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James Oxley
Louisiana State University

Charles Semple
University of Canterbury

Geoff Whittle
Victoria University of Wellington

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An upgraded Wheels-and-Whirls Theorem for 3-connected matroids ☆

James Oxley [a], Charles Semple [b], Geoff Whittle [c]

[a] Department of Mathematics, Louisiana State University, Baton Rouge, LA, USA
[b] Department of Mathematics and Statistics, University of Canterbury, Christchurch, New Zealand
[c] School of Mathematical and Computing Sciences, Victoria University, Wellington, New Zealand

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ABSTRACT

Let M be a 3-connected matroid that is not a wheel or a whirl. In this paper, we prove that M has an element e such that $M\backslash e$ or M/e is 3-connected and has no 3-separation that is not equivalent to one induced by M.

1. Introduction

In both matroid representation theory and matroid structure theory, one frequently encounters situations where connectivity is required to avoid degeneracies. Because 3-connectivity is so well understood, it would be ideal if it always sufficed. However, higher connectivity is often required. Typically, 4-connectivity is too strong a condition since, for example, projective geometries and the cycle matroids of complete graphs are not 4-connected as matroids. Moreover, developing the necessary technology to make inductive arguments possible within the class of 4-connected matroids has proved to be very difficult. What is often required is some type of intermediate connectivity where 3-separations are allowed, but are controlled in some way. The primary motivation for this paper is to develop master theorems that will give as corollaries useful results for many of the connectivities intermediate between 3- and 4-connectivity.

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leqslant k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation*

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E-mail addresses: oxley@math.lsu.edu (J. Oxley), c.semple@math.canterbury.ac.nz (C. Semple), Geoff.Whittle@mcs.vuw.ac.nz (G. Whittle).

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if $|X|, |E - X| \geqslant k$. A k-separating set X, or a k-separating partition $(X, E - X)$, or a k-separation $(X, E - X)$ is *exact* if $\lambda_M(X) = k - 1$. A k-separation $(X, E - X)$ is *minimal* if $\min\{|X|, |E - X|\} = k$.

A set X in a matroid M is *fully closed* if it is closed in both M and M^*, that is, $\mathrm{cl}(X) = X$ and $\mathrm{cl}^*(X) = X$. The *full closure* of X, denoted $\mathrm{fcl}(X)$, is the intersection of all fully closed sets that contain X. One way to obtain $\mathrm{fcl}(X)$ is to take $\mathrm{cl}(X)$ and then $\mathrm{cl}^*(\mathrm{cl}(X))$ and so on until neither the closure nor coclosure operator adds any new elements of M. Two exactly 3-separating partitions (A_1, B_1) and (A_2, B_2) of M are *equivalent*, written $(A_1, B_1) \cong (A_2, B_2)$, if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$. If $\mathrm{fcl}(A_1)$ or $\mathrm{fcl}(B_1)$ is $E(M)$, then (A_1, B_1) is *sequential*. A 3-connected matroid M is *sequentially* 4*-connected* if it has no non-sequential 3-separations.

Let e be an element of a 3-connected matroid M. When $M\backslash e$ is 3-connected, a 3-separation (X, Y) of $M\backslash e$ is *well blocked* by e if, for all exactly 3-separating partitions (X', Y') equivalent to (X, Y), neither $(X' \cup e, Y')$ nor $(X', Y' \cup e)$ is exactly 3-separating in M. An element f of M *exposes* a 3-separation (U, V) of $M\backslash f$ if $M\backslash f$ is 3-connected and (U, V) is a 3-separation of $M\backslash f$ that is well blocked by f. Evidently, if e exposes an exactly 3-separating partition (E_1, E_2) of $M\backslash e$, then e exposes all exactly 3-separating partitions (E_1', E_2') that are equivalent to (E_1, E_2). We shall say that an element g of M *exposes a* 3*-separation in* M/g if g exposes a 3-separation in $M^*\backslash g$.

Next we give a context for the results of this paper. Some of the technical terms used may be unfamiliar to the reader. These terms are formally defined in Sections 2 and 3. For a finite field $GF(q)$ with at least seven elements, Oxley, Vertigan, and Whittle [19] disproved a conjecture of Kahn [11] by showing that the number of inequivalent representations of a 3-connected matroid over $GF(q)$ can be arbitrarily large. By contrast, Geelen and Whittle [5] proved that, when q is prime, the number of inequivalent $GF(q)$-representations of 4-connected matroids is bounded. Due to the difficulty of working with 4-connected matroids, the theorem that is proved in [5] is necessarily somewhat stronger. For fixed $k \geqslant 5$, a 3-connected matroid is *k-coherent* if it has no swirl-like flower of order k. For the uninitiated, k-coherence is nothing more than a condition that places some control on the 3-separations that are allowed in the matroid. The notion of k-coherence is easier to work with than 4-connectivity and it is proved in [5] that, for a fixed $k \geqslant 5$ and prime p, there is a bound on the number of inequivalent $GF(p)$-representations of a k-coherent matroid.

Other intermediate connectivity notions that have also been studied include weak 4-connectivity [4,7], internal 4-connectivity [2,3,6,8,12,22], sequential 4-connectivity [4], and fork-connectivity [10]. We anticipate the need for even more such notions in the future, each one being tailored to the requirements of a specific problem. Thus it may be that it will be required to control flowers other than swirl-like flowers or to control the lengths of paths of inequivalent 3-separations. In each case, theorems will be required to make inductive arguments possible.

In this paper, we prove the following theorem.

Theorem 1.1. *Let M be a* 3*-connected matroid that is not a wheel or a whirl. Then M has an element whose deletion from M or M^* is* 3*-connected but does not expose any* 3*-separations.*

Theorem 1.1 extends the following result of [4, Theorem 1.2].

Theorem 1.2. *Let M be a sequentially* 4*-connected matroid other than a wheel or whirl. Then M has an element e whose deletion from M or M^* is sequentially* 4*-connected.*

Note that Theorem 1.2 in turn generalizes Tutte's Wheels-and-Whirls Theorem [21], which establishes that if M is a 3-connected matroid other than a wheel or a whirl, then M has an element that can be deleted or contracted to maintain 3-connectivity.

On the other hand, Theorem 1.1 implies the following theorem from [5].

Corollary 1.3. *Let k be an integer exceeding four and M be a k-coherent matroid. If M is neither a wheel nor a whirl, then M has an element e such that either $M\backslash e$ or M/e is k-coherent.*

In fact, the main theorem of this paper, Theorem 4.2, is much more powerful than Theorem 1.1. Theorem 4.2 relies on trees of 3-separations that can be associated with a 3-connected matroid M.

It is shown that if S is the set of elements corresponding to a leaf of such a tree, then S contains an element f in its *full closure* whose deletion from M or M^* is 3-connected but does not expose any 3-separations. In many cases, this greatly expands the number of elements that can be removed without exposing 3-separations. Moreover, because this result applies to the tree of 3-separations, it can be applied to all connectivities intermediate between 3- and 4-connectivity.

This paper is the third in a series. In [17], we analyzed when it is not possible to remove an element from a triangle without exposing a 3-separation. We make essential use of the results of [17] in this paper. Moreover, the main result of [18] is, in effect, a lemma for this paper. We also believe that some of the other results of this paper are of independent interest. For example, Theorem 7.1 is applied in several places in [5].

Since we now have a Wheels-and-Whirls Theorem for exposing 3-separations, it is natural to ask if Seymour's Splitter Theorem [20] has a similar strengthening. Let N be a 3-connected minor of a 3-connected matroid M. Then it may be that N has 3-separations that are not equivalent to any induced in M. In moving from M to N via single-element deletions or contractions, such 3-separations must be exposed at some stage. Taking this into account, the following conjecture is best-possible.

Conjecture 1.4. *Let N be a 3-connected minor of a 3-connected matroid M. Then M has an element x such that some M' in $\{M\backslash x, M/x\}$ is 3-connected with the property that if (A, B) is a 3-separation of M' exposed by x, then (A, B) is induced by a non-sequential 3-separation of N.*

2. Preliminaries

Our terminology will follow Oxley [13]. We write $x \in \mathrm{cl}^{(*)}(Y)$ to mean that $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$. A *quad* is a 4-element set in a matroid that is both a circuit and a cocircuit. The set $\{1, 2, \dots, n\}$ will be denoted by $[n]$.

If an exactly 3-separating set X in a matroid M has an ordering $(x_1, x_2, \dots, x_n)$ such that $\{x_1, x_2, \dots, x_i\}$ is 3-separating for all i in $[n]$, then X is *sequential* and $(x_1, x_2, \dots, x_n)$ is a *sequential ordering* of X. Thus an exactly 3-separating partition (X, Y) of M is sequential if X or Y is a sequential 3-separating set. In a 3-connected matroid M, a 3-*sequence* is an ordered partition $(A, x_1, x_2, \dots, x_n, B)$ of $E(M)$ such that $|A|, |B| \geqslant 2$ and $(A \cup \{x_1, x_2, \dots, x_i\}, \{x_{i+1}, x_{i+2}, \dots, x_n\} \cup B)$ is exactly 3-separating for all i in $\{0, 1, \dots, n\}$. If M has a 3-sequence in which $|A| = |B| = 2$, then M is *sequential*.

A triangle T of a 3-connected matroid M is *wild* if, for all t in T, either $M\backslash t$ is not 3-connected, or $M\backslash t$ is 3-connected and t exposes a 3-separation in $M\backslash t$. A subset S of a 3-connected matroid M is a *fan* in M if $|S| \geqslant 3$ and there is an ordering $(s_1, s_2, \dots, s_n)$ of S such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \dots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads beginning with either. We call $(s_1, s_2, \dots, s_n)$ a *fan ordering* of S. If $n \geqslant 4$, then s_1 and s_n, which are the only elements of S that are not in both a triangle and a triad contained in S, are the *ends* of the fan. The remaining elements of S are the *internal elements* of the fan. An *internal triangle* of S is a triangle all of whose elements are internal elements of S.

The connectivity function λ_M of a matroid M has many attractive properties. In particular, $\lambda_M = \lambda_{M^*}$. Moreover, $\lambda_M(X) = \lambda_M(E - X)$. We often abbreviate λ_M as λ. This function is submodular, that is, $\lambda(X) + \lambda(Y) \geqslant \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E(M)$. The next lemma is a consequence of this. We make frequent use of it here and write *by uncrossing* to mean "by an application of Lemma 2.1".

Lemma 2.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geqslant 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geqslant 2$, then $X \cap Y$ is 3-separating.*

Another consequence of the submodularity of λ is the following very useful result for 3-connected matroids known as Bixby's Lemma [1].

Lemma 2.2. *Let M be a 3-connected matroid and e be an element of M. Then either $M\backslash e$ or M/e has no non-minimal 2-separations. Moreover, in the first case, $\mathrm{co}(M\backslash e)$ is 3-connected while, in the second case, $\mathrm{si}(M/e)$ is 3-connected.*

A useful companion function to the connectivity function is the *local connectivity*, $\sqcap(X,Y)$, defined for sets X and Y in a matroid M by

$$\sqcap(X,Y) = r(X) + r(Y) - r(X \cup Y).$$

Clearly $\sqcap(X, E-X) = \lambda_M(X)$. For a field $\mathbb{F}$, when M is simple and $\mathbb{F}$-representable, and hence viewable as a subset of the vector space $V(r(M),\mathbb{F})$, the local connectivity $\sqcap(X,Y)$ is precisely the rank of the intersection of those subspaces in $V(r(M),\mathbb{F})$ that are spanned by X and Y.

An attractive link between connectivity and local connectivity is provided by the following easily verified result [15, Lemma 2.6].

Lemma 2.3. *Let X and Y be disjoint sets in a matroid M. Then*

$$\lambda_M(X \cup Y) = \lambda_M(X) + \lambda_M(Y) - \sqcap_M(X,Y) - \sqcap_{M^*}(X,Y).$$

The first part of the next lemma [15, Lemma 2.3] simply restates [13, Lemma 8.2.10]. The second part, which follows from the first, is the well-known fact that the connectivity function is monotone under taking minors.

Lemma 2.4. *Let M be a matroid.*

(i) *Let X_1, X_2, Y_1 and Y_2 be subsets of $E(M)$. If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then $\sqcap(X_1,X_2) \leqslant \sqcap(Y_1,Y_2)$.*
(ii) *If N is a minor of M and $X \subseteq E(M)$, then*

$$\lambda_N\big(X \cap E(N)\big) \leqslant \lambda_M(X).$$

Next we note a useful consequence of part (i) of the last lemma, along with some basic properties of 3-separating sets.

Lemma 2.5. *In a matroid M, let X, Y, and Z be sets such that $X \subseteq Y$. If $\sqcap(Y,Z) = \sqcap(X,Z)$ and $e \in \mathrm{cl}(Z) \cap \mathrm{cl}(Y)$, then $e \in \mathrm{cl}(Z) \cap \mathrm{cl}(X)$.*

Proof. Since $e \in \mathrm{cl}(Z) \cap \mathrm{cl}(Y)$, we have $\sqcap(Y \cup e, Z \cup e) = \sqcap(Y,Z)$. Thus, by the last lemma,

$$\sqcap(Y,Z) = \sqcap(Y \cup e, Z \cup e) \geqslant \sqcap(X \cup e, Z \cup e) \geqslant \sqcap(X,Z) = \sqcap(Y,Z).$$

Hence $\sqcap(X \cup e, Z \cup e) = \sqcap(X,Z)$. As $e \in \mathrm{cl}(Z)$, it follows that $r(X \cup e) = r(X)$, so $e \in \mathrm{cl}(X)$. □

Lemma 2.6. *In a 3-connected matroid M, suppose that A and B are disjoint sets such that A and $A \cup B$ are 3-separating in M and $B \subseteq \mathrm{fcl}(A) \neq E(M)$. Then there is an ordering $(b_1, b_2, \dots, b_n)$ of B such that $A \cup \{b_1, b_2, \dots, b_i\}$ is 3-separating for all i in $[n]$.*

Proof. There is an ordering $(z_1, z_2, \dots, z_m)$ of $\mathrm{fcl}(A) - A$ such that $A \cup \{z_1, z_2, \dots, z_j\}$ is 3-separating for all j in $[m]$. Then, by Lemma 2.1, the intersection of $A \cup B$ with $A \cup \{z_1, z_2, \dots, z_j\}$ is also 3-separating for each j, and the lemma follows without difficulty. □

Lemma 2.7. *In a 3-connected matroid M, let X and Y be 3-separating sets such that $|E(M) - X| \geqslant 2$ and $Y \subseteq X$. If X is sequential, then so is Y.*

Proof. Take a sequential ordering $(x_1, x_2, \dots, x_n)$ of X. Then, by Lemma 2.1, for all i in $[n]$, the set $Y \cap \{x_1, x_2, \dots, x_i\}$ is 3-separating. □

Lemma 2.8. *Let M be a sequential 3-connected matroid. If M has a quad Q, then, for every sequential ordering $(x_1, x_2, \dots, x_n)$ of $E(M)$, both $|Q \cap \{x_1, x_2, x_3\}|$ and $|Q \cap \{x_{n-2}, x_{n-1}, x_n\}|$ are two.*

Proof. Assume that $|Q \cap \{x_{n-2}, x_{n-1}, x_n\}| \leqslant 1$. Note that if this cardinality is one, we may assume that $x_{n-2} \in Q$. Let x_j be the third element of Q in the ordering $(x_1, x_2, \dots, x_n)$ of X. Then $\{x_1, x_2, \dots, x_j\}$ and Q are 3-separating, so, by uncrossing, their intersection is too. This intersection has three elements, so Q contains a triangle or a triad; a contradiction. $\square$

The next lemma is from [17, Lemma 2.4].

Lemma 2.9. *Let M be a 3-connected matroid. If f exposes a 3-separation (U, V) in M, then (U, V) is non-sequential. In particular, $|U|, |V| \geqslant 4$. Moreover, if $|V| = 4$, then V is a quad of $M \backslash f$.*

Next we show that an element in a sequential 3-separating set does not expose any 3-separations.

Lemma 2.10. *Let M be a 3-connected matroid with ground set E and let X be a sequential 3-separating set with $|X| \geqslant 4$. If $e \in X$ and $M \backslash e$ is 3-connected, then e does not expose any 3-separations in M.*

Proof. Suppose a 3-separation (Y, Z) is exposed in $M \backslash e$. Then, by Lemma 2.9, both Y and Z are non-sequential and $|Y|, |Z| \geqslant 4$. If M is sequential, then, by [9, Lemmas 4.2 and 4.1], $M \backslash e$ is sequential, and Y or Z is sequential; a contradiction. Thus M is non-sequential, so $|E - X| \geqslant 4$.

Now $e \notin \mathrm{cl}(Y) \cup \mathrm{cl}(Z)$. As $(X, E - X)$ and $(X - e, E - X)$ are 3-separations of M and $M \backslash e$, we have $e \in \mathrm{cl}(X - e)$. Thus neither Y nor Z contains $X - e$. As $X - e$ is sequential in $M \backslash e$, Lemma 2.7 implies that neither Y nor Z is contained in $X - e$, so $Y \cap (E - X) \neq \emptyset \neq Z \cap (E - X)$. Suppose that $|Y \cap (E - X)| = 1$. Then $|Y \cap (X - e)| \geqslant 3$ and $|(E - X) \cap Z| \geqslant 3$. Thus, by Lemma 2.1, $Y \cup (X - e)$ is 3-separating in $M \backslash e$. As $X - e$ is sequential, so is $Y \cup (X - e)$. Hence, by Lemma 2.7, so is Y; a contradiction. Thus $|Y \cap (E - X)| \geqslant 2$ and, similarly, $|Z \cap (E - X)| \geqslant 2$.

From above, $Y \cap (X - e) \neq \emptyset$. Suppose $|Y \cap (X - e)| = 1$. Then $|Z \cap (X - e)| \geqslant 2$ so, by Lemma 2.1, $Z \cup (X - e)$ is 3-separating. Moreover, $(Z \cup (X - e), (E - X) \cap Y) \cong (Z, Y)$. But $e \in \mathrm{cl}(Z \cup (X - e))$, so (Z, Y) is not exposed by e. Thus $|Y \cap (X - e)| \geqslant 2$. Hence $(X - e) \cup Y$ is 3-separating. By symmetry, so is $(X - e) \cup Z$.

Now $X - e$ has a sequential ordering $(x_1, x_2, \dots, x_n)$. By interchanging Y and Z if necessary, we may assume that two of x_1, x_2, and x_3 are in Y. Then, by possibly reordering the first three elements, we may assume that $x_1, x_2 \in Y$. Then, by uncrossing, $Y \cup \{x_1, x_2, \dots, x_i\}$ is 3-separating in $M \backslash e$ for all i in $\{0, 1, \dots, n\}$. Hence $(Y, Z) \cong (Y \cup (X - e), (E - X) \cap Z)$, a contradiction as $e \in \mathrm{cl}(Y \cup (X - e))$. $\square$

The next lemma establishes that Theorem 1.1 holds if M has a fan with four or more elements.

