7-1-2013

# Towards a splitter theorem for internally 4-connected binary matroids III

Carolyn Chun
*Victoria University of Wellington*

Dillon Mayhew
*Victoria University of Wellington*

James Oxley
*Louisiana State University*



## Recommended Citation

Chun, C., Mayhew, D., & Oxley, J. (2013). Towards a splitter theorem for internally 4-connected binary matroids III. *Advances in Applied Mathematics, 51* (2), 309-344. https://doi.org/10.1016/j.aam.2013.04.004

Advances in Applied Mathematics 51 (2013) 309–344

Contents lists available at SciVerse ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

# Towards a splitter theorem for internally 4-connected binary matroids III ☆

Carolyn Chun [a,1], Dillon Mayhew [a], James Oxley [b,*]

[a] School of Mathematics, Statistics and Operations Research, Victoria University, Wellington, New Zealand
[b] Mathematics Department, Louisiana State University, Baton Rouge, LA, USA

ARTICLE INFO

*Article history:*
Received 7 June 2012
Accepted 18 March 2013
Available online 8 May 2013

*MSC:*
05B35
05C40

*Keywords:*
Splitter theorem
Binary matroid
Internally 4-connected

ABSTRACT

This paper proves a preliminary step towards a splitter theorem for internally 4-connected binary matroids. In particular, we show that, provided $M$ or $M^*$ is not a cubic Möbius or planar ladder or a certain coextension thereof, an internally 4-connected binary matroid $M$ with an internally 4-connected proper minor $N$ either has a proper internally 4-connected minor $M'$ with an $N$-minor such that $|E(M) - E(M')| \leqslant 3$ or has, up to duality, a triangle $T$ and an element $e$ of $T$ such that $M\backslash e$ has an $N$-minor and has the property that one side of every 3-separation is a fan with at most four elements.



## 1. Introduction

When dealing with matroid connectivity, it is often useful in inductive arguments to be able to remove a small set of elements from a matroid $M$ to obtain a minor $M'$ that maintains the connectivity of $M$. Results that guarantee the existence of such removal sets are referred to as *chain theorems*. Tutte [16] proved that, when $M$ is 2-connected, if $e \in E(M)$, then $M\backslash e$ or $M/e$ is 2-connected. More significantly, when $M$ is 3-connected, Tutte [16] proved the Wheels-and-Whirls Theorem which shows that $M$ has a proper 3-connected minor $M'$ such that $|E(M) - E(M')| = 1$ unless $r(M) \geqslant 3$ and $M$ is a wheel or a whirl. A 3-connected matroid is *internally 4-connected* if, for every 3-separation $(X, Y)$, either $X$ or $Y$ is a triangle or a triad. In [2], we proved a chain theorem for binary internally

☆ The authors were supported by an NSF IRFP Grant, the Marsden Fund, and the National Security Agency, respectively.
* Corresponding author.
*E-mail addresses:* carolyn.chun@brunel.ac.uk (C. Chun), dillon.mayhew@msor.vuw.ac.nz (D. Mayhew), oxley@math.lsu.edu (J. Oxley).
[1] Current address: Department of Mathematical Sciences, Brunel University, Uxbridge, UK.

0196-8858/$ – see front matter 
http://dx.doi.org/10.1016/j.aam.2013.04.004

4-connected matroids showing that such a matroid $M$ has an internally 4-connected proper minor $M'$ with $|E(M) - E(M')| \leqslant 3$ unless $M$ or its dual is the cycle matroid of a planar or Möbius quartic ladder, or a 16-element graphic matroid that is a variant of such a planar ladder.

Seymour's Splitter Theorem [15] extends the Wheels-and-Whirls Theorem for 3-connected matroids by showing that if such a matroid $M$ has a proper 3-connected minor $N$, then $M$ has a proper 3-connected minor $M'$ that has an $N$-minor and satisfies $|E(M) - E(M')| = 1$ unless $r(M) \geqslant 3$ and $M$ is a wheel or a whirl. The current paper is the third in a series whose aim is to extend our chain theorem for binary internally 4-connected matroids to a splitter theorem for such matroids. Our overall goal is to obtain a theorem that says if $M$ and $N$ are internally 4-connected binary matroids, and $M$ has a proper $N$-minor, then $M$ has a minor $M'$ such that $M'$ is internally 4-connected with an $N$-minor, and $M'$ can be produced from $M$ by a bounded number of simple operations.

