbol{\kappa})u + \epsilon\mu\omega^2 u = 0$ in Ω . Then we can multiply the modified Helmholtz equation by $v \in H_{\text{per}}^1(\Omega)$ to obtain

$$\int_{\Gamma} \frac{1}{\mu} \partial_{\mathbf{n}}u\bar{v} - \int_{\Omega} \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa})u \cdot (\nabla - i\boldsymbol{\kappa})\bar{v} + \int_{\Omega} \epsilon\omega^2 u\bar{v} = 0.$$

Comparing it with (2.16), we prove that $\partial_{\mathbf{n}}(u - u^{inc}) = -T(u - u^{inc})$, i.e. u solves the problem 2. \square

A *guided mode* is a solution of the homogeneous problem where there is no source:

$$a(u, v) - \omega^2 b(u, v) = 0, \quad \text{for all } v \in H_{\text{per}}^1(\Omega). \quad (2.17)$$

Note that in the proof of the above equivalence, it is not required that $\omega^2 \in \mathbb{R}$. But if the square frequency is real, we have the following result in particular for the homogeneous problem.

Theorem 5. (Real eigenvalues) If $\omega^2 \in \mathbb{R}$, then a function $u \in H_{\text{per}}^1(\Omega)$ satisfies the homogeneous problem (2.17) if and only if it satisfies the equation

$$a_r(u, v) + ia_i(u, v) - \omega^2 b(u, v) = 0, \quad \text{for all } v \in H_{\text{per}}^1(\Omega), \quad (2.18)$$

and if and only if it satisfies

$$\begin{aligned} a_r(u, v) - \omega^2 b(u, v) &= 0, \quad \text{for all } v \in H_{\text{per}}^1(\Omega), \\ (\widehat{u}|_{\Gamma})_m &= 0, \quad \forall m \in \mathcal{Z}_p. \end{aligned} \quad (2.19)$$

Proof. We only need to show the equivalence of (2.18) and (2.19). If $(\widehat{u|_{\Gamma}})_m = 0, \forall m \in \mathcal{Z}_p$, then $\int_{\Gamma}(T_i u)v = 0$, so (2.19) implies (2.18). Conversely, we take the imaginary part of $a_r(u, u) + ia_i(u, u) - \omega^2 b(u, u)$ to obtain $a_i(u, u) = 0$ for all $u \in H_{\text{per}}^1(\Omega)$. This implies that $(\widehat{u|_{\Gamma}})_m = 0, \forall m \in \mathcal{Z}_p$ because $\eta_m \neq 0$ for $m \in \mathcal{Z}_p$. \square

2.3.3 Existence of Solutions of Scattering Problems

The existence of the solution of the scattering problem can be analyzed by Fredholm alternative theory. To this end, we write the form $a - \omega^2 b$ as

$$a(u, v) - \omega^2 b(u, v) = c_1(u, v) + c_2(u, v),$$

in which $c_1(u, v) = a(u, v) + b(u, v)$ and $c_2(u, v) = -(\omega^2 + 1)b(u, v)$.

Lemma 6. Both c_1 and c_2 are bounded in $H_{\text{per}}^1(\Omega)$.

Proof. The boundedness of c_1 is shown by

$$\begin{aligned} |a(u, v) + b(u, v)| &= \left| \int_{\Omega} \mu^{-1}(\nabla + i\boldsymbol{\kappa})u \cdot (\nabla - i\boldsymbol{\kappa})\bar{v} + \int_{\Omega} \epsilon u \bar{v} \right| \\ &\leq \frac{1}{\mu_-} \left| \int_{\Omega} (\nabla + i\boldsymbol{\kappa})u \cdot (\nabla - i\boldsymbol{\kappa})\bar{v} \right| + \epsilon_+ \left| \int_{\Omega} u \bar{v} \right| \\ &\leq \frac{1}{\mu_-} \|(\nabla + i\boldsymbol{\kappa})u\|_{L^2} \|(\nabla + i\boldsymbol{\kappa})v\|_{L^2} + \epsilon_+ \left| \int_{\Omega} u \bar{v} \right| \\ &\leq \frac{1}{\mu_-} (\|\nabla u\|_{L^2} + |\boldsymbol{\kappa}| \|u\|_{L^2}) (\|\nabla v\|_{L^2} + |\boldsymbol{\kappa}| \|v\|_{L^2}) + \epsilon_+ \|u\|_{L^2} \|v\|_{L^2} \\ &\leq M \|u\|_{H_{\text{per}}^1(\Omega)} \cdot \|v\|_{H_{\text{per}}^1(\Omega)}, \text{ for some } M > 0, \end{aligned}$$

and

$$\int_{\Gamma}(Tu)v \leq \sum_{m \in \mathbb{Z}} |\eta_m^{1/2} \hat{u}_m| \cdot |\eta_m^{1/2} \hat{v}_m| \leq \|u\|_{H^{1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)} \leq \|u\|_{H_{\text{per}}^1(\Omega)} \|v\|_{H_{\text{per}}^1(\Omega)}$$

The boundedness c_2 is shown by

$$\int_{\Omega} \epsilon u \bar{v} \leq \epsilon_+ \int_{\Omega} u \bar{v} \leq \epsilon_+ \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \epsilon_+ \|u\|_{H_{\text{per}}^1(\Omega)} \|v\|_{H_{\text{per}}^1(\Omega)}.$$

\square

These forms are represented by bounded operators C_1 and C_2 in $H_{\text{per}}^1(\Omega)$ by Riesz Representation:

$$(C_1 u, v)_{H_{\text{per}}^1(\Omega)} = c_1(u, v),$$

$$(C_2 u, v)_{H_{\text{per}}^1(\Omega)} = c_2(u, v).$$

Because of the coercivity of c_1 and the compact embedding of $L^2(\Omega)$ into $H_{\text{per}}^1(\Omega)$, we have

Lemma 7. The operator C_1 has a bounded inverse and C_2 is compact.

Proof. The operator C_1 has bounded inverse because the sesquilinear form $c_1(u, v)$ is bounded and coercive:

$$\text{Re}(c_1(u, u) + c_2(u, u)) \geq \min\{\mu_+^{-1}, \epsilon_-\} \|u\|_{H_{\text{per}}^1(\Omega)}^2.$$

The operator C_2 can be written as $C_2 = I_1 I_2$, where $I_2 : H_{\text{per}}^1(\Omega) \rightarrow L^2(\Omega)$ is the natural embedding, and $I_1 : L^2(\Omega) \rightarrow H_{\text{per}}^1(\Omega)$ is defined by $\langle I_1 u, v \rangle_{H_{\text{per}}^1(\Omega)} := -(\omega^2 + 1) \int_{\Omega} u \bar{v}$. The operator I_2 is compact because of the compactness of the injection of $H_{\text{per}}^1(\Omega)$ into $L^2(\Omega)$. The operator I_1 is continuous because

$$\|I_1 u\|_{H_{\text{per}}^1(\Omega)} = \sup_{0 \neq v \in H_{\text{per}}^1(\Omega)} \frac{|\int_{\Omega} u \bar{v}|}{\|v\|_{H_{\text{per}}^1(\Omega)}} \leq \sup_{0 \neq v \in H_{\text{per}}^1(\Omega)} \frac{|\int_{\Omega} u \bar{v}|}{\|v\|_{L^2(\Omega)}} \leq \|u\|_{L^2(\Omega)}.$$

Their composition C_2 is therefore compact. \square

If we denote by w^{inc} the unique element of $H_{\text{per}}^1(\Omega)$ such that $(w^{\text{inc}}, v)_{H_{\text{per}}^1(\Omega)} = \ell(v)$, the scattering problem becomes $(C_1 u, v) + (C_2 u, v) = (w^{\text{inc}}, v)$ for all $v \in H_{\text{per}}^1(\Omega)$, or

$$C_1 u + C_2 u = w^{\text{inc}}. \quad (2.20)$$

The term $\ell(v)$ consists of the incident wave u^{inc} and therefore w^{inc} represent the source term in the equation (2.20). In the operator form, a guided mode is a

nontrivial solution of the homogeneous problem

$$C_1 u + C_2 u = 0. \quad (2.21)$$

By means of the Fredholm alternative one can demonstrate that, even if a slab admits a guided mode for a given real pair (κ, ω) , the problem of scattering of a plane wave always has a solution. Proofs are given in [1, Thm. 3.1] and [23, Thm. 9]; the idea is essentially that plane waves contain only propagating harmonics whereas guided modes contain only evanescent harmonics and are therefore orthogonal to any plane-wave source field.

Theorem 8. For real $\omega > 0$ and $\kappa \in B$, the scattering problem 3 with a plane-wave source field has at least one solution and the set of solutions is a finite dimensional affine space with the dimension equal to the dimension of the space of guided modes.

Proof. By the Fredholm alternative, (2.20) has a solution if and only if $\langle w^{\text{inc}}, v \rangle = 0$ for all $v \in \text{Null}(C_1 + C_2)^\dagger$, i.e. for all v satisfying

$$(w, (C_1 + C_2)^\dagger v) = 0, \forall w \in H_{\text{per}}^1(\Omega),$$

or for all v satisfying

$$a_r(w, v) + ia_i(w, v) - \omega^2 b(w, v) = 0, \forall w \in H_{\text{per}}^1(\Omega),$$

or just

$$a_r(v, w) - ia_i(v, w) - \omega^2 b(v, w) = 0, \forall w \in H_{\text{per}}^1(\Omega).$$

By Theorem 5, the propagating harmonics of v vanish on Γ , and so (w^{inc}, v) vanishes.

The space of solutions is finite-dimensional because C_1 has a bounded inverse and C_2 is compact. □

2.4 Guided Modes

A guided mode is a nonzero solution $u(x, z)$ of the scattering problem without the source originating from the exterior of the slab, that is, it satisfies (2.18) and (2.19). It may also be understood as a nontrivial solution of the problem (2.20) in operator form.

A functional analysis framework for spectral analysis is on one period in \mathbb{R}^2

$$\mathcal{S} = \{(x, z) \in \mathbb{R}^2 : -\pi < x < \pi\}. \quad (2.22)$$

We consider the Hilbert space $L^2(\mathcal{S}, \epsilon dV)$ of square-Lebesgue-integrable complex-valued functions in \mathcal{S} with inner product

$$b(u, v) = \int_{\mathcal{S}} \epsilon u \bar{v} dV \quad (2.23)$$

and the unbounded symmetric nonnegative quadratic form in $L^2(\mathcal{S}, \epsilon dV)$, with form domain $H_{per}^1(\mathcal{S})$, defined by

$$a(u, v) = \int_{\mathcal{S}} \mu^{-1} (\nabla + i\kappa)u \cdot (\nabla - i\kappa)\bar{v} dV, \forall u, v \in H_{per}^1(\mathcal{S}) \quad (2.24)$$

This form defines a positive operator S_κ by

$$S_\kappa u = -\epsilon^{-1} (\nabla + i\kappa) \cdot \mu^{-1} (\nabla + i\kappa)u, u \in \mathcal{D}(S_\kappa) \subset H_{per}^1(\mathcal{S}) \quad (2.25)$$

The spectrum of S_κ can be analyzed by min-max principle ([22], Chapter XIII).

The sequence defined by

$$\lambda_j(\kappa) = \sup_{V^{j-1} < L^2(\mathcal{S})} \inf_{\substack{u \in (V^{j-1}) \setminus \{0\} \\ u \in H_{per}^1(\mathcal{S})}} \frac{a(u, u)}{b(u, u)} \quad (2.26)$$

where the supremum is taken over all $(j - 1)$ -dimensional subspaces, is nondecreasing, and converges to the infimum λ_- of the essential spectrum of S_κ . Let $\epsilon_- = \inf \epsilon$, $\epsilon_+ = \sup \epsilon$, $\mu_- = \inf \mu$, $\mu_+ = \sup \mu$. The following theorem is given by Bonnet-Bendhia and Starling in [1].

Theorem 9. The spectrum of the operator S_κ has the following properties:

- i). $\sigma(S_\kappa) \subset [\frac{\kappa^2}{\epsilon_+, \mu_+}, +\infty)$;
- ii). the essential spectrum $\sigma_{ess}(S_\kappa) \subset [\frac{\kappa^2}{\epsilon_-, \mu_-}, +\infty)$;
- iii) there are finitely many eigenvalues λ_j strictly less than $\frac{\kappa^2}{\epsilon_0 \mu_0}$.

We have seen in Theorem 5 that a function $u \in H_{\text{per}}^1(\Omega)$ is a guided mode if and only if it satisfies

$$\begin{aligned} a_r(u, v) - \omega^2 b(u, v) &= 0, \quad \text{for all } v \in H_{\text{per}}^1(\Omega), \\ (\widehat{u}|_\Gamma)_m &= 0, \quad \forall m \in \mathcal{Z}_p. \end{aligned} \tag{2.27}$$

The second condition requires that all the propagating harmonics vanish and so there are only evanescent harmonics left. The first equation can be analyzed by the min-max principle on the Rayleigh quotient $\frac{a_r(u, u)}{b(u, u)}$ because the sesquilinear form a_r is symmetric. We have the following theorem:

Theorem 10 (Guided-mode frequencies). Assume $\kappa \in B$ and $\eta_m \neq 0$ as above. The equation $a_r^\omega(u, v) - \omega^2 b(u, v) = 0$ has a nontrivial solution $u \in H_{\text{per}}^1(\Omega)$ for positive nondecreasing frequencies $\{\omega_j\}_{j=1}^\infty$ that tends to ∞ . The frequencies for which the slab admits a guided mode with Bloch wavenumber κ is a subset of this sequence that includes the ones that are less than $\frac{|\kappa|}{\sqrt{\epsilon_0 \mu_0}}$.

Moreover, if parameters other than μ_1 are fixed, the eigenvalues α_j and eigenfrequencies ω_j are strictly decreasing in ϵ_1 , and if parameters other than ϵ_1 are fixed, the eigenvalues and eigenfrequencies are strictly decreasing in μ_1 .

The proofs are given in [27][1]. The idea is as follows. We consider the problem $a_r(u, v) - \omega b(u, v), \forall v \in H_{\text{per}}^1(\Omega)$ for any fixed $\omega > 0$. It can be analyzed by the min-max principle. In fact, from the proof in Lemma 6 we can see that $a(u, v), b(u, v)$

are bounded bilinear form. We can define operators A_r^ω, B on $H_{\text{per}}^1(\Omega)$ by

$$\begin{aligned}(A_r^\omega u, v) &= a_r^\omega(u, v) + b(u, v) \\ (Bu, v) &= b(u, v)\end{aligned}$$

The operator A_r^ω is bijective with bounded inverse and the operator B is compact. Therefore the set of α that admit a nontrivial solution to $[A_r^\omega - (\alpha^2 + 1)B]u = 0$ is a sequence converging to infinity. The values $\alpha_j(\omega)$ are constructed by the min-max principle

$$\lambda_j(\kappa) = \sup_{V^{j-1} < L^2(\mathcal{S})} \inf_{\substack{u \in (V^{j-1}) \setminus \{0\} \\ u \in H_{\text{per}}^1(\mathcal{S})}} \frac{a(u, u)}{b(u, u)}.$$

These values are positive functions because $a_r^\omega(u, u) \geq 0$ and in the second term $\frac{1}{\mu_0} \int (Tu)\bar{v}$ of a_r^α , the multipliers $-i\eta_m = \sqrt{\epsilon_0 \mu_0 \omega^2 - (m + \kappa)^2}$ are nondecreasing functions of ω , so the eigenvalues $\alpha_j(\omega)$ are also nondecreasing in ω . They can also be proved continuous in ω (see [1]).

Therefore, the solution of $\alpha_j(\omega_j) = \omega_j^2$ for any j , are the values of frequencies that admit nontrivial solutions to the equation $a_r(u, v) - \omega^2 b(u, v) = 0$, for all $v \in H_{\text{per}}^1(\Omega)$. The set of guided modes frequencies are subset of $\{\omega_j\}_{j=1}^\infty$ for which the second part of (2.19) are satisfied. If the eigenvalue ω is less than $\frac{|\kappa|}{\epsilon_0 \mu_0}$, the second part of (2.19) is automatically satisfied because the set \mathcal{Z}_p is empty. The corresponding eigenfunction u is automatically a guided mode.

The special value $\frac{|\kappa|}{\sqrt{\epsilon_0 \mu_0}}$ is called the *cutoff frequency*. If the eigenfrequency ω_j is less than the cutoff frequency $\frac{|\kappa|}{\sqrt{\epsilon_0 \mu_0}}$, the nontrivial solution u of $a_r(u, v) - \omega_j^2 b(u, v) = 0$, for all $v \in H_{\text{per}}^1(\Omega)$ is naturally a guided mode. Moreover, these ω_j^2 coincide with the eigenvalues of the operator S_κ defined earlier in this section.

If the eigenfrequency $\omega_j \geq \frac{|\kappa|}{\sqrt{\epsilon_0 \mu_0}}$, the second part of (2.19) defines some extra conditions that are in general not satisfied. If the extra conditions are satisfied for some nonzero solution u of $a_r(u, v) - \omega_j^2 b(u, v) = 0$, for all $v \in H_{\text{per}}^1(\Omega)$, the

function u is a guided mode. We call it an *embedded* guided mode to indicate that the associated ω_j^2 is embedded in the continuous spectrum of the operator S_κ .

2.5 Existence and Nonexistence of Guided Modes

We list a few existence results from [1, 27]. We will not give the proof for most of them, but focus on the existence for slabs symmetric in x . This type of guided modes are non-embedded guided modes.

2.5.1 Existence

The following two theorems on the existence are adapted from the proof in [1]. We simply restate the theorems from [23] without providing the proofs. Let $\mathcal{N}(\kappa)$ be the number of eigenvalues λ_j less than $\frac{|\kappa|^2}{\epsilon_0\mu_0}$.

Theorem 11 (Non-embedded guided modes). i). If $\epsilon\mu > \epsilon_0\mu_0$ on a set of positive measure and

$$\int_S \left(\frac{\epsilon}{\epsilon_0} - \frac{\mu_0}{\mu} \right) dV \geq 0, \quad (2.28)$$

then for all $\kappa \in B \setminus \{0\}$, $\mathcal{N}(\kappa) \geq 1$, i.e. there exists a guided mode at a frequency below $\frac{|\kappa|}{\sqrt{\epsilon_0\mu_0}}$.

ii). Let K be an open set in \mathcal{S} , and $\{\beta_j\}_{j=1}^\infty$ be the spectrum of the Dirichlet Laplacian operator in K (i.e. $-\nabla^2$ with Dirichlet boundary condition $u = 0$ on ∂K). If $\kappa \in B \setminus \{0\}$ and $\epsilon > \epsilon_*$, $\mu > \mu_*$ on K , with $\epsilon_*\mu_* > \beta_j \frac{\epsilon_0\mu_0}{|\kappa|^2}$, then $\mathcal{N} \geq j$, and there are at least j independent guided modes with Bloch wavenumber κ and frequency below $\frac{|\kappa|}{\sqrt{\epsilon_0\mu_0}}$.

These guided modes are robust and hold continuous dispersion relations.

Theorem 12 (Dispersion relations). i). The eigenvalues $\lambda_j(\tilde{\kappa})$ are continuous functions of $\tilde{\kappa} \in B$.

ii). If $\epsilon\mu \geq \epsilon_0\mu_0$, then for any $\hat{\kappa} \in \mathbb{R}$, the function $\lambda(s\kappa) - \frac{s^2}{\epsilon_0\mu_0}$ are nonincreasing in s for $s\kappa \in B$, and therefore $\mathcal{N}(s\kappa)$ is nondecreasing.

We are more interested in the existence of embedded guided modes. One special type is for a slab structure symmetric in x in each period.

Theorem 13 (Embedded guided modes). If $\kappa = 0$, then there exist functions $\epsilon(x, z), \mu(x, z)$ symmetric in x that admit a guided-mode frequency above the cutoff frequency $\frac{|\kappa|}{\sqrt{\epsilon_0\mu_0}}$.

Proof. It ϵ, μ are symmetric with respect to x , then the symmetric part $H_\kappa^{1, sym}(\Omega)$ and the antisymmetric part $H_\kappa^{1, ant}(\Omega)$ of the space $H_{\text{per}}^1(\Omega)$ are orthogonal with respect to a^ω and b . A function that is antisymmetric with respect to x and solves $a^\omega(u, v) - b(u, v) = 0, \forall v \in H_{\text{per}}^{1, ant}(\Omega)$ also solves $a^\omega(u, v) - b(u, v) = 0, \forall v \in H_{\text{per}}^1(\Omega)$. We can apply the min-max principle on $H_{\text{per}}^{1, ant}(\Omega)$ to obtain frequencies $\{\omega_j^{ant}\}_{j=1}^\infty$ of antisymmetric modes that form a subset of the frequencies $\{\omega_j\}_{j=1}^\infty$.

Since the eigenfrequencies are strictly decreasing to 0 in ϵ_1 or μ_1 , there exist ϵ_+, μ_+ large enough such that $0 < \epsilon_0\mu_0(\omega_j^{ant})^2 < 1$. With these parameters, we see that $\mathcal{Z}_p = \{0\}, \mathcal{Z}_l = \emptyset$. In this regime of (κ, ω) , the smallest eigenvalue ω_1^{ant} corresponds to an antisymmetric eigenfunction u . The extra condition $(\widehat{u|_\Gamma})_m = 0$ is automatically satisfied because of the antisymmetry of u . Therefore, the function u is an embedded guided mode. \square

The existence of this embedded guided mode requires the symmetry, and it is typically nonrobust: as κ is perturbed from 0, the guided mode loses its antisymmetry and vanishes. It is the interaction of the dissolution of the guided mode and the scattering wave that causes the resonance.

2.5.2 Nonexistence

Certain nonexistence results can be proved for some materials. We introduce two types of nonexistence. The first nonexistence result is from [27], and the second is adapted from [1]. We simply restate the Theorem 13 in [23] in our notations:

Theorem 14. Let ω and κ be real, and one of the following conditions be satisfied:

i). In Ω , $\epsilon_- < \epsilon(x, z) \leq \epsilon_0$, and $\mu_- < \mu(x, z) \leq \mu_0$, and

$$L(\omega^2 \epsilon_0 \mu_0 - \kappa^2)^{1/2} < \pi, \text{ if } \mathcal{Z}_p \neq \emptyset;$$

ii) There is a real number z_0 such that $\epsilon(x, z_0+z), \epsilon(x, z_0-z), \mu(x, z_0+z), \mu(x, z_0-z)$ are nondecreasing functions of z for all $x \in \mathbb{R}^2$.

Then there exists no such field $u(x, z)$ for which the variational homogeneous problem (2.18) holds. The periodic slab does not admit guided modes with the pair (κ, ω) .

2.6 Transmission Anomalies

When a periodic slab admits an embedded true guided mode at a real pair (κ_0, ω_0) , the guided mode is typically nonrobust with respect to the perturbation of parameters. Transmission anomalies can be observed when the wavenumber is perturbed slightly from κ_0 , such as numerically in [25], or when the geometry of the material coefficients ϵ, μ is perturbed. In our study, we focus on real perturbations of κ_0 . This perturbation can cause sharp downward and upward spikes in the graph of the transmission coefficient as a function of frequency ω . The spikes emerge from the frequency ω_0 and become wider as κ deviates more from κ_0 .

In this section, we present some results from [25, 26, 23] for piecewise materials. For the periodic slab discussed above, we assume in addition that in a domain $\Omega_1 \subset [-\pi, \pi] \times [-L, L]$, $\epsilon = \epsilon_1, \mu = \mu_1$ and in the exterior domain $\Omega_0 = \mathcal{S} \setminus \Omega_1$ of

Ω_1 , $\epsilon = \epsilon_0$, $\mu = \mu_0$. While we briefly list previous results from the literature, we will prove them for more general structures for which the materials are not required to be piecewise constant.

Consider the regime in which there is exactly one propagating harmonic. A left incoming incident wave $\ell(\kappa, \omega)e^{i\eta_0 z}e^{i\kappa x}$ is scattered into the reflected propagating harmonic $a(\kappa, \omega)e^{-i\eta_0 z}e^{i\kappa x}$ and a transmitted propagating harmonic $b(\kappa, \omega)e^{i\eta_0 z}e^{i\kappa x}$. Using integral representations of u on the boundary Γ [25, 23], the coefficients of the propagating spatial harmonics of the reflected and transmitted fields are seen to be analytic functions of (κ, ω) at (κ_0, ω_0) .

Let $\tilde{\kappa} = \kappa - \kappa_0$, $\tilde{\omega} = \omega - \omega_0$. Assume $\frac{\partial \ell}{\partial \omega}, \frac{\partial a}{\partial \omega}, \frac{\partial b}{\partial \omega} \neq 0$ at (κ_0, ω_0) . The Weierstraß Preparation Theorem (Theorem 6.4.5 of [13]) can be applied to obtain the following factorizations

$$\begin{aligned} a(\kappa, \omega) &= (\tilde{\omega} + r_1 \tilde{\kappa} + r_2 \tilde{\kappa}^2 + \dots)(r_0 e^{i\gamma} + r_{\tilde{\kappa}} \tilde{\kappa} + r_{\tilde{\omega}} \tilde{\omega} + O(|\tilde{\kappa}|^2 + |\tilde{\omega}|^2)), \\ b(\kappa, \omega) &= (\tilde{\omega} + t_1 \tilde{\kappa} + t_2 \tilde{\kappa}^2 + \dots)(it_0 e^{i\gamma} + t_{\tilde{\kappa}} \tilde{\kappa} + t_{\tilde{\omega}} \tilde{\omega} + O(|\tilde{\kappa}|^2 + |\tilde{\omega}|^2)), \\ \ell(\kappa, \omega) &= (\tilde{\omega} + l_1 \tilde{\kappa} + l_2 \tilde{\kappa}^2 + \dots)(1 + l_{\tilde{\kappa}} \tilde{\kappa} + l_{\tilde{\omega}} \tilde{\omega} + O(|\tilde{\kappa}|^2 + |\tilde{\omega}|^2)), \end{aligned} \quad (2.29)$$

and

$$|\ell(\kappa, \omega)| = [|\tilde{\omega} + l_1 \tilde{\kappa} + l_2 \tilde{\kappa}^2| + O(|\tilde{\kappa}|^3)] \times [1 + c_1 \tilde{\omega} + c_2 \tilde{\kappa} + O(\tilde{\kappa}^2 + \tilde{\omega}^2)].$$

In lossless transient materials, the conservation of energy implies the relation $|\ell|^2 = |a|^2 + |b|^2$, for $(\kappa, \omega) \in \mathbb{R}^2$.

In the special case of embedded guided mode shown in the previous section, the slab symmetric in x admits an antisymmetric guided mode with the pair (κ_0, ω_0) with $\kappa_0 = 0$, lying in the regime for which $\mathcal{Z}_p = \{0\}$. The relation $\ell(\kappa, \omega) = 0$ represents a complex dispersion relation

$$\omega = \omega_0 - l_1(\kappa - \kappa_0) - l_2(\kappa - \kappa_0)^2 - \dots .$$

If we assume that $\text{Im}(\ell_2) \neq 0$, then the pair (κ_0, ω_0) is the only real pair that satisfies the equation $\ell(\kappa, \omega) = 0$ and therefore admits the only true guided mode in a neighborhood of (κ_0, ω_0) .

The transmission rate

$$|T(\kappa, \omega)|^2 = \left| \frac{b(\kappa, \omega)}{\ell(\kappa, \omega)} \right|^2 = \frac{|b|^2}{|a|^2 + |b|^2} \quad (2.30)$$

characterizes the ratio of energy flux passing through the slab. If the coefficients r_n, t_n are all real numbers, at the frequencies $\omega = \omega_0 - r_1(\kappa - \kappa_0) - r_2(\kappa - \kappa_0)^2 - \dots$ and $\omega = \omega_0 - t_1(\kappa - \kappa_0) - t_2(\kappa - \kappa_0)^2 - \dots$, the transmission coefficient $|T|$ reaches the magnitude of 1 or 0, respectively.