Lemma 2.11. *Let M be a 3-connected matroid that is not a wheel or a whirl. Let F be a maximal fan in M having at least four elements and let z be an end of F. Then the deletion of z from M or M^* is 3-connected but does not expose any 3-separations.*

Proof. Let $(z, x_2, \dots, x_n)$ be a fan ordering of F and assume, by switching to the dual if necessary, that $\{z, x_2, x_3\}$ is a triangle. Then, by [14, Lemma 1.5], $M \backslash z$ is 3-connected. But F is a sequential set with at least four elements. Thus, by Lemma 2.10, z does not expose any 3-separations in M. $\square$

Lemma 2.12. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M such that $\{a, b, c\}$ is not in a 4-element fan. If c exposes a 3-separation in M, then $\{a, b, c\}$ is fully closed in M.*

Proof. Let (C_1, C_2) be a 3-separation of $M \backslash c$ that is exposed. If $d \in \mathrm{cl}(\{a, b, c\}) - \{a, b, c\}$, then at least two of a, b, and d are in C_1 or C_2, say C_1. Hence $c \in \mathrm{cl}(C_1)$; a contradiction. Thus $\{a, b, c\}$ is closed. If $e \in \mathrm{cl}^*(\{a, b, c\}) - \{a, b, c\}$, then, as $\{a, b, c\}$ is not in a 4-element fan, $\{a, b, c, e\}$ is a cocircuit of M.

Thus $\{a, b, e\}$ is a cocircuit of $M\backslash c$. Hence we may assume that at least two and therefore all three of a, b, and e are in C_1. Then $c \in \mathrm{cl}(C_1)$; a contradiction. Thus $\{a, b, c\}$ is coclosed. □

By combining the last lemma with [17, Corollary 4.3], we immediately obtain the following.

Corollary 2.13. *If $\{a, b, c\}$ is a wild triangle in a 3-connected matroid and $\{a, b, c\}$ is not in a 4-element fan, then $\{a, b, c\}$ is fully closed.*

We shall use the next lemma [17, Lemma 2.9] in the proof that Theorem 1.1 holds if M has a quad.

Lemma 2.14. *Let Q be a quad in a 3-connected matroid M. If $e \in Q$, then $\mathrm{si}(M/e)$ is 3-connected.*

Two sets A and B in a matroid are a *modular pair* if $r(A) + r(B) = r(A \cup B) + r(A \cap B)$. Such pairs of sets will be useful in proving our main results. The next two lemmas concern such pairs. The first is elementary.

Lemma 2.15. *Let z be an element of the matroid M and let X and Y be a modular pair of sets in $M\backslash z$. If $z \in \mathrm{cl}_M(X)$ and $z \in \mathrm{cl}_M(Y)$, then $z \in \mathrm{cl}_M(X \cap Y)$.*

Lemma 2.16. *Let A and B be sets of elements in a matroid M. If $\lambda(A) + \lambda(B) = \lambda(A \cup B) + \lambda(A \cap B)$, then A and B are a modular pair.*

Proof. Let $A' = E(M) - A$ and $B' = E(M) - B$. Since $\lambda(A) + \lambda(B) = \lambda(A \cup B) + \lambda(A \cap B)$, we have

$$r(A) + r\big(A'\big) + r(B) + r\big(B'\big) = r(A \cup B) + r\big(A' \cap B'\big) + r(A \cap B) + r\big(A' \cup B'\big),$$

so

$$r(A) + r(B) - r(A \cup B) - r(A \cap B) = r\big(A' \cup B'\big) + r\big(A' \cap B'\big) - r\big(A'\big) - r\big(B'\big).$$

The lemma now follows from the submodularity of the rank function. □

The following well-known result is straightforward to prove.

Lemma 2.17. *Let M be a matroid, $X \subseteq E(M)$, and $e \in E(M) - X$. Then*

(i) $\lambda(X \cup e) = \lambda(X)$ *if and only if e is in exactly one of* $\mathrm{cl}(X)$ *and* $\mathrm{cl}^*(X)$; *and*
(ii) $\lambda(X \cup e) < \lambda(X)$ *if and only if e is in both* $\mathrm{cl}(X)$ *and* $\mathrm{cl}^*(X)$.

Let $\{X, Y, \{e\}\}$ be a partition of the ground set of a matroid M. Then *e blocks* (X, Y) if (X, Y) is not induced in M, that is, if $\lambda_M(X \cup e, Y) > \lambda_{M\backslash e}(X, Y)$, and $\lambda_M(X, Y \cup e) > \lambda_{M\backslash e}(X, Y)$. In addition, we say that *e blocks X* if e blocks $(X, E(M) - (X \cup e))$. The element *e coblocks* (X, Y) if $\lambda_{M^*}(X \cup e, Y) > \lambda_{M^*\backslash e}(X, Y)$, and $\lambda_{M^*}(X, Y \cup e) > \lambda_{M^*\backslash e}(X, Y)$. Equivalently, e coblocks (X, Y) if $\lambda_M(X \cup e, Y) > \lambda_{M/e}(X, Y)$, and $\lambda_M(X, Y \cup e) > \lambda_{M/e}(X, Y)$. If U, V, and W are sets in a matroid M such that U and V are disjoint, we say that (U, V) *crosses* W if both $U \cap W$ and $V \cap W$ are nonempty. The next lemma is routine and well known.

Lemma 2.18. *The following are equivalent for a partition $\{X, Y, \{e\}\}$ of the ground set of a matroid M.*

(a) *e blocks (X, Y).*
(b) $e \in \mathrm{cl}^*(X)$ *and* $e \in \mathrm{cl}^*(Y)$.
(c) $e \notin \mathrm{cl}(X)$ *and* $e \notin \mathrm{cl}(Y)$.

Lemma 2.19. *In a matroid M, let $(X, Y, \{s\}, \{t\})$ be a partition of $E(M)$. If $t \in \mathrm{cl}^*_{M\backslash s}(X)$ and $s \in \mathrm{cl}_M(Y)$, then $t \in \mathrm{cl}^*_M(X)$.*

Proof. Under the hypotheses, t is a coloop of $M|(Y \cup \{s, t\})$. □

3. A matroid garden

In this section, we recall some definitions from [15,16]. Let $(P_1, P_2, \ldots, P_n)$ be a *flower* Φ in a 3-connected matroid M, that is, $(P_1, P_2, \ldots, P_n)$ is an ordered partition of $E(M)$ such that $\lambda_M(P_i) = 2 = \lambda_M(P_i \cup P_{i+1})$ for all i in $[n]$, where all subscripts are interpreted modulo n. The sets $P_1, P_2, \ldots, P_n$ are the *petals* of Φ. Each has at least two elements. It is shown in [15, Theorem 4.1] that every flower in a 3-connected matroid is either an *anemone* or a *daisy*. In the first case, all unions of petals are 3-separating; in the second, a union of petals is 3-separating if and only if the petals are consecutive in the cyclic ordering $(P_1, P_2, \ldots, P_n)$. A 3-separation (X, Y) is *displayed* by a flower if X is a union of petals of the flower.

Let Φ_1 and Φ_2 be flowers in a matroid M. A natural quasi-ordering on the set of flowers of M is obtained by setting $\Phi_1 \preccurlyeq \Phi_2$ if every non-sequential 3-separation displayed by Φ_1 is equivalent to one displayed by Φ_2. If $\Phi_1 \preccurlyeq \Phi_2$ and $\Phi_2 \preccurlyeq \Phi_1$, then Φ_1 and Φ_2 are *equivalent* flowers. Such flowers display, up to equivalence of 3-separations, exactly the same non-sequential 3-separations of M. Let Φ be a flower of M. The *order* of Φ is the minimum number of petals in a flower equivalent to Φ. We say that Φ is *maximal* if Φ is equivalent to Φ' whenever $\Phi \preccurlyeq \Phi'$.

An element e of M is *loose* in Φ if $e \in \mathrm{fcl}(P_i) - P_i$ for some petal P_i of Φ; otherwise e is *tight*. A petal P_i is *loose* if all its elements are loose; and P_i is *tight* otherwise. A flower of order at least 3 is *tight* if all of its petals are tight. A flower of order 2 or 1 is *tight* if it has two petals or one petal, respectively.

The next two lemmas exemplify how we will use flowers in this paper. The first corrects [17, Lemma 2.10].

Lemma 3.1. *Let $(P_1, P_2, \ldots, P_k)$ be a flower in a* 3*-connected matroid. If P_2 is loose and P_1 is tight, then $P_2 \subseteq \mathrm{fcl}(P_1)$.*

Proof. Since P_2 is loose, for some $i \neq 2$, there is a sequence $z_1, z_2, \ldots, z_m, z_{m+1}$ where $\{z_1, z_2, \ldots, z_{m+1}\} \cap P_2 = \{z_{m+1}\} = \{z\}$ and $P_i \cup \{z_1, z_2, \ldots, z_j\}$ is 3-separating for all j in $[m+1]$. Now move the elements $z_1, z_2, \ldots, z_m$ one at a time in order from their original petals into P_i. When such a move reduces the size of a petal to one, add that one remaining element to an adjacent petal other than P_2 before continuing. This ensures that, after each step, we still have a flower. Throughout the process, each petal retains its label unless it is absorbed into an adjacent petal in which case the resulting petal takes the name of the absorbing petal. Each petal in the final flower has the same full closure as the petal with the same name in the original flower. Because P_1 was tight originally, it remains tight and so still labels a petal in the final flower.

We relabel this final flower as $(R_1, R_2, \ldots, R_t)$ where $R_s = P_i$ and $(P_1, P_2) = (R_1, R_2)$. Then $z \in \mathrm{cl}^{(*)}(R_s) - R_s$. We argue by induction on $|R_2|$. Suppose $|R_2| = 2$. If $s = 1$, then, by Lemma 5.2 of [15], $R_2 \subseteq \mathrm{fcl}(R_1)$, as required. If $s \neq 1$, then $R_3 \cup R_4 \cup \cdots \cup R_t \cup z$ is 3-separating. Thus so is $R_1 \cup y$ where $R_2 - z = \{y\}$, and Lemma 5.2 of [15] again implies that $R_2 \subseteq \mathrm{fcl}(R_1)$. Now assume the result holds for $|R_2| < n$ and let $|R_2| = n \geqslant 3$. If $s = 1$, then $(R_s \cup z, R_2 - z, R_3, \ldots, R_t)$ is a flower in which $R_2 - z$ is loose and $R_1 \cup z$ is tight so, by the induction assumption, $R_2 - z \subseteq \mathrm{fcl}(R_1 \cup z)$. Hence $R_2 \subseteq \mathrm{fcl}(R_1)$ as $z \in \mathrm{fcl}(R_1)$. Now suppose $s \neq 1$. Then $(R_1, R_2 - z, R_3, \ldots, R_s \cup z, \ldots, R_t)$ is a flower in which $R_2 - z$ is loose and R_1 is tight. Hence, by the induction assumption, $R_2 - z \subseteq \mathrm{fcl}(R_1)$. Moreover, as both $R_2 - z$ and R_2 are 3-separating, $z \in \mathrm{cl}^{(*)}(R_2 - z)$. Hence $z \in \mathrm{fcl}(R_1)$ and so $R_2 \subseteq \mathrm{fcl}(R_1)$. The lemma follows by induction. □

Lemma 3.2. *Let (P, Q) be a* 3*-separation of a* 3*-connected matroid M where P is sequential and Q is a quad. Then M is sequentially* 4*-connected.*

Proof. Let (R, G) be a non-sequential 3-separation of M. Then $|R|, |G| \geqslant 4$, so $P \cap R \neq \emptyset \neq P \cap G$, otherwise Q is R or G. As P is sequential, neither R nor G is contained in P. If R contains a single element of P, then, as $|R| \geqslant 4$ and $|Q| = 4$, but R does not contain Q, we deduce that $|R| = 4$. By Lemma 2.1, $R \cap Q$ is 3-separating. Hence R is sequential; a contradiction. Thus $|R \cap P| \geqslant 2$ and, similarly, $|G \cap P| \geqslant 2$. Again, by Lemma 2.1, $|R \cap Q| \neq 1$ otherwise $G \cap Q$ is a triangle or a triad; a contradiction. Hence, by symmetry, $|R \cap Q| = 2 = |G \cap Q|$. Thus M has a flower $(Q \cap G, P \cap G, P \cap R, Q \cap R)$. Let F be the set consisting of the first three elements in a sequential ordering of P. Then we may assume that $P \cap R$ contains at least two elements of F. As $P \cap R$ is 3-separating, it follows by repeatedly uncrossing that there is a sequential ordering of P whose first $|P \cap R|$ elements are the elements of $P \cap R$. Thus $P \cap G \subseteq \mathrm{fcl}(P \cap R)$. Hence $P \cap G$ is a loose petal. Therefore, by Lemma 3.1,

(i) $P \cap G \subseteq \mathrm{fcl}(Q \cap G)$, or
(ii) $Q \cap G$ is a loose petal of the flower $(Q \cap G, P \cap G, P \cap R, Q \cap R)$.

We show next that

$$E(M) \in \{\mathrm{fcl}(Q \cap G), \mathrm{fcl}(Q \cap R)\}. \tag{1}$$

This holds in case (i) by Lemma 2.6, otherwise G is sequential. In case (ii), $Q \cap G$ is also loose in the flower $(Q \cap G, P, Q \cap R)$. But, as Q is a quad, no element of Q is in $\mathrm{fcl}(P)$, so $\mathrm{fcl}(Q \cap R) \supseteq Q \cap G$. Thus there is a sequence $y_1, y_2, \ldots, y_{t+1}$ such that $(Q \cap R) \cup \{y_1, y_2, \ldots, y_i\}$ is 3-separating for all i in $[t+1]$ where $\{y_1, y_2, \ldots, y_t\} \subseteq P$ while $y_{t+1} \in Q \cap G$. Assume this sequence is chosen to maximize t. Suppose $\{y_1, y_2, \ldots, y_t\} \neq P$. If $P - \{y_1, y_2, \ldots, y_t\} = \{z\}$ for some element z, then $(Q \cap R) \cup \{y_1, y_2, \ldots, y_t, z\}$ is 3-separating and the choice of t is contradicted. Thus $|P - \{y_1, y_2, \ldots, y_t\}| \geqslant 2$. Then, by Lemma 2.1, $[(Q \cap R) \cup \{y_1, y_2, \ldots, y_{t+1}\}] \cap Q$ is a 3-element 3-separating subset of the quad Q; a contradiction. Therefore $\{y_1, y_2, \ldots, y_t\} = P$ and so, in case (ii), $E(M) = \mathrm{fcl}(Q \cap R)$, so (1) holds.

By (1) and symmetry, we may assume that $\mathrm{fcl}(Q \cap G) = E(M)$. Then M has a sequential ordering whose first two elements are in $Q \cap G$. By Lemma 2.8, we may assume that the last two elements in this sequential ordering are in $Q \cap R$. Then G avoids the last two elements of this ordering, so, by Lemma 2.7, G is sequential; a contradiction. □

Next we note a corollary for flowers of Lemma 2.16 together with an extension of this corollary.

Corollary 3.3. *Let (R_1, R_2, R_3, R_4) be a flower in a 3-connected matroid M. Then $R_1 \cup R_2$ and $R_2 \cup R_3$ are a modular pair.*

Lemma 3.4. *Let (R_1, R_2, R_3, R_4) be a flower in a 3-connected matroid M, and let $z \in R_4$. If $z \in \mathrm{cl}^*(R_1 \cup R_2)$ and $z \in \mathrm{cl}^*(R_2 \cup R_3)$, then $z \in \mathrm{cl}^*(R_2)$.*

Proof. Note that (R_1, R_2, R_3, R_4) is a flower in M^*. By Corollary 3.3, $R_1 \cup R_2$ and $R_2 \cup R_3$ are a modular pair. Thus, by Lemma 2.15, $z \in \mathrm{cl}_{M^*}(R_2)$. □

The classes of anemones and daisies can be further refined using local connectivity. Let $(P_1, P_2, \ldots, P_n)$ be a flower Φ with $n \geqslant 3$. If Φ is an anemone, then $\sqcap(P_i, P_j)$ takes a fixed value k in $\{0, 1, 2\}$ for all distinct i, j in $[n]$. We call Φ a *paddle* if $k = 2$, a *copaddle* if $k = 0$, and a *spike-like* flower if $k = 1$ and $n \geqslant 4$. Similarly, if Φ is a daisy, then $\sqcap(P_i, P_j) = 1$ for all consecutive i and j. We say Φ is *swirl-like* if $n \geqslant 4$ and $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and Φ is *Vámos-like* if $n = 4$ and $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$.

If (P_1, P_2, P_3) is a flower Φ and $\sqcap(P_i, P_j) = 1$ for all distinct i and j, we call Φ *ambiguous* if it has no loose elements, *spike-like* if there is an element in $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) \cap \mathrm{cl}(P_3)$ or $\mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) \cap \mathrm{cl}^*(P_3)$, and *swirl-like* otherwise. Every flower with at least three petals is of one of these six *types*: a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or ambiguous [15].

Let Φ be a flower. By replacing two petals P and P' of Φ by their union, we obtain another flower provided that, when Φ is a daisy, P and P' are consecutive petals. Any flower that can be obtained from Φ by repeated application of this process is said to have been obtained from Φ by *concatenating petals* or is called a *concatenation* of Φ. It will also be convenient to view Φ as a concatenation of itself. We shall repeatedly use concatenation of flowers throughout this paper along with the following lemma whose elementary proof we omit.

Lemma 3.5. *If Φ' is a concatenation of Φ, and Φ' has at least three petals, then the local connectivity between pairs of consecutive petals in Φ' equals the local connectivity between pairs of consecutive petals in Φ.*

Flowers provide a way of representing 3-separations in a 3-connected matroid M. It was shown in [15] that, by using a certain type of tree, one can simultaneously display a representative of each equivalence class of non-sequential 3-separations of M. We now describe the type of tree that is used. Let π be a partition of a finite set E. Let T be a tree such that every member of π labels a vertex of T; some vertices may be unlabelled but no vertex is multiply labelled. We say that T is a *π-labelled tree*; labelled vertices are called *bag vertices* and members of π are called *bags*.

Let G be a subgraph of T with components $G_1, G_2, \dots, G_m$. Let X_i be the union of those bags that label vertices of G_i. Then *the subsets of E displayed by G* are $X_1, X_2, \dots, X_m$. In particular, if $V(G) = V(T)$, then $\{X_1, X_2, \dots, X_m\}$ is the *partition of E displayed by G*. Let e be an edge of T. The *partition of E displayed by e* is the partition displayed by $T\backslash e$. If $e = v_1v_2$ for vertices v_1 and v_2, then (Y_1, Y_2) is the (*ordered*) *partition of $E(M)$ displayed by v_1v_2* if Y_1 is the union of the bags in the component of $T\backslash v_1v_2$ containing v_1. Let v be a vertex of T that is not a bag vertex. The *partition of E displayed by v* is the partition displayed by $T - v$. The edges incident with v correspond to the components of $T - v$, and hence to the members of the partition displayed by v. In what follows, if a cyclic ordering $(e_1, e_2, \dots, e_n)$ is imposed on the edges incident with v, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by v.

Let M be a 3-connected matroid with ground set E. An *almost partial 3-tree T* for M is a π-labelled tree, where π is a partition of E such that:

(i) For each edge e of T, the partition (X, Y) of E displayed by e is 3-separating, and, if e is incident with two bag vertices, then (X, Y) is a non-sequential 3-separation.
(ii) Every non-bag vertex v is labelled either D or A; if v is labelled D, then there is a cyclic ordering on the edges incident with v.
(iii) If a vertex v is labelled A, then the partition of E displayed by v is a tight maximal anemone of order at least 3.
(iv) If a vertex v is labelled D, then the partition of E displayed by v, with the cyclic order induced by the cyclic ordering on the edges incident with v, is a tight maximal daisy of order at least 3.

By conditions (iii) and (iv), a vertex v labelled D or A corresponds to a flower of M. The 3-separations displayed by this flower are the 3-separations *displayed by v*. A vertex of a partial 3-tree is referred to as a *daisy vertex* or an *anemone vertex* if it is labelled D or A, respectively. A vertex labelled either D or A is a *flower vertex*. A 3-separation is *displayed by an almost partial 3-tree T* if it is displayed by some edge or some flower vertex of T.