Johnson and Thomas [8] showed that, even for graphs, a splitter theorem in the internally 4-connected case must take account of some special examples. For $n \geqslant 3$, let $G_{2n+2}$ be the *biwheel* with $2n + 2$ vertices, that is, $G$ consists of a $2n$-cycle $v_1, v_2, \ldots, v_{2n}, v_1$, the *rim*, and two additional *hub* vertices, $u$ and $w$, both of which are adjacent to every $v_i$. Thus the dual of $G_{2n+2}$ is a cubic planar ladder. Let $M$ be the cycle matroid of $G_{2n+2}$ for some $n \geqslant 3$ and let $N$ be the cycle matroid of the graph that is obtained by proceeding around the rim of $G_{2n+2}$ and alternately deleting the edges from the rim vertex to $u$ and to $w$. Both $M$ and $N$ are internally 4-connected but there is no internally 4-connected proper minor of $M$ that has a proper $N$-minor. We can modify $M$ slightly and still see the same phenomenon. Let $G^+_{n+2}$ be obtained from $G_{n+2}$ by adding a new edge $a$ joining the hubs $u$ and $w$. Let $\Delta_{n+1}$ be the binary matroid that is obtained from $M(G^+_{n+2})$ by deleting the edge $v_{n-1}v_n$ and adding the third element on the line spanned by $wv_n$ and $uv_{n-1}$. This new element is also on the line spanned by $uv_n$ and $wv_{n-1}$. For $r \geqslant 3$, Mayhew, Royle, and Whittle [9] call $\Delta_r$ the *rank-r triangular Möbius matroid* and note that $\Delta_r \backslash a$ is the dual of the cycle matroid of a cubic Möbius ladder.

In [3], we proved a splitter theorem when $M$ is a 4-connected binary matroid and $N$ is an internally 4-connected proper minor of $M$. In particular, we showed that, unless $M$ is a certain 16-element non-graphic matroid, we can find an internally 4-connected matroid $M'$ with $|E(M) - E(M')| = 1$ such that $M'$ has an $N$-minor. In view of this result, we are now able to assume that $M$ is an internally 4-connected matroid having a triangle or a triad. But we know nothing about how this triangle or triad relates to the $N$-minors of $M$. Our second step towards the desired splitter theorem was to consider the case when $M$ is internally 4-connected and all triangles and triads of $M$ are retained in $N$. In this case, we have proved [4, Theorem 1.2] the following result.

**Theorem 1.1.** *Let $M$ and $N$ be internally 4-connected binary matroids such that $|E(N)| \geqslant 7$, and $N$ is isomorphic to a proper minor of $M$. Assume that if $T$ is a triangle of $M$ and $e \in T$, then $M \backslash e$ does not have an $N$-minor. Dually, assume that if $T^*$ is a triad of $M$ and $f \in T^*$, then $M/f$ does not have an $N$-minor. Then $M$ has an internally 4-connected minor $M'$ of $M$ such that $M'$ has an $N$-minor and $1 \leqslant |E(M) - E(M')| \leqslant 2$.*

In view of this theorem, we are now able to assume, by replacing $M$ by its dual if necessary, that $M$ has a triangle $T$ that contains an element $e$ for which $M \backslash e$ has an $N$-minor. When we were proving our chain theorem for a binary internally 4-connected matroid $M$, we began by finding a triangle that either formed part of a very special type of substructure of $M$, or that had an element whose deletion satisfied a weaker form of connectivity than internal 4-connectivity (see Theorem 3.1). The only 3-separations allowed in an internally 4-connected matroid have a triangle or a triad on one side. A 3-connected matroid $M$ is $(4, 4, S)$-*connected* if, for every 3-separation $(X, Y)$ of $M$, one of $X$ and $Y$ is a triangle, a triad, or a 4-element fan, that is, a 4-element set $\{x_1, x_2, x_3, x_4\}$ that can be ordered so that $\{x_1, x_2, x_3\}$ is a triangle and $\{x_2, x_3, x_4\}$ is a triad.

The following is the main result of the paper.

**Theorem 1.2.** *Let $M$ be an internally 4-connected binary matroid with an internally 4-connected proper minor $N$ such that $|E(M)| \geqslant 15$ and $|E(N)| \geqslant 6$. Then*

(i) *$M$ has a proper minor $M'$ such that $|E(M) - E(M')| \leqslant 3$ and $M'$ is internally 4-connected with an $N$-minor*; *or*
(ii) *for some $(M_0, N_0)$ in $\{(M, N), (M^*, N^*)\}$, the matroid $M_0$ has a triangle $T$ that contains an element $e$ such that $M_0 \backslash e$ is $(4, 4, S)$-connected having an $N$-minor*; *or*
(iii) *$M$ or $M^*$ is isomorphic to $M(G^+_{r+1})$, $M(G_{r+1})$, $\Delta_r$ or $\Delta_r \backslash z$ for some $r \geqslant 5$.*

To complete the derivation of our desired splitter theorem, we begin by building detailed structure around the triangle $T$ in (ii). We have already completed the next step in this process [5] and observe here that, while it has considerable additional difficulties posed by the need to retain an $N$-minor, this process has much in common with the analysis used to prove the chain theorem [2].