The following approximation formula of the transmission rate holds:

$$T^2(\kappa, \omega) = \frac{t_0^2 |\tilde{\omega} + \ell_1 \kappa + t_2 \tilde{\kappa}^2|^2}{|\tilde{\omega} + \ell_1 \tilde{\kappa} + \ell_2 \tilde{\kappa}^2|^2} (1 + c_1 \tilde{\omega})^2 + \mathcal{O}(|\tilde{\kappa}| + \tilde{\omega}^2). \quad (2.31)$$

In Chapter 3, we prove the total transmission and reflection are in fact obtained for structures symmetric in z at nearby frequencies near the pair (κ_0, ω_0) .

Chapter 3

Total Transmission Resonance in Periodic Slabs

The wavenumber and frequency pair (κ, ω) can be extended to the complex domain for the study of the scattering problem near the real pair (κ_0, ω_0) that admits a nonrobust true guided mode, and certain asymptotic formula of the transmission coefficient can be proved in [23, 25, 26]. We start from this idea and prove the analyticity of the scattered field for a general periodic material with bounded measurable parameters ϵ, μ . The main contribution of this chapter is that if the frequency and wavenumber are close to those of the nonrobust guided modes, then 100% and 0% transmission is reached for structures with symmetry about a parallel axis.

3.1 Complex Extension

Assume \mathcal{Z}_ℓ is empty. We consider the modified Helmholtz equation

$$(\nabla + i\boldsymbol{\kappa}) \cdot \mu^{-1}(\nabla + i\boldsymbol{\kappa})u(x, z) + \epsilon\omega^2 u(x, z) = 0,$$

in which $\boldsymbol{\kappa} = (\kappa, 0)$, the number κ is restricted to lie in the Brillouin zone $[-\frac{1}{2}, \frac{1}{2})$, and we have

$$u(x, z) = \sum_{m=-\infty}^{\infty} (A_m^\pm e^{i\eta_m z} + B_m^\pm e^{-i\eta_m z}) e^{imx} \quad \text{for } \pm z > L. \quad (3.1)$$

For real $\omega > 0$ and κ , the square root is chosen with a branch on the negative imaginary axis, and the sign is taken such that $\eta_m = |\eta_m|$ if $\eta_m^2 > 0$ and $\eta_m = i|\eta_m|$ if $\eta_m^2 < 0$.

We will be concerned with the case of one propagating harmonic $m = 0$. This regime corresponds to real pairs (κ, ω) that lie in the diamond

$$\mathcal{D} = \{(\kappa, \omega) \in \mathbb{R}^2 : |\kappa| < \frac{1}{2} \text{ and } |\kappa| < \omega\sqrt{\epsilon_0\mu_0} < 1 - |\kappa|\}.$$

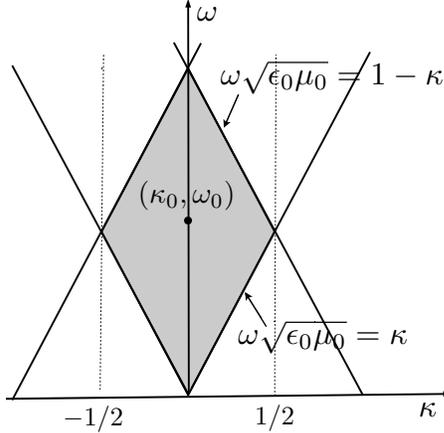


FIGURE 3.1. The diamond \mathcal{D} of one propagating diffractive order within the first Brillouin zone.

The numbers η_m are analytic functions of (κ, ω) in a complex neighborhood \mathcal{D}' of \mathcal{D} ; thus, $\mathbb{R}^2 \supset \mathcal{D} \subset \mathcal{D}' \subset \mathbb{C}^2$.

We now assume that the frequency ω^2 is perturbed from positive real line to the complex plane. We first discuss the case that ω attains a small negative imaginary part $\text{Im}(\omega) < 0$. For $m \in \mathcal{Z}_p$, the number $\eta_m = \sqrt{\epsilon_0\mu_0\omega^2 - (m + \kappa)^2}$ still has a positive real part, but it possesses a negative imaginary part. The propagating harmonic $e^{i\eta_m z} e^{i(m+\kappa)x} e^{-i\omega t}$ of w in (2.1) can be calculated

$$\begin{aligned} e^{i\eta_m z} e^{i(m+\kappa)x} e^{-i\omega t} &= e^{i[\text{Re}(\eta_m) + i\text{Im}(\eta_m)]z} e^{i(m+\kappa)x} e^{-i\omega t} \\ &= e^{-\text{Im}(\eta_m)z} e^{i\text{Re}(\eta_m)z} e^{i(m+\kappa)x} e^{\text{Im}(\omega)t} e^{-i\text{Re}(\omega)t}. \end{aligned}$$

This harmonic decays in time, is exponentially growing away from the slab, and is still outgoing. These modes are associated with leaky modes (see [18, 28, 10]).

For $m \in \mathcal{Z}_e$, by our choice of square root, the number $\eta_m = i\sqrt{(m + \kappa)^2 - \epsilon_0\mu_0\omega^2}$ will still have a positive imaginary part but has a negative real part. The evanescent harmonic $e^{i\eta_m z} e^{i(m+\kappa)x} e^{-i\omega t}$ of w in (2.1) is $e^{-\text{Im}(\eta_m)z} e^{i\text{Re}(\eta_m)z} e^{i(m+\kappa)x} e^{\text{Im}(\omega)t} e^{-i\text{Re}(\omega)t}$, and decays in time and in space but is incoming.

In the other case for which $\text{Im}(\omega) > 0$, for $m \in \mathcal{Z}_p$, η_m has a positive imaginary part; the propagating harmonic $e^{i\eta_m z} e^{i(m+\kappa)x} e^{-i\omega t}$ grows exponentially in time, de-

cays exponentially in space parameter $|z|$ and is outgoing. For $m \in \mathcal{Z}_e$, η_m has a positive real part; the evanescent harmonic $e^{i\eta_m z} e^{i(m+\kappa)x} e^{-i\omega t}$ grows exponentially in time, decays exponentially in $|z|$ and is outgoing.

We do not deal with the cases when ω decreases through a value such that $\eta_m = 0$, in which the transmission coefficient exhibits the Wood anomaly [28] [14]. Moreover, we do not discuss the perturbation of ω remaining real and κ being extended to the complex domain which is treated in [19].

The radiation condition is extended to the following *generalized outgoing condition*.

Condition 15 (Outgoing Condition). A pseudo-periodic function $\tilde{u}(x, z) = u(x, z)e^{i\kappa x}$ is said to satisfy the *outgoing condition* for the complex pair (κ, ω) , with $\text{Re}(\omega) > 0$ if and complex coefficients $\{c_m^\pm\}$ such that

$$u(x, z) = \sum_{m \in \mathcal{Z}} c_m^\pm e^{\pm i\eta_m z} e^{imx} \quad \text{for } \pm z > L.$$

True guided modes are nontrivial solutions to the Helmholtz equation that decays exponentially away from the slab. If the wavenumber κ is kept to be real and ω is allowed to be complex, the imaginary part of ω for the guided mode must be nonnegative. The following theorem is proved in [23][26].

Theorem 16 (Generalized Guided Modes). Suppose (κ, ω) is such that $\mathcal{Z}_\ell = \emptyset$ and u is a periodic equation with real wavevector κ and satisfies the modified Helmholtz equation and the generalized outgoing condition. Then $\text{Im}(\omega) \leq 0$, and $u \rightarrow 0$ as $|z| \rightarrow +\infty$ if and only if ω is real.

Proof. The Helmholtz equation gives

$$\begin{aligned}
0 &= \int_{\Omega} ((\nabla + i\boldsymbol{\kappa}) \cdot \mu^{-1}(\nabla + i\boldsymbol{\kappa})u + \omega^2 \epsilon u) \bar{u} \\
&= \int_{\Omega} (-\mu^{-1}|(\nabla + i\boldsymbol{\kappa})u|^2 + \omega^2 \epsilon |u|^2) + \int_{\Gamma} \mu_0^{-1}(\partial_n u) \bar{u} \\
&= \int_{\Omega} (-\mu^{-1}|(\nabla + i\boldsymbol{\kappa})u|^2 + \omega^2 \epsilon |u|^2) + \frac{2\pi}{\mu_0} \sum_{m \in \mathbb{Z}} i\eta_m (|c_m^-|^2 + |c_m^+|^2) e^{-2\text{Im}(\eta_m)L}, \\
&\quad \forall u \in H_{\text{per}}^1(\Omega).
\end{aligned}$$

Taking the imaginary part of this identity, we have

$$-\text{Im}(\omega^2) \int_{\Omega} \epsilon |u|^2 = \frac{2\pi}{\mu_0} \sum_{m \in \mathbb{Z}} \text{Re}(\eta_m) (|c_m^-|^2 + |c_m^+|^2) e^{-2\text{Im}(\eta_m)L} \quad (3.2)$$

Assume $\kappa \in \mathbb{R}$, ω is perturbed from positive real line to the complex plane and η_m are analytic in ω . If $\text{Im}(\omega) > 0$, then $\text{Im}(\omega^2) > 0$ but $\text{Re}(\eta_m) > 0$. This gives a contradiction on the signs of the two sides of the identity (3.2). Thus $\text{Im}(\omega) \geq 0$.

In particular, if $\text{Im}(\omega) = 0$, the identity 3.2 implies that all the coefficients c_m^-, c_m^+ vanish for all m such that $\text{Re}(\eta_m) > 0$, i.e. $m \in \mathcal{Z}_p$ and so u decays exponentially as $|z| \rightarrow \infty$. Conversely, since $\text{Im}(\omega) \leq 0$, all the harmonics $m \in \mathcal{Z}_p$ exponentially growing as $|z| \rightarrow \infty$ should all vanish, i.e. $c_m^- = c_m^+ = 0, \forall m \in \mathcal{Z}_p$. In (3.2) we let $L \rightarrow \infty$, then $\text{Im}(\omega^2) = 0$ and so $\text{Im}(\omega) = 0$. \square

3.2 Scattering and Guided Modes with Complex Extension

In the extended complex domain of the pair (κ, ω) , the problem of scattering of plane-waves by a periodic slab is the following:

Problem 17 (Scattering problem). Find a function $\tilde{u}(x, z)$ such that

$$\left\{ \begin{array}{l} \tilde{u}(x, z) = e^{i\kappa x} u(x, z), \quad u \text{ is } 2\pi\text{-periodic in } x, \\ (\nabla + i\boldsymbol{\kappa}) \cdot \mu^{-1}(\nabla + i\boldsymbol{\kappa})u(x, z) + \epsilon \omega^2 u(x, z) = 0, \quad \text{for } (x, z) \in \mathbb{R}^2, \\ u(x, z) = u^{\text{inc}}(x, z) + u^{\text{sc}}(x, z), \quad u^{\text{sc}} \text{ satisfies the outgoing condition,} \end{array} \right. \quad (3.3)$$

in which $u^{\text{inc}}(x, z) = A_0^+ e^{i\eta_0 z} + B_0^- e^{-i\eta_0 z}$.

The variational, or weak, formulation of this problem is posed in the truncated period Ω .

Problem 18 (Scattering problem, variational form). Given a pair (κ, ω) , find a function $u \in H_{\text{per}}^1(\Omega)$ such that

$$a(u, v) - \omega^2 b(u, v) = p(v), \quad \text{for all } v \in H_{\text{per}}^1(\Omega), \quad (3.4)$$

where the forms are defined in section 2.3.2.

A **generalized guided mode** is a nontrivial solution of Problem 18 with p set to zero. The condition $p = 0$ means that there is no incident field and hence the outgoing Condition 15 is satisfied. If (κ, ω) is a real pair, then the propagating harmonics in the Fourier expansion (3.1) vanish altogether and the solution is a true guided mode.

As we have introduced in Chapter 2, we write the form $a - \omega^2 b$ as

$$a(u, v) - \omega^2 b(u, v) = c_1(u, v) + c_2(u, v),$$

in which $c_1(u, v) = a(u, v) + b(u, v)$ and $c_2(u, v) = -(\omega^2 + 1)b(u, v)$ are bounded bilinear forms in $H_{\text{per}}^1(\Omega)$. If the pair (κ, ω) is in a sufficiently small neighborhood \mathcal{D}' inside the diamond \mathcal{D} , c_1 can be shown to be coercive for all (κ, ω) in \mathcal{D}' by the same argument in chapter 2. These forms are represented by bounded operators C_1 and C_2 in $H_{\text{per}}^1(\Omega)$:

$$(C_1 u, v)_{H_{\text{per}}^1(\Omega)} = c_1(u, v),$$

$$(C_2 u, v)_{H_{\text{per}}^1(\Omega)} = c_2(u, v).$$

We denote by w^{inc} the unique element of $H_{\text{per}}^1(\Omega)$ such that $(w^{\text{inc}}, v)_{H_{\text{per}}^1(\Omega)} = p(v)$, and the scattering problem becomes $(C_1 u, v) + (C_2 u, v) = (w^{\text{inc}}, v)$ for all $v \in$

$H_{\text{per}}^1(\Omega)$, or in the operator form

$$C_1 u + C_2 u = w^{\text{inc}}. \quad (3.5)$$

We have a similar lemma:

Lemma 19. The operator C_1 has a bounded inverse and C_2 is compact.

The following existence theorem of the scattered wave can be proved using the Fredholm alternative. The reader can also refer [1, Thm. 3.1] and [23, Thm. 9].

Theorem 20. For any pair $(\kappa, \omega) \in \mathcal{D}$, the scattering Problem 18 with a plane-wave source field has at least one solution and the set of solutions is a finite dimensional affine space with the dimension equal to the dimension of the space of guided modes. The far-field behavior of all solutions is identical.

Denote

$$\begin{aligned} A(\kappa, \omega) &= I + C_1(\kappa, \omega)^{-1} C_2(\kappa, \omega), \\ \psi &= u \quad \text{and} \quad \phi = C_1^{-1} w^{\text{inc}}, \end{aligned}$$

and the equation (3.5) can be written as

$$A(\kappa, \omega)\psi(\kappa, \omega) = \phi(\kappa, \omega), \quad (\text{Scattering problem in operator form}) \quad (3.6)$$

in which A is the identity plus a compact operator. A generalized guided mode in operator form is a solution of the following homogeneous problem:

$$A(\kappa, \omega)\psi(\kappa, \omega) = 0. \quad (\text{Guided mode}) \quad (3.7)$$

As proved in chapter 2, if ϵ and μ are large enough and symmetric in the x variable (i.e., about the z -axis normal to the slab), there exists an antisymmetric embedded guided mode at some point $(0, \omega_0)$ in the diamond \mathcal{D} [27]. The operator

associated with the form $a_r(u, v)$ can be viewed as a Dirichlet operator in the strip $\{(x, z) : 0 < x < \pi\}$, and the eigenvalue of the antisymmetric guided mode, which is the smallest guided mode of this operator, is therefore simple. As the wavenumber κ is perturbed from 0, the system loses its symmetry and consequently the antisymmetric guided mode vanishes. Numerical computations have verified the vanishing of this kind of guided mode in periodic cylinders, but we do not have a rigorous proof of the nonrobustness for any embedded guided modes. In fact, there do exist robust embedded guided modes and dispersion relations for structures with smaller periodic cell structures in each period. We will give a proof of such robust guided modes for pillars in the next Chapter.

3.3 Analyticity

In order to analyze the anomaly of the transmission coefficient, we establish the analyticity of the solutions to the scattering problem in (κ, ω) . In this section, we prove the analyticity of the operator and the scattering problem. With the result of analyticity, the solution to the scattering problem can be investigated by analyzing the coefficients in the complex wave number and frequency (κ, ω) within a neighborhood of (κ_0, ω_0) .

Lemma 21. The operators C_1 , C_2 , and A are analytic with respect to ω and κ if $\eta_m^2 \neq 0$.

Proof. To prove C_1 is analytic with respect to ω at (κ_0, ω_0) , we let $\omega = \omega_0 + \Delta\omega$ and show that

$$\lim_{\Delta\omega \rightarrow 0} \left\| \frac{C_1(\kappa_0, \omega_0 + \Delta\omega) - C_1(\kappa_0, \omega_0)}{\Delta\omega} - C_{1\omega} \right\| = 0$$

in operator norm, where $C_{1\omega}(\kappa_0, \omega_0) : H_{\text{per}}^1(\Omega) \rightarrow H_{\text{per}}^1(\Omega)$ is defined by

$$(C_{1\omega}u, v) = \frac{1}{\mu_0} \sum_m \frac{-i\epsilon_0\mu_0\omega_0}{\sqrt{\epsilon_0\mu_0\omega_0^2 - (m + \kappa_0)^2}} \hat{u}_m \bar{\hat{v}}_m, \forall u, v \in H_{\text{per}}^1(\Omega).$$

The partial derivatives of η_m with respect to ω are

$$\eta_{m\omega}(\kappa, \omega) = \frac{\epsilon_0 \mu_0 \omega}{\sqrt{\epsilon_0 \mu_0 \omega^2 - (m + \kappa)^2}},$$

and the joint analyticity of η_m at (κ_0, ω_0) implies that

$$\begin{aligned} \eta_m(\kappa_0, \omega_0 + \Delta\omega) &= \eta_m(\kappa_0, \omega_0) + \eta_{m\omega}(\kappa_0, \omega_0) \Delta\omega + R_m^{(2)}(\Delta\omega) \\ &= \eta_m(\kappa_0, \omega_0) + \frac{\epsilon_0 \mu_0 \omega_0}{\sqrt{\epsilon_0 \mu_0 \omega_0^2 - (m + \kappa_0)^2}} \Delta\omega + R_m^{(2)}(\Delta\omega) \end{aligned}$$

with the remainder term

$$R_m^{(2)}(\Delta\omega) = \frac{(\Delta\omega)^2}{2\pi i} \int_{\mathcal{C}_0} \frac{\eta_m(s)}{(s - \omega)(s - \omega_0)^2} ds$$

in which \mathcal{C}_0 is the circle in the complex plane centered at ω_0 with radius r_0 and $2|\Delta\omega| < r_0$. For any $s \in \mathcal{C}_0$, $|s - \omega| \geq |r_0 - |\Delta\omega|| \geq r_0 - |\Delta\omega| \geq r_0/2$. We estimate that for all $m \neq 0$,

$$|R_m^{(2)}(\Delta\omega)| \leq \frac{|\Delta\omega|^2}{2\pi} \cdot 2\pi r_0 \cdot \frac{\sup_{s \in \mathcal{C}_0} \{|\eta_m(s)|\}}{(r_0 - |\Delta\omega|)r_0^2} \leq \frac{\sup_{s \in \mathcal{C}_0} \{|\eta_m(s)|\} (|\Delta\omega|)^2}{r_0/2 \cdot r_0}.$$

In the last expression, if $m = 0$, $|\eta_m(s)| < C$ for some $C > 0$. If $m \neq 0$ and $\epsilon_0 \mu_0 \omega_0^2 - (m + \kappa)^2 \leq 0$, $|\eta_m(s)| = \sqrt{(m + \kappa)^2 - \epsilon_0 \mu_0 \omega^2} \leq (m + \kappa) \leq 2m < Cm$ for some C ; if $m \neq 0$ and $\epsilon_0 \mu_0 \omega_0^2 - (m + \kappa)^2 \geq 0$, we have $|\eta_m(s)| \leq \sqrt{\epsilon_0 \mu_0 \omega_0^2} \leq C \leq Cm$. To summarize, $|\eta_m(s)| \leq C(m + 1)$.

For some constants $C', C'' > 0$,

$$\begin{aligned} & \left| \left(\left[\frac{C_1(\kappa_0, \omega_0 + \Delta\omega) - C_1(\kappa_0, \omega_0)}{\Delta\omega} - C_{1\omega} \right] u, v \right) \right| \\ &= \left| \frac{1}{\mu_0} \sum_m (-i) \left[\frac{(\eta_m(\kappa_0, \omega_0 + \Delta\omega) - \eta_m(\kappa_0, \omega_0))}{\Delta\omega} - \frac{\epsilon_0 \mu_0 \omega_0}{\sqrt{\epsilon_0 \mu_0 \omega_0^2 - (m + \kappa_0)^2}} \right] \hat{u}_m \bar{\tilde{v}}_m \right| \\ &= \left| \frac{1}{\mu_0} \sum_m \left[\frac{R_m^{(2)}(\Delta\omega)}{\Delta\omega} \right] \hat{u}_m \bar{\tilde{v}}_m \right| \leq \frac{C}{\mu_0} |\Delta\omega| \left| \sum_m (m + 1) \hat{u}_m \bar{\tilde{v}}_m \right| \\ &\leq C' |\Delta\omega| \|u\|_{H^{1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)} \leq C'' |\Delta\omega| \|u\|_{H_{\text{per}}^1(\Omega)} \|v\|_{H_{\text{per}}^1(\Omega)} \end{aligned}$$

and

$$\begin{aligned} & \lim_{\Delta\omega \rightarrow 0} \left\| \frac{C_1(\kappa_0, \omega_0 + \Delta\omega) - C_1(\kappa_0, \omega_0)}{\Delta\omega} - C_{1\omega} \right\| \\ &= \lim_{\Delta\omega \rightarrow 0} \sup_{u, v \neq 0} \frac{\left| \left(\left[\frac{C_1(\kappa_0, \omega_0 + \Delta\omega) - C_1(\kappa_0, \omega_0)}{\Delta\omega} - C_{1\omega} \right] u, v \right) \right|}{\|u\|_{H_{per}^1(\Omega)} \|v\|_{H_{per}^1(\Omega)}} = 0. \end{aligned}$$

and hence C_1 is analytic with respect to ω .

Now we show that the operator C_1 is analytic in κ . We prove the limit

$$\lim_{\Delta\kappa \rightarrow 0} \left\| \frac{C_1(\kappa_0 + \Delta\kappa, \omega_0) - C_1(\kappa_0, \omega_0)}{\Delta\kappa} - C_{1\kappa} \right\| = 0$$

in operator norm, where $C_{1\kappa}$ is defined by

$$(C_{1\kappa}u, v) = \int_{\Omega} \frac{1}{\mu} [iu\bar{v}_x - iu_x\bar{v} + 2\kappa_0u\bar{v}] + \sum_m \frac{1}{\mu_0} \frac{i(m + \kappa_0)}{\sqrt{\epsilon_0\mu_0\omega_0^2 - (m + \kappa_0)^2}} \hat{u}_m \bar{\hat{v}}_m.$$

The partial derivatives of η_m with respect to κ are

$$\eta_{m\kappa}(\kappa, \omega) = \frac{-(m + \kappa)}{\sqrt{\epsilon_0\mu_0\omega^2 - (m + \kappa)^2}},$$

and

$$\eta_m(\kappa_0 + \Delta\kappa, \omega_0) - \eta_m(\kappa_0, \omega_0) = \frac{-(m + \kappa_0)}{\sqrt{\epsilon_0\mu_0\omega_0^2 - (m + \kappa_0)^2}} \Delta\kappa + T_m^{(2)}(\Delta\kappa),$$

with $|T_m^{(2)}(\Delta\kappa)| \leq D(m + 1)|\Delta\kappa|^2$ for some constant $D > 0$ and $|\Delta\kappa|$ sufficiently small. So

$$\begin{aligned} & ((C_1(\kappa_0 + \Delta\kappa, \omega_0)u, v) - (C_1(\kappa_0, \omega_0)u, v)) \\ &= \int_{\Omega} \frac{1}{\mu} [\nabla + i(\kappa_0 + \Delta\kappa)]u \cdot [\nabla - i(\kappa_0 + \Delta\kappa)]\bar{v} - \int_{\Omega} \frac{1}{\mu} (\nabla + i\kappa_0)u \cdot (\nabla - i\kappa_0)\bar{v} \\ & \quad + \sum_m \frac{-i}{\mu_0} [\eta_m(\kappa_0 + \Delta\kappa, \omega_0) - \eta_m(\kappa_0, \omega_0)] \hat{u}_m \bar{\hat{v}}_m \\ &= \int_{\Omega} \frac{1}{\mu} [\Delta\kappa(iu\bar{v}_x - iu_x\bar{v} + 2\kappa_0u\bar{v}) + (\Delta\kappa)^2u\bar{v}] \\ & \quad + \sum_m \frac{-i}{\mu_0} \left[\frac{-(m + \kappa_0)}{\sqrt{\epsilon_0\mu_0\omega_0^2 - (m + \kappa_0)^2}} \Delta\kappa + T_m^{(2)} \right] \hat{u}_m \bar{\hat{v}}_m \end{aligned}$$

and

$$\begin{aligned}
& \left| \left(\frac{(C_1(\kappa_0 + \Delta\kappa, \omega_0)u, v) - (C_1(\kappa_0, \omega_0)u, v)}{\Delta\kappa} \right) - (C_{1\kappa}u, v) \right| \\
&= \left| \int_{\Omega} \frac{1}{\mu} u \bar{v} \Delta\kappa + \sum_m \frac{-i T_m^{(2)}}{\mu_0 \Delta\kappa} \hat{u}_m \bar{\hat{v}}_m \right| \\
&\leq |\Delta\kappa| \left| \int_{\Omega} \frac{1}{\mu} u \bar{v} \right| + \frac{1}{\mu_0 |\Delta\kappa|} \left| \sum_m T_m^{(2)} \hat{u}_m \bar{\hat{v}}_m \right| \\
&\leq D' |\Delta\kappa| \left(\|u\|_{H_{\text{per}}^1(\Omega)} \|v\|_{H_{\text{per}}^1(\Omega)} + \|u\|_{H^{1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)} \right) \\
&\leq D'' |\Delta\kappa| \|u\|_{H_{\text{per}}^1(\Omega)} \|v\|_{H_{\text{per}}^1(\Omega)}
\end{aligned}$$

for some constant $D', D'' > 0$. Therefore

$$\begin{aligned}
& \lim_{\Delta\kappa \rightarrow 0} \left\| \frac{C_1(\kappa_0 + \Delta\kappa, \omega_0) - C_1(\kappa_0, \omega_0)}{\Delta\kappa} - C_{1\kappa} \right\| \\
&= \lim_{\Delta\kappa \rightarrow 0} \sup_{u, v \in H_{\text{per}}^1(\Omega), u, v \neq 0} \frac{\left| \frac{(C_1(\kappa_0 + \Delta\kappa, \omega_0)u, v) - (C_1(\kappa_0, \omega_0)u, v)}{\Delta\kappa} - (C_{1\kappa}u, v) \right|}{\|u\|_{H_{\text{per}}^1(\Omega)} \|v\|_{H_{\text{per}}^1(\Omega)}} = 0
\end{aligned}$$

in the operator norm and so C_1 is analytic with respect to κ .

To prove the analyticity of C_2 with respect to ω at (κ_0, ω_0) , we define an operator $C_{2\omega}(\kappa_0, \omega_0)$ by

$$(C_{2\omega}u, v)_{H_{\text{per}}^1(\Omega)} = -2\omega_0 \int_{\Omega} \epsilon u \bar{v}$$

and we have

$$\begin{aligned}
& \left(\left[\frac{C_2(\kappa_0, \omega_0 + \Delta\omega) - C_2(\kappa_0, \omega_0)}{\Delta\omega} - C_{2\omega} \right] u, v \right) \\
&= \left(-\frac{\omega_0^2 + 2\omega_0 \Delta\omega + \Delta\omega^2 - \omega_0^2}{\Delta\omega} + 2\omega_0 \right) \int_{\Omega} \epsilon u \bar{v} = -\Delta\omega \int_{\Omega} \epsilon u \bar{v}.
\end{aligned}$$

As $\Delta\omega \rightarrow 0$, this tends to 0, and thus C_2 is analytic with respect to ω .

The operator C_2 does not depend upon κ . Because C_1 is an analytic automorphism, it has an analytic inverse and, hence, A is analytic. \square

We can characterize a guided mode nonrobust to the perturbation of κ as follows. We can assume that $A(\kappa, \omega)$ has a unique and simple eigenvalue $\tilde{\ell}(\kappa, \omega)$ for all (κ, ω)

in a complex neighborhood of $(\kappa_0, \omega_0) \in \mathcal{D} \subset \mathbb{R}$, $\tilde{\ell}(\kappa_0, \omega_0) = 0$, and that $\tilde{\ell}(\kappa, \omega) \neq 0$ in any real neighborhood small enough in the real plane of (κ, ω) . The solutions of $\ell(\kappa, \omega) = 0$ are considered the generalized guided modes, but only the pair (κ_0, ω_0) is the real pair that admits a true (evanescent) guided mode. We let $\ell = c\tilde{\ell}$ for a nonzero constant c .