A 3-separation (R, G) of M *conforms* with an almost partial 3-tree T if either (R, G) is equivalent to a 3-separation that is displayed by a flower vertex or an edge of T, or (R, G) is equivalent to a 3-separation (R', G') with the property that either R' or G' is contained in a bag of T.

An almost partial 3-tree for M is a *partial 3-tree* if every non-sequential 3-separation of M conforms with T. We now define a quasi-order on the set of partial 3-trees for M. Let T_1 and T_2 be two partial 3-trees for M. Then $T_1 \preccurlyeq T_2$ if all of the non-sequential 3-separations displayed by T_1 are displayed by T_2. If $T_1 \preccurlyeq T_2$ and $T_2 \preccurlyeq T_1$, then T_1 is *equivalent* to T_2. A partial 3-tree is *maximal* if it is maximal with respect to this quasi-order. We shall call a maximal partial 3-tree a *3-tree*. Note that this terminology differs from that used in [16] where we use the term '3-tree' for a particular type of maximal 3-tree defined in that paper.

The following theorem is the main result of [15, Theorem 9.1].

Theorem 3.6. *Let M be a 3-connected matroid with $|E(M)| \geqslant 9$. Then M has a 3-tree T. Moreover, every non-sequential 3-separation of M is equivalent to a 3-separation displayed by T.*

This paper will rely on the results from [17] that specify how wild triangles can arise. Let $\{a, b, c\}$ be a triangle of a 3-connected matroid M. Then $\{a, b, c\}$ is a *standard* wild triangle if there is a partition $\mathbf{P} = (P_1, P_2, \dots, P_6)$ of $E(M) - \{a, b, c\}$ such that $|P_i| \geqslant 2$ for all i and the following hold:

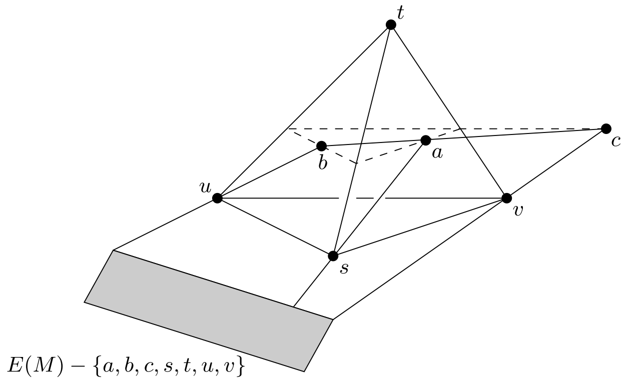

Fig. 1. A trident.

(i) $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected, $M\backslash a,b,c$ is connected, and $\mathrm{co}(M\backslash a,b,c)$ is 3-connected.
(ii) $(P_1 \cup P_2 \cup a, P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup c)$ is a flower in M.
(iii) $(P_2 \cup P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup P_1 \cup c)$, $(P_4 \cup P_5 \cup P_6 \cup c, P_1 \cup P_2 \cup P_3 \cup a)$, and $(P_6 \cup P_1 \cup P_2 \cup a, P_3 \cup P_4 \cup P_5 \cup b)$ are 3-separations exposed in M by a, b, and c, respectively.

A partition $\mathbf{P}$ satisfying these conditions is a partition *associated* to $\{a,b,c\}$.

Now denote the triangle $\{a,b,c\}$ of matroid M by Δ and take a copy of $M(K_4)$ having Δ as a triangle and $\{a',b',c'\}$ as the complementary triad, where e' is the element of $M(K_4)$ that is not in a triangle with e. Let $P_\Delta(M(K_4), M)$ be the *generalized parallel connection* of $M(K_4)$ and M. We write ΔM for $P_\Delta(M(K_4), M)\backslash\Delta$ and say that ΔM is obtained from M by a *$\Delta - Y$ exchange on* Δ. Note that ΔM has ground set $(E(M) - \{a,b,c\}) \cup \{a',b',c'\}$. It is common to relabel a', b', and c' as a, b, and c so that M and ΔM have the same ground set, and we do this unless specified otherwise. We say that Δ is a *costandard* wild triangle in M if Δ is a standard wild triangle in $(\Delta M)^*$. Let $\mathbf{P} = (P_1, P_2, \dots, P_6)$ be a partition of $E(M) - \{a,b,c\}$. Then $\mathbf{P}$ is *associated* to the costandard wild triangle Δ in M if $\mathbf{P}$ is associated to the standard wild triangle Δ in $(\Delta M)^*$.

Let X be a 3-separating set $\{a,b,c,s,t,u,v\}$ in a 3-connected matroid M, where $\{a,b,c\}$ is a triangle. Then X is a *trident* with wild triangle $\{a,b,c\}$ if $\{t,s,u,b\}$, $\{t,u,v,c\}$, and $\{t,s,v,a\}$ are quads exposed in $M\backslash a$, $M\backslash b$, and $M\backslash c$, respectively (see Fig. 1). Observe that $(M/t)|(X-t) \cong M(K_4)$. We remark that what we have called a trident is quite different from what Geelen and Zhou [7] call a trident.

The following is the main result of [17, Theorem 3.1].

Theorem 3.7. *Let $\{a,b,c\}$ be a wild triangle in a 3-connected matroid M, where $|E(M)| \neq 11$, and suppose that $\{a,b,c\}$ is not an internal triangle of a fan of M. Then $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected. Moreover, if (A_1,A_2), (B_1,B_2), and (C_1,C_2) are 3-separations exposed by a, b, and c, respectively, with $a \in B_2 \cap C_1$, $b \in C_2 \cap A_1$, and $c \in A_2 \cap B_1$, then exactly one of the following holds*:

(i) *$\{a,b,c\}$ is a wild triangle in a trident*;
(ii) *$\{a,b,c\}$ is a standard wild triangle and (A_1,A_2), (B_1,B_2), and (C_1,C_2) can be replaced by equivalent 3-separations such that*
  (a) *$(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ is a partition associated to $\{a,b,c\}$*;
  (b) *every 2-element cocircuit of $M\backslash a,b,c$ meets exactly two of $A_2 \cap B_1$, $B_2 \cap C_1$, and $C_2 \cap A_1$*; *and*
  (c) *in $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$, every union of consecutive sets is exactly 3-separating in $M\backslash a,b,c$*;
(iii) *$\{a,b,c\}$ is a costandard wild triangle*; *more particularly, if M' is the matroid that is obtained from M by performing a $\Delta - Y$ exchange on $\{a,b,c\}$ in M and then taking the dual of the result, then M' is 3-connected and $((A_2 - c) \cup b, (A_1 - b) \cup c)$, $((B_2 - a) \cup c, (B_1 - c) \cup a)$, and $((C_2 - b) \cup a$,*

$(C_1 - a) \cup b)$ are 3-separations in M' exposed by a, b, and c, respectively. Moreover, (ii) *holds when $(M, A_1, A_2, B_1, B_2, C_1, C_2)$ is replaced by $(M', (A_2 - c) \cup b, (A_1 - b) \cup c, (B_2 - a) \cup c, (B_1 - c) \cup a, (C_2 - b) \cup a, (C_1 - a) \cup b)$.*

4. A more powerful result

In this section, we state a more powerful result from which Theorem 1.1 will follow when $|E(M)| \geqslant 9$. First we prove Theorem 1.1 when $|E(M)| \leqslant 8$.

Lemma 4.1. *Let M be a 3-connected matroid other than a wheel or a whirl. If $|E(M)| \leqslant 8$, then M has an element whose deletion from M or M^* is 3-connected but does not expose any 3-separations.*

Proof. As M is not a wheel or a whirl, it follows by Tutte's Wheels-and-Whirls Theorem [21] that, by replacing M by its dual if necessary, we have that M has an element e such that $M \backslash e$ is 3-connected. By Lemma 2.9, since $|E(M)| \leqslant 8$, the element e does not expose any 3-separations in M. $\square$

We may assume now that $|E(M)| \geqslant 9$. In that case, Theorem 1.1 is immediate from the following more powerful result. A *terminal bag* in a 3-tree T for a 3-connected matroid M is a degree-one vertex of T. A subset S of $E(M)$ is a *terminal set* if there is a 3-tree T for M such that S labels a terminal bag of T.

Theorem 4.2. *Let M be a 3-connected matroid other than a wheel or a whirl. Suppose $|E(M)| \geqslant 9$ and let S be a terminal bag of some 3-tree for M. Then $\mathrm{fcl}(S)$ contains an element e whose deletion from M or M^* is 3-connected but does not expose any 3-separations.*

The next lemma establishes this theorem when S is a quad.

Lemma 4.3. *Let f be an element of a quad Q in a 3-connected matroid M.*

(i) *When $M \backslash f$ is 3-connected, f does not expose any 3-separations in $M \backslash f$.*
(ii) *There is an element e in $\mathrm{fcl}(Q)$ whose deletion from M or M^* is 3-connected but does not expose any 3-separations.*

Proof. Take an element f in Q. Assume that $M \backslash f$ is 3-connected. Suppose f exposes a 3-separation (X, Y) of $M \backslash f$. By Lemma 2.9, $|X|, |Y| \geqslant 4$. Clearly we may assume that $|X \cap (Q - f)| \geqslant 2$. Since $Q - f$ is a triad of $M \backslash f$, the 3-separation (X, Y) is equivalent to the 3-separation $(X \cup (Q - f), Y - Q)$. But $f \in \mathrm{cl}(Q - f)$, so $(X \cup Q, Y - Q)$ is a 3-separation of M; a contradiction. Hence (i) holds.

By (i) and duality, we may assume that neither M/f nor $M \backslash f$ is 3-connected. Since Lemma 2.14 implies that $\mathrm{si}(M/f)$ and $\mathrm{co}(M \backslash f)$ are 3-connected, we deduce that f is in both a triangle and a triad. Hence f is in a fan F with at least four elements. By orthogonality, $F \subseteq \mathrm{fcl}(Q)$. Hence, by Lemma 2.11, (ii) holds. $\square$

Lemma 4.4. *For a tight flower $(\{a, b\}, P, R)$ in a 3-connected matroid M with $\{a, b\}$ fully closed, $\{a, b\} \cup P$ a quad, and $|E(M)| \geqslant 7$, either*

(i) *for some M_1 in $\{M, M^*\}$, the matroid $M_1 \backslash a$ is 3-connected and does not expose any 3-separations*; *or*
(ii) *R contains distinct elements t and c, and there is a labelling a', b' of the elements of P such that $\{a, a', t\}$ and $\{b, b', t\}$ are triangles and $\{a, a', c\}$ and $\{b, b', c\}$ are triads of M.*

Proof. By (i) of the last lemma, we may assume that neither M/a nor $M \backslash a$ is 3-connected. Thus, by Lemma 2.14, a is in both a triangle T and a triad T^*. As $\{a, b\}$ is fully closed, $b \notin T \cup T^*$. By orthogonality between T and the cocircuit $\{a, b\} \cup P$, we deduce that there is an element a' of P such that $T = \{a, a', t\}$ for some element t of R. Let $P - \{a'\} = \{b'\}$. Suppose $b' \in T^*$. Then, by orthogonality with the circuit T, we must have that $T^* = \{a, b', t\}$. Then, for $X = \{a, b, a', b', t\}$, we have $\lambda_M(X) = r(X) + r^*(X) - |X| \leqslant 3 + 3 - 5 = 1$, so $|E(M) - X| \leqslant 1$. Hence $|E(M)| \leqslant 6$; a contradiction.

We may now assume that $a' \in T^*$. Then $T^* = \{a, a', c\}$ for some element c of R. Moreover, $c \neq t$ as $|E(M)| \neq 4$. By circuit exchange, $(\{a, a', t\} \cup \{a, b, a', b'\}) - a$ contains a circuit C of M. By orthogonality with the cocircuit T^*, we get that $a' \notin C$, so $C = \{t, b, b'\}$. By symmetry, M has $\{c, b, b'\}$ as a cocircuit. We conclude that (ii) holds. □

The next theorem is the main result of [18].

Theorem 4.5. *Let (A, B) be a non-sequential 3-separation in a 3-connected matroid M. Suppose that B is fully closed, A meets no triangle or triad of M, and if (X, Y) is a non-sequential 3-separation of M, then either $A \subseteq \mathrm{fcl}(X)$ or $A \subseteq \mathrm{fcl}(Y)$. Then A contains an element whose deletion from M or M^* is 3-connected but does not expose any 3-separations.*

The following consequence of the last theorem plays an important role in the proof of Theorem 4.2.

Corollary 4.6. *Let S be a non-sequential terminal set in a 3-connected matroid M and let $S' = S - \mathrm{fcl}(E(M) - S)$. If no triangle or triad of M contains at least two elements of S', then S' contains an element e whose deletion from M or M^* is 3-connected but does not expose any 3-separations.*

Proof. Let T be a 3-tree in which S is a terminal set. If M is sequentially 4-connected, then, by Theorem 1.2 and Lemma 2.9, the lemma holds. Thus we may assume that M is not sequentially 4-connected and so T has at least two vertices. Let u be the vertex of T labelled by S and let v be the vertex of T adjacent to u. We next show that $(S, E(M) - S)$ is a non-sequential 3-separation of M. This is certainly true if v is a bag vertex, so assume that v is a flower vertex. Then the partition of $E(M)$ displayed by v is a tight maximal flower with S as a petal. Thus $(S, E(M) - S)$ is non-sequential.

Now let (X, Y) be a non-sequential 3-separation of M. By Theorem 3.6, (X, Y) is equivalent to a 3-separation (X', Y') displayed by T. Since S labels a terminal bag, we may assume without loss of generality that $S \subseteq X'$, so

$$S' \subseteq S \subseteq \mathrm{fcl}\big(X'\big) = \mathrm{fcl}(X).$$

The corollary now holds by Theorem 4.5. □

5. Two elements in the guts

In this section, we prove Theorem 1.1 when M has a non-minimal 3-separation (X, Y) with $|\mathrm{cl}(X) \cap \mathrm{cl}(Y)| \geqslant 2$. In particular, the next lemma will be needed in our treatment of wild triangles.

Lemma 5.1. *In a 3-connected matroid M, let $(X_1, \{a, b\}, X_2)$ be a partition of $E(M)$ such that both $(X_1, \{a, b\} \cup X_2)$ and $(X_1 \cup \{a, b\}, X_2)$ are 3-separations, and $\{a, b\} \subseteq \mathrm{cl}(X_1) \cap \mathrm{cl}(X_2)$. Assume that $M\backslash a$ and $M\backslash b$ are 3-connected. Then either*

(i) *at least one of a and b does not expose any 3-separations in M; or*
(ii) *$|E(M)| = 10$ and, for all e in $E(M) - \{a, b\}$, the matroid $M\backslash e$ is 3-connected but e does not expose any 3-separations in M.*

Proof. Since $(X_1, \{a, b\} \cup X_2) \cong (X_1 \cup \{a, b\}, X_2)$, both of these 3-separations are sequential, or both are non-sequential. In the first case, since $\{a, b\} \subseteq \mathrm{cl}(X_1) \cap \mathrm{cl}(X_2)$, we may assume that $X_1 \cup \{a, b\}$ is sequential. By Lemma 2.10, a does not expose any 3-separations. We may now assume that both $(X_1, \{a, b\} \cup X_2)$ and $(X_1 \cup \{a, b\}, X_2)$ are non-sequential. Thus $|X_1|, |X_2| \geqslant 4$. Moreover, each of $(X_1, a \cup X_2)$ and $(X_1 \cup a, X_2)$ are non-sequential 3-separations of $M\backslash b$.

Assume that $M\backslash a$ and $M\backslash b$ have exposed 3-separations (A_1, A_2) and (B_1, B_2), respectively. Since $b \in \mathrm{cl}(X_1) \cap \mathrm{cl}(X_2)$ but $b \notin \mathrm{cl}(B_1) \cup \mathrm{cl}(B_2)$, all of $X_1 \cap B_1$, $X_1 \cap B_2$, $X_2 \cap B_1$, and $X_2 \cap B_2$ are non-empty.

Suppose $|X_1 \cap B_1| = 1$. Then, as $|X_1|, |B_1| \geqslant 4$, we have $|X_1 \cap B_2|, |X_2 \cap B_1| \geqslant 2$. Thus, by uncrossing, $X_1 \cup B_2$ is 3-separating in $M\backslash b$ and $(X_1 \cup B_2, (X_2 \cup a) \cap B_1) \cong (B_1, B_2)$. But $b \in \mathrm{cl}(X_1 \cup B_2)$ so we contradict the fact that (B_1, B_2) is exposed by b. We deduce that $|X_1 \cap B_1| \geqslant 2$. By symmetry, each of $|X_1 \cap B_2|$, $|X_2 \cap B_1|$, and $|X_2 \cap B_2|$ has at least two elements. Thus $M\backslash b$ has $(X_1 \cap B_1, X_1 \cap B_2, (X_2 \cup a) \cap B_2, (X_2 \cup a) \cap B_1)$ as a flower, Φ. Suppose $X_1 \cap B_1$ is loose and $X_1 \cap B_2$ is tight. Then, by Lemma 3.1, $X_1 \cap B_1 \subseteq \mathrm{fcl}(X_1 \cap B_2)$. From Lemma 2.6, it follows that $((X_2 \cup a) \cap B_1, X_1 \cup B_2) \cong (B_1, B_2)$; a contradiction. By symmetry, it follows that $X_1 \cap B_1$ and $X_1 \cap B_2$ are either both loose or are both tight petals of Φ. In the former case, as $X_2 \cup a$ is not sequential, it is not loose in the flower $(X_1 \cap B_1, X_1 \cap B_2, X_2 \cup a)$ of $M\backslash b$. Thus, by Lemma 3.1, each of $X_1 \cap B_1$ and $X_1 \cap B_2$ is contained in $\mathrm{fcl}_{M\backslash b}(X_2 \cup a)$. Hence X_1 is sequential in $M\backslash b$; a contradiction. We deduce that both $X_1 \cap B_1$ and $X_1 \cap B_2$ are tight petals of Φ.

Now, without loss of generality, $a \in B_1$. By Lemma 2.1, each of $X_1 \cap B_1$ and $(X_1 \cup a) \cap B_1$ is 3-separating in $M\backslash b$. Thus $a \in \mathrm{cl}^{(*)}_{M\backslash b}(X_1 \cap B_1)$. But $a \in \mathrm{cl}_{M\backslash b}(X_2)$, so, by orthogonality, $a \in \mathrm{cl}_{M\backslash b}(X_1 \cap B_1)$.

As $X_1 \cap B_1$ and $X_1 \cap B_2$ are 3-separating in $M\backslash b$ and their complements contain X_2, each is 3-separating in M. Thus $(X_1 \cap B_1, X_1 \cap B_2, X_2 \cup \{a, b\})$ is a flower Ψ in M. As $a \in \mathrm{cl}(X_1 \cap B_1) \cap \mathrm{cl}(X_2 \cup \{a, b\})$, the flower Ψ is not a copaddle. If Ψ is a paddle, then $\sqcap(X_2 \cup \{a, b\}, X_1 \cap B_1) = 2$. But $\sqcap(X_2 \cup \{a, b\}, X_1) = 2$ and $b \in \mathrm{cl}(X_2 \cup \{a, b\}) \cap \mathrm{cl}(X_1)$, so, by Lemma 2.5, $b \in \mathrm{cl}(X_1 \cap B_1)$. Thus $b \in \mathrm{cl}(B_1)$; a contradiction. Hence Ψ is not a paddle. Thus the local connectivity between consecutive petals of Ψ is one.