The proof of Theorem 1.2 will be given in Section 6. Before that, we give some basic definitions and preliminary results in Section 2 where we also state a weaker form of our main theorem (Theorem 2.1) that will be very helpful in deriving the main theorem. We begin to work towards the proof of Theorem 2.1 in Section 3, and we prove a major step towards the theorem in Section 4. We complete the proof of Theorem 2.1 in Section 5 before finishing the proof of the main theorem.

## 2. Preliminaries

The matroid terminology used here will follow Oxley [11]. We shall sometimes write $N \preccurlyeq M$ to indicate that $M$ has an $N$-*minor*, that is, a minor isomorphic to the matroid $N$. We will denote by $C_3^2$ the graph that is obtained from $K_3$ by adding a new edge in parallel to each existing edge. A *quad* in a matroid is a 4-element set that is both a circuit and a cocircuit. The property that a circuit and a cocircuit in a matroid cannot have exactly one common element will be referred to as *orthogonality*. It is well known [11, Theorem 9.1.2] that, in a binary matroid, a circuit and cocircuit must meet in an even number of elements.

Let $M$ be a matroid with ground set $E$ and rank function $r$. The *connectivity function* $\lambda_M$ of $M$ is defined on all subsets $X$ of $E$ by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate $\lambda_M$ as $\lambda$. For a positive integer $k$, a subset $X$ or a partition $(X, E - X)$ of $E$ is $k$-*separating* if $\lambda_M(X) \leqslant k - 1$. A $k$-separating partition $(X, E - X)$ is a $k$-*separation* if $|X|, |E - X| \geqslant k$. If $n$ is an integer exceeding one, a matroid is $n$-*connected* if it has no $k$-separations for all $k < n$. This definition has the attractive property that a matroid is $n$-connected if and only if its dual is. Moreover, this matroid definition of $n$-connectivity is relatively compatible with the graph notion of $n$-connectivity when $n$ is 2 or 3. For example, if $G$ is a graph with at least four vertices and with no isolated vertices, $M(G)$ is a 3-connected matroid if and only if $G$ is a 3-connected simple graph. But the link between $n$-connectivity for matroids and graphs breaks down for $n \geqslant 4$. In particular, a 4-connected matroid with at least six elements cannot have a triangle. Hence, for $r \geqslant 3$, neither $M(K_{r+1})$ nor $PG(r - 1, 2)$ is 4-connected. This motivates the consideration of other types of 4-connectivity in which certain 3-separations are allowed. Let $n$ and $k$ be integers with $n \geqslant 3$ and $k \geqslant 2$. A matroid $M$ is $(n, k)$-*connected* if $M$ is $(n - 1)$-connected and, whenever $(X, Y)$ is an $(n - 1)$-separating partition of $E(M)$, either $|X| \leqslant k$ or $|Y| \leqslant k$. In particular, a matroid is $(4, 3)$-connected if and only if it is internally 4-connected. A graph $G$ without isolated vertices is *internally* 4-*connected* if $M(G)$ is internally 4-connected.

A $k$-separating set $X$ or a $k$-separation $(X, E - X)$ is *exact* if $\lambda_M(X) = k - 1$. A $k$-separation $(X, E - X)$ is *minimal* if $|X| = k$ or $|E - X| = k$. It is well known (see, for example, [11, Corollary 8.2.2]) that if $M$ is $k$-connected having $(X, E - X)$ as a $k$-separation with $|X| = k$, then $X$ is a circuit or a cocircuit of $M$. In a matroid $M$, the *local connectivity* $\sqcap_M(X, Y)$ between sets $X$ and $Y$ is $r(X) + r(Y) - r(X \cup Y)$. In particular, $\sqcap_M(X, E(M) - X) = \lambda_M(X) = \lambda_M(E(M) - X)$.

Let $M$ be a matroid. A subset $S$ of $E(M)$ is a *fan* in $M$ if $|S| \geqslant 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of $S$ such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \ldots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads. We call $(s_1, s_2, \ldots, s_n)$ a *fan ordering* of $S$. We will be mainly concerned with 4-element and 5-element fans. For convenience, we shall always view a fan ordering of a 4-element fan as beginning with a triangle and we shall use the term 4-*fan* to refer to both the 4-element fan and such a fan ordering of it. Moreover, we shall use the terms 5-*fan* and 5-*cofan* to refer to the two different types of 5-element fan where the first contains two triangles and the second two triads. The *central* *element* of a 5-cofan is the element that is in both triads. This element will always be the third element in any fan ordering of the 5-cofan. Fans are examples of sequential 3-separating sets in $M$. A subset $X$ of $E(M)$ is *sequential* if it has a *sequential ordering*, that is, an ordering $(x_1, x_2, \dots, x_k)$ such that $\{x_1, x_2, \dots, x_i\}$ is 3-separating for all $i$ in $\{1, 2, \dots, k\}$. It is straightforward to check that, when $M$ is binary, a sequential set with 3, 4, or 5 elements is a fan. A 3-connected matroid $M$ is $(4, k, S)$-*connected* if $M$ is $(4, k)$-connected and, for every 3-separation $(X, Y)$, at least one of $X$ and $Y$ is sequential.