For any analytic source field $\phi(\kappa, \omega)$ at (κ_0, ω_0) , we now consider the scattering problem

$$A(\kappa, \omega)\psi = \ell\phi. \quad (3.8)$$

The field ψ can be proved to be analytic and the values of $\ell(\kappa, \omega)$ connects generalized guided modes on the complex dispersion relation $\ell(\kappa, \omega) = 0$ with scattered waves at $\ell(\kappa, \omega) \neq 0$. The following proof is adapted from [23, §5.2].

Theorem 22. The simple eigenvalue $\tilde{\ell}$ is analytic at (κ_0, ω_0) , and, for any source field ϕ that is analytic at (κ_0, ω_0) , the solution $\psi(\kappa, \omega)$ is analytic at (κ_0, ω_0) .

Proof. To prove that for any source field, the solution to the scattering problem exists and is analytic, we introduce the Riesz projection

$$P_1(\kappa, \omega) = \frac{1}{2\pi i} \oint_C (\lambda I - A(\kappa, \omega))^{-1} d\lambda, \quad (3.9)$$

where C is a sufficiently small circle centered at 0 in the complex plain. The Riesz projection is jointly analytic in (κ, ω) at (κ_0, ω_0) , it commutes with $A(\kappa, \omega)$, and its image is the one-dimensional eigenspace of the operator $A(\kappa, \omega)$ corresponding to the eigenvalue $\tilde{\ell}(\kappa, \omega)$ if it is in the circle. The identity I can be decomposed to P_1 and its complement $P_2 = I - P_1$ both analytic at (κ_0, ω_0) .

We define another operator $\tilde{A} = P_1 + AP_2$, for which $\tilde{A} = P_1 + (I + C_1^{-1}C_2)P_2 = I + C_1^{-1}C_2P_2$, where $C_1^{-1}C_2P_2$ is compact. In a neighborhood of (κ_0, ω_0) the eigenvalue of $P_1\tilde{A}$ is 1 and all eigenvalues of \tilde{A} are bounded away from zero uniformly.

Note that the analytic Fredholm Theorem (Theorem VI.14 of [22]) guarantees the analyticity of the inverse of the operators of the form $I + C(\kappa, \omega)$ with C compact, provided that there are no singular pairs (κ, ω) in the neighborhood such that $I + C(\kappa, \omega)$ is non-invertible. The map \tilde{A} is hence analytically invertible in a neighborhood of (κ_0, ω_0) , and

$$AP_2\tilde{A}^{-1}P_2 = P_2.$$

Let $\psi(\kappa_0, \omega_0)$ be an eigenfunction of $A(\kappa_0, \omega_0)$ corresponding to the eigenvalue $\tilde{\ell}(\kappa_0, \omega_0)$, and $\hat{\psi}(\kappa, \omega) = P_1(\kappa, \omega)\psi(\kappa_0, \omega_0)$ be an analytic eigenfunction corresponding to the eigenvalue $\tilde{\ell}(\kappa, \omega)$ of A . For any analytic source vector ϕ , we decompose $\phi = P_1\phi + P_2\phi$ and let

$$P_1\phi = \alpha\hat{\psi},$$

$$P_2\phi = \phi_2.$$

By the analyticity of P_1 and P_2 , the fields $\alpha\hat{\psi}$ and ϕ_2 are analytic. If we let

$$\psi = c\alpha\hat{\psi} + \ell\tilde{A}^{-1}\phi_2,$$

we can verify that

$$A\psi = \begin{pmatrix} \tilde{\ell} & 0 \\ 0 & AP_2 \end{pmatrix} \begin{pmatrix} c\alpha\hat{\psi} \\ \ell\tilde{A}^{-1}\phi_2 \end{pmatrix} = \begin{pmatrix} \ell\alpha\hat{\psi} \\ \ell\phi_2 \end{pmatrix} = \ell\phi$$

So ψ is the solution to the scattering problem, and all we need to show is the analyticity of α and ℓ .

To show that α and ℓ are analytic, let $\beta(\kappa, \omega)$ be such that

$$P_1(\kappa_0, \omega_0)\hat{\psi}(\kappa, \omega) = \beta(\kappa, \omega)\psi(\kappa_0, \omega_0)$$

The analyticity of $\hat{\psi}(\kappa, \omega)$ implies that β is analytic, and $\beta(\kappa_0, \omega_0) = 1$. Observing that

$$P_1(\kappa_0, \omega_0)A(\kappa, \omega)\hat{\psi}(\kappa, \omega) = \tilde{\ell}(\kappa, \omega)\beta(\kappa, \omega)\hat{\psi}(\kappa_0, \omega_0),$$

$$P_1(\kappa_0, \omega_0)\alpha(\kappa, \omega)\hat{\psi}(\kappa, \omega) = \alpha(\kappa, \omega)\beta(\kappa, \omega)\hat{\psi}(\kappa_0, \omega_0),$$

we see the analyticity of $\tilde{\ell}\beta$ and $\alpha\beta$. Since β is analytic and nonzero at (κ_0, ω_0) , the functions $\tilde{\ell}(\kappa, \omega)$ and α are analytic and so ℓ is, too. \square

3.4 Main Theorem: Total Resonant Transmission and Reflection for Symmetric Slabs

We assume that the structure is symmetric with respect to the x -axis, or equivalently, $\epsilon(x, -z) = \epsilon(x, z)$, $\mu(x, -z) = \mu(x, z)$ for all x . As we have discussed earlier in this chapter, we assume that $\ell(\kappa, \omega)$ is a simple eigenvalue in a complex neighborhood \mathcal{D}' of the real point at (κ_0, ω_0) and that $\ell(\kappa_0, \omega_0) = 0$.

3.4.1 The Reduced Scattering Matrix

Consider the scattering problem with the analytic incident field $e^{i(\kappa x + \eta_0 z)}$ on the left of the slab. From the existence Theorem 20 of the solution to the scattering problem, the scattered field always exists up to a linear superposition of evanescent harmonics, and if $\ell(\kappa, \omega) \neq 0$, the solution is unique. The propagating components of the periodic part u give rise to the following reflection and transmission coefficients R and T ,

$$u = e^{i\eta_0 z} + Re^{-i\eta_0 z} + \sum_{m \neq 0} c_m^- e^{i(mx - \eta_m z)} \quad \text{for } z \leq L,$$

$$u = Te^{i\eta_0 z} + \sum_{m \neq 0} c_m^+ e^{i(mx + \eta_m z)} \quad \text{for } z \geq L.$$

Since the slab considered here is symmetric about the x -axis, an analytic incident field $e^{i(\kappa x - \eta_0 z)}$ from the right of the slab produces identical reflection and transmission coefficients. Thus, the *reduced scattering matrix* for the structure can be

written as

$$S(\kappa, \omega) = \begin{bmatrix} T(\kappa, \omega) & R(\kappa, \omega) \\ R(\kappa, \omega) & T(\kappa, \omega) \end{bmatrix},$$

which gives the outward propagating components in terms of the inward propagating components in the expression (3.1) via $S(A_0^-, B_0^+)^T = (B_0^-, A_0^+)^T$. To do the analysis in this section, we take the incident field from the left to be $\ell(\kappa, \omega)e^{i(\kappa x + \eta_0 z)}$, which renders a reflected far field $a(\kappa, \omega)e^{i(\kappa x - \eta_0 z)}$ for $z \rightarrow -\infty$ and a transmitted far field $b(\kappa, \omega)e^{i(\kappa x + \eta_0 z)}$ for $z \rightarrow \infty$, with coefficients

$$a = \ell R, \quad b = \ell T. \quad (3.10)$$

The incident field $\ell(\kappa, \omega)e^{i(\kappa x - \eta_0 z)}$ from the right results in a reflected far field $a(\kappa, \omega)e^{i(\kappa x + \eta_0 z)}$ for $z \rightarrow \infty$ and a transmitted far field $b(\kappa, \omega)e^{i(\kappa x - \eta_0 z)}$ for $z \rightarrow -\infty$, with coefficients also given by (3.10).

Lemma 23. The coefficients $a(\kappa, \omega)$ and $b(\kappa, \omega)$ are analytic in κ and ω .

Proof. The analyticity of the incident field $\ell(\kappa, \omega)e^{i(\kappa x + \eta_0 z)}$ implies the analyticity of the source field $\ell(\kappa, \omega)\phi(\kappa, \omega)$ in the equation $A(\kappa, \omega)\psi = \ell\phi$ and hence, by Theorem 22, also the analyticity of the solution field $\psi(\kappa, \omega) = u(x, z; \kappa, \omega)$ in $H_{\text{per}}^1(\Omega)$. The coefficients $a(\kappa, \omega)$ and $b(\kappa, \omega)$ of ψ are given by

$$\begin{aligned} a(\kappa, \omega) &= \frac{e^{i\eta_0(\kappa, \omega)L}}{2\pi} \int_0^{2\pi} u(x, -L; \kappa, \omega) dx, \\ b(\kappa, \omega) &= \frac{e^{-i\eta_0(\kappa, \omega)L}}{2\pi} \int_0^{2\pi} u(x, L; \kappa, \omega) dx, \end{aligned}$$

and since $\eta_0(\kappa, \omega)$ is analytic and $u \mapsto \int_0^{2\pi} u(x, \pm L) dx$ are bounded linear functionals on $H_{\text{per}}^1(\Omega)$, both a and b are analytic. \square

If $\ell \neq 0$, we can represent the scattering matrix by the ratios of functions a, b, ℓ analytic at (κ_0, ω_0) in \mathcal{D} .

$$S(\kappa, \omega) = \frac{1}{\ell(\kappa, \omega)} \begin{bmatrix} b(\kappa, \omega) & a(\kappa, \omega) \\ a(\kappa, \omega) & b(\kappa, \omega) \end{bmatrix}. \quad (3.11)$$

For real wavenumber and frequency (κ, ω) , the scattering matrix is well known in scattering theory to be unitary

$$\begin{cases} |\ell|^2 = |a|^2 + |b|^2, \\ a\bar{b} + \bar{a}b = 0. \end{cases} \quad (3.12)$$

At the point (κ_0, ω_0) , this also implies that

$$\ell(\kappa_0, \omega_0) = a(\kappa_0, \omega_0) = b(\kappa_0, \omega_0) = 0,$$

which corresponds to the guided modes at (κ_0, ω_0) with no propagating harmonics at present.

In this section, we analyze the generic case

$$\frac{\partial \ell}{\partial \omega} \neq 0, \quad \frac{\partial a}{\partial \omega} \neq 0, \quad \frac{\partial b}{\partial \omega} \neq 0 \quad \text{at } (\kappa_0, \omega_0). \quad (3.13)$$

Let $\tilde{\kappa} = \kappa - \kappa_0$ and $\tilde{\omega} = \omega - \omega_0$. With the appropriate choice of c in $\ell = c\tilde{\ell}$, by the Weierstraß Preparation Theorem, the coefficients have the following factorizations:

$$\begin{aligned} a(\kappa, \omega) &= (\tilde{\omega} + r_1\tilde{\kappa} + r_2\tilde{\kappa}^2 + \cdots)(r_0e^{i\gamma} + r_{\tilde{\kappa}}\tilde{\kappa} + r_{\tilde{\omega}}\tilde{\omega} + O(|\tilde{\kappa}|^2 + |\tilde{\omega}|^2)), \\ b(\kappa, \omega) &= (\tilde{\omega} + t_1\tilde{\kappa} + t_2\tilde{\kappa}^2 + \cdots)(it_0e^{i\gamma} + t_{\tilde{\kappa}}\tilde{\kappa} + t_{\tilde{\omega}}\tilde{\omega} + O(|\tilde{\kappa}|^2 + |\tilde{\omega}|^2)), \\ \ell(\kappa, \omega) &= (\tilde{\omega} + \ell_1\tilde{\kappa} + \ell_2\tilde{\kappa}^2 + \cdots)(1 + \ell_{\tilde{\kappa}}\tilde{\kappa} + \ell_{\tilde{\omega}}\tilde{\omega} + O(|\tilde{\kappa}|^2 + |\tilde{\omega}|^2)), \end{aligned} \quad (3.14)$$

in which $0 < r_0 < 1$ and either $0 < t_0 < 1$ or $-1 < t_0 < 0$. Notice that in this factorization, the same unitary number $e^{i\gamma}$ appears in the second factors of both a and b . This rises from the second expression of the unitarity property (3.12).

There are some basic properties of the coefficients from [23, 26].

Lemma 24. The following relations hold among the coefficients in the form (3.14):

- i) $r_0^2 + t_0^2 = 1$,
- ii) $\ell_1 = r_1 = t_1 \in \mathbb{R}$,
- iii) $\text{Im}(\ell_2) \geq 0$,
- iv) $\ell_2 \in \mathbb{R} \iff r_2 = t_2 \in \mathbb{R} \iff \ell_2 = r_2 = t_2 \in \mathbb{R}$.

We assume that $\text{Im}(\ell_2) > 0$. This assumption is a sufficient condition for the nonrobustness of the guided mode, because (κ_0, ω_0) is an isolated point of $\ell(\kappa, \omega) = 0$ in the real region \mathcal{D} .

Lemma 25. Under the assumptions (3.13) and $\text{Im}(\ell_2) > 0$, one of the following alternatives is satisfied:

- i) r_2 and t_2 are distinct real numbers;
- ii) $r_2 = t_2 \notin \mathbb{R}$ and either $\ell_2 = r_2 = t_2$ or $\ell_2 = \bar{r}_2 = \bar{t}_2$.

Proof. We first compare the coefficients in $|\ell|^2 - |a|^2 - |b|^2 = 0$ using the expansions (3.14) and keeping in mind that $\ell_1 = r_1 = t_1$. The coefficients of $\tilde{\kappa}^2$, $\tilde{\kappa}^2\tilde{\omega}$, $\tilde{\kappa}\tilde{\omega}^2$ and $\tilde{\omega}^3$ are

$$\begin{aligned} & \ell_1[(\ell_2 + \bar{\ell}_2) + \ell_1(\ell_{\tilde{\kappa}} + \bar{\ell}_{\tilde{\kappa}}) - r_0^2(r_2 + \bar{r}_2) \\ & \quad - \ell_1 r_0(r_{\tilde{\kappa}} e^{-i\gamma} + \bar{r}_{\tilde{\kappa}} e^{i\gamma}) - t_0^2(t_2 + \bar{t}_2) + i\ell_1 t_0(t_{\tilde{\kappa}} e^{-i\gamma} - \bar{t}_{\tilde{\kappa}} e^{i\gamma})], \end{aligned}$$

$$\begin{aligned} & (\ell_2 + \bar{\ell}_2) + 2\ell_1(\ell_{\tilde{\kappa}} + \bar{\ell}_{\tilde{\kappa}}) + \ell_1^2(\ell_{\tilde{\omega}} + \bar{\ell}_{\tilde{\omega}}) - r_0^2(r_2 + \bar{r}_2) - 2\ell_1 r_0(r_{\tilde{\kappa}} e^{-i\gamma} + \bar{r}_{\tilde{\kappa}} e^{i\gamma}) - \ell_1^2 r_0 \\ & \quad \times (r_{\tilde{\omega}} e^{-i\gamma} + \bar{r}_{\tilde{\omega}} e^{i\gamma}) - t_0^2(t_2 + \bar{t}_2) - 2i\ell_1 t_0(t_{\tilde{\kappa}} e^{-i\gamma} - \bar{t}_{\tilde{\kappa}} e^{i\gamma}) + i\ell_1^2 t_0(t_{\tilde{\omega}} e^{-i\gamma} - \bar{t}_{\tilde{\omega}} e^{i\gamma}), \end{aligned}$$

$$\begin{aligned} & (\ell_{\tilde{\kappa}} + \bar{\ell}_{\tilde{\kappa}}) + 2\ell_1(\ell_{\tilde{\omega}} + \bar{\ell}_{\tilde{\omega}}) - r_0(r_{\tilde{\kappa}} e^{-i\gamma} + \bar{r}_{\tilde{\kappa}} e^{i\gamma}) - 2\ell_1 r_0(r_{\tilde{\omega}} e^{-i\gamma} + \bar{r}_{\tilde{\omega}} e^{i\gamma}) \\ & \quad + it_0(t_{\tilde{\kappa}} e^{-i\gamma} - \bar{t}_{\tilde{\kappa}} e^{i\gamma}) + 2i\ell_1 t_0(t_{\tilde{\omega}} e^{-i\gamma} - \bar{t}_{\tilde{\omega}} e^{i\gamma}), \end{aligned}$$

and

$$(\ell_{\tilde{\omega}} + \bar{\ell}_{\tilde{\omega}}) - r_0(r_{\tilde{\omega}} e^{-i\gamma} + \bar{r}_{\tilde{\omega}} e^{i\gamma}) + it_0(t_{\tilde{\omega}} e^{-i\gamma} - \bar{t}_{\tilde{\omega}} e^{i\gamma}).$$

Define the real quantities $A = r_{\bar{\kappa}}e^{-i\gamma} + \bar{r}_{\bar{\kappa}}e^{i\gamma}$, $B = i(t_{\bar{\kappa}}e^{-i\gamma} - \bar{t}_{\bar{\kappa}}e^{i\gamma})$, $C = r_{\bar{\omega}}e^{-i\gamma} + \bar{r}_{\bar{\omega}}e^{i\gamma}$, $D = i(t_{\bar{\omega}}e^{-i\gamma} - \bar{t}_{\bar{\omega}}e^{i\gamma})$. Since $\text{Re}(|\ell|^2 - |a|^2 - |b|^2) = 0$, the real parts of the coefficients of $\tilde{\kappa}^3$, $\tilde{\kappa}^2\tilde{\omega}$, $\tilde{\kappa}\tilde{\omega}^2$, and $\tilde{\omega}^3$ are all 0, that is,

$$\begin{aligned} \ell_1[2\text{Re}(\ell_2) + 2\ell_1\text{Re}(\ell_{\bar{\kappa}}) - 2r_0^2\text{Re}(r_2) - \ell_1r_0A - 2t_0^2\text{Re}(t_2) + \ell_1t_0B] &= 0, \\ 2\text{Re}(\ell_2) + 2\ell_1\text{Re}(\ell_{\bar{\kappa}}) + 2\ell_1^2\text{Re}(\ell_{\bar{\omega}}) - 2r_0^2\text{Re}(r_2) - 2\ell_1r_0A + \\ &\quad - \ell_1^2r_0C - 2t_0^2\text{Re}(t_2) + 2\ell_1t_0B + \ell_1^2t_0D = 0, \\ 2\text{Re}(\ell_{\bar{\kappa}}) + 4\ell_1\text{Re}(\ell_{\bar{\omega}}) - r_0A - 2\ell_1r_0C + t_0B + 2\ell_1t_0D &= 0, \\ 2\text{Re}(\ell_{\bar{\omega}}) - r_0C + t_0D &= 0. \end{aligned}$$

This linear system can be reduced to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r_0 & t_0 & 0 & 0 \\ 0 & 0 & r_0 & t_0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} \ell_1[2\text{Re}(\ell_2) - 2r_0^2\text{Re}(r_2) - 2t_0^2\text{Re}(t_2)] \\ 2\text{Re}(\ell_2) - 2r_0^2\text{Re}(r_2) - 2t_0^2\text{Re}(t_2) \\ 2\text{Re}(\ell_{\bar{\kappa}}) \\ 2\text{Re}(\ell_{\bar{\omega}}) \end{pmatrix},$$

which implies $\text{Re}(\ell_2) - r_0^2\text{Re}(r_2) - t_0^2\text{Re}(t_2) = 0$.

The coefficients of k^3 , $k^2\tilde{\omega}$, $k\tilde{\omega}^2$, and $\tilde{\omega}^3$ in $a\bar{b}$ are

$$\begin{aligned} &-i\ell_1r_0t_0(r_2 + \bar{t}_2) + \ell_1(r_0\bar{t}_{\bar{\kappa}}e^{i\gamma} - it_0r_{\bar{\kappa}}e^{-i\gamma}), \\ &-ir_0t_0(r_2 + \bar{t}_2) + 2\ell_1(r_0\bar{t}_{\bar{\kappa}}e^{i\gamma} - it_0r_{\bar{\kappa}}e^{-i\gamma}) + \ell_1^2(r_0\bar{t}_{\bar{\omega}}e^{i\gamma} - it_0r_{\bar{\omega}}e^{-i\gamma}), \\ &(r_0\bar{t}_{\bar{\kappa}}e^{i\gamma} - it_0r_{\bar{\kappa}}e^{-i\gamma}) + 2\ell_1(r_0\bar{t}_{\bar{\omega}}e^{i\gamma} - it_0r_{\bar{\omega}}e^{-i\gamma}), \\ &r_0\bar{t}_{\bar{\omega}}e^{i\gamma} - it_0r_{\bar{\omega}}e^{-i\gamma}. \end{aligned}$$

Since $a\bar{b}$ is purely imaginary, so are these coefficients, and from the second and third, we obtain $r_2 + \bar{t}_2 \in \mathbb{R}$.

Along the curve $\{(\tilde{\kappa}, \tilde{\omega}) \in \mathbb{R}^2 : \tilde{\omega} + \ell_1\tilde{\kappa} = 0\}$, the coefficients of $\tilde{\kappa}^4$ in $|\ell|^2 - |a|^2 - |b|^2$ and in $a\bar{b}$ are $|\ell_2|^2 - r_0^2|r_2|^2 - t_0^2|t_2|^2$ and $-ir_0r_2t_0\bar{t}_2$. This yields $|\ell_2|^2 = r_0^2|r_2|^2 + t_0^2|t_2|^2$ and $r_2\bar{t}_2 \in \mathbb{R}$.

The relations $r_2 + \bar{t}_2 \in \mathbb{R}$ and $r_2 \bar{t}_2 \in \mathbb{R}$ imply that either $r_2, t_2 \in \mathbb{R}$ or $r_2 = t_2$. Because, by Lemma 24, $\ell_2 \in \mathbb{R} \Leftrightarrow r_2 = t_2 \in \mathbb{R}$ and because we assume $\text{Im}(\ell_2) > 0$, the numbers r_2, t_2 cannot be identical real numbers. In the second case with $r_2 = t_2 \notin \mathbb{R}$, from $\text{Re}(\ell_2) = r_0^2 \text{Re}(r_2) + t_0^2 \text{Re}(t_2)$, $|\ell_2|^2 = r_0^2 |r_2|^2 + t_0^2 |t_2|^2$, and $r_0^2 + t_0^2 = 1$, we find that $\text{Re}(\ell_2) = \text{Re}(r_2) = \text{Re}(t_2)$ and $|\text{Im}(\ell_2)| = |\text{Im}(r_2)| = |\text{Im}(t_2)|$. \square

The second alternative of this Lemma seems like an unlikely case. We do not have a proof to rule out this situation but have not observed any numerical examples that support it.

3.4.2 Resonant Transmission

The main result given in the Theorem 26 in this section of this chapter is that, in the first alternative of Lemma 25, all the coefficients r_n and t_n are real numbers. Thus the coefficients a and b vanish along real-analytic curves in \mathcal{D} given by

$$\omega = \omega_0 - \ell_1(\kappa - \kappa_0) - r_2(\kappa - \kappa_0)^2 - \dots, \quad (a = 0) \quad (3.15)$$

$$\omega = \omega_0 - \ell_1(\kappa - \kappa_0) - t_2(\kappa - \kappa_0)^2 - \dots. \quad (b = 0) \quad (3.16)$$

These curves are the loci of 100% and 0% transmission, respectively. In the factorizations and the above real-analytic curves, the first order coefficient are both ℓ_1 but $r_2 \neq t_2 \in \mathbb{R}$, and therefore the loci intersect with each other tangentially at (κ_0, ω_0) but differ quadratically. In the graph of transmission coefficient *vs.* frequency ω , these curves define the two frequencies that are quadratically close to each other and produce the transmission spikes on the graph.

Theorem 26 (Total transmission and reflection). Given a two-dimensional lossless periodic slab that is symmetric about a line parallel to it, let (κ_0, ω_0) be a wavenumber-frequency pair in the regime \mathcal{D} of exactly one propagating harmonic at which the slab admits a guided mode, that is $\ell(\kappa_0, \omega_0) = 0$. Assume in addi-

tion the generic condition (3.13) and that $\text{Im}(\ell_2) > 0$ in the expansion of ℓ in (3.14). Then either the transmission coefficient is continuous at (κ_0, ω_0) or it attains the magnitudes of 0 and 1 on two distinct real-analytic curves that intersect quadratically at (κ_0, ω_0) . Specifically,

i) If $r_2 \neq t_2$, then r_n and t_n are real for all n . Moreover, $a(\kappa_0, \omega)/\ell(\kappa_0, \omega)$ and $b(\kappa_0, \omega)/\ell(\kappa_0, \omega)$ can be extended to continuous functions of ω in a real neighborhood of ω_0 with values $r_0 e^{i\gamma}$ and $it_0 e^{i\gamma}$ at ω_0 .

ii) If $r_2 \neq t_2$, let c denote either a or b and let f denote the corresponding function from the pair (3.15,3.16). Then f can be extended to a real analytic function $g(\kappa)$ on an interval (κ_1, κ_2) containing κ_0 such that the graph of g is in \mathcal{D} and for each $i = 1, 2$, the limit $g(\kappa_i) := \lim_{\kappa \rightarrow \kappa_i} g(\kappa)$ exists and either $(\kappa_i, g(\kappa_i))$ is on the boundary of \mathcal{D} or $\frac{\partial c}{\partial \omega}(\kappa_i, g(\kappa_i)) = 0$.

iii) If $r_2 = t_2$, then $|a/\ell|$ and $|b/\ell|$ can be extended to continuous functions in a real neighborhood of (κ_0, ω_0) with values $|r_0|$ and $|t_0|$ at (κ_0, ω_0) .

Proof. i) Assume $r_2 \neq t_2 \in \mathbb{R}$. Assuming $r_2, \dots, r_n \in \mathbb{R}$ for $n \geq 2$, we will show that $r_{n+1} \in \mathbb{R}$. For $(\tilde{\kappa}, \tilde{\omega})$ subject to the relation $\tilde{\omega} + \ell_1 \tilde{\kappa} + r_2 \tilde{\kappa}^2 + r_3 \tilde{\kappa}^3 + \dots + r_n \tilde{\kappa}^n = 0$,

$$\begin{aligned} \frac{a}{b} &= \frac{r_{n+1} \tilde{\kappa}^{n+1} + r_{n+2} \tilde{\kappa}^{n+2} + \dots}{(t_2 - r_2) \tilde{\kappa}^2 + (t_3 - r_3) \tilde{\kappa}^3 + \dots + (t_n - r_n) \tilde{\kappa}^n + t_{n+1} \tilde{\kappa}^{n+1} + \dots} \left[\frac{r_0 e^{i\gamma} + O(|\tilde{\kappa}|)}{it_0 e^{i\gamma} + O(|\tilde{\kappa}|)} \right] \\ &= \tilde{\kappa}^{(n-1)} \frac{r_{n+1} + r_{n+2} \tilde{\kappa} + \dots}{(t_2 - r_2) - (t_3 - r_3) \tilde{\kappa} + \dots} \left[\frac{r_0 e^{i\gamma} + O(|\tilde{\kappa}|)}{it_0 e^{i\gamma} + O(|\tilde{\kappa}|)} \right]. \end{aligned}$$

Because $a\bar{b} + \bar{a}b = 0$, $a/b \in i\mathbb{R}$, so that

$$\left[\frac{r_{n+1} + r_{n+2} \tilde{\kappa} + \dots}{(t_2 - r_2) + (t_3 - r_3) \tilde{\kappa} + \dots} \right] \left[\frac{r_0 e^{i\gamma} + O(\tilde{\kappa})}{it_0 e^{i\gamma} + O(\tilde{\kappa})} \right] \in i\mathbb{R}.$$

Letting $\tilde{\kappa} \rightarrow 0$ yields

$$\frac{r_{n+1}}{t_2 - r_2} \cdot \frac{r_0}{it_0} \in i\mathbb{R},$$

which implies that $r_{n+1} \in \mathbb{R}$. The proof that $t_n \in \mathbb{R}$ for all n is analogous.