Since (A_1, A_2) is a 3-separation of $M\backslash a$ exposed by a, a symmetric argument to that just given establishes that $(X_1 \cap A_1, X_1 \cap A_2, X_2 \cup \{a, b\})$ is a flower in M in which the local connectivity between petals is one. Without loss of generality, $b \in A_1$. Note that this means that we have symmetry between $(b, a, B_1, B_2, A_1, A_2)$ and $(a, b, A_1, A_2, B_1, B_2)$. Thus $b \in \mathrm{cl}(X_1 \cap A_1)$.

Let $A_1 \cap X_1 = R$ and $A_2 \cap X_1 = G$ and colour the elements of R and G *red* and *green*, respectively. Note that we are only colouring elements of X_1. Since $b \notin \mathrm{cl}(B_1) \cup \mathrm{cl}(B_2)$ but $b \in \mathrm{cl}(R)$, it follows that $R \nsubseteq B_1 \cap X_1$ and $R \nsubseteq B_2 \cap X_1$. Likewise, as $a \notin \mathrm{cl}(A_1)$, we deduce that R does not contain $B_1 \cap X_1$. We have just noted that $B_1 \cap X_1$ is not monochromatic. From above, we deduce that we have the following two cases:

(I) $B_2 \cap X_1$ is all red; or
(II) $B_2 \cap X_1$ contains both red and green elements.

Consider case (I). We have $A_2 \cap X_1 = G \subseteq B_1 \cap X_1$ and $\sqcap(B_1 \cap X_1, X_2 \cup \{a, b\}) = 1 = \sqcap(A_2 \cap X_1, X_2 \cup \{a, b\})$. Since $a \in \mathrm{cl}(B_1 \cap X_1) \cap \mathrm{cl}(X_2 \cup \{a, b\})$, we deduce by Lemma 2.5 that $a \in \mathrm{cl}(A_2 \cap X_1)$, so $a \in \mathrm{cl}(A_2)$; a contradiction.

We may now assume that case (II) occurs.

5.1.1. *At least one of* $|X_1 \cap B_1 \cap A_1|$, $|X_1 \cap B_1 \cap A_2|$, $|X_1 \cap B_2 \cap A_1|$, *and* $|X_1 \cap B_2 \cap A_2|$ *is one.*

Assume that all of these sets have at least two elements. Then, by applying [15, 8.2.2] to the flower $(X_1 \cap B_1, X_1 \cap B_2, X_2 \cup \{a, b\})$ and the 3-separation $(A_1 \cap X_1, E(M) - (A_1 \cap X_1))$, we get that $(X_1 \cap B_1 \cap A_1, X_1 \cap B_1 \cap A_2, X_1 \cap B_2, X_2 \cup \{a, b\})$ is a flower in which the local connectivity between consecutive petals is 1. Thus $\sqcap(X_1 \cap B_1 \cap A_1, X_2 \cup \{a, b\}) = \sqcap(X_1 \cap B_1, X_2 \cup \{a, b\}) = 1$ and $a \in \mathrm{cl}(B_1 \cap X_1) \cap \mathrm{cl}(X_2 \cup \{a, b\})$, so, by Lemma 2.5, $a \in \mathrm{cl}(X_1 \cap B_1 \cap A_1)$. But $\sqcap(X_1 \cap B_1 \cap A_1, X_2 \cup \{a, b\}) = \sqcap(A_1 \cap X_1, X_2 \cup \{a, b\}) = 1$ and $b \in \mathrm{cl}(A_1 \cap X_1) \cap \mathrm{cl}(X_2 \cup \{a, b\})$, so $b \in \mathrm{cl}(X_1 \cap B_1 \cap A_1)$. Hence $\sqcap(X_1 \cap B_1 \cap A_1, X_2 \cup \{a, b\}) \geqslant 2$; a contradiction.

The next three assertions establish that all of $|X_1 \cap B_1 \cap A_1|$, $|X_1 \cap B_1 \cap A_2|$, $|X_1 \cap B_2 \cap A_1|$, and $|X_1 \cap B_2 \cap A_2|$ are one.

5.1.2. *If* $|X_1 \cap B_1 \cap A_1| = 1$, *then* $|X_1 \cap B_2 \cap A_1| = |X_1 \cap B_1 \cap A_2| = |X_1 \cap B_2 \cap A_2| = 1$.

Suppose that $|X_1 \cap B_1 \cap A_1| = 1$. Let $X_1 \cap B_1 \cap A_1 = \{x_1\}$. Suppose $|X_1 \cap B_2 \cap A_1| \geqslant 2$. Then, as $A_1 \cap X_1$ is 3-separating in M and hence in $M\backslash b$, and B_2 is 3-separating in $M\backslash b$, by Lemma 2.1,

$B_2 \cup x_1$ is 3-separating in $M\backslash b$. Thus $(B_1, B_2) \cong (B_1 - x_1, B_2 \cup x_1)$. Also, as $X_1 \cap B_1$ is a tight petal of Φ, it follows that $((X_1 \cap B_1) - x_1, (X_1 \cap B_2) \cup x_1, (X_2 \cup a) \cap B_2, (X_2 \cup a) \cap B_1)$ is a flower, Φ', in $M\backslash b$. Because $\sqcap(X_1 \cap B_1, X_1 \cap B_2) = 1$, the flowers Φ and Φ' imply that $\sqcap(X_1 \cap B_1, (X_2 \cup a) \cap B_1) = 1$ and $\sqcap((X_1 \cap B_1) - x_1, (X_2 \cup a) \cap B_1) = 1$. As $a \in \mathrm{cl}(X_1 \cap B_1) \cap \mathrm{cl}((X_2 \cup a) \cap B_1)$, it follows, by Lemma 2.5, that $a \in \mathrm{cl}((X_1 \cap B_1) - x_1)$. But $(X_1 \cap B_1) - x_1 \subseteq A_2$; a contradiction. Thus $|X_1 \cap B_2 \cap A_1| = 1$.

Suppose that $|X_1 \cap B_1 \cap A_2| \geqslant 2$. Then, as $B_1 \cap X_1$ and $A_2 \cap X_1$ are 3-separating in $M\backslash a$, it follows by Lemma 2.1 that $(A_2 \cap X_1) \cup x_1$ is 3-separating in $M\backslash a$, so $A_1 \cap X_1$ is loose in $(A_1 \cap X_1, A_2 \cap X_1, (X_2 \cup a) \cap A_2, (X_2 \cup a) \cap A_1)$, which, by symmetry, is a contradiction. Thus $|X_1 \cap B_1 \cap A_2| = 1$ and, by symmetry, $|X_1 \cap B_2 \cap A_2| = 1$. Hence 5.1.2 holds.

5.1.3. *If* $|X_1 \cap B_2 \cap A_1| = 1$, *then* $|X_1 \cap B_1 \cap A_1| = |X_1 \cap B_1 \cap A_2| = 1 = |X_1 \cap B_2 \cap A_2|$.

Suppose $X_1 \cap B_2 \cap A_1$ contains a single element, x_2 say. By 5.1.2, we may assume that $|X_1 \cap B_1 \cap A_1| \geqslant 2$. Then $(B_1, B_2) \cong (B_1 \cup x_2, B_2 - x_2)$ and $(X_1 \cap (B_1 \cup x_2), (X_1 \cap B_2) - x_2, X_2 \cup a)$ is a flower that is equivalent to the flower $(X_1 \cap B_1, X_1 \cap B_2, X_2 \cup a)$ in $M\backslash b$. But $X_1 \cap A_1$ and $X_1 \cap B_1$ are contained in $X_1 \cap (B_1 \cup x_2)$. Thus $\{a, b\} \subseteq \mathrm{cl}(X_1 \cap (B_1 \cup x_2)) \cap \mathrm{cl}(X_2 \cup a)$, so $1 = \sqcap(X_1 \cap (B_1 \cup x_2), X_2 \cup a) \geqslant 2$; a contradiction. Hence 5.1.3 holds.

5.1.4. *If* $|X_1 \cap B_2 \cap A_2| = 1$, *then* $|X_1 \cap B_1 \cap A_1| = |X_1 \cap B_1 \cap A_2| = 1 = |X_1 \cap B_2 \cap A_1|$.

Suppose $X_1 \cap B_2 \cap A_2 = \{y_2\}$. By 5.1.2 and symmetry, we may assume that $|X_1 \cap A_2 \cap B_1| \geqslant 2$. Then $(B_1, B_2) \cong (B_1 \cup y_2, B_2 - y_2)$ and $X_1 \cap A_2 = G \subseteq X_1 \cap (B_1 \cup y_2)$. Replacing (B_1, B_2) by $(B_1 \cup y_2, B_2 - y_2)$, we have reduced to case (I), so we have a contradiction that establishes 5.1.4.

By combining the last four sublemmas and using the symmetry between $(b, a, B_1, B_2, A_1, A_2)$ and $(a, b, A_1, A_2, B_1, B_2)$, we deduce that all of $|X_1 \cap B_1 \cap A_1|$, $|X_1 \cap B_1 \cap A_2|$, $|X_1 \cap B_2 \cap A_1|$, and $|X_1 \cap B_2 \cap A_2|$ are one. By symmetry, all of these cardinalities are still one when we replace X_1 by X_2. Hence $|E(M)| = 10$. Thus X_1 and X_2 are both quads in M, so $r(M) = 4$. Hence, if $e \in X_i$, one easily checks that $M\backslash e$ is 3-connected. If e exposes a 3-separation (X, Y) of $M\backslash e$, then we may assume that X contains $X_i - e$. Hence X spans e; a contradiction. □

Lemma 5.2. *In a* 3*-connected matroid* M *other than a wheel or a whirl, let* $(X_1, \{a, b\}, X_2)$ *be a partition of* $E(M)$ *such that both* $(X_1, \{a, b\} \cup X_2)$ *and* $(X_1 \cup \{a, b\}, X_2)$ *are* 3*-separations, and* $\{a, b\} \subseteq \mathrm{cl}(X_1) \cap \mathrm{cl}(X_2)$*. Then* M *has an element* e *whose deletion from* M *or* M^* *is* 3*-connected and does not expose any* 3*-separations.*

Proof. The result is immediate from the preceding lemma if both $M\backslash a$ and $M\backslash b$ are 3-connected. Thus we may assume that $M\backslash a$ is not 3-connected. As $a \in \mathrm{cl}(X_1) \cap \mathrm{cl}(X_2)$, the matroid M/a has $(X_1, X_2 \cup b)$ as a non-minimal 2-separation. Thus, by Lemma 2.2, $\mathrm{co}(M\backslash a)$ is 3-connected. Since $M\backslash a$ is not 3-connected, it follows that a is in a triad $\{a, x, y\}$ of M. Now either $\{a, x, y\}$ is a wild triangle of M^*, or, for some z in $\{x, y\}$, the matroid $M^*\backslash z$ is 3-connected and z does not expose any 3-separations of M^*. We may assume that the former holds. Since $M^*\backslash a$ is not 3-connected, it follows by [17, Corollary 4.3] that $\{a, x, y\}$ is in a 4-element fan of M. Then, by Lemma 2.11, the required result holds. □

6. Tridents

In this section, we show that, when a triangle Z of M is contained in a trident X, if $e \in X - Z$, then e fails to expose a 3-separation in at least one of $M\backslash e$ and M/e. Throughout the section, we shall assume that the trident is labelled as in Fig. 1.

Lemma 6.1. *In a trident* X *in a* 3*-connected matroid* M*, for all pairs of distinct elements* e *and* f *of* $\{t, s, u, v\}$*, the set* $(E(M) - X) \cup \{e, f\}$ *spans* M*, and* $r(X) = r(\{t, s, u, v\}) = 4$.

Proof. Let $Y = E(M) - X$. First observe that $\mathrm{cl}(Y \cup s)$ avoids the cocircuit $\{t, u, v, c, b\}$ of M, so

$$\mathrm{cl}(Y \cup s) \cap \{u, v, t\} = \emptyset. \tag{2}$$

Symmetry between the triples (b, c, s), (c, a, u), and (a, b, v) implies that $\mathrm{cl}(Y \cup u) \cap \{v, s, t\} = \emptyset = \mathrm{cl}(Y \cup v) \cap \{s, u, t\}$. As $t \notin \mathrm{cl}(Y \cup s)$ and $s \notin \mathrm{cl}(Y)$, the Mac Lane–Steinitz condition implies that $s \notin \mathrm{cl}(Y \cup t)$. By symmetry,

$$\mathrm{cl}(Y \cup t) \cap \{u, v, s\} = \emptyset. \tag{3}$$

Now $r(Y) + 1 = r(Y \cup t) \leqslant r(M) - 1$ and $r(Y) + r(X) = r(M) + 2$. But $\{t, s, u, v\}$ spans X. Hence $r(\{t, s, u, v\}) = r(X) \leqslant 4$. Combining these observations, we get $r(Y) = r(M) - 2$ and $r(X) = 4$. The lemma follows from (2) and (3) using symmetry. $\square$

Lemma 6.2. *In a trident X in a 3-connected matroid M, for all x in $X - \{a, b, c\}$, either M/x or $M\backslash x$ is 3-connected having no exposed 3-separations.*

Proof. Let $Y = E(M) - X$. By symmetry, it suffices to show that M/t and at least one of M/s and $M\backslash s$ are 3-connected having no exposed 3-separations. We show first that:

6.2.1. *M/t is simple; and either M/s is simple or M has a circuit $\{s, a, y\}$ for some y in Y.*

The circuits $\{t, s, u, b\}$, $\{t, u, v, c\}$, and $\{t, s, v, a\}$ of M imply that $(M/t)|\{a, b, c, s, u, v\}$ and $(M/s)|\{a, b, c, t, u, v\}$ are isomorphic to the rank-3 wheel and whirl, respectively. Moreover, $\mathrm{cl}(Y)$ avoids $\{t, s\}$, so $(M/t)|Y \cong M|Y$ and $(M/s)|Y \cong M|Y$. Hence a 2-circuit of M/t or of M/s must contain one element of Y and one element of $\{a, b, c, s, t, u, v\}$. Using orthogonality with the cocircuits $\{t, s, u, b, a\}$, $\{t, u, v, c, b\}$, and $\{t, s, v, a, c\}$, we deduce that 6.2.1 holds.

We show next that:

6.2.2. *If $w \in \{s, t\}$ and M/w is simple, then M/w is 3-connected having no exposed 3-separations.*

Suppose (U, V) is either a 2-separation or an exposed 3-separation of $M^*\backslash w$. Then neither U nor V is spanning in M/w. Since we may assume at least two of a, b, and c are in U, we may assume that U contains $\{a, b, c\}$. We show next that we may assume that U contains $X - w$. This is certainly true if U meets $\{s, t, u, v\} - w$ for then U spans $X - w$ in M/w. If U avoids $\{s, t, u, v\} - w$, then V spans $X - w$ in M/w and, by interchanging U and V, we again get that we may assume U contains $X - w$. The known cocircuits of M imply that w is a coloop of $M|(V \cup w)$. Hence $\lambda_{M/w}(V) = \lambda_M(V) = k$, say. But M is 3-connected, so $k \neq 1$; and $k \neq 2$ as (U, V) is exposed in $M^*\backslash w$. Thus 6.2.2 holds.

By 6.2.1, to complete the proof of the lemma, we need to show that:

6.2.3. *If M has a triangle $\{s, a, y\}$ for some y in Y, then $M\backslash s$ is 3-connected having no exposed 3-separations.*

We show first that $M\backslash s$ has no minimal 2-separations. Certainly $M\backslash s$ has no 2-circuits. Suppose $M\backslash s$ has a 2-cocircuit C^*. Then $C^* \cup s$ is a triad of M meeting the triangle $\{s, a, y\}$. Thus $|C^* \cap \{a, y\}| = 1$. If C^* meets $\{a, b, c\}$, then, by orthogonality, C^* contains exactly two elements of $\{a, b, c\}$. In that case, $E(M) - (C^* \cup s)$ avoids $(E(M) - X) \cup \{t, u\}$ and, by Lemma 6.1, the latter spans M; a contradiction. Thus C^* avoids $\{a, b, c\}$ so $y \in C^*$. Orthogonality between $C^* \cup s$ and the circuits $\{t, s, v, a\}$ and $\{t, s, u, b\}$ implies that $C^* - y = \{t\}$. But then $C^* \cup s$ meets the circuit $\{t, u, v, c\}$ in a single element; a contradiction. We conclude that $M\backslash s$ has no 2-cocircuits, so $M\backslash s$ has no minimal 2-separations.

Now let (U, V) be either a 2-separation or an exposed 3-separation of $M\backslash s$. Then neither U nor V spans s. Moreover, since $M\backslash s$ has no minimal 2-separations, if (U', V') is a partition of $E(M\backslash s)$ such that $\mathrm{fcl}_{M\backslash s}(U') = \mathrm{fcl}_{M\backslash s}(U)$ and $\mathrm{fcl}_{M\backslash s}(V') = \mathrm{fcl}_{M\backslash s}(V)$, then (U', V') is a 2-separation or an exposed

3-separation of $M\backslash s$. Thus we may assume that $\{a,b,c\} \subseteq U$. Then $y \in V$ otherwise $s \in \mathrm{cl}(U)$. Now $|U \cap \{t,u,v\}| \leqslant 1$ otherwise U spans s. If $t \in U$, then the cocircuit $\{t,u,b,a\}$ of $M\backslash s$ means that we can move u from V into U. Likewise, if u or v is in U, the cocircuit $\{t,v,a,c\}$ of $M\backslash s$ allows us to move t from V into U. After these moves, $|U \cap \{t,u,v\}| \geqslant 2$; a contradiction. We deduce that neither of these moves is possible, so $\{t,u,v\} \subseteq V$. Then, in $M\backslash s$, using the circuit $\{t,u,v,c\}$ and the cocircuit $\{t,v,a,c\}$, we can move c and then a from U to V. After these moves, V contains $\{a,y\}$, and so $s \in \mathrm{cl}(V)$; a contradiction. We conclude that 6.2.3 holds and hence so does the lemma. □

7. Two-element petals in tight flowers

The goal of this section is to prove the next theorem, which will be crucial in the proof of Theorem 4.2 and is also of some independent interest.

Theorem 7.1. *Let $(P,\{a,b\},Q)$ be a tight flower of a 3-connected matroid M where $\{a,b\}$ is fully closed and both P and Q have at least three elements. Then the following hold.*

(i) *If a is in a triangle, then $M\backslash a$ is 3-connected and has no 3-separations exposed by a.*
(ii) *If a is in a triad, then M/a is 3-connected and has no 3-separations exposed by a.*
(iii) *If a is in neither a triangle nor a triad, then both $M\backslash a$ and M/a are 3-connected.*

Moreover, if a is in neither a triangle nor a triad and both $M\backslash a$ and M/a have 3-separations exposed by a, then $|P| = |Q| = 4$, both $M\backslash b$ and M/b are 3-connected, and neither $M\backslash b$ nor M/b has a 3-separation exposed by b.

Proof. From the fact that $(P,\{a,b\},Q)$ is tight, we immediately obtain

7.1.1. $\mathrm{fcl}(P) \cap \{a,b\} = \emptyset = \mathrm{fcl}(Q) \cap \{a,b\}$.

Next we show that:

7.1.2. *If a is in a triangle, then a is not in a triad.*

Let $\{a,p,q\}$ be a triangle T. If $b \in T$, then $\{a,b\}$ is not closed, so $b \notin T$. If $\{p,q\} \subseteq P$, then $a \in \mathrm{cl}(P)$ contradicting 7.1.1, so $\{p,q\} \nsubseteq P$. Hence we may assume that $p \in P$ and $q \in Q$.