At this point, we introduce yet another form of connectivity. To motivate this, we return to the example in the introduction letting $M$ be the cycle matroid of the biwheel $G_{2n+2}$ and $N$ be the cycle matroid of the graph that is obtained by proceeding around the rim of $G_{2n+2}$ and alternately deleting the edges from the rim vertex to $u$ and to $w$. Each triangle of $M$ has an element whose deletion has an $N$-minor but every such deletion has a 5-fan. We call a 3-connected matroid $(4, 5, S, +)$-*connected* if, whenever it has $(X, Y)$ as a 3-separation, one of $X$ and $Y$ is a triangle or a triad, a 4-fan, or a 5-fan. As a very useful preliminary step towards Theorem 1.2, we shall first prove the following result.

**Theorem 2.1.** *Let $M$ be an internally 4-connected binary matroid with an internally 4-connected proper minor $N$ such that $|E(M)| \geqslant 15$ and $|E(N)| \geqslant 6$. Then*

(i) *$M$ has a proper minor $M'$ such that $|E(M) - E(M')| \leqslant 3$ and $M'$ is internally 4-connected with an $N$-minor; or*
(ii) *for some $(M_0, N_0)$ in $\{(M, N), (M^*, N^*)\}$, the matroid $M_0$ has a triangle $T$ that contains an element $e$ such that $M_0\backslash e$ is $(4, 5, S, +)$-connected having an $N$-minor.*

Although we do not retain internal 4-connectivity in (ii), the example described above means that we cannot expect to do better than get $(4, 5, S, +)$-connectivity. Let $(X, Y)$ be a 3-separation of a 3-connected matroid $M$. We shall frequently be interested in 3-separations that indicate that $M$ is, for example, not internally 4-connected. We call $(X, Y)$ a $(4, k)$-*violator* if $|X|, |Y| \geqslant k + 1$. In this case, we may also refer to $X$ as a $(4, k)$-violator. Similarly, $(X, Y)$ is a $(4, 4, S)$-*violator* if, for each $Z$ in $\{X, Y\}$, either $|Z| \geqslant 5$, or $Z$ is non-sequential. Evidently $M$ is internally 4-connected if and only if it has no $(4, 3)$-violators; and $M$ is $(4, 4, S)$-connected if and only if it has no $(4, 4, S)$-violators. It is well known and easy to check that if $(X, Y)$ is a $(4, 3)$-violator in a 3-connected binary matroid, and $|X| = 4$, then $X$ is either a quad or a 4-fan. If a matroid $M$ is $(4, k)$-connected or $(4, k, S)$-connected, then $M^*$ is, respectively, $(4, k)$-connected or $(4, k, S)$-connected. However, $(4, 5, S, +)$-connectivity allows the presence of 5-fans but not 5-cofans, so a matroid $M$ may be $(4, 5, S, +)$-connected even if $M^*$ is not. A $(4, 5, S, +)$-*violator* is a 3-separation $(X, Y)$ of $M$ such that either $\min\{|X|, |Y|\} \geqslant 6$, or $\min\{|X|, |Y|\} \leqslant 5$ and neither $X$ nor $Y$ is a triangle, a triad, a 4-fan, or a 5-fan.

Johnson and Thomas's [8] work towards finding a splitter theorem for internally 4-connected graphs revealed, using the example given in the introduction, that we can be forced to remove arbitrarily many elements to recover internal 4-connectivity while maintaining a copy of a specified minor. By controlling the presence of biwheels and ladders, Johnson and Thomas [8] were able to prove a type of splitter theorem for internally 4-connected graphs. In their result, each intermediate graph is obtained from its predecessor by removing, via deletion or contraction, at most two edges, and the cycle matroid of each such intermediate graph is $(4, 4)$-connected satisfying some additional constraints.

