When excluding one matroid prevents infinite antichains

Natalie Hine  
*Louisiana State University*

James Oxley  
*Louisiana State University*

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When excluding one matroid prevents infinite antichains

Natalie Hine, James Oxley∗,1

Department of Mathematics, Louisiana State University, Baton Rouge, LA, United States

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Geelen, Gerards, and Whittle have announced that there are no infinite sets of binary matroids none of which is isomorphic to a minor of another. In this paper, we use this result to determine precisely when a minor-closed class of matroids with a single excluded minor does not contain such an infinite antichain.

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1. Introduction

The matroid terminology used here will follow Oxley [9]. For a matroid \( N \), let \( EX(N) \) denote the class of matroids having no minor isomorphic to \( N \). Tutte [12] proved that \( EX(U_{2,4}) \) is the class of binary matroids. Robertson and Seymour [11] proved a conjecture of Wagner that there are no infinite antichains of graphs. They also conjectured, though apparently not in print [4,5], that, for all prime powers \( q \), this theorem can be extended to the class of matroids representable over \( GF(q) \). Geelen, Gerards, and Whittle [6] have announced that they have proved this conjecture for \( q = 2 \); that is, under the minor ordering, \( EX(U_{2,4}) \) does not contain an infinite antichain. This theorem prompts the question as to precisely when \( EX(N) \) does not contain an infinite antichain. The purpose of this note is to answer this question. The following theorem is our main result.

**Theorem 1.1.** Under the minor ordering, \( EX(N) \) does not contain an infinite antichain if and only if \( N \) is a minor of \( U_{2,4} \oplus U_{1,3} \) or \( U_{2,4} \oplus U_{2,3} \).

* Corresponding author.
E-mail addresses: nhine1@math.lsu.edu (N. Hine), oxley@math.lsu.edu (J. Oxley).

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2. Infinite antichains

The proof that certain classes $EX(N)$ contain infinite antichains will use three examples of such antichains.

Example 2.1. For all $n \geq 3$, let $P_n$ be the rank-3 matroid consisting of a ring of $n$ three-point lines, that is, $P_n$ has ground set $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and its only non-spanning circuits are $\{x_1, y_1, x_2\}, \{x_2, y_2, x_3\}, \ldots, \{x_n, y_n, x_1\}$. The set $\{P_n: n \geq 3\}$ is an infinite antichain [2, p. 155].

Example 2.2. For all $k \geq 2$, let $T_k$ be the matroid that is obtained by taking the direct sum of two $k$-element circuits and truncating this to rank $k$. Oxley, Prendergast, and Row [10] proved that the set $\{T_k: k \geq 2\}$ is an infinite antichain.

Example 2.3. For all $r \geq 2$, let $N_r$ be the $r$-tuple binary spike of rank $2r$, that is, the vector matroid of the matrix $[I_{2r}, I_{2r} - I_{2r}]$ over $GF(2)$ where $I_{2r}$ is the $2r \times 2r$ matrix of all ones. Let $M_r$ be a matroid obtained from $N_r$ by relaxing a pair of complementary circuit-hyperplanes. Kahn (in [9, p. 471]) proved that the set $\{M_r: r \geq 2\}$ is an infinite antichain no member of which has a $U_{2,5}$- or $U_{3,5}$-minor.

A binary relation $\leq$ on a set $Q$ is a quasi-order if it is reflexive and transitive. A well-quasi-order is a quasi-order such that, for every infinite sequence $q_1, q_2, \ldots$ of members of $Q$, there are indices $i$ and $j$ such that $i < j$ and $q_i \leq q_j$. For example, the set $\mathbb{N}$ of natural numbers under the usual ordering is a well-quasi-order. If $\mathcal{M}$ is a class of matroids that is closed under isomorphism and minors, then $\mathcal{M}$ is a quasi-order under the minor relation $\leq_m$. The examples above show that, when $\mathcal{M}$ is the class of all matroids, $(\mathcal{M}, \leq_m)$ is not a well-quasi-order. This paper determines precisely when $(EX(N), \leq_m)$ is a well-quasi-order.

