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On bipartite restrictions of binary matroids
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ABSTRACT

In a 1965 paper, Erdős remarked that a graph $G$ has a bipartite subgraph that has at least half as many edges as $G$. The purpose of this note is to prove a matroid analogue of Erdős's original observation. It follows from this matroid result that every loopless binary matroid has a restriction that uses more than half of its elements and has no odd circuits; and, for $2 \leq k \leq 5$, every bridgeless graph $G$ has a subgraph that has a nowhere-zero $k$-flow and has more than $\frac{k-1}{k} |E(G)|$ edges.

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1. Introduction

The matroid terminology used in this note will follow Oxley [9]. The results considered here relate to the critical problem of Crapo and Rota [5]. The reader is referred to the survey paper of Brylawski and Oxley [4] for the theoretical background to these results that is not included here. We shall require only a minimal amount of this theory. Let $M$ be a matroid. Its simplification is denoted by $si(M)$. When $M$ is $GF(q)$-representable and loopless having rank $r$, we say that $M$ is affine over $GF(q)$ or $q$-affine if $si(M)$ is isomorphic to a restriction of the affine geometry $AG(r - 1, q)$. Equivalently, the loopless $GF(q)$-representable rank-$r$ matroid $M$ is affine over $GF(q)$ if, whenever there is a subset $T$ of the projective geometry $PG(r - 1, q)$ such that $si(M)$ is isomorphic to $PG(r - 1, q) \cap T$, there is a hyperplane $H$ of $PG(r - 1, q)$ such that $H$ avoids $T$. It is well known that if $G$ is a graph, then $M(G)$ is affine over $GF(q)$ if and only if $G$ is $q$-colourable. In particular, $G$ is bipartite if and only if $M(G)$ is affine over $GF(2)$. On the other hand, $M^*(G)$ is affine over $GF(q)$ if and only if $G$ has a nowhere-zero $q$-flow. An arbitrary binary matroid is affine if and only if all of its circuits have even cardinality. We follow Welsh [13] in calling such a matroid bipartite.

2. The theorem and some consequences

Erdős's observation [7] that every loopless graph has a bipartite subgraph that has at least half as many edges as $G$ was sharpened by Edwards [6], and Erdős et al. [8] gave simpler proofs of the results
of Edwards. The following theorem generalizes Erdős's original result to matroids. For a matroid \( M \) and for \( k \) in \( \{0, 1, \ldots, r(M)\} \), let \( h_k(M) \) denote the number of flats of \( M \) of rank \( r(M) - k \).

**Theorem 2.1.** Let \( k \) be a non-negative integer, \( P \) be a matroid of rank at least \( k + 1 \), and \( T \) be a subset of \( E(P) \) that contains no loops. Let \( d = \max_{e \in T} h_k(P/e) \). Then \( P \) has a flat \( F \) of rank \( r(P) - k \) such that

\[
\frac{|T - F|}{|T|} \geq 1 - \frac{d}{h_k(P)}.
\]

**Proof.** Let \( r(P) = r \). Construct the bipartite graph \( G \) with vertex classes \( V_1 \) and \( V_2 \) where \( V_1 \) is the set of rank-\((r - k)\) flats of \( P \), and \( V_2 = T \). An element \( e \) of \( T \) is adjacent to a rank-\((r - k)\) flat \( F \) of \( P \) if and only if \( e \in F \).

If \( e \in T \), then the number of rank-\((r - k)\) flats of \( P \) containing \( e \) is the same as the number of rank-\((r - k)\) flats of \( P/e \). As \( d = \max_{e \in T} h_k(P/e) \), we deduce that the vertex \( e \) of \( V_2 \) has degree at most \( d \) in \( G \). Hence \( |E(G)| \leq d|T| \). Since \( |V_1| = h_k(P) \), it follows that \( V_1 \) contains a vertex of degree at most \( \frac{d|T|}{h_k(P)} \). Hence \( P \) has a rank-\((r - k)\) flat \( F \) that avoids at least \( |T| - \frac{d|T|}{h_k(P)} \) elements of \( T \). Thus

\[
\frac{|T - F|}{|T|} \geq 1 - \frac{d}{h_k(P)}. \quad \square
\]

**Corollary 2.2.** Let \( M \) be a loopless non-empty GF\((q)\)-representable matroid. Then \( M \) has a \( q \)-affine restriction \( N \) such that

\[
|E(N)| > q - 1 \frac{1}{q} |E(M)|.
\]

**Proof.** Let the size of a largest parallel class of \( M \) be \( t \) and let \( r = r(M) \). Replace each element of \( PG(r - 1, q) \) by \( t \) elements in parallel, letting the resulting matroid be \( P \). Then \( M \) can be viewed as a restriction of \( P \). Let \( T = E(M) \). Applying the theorem with \( k = 1 \), we get that there is a hyperplane \( H \) of \( P \) such that

\[
\frac{|T - H|}{|T|} \geq \frac{h_1(P) - \max_{e \in T} h_1(P/e)}{h_1(P)}.
\]