Geelen and Zhou [6] proved an analogue of Johnson and Thomas's theorem for internally 4-connected binary matroids. Subsequently, Zhou [18] proved a stronger theorem showing that, with the exception of various matroids related to biwheels and ladders, when one begins with an internally 4-connected binary matroid $M$ having an internally 4-connected minor $N$, one can remove at most two elements from $M$ to get a matroid that has an $N$-minor and is $(4, 4)$-connected. Both this theorem and the graph result of Johnson and Thomas have the advantage that, except in known special cases, each step involves removing only one or two elements. But the major disadvantage of each is that, in removing these elements, one may lose internal 4-connectivity. We have already seen that we may be forced to remove arbitrarily many elements to recover internal 4-connectivity while maintaining a copy of a certain minor. Consider a modification of the example given earlier.

Begin with two non-adjacent edges $u_0v_0$ and $u_nv_n$ in a large complete graph. Add disjoint paths $u_0, u_1, \dots, u_n$ and $v_0, v_1, \dots, v_n$ together with the edges $u_1v_1, u_2v_2, \dots, u_{n-1}v_{n-1}$. This produces an internally 4-connected graph $H$. Now add the edges $u_0v_1, u_1v_2, \dots, u_{n-1}v_n$ to produce another internally 4-connected graph $G$. Certainly $H$ is a minor of $G$, but there is no internally 4-connected graph that lies properly between $G$ and $H$ in the minor order. To get a splitter theorem for internally 4-connected matroids, Geelen (private communication) proposed that one should allow, as a single move, the conversion of a quartic ladder into a cubic ladder as occurs when one goes from $G$ to $H$. We know of another related move that will also be required to get the desired theorem.

The paper of Zhou cited above contains three results that will be very useful here. The first is the following lemma [18, Lemma 2.13]. The second is the subsequent lemma, and the third is stated as Lemma 4.1.

**Lemma 2.2.** *Let $N$ be an internally 4-connected minor of a 3-connected binary matroid $M$ with $|E(N)| \geqslant 10$. Let $(X, Y)$ be a 3-separation of $M$ such that $X$ contains a triangle $T$ and $X - T$ is either a triangle or a triad.*

(i) *If $X - T$ is a triangle, then $M\backslash x$ has an $N$-minor for all $x$ in $X$.*
(ii) *If $X - T$ is a triad and $\sqcap_M(T, X - T) = 2$, then $M\backslash t$ has an $N$-minor for all $t$ in $T$.*

The case when $|E(N)| \geqslant 7$ in the next result is implicit in Zhou [18]. We include the proof here for completeness.

**Lemma 2.3.** *Let $M$ be an internally 4-connected binary matroid having a proper internally 4-connected minor $N$ where $|E(M)| \geqslant 15$ and $|E(N)| \geqslant 6$. Then $M$ has a proper internally 4-connected minor $N'$ with an $N$-minor such that $|E(N')| \geqslant 10$.*

**Proof.** The result is immediate if $|E(N)| \geqslant 10$ as we may take $N' = N$. Now assume that $7 \leqslant |E(N)| \leqslant 9$. In that case, by [6, Lemma 2.1], $N$ is isomorphic to one of $F_7, F_7^*, M(K_{3,3})$, or $M^*(K_{3,3})$. By taking duals when necessary, we may assume that $N$ is isomorphic to $F_7$ or $M(K_{3,3})$. Zhou [17] proved that an internally 4-connected binary matroid with a proper $F_7$-minor has a minor isomorphic to one of five internally 4-connected binary matroids each of which has ten or eleven elements. Thus if $N \cong F_7$, then we can find $N'$ with $|E(N')|$ in $\{10, 11\}$. Geelen and Zhou [6] proved that an internally 4-connected binary matroid with a proper $M(K_{3,3})$-minor has a minor isomorphic to one of eight internally 4-connected binary matroids each of which has at least ten and at most fourteen elements. Thus if $N \cong M(K_{3,3})$, then we can find $N'$ with $|E(N')|$ in $\{10, 11, 12, 13, 14\}$. Since $|E(M)| \geqslant 15$, the lemma follows when $7 \leqslant |E(N)| \leqslant 9$.

Finally, suppose that $|E(N)| = 6$. Then $N \cong M(K_4)$. Now every 3-connected binary matroid with at least six elements has an $M(K_4)$-minor. Hence, to prove the result in this case, we need only show that $M$ has an internally 4-connected minor $M'$ with $|E(M')|$ in $\{10, 11, 12, 13, 14\}$. This follows by repeatedly applying the main result of [2]. □

We close this section with one final lemma whose elementary proof is omitted.

**Lemma 2.4.** *Let $(X, Y)$ be a $(4, 3)$-violator of a 3-connected binary matroid $M$ that has no 4-fans. Then neither $X$ nor $Y$ is sequential. Moreover, if some element $x$ of $X$ is in the closure or coclosure of $Y$, then $(X - x, Y \cup x)$ is a $(4, 3)$-violator of $M$.*

## 3. Developing structure

In this section, we develop some more tools that will be needed in the proof of the main theorem.