Now assume that a is in a triad. Without loss of generality we may assume that $\{a,p,s\}$ is a triad. As $\{a,b\}$ is coclosed, $s \neq b$. If $s \in P$, then $a \in \mathrm{cl}^*(P)$; a contradiction. Thus $s \in Q$.

The triangle T and the triad $\{a,p,s\}$ imply that $p \in \mathrm{cl}(Q \cup \{a,b\})$ and $p \in \mathrm{cl}^*(Q \cup \{a,b\})$. Thus, by Lemma 2.17(ii), $\lambda(Q \cup \{a,b,p\}) < \lambda(Q \cup \{a,b\}) = 2$. This is a contradiction since $|E - (Q \cup \{a,b,p\})| \geqslant 2$. Hence 7.1.2 holds.

By replacing M by its dual if necessary, we may now assume that a is not in a triad of M. Since $a \notin \mathrm{cl}^*(P)$ and $a \notin \mathrm{cl}^*(Q)$, it follows that:

7.1.3. $a \in \mathrm{cl}(Q \cup b)$ *and* $a \in \mathrm{cl}(P \cup b)$.

Similarly, since $b \notin \mathrm{cl}(Q)$ and $b \notin \mathrm{cl}(P)$, it follows that:

7.1.4. $b \in \mathrm{cl}^*_{M\backslash a}(P)$ *and* $b \in \mathrm{cl}^*_{M\backslash a}(Q)$.

We show next that:

7.1.5. *$M\backslash a$ is 3-connected.*

Let (X, Y) be a 2-separation of $M\backslash a$, where $b \in X$. As a is not in a triad, $|X|, |Y| \geqslant 3$. Assume that $P \subseteq Y$. By 7.1.4, $b \in \mathrm{cl}^*_{M\backslash a}(P)$ and hence $b \in \mathrm{cl}^*_{M\backslash a}(Y)$. Thus $(X - b, Y \cup b)$ is a 2-separation of $M\backslash a$. But $P \cup b \subseteq Y \cup b$ and, by 7.1.3, $a \in \mathrm{cl}_M(Y \cup b)$. Therefore $\lambda_M(X - b) = 1$. But $|X - b| \geqslant 2$ and we have contradicted the fact that M is 3-connected. Thus $X \cap P \neq \emptyset$. On the other hand, if $P \subseteq X$, then $P \cup b \subseteq X$ and $a \in \mathrm{cl}(X)$ again contradicting the fact that M is 3-connected. Thus every 2-separation (X, Y) of $M\backslash a$ crosses both P and Q.

Now $\lambda_{M\backslash a}(Y) = 1$ and $\lambda_{M\backslash a}(P \cup b) = \lambda_M(P \cup \{a, b\}) = 2$. Thus, by the submodularity of λ, we deduce that either $\lambda_{M\backslash a}(Y \cap (P \cup b)) = 1$ or $\lambda_{M\backslash a}(Y \cup P \cup b) = 1$. Hence $\lambda_{M\backslash a}(X \cup Q) = 1$ or $\lambda_{M\backslash a}(Y \cup P \cup b) = 1$. As $b \in X$, this means that either $\lambda_M(X \cup Q \cup a) = 1$ or $\lambda_M(Y \cup P \cup b \cup a) = 1$. As M is 3-connected, we deduce that either $|Y \cap P| = 1$ or $|X \cap Q| = 1$.

Assume that $|Y \cap P| = 1$, say $Y \cap P = \{y\}$. If $\lambda_{M\backslash a}(P - y) = 1$, then, as $|P - y| > 1$ and $a \in \mathrm{cl}_M(Q \cup b)$, we again contradict the 3-connectivity of M. Thus $\lambda_{M\backslash a}(P - y) > 1$. But $\lambda_{M\backslash a}(P) = 2$. Therefore $y \in \mathrm{cl}^{(*)}_{M\backslash a}(P - y)$, so $y \in \mathrm{cl}^{(*)}_{M\backslash a}(X)$ and $(X \cup y, Y - y)$ is also a 2-separation of $M\backslash a$. But $Y - y$ avoids P, contradicting the fact that $(X \cup y, Y - y)$ crosses P. An identical argument holds in the case that $|X \cap Q| = 1$ and we conclude that $M\backslash a$ is indeed 3-connected, that is, 7.1.5 holds.

It follows from 7.1.5 and duality that (iii) of the theorem holds.

7.1.6. *Suppose $Z \cup e \subseteq P$ or $Z \cup e \subseteq Q$. Then*

(i) $\lambda_{M\backslash a}(Z) = \lambda_{M/a}(Z) = \lambda_M(Z)$; *and*
(ii) *the following are equivalent*:
  (a) $e \in \mathrm{cl}^{(*)}_{M\backslash a}(Z)$;
  (b) $e \in \mathrm{cl}^{(*)}_{M/a}(Z)$;
  (c) $e \in \mathrm{cl}^{(*)}_{M}(Z)$.

To show this, suppose that $Z \subseteq P$. By 7.1.1, $a \notin \mathrm{cl}_M(Z)$ and, by 7.1.3, the element a is not a coloop of $E(M) - Z$. Part (i) of 7.1.6 follows from these facts and elementary rank calculations. Part (ii) follows from (i) and Lemma 2.17. Thus 7.1.6 holds.

Now assume that a exposes a 3-separation (D_1, D_2) in $M\backslash a$, where $b \in D_1$. Let $(P_1, P_2, Q_2, Q_1) = (D_1 \cap P, D_2 \cap P, D_2 \cap Q, D_1 \cap Q)$.

7.1.7. $(P_1 \cup b, P_2, Q_2, Q_1)$ *is a flower in $M\backslash a$. Moreover, $b \in \mathrm{cl}^*_{M\backslash a}(P_1)$ and $b \in \mathrm{cl}^*_{M\backslash a}(Q_1)$.*

To see this, suppose first that $P \subseteq D_2$. Then, as $b \in \mathrm{cl}^*_{M\backslash a}(P)$, we have $b \in \mathrm{cl}^*_{M\backslash a}(D_2)$, so $(D_1 - b, D_2 \cup b)$ is a 3-separation of $M\backslash a$ equivalent to (D_1, D_2). But $a \in \mathrm{cl}_M(D_2 \cup b)$, so we have contradicted the fact that (D_1, D_2) is exposed by a. Hence $P \nsubseteq D_2$. If $P \subseteq D_1$, then $P \cup b \subseteq D_1$, so $a \in \mathrm{cl}_M(D_1)$. Hence (D_1, D_2) is not exposed by a. Thus $P \nsubseteq D_1$. By symmetry, it follows that (D_1, D_2) crosses both P and Q.

Assume that $|P \cap D_1| = 1$. Now $|P| \geqslant 3$ and, by Lemma 2.9, $|D_1| \geqslant 4$. By two applications of uncrossing, we get that $\lambda_{M\backslash a}(P \cup D_2) = 2$ and $\lambda_{M\backslash a}(P \cup D_2 \cup b) = 2$. Thus, in $M\backslash a$, the 3-separation (D_1, D_2) is equivalent to $(D_1 - P, D_2 \cup P)$ and hence to $(D_1 - (P \cup b), D_2 \cup P \cup b)$. But $a \in \mathrm{cl}(D_2 \cup P \cup b)$. Hence a does not expose (D_1, D_2); a contradiction. We deduce that $|P \cap D_1| > 1$. By symmetry, $|Q \cap D_1| > 1$.

Next assume that $|P \cap D_2| = 1$. Then (D_1, D_2) is equivalent to $(D_1 \cup P, D_2 - P)$. As $b \in D_1$, it follows that $a \in \mathrm{cl}(D_1 \cup P)$, so (D_1, D_2) is not exposed by a. Hence $|P \cap D_2| > 1$ and, by symmetry, $|Q \cap D_2| > 1$. We conclude that $(P_1 \cup b, P_2, Q_2, Q_1)$ is a flower in $M\backslash a$. By symmetry, $(P_1, P_2, Q_2, Q_1 \cup b)$ is also a flower in $M\backslash a$. Thus b is a loose element of this flower. Hence $b \in \mathrm{cl}^{(*)}_{M\backslash a}(P_1)$. But $b \notin \mathrm{cl}_M(P)$, so $b \notin \mathrm{cl}_{M\backslash a}(P_1)$. Hence $b \in \mathrm{cl}^*_{M\backslash a}(P_1)$. By symmetry, $b \in \mathrm{cl}^*_{M\backslash a}(Q_1)$. We conclude that 7.1.7 holds.

Next we show that:

7.1.8. *If $z \in Q_2$ and $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Q_1 \cup b)$, then $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Q_1)$.*

To see this, suppose first that $z \in \mathrm{cl}^*_{M\backslash a}(Q_1 \cup b)$. Then, since, by 7.1.7, $b \in \mathrm{cl}^*_{M\backslash a}(Q_1)$, it follows that $\mathrm{cl}^*_{M\backslash a}(Q_1 \cup b) = \mathrm{cl}^*_{M\backslash a}(Q_1)$, so $z \in \mathrm{cl}^*_{M\backslash a}(Q_1)$. Next suppose that $z \in \mathrm{cl}_{M\backslash a}(Q_1 \cup b)$. Then, as $b \notin \mathrm{cl}(Q)$, it follows that $b \notin \mathrm{cl}(Q_1 \cup z)$. Hence $z \in \mathrm{cl}(Q_1)$. We conclude that 7.1.8 holds.

7.1.9. *The element a is not in a triangle of M.*

Assume that $\{p, a, q\}$ is a triangle of M. As noted in 7.1.2, we may assume that $p \in P$ and $q \in Q$. By 7.1.3, $a \in \mathrm{cl}(Q \cup b)$. Thus $p \in \mathrm{cl}(Q \cup b)$. By applying [15, Lemma 5.5(ii)] in the flower $(P_1, P_2, Q_2, Q_1 \cup b)$ in $M\backslash a$, we get that either $p \in \mathrm{cl}(Q_2)$ or $p \in \mathrm{cl}(Q_1 \cup b)$. The former implies that $a \in \mathrm{cl}(Q)$, contradicting 7.1.1. Thus the latter holds. By symmetry, $q \in \mathrm{cl}(P_1 \cup b)$. Thus $\{p, q\} \subseteq \mathrm{cl}(D_1)$ so that $a \in \mathrm{cl}(D_1)$ contradicting the fact that (D_1, D_2) is blocked by a. Hence 7.1.9 holds.

It follows from 7.1.9 that (i) of the theorem holds and, by duality, so does (ii). We now assume that a is in neither a triangle nor a triad. Then, by (iii), both $M\backslash a$ and M/a are 3-connected. As above, assume (D_1, D_2) is a 3-separation of $M\backslash a$ exposed by a, where $b \in D_1$, and let P_1, P_2, Q_1, and Q_2 be as before. Assume too that M/a has a 3-separation (R, G) that is exposed by a where $b \in R$. Then $|R|, |G| \geqslant 4$. Note that, up to duality, we have symmetry between (D_1, D_2) and (R, G). We make frequent use of this fact as, for example, in the following.

7.1.10. *In the matroid M,*

(i) *a blocks $P_2 \cup Q_2$;*
(ii) *a blocks $P_1 \cup b$;*
(iii) *a blocks $Q_1 \cup b$;*
(iv) *a coblocks G;*
(v) *a coblocks $(R \cap P) \cup b$;*
(vi) *a coblocks $(R \cap Q) \cup b$.*

Part (i) follows from the fact that a blocks (D_1, D_2), and $D_2 = P_2 \cup Q_2$. Consider (ii). As a blocks (D_1, D_2), we have $a \in \mathrm{cl}^*(D_2)$, so $a \in \mathrm{cl}^*(Q \cup P_2)$. Now suppose that $a \notin \mathrm{cl}^*_M(P_1 \cup b)$. Then $a \in \mathrm{cl}_M(Q \cup P_2)$. But $a \in \mathrm{cl}_M(Q \cup b)$. By considering the flowers $(P_1 \cup b, P_2, Q_2, Q_1)$ and $(P_1, P_2, Q_2, Q_1 \cup b)$ of $M\backslash a$ and using Lemma 2.16, we see that $Q \cup P_2$ and $Q \cup b$ are a modular pair in $M\backslash a$. It follows by Lemma 2.15 that $a \in \mathrm{cl}_M(Q)$, contradicting 7.1.1. Thus (ii) holds. Part (iii) follows by the symmetry between P and Q. Parts (iv), (v), and (vi) hold by the symmetry between (D_1, D_2) and (R, G) under duality.

The next assertion follows from 7.1.7 by duality.

7.1.11. *$((P \cap R) \cup b,\ P \cap G,\ Q \cap G,\ Q \cap R)$ is a flower in M/a. Moreover, $b \in \mathrm{cl}_{M/a}(P \cap R)$ and $b \in \mathrm{cl}_{M/a}(Q \cap R)$.*

A consequence of 7.1.11 and 7.1.6 is

7.1.12. $\lambda_{M\backslash a}(R \cap P) = \lambda_{M\backslash a}(G \cap P) = \lambda_{M\backslash a}(R \cap Q) = \lambda_{M\backslash a}(G \cap Q) = 2$.

We show next that:

7.1.13. $b \notin \mathrm{cl}^{(*)}_{M\backslash a}(R \cap P)$.

If $b \in \mathrm{cl}_{M\backslash a}(R \cap P)$, then $b \in \mathrm{cl}_M(P)$, so this case does not occur. Assume that $b \in \mathrm{cl}^*_{M\backslash a}(R \cap P)$. By 7.1.10(iv), $a \in \mathrm{cl}_M(G)$. Thus, by Lemma 2.19, $b \in \mathrm{cl}^*_M(R \cap P)$ so $b \in \mathrm{cl}^*_M(P)$, contradicting 7.1.1. Hence 7.1.13 holds.

Assume from now on that among 3-separations exposed by a in $M\backslash a$ and 3-separations exposed by a in M/a, we have chosen $(P_1 \cup Q_1 \cup b, P_2 \cup Q_2)$ and (R, G) such that the number of non-empty sets amongst $P_1 \cap R$, $P_1 \cap G$, $P_2 \cap R$, $P_2 \cap G$, $Q_1 \cap R$, $Q_1 \cap G$, $Q_2 \cap R$, $Q_2 \cap G$ is minimized. We call this assumption the *minimality assumption*.

7.1.14. *If $1 \in \{|R \cap P_1|, |R \cap P_2|, |G \cap P_1|, |G \cap P_2|\}$, then $|P| = 4$, and $|R \cap P_1| = |R \cap P_2| = |G \cap P_1| = |G \cap P_2| = 1$.*

Let (R_1, R_2) be a permutation of (P_1, P_2) and let (Y, B) be a permutation of (R, G). Assume that $|R_1 \cap Y| = 1$, letting $R_1 \cap Y = \{y_1\}$, say. Assume that $|R_1| > 2$. Then $\lambda_{M\backslash a}(R_1 \cap B) \geqslant \lambda_{M\backslash a}(R_1) = 2$. Thus, by Lemma 2.17, $y_1 \in \mathrm{cl}^{(*)}_{M\backslash a}(R_1 \cap B)$. Therefore, by 7.1.6(ii), $y_1 \in \mathrm{cl}^{(*)}_{M/a}(R_1 \cap B)$. Hence $y_1 \in \mathrm{cl}^{(*)}_{M/a}(B)$. This means that $(B \cup y_1, Y - y_1)$ is also a 3-separation of M/a. Using $(B \cup y_1, Y - y_1)$ to replace (B, Y), we get a contradiction to the minimality assumption. Therefore $|R_1 \cap B| = 1$ and $|R_1| = 2$.

It is now clear that either 7.1.14 holds, or we may assume, up to symmetry, that $|Y \cap R_2| \geqslant 2$. Assume the latter holds. Then uncrossing the 3-separating sets R_2 and $Y \cap P$ shows that $\lambda_{M\backslash a}(R_2 \cup y_1) = 2$, so $y_1 \in \mathrm{cl}^{(*)}_{M\backslash a}(R_2)$. If $R_2 = P_1$, then we deduce that $y_1 \in \mathrm{cl}^{(*)}_{M\backslash a}(P_1 \cup Q_1 \cup b)$, and we can replace $(P_1 \cup Q_1 \cup b, P_2 \cup Q_2)$ by $(P_1 \cup Q_1 \cup b \cup y_1, (P_2 \cup Q_2) - y_1)$. If, instead, $R_2 = P_2$, we deduce that $y_1 \in \mathrm{cl}^{(*)}_{M\backslash a}(P_2 \cup Q_2)$ and we can replace $(P_1 \cup Q_1 \cup b, P_2 \cup Q_2)$ by $((P_1 \cup Q_1 \cup b) - y_1, P_2 \cup Q_2 \cup y_1)$. In both cases, these replacements contradict the minimality assumption. We deduce that 7.1.14 holds.

The next assertion will require several steps to establish it.

7.1.15. *If $1 \notin \{|R \cap P_1|, |R \cap P_2|, |G \cap P_1|, |G \cap P_2|\}$, then $(P_1, P_2) = (P \cap G, P \cap R)$, and the flower $(P_1, P_2, Q_2, Q_1 \cup b)$ in $M\backslash a$ is swirl-like.*

Assume first that (R, G) crosses both P_1 and P_2. Then $|R \cap P_1|, |R \cap P_2|, |G \cap P_1|, |G \cap P_2| \geqslant 2$. Moreover, $\lambda_{M\backslash a}(R \cap P) = 2$. Then the flower $(P_1, P_2, Q_2, Q_1 \cup b)$ of $M\backslash a$ refines to

$$(P_1 \cap G, P_1 \cap R, P_2 \cap R, P_2 \cap G, Q_2, Q_1 \cup b),$$

which is a flower Φ_1 that displays $R \cap P$. This follows by repeated uncrossing arguments. In particular, $\lambda_{M\backslash a}(P_1 \cap R) = 2$ as $\lambda_{M\backslash a}(P_1) = 2 = \lambda_{M\backslash a}(P \cap R)$ and $P_1 \cap R = P_1 \cap (P \cap R)$. Also $\lambda_{M\backslash a}((P_2 \cap G) \cup Q_2) = 2$ since $(P_2 \cap G) \cup Q_2$ is the complement of the union of the 3-separating sets $P_1 \cup Q_1 \cup b$ and $P \cap R$.

Now $\lambda_{M\backslash a}(G \cap P) = 2$, and $G \cap P$ is the union of two non-adjacent petals of Φ_1. Thus Φ_1 is an anemone. Hence, by Lemma 3.5, $(P_1, P_2 \cap R, P_2 \cap G, Q_2, Q_1 \cup b)$ is also an anemone Φ_1'. As $b \in \mathrm{cl}^*_{M\backslash a}(P_1)$, by applying [15, Lemma 6.4] to Φ_1' and then to Φ_1, we deduce that $b \in \mathrm{cl}^*_{M\backslash a}(P_1 \cap R)$; a contradiction to 7.1.13.

Assume next that $P_1 \subseteq R$. As $a \in \mathrm{cl}_M(G)$, we have $a \in \mathrm{cl}_M(G \cup Q \cup (P_2 \cap R))$. Since $b \in \mathrm{cl}^*_{M\backslash a}(P_1)$, we can apply Lemma 2.19 to contradict the fact that $b \notin \mathrm{cl}^*_M(P)$. We conclude that $P_1 \cap G \neq \emptyset$.