For a quasi-order $(Q, \leq)$, let $Q^{<w}$ be the set of all finite sequences of members of $Q$. For $(p_1, p_2, \ldots, p_m)$ and $(q_1, q_2, \ldots, q_n)$ in $Q^{<w}$, define $(p_1, p_2, \ldots, p_m) \leq^{<w} (q_1, q_2, \ldots, q_n)$ if there are indices $i_1, i_2, \ldots, i_m$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $p_j \leq q_j$ for all $j$ in $\{1, 2, \ldots, m\}$. Higman [7] proved the following fundamental result.

Lemma 2.4. If $(Q, \leq)$ is a well-quasi-order, then $(Q^{<w}, \leq^{<w})$ is a well-quasi-order.

Let $(Q_1, \leq_1), (Q_2, \leq_2), \ldots, (Q_k, \leq_k)$ be quasi-orders. For $(p_1, p_2, \ldots, p_k)$ and $(q_1, q_2, \ldots, q_k)$ in $Q_1 \times Q_2 \times \cdots \times Q_k$, define $(p_1, p_2, \ldots, p_k) \leq (q_1, q_2, \ldots, q_k)$ if $p_j \leq_j q_j$ for all $j$ in $\{1, 2, \ldots, k\}$. As noted, for example, in [3], the following is a well-known consequence of Lemma 2.4.

Corollary 2.5. If $(Q_i, \leq_i)$ is a well-quasi-order for all $i$ in $\{1, 2, \ldots, k\}$, then $(Q_1 \times Q_2 \times \cdots \times Q_k, \leq_1 \times \leq_2 \times \cdots \times \leq_k)$ is a well-quasi-order.

Let $M$ be a uniform matroid with ground set $\{x_1, x_2, \ldots, x_n\}$. Replace each element $x_i$ by $k_i$ parallel elements for some $k_i \geq 1$ where if $r(M) = 0$, each $k_i = 1$. We call the resulting matroid a parallel extension of a uniform matroid. Its dual is a series extension of a uniform matroid. Note that this terminology differs from Oxley [9] where parallel and series extensions require the addition of a single element.

Lemma 2.6. There are no infinite antichains of series extensions of uniform matroids.

Proof. Associate the pair $(r, s - r)$ and the $s$-tuple $(k_1, k_2, \ldots, k_s)$ with $k_1 \leq k_2 \leq \cdots \leq k_s$ to each series extension of a non-empty uniform matroid $U_{r,s}$. From above, $\mathbb{N}^2 \times \mathbb{N}^{<w}$ is a well-quasi-order. Thus the class of series extensions of uniform matroids is a well-quasi-order. □
3. EX(N)

In the next lemma, \( \mathcal{W}^3 \) denotes the rank-3 whirl, while \( Q_6 \) and \( P_6 \) are obtained from \( \mathcal{W}^3 \) by relaxing one and two circuit-hyperplanes, respectively.

**Lemma 3.1.** The class \( \text{EX}(U_{2,4} \oplus 2 U_{1,3}) \) consists of direct sums of binary matroids and series extensions of uniform matroids.