To prove the second statement, one sets $\tilde{\kappa} = \kappa - \kappa_0 = 0$ and observes that the ratios a/ℓ and b/ℓ have limiting values of $r_0 e^{i\gamma}$ and $it_0 e^{i\gamma}$, respectively, as $\tilde{\omega} \rightarrow 0$, or $\omega \rightarrow \omega_0$.

ii) Define the set

$$\mathcal{G} := \{g : (\kappa_-, \kappa_+) \rightarrow \mathbb{R} \mid \kappa_0 \in (\kappa_-, \kappa_+), g \text{ is real analytic, } a(\kappa, g(\kappa)) = 0, \Gamma(g) \in \mathcal{D}\},$$

in which $\Gamma(g)$ is the graph of g . are the lower and upper sides of the diamond \mathcal{D} , and the numbers

$$\kappa_1 := \inf\{\kappa_- \mid g \in \mathcal{G}\},$$

$$\kappa_2 := \sup\{\kappa_+ \mid g \in \mathcal{G}\}.$$

By virtue of the function (3.15), which belongs to \mathcal{G} , $\kappa_1 < \kappa_0 < \kappa_2$. Standard arguments show that any two functions from \mathcal{G} coincide on the intersection of their domains, and one obtains thereby a maximal extension $g \in \mathcal{G}$ of (3.15) with domain (κ_1, κ_2) . We now show that $\lim_{\kappa \nearrow \kappa_2} g(\kappa)$ exists. Set $\omega_- = \liminf_{\kappa \nearrow \kappa_2} g(\kappa)$ and $\omega_+ = \limsup_{\kappa \nearrow \kappa_2} g(\kappa)$. Because of the continuity of g , the segment $(\kappa_2, [\omega_-, \omega_+])$ in $\overline{\mathcal{D}}$ is in the closure of the graph of g , on which a vanishes. Thus $a(\kappa_2, \omega) = 0 \forall \omega \in [\omega_-, \omega_+]$. Moreover, for each $\omega \in (\omega_-, \omega_+)$, there is a sequence of points (κ_j, ω) with $\kappa_j \nearrow \kappa_2$ and $a(\kappa_j, \omega) = 0$ from which we infer that $\partial^n a / \partial \kappa^n(\kappa_2, \omega) = 0 \forall n \in \mathbb{N}$ and hence that $\partial^{m+n} a / \partial \omega^m \partial \kappa^n(\kappa_2, \omega) = 0 \forall m, n \in \mathbb{N} \forall \omega \in (\omega_-, \omega_+)$. If (ω_-, ω_+) is nonempty, then a must vanish in \mathcal{D} , which is untenable in view of the assumption that $\partial a / \partial \omega(\kappa_0, \omega_0) \neq 0$. This proves that $\omega_- = \omega_+$ so that $\omega_2 := \lim_{\kappa \nearrow \kappa_2} g(\kappa)$ exists. If $|\kappa| / \sqrt{\epsilon_0 \mu_0} < \omega_2 < (1 - |\kappa|) / \sqrt{\epsilon_0 \mu_0}$ and $\partial a / \partial \omega(\kappa_2, \omega_2) \neq 0$, the implicit function theorem provides an element of \mathcal{G} with $\kappa_+ > \kappa_2$, which is not compatible with the definition of κ_2 . Analogous arguments apply to the endpoint κ_1 and to the function b .

iii) If $r_2 = t_2$, then by Lemma 25, $t_2 = \ell_2$ or $t_2 = \bar{\ell}_2$. Keeping in mind that $\text{Im}(\ell_2) > 0$ and $\ell_1 \in \mathbb{R}$ and restricting to $(\tilde{\kappa}, \tilde{\omega}) \in \mathbb{R}^2$,

$$\begin{aligned} \lim_{\tilde{\kappa}, \tilde{\omega} \rightarrow 0} \left| \frac{b}{\ell} \right| &= \lim_{\tilde{\kappa}, \tilde{\omega} \rightarrow 0} \left| \frac{\tilde{\omega} + \ell_1 \tilde{\kappa} + t_2 \tilde{\kappa}^2 + t_3 \tilde{\kappa}^3 + \dots}{\tilde{\omega} + \ell_1 \tilde{\kappa} + \ell_2 \tilde{\kappa}^2 + \ell_3 \tilde{\kappa}^3 + \dots} \right| \left| \frac{it_0 e^{i\gamma} + O(|\tilde{\omega}| + |\tilde{\kappa}|)}{1 + O(|\tilde{\omega}| + |\tilde{\kappa}|)} \right| \\ &= \lim_{\tilde{\kappa}, \tilde{\omega} \rightarrow 0} |t_0| \left| \frac{\tilde{\omega} + \ell_1 \tilde{\kappa} + t_2 \tilde{\kappa}^2}{\tilde{\omega} + \ell_1 \tilde{\kappa} + \ell_2 \tilde{\kappa}^2} \right|. \end{aligned}$$

Whether t_2 is equal to ℓ_2 or $\bar{\ell}_2$, the second factor of the last expression is equal to unity, and we obtain $\lim_{\tilde{\kappa}, \tilde{\omega} \rightarrow 0} |b/\ell| = |t_0|$. Similarly, one shows that $\lim_{\tilde{\kappa}, \tilde{\omega} \rightarrow 0} |a/\ell| = |r_0|$. \square

The last part of this Theorem shows that the transmission coefficient is continuous at (κ_0, ω_0) and therefore has no spike on the graph for the alternative $r_2 = t_2 \notin \mathbb{R}$ of Lemma 25. However, no existing numerical examples lie in this alternative and so it is hoped that this alternative can be ruled out. Here we just state that if the transmission coefficient is discontinuous at the special value (κ_0, ω_0) , then it must attain the magnitude of 1 and 0, achieving full and zero transmission, along the real-analytic curves (3.15,3.16).

In our example of antisymmetric nonrobust guided modes, the wavenumber $\kappa = 0$ and the slab is symmetric in the x direction. The guided mode is a standing wave, and the symmetry of $\ell(\kappa, \omega)$ and the Helmholtz equation in κ implies that $\ell_1 = 0$. We have not seen any guided mode nonrobust to the perturbation of κ for $\kappa_0 \neq 0$ or for structures nonsymmetric in x . But for discrete models in [20] and [24], the existence of certain guided modes with ℓ_1 not necessarily zero have been proved. These guided modes are all amenable to our analysis and the Theorem 26 in this chapter. In the transmission graph shown later in this chapter, we compare the two cases in Fig. 3.2 and 3.3.

3.5 Nongeneric Resonant Transmission

In the generic case discussed in the previous section, we assume that the partial derivatives $\frac{\partial a}{\partial \omega}, \frac{\partial b}{\partial \omega} \neq 0$. Let $\kappa = \kappa_0$ and $\omega \rightarrow \omega_0$, then the coefficients have limit norms $|r_0|$ and $|t_0|$, lying strictly between 0 and 1. The positive numbers $|r_0|^2$ and $|t_0|^2$ are the ratios of the energy flux reflected or transmitted by the slab.

In more general cases, these “background” reflection and transmission values lie between 0 and 1, possibly equal to 0 or 1. If we allow one of the derivatives $\frac{\partial a}{\partial \omega}$ or $\frac{\partial b}{\partial \omega}$ to be 0 at (κ_0, ω_0) , then the form of the factorization by the Weierstraß Preparation Theorem is modified accordingly and these are precisely the conditions that correspond to the limits $r_0 \rightarrow 0$ and $t_0 \rightarrow 0$. In the former case, the transmission anomaly has a sharp dip down to 0 transmission, while in the second case, the transmission anomaly is formed by one peak.

If we allow the partial derivative $\frac{\partial \ell}{\partial \omega} = 0$, but $\frac{\partial^2 \ell}{\partial \omega^2} \neq 0$ at (κ_0, ω_0) , then the perturbation of κ can excite two spikes from the guided mode frequency. This is discussed in the last part of this section.

At (κ_0, ω_0) , if we assume $\frac{\partial \ell}{\partial \omega} \neq 0$, there are three possibilities: i) $\frac{\partial a}{\partial \omega} \neq 0, \frac{\partial b}{\partial \omega} \neq 0$, ii) $\frac{\partial a}{\partial \omega} \neq 0, \frac{\partial b}{\partial \omega} = 0$, iii) $\frac{\partial a}{\partial \omega} = 0, \frac{\partial b}{\partial \omega} \neq 0$. The first generic case was analyzed in the previous section. The second and the third cases are similar, and we analyze the second case in this following subsection. We analyze the higher order case $\frac{\partial \ell}{\partial \omega}$ in the next subsection.

3.5.1 Total Background Reflection and Transmission

Here we assume that $\partial \ell / \partial \omega \neq 0$ at (κ_0, ω_0) , then only one of the functions a and b can be degenerate at (κ_0, ω_0) .

Proposition 27. Suppose that at $(\kappa_0, \omega_0) \in \mathcal{D}$,

$$\ell(\kappa_0, \omega_0) = 0, \quad \text{and} \quad \frac{\partial \ell}{\partial \omega}(\kappa_0, \omega_0) \neq 0.$$

Then

$$\frac{\partial a}{\partial \omega}(\kappa_0, \omega_0) \neq 0 \quad \text{or} \quad \frac{\partial b}{\partial \omega}(\kappa_0, \omega_0) \neq 0.$$

Proof. By conservation of energy $|\ell|^2 = |a|^2 + |b|^2$ for $(\kappa, \omega) \in \mathcal{D}$, we have $a(\kappa_0, \omega_0) = 0$ and $b(\kappa_0, \omega_0) = 0$ and therefore the representations $\ell(\kappa_0, \omega) = (\omega - \omega_0)h_1(\omega)$, $a(0, \omega) = (\omega - \omega_0)^m h_2(\omega)$, and $b(0, \omega) = (\omega - \omega_0)^n h_3(\omega)$, with h_i analytic and nonzero at ω_0 . This is consistent with $|\ell|^2 = |a|^2 + |b|^2$ only if $m = 1$, that is $\partial a / \partial \omega \neq 0$, or $n = 1$, that is $\partial b / \partial \omega \neq 0$, at (κ_0, ω_0) . \square

Without loss of generality, we only discuss the case of 100% background transmission: $\partial \ell / \partial \omega$, $\partial^2 a / \partial \omega^2$, $\partial b / \partial \omega \neq 0$ and $\partial a / \partial \omega = 0$ at (κ_0, ω_0) . The Weierstraß Preparation Theorem gives the following factorizations:

$$\begin{aligned} \ell(\kappa, \omega) &= (\tilde{\omega} + \ell_1 \tilde{\kappa} + \ell_2 \tilde{\kappa}^2 + \dots) (1 + O(|\tilde{\kappa}| + |\tilde{\omega}|)), \\ a(\kappa, \omega) &= (\tilde{\omega}^2 + \tilde{\omega} \alpha^1(\tilde{\kappa}) + \alpha^0(\tilde{\kappa})) (r_0 e^{i\alpha} + O(|\tilde{\kappa}| + |\tilde{\omega}|)), \\ b(\kappa, \omega) &= (\tilde{\omega} + t_1 \tilde{\kappa} + t_2 \tilde{\kappa}^2 + \dots) (t_0 e^{i\beta} + O(|\tilde{\kappa}| + |\tilde{\omega}|)), \end{aligned}$$

where $r_0, t_0 > 0$. We also suppose that $\tilde{\omega}^2 + \tilde{\omega} \alpha^1(\tilde{\kappa}) + \alpha^0(\tilde{\kappa})$ has distinct roots at $\tilde{\kappa} = 0$, so that it can be factored analytically,

$$a = \left(\tilde{\omega} + r_1^{(1)} \tilde{\kappa} + r_2^{(1)} \tilde{\kappa}^2 + \dots \right) \left(\tilde{\omega} + r_1^{(2)} \tilde{\kappa} + r_2^{(2)} \tilde{\kappa}^2 + \dots \right) (r_0 e^{i\alpha} + O(|\tilde{\kappa}| + |\tilde{\omega}|)).$$

Lemma 28. The coefficients of the expansions satisfy the following properties.

- i) $t_0 = 1$, $t_1 = \ell_1 \in \mathbb{R}$, $\text{Im}(\ell_2) \geq 0$;
- ii) $e^{i\beta} = \pm i e^{i\alpha}$;
- iii) $r_1^{(1)} + r_1^{(2)}$, $r_1^{(1)} r_1^{(2)}$, and $(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1)$ are real-valued. Therefore, either $r_1^{(1)}, r_1^{(2)}$ are both real or they are conjugate complex numbers.

Proof. Since $\text{Im}(\omega) \leq 0$ whenever $\ell(\tilde{\kappa}, \tilde{\omega}) = 0$ for real κ near κ_0 , the relation $\ell(\tilde{\kappa}, \tilde{\omega}) = 0 \Leftrightarrow \omega = \omega_0 - \ell_1(\kappa - \kappa_0) - \ell_2(\kappa - \kappa_0)^2 - \dots$ implies that $\ell_1 \in \mathbb{R}$, $\text{Im}(\ell_2) \geq 0$.

The $\tilde{\omega}^2$ terms coefficients in $|\ell|^2 - |a|^2 - |b|^2 = 0$ is $\tilde{\omega}^2 - t_0^2 \tilde{\omega}^2 = 0$ so we know $t_0 = 1$. The $\tilde{\kappa}^2$ terms coefficient in $|\ell|^2 - |a|^2 - |b|^2$ is $\ell_1^2 - |t_1|^2 t_0^2$ which implies $|t_1|^2 = \ell_1^2$. The $\tilde{\omega}^3$ coefficient in $|\ell|^2 - |a|^2 - |b|^2$ is $2\ell_1 - \text{Re}(t_1)t_0^2 = 0$ which implies $\text{Re}(t_1) = \ell_1$. Combining these relations, we have $t_1 = \ell_1 \in \mathbb{R}$. Part i) is proved.

To prove ii), we calculate that the coefficients of $\tilde{\omega}^3$ in $a\bar{b}$ is $r_0 e^{i\alpha} t_0 e^{-i\beta} \in i\mathbb{R}$, and so $e^{i\alpha} = \pm i e^{i\beta}$.

The $\tilde{\omega}^2 \tilde{\kappa}$ coefficient of $a\bar{b}$ is

$$r_0 t_0 e^{i\alpha} e^{-i\beta} \left[\bar{t}_1 + r_1^{(1)} + r_1^{(2)} \right] \in i\mathbb{R},$$

and since $t_1 = \ell_1 \in \mathbb{R}$ and $e^{i\alpha} e^{-i\beta} \in i\mathbb{R}$, we have $r_1^{(1)} + r_1^{(2)} \in \mathbb{R}$. The $\tilde{\omega} \tilde{\kappa}^2$ coefficient is

$$r_0 t_0 e^{i\alpha} e^{-i\beta} \left[r_1^{(2)} \bar{t}_1 + r_1^{(1)} \bar{t}_1 + r_1^{(1)} r_1^{(2)} \bar{t}_1 \right] \in i\mathbb{R}.$$

Using $r_1^{(1)} + r_1^{(2)} \in \mathbb{R}$ and $t_1 \in \mathbb{R}$, we know $r_1^{(1)} r_1^{(2)} \in \mathbb{R}$. So $(\ell - r_1^{(1)})(\ell - r_1^{(2)}) = \ell^2 - (r_1^{(1)} + r_1^{(2)})\ell + r_1^{(1)} r_1^{(2)}$ is also real and iii) is proved. \square

In order to analyze the transmission anomaly for this case, we need the following lemma.

Lemma 29. One of the following alternatives holds.

- i) If $(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1) \neq 0$, then $t_2 = \text{Re}(\ell_2)$ and $|\text{Im}(\ell_2)|^2 = r_0^2 \left| r_1^{(1)} - \ell_1 \right|^2 \cdot \left| r_1^{(2)} - \ell_1 \right|^2 \neq 0$,
- ii) If $(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1) = 0$, then $\text{Re}(\ell_2) = \text{Re}(t_2)$ and $|\text{Im}(\ell_2)| = |\text{Im}(t_2)|$.

Proof. We let $\tilde{\kappa} \rightarrow 0$ along the set $\{(\tilde{\kappa}, \tilde{\omega}) : \tilde{\omega} + \ell_1 \tilde{\kappa} = 0\} \subset \mathbb{R}^2$ and calculate that the coefficient of $\tilde{\kappa}^4$ in $|\ell|^2 - |a|^2 - |b|^2 = 0$ is

$$|\ell_2|^2 - r_0^2 \left| (r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1) \right|^2 - |t_2|^2 = 0,$$

so

$$|\ell_2|^2 = r_0^2 \left| r_1^{(1)} - \ell_1 \right|^2 \cdot \left| r_1^{(2)} - \ell_1 \right|^2 + |t_2|^2. \quad (3.17)$$

The coefficient of $\tilde{\kappa}^4$ in $a\bar{b}$ is

$$-r_0 t_0 \bar{t}_2 i \left[\ell_1^2 - \ell_1 r_1^{(1)} - \ell_1 r_1^{(2)} + r_1^{(1)} r_1^{(2)} \right] \in i\mathbb{R}.$$

So

$$\bar{t}_2 \left(r_1^{(1)} - \ell_1 \right) \left(r_1^{(2)} - \ell_1 \right) \in \mathbb{R}. \quad (3.18)$$

If $(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1) \neq 0$, since $(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1) \in \mathbb{R}$ as proved in Lemma 28, $t_2 \in \mathbb{R}$.

Similarly, if we let $\tilde{\kappa} \rightarrow 0$ along the set $\{(\tilde{\kappa}, \tilde{\omega}) : \tilde{\omega} + \ell_1 \tilde{\kappa} + \operatorname{Re}(\ell_2) \tilde{\kappa}^2 = 0\} \subset \mathbb{R}^2$, calculating $\tilde{\kappa}^4$ coefficients in $|\ell|^2 - |a|^2 - |b|^2 = 0$ gives

$$\operatorname{Re}(\ell_2)(\operatorname{Re}(\ell_2) - \operatorname{Re}(t_2)) = 0. \quad (3.19)$$

If we let $\tilde{\kappa} \rightarrow 0$ along the set $\{(\tilde{\kappa}, \tilde{\omega}) : \tilde{\omega} + \ell_1 \tilde{\kappa} + \operatorname{Re}(t_2) \tilde{\kappa}^2 = 0\}$, calculating the $\tilde{\kappa}^4$ coefficients in $|\ell|^2 - |a|^2 - |b|^2 = 0$ gives $(\operatorname{Re}(\ell_2) - \operatorname{Re}(t_2))^2 + \operatorname{Im}(\ell_2)^2 - \left| r_1^{(1)} - \ell_1 \right|^2 \cdot \left| r_1^{(2)} - \ell_1 \right|^2 r_0^2 - \operatorname{Im}(t_2)^2 = 0$. Using (3.17), we get

$$\operatorname{Re}(t_2)(\operatorname{Re}(t_2) - \operatorname{Re}(\ell_2)) = 0. \quad (3.20)$$

Therefore, (3.19) and (3.18) imply that $\operatorname{Re}(\ell_2) = \operatorname{Re}(t_2)$.

Use this property in (3.17), we have

$$|\operatorname{Im}(\ell_2)|^2 = r_0^2 \left| r_1^{(1)} - \ell_1 \right|^2 \cdot \left| r_1^{(2)} - \ell_1 \right|^2 + |\operatorname{Im}(t_2)|^2$$

If $(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1) \neq 0$ and $t_2 \in \mathbb{R}$, we have $|\operatorname{Im}(\ell_2)|^2 = r_0^2 \left| r_1^{(1)} - \ell_1 \right|^2 \cdot \left| r_1^{(2)} - \ell_1 \right|^2$.

If $(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1) = 0$, $|\operatorname{Im}(\ell_2)| = |\operatorname{Im}(t_2)|$. \square

We obtain the factorizations

$$\begin{aligned} \ell &= (\tilde{\omega} + \ell_1 \tilde{\kappa} + \ell_2 \tilde{\kappa}^2 + \cdots) (1 + O(|\tilde{\kappa}| + |\tilde{\omega}|)), \\ a &= \left(\tilde{\omega} + r_1^{(1)} \tilde{\kappa} + r_2^{(1)} \tilde{\kappa}^2 + \cdots \right) \left(\tilde{\omega} + r_1^{(2)} \tilde{\kappa} + r_2^{(2)} \tilde{\kappa}^2 + \cdots \right) (r_0 e^{i\gamma} + O(|\tilde{\kappa}| + |\tilde{\omega}|)), \\ b &= (\tilde{\omega} + \ell_1 \tilde{\kappa} + t_2 \tilde{\kappa}^2 + \cdots) (\pm i e^{i\gamma} + O(|\tilde{\kappa}| + |\tilde{\omega}|)). \end{aligned} \quad (3.21)$$

Theorem 30. Suppose that ϵ and μ are symmetric in z and that $\ell(\kappa_0, \omega_0) = 0$ at $(\kappa_0, \omega_0) \in \mathcal{D}$. Let $\frac{\partial \ell}{\partial \omega}$, $\frac{\partial^2 a}{\partial \omega^2}$, and $\frac{\partial b}{\partial \omega}$ be nonzero and $\frac{\partial a}{\partial \omega} = 0$ at (κ_0, ω_0) . Suppose in addition that $(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1) \neq 0$. Then

i) t_n is real for all n and therefore the coefficient b of the transmitted field vanishes on the real-analytic curve in a neighborhood of (κ_0, ω_0) given by (3.22);

ii) if $r_1^{(1)}$ and $r_1^{(2)}$ are distinct real numbers, then $r_n^{(1)}$ and $r_n^{(2)}$ are real for all n and therefore the coefficient a of the reflected field vanishes on the real analytic curves given by (3.23).

Proof. Assume $(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1) \neq 0$ and $t_2 \in \mathbb{R}$. Assuming $t_2, t_3, \dots, t_n \in \mathbb{R}$, we show that $t_{n+1} \in \mathbb{R}$. For real (κ, ω) subject to the relation $\tilde{\omega} + \ell_1 \tilde{\kappa} + t_2 \tilde{\kappa}^2 + \dots + t_n \tilde{\kappa}^n = 0$, the property $\frac{a}{b} \in i\mathbb{R}$ implies that

$$\begin{aligned} \frac{a}{b} &= [(r_1^{(1)} - \ell_1)\tilde{\kappa} + \dots + (r_n^{(1)} - t_n)\tilde{\kappa}^n + r_{n+1}^{(1)}\tilde{\kappa}^{n+1} + \dots] \\ &\quad \cdot [(r_1^{(2)} - \ell_1)\tilde{\kappa} + \dots + (r_n^{(2)} - t_n)\tilde{\kappa}^n + r_{n+1}^{(2)}\tilde{\kappa}^{n+1} + \dots] \\ &\quad \cdot \frac{[r_0 e^{i\gamma_0} + O(|\tilde{\kappa}|)]}{(t_{n+1}\tilde{\kappa}^{n+1} + t_{n+2}\tilde{\kappa}^{n+2} + \dots) [i e^{i\gamma_0} + O(|\tilde{\kappa}|)]} \\ &= \frac{1}{\tilde{\kappa}^{n-1}} [(r_1^{(1)} - \ell_1) + \dots + (r_n^{(1)} - t_n)\tilde{\kappa}^{n-1} + r_{n+1}^{(1)}\tilde{\kappa}^n + \dots] \\ &\quad \cdot [(r_1^{(2)} - \ell_1) + \dots + (r_n^{(2)} - t_n)\tilde{\kappa}^{n-1} + r_{n+1}^{(2)}\tilde{\kappa}^n + \dots] \\ &\quad \cdot \frac{[r_0 e^{i\gamma_0} + O(|\tilde{\kappa}|)]}{(t_{n+1} + t_{n+2}\tilde{\kappa} + \dots) [i e^{i\gamma_0} + O(|\tilde{\kappa}|)]} \\ &\in i\mathbb{R} \end{aligned}$$

Multiplying this expression by $\tilde{\kappa}^{n-1}$ and letting $\tilde{\kappa} \rightarrow 0$, we obtain

$$\frac{(r_1^{(1)} - \ell_1)(r_1^{(2)} - \ell_1)}{t_{n+1}} \cdot \frac{r_0 e^{i\gamma_0}}{i e^{i\gamma_0}} \in i\mathbb{R}$$

and so $t_{n+1} \in \mathbb{R}$. Then all the coefficients are real by induction. For any real κ the reflective coefficient a becomes 0 for $\tilde{\omega}$ given by (3.22), and hence the transmission coefficient reaches the magnitude of 0%.

If in addition $r_1^{(1)}, r_1^{(2)}$ are real and $r_1^{(1)} \neq r_1^{(2)}$, we let $\tilde{\kappa} \rightarrow 0$ along the curve $\tilde{\omega} + r_1^{(1)}\tilde{\kappa} = 0$ to obtain

$$\frac{a}{b} = \frac{[r_2^{(1)}\tilde{\kappa}^2 + \dots][(r_1^{(2)} - r_1^{(1)})\tilde{\kappa} + r_2^{(2)}\tilde{\kappa}^2 + \dots](r_0e^{i\gamma_0} + O(|\tilde{\kappa}|))}{[(\ell_1 - r_1^{(1)})\tilde{\kappa} + t_2\tilde{\kappa} + \dots](ie^{i\gamma_0} + O(|\tilde{\kappa}|))} \in i\mathbb{R}$$

so

$$\tilde{\kappa} \cdot \frac{[r_2^{(1)} + O(|\tilde{\kappa}|)][(r_1^{(2)} - r_1^{(1)}) + O(|\tilde{\kappa}|)](r_0e^{i\gamma_0} + O(|\tilde{\kappa}|))}{[(\ell_1 - r_1^{(1)}) + O(|\tilde{\kappa}|)](ie^{i\gamma_0 + O(|\tilde{\kappa}|)})} \in i\mathbb{R}$$

and $r_2^{(1)}(r_1^{(2)} - r_1^{(1)}) \in \mathbb{R}$ and so $r_2^{(1)} \in \mathbb{R}$. The induction arguments can be applied to show $r_3^{(1)}, r_4^{(1)}, \dots \in \mathbb{R}$. Similarly, $r_2^{(2)}, r_3^{(2)}, \dots \in \mathbb{R}$, and the transmission coefficient obtains the magnitude of 100%. \square

In the first alternative of Lemma 29, Theorem 30 proves that t_n are all real numbers, and so the transmission vanishes along the real-analytic curve

$$\omega = \omega_0 - \ell_1(\kappa - \kappa_0) - t_2(\kappa - \kappa_0)^2 - \dots \quad (b = 0) \quad (3.22)$$

If $r_n^{(1)}$ and $r_n^{(2)}$ are real numbers as in the part ii) of this Theorem, then the transmission achieves 100% along two real-analytic curves

$$\omega = \omega_0 - r_1^{(i)}(\kappa - \kappa_0) - r_2^{(i)}(\kappa - \kappa_0)^2 - \dots, \quad i = 1, 2. \quad (a = 0) \quad (3.23)$$

These frequencies of total transmission and total reflection move apart from ω_0 as κ is perturbed. The transmission rate at other frequencies are close to 1 and the maximal transmission rate are difficult to detect, as in Fig. 3.4 and 3.6. We show this by a magnified view on the right of Fig. 3.6. We also believe that t_1 must lie between $r_1^{(1)}$ and $r_1^{(2)}$ but so far we do not have a proof.