Clearly \( h_1(P) = h_1(PG(r - 1, q)) = q^{r-1} \) while, for all elements \( e \) of \( P \), by symmetry, \( h_1(P/e) = h_1(PG(r - 2, q)) = \frac{q^{r-1}}{q-1} \). Thus

\[
\frac{h_1(P) - \max_{e \in T} h_1(P/e)}{h_1(P)} = \frac{q^r - q^{r-1}}{q^r - 1} = \frac{(q - 1)}{q} \left( \frac{q^r}{q^{r-1}} - 1 \right) > q - 1 \frac{1}{q}.
\]

As \( \text{si}(M|T - H) \) is the ground set of a restriction of \( AG(r - 1, q) \), we get the required result. \( \square \)

The bound in Corollary 2.2 can be restated as

\[
|E(N)| \geq \frac{q - 1}{q} |E(M)| + \frac{1}{q}.
\]

This bound is sharp, with equality being attained when \( M \cong PG(r - 1, q) \) since a largest affine restriction of this matroid is isomorphic to \( AG(r - 1, q) \).

Corollary 2.2 has some interesting consequences. Taking \( q = 2 \), we immediately get the following result which implies Erdős’s result. Welsh [13] showed that a binary matroid is bipartite if and only if its ground set can be written as a disjoint union of cocircuits.

**Corollary 2.3.** Let \( M \) be a loopless non-empty binary matroid. Then \( M \) has a bipartite restriction that has more than half of the elements of \( M \).
**Corollary 2.4.** Let $G$ be a loopless non-empty graph and $k$ be a positive integer. Then $G$ has a $k$-colourable subgraph that has more than $\frac{k-1}{k}|E(G)|$ edges.

**Proof.** Suppose that $G$ has $n$ vertices and that its largest parallel class has size $t$. Then we can view $G$ as a subgraph of the graph $K^*_{nt}$ that is obtained from $K_n$ by replacing each edge by $t$ parallel edges. There is an obvious one-to-one correspondence between the flats of $M(K^*_{nt})$ and those of $M(K_n)$. Moreover, the flats of $M(K_n)$ of rank $r(M(K_n)) - k$ correspond to the partitions of $V(K_n)$ into exactly $k$ classes where an edge of $K_n$ is in the flat if and only if both ends lie in the same class. The number of such partitions is $S(n, k)$, the Stirling number of the second kind. Moreover, the complement of a flat of rank $r(M(K_n)) - k$ is a complete $k$-partite graph. In this case, the right-hand side of the inequality in Theorem 2.1 is

$$\frac{S(n, k) - S(n - 1, k)}{S(n, k)} = \frac{S(n, k - 1) + (k - 1)S(n - 1, k)}{S(n, k - 1) + kS(n - 1, k)} > \frac{k - 1}{k}. \quad \Box$$

Since the last result is derived from such a general matroid result, it is not surprising that a stronger graph result is known. Andersen et al. [1] showed that, for all positive integers $k$, every loopless graph $G$ has a $k$-colourable subgraph $H$ with

$$|E(H)| \geq \frac{k-1}{k}|E(G)| + \alpha_k(|V(G)| - 1)$$

where $\alpha_k = 1/k$ when $k \geq 3$, while $\alpha_2 = 1/4$.

Next we consider some variants on the results above that use the following special case of a result of Asano et al. [2].

**Lemma 2.5.** Let $M$ be a GF $q$-representable matroid and $X$ be a subset of $E(M)$. Then $X$ is minimal with the property that $M \setminus X$ is $q$-affine if and only if $X$ is minimal with the property that $M/X$ is $q$-affine.

Applying the last result to the theorem, we immediately obtain the following.

**Corollary 2.6.** Let $M$ be a loopless non-empty GF $q$-representable matroid. Then there is a contraction $N$ of $M$ that is $q$-affine and satisfies

$$|E(N)| > \frac{q-1}{q}|E(M)|.$$

The next result is obtained from the last corollary by taking $q = 2$ and using duality.

**Corollary 2.7.** Let $M$ be a non-empty binary matroid without coloops. Then $M$ has a restriction $N$ having more than half of the elements of $M$ such that every cocircuit of $N$ has even cardinality.

Finally, we note another consequence of Corollary 2.7 and duality, this one involving nowhere-zero $k$-flows. We state it only for $k$ at most 5 because Seymour [11] has proved that every bridgeless graph has a nowhere-zero 6-flow. That result is the best partial result towards Tutte’s 5-Flow Conjecture [12], that every bridgeless graph has a nowhere-zero 5-flow.

**Corollary 2.8.** Let $G$ be a bridgeless graph and suppose that $k \in \{2, 3, 4, 5\}$. Then $G$ has a subgraph $H$ that has a nowhere-zero $k$-flow and has more than $\frac{k-1}{k}|E(G)|$ edges.

A bridgeless graph has a nowhere-zero 2-flow if and only if its edge set is a disjoint union of cycles or, equivalently, every vertex has even degree. By the last result, every bridgeless graph $G$ has a subgraph $H$ that is a disjoint union of cycles such that $|E(H)| > \frac{1}{2}|E(G)|$. This bound can be significantly strengthened. Indeed, Bermond et al. [3, Lemma 3.2] have proved that one can always find such a subgraph $H$ with $|E(H)| \geq \frac{2}{7}|E(G)|$. The latter bound is sharp since, by Petersen’s Theorem [10], equality holds for every cubic bridgeless graph $G$. 

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