**Proof.** Let \( M \in \text{EX}(U_{2,4} \oplus 2 U_{1,3}) \). Assume \( M \) is 3-connected. Observe that \( M \in \text{EX}(\mathcal{W}^3, Q_6, P_6) \). Thus, by [8, Theorem 1.5], \( M \) is binary or uniform. Now assume \( M \) is connected, but not 3-connected. Then \( M = M_1 \oplus 2 M_2 \) for some connected matroids \( M_1 \) and \( M_2 \). Suppose \( M \) is non-binary. Then, without loss of generality, \( M_1 \) is non-binary. Hence, \( M_1 \) has a \( U_{2,4} \)-minor. Furthermore, Bixby [1] proved that every element of \( M_1 \), so, in particular, the basepoint \( p \) of the 2-sum, is in a \( U_{2,4} \)-minor of \( M_1 \). Thus, no cocircuit of \( M_2 \) containing \( p \) has more than two elements. Hence, \( M_2 \) is a circuit. Thus, every 2-sum decomposition of \( M \) has a circuit as one part. It follows without difficulty that \( M \) is a series extension of a uniform matroid, and it is straightforward to complete the proof of the lemma.

**Corollary 3.2.** The classes \( \text{EX}(U_{2,4} \oplus 2 U_{1,3}) \) and \( \text{EX}(U_{2,4} \oplus 2 U_{2,3}) \) do not contain infinite antichains.

**Proof.** By duality, it suffices to prove the result for \( \text{EX}(U_{2,4} \oplus 2 U_{1,3}) \). If \( M \in \text{EX}(U_{2,4} \oplus 2 U_{1,3}) \), then, by the previous lemma, we can write \( M \) as \( M_0 \oplus M_1 \oplus \cdots \oplus M_k \) for some \( k \geq 0 \) where \( M_0 \) is binary and every other \( M_i \) is a series extension of a uniform matroid. Note that we shall allow \( M_0 \) to be \( U_{0,0} \). Let \( Q_6 \) denote the class of binary matroids and let \( Q_5 \) denote the class of series extensions of uniform matroids. By [6] and Lemma 2.6, neither \( Q_6 \) nor \( Q_5 \) contains any infinite antichains. By Lemma 2.4 and Corollary 2.5, \( Q_6 \times Q_5 \) is a well-quasi-order. Thus \( \text{EX}(U_{2,4} \oplus 2 U_{1,3}) \) is a well-quasi-order.

We now prove the main theorem.

**Proof of Theorem 1.1.** Assume \( \text{EX}(N) \) contains an infinite antichain. Then, by Corollary 3.2, \( N \) is not a minor of \( U_{2,4} \oplus 2 U_{1,3} \) or \( U_{2,4} \oplus 2 U_{2,3} \).

Assume \( N \) is not a minor of \( U_{2,4} \oplus 2 U_{1,3} \) or \( U_{2,4} \oplus 2 U_{2,3} \), so \( |E(N)| \geq 3 \). If \( r(N) \geq 4 \) or \( r(N^*) \geq 4 \), then \( \text{EX}(N) \) contains \( \{P_n^* : n \geq 3\} \) or \( \{P_n : n \geq 3\} \), respectively. Hence, \( r(N) \leq 3 \) and \( r(N^*) \leq 3 \). Thus \( |E(N)| \leq 6 \). Observe that \( \text{EX}(U_{0,2} \oplus U_{1,1}) \) and \( \text{EX}(U_{2,2} \oplus U_{0,1}) \) contain \( \{P_n : n \geq 3\} \) and \( \{P_n^* : n \geq 3\} \), respectively; both \( \text{EX}(U_{1,2} \oplus U_{1,1}) \) and \( \text{EX}(U_{2,4} \oplus U_{2,4}) \) contain \( \{T_k : k \geq 4\} \); and \( \text{EX}(U_{3,5}) \) and \( \text{EX}(U_{2,5}) \) contain \( \{M_r^+ : r \geq 2\} \) and \( \{M_r^+ : r \geq 2\} \), respectively. Hence we may assume that \( N \) has no minor isomorphic to \( U_{0,2} \oplus U_{1,1} \), \( U_{1,2} \oplus U_{0,1} \), \( U_{1,2} \oplus U_{1,2} \), \( U_{2,5} \), \( U_{3,5} \), or \( U_{2,4} \oplus 2 U_{2,4} \). It is not difficult to check that this leaves no remaining choices for \( N \).

**References**