3.5.2 Multiple Anomalies

If $\frac{\partial \ell}{\partial \omega} = 0$ at $(\kappa_0, \omega_0) \in \mathcal{D}$, then since $|\ell|^2 = |a|^2 + |b|^2$ for real (κ, ω) , we have $\frac{\partial a}{\partial \omega} = \frac{\partial b}{\partial \omega} = 0$. In this section, we consider the case in which

$$\begin{aligned} \frac{\partial \ell}{\partial \omega} &= 0, \quad \frac{\partial a}{\partial \omega} = 0, \quad \frac{\partial b}{\partial \omega} = 0; \\ \frac{\partial^2 \ell}{\partial \omega^2} &\neq 0, \quad \frac{\partial^2 a}{\partial \omega^2} \neq 0, \quad \frac{\partial^2 b}{\partial \omega^2} \neq 0. \end{aligned} \tag{3.24}$$

The zero loci of the above a, b, ℓ are given locally by the roots of quadratic functions in $\tilde{\omega}$ with coefficients analytic in $\tilde{\kappa}$ and vanish at $\tilde{\kappa} = 0$. We have

$$\begin{aligned} \ell(\kappa, \omega) &= (\tilde{\omega}^2 + \tilde{\omega}\lambda^1(\tilde{\kappa}) + \lambda^0(\tilde{\kappa})) (1 + O(|\tilde{\kappa}| + |\tilde{\omega}|)), \\ a(\kappa, \omega) &= (\tilde{\omega}^2 + \tilde{\omega}\alpha^1(\tilde{\kappa}) + \alpha^0(\tilde{\kappa})) (r_0 e^{i\gamma} + O(|\tilde{\kappa}| + |\tilde{\omega}|)), \\ b(\kappa, \omega) &= (\tilde{\omega}^2 + \tilde{\omega}\beta^1(\tilde{\kappa}) + \beta^0(\tilde{\kappa})) (it_0 e^{i\gamma} + O(|\tilde{\kappa}| + |\tilde{\omega}|)), \end{aligned}$$

where $\lambda^i(0) = \alpha^i(0) = \beta^i(0) = 0$, $0 < r_0 < 1$, and t_0 is real with $0 < |t_0| < 1$. The common unitary factor $e^{i\gamma}$ comes from the second equation of 3.12, which is due to the symmetry of the structure. Similar to the total background transmission, we assume again that the first factors of a, b, ℓ have distinct roots so that they can be factored analytically. Then we have the following factorization

$$\begin{aligned} \ell(\kappa, \omega) &= \left(\tilde{\omega} + \ell_1^{(1)} \tilde{\kappa} + \ell_2^{(1)} \tilde{\kappa}^2 + \dots \right) \left(\tilde{\omega} + \ell_1^{(2)} \tilde{\kappa} + \ell_2^{(2)} \tilde{\kappa}^2 + \dots \right) (1 + O(|\tilde{\kappa}| + |\tilde{\omega}|)), \\ a(\kappa, \omega) &= \left(\tilde{\omega} + r_1^{(1)} \tilde{\kappa} + r_2^{(1)} \tilde{\kappa}^2 + \dots \right) \left(\tilde{\omega} + r_1^{(2)} \tilde{\kappa} + r_2^{(2)} \tilde{\kappa}^2 + \dots \right) (r_0 e^{i\gamma} + r_{\tilde{\kappa}} \tilde{\kappa} + r_{\tilde{\omega}} \tilde{\omega} + \dots), \\ b(\kappa, \omega) &= \left(\tilde{\omega} + t_1^{(1)} \tilde{\kappa} + t_2^{(1)} \tilde{\kappa}^2 + \dots \right) \left(\tilde{\omega} + t_1^{(2)} \tilde{\kappa} + t_2^{(2)} \tilde{\kappa}^2 + \dots \right) (it_0 e^{i\gamma} + t_{\tilde{\kappa}} \tilde{\kappa} + t_{\tilde{\omega}} \tilde{\omega} + \dots). \end{aligned} \tag{3.25}$$

Lemma 31. Assume $\ell_1^{(1)} \neq \ell_1^{(2)}$. In these forms, the coefficients satisfy the following properties:

- i) $\ell_1^{(1)}, \ell_1^{(2)} \in \mathbb{R}$ and $\text{Im}(\ell_2^{(1)}), \text{Im}(\ell_2^{(2)}) \geq 0$;
- ii) After possibly reindexing the coefficients $r_1^{(i)}$ and $t_1^{(i)}$, $\ell_1^{(1)} = r_1^{(1)} = t_1^{(1)}$, $\ell_1^{(2)} =$

$$r_1^{(2)} = t_1^{(2)};$$

$$\text{iii) } |\ell_2^{(1)}|^2 = |r_2^{(1)}|^2 r_0^2 + |t_2^{(1)}|^2 t_0^2, \text{ and } |\ell_2^{(2)}|^2 = |r_2^{(2)}|^2 r_0^2 + |t_2^{(2)}|^2 t_0^2.$$

Proof. The relation $\ell(\tilde{\kappa}, \tilde{\omega}) = 0, \forall (\kappa, \omega) \in \mathbb{R}^2$ gives $\omega = \omega_0 - \ell_1^{(1)}(\kappa - \kappa_0) + \ell_2^{(1)}(\kappa - \kappa_0)^2 + \dots$ or $\omega = \omega_0 - \ell_1^{(2)}(\kappa - \kappa_0) + \ell_2^{(2)}(\kappa - \kappa_0)^2 + \dots$. The condition that $\text{Im}(\tilde{\omega}) \leq 0, \forall \tilde{\kappa} \in \mathbb{R}$ for generalized guided modes implies (i).

From the property $|\ell|^2 = |a|^2 + |b|^2$ for all $(\tilde{\kappa}, \tilde{\omega}) \in \mathbb{R}^2$, we take $\tilde{\kappa} \rightarrow 0$ along $\{(\tilde{\kappa}, \tilde{\omega}) : \tilde{\omega} + \ell_1^{(1)}\tilde{\kappa} = 0\} \subset \mathbb{R}^2$ to obtain

$$\begin{aligned} |\ell|^2 &= \left| \ell_2^{(1)}\tilde{\kappa}^2 + o(|\tilde{\kappa}|^3) \right|^2 \left| (\ell_1^{(2)} - \ell_1^{(1)})\tilde{\kappa} + \ell_2^{(2)}\tilde{\kappa}^2 + \dots \right|^2 (1 + O(|\tilde{\kappa}|)), \\ |a|^2 &= \left| (r_1^{(1)} - \ell_1^{(1)})\tilde{\kappa} + r_2^{(1)}\tilde{\kappa}^2 + \dots \right|^2 \left| (r_1^{(2)} - \ell_1^{(1)})\tilde{\kappa} + r_2^{(2)}\tilde{\kappa}^2 + \dots \right|^2 (r_0^2 + O(|\tilde{\kappa}|)), \\ |b|^2 &= \left| (t_1^{(1)} - \ell_1^{(1)})\tilde{\kappa} + t_2^{(1)}\tilde{\kappa}^2 + \dots \right|^2 \left| (t_1^{(2)} - \ell_1^{(1)})\tilde{\kappa} + t_2^{(2)}\tilde{\kappa}^2 + \dots \right|^2 (t_0^2 + O(|\tilde{\kappa}|)). \end{aligned}$$

Compare $\tilde{\kappa}^4$ terms in $|\ell|^2 = |a|^2 + |b|^2$, then we get

$$\begin{aligned} 0 &= \left| r_1^{(1)} - \ell_1^{(1)} \right|^2 \left| r_1^{(2)} - \ell_1^{(1)} \right|^2 r_0^2 + \left| t_1^{(1)} - \ell_1^{(1)} \right|^2 \left| t_1^{(2)} - \ell_1^{(1)} \right|^2 t_0^2 \\ &= \left[(\text{Re}(r_1^{(1)}) - \ell_1^{(1)})^2 + \text{Im}(r_1^{(1)})^2 \right] \left[(\text{Re}(r_1^{(2)}) - \ell_1^{(1)})^2 + \text{Im}(r_1^{(2)})^2 \right] r_0^2 \\ &\quad + \left[(\text{Re}(t_1^{(1)}) - \ell_1^{(1)})^2 + \text{Im}(t_1^{(1)})^2 \right] \left[(\text{Re}(t_1^{(2)}) - \ell_1^{(1)})^2 + \text{Im}(t_1^{(2)})^2 \right] t_0^2. \end{aligned}$$

This implies that

$$\begin{cases} \ell_1^{(1)} = \text{one of } r_1^{(1)}, r_1^{(2)}, \text{ and} \\ \ell_1^{(1)} = \text{one of } t_1^{(1)}, t_1^{(2)} \end{cases}$$

Similarly we can also prove

$$\begin{cases} \ell_1^{(2)} = \text{one of } r_1^{(1)}, r_1^{(2)}, \text{ and} \\ \ell_1^{(2)} = \text{one of } t_1^{(1)}, t_1^{(2)} \end{cases}$$

Assuming $\ell_1^{(1)} \neq \ell_1^{(2)}$, without loss of generality, we have ii). Comparing the $\tilde{\kappa}^6$ terms, we get

$$|\ell_2^{(1)}|^2 \left(\ell_1^{(2)} - \ell_1^{(1)} \right)^2 = |r_2^{(1)}|^2 |r_1^{(2)} - \ell_1^{(1)}|^2 r_0^2 + |t_2^{(1)}|^2 |t_1^{(2)} - \ell_1^{(1)}|^2 t_0^2,$$

which simplifies to

$$|\ell_2^{(1)}|^2 = |r_2^{(1)}|^2 r_0^2 + |t_2^{(1)}|^2 t_0^2.$$

Similarly, along $\{(\tilde{\kappa}, \tilde{\omega}) : \tilde{\omega} + \ell_1^{(2)} \tilde{\kappa} = 0\}$, we have

$$|\ell_2^{(2)}|^2 = |r_2^{(2)}|^2 r_0^2 + |t_2^{(2)}|^2 t_0^2.$$

□

We can rewrite a, b as

$$\begin{aligned} a(\kappa, \omega) &= \left(\tilde{\omega} + \ell_1^{(1)} \tilde{\kappa} + r_2^{(1)} \tilde{\kappa}^2 + \dots \right) \left(\tilde{\omega} + \ell_1^{(2)} \tilde{\kappa} + r_2^{(2)} \tilde{\kappa}^2 + \dots \right) (r_0 e^{i\gamma} + r_{\tilde{\kappa}} \tilde{\kappa} + r_{\tilde{\omega}} \tilde{\omega} + \dots), \\ b(\kappa, \omega) &= \left(\tilde{\omega} + \ell_1^{(1)} \tilde{\kappa} + t_2^{(1)} \tilde{\kappa}^2 + \dots \right) \left(\tilde{\omega} + \ell_1^{(2)} \tilde{\kappa} + t_2^{(2)} \tilde{\kappa}^2 + \dots \right) (it_0 e^{i\gamma} + t_{\tilde{\kappa}} \tilde{\kappa} + t_{\tilde{\omega}} \tilde{\omega} + \dots). \end{aligned}$$

Lemma 32. If we assume that $\text{Im}(\ell_2^{(1)}), \text{Im}(\ell_2^{(2)}) > 0$, then for each $i \in \{1, 2\}$, either $r_2^{(i)}$ and $t_2^{(i)}$ are distinct real numbers, or they are equal and not real.

Proof. By $\bar{a}b + a\bar{b} = 0, \forall (\tilde{\kappa}, \tilde{\omega}) \in \mathbb{R}^2$, we compute that the $\tilde{\kappa}^5, \tilde{\kappa}^4 \tilde{\omega}, \tilde{\kappa} \tilde{\omega}^4, \tilde{\kappa}^2 \tilde{\omega}^3, \tilde{\kappa}^3 \tilde{\omega}^2, \tilde{\omega}^5$ coefficients of $\bar{a}b + a\bar{b}$ are

$$\begin{aligned} & -r_0 t_0 i \left(r_2^{(1)} \ell_1^{(2)} + r_2^{(2)} \ell_1^{(1)} + \ell_1^{(2)} \bar{t}_2^{(1)} + \ell_1^{(1)} \bar{t}_2^{(2)} \right) + \ell_1^{(1)} \ell_1^{(2)} \left(-r_{\tilde{\kappa}} t_0 e^{-i\gamma} + \bar{t}_{\tilde{\kappa}} r_0 e^{i\gamma} \right), \\ & -r_0 t_0 i \left[2\ell_1^{(1)} \ell_2^{(2)} (r_2^{(2)} + \bar{t}_2^{(2)} + r_2^{(1)} + \bar{t}_2^{(1)}) \right] \\ & + 2\ell_1^{(1)} \ell_1^{(2)} \left[-ir_{\tilde{\kappa}} t_0 e^{-i\gamma} + r_0 \bar{t}_{\tilde{\kappa}} e^{i\gamma} \right] (\ell_1^{(1)} + \ell_1^{(2)}) \\ & - r_0 t_0 \left[(\ell_1^{(1)})^2 (\bar{t}_2^{(2)} + r_2^{(2)}) + (\ell_1^{(2)})^2 (\bar{t}_2^{(1)} + r_2^{(1)}) \right], \\ & 2(\ell_1^{(1)} + \ell_1^{(2)}) \left[r_{\tilde{\omega}} t_0 (-i) e^{-i\gamma} + r_0 \bar{t}_{\tilde{\omega}} \right] \\ & + \left[r_{\tilde{\kappa}} t_0 (-i) e^{-i\gamma} + \bar{t}_{\tilde{\kappa}} r_0 e^{i\gamma} \right], \\ & -ir_0 t_0 \left[r_2^{(1)} + r_2^{(2)} + \bar{t}_2^{(1)} + \bar{t}_2^{(2)} \right] + 2(\ell_1^{(1)} + \ell_1^{(2)}) \left[r_{\tilde{\kappa}} t_0 (-i) e^{-i\gamma} + r_0 \bar{t}_{\tilde{\kappa}} e^{i\gamma} \right] \\ & + ((\ell_1^{(1)})^2 + (\ell_1^{(2)})^2) \left[r_{\tilde{\omega}} t_0 (-i) e^{-i\gamma} + r_0 \bar{t}_{\tilde{\omega}} e^{i\gamma} \right] \\ & + 4\ell_1^{(1)} \ell_2^{(2)} \left[r_{\tilde{\kappa}} t_0 (-i) e^{-i\gamma} + r_0 \bar{t}_{\tilde{\kappa}} e^{i\gamma} \right], \end{aligned}$$

$$\begin{aligned}
& -ir_0t_0 \left[\ell_1^{(1)}r_2^{(1)} + 2\ell_1^{(2)}r_2^{(1)} + 2\ell_1^{(1)}r_2^{(2)} + \ell_1^{(2)}r_2^{(2)} \right] \\
& + \left[(\ell_1^{(1)})^2 + (\ell_1^{(2)})^2 \right] \left[r_{\bar{\kappa}}t_0(-i)e^{-i\gamma} + r_0\bar{t}_{\bar{\kappa}}e^{i\gamma} \right] \\
& + 4\ell_1^{(1)}\ell_1^{(2)} \left[r_{\bar{\kappa}}t_0(-i)e^{-i\gamma} + r_0\bar{t}_{\bar{\kappa}}e^{i\gamma} \right] \\
& + 2\ell_1^{(1)}\ell_1^{(2)}(\ell_1^{(1)} + \ell_1^{(2)}) \left[r_{\bar{\omega}}t_0(-i)e^{-i\gamma} + r_0\bar{t}_{\bar{\omega}}e^{i\gamma} \right] \\
& - ir_0t_0 \left[\ell_1^{(1)}\bar{t}_2^{(1)} + 2\ell_1^{(2)}\bar{t}_2^{(1)} + 2\ell_1^{(1)}\bar{t}_2^{(2)} + \ell_1^{(2)}\bar{t}_2^{(2)} \right], \\
& \quad r_{\bar{\omega}}t_0(-i)e^{-i\gamma} + r_0\bar{t}_{\bar{\omega}}.
\end{aligned}$$

In total, from the fact that $a\bar{b} \in i\mathbb{R}, \forall(\kappa, \omega) \in \mathbb{R}^2$ we have

$$\begin{aligned}
& (-r_0t_0i)A + \ell_1^{(1)}\ell_1^{(2)}B \in i\mathbb{R}, \\
& (-r_0t_0i)C + 2\ell_1^{(1)}\ell_1^{(2)}B(\ell_1^{(1)} + \ell_1^{(2)}) - r_0t_0iD \in i\mathbb{R}, \\
& 2(\ell_1^{(1)} + \ell_1^{(2)})E + B \in i\mathbb{R}, \\
& -ir_0t_0\frac{C}{2\ell_1^{(1)}\ell_1^{(2)}} + 2(\ell_1^{(1)} + \ell_1^{(2)})B + [(\ell_1^{(1)})^2 + (\ell_1^{(2)})^2]E + 4\ell_1^{(1)}\ell_1^{(2)}E \in i\mathbb{R}, \\
& -ir_0t_0G + [(\ell_1^{(1)})^2 + (\ell_1^{(2)})^2]B + 4\ell_1^{(1)}\ell_1^{(2)}B + 2\ell_1^{(1)}\ell_1^{(2)}(\ell_1^{(1)} + \ell_1^{(2)})E - ir_0t_0H \in i\mathbb{R}, \\
& \quad E \in i\mathbb{R},
\end{aligned}$$

where

$$\begin{aligned}
A &= r_2^{(1)}\ell_1^{(2)} + r_2^{(2)}\ell_1^{(1)} + \ell_1^{(2)}\bar{t}_2^{(1)} + \ell_1^{(1)}\bar{t}_2^{(2)}, \\
B &= r_{\bar{\kappa}}t_0(-i)e^{-i\gamma} + \bar{t}_{\bar{\kappa}}r_0e^{i\gamma}, \\
C &= 2\ell_1^{(1)}\ell_1^{(2)} \left(r_2^{(2)} + \bar{t}_2^{(2)} + r_2^{(1)} + \bar{t}_2^{(1)} \right), \\
D &= (\ell_1^{(2)})^2(\bar{t}_2^{(2)} + r_2^{(2)}) + (\ell_1^{(1)})^2(\bar{t}_2^{(1)} + r_2^{(1)}), \\
E &= r_{\bar{\omega}}t_0(-i)e^{-i\gamma} + r_0\bar{t}_{\bar{\omega}},
\end{aligned}$$

and

$$F = \frac{C}{2\ell_1^{(1)}\ell_1^{(2)}},$$

$$\begin{aligned}
G &= \ell_1^{(1)}\bar{t}_2^{(1)} + 2\ell_1^{(2)}\bar{t}_2^{(1)} + \ell_1^{(1)}\bar{t}_2^{(2)} + \ell_1^{(2)}\bar{t}_2^{(2)}, \\
&= (\ell_1^{(1)} + \ell_1^{(2)})(r_2^{(1)} + r_2^{(2)}) + \ell_1^{(2)}r_r^{(1)} + \ell_1^{(1)}r_r^{(2)},
\end{aligned}$$

$$\begin{aligned}
H &= \ell_1^{(1)}\bar{t}_2^{(1)} + 2\ell_1^{(2)}\bar{t}_2^{(1)} + 2\ell_1^{(1)}\bar{t}_2^{(2)} + \ell_1^{(2)}\bar{t}_2^{(2)} \\
&= (\ell_1^{(1)} + \ell_1^{(2)})(\bar{t}_2^{(1)} + \bar{t}_2^{(2)}) + \ell_1^{(2)}\bar{t}_2^{(1)} + \ell_1^{(1)}\bar{t}_2^{(2)},
\end{aligned}$$

$$\begin{aligned}
G + H &= \ell_1^{(1)}(r_2^{(1)} + \bar{t}_2^{(1)}) + 2\ell_1^{(2)}(r_2^{(1)} + \bar{t}_2^{(1)}) + 2\ell_1^{(1)}(r_2^{(2)} + \bar{t}_2^{(2)}) + \ell_1^{(2)}(r_2^{(2)} + \bar{t}_2^{(2)}) \\
&= (\ell_1^{(1)} + \ell_1^{(2)})\frac{C}{2\ell_1^{(1)}\ell_1^{(2)}} + A.
\end{aligned}$$

These conditions can be written as a linear system

$$\begin{pmatrix}
r_0 t_0 & \ell_1^{(1)} \ell_1^{(2)} & 0 & 0 & 0 \\
0 & 2\ell_1^{(1)} \ell_1^{(2)} (\ell_1^{(1)} + \ell_1^{(2)}) & r_0 t_0 & r_0 t_0 & 0 \\
0 & 1 & 0 & 0 & 2(\ell_1^{(1)} + \ell_1^{(2)}) \\
0 & 2(\ell_1^{(1)} + \ell_1^{(2)}) & \frac{r_0 t_0}{2\ell_1^{(1)} \ell_1^{(2)}} & 0 & [(\ell_1^{(1)})^2 + (\ell_1^{(2)})^2 + 4\ell_1^{(1)} \ell_1^{(2)}] \\
r_0 t_0 & [(\ell_1^{(1)})^2 + (\ell_1^{(2)})^2 + 4\ell_1^{(1)} \ell_1^{(2)}] & \frac{r_0 t_0 (\ell_1^{(1)} + \ell_1^{(2)})}{2\ell_1^{(1)} \ell_1^{(2)}} & 0 & 2\ell_1^{(1)} \ell_1^{(2)} (\ell_1^{(1)} + \ell_1^{(2)})
\end{pmatrix} \cdot \begin{pmatrix} \text{Im}(A) \\ \text{Re}(B) \\ \text{Im}(C) \\ \text{Im}(D) \\ \text{Re}(E) \end{pmatrix} = 0.$$

The determinant of this matrix is

$$\begin{aligned}
&\frac{1}{2}\ell_1^{(1)}r_0^3t_0^3 + \frac{(\ell_1^{(1)})^3}{2\ell_1^{(2)}}r_0^3t_0^3 + \frac{1}{2}\ell_1^{(2)}r_0^3t_0^3 + \frac{1}{2}\frac{(\ell_1^{(2)})^2r_0^3t_0^3}{\ell_1^{(1)}} \\
&= \frac{r_0^3t_0^3}{2} \left[\ell_1^{(1)} + \ell_1^{(2)} + \frac{(\ell_1^{(1)})^2}{\ell_1^{(2)}} + \frac{(\ell_1^{(2)})^2}{\ell_1^{(1)}} \right] \\
&= \frac{r_0^3t_0^3}{2} \frac{(\ell_1^{(1)} + \ell_1^{(2)}) \left((\ell_1^{(1)})^2 + (\ell_1^{(2)})^2 \right)}{\ell_1^{(1)}\ell_1^{(2)}}.
\end{aligned}$$

If $\ell_1^{(1)} + \ell_1^{(2)} \neq 0$, the determinant is nonzero and so

$$\begin{pmatrix} \text{Im}(A) \\ \text{Re}(B) \\ \text{Im}(C) \\ \text{Im}(D) \\ \text{Re}(E) \end{pmatrix} = 0.$$

Here $\text{Im}(A) = 0$ implies that $r_2^{(1)}\ell_1^{(2)} + r_2^{(2)}\ell_1^{(1)} + \ell_1^{(2)}\bar{t}_2^{(1)} + \ell_1^{(1)}\bar{t}_2^{(2)} = \ell_1^{(1)}(r_2^{(2)} + \bar{t}_2^{(2)}) + \ell_1^{(2)}(r_2^{(1)} + \bar{t}_2^{(1)}) \in \mathbb{R}$, $\text{Im}(C) = 0$ implies that $r_2^{(2)} + \bar{t}_2^{(2)} + r_2^{(1)} + \bar{t}_2^{(1)} \in \mathbb{R}$, and $\text{Im}(D) = 0$ implies that $(\ell_1^{(1)})^2 (r_2^{(2)} + \bar{t}_2^{(2)}) + (\ell_1^{(2)})^2 (r_2^{(1)} + \bar{t}_2^{(1)}) \in \mathbb{R}$. Since we assume $\ell_1^{(1)} \neq \ell_1^{(2)}$, it follows that

$$\begin{cases} r_2^{(1)} + \bar{t}_2^{(1)} \in \mathbb{R}, \\ r_2^{(2)} + \bar{t}_2^{(2)} \in \mathbb{R}. \end{cases}$$

If $\ell_1^{(1)} + \ell_1^{(2)} = 0$, then the system becomes

$$\begin{pmatrix} r_0 t_0 & -(\ell_1^{(1)})^2 & 0 & 0 & 0 \\ 0 & 0 & r_0 t_0 & r_0 t_0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{r_0 t_0}{-2(\ell_1^{(1)})^2} & 0 & 0 \\ r_0 t_0 & -2(\ell_1^{(1)})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Im}(A) \\ \text{Re}(B) \\ \text{Im}(C) \\ \text{Im}(D) \\ \text{Re}(E) \end{pmatrix} = 0.$$

This also implies that $\text{Im}(A) = \text{Re}(B) = \text{Im}(C) = \text{Im}(D) = \text{Re}(E) = 0$, and we also have

$$\begin{cases} r_2^{(1)} + \bar{t}_2^{(1)} \in \mathbb{R}, \\ r_2^{(2)} + \bar{t}_2^{(2)} \in \mathbb{R}. \end{cases}$$

To prove $r_2^{(1)}\bar{t}_2^{(1)} \in \mathbb{R}$ and $r_2^{(2)}\bar{t}_2^{(2)} \in \mathbb{R}$, we let $\tilde{\kappa} \rightarrow 0$ along the set $\{(\tilde{\kappa}, \tilde{\omega}) : \tilde{\omega} + \ell_1^{(1)}\tilde{\kappa} = 0\} \subset \mathbb{R}^2$. The condition $a\bar{b} \in i\mathbb{R}$ implies that the coefficient of $\tilde{\kappa}^6$ in $a\bar{b}$

$$-ir_0t_0r_1^{(1)}\bar{t}_2^{(1)} \left[(\ell_1^{(1)})^2 - 2\ell_1^{(1)}\ell_1^{(2)} + (\ell_1^{(2)})^2 \right] \in i\mathbb{R},$$

which implies $r_2^{(1)}\bar{t}_2^{(1)} \in \mathbb{R}$ because $\ell_1^{(1)} \neq \ell_1^{(2)}$. Similarly, along $\{(\tilde{\kappa}, \tilde{\omega}) : \tilde{\omega} + \ell_1^{(2)}\tilde{\kappa} = 0\}$, we obtain $r_2^{(2)}\bar{t}_2^{(2)} \in \mathbb{R}$.

In conclusion, $r_2^{(1)}, \bar{t}_2^{(1)}$ are two roots of a real quadratic equation, which implies the statement in this lemma. \square

The relation between $r_2^{(i)}$ and $t_2^{(i)}$ can be categorized into three cases: i) all $r_2^{(1)}, t_2^{(1)}, r_2^{(2)}, t_2^{(2)}$ are real; ii) only one pair of $r_2^{(i)}, t_2^{(i)}$ is real, and the members of the other pair are equal; iii) none of them are real numbers. Subject to structural symmetry with respect to z , one can prove a theorem analogous to those for the previous cases. It says in particular that, if the first alternative in the lemma holds for $i = 1$ and $i = 2$, then two peak-dip anomalies emerge from ω_0 as κ is perturbed from κ_0 , each of which attains the values 0 and 1 along real-analytic curves passing through (κ_0, ω_0) .

Theorem 33. i) If

$$\begin{cases} r_2^{(1)} \neq t_2^{(1)} \in \mathbb{R} \\ r_2^{(2)} \neq t_2^{(2)} \in \mathbb{R}, \end{cases}$$

then all the coefficients $r_n^{(1)}, r_n^{(2)}, t_n^{(1)}$, and $t_n^{(2)}$ are real. For κ near κ_0 , there exist two values of ω for which the transmission coefficient attains the magnitude of 1 along the real-analytic curves given by

$$\omega = \omega_0 - r_1^{(1)}(\kappa - \kappa_0) - r_2^{(1)}(\kappa - \kappa_0)^2 - \dots,$$

$$\omega = \omega_0 - r_1^{(2)}(\kappa - \kappa_0) - r_2^{(2)}(\kappa - \kappa_0)^2 - \dots.$$

and the transmission coefficient attains the magnitude of 0 along the real-analytic curve given by

$$\omega = \omega_0 - t_1^{(1)}(\kappa - \kappa_0) - t_2^{(1)}(\kappa - \kappa_0)^2 - \dots,$$

$$\omega = \omega_0 - t_1^{(2)}(\kappa - \kappa_0) - t_2^{(2)}(\kappa - \kappa_0)^2 - \dots.$$

ii) If only one pair of them are real, for example, $r_2^{(1)}, t_2^{(1)} \in \mathbb{R}$, then there is only one value of ω for which the transmission coefficient reaches the magnitude 100% and one value for which the transmission is 0%.

iii) If $r_2^{(i)} = t_2^{(i)}$ for all $i \in \{1, 2\}$, then a/ℓ and b/ℓ are continuous at (κ_0, ω_0) . The transmission coefficient is continuous with respect to κ and does not reach 100% or 0%.