Now assume that $P_1 \subseteq G$ and that $G \cap P_2 \neq \emptyset$. Then, by the hypothesis of 7.1.15, $|P_2 \cap G| > 1$. Moreover, by 7.1.12, $|P_2 \cap R| > 1$. Then, arguing as for Φ_1 and noting that $Q_2 \cup (P_2 \cap R)$ is the intersection of the two 3-separating sets $Q \cup b \cup R$ and $P_2 \cup Q_2$, we get that the partition

$$(P_1, P_2 \cap G, P_2 \cap R, Q_2, Q_1 \cup b)$$

is a flower Φ_2 in $M\backslash a$. In particular, $(P_1 \cup (P_2 \cap G) \cup Q_1 \cup b, (P_2 \cap R) \cup Q_2)$ is a 3-separation (Z_1, Z_2) in $M\backslash a$.

Next we shall show that (Z_1, Z_2) is well blocked by a. First assume (Z_1, Z_2) is not blocked by a. Then $a \in \mathrm{cl}_M(P_1 \cup (P_2 \cap G) \cup b \cup Q_1)$ since $a \notin \mathrm{cl}(P_2 \cup Q_2)$. Also $a \in \mathrm{cl}_M(Q_1 \cup Q_2 \cup b)$. By concatenating petals in Φ_2 and applying Lemma 2.15, we get that $a \in \mathrm{cl}_M(Q_1 \cup b)$; a contradiction as $a \notin \mathrm{cl}(D_1)$. Thus (Z_1, Z_2) is blocked by a.

We now need to show that every 3-separation of $M\backslash a$ equivalent to (Z_1, Z_2) is also blocked by a. We do this by showing that each 3-separation (Z_1', Z_2') of $M\backslash a$ that is of one of the forms $(Z_1 \cup z, Z_2 - z)$ or $(Z_1 - z, Z_2 \cup z)$ is blocked by a. Moreover, for each such choice of (Z_1', Z_2'), we can replace (R, G) or (D_1, D_2) by equivalent 3-separations for which it can be easily checked that the minimality assumptions continue to hold along with the other assumptions governing both 7.1.15 and this case, namely that $1 \notin \{|R \cap P_1|, |R \cap P_2|, |G \cap P_1|, |G \cap P_2|\}$, that $P_1 \subseteq G$, and that $G \cap P_2 \neq \emptyset$. Once we have established this, we can replace (Z_1, Z_2) by (Z_1', Z_2') and repeat the argument.

Assume first that $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Z_1) \cap Z_2$. We now use the flower $(P_1 \cup (P_2 \cap G), P_2 \cap R, Q_2, Q_1 \cup b)$ of $M\backslash a$ obtained by uncrossing the 3-separations (Z_1, Z_2) and $(P, Q \cup b)$. This flower is a concatenation of Φ_2. By [15, Lemma 5.5], we see that either (i) $z \in P_2 \cap R$ and $z \in \mathrm{cl}^{(*)}_{M\backslash a}(P_1 \cup (P_2 \cap G))$; or (ii) $z \in Q_2$ and $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Q_1 \cup b)$. If (i) holds, then, by 7.1.6, $z \in \mathrm{cl}^{(*)}_{M/a}(P_1 \cup (P_2 \cap G))$, so $z \in \mathrm{cl}^{(*)}_{M/a}(G)$. Then, by using 7.1.14, we see that by replacing (R, G) with the equivalent 3-separation $(R - z, G \cup z)$ and repeating the argument from the second last paragraph, we obtain that in this case $(Z_1 \cup z, Z_2 - z)$ is blocked by a.

Suppose that (ii) holds. Then, by 7.1.8, $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Q_1)$, so $z \in \mathrm{cl}^{(*)}_{M\backslash a}(D_1)$. We now replace (D_1, D_2) by the equivalent $(D_1 \cup z, D_2 - z)$ noting that, by 7.1.7, $|Q_2 - z| \geqslant 2$. The minimality assumption still holds unless $z \in Q_2 \cap R$ and $Q_1 \subseteq G$, or $z \in Q_2 \cap G$ and $Q_1 \subseteq R$. By 7.1.6, $z \in \mathrm{cl}^{(*)}_{M/a}(Q_1)$. Thus, in the exceptional cases, $z \in \mathrm{cl}^{(*)}_{M/a}(G)$ or $z \in \mathrm{cl}^{(*)}_{M/a}(R)$, respectively. In these exceptional cases, in addition to replacing (D_1, D_2) by $(D_1 \cup z, D_2 - z)$, we also replace (R, G) by $(R - z, G \cup z)$ and $(R \cup z, G - z)$, respectively. After making these replacements, we can apply the argument from the previous paragraph to get that $(Z_1 \cup z, Z_2 - z)$ is blocked by a.

We now need to establish that $(Z_1 - z, Z_2 \cup z)$ is blocked by a when $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Z_2) \cap Z_1$. In this case, the argument is similar to that given in the last two paragraphs except in the case that $z = b$ which we now consider. By 7.1.7, $b \in \mathrm{cl}^*_{M\backslash a}(Q_1)$. Thus b is loose in the flower $(b \cup P_1 \cup (P_2 \cap G), P_2 \cap R, Q_2, Q_1)$. Since $b \in \mathrm{cl}^*_{M\backslash a}(P)$ and $b \in \mathrm{cl}^{(*)}_{M\backslash a}(Z_2)$, it follows that $b \in \mathrm{cl}^*_{M\backslash a}(Z_2)$. By Lemma 3.4, $b \in \mathrm{cl}^*_{M\backslash a}(P_2 \cap R)$. This contradicts 7.1.13.

This proves that (Z_1, Z_2) is indeed well blocked by a. But the existence of (Z_1, Z_2) contradicts the minimality assumption so this case does not occur.

Next assume that $P_2 \subseteq R$ and that $R \cap P_1 \neq \emptyset$. Recall that $P_1 \cap G \neq \emptyset$. Then the hypotheses of 7.1.15 imply that $|R \cap P_1| \geqslant 2$ and $|G \cap P_1| \geqslant 2$. Let $Z_1 = (P_1 \cap G) \cup b \cup Q_1$ and $Z_2 = (P \cap R) \cup Q_2$. Arguing as in the earlier case, we deduce that $(P_1 \cap G, P \cap R, Q_2, Q_1 \cup b)$ and $((P_1 \cap G) \cup b, P \cap R, Q_2, Q_1)$ are flowers in $M\backslash a$. Assume that (Z_1, Z_2) is not blocked by a. Then either $a \in \mathrm{cl}_M(Z_1)$ or $a \in \mathrm{cl}_M(Z_2)$. Assume that $a \in \mathrm{cl}_M(Z_1)$. We know that $a \in \mathrm{cl}_M(Q \cup b)$. So, by Lemma 2.15 and Corollary 3.3, $a \in \mathrm{cl}_M(Q_1 \cup b)$, so a does not block $Q_1 \cup b$, which contradicts 7.1.10(iii). Assume that $a \in \mathrm{cl}_M(Z_2)$. Recall that $a \in \mathrm{cl}_M(P \cup b)$. In this case, by Lemma 2.15 and Corollary 3.3, we deduce that $a \in \mathrm{cl}_M(Z_2 \cap (P \cup b))$. This gives the contradiction that $a \in \mathrm{cl}_M(P)$. Thus a blocks (Z_1, Z_2).

As before, we need to show that every 3-separation of $M\backslash a$ equivalent to (Z_1, Z_2) is also blocked by a, and the strategy used here is the same as that described in detail above. Assume that $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Z_1) \cap Z_2$. Then, by [15, Lemma 5.5], either (i) $z \in P \cap R$ and $z \in \mathrm{cl}^{(*)}_{M\backslash a}(P_1 \cap G)$, or (ii) $z \in Q_2$ and $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Q_1 \cup b)$. If (i) holds, then, by 7.1.6, $z \in \mathrm{cl}^{(*)}_{M/a}(P_1 \cap G)$, so $z \in \mathrm{cl}^{(*)}_{M/a}(G)$. In this case, by using 7.1.14 again, we see that we can replace (R, G) by the equivalent $(R - z, G \cup z)$ and preserve all the assumptions governing this case. Now $z \in P_1 \cap R$ or $z \in P_2$. In the former case, we leave (D_1, D_2) unchanged. In the latter case, we replace it by $(D_1 \cup z, D_2 - z)$. In both cases, by arguing as in the last paragraph, we get that $(Z_1 \cup z, Z_2 - z)$ is blocked by a. If (ii) holds, then, by 7.1.8, $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Q_1)$, so, by 7.1.6, $z \in \mathrm{cl}^{(*)}_{M/a}(Q_1)$. We replace (D_1, D_2) by the equivalent $(D_1 \cup z, D_2 - z)$. This will not produce a violation of the minimality condition unless either $Q_1 \subseteq G$ and $z \in R$, or $Q_1 \subseteq R$ and $z \in G$. In the exceptional cases, we again replace (R, G) by $(R - z, G \cup z)$ or $(R \cup z, G - z)$, respectively. In both cases, the argument from the previous paragraph establishes that $(Z_1 \cup z, Z_2 - z)$ is blocked by a.

Next assume that $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Z_2) \cap Z_1$. Then either (i) $z \in P_1 \cap G$ and $z \in \mathrm{cl}^{(*)}_{M\backslash a}(P \cap R)$, or (ii) $z \in Q_1 \cup b$ and $z \in \mathrm{cl}^{(*)}_{M\backslash a}(Q_2)$. If (i) occurs, then $z \in \mathrm{cl}^{(*)}_{M/a}(P \cap R)$, so $z \in \mathrm{cl}^{(*)}_{M/a}(R)$. Hence we can replace (R, G) by $(R \cup z, G - z)$ to get that $(Z_1 - z, Z_2 \cup z)$ is blocked by a. Now suppose that (ii) occurs. Assume that $z = b$. Then, by considering the flower $((P_1 \cap G) \cup b, Q_1, Q_2, P \cap R)$, we have, since $b \in \mathrm{cl}^{(*)}_{M\backslash a}(Q_2)$ and $b \in \mathrm{cl}^*_{M\backslash a}(Q_1)$, that $b \in \mathrm{cl}^*_{M\backslash a}(Q_2)$. Then, by [15, Lemma 5.5], as $b \in \mathrm{cl}^*_{M\backslash a}(Q_2 \cup (P \cap R))$, we have $b \in \mathrm{cl}^*_{M\backslash a}(P \cap R)$. As $a \in \mathrm{cl}(G)$, Lemma 2.19 implies that $b \in \mathrm{cl}^*(P \cap R)$, so $b \in \mathrm{cl}^*(P)$, contradicting the fact that $(P, \{a, b\}, Q)$ is tight. We deduce that $z \neq b$. Then $(D_1 - z, D_2 \cup z)$ is equivalent to (D_1, D_2). As before, the minimality assumption is preserved unless either $Q_2 \subseteq R$ and $z \in G$, or $Q_2 \subseteq G$ and $z \in R$. In each case, $z \in \mathrm{cl}^{(*)}_{M/a}(Q_2)$, so we can replace (R, G) by $(R \cup z, G - z)$ or $(R - z, G \cup z)$, respectively. In both cases, the same argument that was used above for (Z_1, Z_2) shows that $(Z_1 - z, Z_2 \cup z)$ is blocked by a.

Finally, we do indeed have $P_1 = G \cap P$ and $P_2 = R \cap P$. Assume that $(P_1, P_2, Q_2, Q_1 \cup b)$ is an anemone. Then $(P_2, P_1, Q_2, Q_1 \cup b)$ is a flower. By 7.1.4, $b \in \mathrm{cl}^*_{M\backslash a}(P)$, so, by [15, Lemma 5.5], $b \in \mathrm{cl}^*_{M\backslash a}(P_2)$. As $G \subseteq P_1 \cup Q$, and $a \in \mathrm{cl}_M(G)$, it follows by Lemma 2.19 that $b \in \mathrm{cl}^*_M(P_2)$, so $b \in \mathrm{cl}^*_M(P)$; a contradiction. We conclude that 7.1.15 holds.

7.1.16. *If* $1 \notin \{|R \cap P_1|, |R \cap P_2|, |G \cap P_1|, |G \cap P_2|\}$, *then* $|Q_1 \cap R| = |Q_1 \cap G| = |Q_2 \cap R| = |Q_2 \cap G| = 1$.

Assume otherwise. Then, by 7.1.14 and symmetry, $1 \notin \{|R \cap Q_1|, |R \cap Q_2|, |G \cap Q_1|, |G \cap Q_2|\}$. Now, by 7.1.15 and symmetry, we have $R = P_2 \cup Q_2 \cup b$ and $G = P_1 \cup Q_1$. By 7.1.10(iv), $a \in \mathrm{cl}_M(G)$, so $a \in \mathrm{cl}_M(P_1 \cup b \cup Q_1)$. By 7.1.3, $a \in \mathrm{cl}_M(P_1 \cup b \cup P_2)$. Thus, by Lemma 2.15, $a \in \mathrm{cl}(P_1 \cup b)$ contradicting 7.1.10(ii). Hence 7.1.16 holds.

7.1.17. $|P| = |Q| = 4$ *and if* $X \in \{P_1, P_2, Q_1, Q_2\}$ *and* $Y \in \{R, G\}$, *then*, $|X \cap Y| = 1$.

Assume this does not hold. Then we may assume that $1 \notin \{|R \cap P_1|, |R \cap P_2|, |G \cap P_1|, |G \cap P_2|\}$, so $P_1 = G \cap P$ and $P_2 = R \cap P$. Moreover, $|Q_1 \cap R| = |Q_1 \cap G| = |Q_2 \cap R| = |Q_2 \cap G| = 1$ and the flower $(P_1, P_2, Q_2, Q_1 \cup b)$ of $M\backslash a$ is swirl-like. Let $\{q_1\} = R \cap Q_1$ and $\{q_2\} = R \cap Q_2$.

By duality, the flower $(R \cap P, G \cap P, G \cap Q, (R \cap Q) \cup b)$ of M/a is swirl-like. Moreover, by 7.1.11, b is in the closure of both $R \cap P$ and $(R \cap Q) \cup b$. This means that $\sqcap_{M/a}(P_2, \{q_1, q_2\}) = 1$. But $\sqcap_M(P_2, \{q_1, q_2\}) = 0$ as otherwise, $r(P_2 \cup \{q_1, q_2\}) = r(P_2) + 1$, so $q_1 \in \mathrm{cl}(P_2 \cup \{q_2\})$. Then, by replacing (D_1, D_2) by the equivalent $(D_1 - q_1, D_2 \cup q_2)$, we find that the new Q_1 has just a single element; a contradiction to 7.1.7. We conclude that $a \in \mathrm{cl}_M(P_2 \cup \{q_1, q_2\})$ contradicting the fact that a blocks $P_1 \cup b$. Hence 7.1.17 holds.

We may now assume that $|P| = |Q| = 4$. Then, by 7.1.12, each of $|R \cap P|$, $|G \cap P|$, $|R \cap Q|$, and $|G \cap Q|$ is 2.

7.1.18. *Both* P *and* Q *are quads in* M.

Assume P is not a quad. Then it is sequential. By 7.1.6, a sequential ordering (x_1, x_2, x_3, x_4) of P in M is also a sequential ordering of P in M/a. Now $\{x_1, x_2, x_3\}$ contains either a unique element z of R or a unique element z' of G. Then (R, G) is equivalent to $(R - z, G \cup z)$ or $(R \cup z', G - z')$, respectively. Hence $|(R - z) \cap P| = 1$ or $|(G - z') \cap P| = 1$, so we have a contradiction to 7.1.12. Thus 7.1.18 holds.

By 7.1.15, the flower $(P_1, P_2, Q_2, Q_1 \cup b)$ of $M\backslash a$ is swirl-like. By 7.1.7, $b \in \mathrm{cl}^*_{M\backslash a}(P_1) \cap \mathrm{cl}^*_{M\backslash a}(Q_1)$. Thus $(P_1 \cup b, P_2, Q_2, Q_1)$ is also a swirl-like flower of $M\backslash a$, and both $P_1 \cup b$ and $Q_1 \cup b$ are triads of $M\backslash a$. Moreover, $M\backslash a, b$ has (P, Q) as a 2-separation. Hence $M|(P \cup Q)$ is the 2-sum, with basepoint x say, of matroids M_P and M_Q that have P and Q respectively as spanning circuits. In particular, $r(M) = 5$.

7.1.19. *In* M_P, *the element* x *is freely placed on the line spanned by* P_2.

Observe from the flowers $(P_1 \cup b, Q_1, Q_2, P_2)$ and $(P_1, Q_1 \cup b, Q_2, P_2)$ of $M\backslash a$ that $\sqcap(P_2, Q_2) = 1$ and $\sqcap(P_1, Q_2) = 0$. Thus $P_2 \cup Q_2$ contains a circuit of M. As P and Q are cocircuits of M, it follows by orthogonality that $P_2 \cup Q_2$ is a circuit of M. Hence $P_2 \cup x$ and $Q_2 \cup x$ are circuits of M_P and M_Q, respectively. As $\sqcap(P_1, Q_2) = 0$, we deduce that 7.1.19 holds.

It follows immediately from 7.1.19 and symmetry that:

7.1.20. $P_2 \cup Q_2$ *is the only circuit of* $M|(P \cup Q)$ *that meets both* P *and* Q *and has at most four elements.*

By orthogonality and the fact that $(P, \{a, b\}, Q)$ is tight, it follows that b is in neither a triangle nor a triad of M. Thus, by (iii), both $M\backslash b$ and M/b are 3-connected. We show next that:

7.1.21. $M\backslash b$ *has no* 3*-separation exposed by* b.

Assume $M\backslash b$ has a 3-separation (Y_1, Y_2) exposed by b and let $|Y_1| \leqslant |Y_2|$. As $|E(M\backslash b)| = 9$ and (Y_1, Y_2) is non-sequential, we deduce that Y_1 is a quad of $M\backslash b$. Suppose $a \in Y_1$. Then $M\backslash b, a$ has $Y_1 - a$ as a triad. As P and Q are both circuits, it follows by orthogonality that $Y_1 - a$ is contained in P or Q. Thus a is in $\mathrm{cl}(P)$ or $\mathrm{cl}(Q)$; a contradiction. Hence $a \notin Y_1$.

Since Y_1 is a circuit of M contained in $P \cup Q$, and P and Q are both cocircuits of M, either $Y_1 \in \{P, Q\}$, or Y_1 meets each of P and Q in exactly two elements. In the first case, (Y_1, Y_2) is not exposed by b; a contradiction. In the second case, we deduce from 7.1.20 that $Y_1 = P_2 \cup Q_2$. Then $P_2 \cup Q_2 \cup b$ is a cocircuit of M. As $r(M) = 5$, it follows that $r(P_1 \cup Q_1 \cup a) = 4$. Now $a \notin \mathrm{cl}(P_1 \cup Q_1)$, so $r(P_1 \cup Q_1) = 3$. Hence $P_1 \cup Q_1$ contains a circuit of M that contradicts 7.1.20.

We conclude that 7.1.21 holds. By duality, M/b has no 3-separation exposed by b, and this completes the proof of Theorem 7.1. $\square$

8. Wild triangles

In this section, we establish several results for wild triangles that will be used in the proofs of Theorems 1.1 and 4.2, which will be given in the last section. In particular, we shall require a property of standard and costandard wild triangles, which will be proved in Lemma 8.2. The proof of that lemma will use the next lemma, which considers a matroid M and a matroid obtained from M by a $\Delta - Y$ exchange, and relates both closure and coclosure in these two matroids.