In [7], Geelen and Zhou introduced the following structure. Let $M$ be an internally 4-connected matroid. A *rotor* with *central triangle* $\{a, b, c\}$ is a 9-tuple $(a, b, c, d, e, T_a, T_c, A, Z)$ such that the following hold:

(i) $E(M) = \{a, b, c, d, e\} \cup T_a \cup T_c \cup A \cup Z$ and $A \cup Z = E(M) - (\{a, b, c, d, e\} \cup T_a \cup T_c)$;
(ii) $a, b, c, d$, and $e$ are distinct, and $T_a$, $T_c$, and $\{a, b, c\}$ are disjoint triangles with $d$ in $T_a$ and $e$ in $T_c$;
(iii) $T_a \cup \{b, e\}$ and $T_c \cup \{b, d\}$ are 3-separating in $M\backslash a$ and $M\backslash c$, respectively;
(iv) $T_a$ and $T_c$ are 2-separating in $M\backslash a, b$ and $M\backslash b, c$, respectively; and
(v) $A$ and $Z$ are disjoint and non-empty, and $T_a \cup a \cup A$ is 3-separating in $M\backslash b$.

We use the following result [2, Theorem 5.1].

**Theorem 3.1.** *Let $T$ be a triangle of an internally 4-connected binary matroid $M$ with $|E(M)| \geqslant 13$. Then either*

(i) *$T$ is the central triangle of a rotor; or*
(ii) *$T$ contains an element $e$ such that $M\backslash e$ is $(4, 4, S)$-connected.*

Let $(X, Y)$ be an exact 3-separation of a simple binary matroid $M$. Since binary matroids are uniquely representable over $GF(2)$, we can view $E(M)$ as a restriction of $PG(r-1, 2)$, where $r = r(M)$. Let $\text{cl}_P$ be the closure operator of $PG(r-1, 2)$. Then

$$r(X \cup Y) + r\big(\text{cl}_P(X) \cap \text{cl}_P(Y)\big) = r(X) + r(Y) = r(M) + 2 = r(X \cup Y) + 2.$$

Thus $\text{cl}_P(X) \cap \text{cl}_P(Y)$ is a line of $PG(r-1, 2)$, that is, a triangle with some element set $\{a, b, c\}$. We call $\{a, b, c\}$ the *guts line* of the 3-separation $(X, Y)$. Let $M$ and $M'$ be matroids such that $E(M) \cap E(M') = \{a, b, c\}$. Suppose that $\{a, b, c\}$ is a triangle of both matroids and that $M'$ is isomorphic to $M(K_4)$. Then $\Delta_{\{a,b,c\}}(M)$ denotes the matroid obtained from $M$ *by performing a* $\Delta$-$Y$ *exchange on* $\{a, b, c\}$, that is, $\Delta_{\{a,b,c\}}(M)$ is obtained by deleting $\{a, b, c\}$ from $P_{\{a,b,c\}}(M', M)$, the generalized parallel connection of $M'$ and $M$ across the triangle $\{a, b, c\}$ [11, p. 449].

**Lemma 3.2.** *Let $(X, Y)$ be an exact 3-separation of a simple, cosimple binary matroid $M$ of rank $r$ and let $N$ be an internally 4-connected minor of $M$ with at least seven elements. Then $\min\{|E(N) \cap X|, |E(N) \cap Y|\} \leqslant 3$. Suppose $|E(N) \cap X| \leqslant 3$ and $\{a, b, c\}$ is the guts line of $(X, Y)$. Then $N$ is isomorphic to a minor of either $PG(r-1, 2)|(Y \cup \{a, b, c\})$ or the matroid obtained from this matroid by performing a $\Delta$-$Y$ exchange on the triangle $\{a, b, c\}$.*

**Proof.** Suppose first that $|X| = 3$. Then $X$ is a triangle or triad in $M$. In the first case, $X = \{a, b, c\}$ and $M = PG(r-1, 2)|(Y \cup \{a, b, c\})$, as required. In the second case, $PG(r-1, 2)|(X \cup \{a, b, c\}) \cong M(K_4)$. Suppose $|X \cap E(N)| = 3$. Then $X$ is a triad of $N$. Moreover, no element of $\{a, b, c\}$ is in $E(N)$, otherwise $N$ contains a 4-element fan; a contradiction. In this case, $N$ is isomorphic to a minor of the matroid obtained from $PG(r-1, 2)|(Y \cup \{a, b, c\})$ by performing a $\Delta$-$Y$ exchange on $\{a, b, c\}$. Hence we may assume that $|X \cap E(N)| < 3$. Then some element $x$ of $X$ is not in $E(N)$. Now $N$ is a minor of $M\backslash x$ or $M/x$. In the former case, if $X - x = \{y, z\}$, then $\{y, z\}$ is a cocircuit of $M\backslash x$, so, without loss of generality, $y$ is not in $E(N)$ and $N$ is a minor of $M\backslash x/y$. Thus we may assume that $N$ is a minor of $M/x$. Then $PG(r-1, 2)|(Y \cup \{a, b, c\})$ has an $N$-minor, as required.