Proof. i) Letting $\tilde{\kappa} \rightarrow 0$ along the real curve $\{(\tilde{\kappa}, \tilde{\omega}) \in \mathbb{R}^2 : \tilde{\omega} + \ell_1^{(1)}\tilde{\kappa} + r_2^{(1)}\tilde{\kappa}^2 = 0\}$, we have

$$a = (r_1^{(1)}\tilde{\kappa}^3 + \dots) \left[(r_2^{(2)} - r_2^{(1)})\tilde{\kappa}^2 + \dots \right] (r_0 e^{i\gamma} + \dots)$$

$$b = \left[(t_2^{(1)} - r_2^{(1)})\tilde{\kappa}^2 + \dots \right] \left[(t_2^{(2)} - r_2^{(1)})\tilde{\kappa}^2 + \dots \right] (t_0 i e^{i\gamma} + \dots)$$

Since $\frac{a}{b} \in i\mathbb{R}$ for real $\tilde{\kappa}$, the expression $\frac{a}{\tilde{\kappa}b}$ is also in \mathbb{R} . So

$$\frac{(r_1^{(1)}\tilde{\kappa}^3 + \dots) \left[(r_2^{(2)} - r_2^{(1)})\tilde{\kappa}^2 + \dots \right] (r_0 e^{i\gamma} + \dots)}{\tilde{\kappa} \left[(t_2^{(1)} - r_2^{(1)})\tilde{\kappa}^2 + \dots \right] \left[(t_2^{(2)} - r_2^{(1)})\tilde{\kappa}^2 + \dots \right] (t_0 i e^{i\gamma} + \dots)} \in i\mathbb{R}.$$

Let $\tilde{\kappa} \rightarrow 0$, and we have $r_3^{(1)} \in \mathbb{R}$. Similarly, we can get $t_3^{(1)}, r_3^{(2)}, t_3^{(2)} \in \mathbb{R}$.

Inductively, if $r_2^{(2)} \neq r_2^{(1)}$, and $r_2^{(1)}, r_2^{(2)}, t_2^{(1)}, t_2^{(2)}$ are real, we can take $\tilde{\kappa} \rightarrow 0$ along

$$\{(\tilde{\kappa}, \tilde{\omega}) \in \mathbb{R}^2 : \tilde{\omega} + \ell_1^{(1)}\tilde{\kappa} + r_2^{(1)}\tilde{\kappa}^2 + r_3^{(1)}\tilde{\kappa}^3 + \dots + r_n^{(1)}\tilde{\kappa}^n = 0\}$$

to prove that $r_{n+1}^{(1)} \in \mathbb{R}$, and therefore, $r_n^{(1)} \in \mathbb{R}, \forall n \in \mathbb{N}$. Also it is similar to prove $t_n^{(1)}, r_n^{(2)}, t_n^{(2)} \in \mathbb{R}, \forall n \in \mathbb{N}$.

ii) Similarly, if $r_2^{(2)} \neq t_2^{(2)}$ and are both in \mathbb{R} , then one can use the same process to prove $r_n^{(2)} \in \mathbb{R}, \forall n \in \mathbb{N}$ and so we have two real factors of $a(\tilde{\kappa}, \tilde{\omega})$

$$\tilde{\omega} + \ell_1^{(2)} \tilde{\kappa} + r_2^{(2)} \tilde{\kappa}^2 + r_3^{(2)} \tilde{\kappa}^3 + \dots$$

and

$$\tilde{\omega} + \ell_1^{(2)} \tilde{\kappa} + t_2^{(2)} \tilde{\kappa}^2 + t_3^{(2)} \tilde{\kappa}^3 + \dots .$$

For any value of $\tilde{\kappa}$ near 0, there is a value $\tilde{\omega}$ such that the transmission coefficient vanishes and another value $\tilde{\omega}$ for which the reflective coefficient vanishes.

iii) If we assume $r_2^{(1)} = t_2^{(1)}$ and $r_2^{(2)} = t_2^{(2)}$ but not real, then we can prove that there is no spike. In fact,

$$\begin{cases} \tilde{\omega} + \ell_1^{(1)} \tilde{\kappa} + r_2^{(1)} \tilde{\kappa}^2 \neq 0, \\ |\tilde{\omega} + \ell_1^{(1)} \tilde{\kappa} + r_2^{(1)} \tilde{\kappa}^2| \geq |\text{Im}(r_2^{(1)})| \tilde{\kappa}^2, \end{cases}$$

so

$$\begin{aligned} \frac{b}{\ell} &= \frac{\left[\tilde{\omega} + t_1^{(1)} \tilde{\kappa} + t_2^{(1)} \tilde{\kappa}^2 + \dots \right] \left[\tilde{\omega} + t_1^{(2)} \tilde{\kappa} + t_2^{(2)} \tilde{\kappa}^2 + \dots \right] (it_0 e^{i\gamma} + t_{\tilde{\kappa}} \tilde{\kappa} + t_{\tilde{\omega}} \tilde{\omega} + \dots)}{\left[\tilde{\omega} + \ell_1^{(1)} \tilde{\kappa} + \ell_2^{(1)} \tilde{\kappa}^2 + \dots \right] \left[\tilde{\omega} + \ell_1^{(2)} \tilde{\kappa} + \ell_2^{(2)} \tilde{\kappa}^2 + \dots \right] (1 + O(|\tilde{\kappa}| + |\tilde{\omega}|))} \\ &= \frac{\left(1 + \frac{t_3^{(1)} \tilde{\kappa}^3}{\tilde{\omega} + \ell_1^{(1)} \tilde{\kappa} + r_2^{(1)} \tilde{\kappa}^2} + \dots\right) \left(1 + \frac{t_3^{(2)} \tilde{\kappa}^3}{\tilde{\omega} + \ell_1^{(2)} \tilde{\kappa} + r_2^{(2)} \tilde{\kappa}^2} + \dots\right) (it_0 e^{i\gamma} + \dots)}{\left(1 + \frac{\ell_3^{(1)} \tilde{\kappa}^3}{\tilde{\omega} + \ell_1^{(1)} \tilde{\kappa} + r_2^{(1)} \tilde{\kappa}^2} + \dots\right) \left(1 + \frac{\ell_3^{(2)} \tilde{\kappa}^3}{\ell_3^{(2)} + \ell_1^{(2)} \tilde{\kappa} + r_2^{(2)} \tilde{\kappa}^2} + \dots\right) (1 + \dots)} \end{aligned}$$

Therefore,

$$\frac{b}{\ell} = \frac{(1 + O(|\tilde{\kappa}|))(1 + O(|\tilde{\kappa}|))(it_0 e^{i\gamma} + O(|\tilde{\omega}| + |\tilde{\kappa}|))}{(1 + O(|\tilde{\kappa}|))(1 + O(|\tilde{\kappa}|))(1 + O(|\tilde{\omega}| + |\tilde{\kappa}|))}$$

is continuous with respect to κ , nonzero and finite. In this case, the transmission coefficient does not attain the magnitude of 0 and 1. \square

3.6 Transmission Graphs

In this section, we demonstrate different forms of transmission anomalies by choosing different values of the coefficients in the expansions (3.14, 3.21, 3.25) of ℓ , a ,

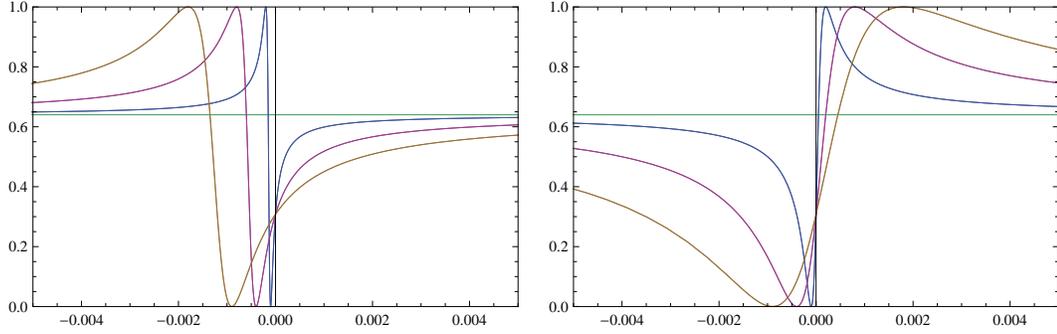


FIGURE 3.2. $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. The generic conditions $\frac{\partial \ell}{\partial \omega}, \frac{\partial a}{\partial \omega}, \frac{\partial b}{\partial \omega} \neq 0$ are satisfied at the bound-state pair (κ_0, ω_0) . In (3.15,3.16), $\ell_1 = 0$ so that there is no linear detuning of the anomaly with $\tilde{\kappa}$. Left: $0 < t_2 = 1 < r_2 = 2$ so that the peak is to the left of the dip and both are to the left of ω_0 . Right: $r_2 = -2 < 0 < t_2 = 1$. In both graphs, $r_0 = 0.6, t_0 = 0.8$. The transmission is symmetric in $\tilde{\kappa}$, and the curve without an anomaly is the transmission graph for $\tilde{\kappa} = 0$.

and b . More specifically, we graph the transmission coefficient

$$|T(\kappa, \omega)|^2 = \left| \frac{b(\kappa, \omega)}{\ell(\kappa, \omega)} \right|^2 = \frac{|b|^2}{|a|^2 + |b|^2}, \quad (3.26)$$

as a function of frequency $\tilde{\omega} = \omega - \omega_0$, keeping only terms up to quadratic order in $\tilde{\kappa} = \kappa - \kappa_0$ in the first factors and only the constant terms in the nonzero factor. This approximation has the accuracy of $\mathcal{O}(|\tilde{\kappa}| + \tilde{\omega}^2)$ in the generic case (see [20, Thm. 16]).

Figures 3.2, 3.3 show the generic case of Section 3.4.2, in which r_2 and t_2 are distinct real numbers. For $\kappa = \kappa_0$ ($\tilde{\kappa} = 0$) the anomaly is absent. As κ is perturbed, i.e. $\tilde{\kappa} \neq 0$, the anomaly appears and widens with width $|t_2 - r_2|\tilde{\kappa}^2$, as shown in Fig. 3.2 for $\ell_1 = 0$. If $\ell_1 \neq 0$, as in Fig. 3.3, then the anomaly is detuned from ω_0 at a rate of $O(\tilde{\kappa})$, whereas it widens with quadratic width $|t_2 - r_2|\tilde{\kappa}^2$.

Figures 3.4 show the degenerate case in which the anomaly pattern is one single dip descending to 0 from a full background transmission or a single peak rising to 1 from a null background transmission (see Section 3.5.1). The peaks reside on two sides of the dip. We show another possibility when the peaks are located on one side of the dip in another Figure 3.5. In particular, if $\ell_1 = 0$, we show the anomaly

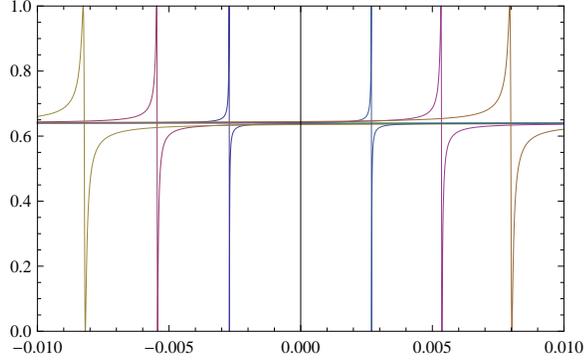


FIGURE 3.3. $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.003, \pm 0.006, \pm 0.009$. The generic conditions $\frac{\partial \ell}{\partial \omega}, \frac{\partial a}{\partial \omega}, \frac{\partial b}{\partial \omega} \neq 0$ are satisfied at the bound-state pair (κ_0, ω_0) . In (3.15,3.16), $\ell_1 = 0.9 \neq 0$, so the anomaly is detuned from $\omega = \omega_0$ ($\tilde{\omega} = 0$) in a linear manner in $\tilde{\kappa}$. The coefficients $r_2 = 2$ and $t_2 = 1$ of $\tilde{\kappa}^2$ are distinct real numbers, and $(r_0, t_0) = (0.6, 0.8)$.

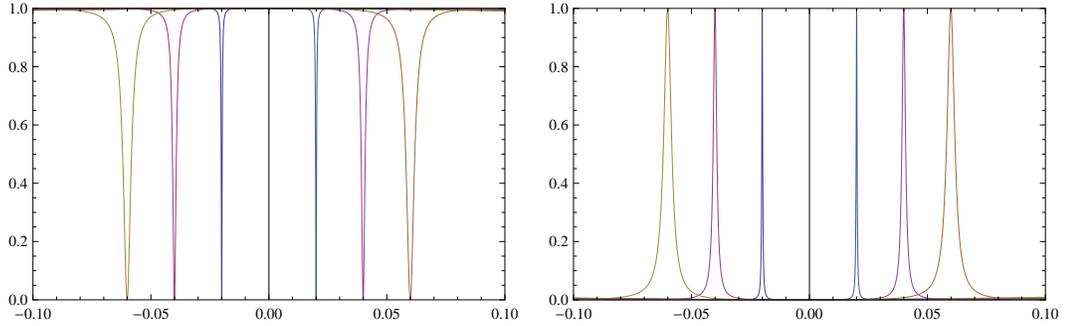


FIGURE 3.4. $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Left: Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} = 0$, and $\frac{\partial b}{\partial \omega} \neq 0$ at (κ_0, ω_0) . In (3.21), $0 < r_1^{(1)} = 0.2 < t_1 = \ell_1 = 2 < r_1^{(2)} = 4$, $(r_2^{(1)}, r_2^{(2)}, t_2) = (7, 7, 0.1)$, and $r_0 = 0.6$. Right: $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} \neq 0$, and $\frac{\partial b}{\partial \omega} = 0$ at (κ_0, ω_0) and $0 < t_1^{(1)} < r_1 = \ell_1 < t_1^{(2)}$.

with single dip and two peaks, as well as two dips and single peak in Fig. 3.6,3.7, and 3.8. In Fig. 3.6 and 3.7, the peaks are on two sides of the dip, or two dips lie on two sides of the single peak, respectively, while in Fig. 3.8, two peaks reside on the same side of the single dip. In the first case, we see that full transmission is actually achieved at precisely two frequencies near $\omega = \omega_0$ ($\tilde{\omega} = 0$), as shown in the magnified, right-hand image of Fig. 3.6.

Without the assumption $r_1^{(1)}, r_1^{(2)} \in \mathbb{R}$, one can still show that the single dip is reached. We also give the figures for the case that $r_1^{(1)}, r_1^{(2)}$ are conjugate for $\ell_1 = 0$ and $\ell_1 \neq 0$ in Fig. 3.10 and Fig. 3.9.

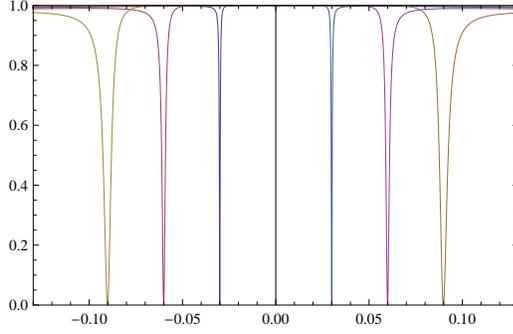


FIGURE 3.5. $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} = 0$, and $\frac{\partial b}{\partial \omega} \neq 0$ at (κ_0, ω_0) . In (3.21), $0 < r_1^{(1)} = 0.6 < r_1^{(2)} = 1.2 < \ell_1 = 3$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, 1, 1, 0.2)$.

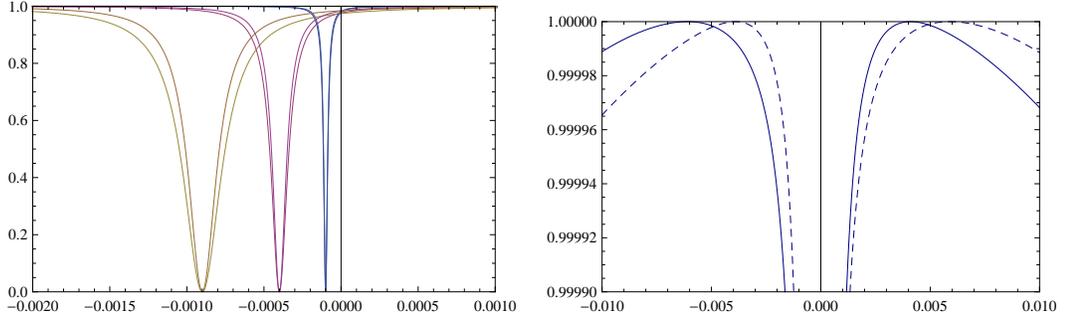


FIGURE 3.6. $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Left: Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} = 0$, and $\frac{\partial b}{\partial \omega} \neq 0$ at (κ_0, ω_0) . In (3.21), $r_1^{(1)} = -0.04 < t_1 = \ell_1 = 0 < r_1^{(2)} = 0.06$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, -1, 1, 1)$. Right: Magnification of the graphs for $\tilde{\kappa} = \pm 0.01$, bringing into view the frequencies of total transmission.

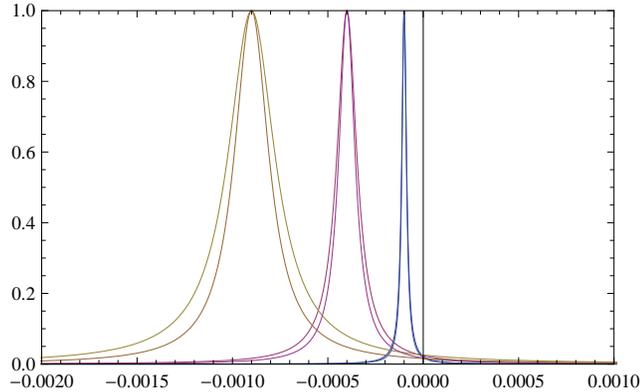


FIGURE 3.7. $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} \neq 0$, and $\frac{\partial b}{\partial \omega} = 0$ at (κ_0, ω_0) and $0 < t_1^{(1)} < r_1 = \ell_1 < t_1^{(2)}$. In (3.21), $t_1^{(1)} = -0.04 < r_1 = \ell_1 = 0 < t_1^{(2)} = 0.06$; $(t_0, t_2^{(1)}, t_2^{(2)}, t_2) = (0.6, -1, 1, 1)$.

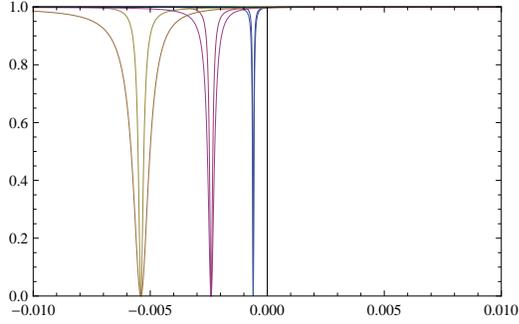


FIGURE 3.8. $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. Full background transmission occurs when $\frac{\partial \ell}{\partial \omega} \neq 0$, $\frac{\partial a}{\partial \omega} = 0$, and $\frac{\partial b}{\partial \omega} \neq 0$ at (κ_0, ω_0) . In (3.21), $t_1 = \ell_1 = 0 < r_1^{(1)} = 0.6 < r_1^{(2)} = 0.8$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, -4, 5, 6)$.

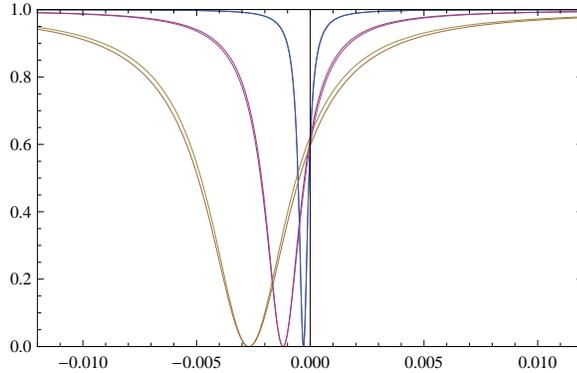


FIGURE 3.9. $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. $\frac{\partial \ell}{\partial \omega} \frac{\partial a}{\partial \omega} \neq 0$, $\frac{\partial b}{\partial \omega} = 0$ at (κ_0, ω_0) . $(r_1^{(1)}, r_1^{(2)}, \ell_1) = (2i, -2i, 0)$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, 2i, 4i, 3)$.

If the first derivatives of ℓ, a, b with respect to ω are zero at (κ_0, ω_0) , under the assumptions in Section 3.5.2, the anomaly has double spikes with peaks and dips. This situation is shown in Figures 3.11 and 3.12 for two possible choices of constants in the Weierstraß expansions. Either sets of the constant choices are possible but we wish to find more properties to refine our results.

The vertical line in all graphs shows the location of $\tilde{\omega} = 0$, or $\omega = \omega_0$.

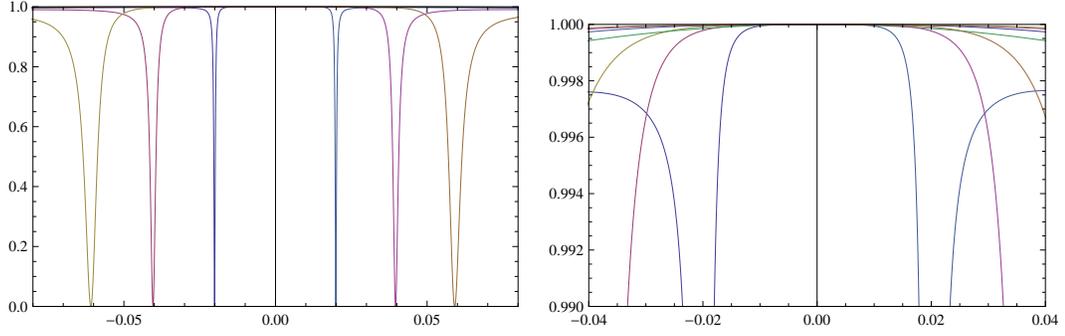


FIGURE 3.10. $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.01, \pm 0.02, \pm 0.03$. $\frac{\partial \ell}{\partial \omega} \frac{\partial a}{\partial \omega} \neq 0$, $\frac{\partial b}{\partial \omega} = 0$ at (κ_0, ω_0) . $(r_1^{(1)}, r_1^{(2)}, \ell_1) = (0.5i, -0.5i, 2)$; $(r_0, r_2^{(1)}, r_2^{(2)}, t_2) = (0.6, i, i, 1)$.

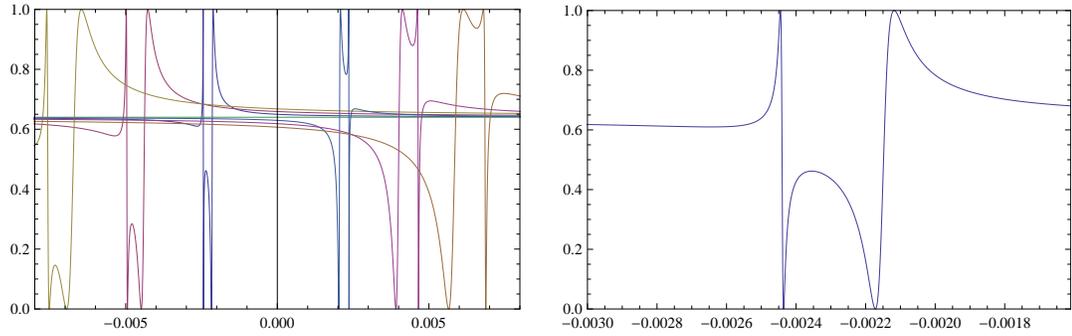


FIGURE 3.11. Left: $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.003, \pm 0.006, \pm 0.009$. The partial derivatives of ℓ , a , and b all vanish at (κ_0, ω_0) , whereas their second derivatives are nonzero. In (3.25), $(\ell_1^{(1)}, \ell_1^{(2)}) = (0.7, 0.8)$, $(r_0, t_0) = (0.6, 0.8)$, $r_2^{(1)} = 2 < t_2^{(1)} = 8$ and $r_2^{(2)} = 4 < r_2^{(2)} = 5$. Right: $\tilde{\kappa} = 0.003$.

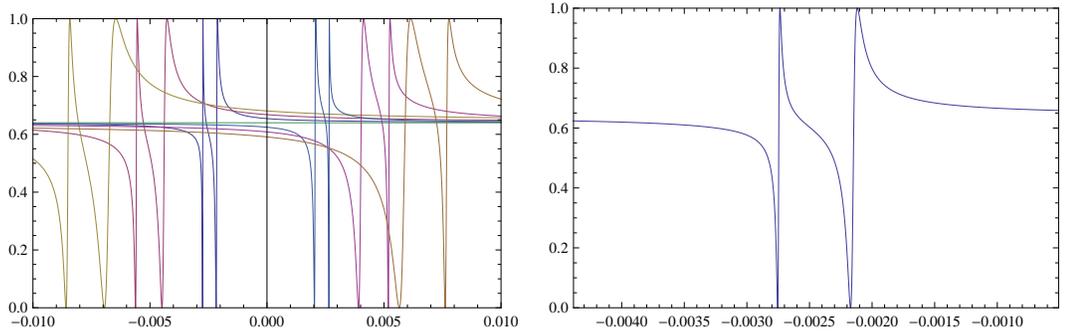


FIGURE 3.12. Left: $|T|^2$ as a function of $\tilde{\omega}$ for $\tilde{\kappa} = 0, \pm 0.003, \pm 0.006, \pm 0.009$. The partial derivatives of ℓ , a , and b all vanish at (κ_0, ω_0) , whereas their second derivatives are nonzero. In (3.25), $(\ell_1^{(1)}, \ell_1^{(2)}) = (0.7, 0.8)$, $(r_0, t_0) = (0.6, 0.8)$, $r_2^{(1)} = 2 < t_2^{(1)} = 8$ and $r_2^{(2)} = 4 < t_2^{(2)} = 6$. Right: $\tilde{\kappa} = 0.003$.

Chapter 4

Guided Modes in Periodic Pillars

This chapter deals with scattering problems and guided modes in periodic pillars. We establish a systematic framework to study plane-wave scattering problems and guided modes in periodic pillars. Existence and nonexistence results are established, among which there is a main new theorem proving the existence of a nontrivial embedded guided mode robust in the wavenumber κ .

4.1 Bessel Functions

We introduce some important properties of Bessel functions to be used in later sections. More details for the properties of Bessel functions can be found in [29].

The Bessel equation

$$\frac{d^2 f}{dz^2} + \frac{1}{z} \frac{df}{dz} + \left(1 - \frac{\ell^2}{z^2}\right) f = 0 \quad (4.1)$$

admits two linearly independent solutions $J_\ell(z)$ and $Y_\ell(z)$. They are called the first and second kind of *Bessel functions*. The third kind of Bessel functions are Hankel functions defined by $H_\ell^1(z) = J_\ell(z) + iY_\ell(z)$, $H_\ell^2(z) = J_\ell(z) - iY_\ell(z)$.

By a change of variables, one sees that a more general form of Bessel equation

$$\frac{d^2 f}{dz^2} + \frac{1}{z} \frac{df}{dz} + \left(\lambda^2 - \frac{\ell^2}{z^2}\right) f = 0, \quad \lambda \in \mathbb{R}$$

has linear independent solutions $J_\ell(\lambda z)$ and $Y_\ell(\lambda z)$. The modified Bessel functions of the first kind and the second kind are $I_\ell(z)$ and $K_\ell(z)$, which solve $\frac{d^2 f}{dz^2} + \frac{1}{z} \frac{df}{dz} - \left(1 + \frac{\ell^2}{z^2}\right) f = 0$. Similar to the Bessel equation, the modified Bessel equation can be generalized to

$$\frac{d^2 f}{dz^2} + \frac{1}{z} \frac{df}{dz} - \left(\lambda^2 + \frac{\ell^2}{z^2}\right) f = 0, \quad \lambda \in \mathbb{R}$$

with independent solutions $I_\ell(\lambda z)$ and $K_\ell(\lambda z)$.