Lemma 8.1. *Let $\{a, b, c\}$ be a triangle Δ in a matroid M. Let K be a copy of $M(K_4)$ having each of $\{a, b, c\}$, $\{a, a'\}$, $\{b, b'\}$, and $\{c, c'\}$ as flats. Let $M' = P_\Delta(K, M)\backslash\Delta$. Then, for $X \subseteq E(M) - \Delta$ and $e \in E(M) - \Delta$,*

(i) *$e \in \mathrm{cl}_M(X)$ if and only if $e \in \mathrm{cl}_{M'}(X)$;*
(ii) *$e \in \mathrm{cl}_M(X \cup \{a, b, c\})$ if and only if $e \in \mathrm{cl}_{M'}(X \cup \{a', b', c'\})$; and*
(iii) *$e \in \mathrm{cl}^*_M(X)$ if and only if $e \in \mathrm{cl}^*_{M'}(X)$.*

Proof. As $M\backslash\Delta = M'\backslash\{a', b', c'\}$, part (i) is immediate. For (ii), note that the flats of $P_\Delta(K, M)$ consist of those sets F such that $F \cap E(M)$ is a flat of M and $F \cap E(K)$ is a flat of K [13, p. 419]. As $\{a, b, c\}$ is a flat of K, we have $\mathrm{cl}_M(X \cup \{a, b, c\}) = \mathrm{cl}_{P_\Delta(K,M)}(X \cup \{a, b, c\})$. Now

$$\begin{aligned}
\mathrm{cl}_{M'}\bigl(X \cup \{a', b', c'\}\bigr) - \{a', b', c'\} &= \mathrm{cl}_{P_\Delta(K,M)\backslash\Delta}\bigl(X \cup \{a', b', c'\}\bigr) - \{a', b', c'\} \\
&= \mathrm{cl}_{P_\Delta(K,M)}\bigl(X \cup \{a', b', c'\}\bigr) - \Delta - \{a', b', c'\} \\
&= \mathrm{cl}_{P_\Delta(K,M)}\bigl(X \cup \{a', b', c'\} \cup \Delta\bigr) - \Delta - \{a', b', c'\} \\
&= \mathrm{cl}_M\bigl(X \cup \{a, b, c\}\bigr) - \{a, b, c\}.
\end{aligned}$$

Thus (ii) holds. Part (iii) follows from (i) because $e \in \mathrm{cl}^*_M(X)$ if and only if $e \notin \mathrm{cl}_M(E(M) - X - e)$. $\square$

Lemma 8.2. *Let $\{a, b, c\}$ be a standard or costandard wild triangle in a 3-connected matroid M. Then there is a partition $(P, Q, R, \{a, b, c\})$ of $E(M)$ such that each of P, Q, and R is a non-sequential 3-separating set and none of $\mathrm{fcl}(P)$, $\mathrm{fcl}(Q)$, or $\mathrm{fcl}(R)$ contains $\{a, b, c\}$.*

Proof. Suppose first that $\{a, b, c\}$ is a standard wild triangle and let $(P_1, P_2, \dots, P_6)$ be a partition of $E(M) - \{a, b, c\}$ associated to $\{a, b, c\}$. Let $(P, Q, R) = (P_1 \cup P_2, P_3 \cup P_4, P_5 \cup P_6)$. Then $|E(M)| \geqslant 15$ as $|P_i| \geqslant 2$ for all i. As P and $P \cup a$ are 3-separating in M, if P is sequential, then so is $P \cup a$. In that case, by Lemma 2.10, a does not expose any 3-separations of M; a contradiction. Thus P is non-sequential and, by symmetry, so are Q and R. It follows that none of $\mathrm{fcl}(P)$, $\mathrm{fcl}(Q)$, or $\mathrm{fcl}(R)$ is $E(M)$.

Now suppose that $\mathrm{fcl}(P)$ contains b. Then it also contains c. Thus M has a 3-sequence of the form $(P, a, e_1, e_2, \dots, e_m, E(M) - \mathrm{fcl}(P))$ and we may assume that $(b, c) = (e_i, e_{i+1})$ for some i. If $e_i \in \mathrm{cl}^*(P \cup \{a, e_1, e_2, \dots, e_{i-1}\})$, then $P \cup \{a, e_1, e_2, \dots, e_{i-1}\}$ is 2-separating in $M\backslash e_i$; a contradiction. Thus $e_i \in \mathrm{cl}(P \cup \{a, e_1, e_2, \dots, e_{i-1}\})$, so $\{b, c\} \subseteq \mathrm{cl}(P \cup \{a, e_1, e_2, \dots, e_{i-1}\})$. Moreover, $\{b, c\} \subseteq \mathrm{cl}(E(M) - (P \cup \{a, e_1, e_2, \dots, e_{i-1}, b, c\}))$. Therefore, by Lemma 5.1, at least one of b and c does not expose any 3-separations in M; a contradiction. We conclude that $\mathrm{fcl}(P)$ avoids $\{b, c\}$. By symmetry, $\mathrm{fcl}(Q)$ avoids $\{a, c\}$, and $\mathrm{fcl}(R)$ avoids $\{a, b\}$. Thus the lemma holds when $\{a, b, c\}$ is a standard wild triangle. Note too that, in this case, a, b, and c are in $\mathrm{cl}(P)$, $\mathrm{cl}(Q)$, and $\mathrm{cl}(R)$, respectively.

Now assume that $\{a, b, c\}$ is a costandard wild triangle in M. Then $\{a, b, c\}$ is a standard wild triangle in $(\Delta M)^*$. Clearly the full closure of a set equals its full closure in the dual matroid. As the lemma holds for standard wild triangles, there is a partition $(P, Q, R, \{a, b, c\})$ of $E(M)$ such that each of P, Q, and R is a non-sequential 3-separating set of ΔM and none of $\mathrm{fcl}_{\Delta M}(P)$, $\mathrm{fcl}_{\Delta M}(Q)$, or $\mathrm{fcl}_{\Delta M}(R)$ contains $\{a, b, c\}$. Moreover, a, b, and c are in $\mathrm{cl}^*_{\Delta M}(P)$, $\mathrm{cl}^*_{\Delta M}(Q)$, and $\mathrm{cl}^*_{\Delta M}(R)$, respectively. By Lemma 8.1, each of P, Q, and R is a non-sequential 3-separating set of M since a sequential ordering of such a set in M is a sequential ordering of it in ΔM.

It remains to show that none of $\mathrm{fcl}_M(P)$, $\mathrm{fcl}_M(Q)$, and $\mathrm{fcl}_M(R)$ contains $\{a, b, c\}$. To avoid confusion, we shall work with the matroid M' defined in the last lemma. Assume that $\mathrm{fcl}_M(P) \supseteq \{a, b, c\}$. We know that $\mathrm{fcl}_{M'}(P) \not\supseteq \{a', b', c'\}$ but $a' \in \mathrm{cl}^*_{M'}(P)$.

Consider the 3-sequence $(P, z_1, z_2, \dots, z_n, E(M) - \mathrm{fcl}_M(P))$. As $\mathrm{fcl}_M(P)$ contains $\{a, b, c\}$, we may assume that $\{a, b, c\} = \{z_i, z_j, z_k\}$ where $i < j < k$. As $\{a, b, c\}$ is a triangle, we can move the first and last members of $\{a, b, c\}$ in the sequence $(z_1, z_2, \dots, z_n)$ so that we maintain a 3-sequence and get $\{a, b, c\} = \{z_{j-1}, z_j, z_{j+1}\}$. By Lemma 8.1, $P \cup \{z_1, z_2, \dots, z_h\}$ is 3-separating in M' for all h in $[j-2]$. As $\{a, b, c\}$ is a triangle of M, we must have that $z_{j-1} \in \mathrm{cl}_M(P \cup \{z_1, z_2, \dots, z_{j-2}\})$.

Suppose $b \in \mathrm{cl}_M(P \cup \{z_1, z_2, \dots, z_{j-2}\})$. Then M has a circuit C such that $b \in C$ and $C \subseteq b \cup P \cup \{z_1, z_2, \dots, z_{j-2}\}$. In $P_\Delta(K, M)$, the set $\{a', b, c'\}$ is a circuit, so $(C - b) \cup \{a', c'\}$ is a circuit of $P_\Delta(K, M)$ and hence of M'. As $a' \in \mathrm{cl}^*_{M'}(P)$, we deduce that $c' \in \mathrm{fcl}_{M'}(P)$. Then, as $\{a', b', c'\}$ is a cocircuit of M', we have $\{a', b', c'\} \subseteq \mathrm{fcl}_{M'}(P)$; a contradiction. Thus $b \notin \mathrm{cl}_M(P \cup \{z_1, z_2, \dots, z_{j-2}\})$. By symmetry, $c \notin \mathrm{cl}_M(P \cup \{z_1, z_2, \dots, z_{j-2}\})$. Hence $a = z_{j-1}$ and we may assume that $(z_j, z_{j+1}) = (b, c)$. Moreover, $b \in \mathrm{cl}^*_M(P \cup \{z_1, z_2, \dots, z_{j-2}, a\})$ otherwise we can interchange a and b in the 3-sequence to get a contradiction. The circuit $\{a, b, c\}$ of M implies that the cocircuit C^* of M that contains b and is contained in $P \cup \{z_1, z_2, \dots, z_{j-2}, a\}$ must contain a. Thus the hyperplane H of M that equals $E(M) - C^*$ contains c and avoids $\{a, b\}$. Hence $P_\Delta(K, M)$ has $H \cup \{a', b'\}$ as a hyperplane and so $c' \cup (C^* - \{a, b\})$ is a union of cocircuits of M'. Thus $c' \in \mathrm{fcl}_{M'}(P \cup \{z_1, z_2, \dots, z_{j-2}\}) = \mathrm{fcl}_{M'}(P)$. But $a' \in \mathrm{fcl}_{M'}(P)$. Thus $\{a', b', c'\} \subseteq \mathrm{fcl}_{M'}(P)$; a contradiction. We conclude, using symmetry, that none of $\mathrm{fcl}_M(P)$, $\mathrm{fcl}_M(Q)$, and $\mathrm{fcl}_M(R)$ contains $\{a, b, c\}$. $\square$

The next lemma will be useful in both the proof of the subsequent lemma and the proof of Theorem 4.2.

Lemma 8.3. *Let M be a 3-connected matroid and R be a petal of a tight flower Φ of M whose order is at least three. Then there is a tight flower (P, R, Q) that is a concatenation of Φ and has $|P| \geqslant 3$. Moreover, either*

(i) $|Q| \geqslant 3$; *or*
(ii) *Φ has at least four petals and has a tight concatenation (P_1, P_2, R, Q) where $|P_2| = 2 = |Q|$.*

Proof. Let $\Phi = (P_1, P_2, \dots, P_{n-2}, R, Q)$. If $n = 3$, then, as Φ has order at least three, $|P_1|, |Q| \geqslant 3$ and (i) holds. Hence we may suppose that $n \geqslant 4$. Let $\Phi' = (P, R, Q)$ where $P = P_1 \cup P_2 \cup \dots \cup P_{n-2}$. Then P is certainly tight. Suppose some Q' in $\{R, Q\}$ is loose in Φ'. Then, by Lemma 3.1, $Q' \subseteq \mathrm{fcl}(P_1 \cup P_2 \cup \dots \cup P_{n-2})$. Hence, by [15, Lemma 5.9], $Q' \subseteq \mathrm{fcl}(P_1) \cup \mathrm{fcl}(P_{n-2})$, so Q' is loose in Φ; a contradiction. Therefore Φ' is a tight flower. Thus if $|Q| \geqslant 3$, then (i) holds. By symmetry, if $|P_{n-2}| \geqslant 3$, then (i) holds. Therefore, we may assume that $|Q| = 2 = |P_{n-2}|$. In that case, we let $\Phi'' = (P_1 \cup P_2 \cup \dots \cup P_{n-3}, P_{n-2}, R, Q)$. Then, arguing as for Φ', we deduce that Φ'' is tight. $\square$

In the next lemma, the hypothesis that Φ is tight is not needed. But we do not need the stronger result here, so we prove only the weaker result.

Lemma 8.4. *Let M be a 3-connected matroid and let $\{a, b, c\}$ be a triangle in M that is not in a 4-element fan. Suppose $\{a, b, c\}$ is a petal of a tight flower Φ of M whose order is at least three. Then $\{a, b, c\}$ is not wild.*

Proof. Assume that $\{a, b, c\}$ is wild. Then, by [17, Corollary 4.3], we may assume that all of $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected.

By Theorem 3.7, $|E(M)| = 11$, or $\{a, b, c\}$ is a wild triangle in a trident, or $\{a, b, c\}$ is a standard or costandard wild triangle. If $\{a, b, c\}$ is standard or costandard, then $|E(M)| \geqslant 15$. Moreover, if $\{a, b, c\}$ is in a trident, then $|E(M)| \geqslant 9$. Thus we may assume that either $|E(M)| \geqslant 10$, or $|E(M)| = 9$ and $\{a, b, c\}$ is a wild triangle in a trident.

8.4.1. *The local connectivity between pairs of consecutive petals of Φ is* 1.

By Lemma 8.3, Φ has a concatenation $(P, \{a, b, c\}, Q)$ in which $|P| \geqslant 3$ and $|Q| \geqslant 2$. Suppose first that $(P, \{a, b, c\}, Q)$ is a paddle. Then the partition $(P, \{a, b\}, c \cup Q)$ satisfies the hypotheses of Lemma 5.1 and that lemma implies that $\{a, b, c\}$ is not wild; a contradiction.

Next assume that $(P, \{a, b, c\}, Q)$ is a copaddle. Let ΔM be the matroid that is obtained by performing a $\Delta - Y$ exchange on $\{a, b, c\}$ and then relabelling the resulting matroid in the natural way so that $E(\Delta M) = E(M)$. Then, by [17, Lemmas 8.2 and 8.3], $\{a, b, c\}$ is a wild triangle in $(\Delta M)^*$, and all of $(\Delta M)^*\backslash a$, $(\Delta M)^*\backslash b$, and $(\Delta M)^*\backslash c$ are 3-connected. Since $(P, \{a, b, c\}, Q)$ is a paddle in $(\Delta M)^*$, we get a contradiction as in the last paragraph. We deduce that $(P, \{a, b.c\}, Q)$, and hence Φ, is neither a paddle nor a copaddle. Thus 8.4.1 holds.

Now assume that $|Q| \geqslant 3$. If $b \in \mathrm{cl}(P)$ and $c \in \mathrm{cl}(Q)$, then $(P \cup b, Q \cup c)$ is a 2-separation of $M\backslash a$, contradicting the assumption that this matroid is 3-connected. As the local connectivity between distinct pairs of petals is 1, it follows that we may assume that neither b nor c is in $\mathrm{cl}(P) \cup \mathrm{cl}(Q)$. Moreover,

8.4.2. $r(P \cup b) = r(P \cup c) = r(P \cup \{b, c\}) = r(P) + 1$, *and* $r(Q \cup b) = r(Q \cup c) = r(Q \cup \{b, c\}) = r(Q) + 1$.

Using this, we deduce that

8.4.3. $\lambda_{M\backslash a}(P \cup b) = \lambda_{M\backslash a}(P \cup c) = 3$.

Let (B, C) be a 3-separation of $M\backslash a$ that is exposed by a. Then we may assume that $b \in B$ and $c \in C$.

8.4.4. (B, C) *crosses both P and Q.*

Suppose $P \subseteq B$. By 8.4.2, $c \in \mathrm{cl}(P \cup b)$ so $c \in \mathrm{cl}(B)$. But then (B, C) is equivalent to $(B \cup c, C - c)$, and $a \in \mathrm{cl}(B \cup c)$. This contradicts the assumption that (B, C) is exposed in $M\backslash a$. A symmetric argument shows that P is not contained in C. Thus 8.4.4 holds.

8.4.5. $|P \cap B| \geqslant 2$ *and* $|P \cap C| \geqslant 2$; *and* $|Q \cap B| \geqslant 2$ *and* $|Q \cap C| \geqslant 2$.

By symmetry, it suffices to prove the inequalities involving P. Suppose that $P \cap B = \{p\}$. As $|P| \geqslant 3$, we have $\lambda_{M\backslash a}(P \cap C) \geqslant 2$. Thus $\lambda_{M\backslash a}((P \cap C) \cup p) = \lambda_{M\backslash a}(P) \leqslant \lambda_{M\backslash a}(P \cap C)$. Therefore $p \in \mathrm{cl}^{(*)}_{M\backslash a}(P \cap C)$, so $p \in \mathrm{cl}^{(*)}_{M\backslash a}(C)$. Thus $(B - p, C \cup p)$ is a 3-separation in $M\backslash a$ that is equivalent to (B, C). Hence $(B - p, C \cup p)$ is exposed by a yet it contradicts 8.4.4. Thus $|P \cap B| \geqslant 2$ and, by symmetry, $|P \cap C| \geqslant 2$.

By 8.4.5, $|C \cap (\{b, c\} \cup Q)| \geqslant 2$ and $|C \cap Q| \geqslant 2$. Hence, by uncrossing and 8.4.5, we have $\lambda_{M\backslash a}(P \cap B) = \lambda_{M\backslash a}((P \cap B) \cup b) = 2$. Therefore $b \in \mathrm{cl}^{(*)}_{M\backslash a}(P \cap B)$, so $b \in \mathrm{cl}^{(*)}_{M\backslash a}(P)$. Hence $\lambda_{M\backslash a}(P \cup b) = 2$. This contradiction to 8.4.3 completes the proof of the lemma when $|Q| \geqslant 3$.

By Lemma 8.3, we may now assume that Φ has a tight concatenation $(P_1, P_2, \{a, b, c\}, Q)$ where $|P_2| = 2 = |Q|$.

8.4.6. $|P_1| \geqslant 3$.

Suppose that $|P_1| = 2$. Then $|E(M)| = 9$, so $\{a,b,c\}$ is in a trident Z of M. We shall assume that this trident is labelled as in Fig. 1 and let $E(M) - Z = \{j,k\}$. As the local connectivity between consecutive petals of Φ is 1, each of $P_1 \cup P_2$ and $P_1 \cup Q$ has rank 3. By Lemma 6.1, if one of $P_1 \cup P_2$ and $P_1 \cup Q$ contains $\{j,k\}$, then the relevant set has rank 4. Thus $\{j,k\}$ meets both P_2 and Q. By symmetry, we may assume that $P_2 = \{j,u\}$. Then, in $M\backslash c$, the set $\{t,s,v,a\}$ is a quad meeting the 3-separating set $\{j,u,a,b\}$ in a single element. But each of the possible structures of this 3-separating set produces a violation to orthogonality. We conclude that 8.4.6 holds.

Now, as before, we let (B,C) be a 3-separation in $M\backslash a$ exposed by a, where $b \in B$ and $c \in C$. Suppose first that $P_2 \subseteq B$. Then, as $\{b,c\} \cup P_2$ is 3-separating in $M\backslash a$ and $\sqcap(\{b,c\},P_2) = 1$, it follows that $c \in \mathrm{cl}^{(*)}(P_2 \cup b)$, so $c \in \mathrm{cl}^{(*)}(B)$. Hence $(B \cup c, C - c)$ is a 3-separation equivalent to (B,C) that is not exposed by a; a contradiction. We conclude that $P_2 \cap C \neq \emptyset$. By symmetry, $P_2 \cap B \neq \emptyset$. Hence $|P_2 \cap C| = 1 = |P_2 \cap B|$ and, similarly, $|Q \cap C| = 1 = |Q \cap B|$.