We may now assume that $|X|, |Y| \geqslant 4$. Then, by [11, Propositions 9.3.4 and 11.4.16], letting $PG(r-1, 2)|(E(M) \cup \{a, b, c\}) = M'$, we have that $M'$ is $P_{\{a,b,c\}}(M_X, M_Y)$, the generalized parallel connection of $M_X$ and $M_Y$ across the triangle $\{a, b, c\}$ where $M_X = M'|(X \cup \{a, b, c\})$ and $M_Y = M'|(Y \cup \{a, b, c\})$. Since $N$ is a minor of $M$, it is also a minor of $M'$. Now $|X \cap E(N)| \leqslant 3$. Each element of $X - E(N)$ is deleted or contracted from $M'$ to produce $N$. Let $D$ be the set of such elements that are deleted and $C$ be the set of such elements that are contracted. Then $M'\backslash D = P_{\{a,b,c\}}(M_X\backslash D, M_Y)$. If $\text{cl}_{M_X}(C)$ meets $\{a, b, c\}$, then it is not difficult to see that $N$ is isomorphic to a minor of $M_Y$. Thus we may assume that $\text{cl}_{M_X}(C)$ avoids $\{a, b, c\}$. Then $M'\backslash D/C = P_{\{a,b,c\}}(M_X\backslash D/C, M_Y)$.

Now, $|E(N) \cap E(M_X\backslash D/C)| \leqslant 3$, so $|E(M_X\backslash D/C)| \leqslant 6$. No element of $E(N) \cap E(M_X\backslash D/C)$ is in a 1- or 2-element cocircuit of $M'\backslash D/C$. It follows that either $r(M_X\backslash D/C) = 2$ or $M_X\backslash D/C \cong M(K_4)$. In

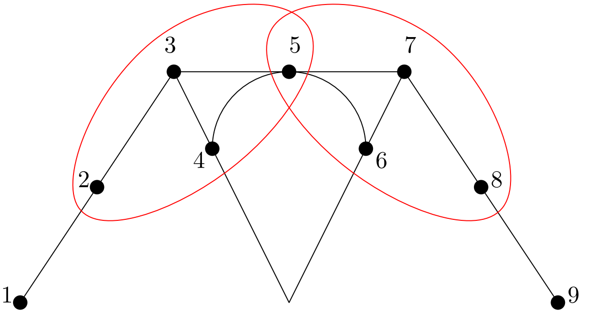


**Fig. 1.** A rotor structure, where $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ are cocircuits.

the first case, $N$ is isomorphic to a minor of $M_Y$. In the second, $N$ is isomorphic to a minor of the matroid that is obtained from $M_Y$ by performing a $\Delta$-$Y$ exchange on $\{a, b, c\}$. □

Let $N$ be an internally 4-connected minor of a simple, cosimple binary matroid $M$ and $(X, Y)$ be an exact 3-separation of $M$ with $|X \cap E(N)| \leqslant 3$. Let $\{a, b, c\}$ be the guts line of $(X, Y)$. By the last result, $N$ is isomorphic to a minor of either $PG(r-1, 2)|(Y \cup \{a, b, c\})$ or the matroid obtained from $PG(r-1, 2)|(Y \cup \{a, b, c\})$ by performing a $\Delta$-$Y$ exchange on $\{a, b, c\}$. In these cases, we say that $N$ is isomorphic to a minor of the matroid obtained by *replacing* $X$ *by a triangle or a triad on the guts line of* $(X, Y)$. We also say that we can get an $N$-minor of the matroid obtained by *putting a triangle or a triad on the guts of* $(X, Y)$.

The next two lemmas establish properties of $M$ when $M$ has a 4-fan or a quad. The first is [4, Lemma 2.2]; the second follows easily from [3, Lemma 2.2].