We will use $H_\ell^1(z) = J_\ell(z) + iY_\ell(z)$ and $H_\ell^2(z) = J_\ell(z) - iY_\ell(z)$ as the two complex valued independent solutions in the following. Observing that $-\lambda^2 = (i\lambda)^2$, we can also use $H_\ell^{1,2}(\lambda x)$ as two independent solutions of the unifying equation

$$\frac{d^2 f}{dz^2} + \frac{1}{z} \frac{df}{dz} + \left(\lambda^2 - \frac{\ell^2}{z^2}\right) f = 0, \quad \lambda \in \mathbb{R} \text{ or } i\mathbb{R}. \quad (4.2)$$

The Bessel function $J_\ell(z)$ has a sequence of zeros $j_{\ell,n} \rightarrow \infty$ as $z \rightarrow \infty$, and $j_{\ell,n} > \ell$. The modified function K_ℓ is strictly decreasing. If $\ell \neq 0$, the function $I_\ell(z)$ is nonzero except at 0, and $I_\ell(z)$ is strictly increasing. If $\ell = 0$, $I_0(0) > 0$ and $I_0(z)$ increases to ∞ as $z \rightarrow \infty$.

For Hankel functions we have the asymptotic expansions for large arguments

$$H_\ell^1(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\ell\pi}{2} - \frac{\pi}{4})} (1 + o(z^{-1})), \quad z > 0, \quad (4.3)$$

$$H_\ell^2(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\ell\pi}{2} - \frac{\pi}{4})} (1 + o(z^{-1})), \quad z > 0. \quad (4.4)$$

If $0 < z \in \mathbb{R}$, then H_ℓ^1 is outgoing and H_ℓ^2 is incoming (given $\omega > 0$). If $z \in i\mathbb{R}$ and $z = i|z|$, then H_ℓ^1 is exponentially decaying as $|z| \rightarrow \infty$, and H_ℓ^2 is exponentially growing as $|z| \rightarrow \infty$.

The modified Bessel function K_ℓ has the following relation with H_ℓ^1 :

$$K_\ell(z) = \frac{1}{2} \pi i e^{\frac{1}{2} \ell \pi i} H_\ell^1(iz).$$

The following results hold for the multiplier $\gamma_{m\ell}$ used in the definition of the Dirichlet-to-Neumann map in the next section.

Lemma 34. If $\eta_m = i|\eta_m|$, $R > 0$, then the multiplier $\gamma_{m\ell} = -\eta_m \frac{H_\ell^1(\eta_m R)}{H_\ell^1(\eta_m R)} > 0$. If $\eta_m > 0$, $R > 0$, then the imaginary part of the multiplier $\text{Im} \left(-\eta_m \frac{H_\ell^1(\eta_m R)}{H_\ell^1(\eta_m R)} \right) \neq 0$.

Proof. Taking $z = |\eta_m|R$, if $\eta_m = i|\eta_m|$, we have

$$\begin{aligned} -\eta_m \frac{H_\ell^{1'}(\eta_m R)}{H_\ell^1(\eta_m R)} &= -|\eta_m| i \frac{-iK_\ell'(|\eta_m|R)/(\frac{\pi}{2}ie^{\ell\pi i/2})}{K_\ell(|\eta_m|R)/(\frac{\pi}{2}ie^{\ell\pi i/2})} \\ &= -|\eta_m| \frac{K_\ell'(|\eta_m|R)}{K_\ell(|\eta_m|R)} \\ &> 0. \end{aligned}$$

If $\eta_m = |\eta_m| > 0, R > 0$, then

$$\text{Im} \left(-\eta_m \frac{H_\ell^{1'}(\eta_m R)}{H_\ell^1(\eta_m R)} \right) = \text{Im} \left(-|\eta_m| \frac{(J_\ell' + iY_\ell')(J_\ell - iY_\ell)}{J_\ell^2 + Y_\ell^2} \right) = -|\eta_m| \frac{J_\ell'Y_\ell - J_\ell Y_\ell'}{J_\ell^2 + Y_\ell^2}$$

The numerator of the last fraction is the Wronskian determinant $\begin{vmatrix} J_\ell & Y_\ell \\ J_\ell' & Y_\ell' \end{vmatrix}$ and is therefore nonzero. \square

The Bessel function $J_\ell(Z)$ is the *generating function* of $e^{\frac{1}{2}Z(t-\frac{1}{t})}$:

$$e^{\frac{1}{2}Z(t-\frac{1}{t})} = \sum_{\ell=-\infty}^{\infty} t^\ell J_\ell(Z). \quad (4.5)$$

If we let $t = e^{i(\theta+\theta_0)}$ to obtain $e^{iZ \sin(\theta+\theta_0)} = \sum_\ell J_\ell(Z) e^{i\ell(\theta+\theta_0)}$. then with $\sin \theta_0 = \frac{\kappa_1}{\eta_m}$, $\cos \theta_0 = \frac{\kappa_2}{\eta_m}$, and $Z = \eta_m r$. The incident wave can be written as a superposition of Hankel functions:

$$\begin{aligned} e^{i(\kappa_1 x + \kappa_2 y + \kappa_3 z)} &= e^{i(\eta_m r \cos \theta \sin \theta_0 + \eta_m r \sin \theta \cos \theta_0)} e^{i\kappa_3 z} \\ &= e^{i\eta_m r \sin(\theta+\theta_0)} e^{i(m+\kappa)z} \\ &= \sum_{\ell \in \mathbb{Z}} J_\ell(\eta_m r) e^{i\ell(\theta+\theta_0)} e^{i(m+\kappa)z} \\ &= \sum_{\ell \in \mathbb{Z}} \frac{1}{2} [H_\ell^1(\eta_m r) + H_\ell^2(\eta_m r)] e^{i\ell(\theta+\theta_0)} e^{i(m+\kappa)z}. \end{aligned} \quad (4.6)$$

As a result, the scattering problem of plane waves can be reduced to the linear superposition of propagating Fourier harmonics with Hankel functions given in the next section.

4.2 Media Structure and Scattering Problem

4.2.1 Pillar Structure and Radiation Condition

We consider an infinitely long pillar that is periodic in the z -direction with period 2π and bounded in the x, y directions.

$$\epsilon(x, y, z + 2\pi) = \epsilon(x, y, z), \quad \mu(x, y, z + 2\pi) = \mu(x, y, z), \quad \forall x, y, z.$$

We use $\Omega = \{(x, y, z) : -\pi < z < \pi\}$ to denote one period. Suppose that $\epsilon = \epsilon_0, \mu = \mu_0$ for $r > R$ and we denote the restricted domain $\Omega_R = \{(x, y, z) : -\pi < z < \pi, r = \sqrt{x^2 + y^2} < R\}$, which is a cylinder whose boundary consists of $\Gamma_R = \{(x, y, z) \in \Omega : -\pi < x, y < \pi, r = R\}$ plus the upper and lower horizontal disks.

The spatial factor of a time-harmonic acoustic or electromagnetic wave is governed by the Helmholtz equation

$$\nabla \cdot \frac{1}{\mu} \nabla u(x, y, z) + \epsilon \omega^2 u(x, y, z) = 0. \quad (4.7)$$

By the κ -pseudo-periodicity, u can be expanded as an infinite superposition $u(x, y, z) = \sum_{m=-\infty}^{\infty} u_m(x, y) e^{i(m+\kappa)z}$. where $\kappa \in B = [-1/2, 1/2)$. Let $(x, y) = (r \cos \theta, r \sin \theta)$.

If $r = \sqrt{x^2 + y^2} > R$, then

$$\Delta u_m(x, y) + \eta_m^2 u_m(x, y) = 0$$

where $\eta_m = \mu_0 \epsilon_0 \omega^2 - (m + \kappa)^2$. Using polar coordinates, we have

$$\frac{\partial^2 u_m}{\partial r^2} + \frac{1}{r} \frac{\partial u_m}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_m}{\partial \theta^2} + \eta_m^2 u_m = 0.$$

The function u_m can be written as an expansion of separable solutions

$$u_m = \sum_{\ell=-\infty}^{\infty} R_{m,\ell}(r) e^{i\ell\theta},$$

where $R_{m,\ell}(r)$ satisfies

$$R''_{m,\ell} + \frac{1}{r}R'_{m,\ell} - \frac{\ell^2}{r^2}R_{m,\ell} + \eta_m^2 R_{m,\ell} = 0.$$

This equation has solutions

$$R_{m,\ell}(r) = \begin{cases} a_{m\ell}H_\ell^1(\eta_m r) + b_{m\ell}H_\ell^2(\eta_m r), & \text{if } \eta_m \neq 0, \\ c_{m1} + c_{m2} \ln |r|, & \text{if } \eta_m = 0, \ell = 0, \\ c_{m\ell 1}|r|^\ell + c_{m\ell 2}|r|^{-\ell}, & \text{if } \eta_m = 0, \ell \neq 0. \end{cases} \quad (4.8)$$

Therefore, the spatial field u can be expanded as an infinite superposition of Fourier harmonics in $\Omega \setminus \Omega_R$:

$$u(x, y, z) = \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} R_{m,\ell}(r) e^{i\ell\theta} e^{i(m+\kappa)z}. \quad (4.9)$$

In this expansion, the Hankel functions $H_\ell^1(\eta_m r)$ are outgoing or exponentially decaying, depending on whether η_m is imaginary or real, as $r \rightarrow \infty$, and the Hankel functions $H_\ell^2(\eta_m r)$ are incoming or exponentially growing.

The following radiation condition is required for the problem of scattering by a periodic pillar.

Condition 35 (Radiation condition). A field $u(r, \theta, z)$ satisfies the radiation condition if it admits the following Fourier-Bessel representation for $r > R$:

$$u(r, \theta, z) = \sum_{m \in \mathcal{Z}_p \cup \mathcal{Z}_e} \sum_{\ell \in \mathbb{Z}} a_{m\ell} H_\ell^1(\eta_m r) e^{i\ell\theta} e^{i(m+\kappa)z} + \sum_{m \in \mathcal{Z}_\ell} \left[\sum_{\ell > 0} c_{m\ell 2} |r|^{-\ell} e^{i\ell\theta} + \sum_{\ell < 0} c_{m\ell 1} |r|^\ell e^{i\ell\theta} \right] e^{i(m+\kappa)z} \quad (4.10)$$

where the sets $\mathcal{Z}_{p,a,e}$ of \mathcal{Z} depend on κ and are defined by

$$m \in \mathcal{Z}_p \Leftrightarrow \eta_m^2 > 0, \eta_m > 0 \text{ (propagating harmonics)}$$

$$m \in \mathcal{Z}_a \Leftrightarrow \eta_m^2 = 0, \eta_m = 0 \text{ (algebraic harmonics)}$$

$$m \in \mathcal{Z}_e \Leftrightarrow \eta_m^2 < 0, -i\eta_m > 0 \text{ (evanescent harmonics)} .$$

4.2.2 Scattering Problems

Before studying the guided modes, we first consider the scattering of a plane wave.

Problem 36 (Scattering problem, strong form). Given $\epsilon_0, \mu_0 > 0$, find u on Ω such that

$$\left\{ \begin{array}{l} \nabla \cdot \frac{1}{\mu} \nabla u + \epsilon \omega^2 u = 0 \text{ in } \Omega, \\ u \text{ is continuous on } \partial\Omega, \\ \frac{1}{\mu} \frac{\partial u}{\partial n} \text{ is continuous on } \partial\Omega, \\ u^{inc} = \sum_{m \in \mathcal{Z}_p} u_m^{inc} e^{i(\kappa_1 x + \kappa_2 y + (m+\kappa)z)}, \\ u^{sc} = u - u^{inc} \text{ and its derivatives are } \kappa\text{-periodic in } z, \\ u^{sc} = u - u^{inc} \text{ satisfies the radiation condition.} \end{array} \right. \quad (4.11)$$

On the truncated domain Ω_R , define the pseudo-periodic field space $H_\kappa^1(\Omega_R) = \{u \in H^1(\Omega_R) : u(x, y, \pi) = u(x, y, -\pi)e^{2\pi\kappa i}\}$. On the vertical boundary Γ_R , the radiation condition is characterized by a Dirichlet-to-Neumann map $T : H_\kappa^{\frac{1}{2}}(\Gamma_R) \rightarrow H_\kappa^{-\frac{1}{2}}(\Gamma_R)$ (as in the Definition 5.19 of [2])

$$T : \sum_{m, \ell} \hat{u}_{m\ell} e^{i\ell\theta} e^{i(m+\kappa)z} \mapsto \sum_{m, \ell} \gamma_{m\ell} \hat{u}_{m\ell} e^{i\ell\theta} e^{i(m+\kappa)z}, \quad (4.12)$$

where

$$\gamma_{m\ell} = \begin{cases} \frac{-\eta_m H_\ell'(\eta_m R)}{H_\ell^1(\eta_m R)}, & \text{if } m \notin \mathcal{Z}_a, \\ |\ell| R^{-1}, & \text{if } m \in \mathcal{Z}_a \text{ and } \ell \neq 0, \\ 0, & \text{if } m \in \mathcal{Z}_a \text{ and } \ell = 0 \end{cases}$$

To satisfy the radiation condition, the harmonics in (4.9) with $H_\ell^2(\eta_m r)$ for $m \in \mathcal{Z}_p \cup \mathcal{Z}_e$, harmonics $(c_{m1} + c_{m2} \ln |r|) e^{i\ell\theta} e^{i(m+\kappa)z}$ for $m \in \mathcal{Z}_a, \ell = 0$, and harmonics with $|r|^\ell$ for $m \in \mathcal{Z}_a, \ell > 0$ all vanish. The radiation condition is hence enforced by

$$\partial_n u + Tu = 0 \text{ on } \Gamma_R. \quad (4.13)$$

The operator T is split into two parts

$$T = T_e + T_p, \quad (4.14)$$

$$\widehat{(T_e f)}_{m\ell} = \begin{cases} \frac{-\eta_m H_\ell^{1'}(\eta_m R)}{H_\ell^1(\eta_m R)} \hat{f}_{m\ell}, & \text{if } m \in \mathcal{Z}_e, \\ |\ell| R^{-1} \hat{f}_{m\ell}, & \text{if } m \in \mathcal{Z}_a \text{ and } \ell \neq 0, \\ 0, & \text{otherwise} \end{cases} \quad (4.15)$$

$$\widehat{(T_p f)}_{m\ell} = \begin{cases} \frac{-\eta_m H_\ell^{1'}(\eta_m R)}{H_\ell^1(\eta_m R)} \hat{f}_{m\ell}, & \text{if } m \in \mathcal{Z}_p, \\ 0, & \text{otherwise} \end{cases} \quad (4.16)$$

Note that the multipliers $\gamma_{m\ell}$ in T_e are nonnegative. In T_p , the multipliers have nonzero imaginary parts for any $m \in \mathcal{Z}_p$ (see Lemma 34).

The variational form of the scattering problem in the truncated domain is

Problem 37 (Scattering problem, variational form).

$$\begin{cases} u \in H_\kappa^1(\Omega_R) \\ a(u, v) - \omega^2 b(u, v) = f(v), \forall v \in H_\kappa^1(\Omega_R) \end{cases} \quad (4.17)$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega_R} \frac{1}{\mu} \nabla u \cdot \nabla \bar{v} + \frac{1}{\mu_0} \int_{\Gamma_R} (Tu) \bar{v} \\ b(u, v) &= \int_{\Omega_R} \epsilon u \bar{v}, \\ f(v) &= \frac{1}{\mu_0} \int_{\Gamma_R} [(\partial_n u^{inc} + Tu^{inc}) \bar{v}]. \end{aligned}$$

Similar to the analysis in Chapter 2, we can prove the existence of the scattered wave by Fredholm alternative theory. The weak form PDE in problem 37 can be written as

$$a(u, v) - \omega^2 b(u, v) = c_1(u, v) + c_2(u, v)$$

with $c_1(u, v) = \int_{\Omega_R} (\frac{1}{\mu} \nabla u \cdot \nabla \bar{v} + \epsilon u \bar{v}) + \frac{1}{\mu_0} \int_{\Gamma_R} (Tu) \bar{v}$ and $c_2(u, v) = -\epsilon(\omega^2 + 1) \int_{\Omega_R} u \bar{v}$. Define operators C_1 and C_2 on $H_\kappa^1(\Omega_R)$ by $(C_1 u, v)_{H_\kappa^1(\Omega_R)} = c_1(u, v)$ and $(C_2 u, v)_{H_\kappa^1(\Omega_R)} = c_2(u, v)$. Because of the coercivity of c_1 and the compact embedding of $L^2(\Omega)$ into $H_\kappa^1(\Omega_R)$, the operator C_1 is an automorphism and C_2 is compact.

If we denote by f^{inc} the unique element of $H_\kappa^1(\Omega_R)$ such that $(f^{inc}, v)_{H_\kappa^1(\Omega_R)} = f(v)$, the variational form of the scattering problem can be characterized by the following operator form

$$C_1 u + C_2 u = f^{inc}.$$

The Fredholm alternative theory implies that the nonuniqueness of the solution of this problem is equivalent to the singularity of the corresponding homogeneous problem $C_1 u + C_2 u = 0$, whose weak form is given by

$$a(u, v) - \omega^2 b(u, v) = 0, \forall v \in H_\kappa^1(\Omega_R) \quad (4.18)$$

Theorem 38. The plane wave scattering problem has at least one solution, and the set of solutions is at most finite dimensional.

Proof. From equation (4.6), we can express the incident plane wave as a superposition of harmonics $\sum_\ell \frac{1}{2} [H_\ell^1(\eta_m r) + H_\ell^2(\eta_m r)] e^{i\ell(\theta+\theta_0)} e^{i(m+\kappa)z}$, with $m \in \mathcal{Z}_p$. By the Fredholm alternative, the scattering problem has a solution if and only if

$$(f^{inc}, w) = 0, \text{ for all } w \in \text{Null}(C_1 + C_2)^\dagger,$$

i.e. for all w such that

$$a(v, w) - \omega^2 b(v, w) = 0, \forall v \in H_\kappa^1(\Omega_R)$$

This w satisfies

$$\overline{a(w, v)} - \omega^2 b(w, v) = 0, \forall v \in H_\kappa^1(\Omega_R)$$

and by the decomposition of T , we know that for all $m \in \mathcal{Z}_p$, $\hat{w}_m = 0$. By the definition of f^{inc} , showing $(f^{inc}, w) = 0$ is equivalent to showing that $\int_{\Gamma_R} (\partial_n + T)u^{inc}\bar{w} = 0$. This is satisfied by the function w above.

The space of solutions is finite-dimensional because C_1 is invertible and C_2 is compact. \square

4.3 Guided Modes

A guided mode is a solution to the Helmholtz equation in the periodic domain in the absence of any source field. In the weak form, it is a solution to the homogeneous equation (4.18).

The sesquilinear form

$$a^\omega(u, v) = \int_{\Omega_R} \frac{1}{\mu} \nabla u \cdot \nabla \bar{v} + \frac{1}{\mu_0} \int_{\Gamma_R} (T^\omega u) \bar{v}$$

can be split into evanescent and propagating parts,

$$\begin{aligned} a_e^\omega(u, v) &= \int_{\Omega_R} \frac{1}{\mu} \nabla u \cdot \nabla \bar{v} + \frac{1}{\mu_0} \int_{\Gamma_R} (T_e^\omega u) \bar{v}, \\ a_p^\omega(u, v) &= \frac{1}{\mu_0} \int_{\Gamma_R} (T_p^\omega u) \bar{v}. \end{aligned}$$

In this chapter we assume that the frequency and the wavenumber are real. Note that in the decomposition (4.14), the multiplier $\gamma_{m\ell}$ has a nonzero imaginary part for $m \in \mathcal{Z}_p$, thus $a(u, u) = 0$ if and only if $\widehat{(u|_{\Gamma_R})}_m = 0$, for all $m \in \mathcal{Z}_p$.

Theorem 39. (Real eigenvalues) If the frequency ω is real, then $u \in H_\kappa^1(\Omega_R)$ solves the equation (4.18) if and only if

$$a_e^\omega(u, v) - a_p^\omega(u, v) - \omega^2 b(u, v) = 0, \forall v \in H_\kappa^1(\Omega_R), \quad (4.19)$$

and if and only if

$$\begin{cases} a_e^\omega(u, v) - \omega^2 b(u, v) = 0, \forall v \in H_\kappa^1(\Omega_R), \\ \widehat{(u|_{\Gamma_R})}_m = 0, \forall m \in \mathcal{Z}_p. \end{cases} \quad (4.20)$$

The eigenfrequencies can be obtained by applying the min-max principle to the real form in (4.20). When $\omega < \sqrt{\frac{\kappa^2}{\epsilon_0\mu_0}}$, the solutions u of $a_r^\omega(u, v) - \omega^2 b(u, v) = 0, \forall v \in H_\kappa^1(\Omega_R)$ are guided modes since this regime admits no propagating harmonics and so the second conditions in (4.20) are automatically satisfied. When $\omega \geq \sqrt{\frac{\kappa^2}{\epsilon_0\mu_0}}$, to be guided modes, these solutions u must satisfy the extra conditions $(\widehat{u|_{\Gamma_R}})_m = 0, \forall m \in \mathcal{Z}_p$ where \mathcal{Z}_p is nonempty. We will design some periodic structures that admit guided modes in the next section. We have the following theorem on properties of the frequencies. The proof is similar to that for periodic slabs, for which one may refer to [27] [1].

Theorem 40. (Eigenvalues and characteristic frequencies) The problem $a_e^\omega(u, v) - \lambda b(u, v) = 0, \forall v \in H_\kappa^1(\Omega_R)$ has a nondecreasing sequence of eigenvalues $\{\lambda_j\}_{j=1}^\infty$, obtained through the min-max principle,

$$\lambda_j = \sup_{\dim V=j-1, V \subset H_\kappa^1(\Omega_R)} \inf_{u \in V^\perp \setminus \{0\}} \frac{a_e(u, u)}{b(u, u)}, \quad (4.21)$$

which tend to $+\infty$ as $j \rightarrow \infty$. Moreover, the homogeneous problem $a_e^\omega(u, v) - \omega^2 b(u, v) = 0, \forall v \in H_\kappa^1(\Omega_R)$ has a nontrivial solution if and only if $\omega^2 = \lambda_j(\omega)$, in which we denote it by ω_j .

If the material is piecewise, i.e.,

$$\begin{cases} \epsilon = \epsilon_1, & \mu = \mu_1 \text{ in } \Omega_1 \subset \Omega_R \\ \epsilon = \epsilon_0, & \mu = \mu_0 \text{ in } \Omega \setminus \Omega_1 \end{cases} \quad (4.22)$$

then each frequency ω_j is a continuous function of ϵ_1 that decreases from $+\infty$ to 0, as ϵ_1 increases from 0 to $+\infty$ and μ_1 is fixed. Similarly, the frequency ω_j is a continuous function of μ_1 that decreases from $+\infty$ to 0, as μ_1 is increased from 0 to $+\infty$ and ϵ_1 is fixed.

The functional framework can be applied to determine the associated spectrum.

We can derive the weak form of the guided modes problem

$$a_S(u, v) = \omega^2 b_S(u, v), \forall v \in H_\kappa^1(\Omega) \quad (4.23)$$

where

$$a_S(u, v) = \int_\Omega \frac{1}{\mu} \nabla u \cdot \nabla \bar{v}, \quad (4.24)$$

$$b_S(u, v) = \int_\Omega \epsilon u \bar{v}. \quad (4.25)$$

The associated operator is the unbounded operator

$$S_\kappa u = -\frac{1}{\epsilon} \nabla \cdot \frac{1}{\mu} \nabla u. \quad (4.26)$$

It is defined on the domain $D(S_\kappa) = \{u \in H_\kappa^1(\Omega) : \exists C \text{ such that } |a_S(u, v)| \leq C \sqrt{b_S(v, v)}, \forall v \in H_\kappa^1(\Omega)\}$. This operator is positive self-adjoint and its eigenvectors and eigenvalues are solutions of the guided modes problem. We denote the spectrum of S_κ as σ , and its essential spectrum as σ_{ess} . The following theorem is an adaptation of Theorem 4.1 of [1] to periodic pillars.

Theorem 41. i) $\sigma \subset [\frac{\kappa^2}{\mu_+ \epsilon_+}, +\infty)$, where $\mu_+ = \sup_\Omega \mu, \epsilon_+ = \sup_\Omega \epsilon$;

ii) $\sigma^{ess} = [\frac{\kappa^2}{\mu_0 \epsilon_0}, +\infty)$;

iii) there are finitely many eigenvalues $\tilde{\lambda}_j(\kappa)$ below $\frac{\kappa^2}{\mu_0 \epsilon_0}$, and $\{\tilde{\lambda}_j(\kappa)\}$ is an increasing sequence that converges to $\frac{\kappa^2}{\mu_0 \epsilon_0}$.

4.4 Existence and Nonexistence of Guided Modes

4.4.1 Existence

The focus of this section is to find guided modes with frequency ω such that ω^2 is embedded in the continuous spectrum of S_κ . As discussed in the previous section, certain extra conditions should be satisfied and hence bring the difficulty.

In [1], guided modes are proved to exist in a symmetric structure and a periodic slab with a finer periodicity. The idea is to consider a closed subspace F on which the operator S_κ has a cutoff frequency that is greater the cutoff frequency on $H_\kappa^1(\Omega_R)$, and prove the existence of guided modes corresponding to eigenfrequencies lying between these two cutoff frequencies. These eigenfunctions are automatically guided modes lying in F because their frequencies are below the cutoff frequency for F , but the frequencies are embedded in the essential spectrum of S_κ on $H_\kappa^1(\Omega_R)$. In their proof, the embedded guide modes retain the original pseudo-periodicity, but they are simply non-embedded guided modes with a smaller pseudo-period. By artificially choosing a larger period, any guided modes with frequencies below the cutoff frequency can be seen as embedded guided modes in the same structure with the larger period. In this chapter, we present a proof of the existence of non-artificial guided modes with frequencies embedded in the essential spectrum the operator S_κ . We only need the parameters ϵ, μ to have smaller period, but the guided modes do not have smaller pseudo-period.

Our newly designed pillar is a periodic structure with period $\frac{2\pi}{L}$ for $L \geq 2$ in \mathbb{Z} that supports guided modes with pseudo-period strictly greater than $\frac{2\pi}{L}$.

Theorem 42. For any κ in the first Brillouin zone of the structure of period 2π , there exists ϵ, μ with period $\frac{2\pi}{L}$ for $L \geq 2$ that admits a guided mode with frequencies ω lying above the cutoff frequency, and with smallest pseudo-period strictly greater than $\frac{2\pi}{L}$.

Proof. Write $u \in H_\kappa^1(\Omega_R)$ as a Fourier expansion $u(r, \theta, z) = \sum_m u_m(r, \theta) e^{i(m+\kappa)z}$. Given $M, N \in \mathbb{N}$ with $2M + N + 2 = L$, define a nontrivial subspace of $H_k^1(\Omega)$:

$$V = \{u \in H_k^1(\Omega) : u_m(r, \theta) \equiv 0, \text{ if } |m - j(2M + N + 2)| \leq M \text{ for some } j \in \mathbb{Z}\} \quad (4.27)$$

Therefore, for $-M + j(2M + N + 2) \leq m \leq M + j(2M + N + 2)$, the coefficients $u_m(r, \theta)$ are 0, and for $M + 1 + j(2M + N + 2) \leq m \leq M + N + 1 + j(2M + N + 2)$, the coefficients $u_m(r, \theta)$ are possibly nonzero.

We claim that $\epsilon V \subseteq V$, $\mu^{-1}V \subseteq V$. In fact, let $(\epsilon)_m(r, \theta)$ be the Fourier coefficients of ϵ . The periodicity of the structure implies that $(\epsilon)_m(r, \theta) \equiv 0, \forall r, \theta$, except when $m = j(2M + N + 2)$ for some integer j . For any $u \in V$, if $|m - j(2M + N + 2)| \leq M$ for some $j \in \mathbb{Z}$, we calculate the m^{th} Fourier coefficient of ϵu :

$$\begin{aligned} (\epsilon u)_m &= \sum_{\ell} (\epsilon)_{\ell} u_{m-\ell} \\ &= \sum_j (\epsilon)_{j(2M+N+2)} u_{m-j(2M+N+2)} \\ &= 0, \text{ because } u_{m-j(2M+N+2)} = 0 \text{ for the field } u \in V. \end{aligned}$$

Therefore, $\epsilon u \in V$. Similarly, $\mu^{-1}V \subseteq V$.