As $|P_1| \geqslant 3$, without loss of generality, we have $|P_1 \cap B| \geqslant 2$. Since $|E(M\backslash a) - (P_1 \cup B)| \geqslant 2$, it follows, by uncrossing, that $\lambda_{M\backslash a}(P_1 \cap B) = 2$, so $\lambda_M(P_1 \cap B) = 2$. Let $P_2 \cap B = \{b'\}$. Then, by uncrossing again, $\lambda_{M\backslash a}((P_1 \cap B) \cup b') = 2$, so $\lambda_M((P_1 \cap B) \cup b') = 2$. We deduce that $b' \in \mathrm{cl}^{(*)}(P_1 \cap B)$, so $b' \in \mathrm{cl}^{(*)}(P_1)$. Hence $P_2 \subseteq \mathrm{fcl}(P_1)$, so P_2 is loose. This contradiction completes the proof of the lemma. □

Lemma 8.5. *In a 3-connected matroid M with $|E(M)| \neq 11$, let $\{a,b,c\}$ be a wild triangle that is not in a trident or a 4-element fan. If T is a 3-tree for M and S labels a terminal bag of T, then $|S \cap \{a,b,c\}| \leqslant 1$.*

Proof. Assume that $|S \cap \{a,b,c\}| \geqslant 2$. Suppose first that S is non-sequential. Because $|E(M)| \neq 11$ and $\{a,b,c\}$ is not in a trident or a 4-element fan, Theorem 3.7 implies that $\{a,b,c\}$ is a standard or costandard wild triangle of M. By Lemma 8.2, there is a partition $(P,Q,R,\{a,b,c\})$ of $E(M)$ such that each of P, Q, and R is a non-sequential 3-separating set and none of $\mathrm{fcl}(P)$, $\mathrm{fcl}(Q)$, or $\mathrm{fcl}(R)$ contains $\{a,b,c\}$. Thus T displays 3-separations $(P', E(M) - P')$, $(Q', E(M) - Q')$, and $(R', E(M) - R')$ that are equivalent to the non-sequential 3-separations $(P, E(M) - P)$, $(Q, E(M) - Q)$, and $(R, E(M) - R)$ of M. Now $\mathrm{fcl}(U') = \mathrm{fcl}(U)$ for all U in $\{P,Q,R\}$, so, for all such U, the set $\mathrm{fcl}(U')$ contains at most one element of $\{a,b,c\}$. Because the terminal bag S contains at least two elements of $\{a,b,c\}$, all of P', Q', and R' avoid S, so $S \subseteq E(M) - (P' \cup Q' \cup R')$. But $\mathrm{fcl}(P' \cup Q' \cup R') \supseteq \mathrm{fcl}(P') \cup \mathrm{fcl}(Q') \cup \mathrm{fcl}(R') \supseteq E(M) - \{a,b,c\}$. Thus $\mathrm{fcl}(P' \cup Q' \cup R') = E(M)$, so S is sequential; a contradiction.

We may now assume that S is sequential. Then so is $S \cup \{a,b,c\}$. Also the neighbour of the vertex S in T is a flower vertex. Suppose that $|S \cup \{a,b,c\}| \geqslant 4$. If $M\backslash b$ is not 3-connected, then, by [17, Theorem 4.2], $\{a,b,c\}$ is not wild; a contradiction. Hence $M\backslash b$ is 3-connected. Then, by Lemma 2.10, b does not expose any 3-separations, contradicting the fact that $\{a,b,c\}$ is wild. We may now assume that $|S \cup \{a,b,c\}| = 3$, so $S = \{a,b,c\}$. Thus M has a tight flower of order at least three having $\{a,b,c\}$ as one of its petals. But Lemma 8.4 implies that $\{a,b,c\}$ is not wild; a contradiction. □

9. Proofs of the main results

In this section, we prove Theorems 4.2 and 1.1.

Proof of Theorem 4.2. Assume first that S is sequential. Suppose that $|\mathrm{fcl}(S)| \geqslant 4$. By switching to the dual if necessary, we may assume that $\mathrm{fcl}(S)$ contains a triangle X. If $M\backslash e$ is 3-connected for some e in X, then, by Lemma 2.10, e does not expose any 3-separations in $M\backslash e$. Thus we may assume that $M\backslash e$ is not 3-connected for all e in X. Then, by Tutte's Triangle Lemma [21], X is contained in a maximal fan F having at least four elements. Thus, by Lemma 2.11, the deletion of an end f of F from M or M^* is 3-connected but does not expose any 3-separations. As $F \subseteq \mathrm{fcl}(S)$, the theorem holds.

We may now assume that $|\mathrm{fcl}(S)| \leqslant 3$. In the 3-tree T, the set S labels a degree-one vertex v. The unique neighbour u of v is a flower vertex. Thus the corresponding tight flower Ψ of M has S as a petal. Then Ψ is equivalent to a tight flower Φ having $\mathrm{fcl}(S)$ as a petal and having order at least three. By Lemma 8.3, Φ has a tight concatenation $(P, \mathrm{fcl}(S), Q)$ where $|P| \geqslant 3$.

Suppose $\mathrm{fcl}(S) = \{a,b\}$. If $|Q| \geqslant 3$, then the result follows by Theorem 7.1. Thus we may assume that $|Q| = 2$. Then, by Lemma 8.3, Φ has a tight concatenation $(P_1, P_2, \{a,b\}, Q)$ where

$|P_2| = 2 = |Q|$. Now $Q \cup \{a, b\}$ is 3-separating but it does not contain a triangle or a triad otherwise one of Q and $\{a, b\}$ is loose. Thus $\{a, b\} \cup Q$ is a quad in M. We can now apply Lemma 4.4 to get that either the theorem holds, or there is a labelling $\{a', b'\}$ of Q and there is an element t of $P_1 \cup P_2$ such that $\{a, a', t\}$ and $\{b, b', t\}$ are triangles of M. By symmetry, $P_2 \cup \{a, b\}$ is a quad in M. Thus, by orthogonality, $t \in P_2$. Hence $\{a, b, t\}$ spans $P_2 \cup \{a, b\} \cup Q$, so $\lambda(P_2 \cup \{a, b\} \cup Q) \leqslant 1$; a contradiction as $|E(M)| \geqslant 9$.

When S is sequential, it remains to consider the case when $|\mathrm{fcl}(S)| = 3$. By duality, we may assume that $\mathrm{fcl}(S)$ is a triangle $\{a, b, c\}$ of M. This triangle is certainly not contained in a 4-element fan. By Lemma 8.4, $\{a, b, c\}$ is not wild, so some e in $\{a, b, c\}$ does not expose any 3-separations in $M \backslash e$. We conclude that the theorem holds when S is sequential.

We may now assume that S is non-sequential. Let $S' = S - \mathrm{fcl}(E(M) - S)$. If there are no triangles or triads of M that have at least two elements in S', then, by Corollary 4.6, the theorem holds. By switching to the dual if necessary, we may assume that M has a triangle Y containing at least two elements of S'. Then $Y \subseteq \mathrm{fcl}(S)$. Now, for y in Y, if $M \backslash y$ is 3-connected but y does not expose any 3-separations in $M \backslash y$, then the theorem holds. Thus we may assume that Y is a wild triangle of M. Then, by Lemma 8.5, one of the following holds: Y is contained in a 4-element fan, Y is contained in a trident, or $|E(M)| = 11$. In the first case, by Lemma 2.11, the theorem holds. Thus we may assume that Y is not contained in a 4-element fan.

Now suppose that Y is a wild triangle in a trident X. We may assume that $\mathrm{fcl}(S) \cap (X - Y)$ is empty otherwise the theorem holds by Lemma 6.2. As S is non-sequential, $|S| \geqslant 4$. If $|S| = 4$, then S is a quad and the theorem holds by Lemma 4.3. Hence we may assume that $|S| \geqslant 5$. Now, by uncrossing X and $E - \mathrm{fcl}(S)$, we get that their intersection, $X - Y$, is 3-separating; a contradiction since $r^*(X - Y) \geqslant 3$ and, by Lemma 6.1, $r(X - Y) = 4$. Hence we may assume that Y is not contained in a trident.

Finally suppose that $|E(M)| = 11$ and let $Y = \{a, b, c\}$. The argument here will require a more detailed analysis of the proof of Theorem 3.7, which appears in [17, Theorem 3.1]. We shall make frequent reference to that proof assuming the reader has access to the paper. Since the triangle $\{a, b, c\}$ is wild but is not contained in a 4-element fan, it follows by [17, Corollary 4.3] that there are 3-separations, (A_1, A_2) and (B_1, B_2), that are exposed by a and b, respectively. Following [17, Theorem 3.1], we assume that a and b are in B_2 and A_1, respectively. Then $c \in A_2 \cap B_1$. From [17, (5.0.5)], $|A_i \cap B_i| \geqslant 2$ for each i in $\{1, 2\}$. Moreover, from [17, (5.0.10), (5.0.9)], $|A_1 \cap B_2| \geqslant 1$ and $|A_2 \cap B_1| \geqslant 2$. Then, as explained in [17, Section 6 and p. 307, l. 10], by [17, Lemmas 7.3, 7.4, and 7.9], we may assume that either $|A_1 \cap B_2| \geqslant 2$ and $|A_2 \cap B_1| \geqslant 3$; or $\{a, b, c\}$ is in a trident. Since the latter does not occur and $|E(M)| = 11$, we have

$$|A_1 \cap B_1| = |A_2 \cap B_2| = |A_1 \cap B_2| = 2 \quad \text{and} \quad |A_2 \cap B_1| = 3. \tag{4}$$

Hence

$$|A_1| = |A_2| = 5 = |B_1| = |B_2|. \tag{5}$$

Now $(\mathrm{fcl}(S), E - \mathrm{fcl}(S))$ is a non-sequential 3-separation of M. Choose (U, V) to be a non-sequential 3-separation of M with $\{a, b, c\} \subseteq U$ and with $|V|$ maximized. Since $|E(M)| = 11$, it follows that $(|U|, |V|)$ is in $\{(5, 6), (6, 5), (7, 4)\}$. We have $(U - a, V)$ and (A_1, A_2) as 3-separations of $M \backslash a$. Clearly b and c in $(U - a) \cap A_1$ and $(U - a) \cap A_2$, respectively. Now V meets both A_1 and A_2 otherwise, since $|A_1| = |A_2| = 5$, we have $|V| = 4$ and we obtain the contradiction that $(A_1 - b, A_2 \cup b)$ or $(A_1 \cup c, A_2 - c)$ is equivalent to (A_1, A_2).

Suppose that $|A_2 \cap V| = 1$. Then, by uncrossing, $A_2 \cap (U - a)$ is 3-separating in $M \backslash a$ and $(A_1, A_2) \cong (A_1 \cup V, A_2 \cap (U - a))$. Thus we can replace (A_1, A_2) by $(A_1 \cup V, A_2 \cap (U - a))$. Since (5) holds for all potential choices of (A_1, A_2), we have a contradiction. Thus $|A_2 \cap V| \geqslant 2$ and, by symmetry, $|A_1 \cap V| \geqslant 2$. Likewise, $|A_2 \cap (U - a)| \geqslant 2$ and $|A_1 \cap (U - a)| \geqslant 2$. By uncrossing, both $A_1 \cap V$ and $A_2 \cap V$ are 3-separating in $M \backslash a$. Since $\{b, c\}$ is in the complement of both of these sets in $E(M \backslash a)$, both sets are 3-separating in M. But the cardinality constraints on the various sets mean that either (i) $|V| = 4$, or (ii) $|V| \geqslant 5$ and $|A_1 \cap V|$ or $|A_2 \cap V|$ is 3. Assume (ii) holds. Then $A_1 \cap V$ or $A_2 \cap V$ is a triangle or a triad of M avoiding $\{a, b, c\}$. If $A_1 \cap V$ is a triangle or triad, then it must contain the two elements of $A_1 \cap B_2$ or the two elements of $A_1 \cap B_1$. Thus we can replace (B_1, B_2) by an equivalent 3-separation of $M\backslash b$ for which (4) fails. If $A_2 \cap V$ is a triangle or triad, this set must contain the two elements of $A_2 \cap B_2$ or the two elements of $(A_2 \cap B_1) - c$. Again we can replace (B_1, B_2) by an equivalent 3-separation of $M\backslash b$ for which (4) fails. We conclude that (ii) does not hold. Hence (i) holds and V is a quad, so $|A_1 \cap V| = 2 = |A_2 \cap V|$. Moreover, by the choice of (U, V), no element of $U - \{a, b, c\}$ is in $\mathrm{cl}^{(*)}(V)$.

To complete the argument, we shall again follow [17] and take a 3-separation (C_1, C_2) of $M\backslash c$ exposed by c where $a \in C_1$ and $b \in C_2$. We shall exploit symmetry a lot in what follows. In particular, we have $|C_1| = |C_2| = 5$, and $|V \cap B_1| = |V \cap B_2| = |V \cap C_1| = |V \cap C_2| = 2$.

By uncrossing, each of $(U - a) \cap A_1$ and $(U - a) \cap A_2$ is a triangle or a triad of $M\backslash a$. By symmetry, each of $(U - b) \cap B_1$ and $(U - b) \cap B_2$ is a triangle or a triad of $M\backslash b$. Suppose that both $(U - a) \cap A_1$ and $(U - a) \cap A_2$ are triangles. Then $r(A_1) = r(A_2) = 3$, otherwise we can replace (A_1, A_2) by an equivalent 3-separation for which (5) fails. Thus $r(M) = 4$. As $r(V) = 3$, we deduce that $r(U) = 3$. It follows that each of $(U - b) \cap B_1$ and $(U - b) \cap B_2$ is a triangle of M. Thus $U - a$ is the disjoint union of a triangle containing b and a triangle containing c, while $U - b$ is the disjoint union of triangles containing a and c. It is not difficult to check that this cannot occur. Hence at most one of $(U - a) \cap A_1$ and $(U - a) \cap A_2$ is a triangle.

Let $((U - a) \cap A_1, (U - a) \cap A_2, V \cap A_2, V \cap A_1) = (\{b, 1, 2\}, \{c, 3, 4\}, \{7, 8\}, \{5, 6\})$. By symmetry, there are three possibilities for $(A_2 \cap B_1) - c$, namely, $\{7, 8\}$, $\{3, 4\}$, and $\{3, 7\}$.

Suppose that $(A_2 \cap B_1) - c = \{7, 8\}$. Then, by (4), $((U - b) \cap B_1, (U - b) \cap B_2, V \cap B_2, V \cap B_1) = (\{c, 1, 2\}, \{a, 3, 4\}, \{5, 6\}, \{7, 8\})$. Clearly at most one of $\{a, 3, 4\}$ and $\{c, 3, 4\}$ is a triangle. If neither is, then $\{a, b, 3, 4\}$ and $\{a, c, 3, 4\}$ are cocircuits of M. Since $\{a, b, c\}$ is not contained in a 4-element fan, it follows by elimination that $\{a, b, c, 3\}$ is a cocircuit of M contradicting Corollary 2.13. If $\{a, c, 3, 4\}$ is a cocircuit and $\{a, 3, 4\}$ is a circuit, then $\{a, c, 3, 4\}$ is a sequential 3-separating set and we obtain a contradiction using Lemma 2.10. Thus $\{a, b, 3, 4\}$ is a cocircuit and $\{c, 3, 4\}$ is a circuit. As $\{a, b, c\}$ is also a circuit, it follows that $\{a, b, 3, 4\}$ is a quad of M. This contradicts Lemma 4.3(i).

Next assume that $(A_2 \cap B_1) - c = \{3, 4\}$. Then, by (4), we have, without loss of generality, that $((U - b) \cap B_1, (U - b) \cap B_2, V \cap B_2, V \cap B_1) = (\{c, 3, 4\}, \{a, 1, 2\}, \{7, 8\}, \{5, 6\})$. If $\{c, 3, 4\}$ is not a triangle of M, then both $\{a, c, 3, 4\}$ and $\{b, c, 3, 4\}$ are cocircuits, so $\{a, b, c, 3\}$ is a cocircuit; a contradiction to Corollary 2.13. Hence $\{c, 3, 4\}$ is a circuit and so $\{a, b, 1, 2\}$ is a cocircuit. Thus $r(B_1) = 3$ and $r(B_2) = 4$, so $r(M) = 5$. Hence $r(C_1)$ or $r(C_2)$ is 3. The circuit $\{c, 3, 4\}$ implies that neither C_1 nor C_2 contains $\{3, 4\}$. Hence, without loss of generality, $(U - c) \cap C_1 = \{a, 1, 3\}$ and $(U - c) \cap C_2 = \{b, 2, 4\}$. As above, exactly one of $\{a, 1, 3\}$ and $\{b, 2, 4\}$ is a circuit. Then $\{a, b, c, 1, 3, 4\}$ or $\{a, b, c, 2, 3, 4\}$ has rank 3. As $r(U) = 4$, it follows that 2 or 1 is in $\mathrm{cl}^*(V)$; a contradiction.

Finally, assume that $(A_2 \cap B_1) - c = \{3, 7\}$. Then, without loss of generality, $((U - b) \cap B_1, (U - b) \cap B_2, V \cap B_2, V \cap B_1) = (\{c, 1, 3\}, \{a, 2, 4\}, \{6, 8\}, \{5, 7\})$. Assume that both $(U - a) \cap A_1$ and $(U - a) \cap A_2$ are triads of $M\backslash a$. Then $\{a, b, 1, 2\}$ and $\{a, c, 3, 4\}$ are cocircuits of M. Thus $r(A_1) = 4 = r(A_2)$, so $r(B_1) = 4 = r(B_2)$. Hence both $\{b, c, 1, 3\}$ and $\{a, b, 2, 4\}$ are cocircuits of M. The cocircuits $\{a, b, 1, 2\}$ and $\{a, b, 2, 4\}$ imply that $\{b, 1, 2, 4\}$ contains a cocircuit. By orthogonality, $\{1, 2, 4\}$ is a cocircuit. Thus $(A_1 \cup 4, A_2 - 4) \cong (A_1, A_2)$; a contradiction. We deduce that either (i) $\{a, b, 1, 2\}$ is a cocircuit and $\{c, 3, 4\}$ is a circuit; or (ii) $\{a, c, 3, 4\}$ is a cocircuit and $\{b, 1, 2\}$ is a circuit. Hence $r(A_1) + r(A_2) = 7$, so $r(U) = 4$. Moreover, either (iii) $\{b, c, 1, 3\}$ is a cocircuit and $\{a, 2, 4\}$ is a circuit; or (iv) $\{a, b, 2, 4\}$ is a cocircuit and $\{c, 1, 3\}$ is a circuit. We have already eliminated the possibility of both (i) and (iv) holding. If (i) and (iii) hold, or (ii) and (iv) hold, then $\{a, b, c, 2, 3, 4\}$ or $\{a, b, c, 1, 2, 3\}$ has rank 3. Since $r(U) = 4$, we deduce that 1 or 4 is in $\mathrm{cl}^*(V)$; a contradiction. Hence we may assume that (ii) and (iii) hold. Then $\{a, 3, 4\}$ and $\{b, 1, 3\}$ are triads of $M\backslash c$. Now $a \in C_1$ and $b \in C_2$. Since 3 is in C_1 or C_2, we must have that either $C_1 \cap (U - c) = \{a, 3, 4\}$, or $C_2 \cap (U - c) = \{b, 1, 3\}$, otherwise we can replace (C_1, C_2) by an equivalent 3-separation that does not have five elements on each side. But now the triangles $\{a, 2, 4\}$ and $\{b, 1, 2\}$ imply that (C_1, C_2) is equivalent to $(C_1 \cup 2, C_1 - 2)$ or $(C_1 - 2, C_2 \cup 2)$. This contradiction completes the proof of the theorem. $\square$

Theorem 1.1 is now a straightforward consequence of earlier results.

Proof of Theorem 1.1. By Lemma 4.1, we may assume that $|E(M)| \geqslant 9$. In that case, the theorem follows immediately from Theorem 4.2. $\square$

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