Therefore the subspace V is also invariant under the operator $\nabla \cdot \frac{1}{\mu} \nabla$. Thanks to the invariance properties, we can consider the Helmholtz equation in the subspace V . The solution $u \in V$ to the weak formulation $a_r^{\omega}(u, v) - \omega^2 b(u, v) = 0, \forall v \in V$ is also a solution to $a_r^{\omega}(u, v) - \omega^2 b(u, v) = 0, \forall v \in H_{\kappa}^1(\Omega_R)$. In fact, for any field $u \in V$ and $v \in V^{\perp}$, $\nabla \cdot \frac{1}{\mu} \nabla u + \omega^2 \epsilon u \in V$ implies that $\nabla \cdot \frac{1}{\mu} \nabla u \bar{v} + \omega^2 \epsilon u \bar{v} = 0$ for all $v \in V^{\perp}$. Integrating it we obtain

$$\begin{aligned} \int_{\Omega} \nabla \cdot \frac{1}{\mu} \nabla u \bar{v} + \int_{\Omega} \epsilon u \bar{v} &= \frac{1}{\mu_0} \int_{\Gamma_R} \partial_n u \bar{v} - \int_{\Omega} \frac{1}{\mu} \nabla u \cdot \nabla \bar{v} + \int_{\Omega} \epsilon u \bar{v} \\ &= -a_r^{\omega}(u, v) + b(u, v) \\ &= 0. \end{aligned}$$

We can obtain a pair (ω, u) by applying the min-max principle to the Rayleigh quotient $\frac{a_r(u, u)}{b(u, u)}$ on the subspace V to obtain $\lambda_j(\omega)$ and solving the equation $\lambda_j(\omega) = \omega^2$. Since ω is continuous and decreasing from $+\infty$ to 0 in ϵ_1, μ_1 separately, one can choose the material parameters such that $\epsilon_0 \mu_0 \omega^2 - (M + 1 + \kappa)^2 < 0 < \epsilon_0 \mu_0 \omega^2 -$

$(M+\kappa)^2$, i.e. for any pair (κ, ω) there are $2M+1$ values $-M, -M+1, \dots, M-1, M$ of m corresponding to propagating harmonics.

The field u obtained in the space V is automatically a guided mode, as the propagating harmonics automatically vanish in the subspace V . \square

As an example, if we let $M = N = 0$, then $2M + N + 2 = 2$, $2M + 1 = 1$, $N + 1 = 1$. The pillar has period π and $\epsilon_{2j+1} = 0$ for all j , and we can allow one propagating harmonic. We apply the min-max principle on the space $V = \{u \in H_\kappa^1(\Omega_R) : u_{2j} = 0, \forall j\}$ and by choosing proper ϵ_1 we can obtain an eigenfunction of smallest period 2π that is automatically a guided mode.

If we take $M = 1, N = 0$, then $2M + N + 2 = 4$, $2M + 1 = 3$ and $N + 1 = 1$. Let ϵ, μ have period $\pi/2$ and so $\epsilon_j = 0$ for $j \notin 4\mathbb{Z}$, or say $\forall j$, and we can allow to have up to $2M + 1 = 3$ propagating harmonics. One can minimize the Rayleigh quotient on the space $V = \{u \in H_\kappa^1(\Omega_R) : u_{4j-1} = u_{4j} = u_{4j+1} = 0, \forall j\}$. If we take $M = N = 1$, then $2M + N + 2 = 5$, $2M + 1 = 3$, and $N + 1 = 2$. The parameters ϵ and μ have period $2\pi/5$ and can be allowed to have up to $2M + 1 = 3$ propagating harmonics. We apply the min-max principle on the space $V = \{u \in H_\kappa^1(\Omega_R) : u_{5j+1} = u_{5j+2} = u_{5j+3} = u_{5j+4} = 0, \forall j\}$. The pseudo-period of the embedded guided mode is 2π .

In our design, the wave number κ can be nonzero and there exists a continuous embedded dispersion relation $\omega(\kappa)$. The guided mode is robust with respect to κ . It is also noticed that the modes are subject to the periodicity $\frac{2\pi}{2M+N+2}$. If the material is perturbed in a way that destroys the smaller periodicity while retaining the period 2π , the guided mode typically vanishes.

This design can also be understood as an existence proof of a guided mode with a larger pseudo-periodicity. If we assume the smallest period of the pillar is 2π , embedded guided modes with period $(2M + N + 2)2\pi$ can exist.

4.4.2 Nonexistence

Nonexistence results for slabs can be found in [27][1]. In [27], the nonexistence of guided modes in inverse structures is discussed. Consider the piecewise constant material as in Theorem 40. An inverse structure is a periodic structure with the material parameters ϵ_1, μ_1 less than the corresponding parameters ϵ_0, μ_0 in the exterior of the material. The proof in [27] requires that the slab satisfy a certain restriction. The proof of the nonexistence includes introducing the subspace X in which the propagating and linear harmonics vanish then estimating the minimum of the Rayleigh quotient. With the restriction on the slab width, it is shown that the Rayleigh quotient a^ω/b is strictly bounded below by ω^2 in inverse structures, and hence the weak problem has no solution in X . We use an analogous restriction on the radius of the pillar in our proof, and whether this restriction is necessary remains an open problem.

In [1], the assumption for the nonexistence proof is on the parameters only. It is assumed that there exists one plane parallel to the slab such that the material parameters ϵ, μ are nondecreasing in the direction perpendicular to the slab. In Theorem 44, we present an analogous condition that the material parameters are nondecreasing in the radial direction. The proof involves an appropriate Rayleigh identity.

Theorem 43. Assume the material is a piecewise constant pillar defined in (4.22) and $\epsilon_1 < \epsilon_0$ and $\mu_1 < \mu_0$. Let the frequency ω and the wave number κ be given in

the first Brillouin zone $[-\frac{1}{2}, \frac{1}{2})$. Suppose that the radius R of the pillar satisfies

$$R \leq \frac{1}{\sqrt{\epsilon_0 \mu_0 \omega^2 - \kappa^2}} \quad (4.28)$$

Then the periodic pillar does not admit any guided modes at the given frequency and wavenumber.

Proof. We restrict to the subspace $X \subset H_\kappa^1(\Omega_R)$ with

$$X = \left\{ u \in H_\kappa^1(\Omega_R) : \int_{\Gamma_R} u(x, y, z) e^{-i\ell\theta} e^{-i(m+\kappa)z} = 0, \right. \\ \left. \text{if either } m \in \mathcal{Z}_p, \text{ or } m \in \mathcal{Z}_a \text{ and } \ell = 0 \right\}$$

The form $a^\omega(\cdot, \cdot)$ is conjugate symmetric in X , and the weak problem (4.19) is equivalent to $a^\omega(u, v) - \omega^2 b(u, v) = 0$ on X , as well as $a^\omega(u, v) - \omega^2 b(u, v) = 0$ for all $v \in X^\perp$. This gives rise to a finite number of extra conditions $(\widehat{\partial_n u|_{\Gamma_R}})_{m\ell} = 0, \forall m \in \mathcal{Z}_p$ or $m \in \mathcal{Z}_a, \ell = 0$.

Consider the eigenvalue problem $a^\omega(u, v) - \alpha\omega^2 b(u, v) = 0$ on X . On X , $a^\omega(u, v) = a_r^\omega(u, v)$. The problem of guided modes is solved by minimizing the quotient $\frac{a(u, u)}{b(u, u)}$ on X . Of course, the field u should satisfy the following radiation condition Pillar:

$$(\widehat{\partial_n u|_{\Gamma_R}})_{m\ell} + \gamma_{m\ell} (\widehat{u|_{\Gamma_R}})_{m\ell} = 0, \quad \forall m \in \mathcal{Z}_e \text{ or } m \in \mathcal{Z}_a \text{ and } \ell \neq 0. \quad (4.29)$$

We first let $\epsilon_1 = \epsilon_0, \mu_1 = \mu_0$. The eigenfunctions satisfy a strong form of the Helmholtz equation

$$\begin{cases} (\nabla + i\boldsymbol{\kappa})^2 \psi + \alpha \epsilon_0 \mu_0 \omega^2 \psi = 0 \text{ in } \Omega_R \\ \psi \in X, \quad T\psi + \partial_n \psi|_\Gamma = 0 \\ \psi \text{ satisfies periodic boundary conditions in } X. \end{cases} \quad (4.30)$$

In Ω_R , the separable solutions are in the form of

$$\begin{aligned}
& A_{m\ell} J_\ell(|\zeta_m| r) e^{i\ell\theta} e^{i(m+\kappa)z}, \text{ if } \zeta_m^2 > 0, \\
& A_{m\ell} I_\ell(|\zeta_m| r) e^{i\ell\theta} e^{i(m+\kappa)z}, \text{ if } \zeta_m^2 < 0, \\
& [C_{m1} + C_{m2} \ln |r|] e^{i\ell\theta} e^{i(m+\kappa)z}, \text{ if } \zeta_m^2 = 0, \text{ and } \ell = 0, \\
& [C_{m\ell 1} |r|^\ell + C_{m\ell 2} |r|^{-\ell}] e^{i\ell\theta} e^{i(m+\kappa)z}, \text{ if } \zeta_m^2 = 0 \text{ and } \ell \neq 0,
\end{aligned} \tag{4.31}$$

where $\zeta_m^2 = \alpha \epsilon_0 \mu_0 \omega^2 - (m + \kappa)^2$.

We treat the cases for m separately.

Case I: $m \in \mathcal{Z}_p$, i.e. $\eta_m^2 > 0$. In this case, the propagating harmonics should vanish, and $\widehat{(u|_{\Gamma_R})}_{m\ell} = 0$. If $\zeta_m^2 > 0$, and we assume $\zeta_m > 0$, then

$$J_\ell(|\zeta_m| R) = 0,$$

so $j_\ell = \zeta_m R = \sqrt{\alpha \epsilon_0 \mu_0 - (m + k)^2} R$, where j_ℓ is a zero of $J_\ell(x)$. The eigenvalues are given by

$$\alpha = \frac{\frac{j_\ell^2}{R^2} + (m + k)^2}{\epsilon_0 \mu_0 \omega^2}$$

The Bessel function $J_\ell(z)$ has a sequence of zeros, and the corresponding α form a sequence of eigenvalues $\{\alpha_j\}_{j=1}^\infty$ with all possible j_ℓ and $m \in \mathbb{Z}$. According to our assumption of the radius of the pillar, the eigenvalues

$$\begin{aligned}
\alpha_{m\ell} &= \frac{1}{\epsilon_0 \mu_\omega^2} \left[\frac{j_\ell^2}{R^2} + (m + \kappa)^2 \right] \\
&\geq \frac{1}{\epsilon_0 \mu_0 \omega^2} \left[\frac{j_\ell^2}{R^2} + \kappa^2 \right] \\
&\geq \frac{1}{\epsilon_0 \mu_0 \omega^2} \left[\frac{\ell^2}{R^2} + \kappa^2 \right] \\
&\geq \frac{1}{\epsilon_0 \mu_0 \omega^2} \left[\frac{1}{R^2} + \kappa^2 \right] \\
&\geq 1.
\end{aligned}$$

If $\zeta_m^2 = 0$, the pillar does not support such harmonics for $\ell = 0$. For $\ell \neq 0$, the separable solution $C_{m\ell 1} r^\ell + C_{m\ell 2} r^{-\ell}$ should satisfy $C_{m\ell 1} R^\ell + C_{m\ell 2} R^{-\ell} = 0$ which is

not possible. If $\zeta_m^2 < 0$, we assume $\zeta_m = i|\zeta_m|$ and

$$I_\ell(|\zeta_m|R) = 0.$$

It is not possible since the modified Bessel functions I_ℓ have no zeros except at 0.

Case II: $m \in \mathcal{Z}_e$, i.e. $\eta_m^2 < 0$, and $\eta_m = i|\eta_m|$. In this case, the conditions in (4.29) for m should be satisfied. If $\zeta_m^2 > 0$, and we assume $\zeta_m > 0$, then

$$\frac{d}{dr} J_\ell(\zeta_m r)|_{r=R} = -\gamma_{m\ell} J_\ell(\zeta_m R),$$

where $\gamma_{m\ell} = -\frac{\eta_m H'_\ell(\eta_m R)}{H'_\ell(\eta_m R)}$. The value of R can be solved, and by comparing ζ_m^2 and η_m^2 , one knows that $\alpha > 1$. If $\zeta_m^2 = 0$, we also have $\alpha > 1$. If $\zeta_m^2 < 0$, we assume $\zeta_m = i|\zeta_m|$. Then

$$|\zeta_m| I'_\ell(|\zeta_m|R) + \gamma_{m\ell} I_\ell(|\zeta_m|R) = 0.$$

However we know that $I'_\ell(|\zeta_m|R) > 0$, $\gamma_{m\ell} > 0$ and $I_\ell(|\zeta_m|R) > 0$, and consequently the left hand side cannot be 0.

Case III: $\eta_m^2 = 0$ and $\ell \neq 0$. The condition $(\widehat{\partial_n u|_{\Gamma_R}})_{m\ell} + \gamma_{m\ell} (\widehat{u|_{\Gamma_R}})_{m\ell} = 0$ should be satisfied. If $\zeta_m^2 \geq 0$, then $\alpha \geq 1$. If $\zeta_m^2 < 0$, $|\zeta_m| I'_\ell(|\zeta_m|R) + \gamma_{m\ell} I_\ell(\zeta_m R) = 0$. It is not possible.

Case IV: $\eta_m^2 = 0$ and $\ell = 0$. The guided modes satisfy $(\widehat{u|_{\Gamma_R}})_{m\ell} = 0$. If $\zeta_m^2 \geq 0$, then $\alpha \geq 1$. If $\zeta_m^2 < 0$, $I_\ell(|\zeta_m|R) = 0$. It is not possible because the Bessel function I_0 has no zero.

In general, when $\epsilon_1 = \epsilon_0$, any eigenvalue $\alpha \geq 1$.

We can now prove the nonexistence of guided modes for inverse structures when $\mu_1 = \mu_0$ and $\epsilon_1 < \epsilon_0$. As we decrease the ϵ_1 from ϵ_0 , the parameter α becomes always > 1 by observing that the quotient $\frac{a_r(u,u)}{b(u,u)}$ is increasing with respect to ϵ_1 . Under our assumption of the size of the pillar, the number $\alpha > 1$. As a result, there exists no guided mode because the number $\alpha > 1$ does not correspond to a guided mode. \square

Theorem 44. Assume there is a pair x_0, y_0 such that for all $-\pi \leq z \leq \pi$ and any vector $\mathbf{r}_0 = (r_{0x}, r_{0y}, 0)$, the material parameters ϵ, μ are nondecreasing along the direction of \mathbf{r}_0 , that is, the weak directional derivatives $\nabla \epsilon \cdot \mathbf{r}_0$, and $\nabla \mu \cdot \mathbf{r}_0$ are nonnegative. Then there exists no guided mode.

Proof. Using polar coordinates, we observe that

$$\begin{aligned} \nabla \cdot \left(r \frac{\partial u}{\partial r} \mu^{-1} \nabla \bar{u} \right) &= \nabla(r u_r) \cdot \mu^{-1} \nabla \bar{u} + r u_r (\nabla \cdot \mu^{-1} \nabla \bar{u}) \\ &= r u_r (\nabla \cdot \mu^{-1} \nabla \bar{u}) + u_r (\nabla r \cdot \mu^{-1} \nabla \bar{u}) + r \nabla u_r \cdot \mu^{-1} \nabla \bar{u}. \end{aligned}$$

We integrate this to obtain

$$\begin{aligned} \int_{\Gamma_R} r u_r \mu_0^{-1} \frac{\partial \bar{u}}{\partial n} &= \int_{\Omega_R} r u_r (\nabla \cdot \mu^{-1} \nabla \bar{u}) + \int_{\Omega_R} u_r (\nabla r \cdot \mu^{-1} \nabla \bar{u}) + \int_{\Omega_R} r \nabla u_r \cdot \mu^{-1} \nabla \bar{u} \\ &= -\omega^2 \int_{\Omega_R} r \epsilon u_r \bar{u} + \int_{\Omega_R} u_r (\nabla r \cdot \mu^{-1} \nabla \bar{u}) + \int_{\Omega_R} r \nabla u_r \cdot \mu^{-1} \nabla \bar{u}. \end{aligned}$$

Adding its complex conjugate, we have

$$\begin{aligned} 2 \int_{\Gamma_R} \mu_0^{-1} R \left| \frac{\partial u}{\partial r} \right|^2 &= -\omega^2 \int_{\Omega_R} \epsilon r \frac{\partial}{\partial r} |u|^2 + \int_{\Omega_R} u_r (\nabla r \cdot \mu^{-1} \nabla \bar{u}) \\ &\quad + \int_{\Omega_R} \bar{u}_r (\nabla r \cdot \mu^{-1} \nabla u) + \int_{\Omega_R} \mu^{-1} r \frac{\partial}{\partial r} |\nabla u|^2. \end{aligned}$$

Use integrate by parts in r for terms including $r \frac{\partial}{\partial r}$,

$$\begin{aligned} \int_{\Omega_R} \epsilon r \frac{\partial |u|^2}{\partial r} &= \int_0^{2\pi} \int_{-\pi}^{\pi} \int_0^R \epsilon r \frac{\partial |u|^2}{\partial r} r dr dz d\theta \\ &= \int_0^{2\pi} \int_{-\pi}^{\pi} \int_0^R \epsilon r^2 \frac{\partial |u|^2}{\partial r} dr dz d\theta \\ &= \int_0^{2\pi} \int_{-\pi}^{\pi} \epsilon r^2 |u|^2 \Big|_0^R dz d\theta - \int_{\Omega_R} 2\epsilon r |u|^2 dr dz d\theta - \int_{\Omega_R} r^2 \frac{\partial \epsilon}{\partial r} |u|^2 dr dz d\theta \\ &= \int_0^{2\pi} \int_{-\pi}^{\pi} \epsilon R^2 |u|^2 \Big|_0^R dz d\theta - \int_{\Omega_R} 2\epsilon |u|^2 - \int_{\Omega_R} r \frac{\partial \epsilon}{\partial r} |u|^2, \end{aligned}$$

and

$$\int_{\Omega_R} \mu^{-1} r \frac{\partial |u|^2}{\partial r} = \int_0^{2\pi} \int_{-\pi}^{\pi} \mu^{-1} R^2 |u|^2 \Big|_0^R dz d\theta - \int_{\Omega_R} 2\mu^{-1} |u|^2 - \int_{\Omega_R} r \frac{\partial \mu^{-1}}{\partial r} |u|^2.$$

The previous identity becomes

$$\begin{aligned}
2 \int_{\Gamma_R} \mu_0^{-1} R \left| \frac{\partial u}{\partial r} \right|^2 &= -\omega^2 \left[\int_0^{2\pi} \int_{-\pi}^{\pi} R^2 \epsilon_0 |u(R)|^2 dz d\theta - \int_{\Omega_R} 2\epsilon |\nabla u|^2 - \int_{\Omega_R} r \frac{\partial \epsilon}{\partial r} |u|^2 \right] \\
&\quad + \int_{\Omega_R} u_r (\nabla r \cdot \mu^{-1} \nabla \bar{u}) + \int_{\Omega_R} \bar{u}_r (\nabla r \cdot \mu^{-1} \nabla u) \\
&\quad + \left[\int_0^{2\pi} \int_{-\pi}^{\pi} R^2 \mu_0^{-1} |u(R)|^2 dz d\theta - \int_{\Omega_R} 2\mu^{-1} |\nabla u|^2 - \int_{\Omega_R} r \frac{\partial \mu^{-1}}{\partial r} |u|^2 \right].
\end{aligned}$$

Since the field satisfies the Helmholtz equation, we can replace $-\int_{\Omega_R} \mu^{-1} |\nabla u|^2$ by

$-\omega^2 \int_{\Omega_R} \epsilon |u|^2 + \mu_0^{-1} \int_{\Gamma_R} \bar{u} T_r u$ to obtain

$$\begin{aligned}
2 \int_{\Gamma_R} \mu_0^{-1} R \left| \frac{\partial u}{\partial r} \right|^2 &= \left[-\omega^2 \int_0^{2\pi} \int_{-\pi}^{\pi} R^2 \epsilon_0 |u(R)|^2 dz d\theta + \omega^2 \int_{\Omega_R} 2\epsilon |\nabla u|^2 + \omega^2 \int_{\Omega_R} r \frac{\partial \epsilon}{\partial r} |u|^2 \right] \\
&\quad + \int_{\Omega_R} u_r (\nabla r \cdot \mu^{-1} \nabla \bar{u}) + \int_{\Omega_R} \bar{u}_r (\nabla r \cdot \mu^{-1} \nabla u) \\
&\quad + \left[\int_0^{2\pi} \int_{-\pi}^{\pi} R^2 \mu_0^{-1} |u(R)|^2 dz d\theta - 2\omega^2 \int_{\Omega_R} \epsilon |u|^2 + 2\mu_0^{-1} \int_{\Gamma_R} \bar{u} T_r u \right. \\
&\quad \left. - \int_{\Omega_R} r \frac{\partial \mu^{-1}}{\partial r} |u|^2 \right],
\end{aligned}$$

and so

$$\begin{aligned}
2 \int_{\Gamma_R} \mu_0^{-1} R \left| \frac{\partial u}{\partial r} \right|^2 &+ \omega^2 \int_0^{2\pi} \int_{-\pi}^{\pi} R^2 \epsilon_0 |u(R)|^2 dz d\theta - \int_0^{2\pi} \int_{-\pi}^{\pi} R^2 \mu_0^{-1} |u(R)|^2 dz d\theta \\
&= \omega^2 \int_{\Omega_R} r \frac{\partial \epsilon}{\partial r} |u|^2 + \int_{\Omega_R} u_r (\nabla r \cdot \mu^{-1} \nabla \bar{u}) + \int_{\Omega_R} \bar{u}_r (\nabla r \cdot \mu^{-1} \nabla u) \\
&\quad - \int_{\Omega_R} r \frac{\partial \mu^{-1}}{\partial r} |u|^2 + 2\mu_0^{-1} \int_{\Gamma_R} \bar{u} T_r u.
\end{aligned}$$

In this identity,

$$u_r (\nabla r \cdot \mu^{-1} \nabla \bar{u}) = \mu^{-1} u_r (\mathbf{r} \cdot \nabla \bar{u}) = (u_r \mathbf{r}) \cdot \nabla \bar{u} \mu^{-1} = \left| \frac{\partial u}{\partial r} \mathbf{r} \right|^2 \mu^{-1},$$

where $\nabla u = \frac{\partial u}{\partial z} \mathbf{z} + \frac{\partial u}{\partial r} \mathbf{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \boldsymbol{\theta}$, $\mathbf{r} = (\cos \theta, \sin \theta, 0)$, $\boldsymbol{\theta} = (-\sin \theta, \cos \theta, 0)$, $\mathbf{z} = (0, 0, 1)$, and $|\nabla u|^2 = |u_r|^2 + \frac{1}{r^2} |u_\theta|^2 + |u_z|^2$. Simplify it to obtain

$$\begin{aligned}
\omega^2 \int_{\Omega_R} r \frac{\partial \epsilon}{\partial r} |u|^2 &+ 2 \int_{\Omega_R} \mu^{-1} \left| \frac{\partial u}{\partial r} \mathbf{r} \right|^2 - \int_{\Omega_R} r \frac{\partial \mu^{-1}}{\partial r} |\nabla u|^2 + 2\mu_0^{-1} \int_{\Gamma_R} \bar{u} T_r u \\
&= 2 \int_{\Gamma_R} \mu_0^{-1} R \left| \frac{\partial u}{\partial r} \right|^2 + \omega^2 \int_0^{2\pi} \int_{-\pi}^{\pi} R^2 \epsilon_0 |u(R)|^2 dz d\theta \\
&\quad - \int_0^{2\pi} \int_{-\pi}^{\pi} R^2 \mu_0^{-1} |\nabla u(R)|^2 dz d\theta.
\end{aligned} \tag{4.32}$$

The left hand side of the identity (4.32) is nonnegative by our condition on the material parameters, and it vanishes if and only if $\|u\|_{H_\kappa^1(\Omega_R)} = 0$. If we assume u is a guided mode, and u has the expansion

$$u(r, \theta, z) = \sum_{m \in \mathcal{Z}_e} \sum_{\ell} a_{m\ell} H_\ell^1(\eta_m r) e^{i\ell\theta} e^{i(m+k)z} + \sum_{m \in \mathcal{Z}_a} \sum_{\ell \neq 0} c r^{-|\ell|} e^{i\ell\theta} e^{i(m+k)z},$$

then the terms with $m \in \mathcal{Z}_a$ of the right hand side of (4.32) are a sum of multiples of

$$\omega^2 \epsilon_0 R^{-2|\ell|+2} - \mu_0^{-1} (m + \kappa)^2 R^{-2|\ell|+2} = 0.$$

Since $H_\ell^1(\eta_m R)$ and $H_\ell^{1'}(\eta_m R)$ are exponentially decaying as $R \rightarrow \infty$, in this limit, the limit of the right hand side is 0. On the other hand, the left hand side does not converge to 0 if $u \neq 0$. Therefore $u = 0$. \square

Chapter 5

Open Problems and Future Work

As discussed in the previous chapters, there are some assumptions made throughout the discussion. We summarize some important ones and specify a few related open problems. We also discuss some issues that are closely related to my current work and can form future projects that involve broader interests.

Some generic assumptions are made in the proofs in Chapter 3. The first important one is that in the discussion of the Weierstraß factorization, we assume $\text{Im}(\ell_2) > 0$. This condition is sufficient to guarantee that the mentioned guided mode is nonrobust with respect to the perturbation of the wavenumber κ . This brings two open problems: prove the nonrobustness of the antisymmetric guided modes in Theorem 13 rigorously, and show $\text{Im}(\ell_2) > 0$ for that guided mode. Another important assumption is in the proof of the total transmission and reflection, the second alternative in Lemma 25, as an extremal case, is hoped to be ruled out. We also hope to gain more understanding of the behavior of anomalies in nongeneric cases.

In the proof of the nonexistence theorem 43 in Chapter 4, we need a restriction condition on the geometry and the parameters. We hope find a proof in a larger regime without the restriction on the size of the pillar. Whether or not this kind of restriction can be removed is one of the challenging open problems we are interested in working on.

There are interesting open questions concerning the detailed nature of transmission resonances. In passing from two-dimensional slabs (with one direction of periodicity) to three-dimensional slabs (with two directions of periodicity), both

the additional dimension of the wavevector parallel to the slab as well as various modes of polarization of the incident field that arise impart considerable complexity to the guided-mode structure of the slab and its interaction with plane waves. The role of structural perturbations is a mechanism for initiating coupling between guided modes and radiation [5] [9, §4.4] that deserves a rigorous mathematical treatment. A practical understanding of the correspondence between structural parameters and salient features of transmission anomalies, such as central frequency and width, would be valuable in applications.

Other future work is to use numerical methods to track guided modes as functions of both wavenumber and structural parameters. One may begin with an antisymmetric embedded guided mode in a symmetric slab for wavenumber $\kappa = 0$. If we consider the slab consisting of an array of circular cylinders, as the wavenumber κ is perturbed from 0, the field loses its antisymmetry and the structure must be perturbed from being symmetric to nonsymmetric to match the perturbation of κ in order to retain the guided mode. One method to track the guided mode is to perturb the position of one cylinder for every N cylinders in the direction parallel to the slab, and to determine the displacement of this cylinder that preserves the guided mode at nonzero κ . The displacement analysis is useful in slabs with periodic defects, when the displacement of one cylinder can be viewed as a defect and the corresponding wavenumber and frequency represent those of a perturbed guided mode in the defective structure.

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