**Lemma 3.3.** *Let* $(1, 2, 3, 4)$ *be a* 4*-element fan in a binary matroid* $M$ *that has an internally* 4*-connected minor* $N$ *such that* $N$ *has at least eight elements. Then* $M\backslash 1$ *or* $M/4$ *has an* $N$*-minor. Also, if* $(1, 2, 3, 4, 5)$ *is a* 5*-fan in* $M$*, then either* $M\backslash 1, 5$ *has an* $N$*-minor, or both* $M/2\backslash 1$ *and* $M/4\backslash 5$ *have* $N$*-minors. In particular, both* $M\backslash 1$ *and* $M\backslash 5$ *have* $N$*-minors.*

**Lemma 3.4.** *Let* $Q$ *be a quad in a* 3*-connected binary matroid* $M$ *that has an internally* 4*-connected minor* $N$ *such that* $N$ *has at least eight elements. Then either* $M\backslash x$ *has an* $N$*-minor for all* $x$ *in* $Q$*, or* $M/x$ *has an* $N$*-minor for all* $x$ *in* $Q$*. Moreover, if* $M\backslash y$ *has an* $N$*-minor for some* $y$ *in* $Q$*, then* $M\backslash y$ *has an* $N$*-minor for all* $y$ *in* $Q$*; and if* $M/y$ *has an* $N$*-minor for some* $y$ *in* $Q$*, then* $M/y$ *has an* $N$*-minor for all* $y$ *in* $Q$*.*

The next theorem proves a strengthening of the main result in the case that $M$ has a triangle $T$ such that $M\backslash e$ has an $N$-minor for all $e$ in $T$.

**Theorem 3.5.** *Let* $T$ *be a triangle of an internally* 4*-connected binary matroid* $M$ *with* $|E(M)| \geqslant 13$*. Let* $N$ *be an internally* 4*-connected minor of* $M$*. If, for all* $t$ *in* $T$*, the matroid* $M\backslash t$ *has an* $N$*-minor, then either*

(i) *$M$ has an internally* 4*-connected proper minor* $M'$ *with* $|E(M) - E(M')| \leqslant 3$ *such that* $M'$ *has an* $N$*-minor; or*
(ii) *for some element* $e$ *of* $T$*, the matroid* $M\backslash e$ *is* $(4, 4, S)$*-connected having an* $N$*-minor.*

**Proof.** Assume that the theorem fails. Then, by Theorem 3.1, $T$ is the central triangle of a rotor. By [7], this means that the rotor can be labelled as in Fig. 1 where $T = \{4, 5, 6\}$, and both $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ are cocircuits of $M$. We call 5 the *central element* of the rotor. Now, as $M\backslash 4$ has an $N$-minor and has $(1, 2, 3, 5, 7)$ as a fan, it follows from Lemma 3.3 that each of $M\backslash 1$ and $M\backslash 7$ have $N$-minors. By symmetry, so do each of $M\backslash 9$ and $M\backslash 3$. As $M\backslash e$ has an $N$-minor for all $e$ in $\{3, 5, 7\}$, by using Theorem 3.1, we may assume that $\{3, 5, 7\}$ is the central triangle of a rotor. Then $M$ has triangles $X$

and $Y$ where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ such that $X$, $Y$, and $\{3, 5, 7\}$ are disjoint. Moreover, by [3, Lemma 2.10], for some labelling $\{a, b, c\}$ of $\{3, 5, 7\}$, we have $\{x_2, x_3, a, b\}$ and $\{b, c, y_1, y_2\}$ as cocircuits, while $\{x_3, b, y_1\}$ is a triangle.

The following is shown in [2, Lemma 6.4].

**3.5.1.** *The only triangles of M containing* 5 *are* $\{4, 5, 6\}$ *and* $\{3, 5, 7\}$, *while the only* 4*-cocircuits of M contained in* $\{1, 2, \dots, 9\}$ *are* $\{2, 3, 4, 5\}$ *and* $\{5, 6, 7, 8\}$.

We show next that

**3.5.2.** $b = 5$.

Assume $b \neq 5$. Then we may assume that $(a, b, c) = (5, 3, 7)$. Applying 3.5.1 to the rotor with central triangle $\{5, 3, 7\}$ and central element 3 gives that the only triangles containing 3 are $\{5, 3, 7\}$ and $\{x_3, 3, y_1\}$. But $\{1, 2, 3\}$ is a triangle, so $(x_3, y_1)$ is either $(1, 2)$ or $(2, 1)$. In the first case, $\{x_2, 1, 3, 5\}$ is a cocircuit of $M$. By orthogonality with the triangle $\{4, 5, 6\}$, we see that $x_2 \in \{4, 6\}$, so $M$ has a 4-cocircuit contained in $\{1, 2, \dots, 9\}$ other than $\{2, 3, 4, 5\}$ or $\{5, 6, 7, 8\}$; a contradiction. We may now assume that $(x_3, y_1) = (2, 1)$. Then the cocircuit $\{1, 3, 7, y_2\}$ implies, by orthogonality with the triangle $\{7, 8, 9\}$, that $y_2 \in \{8, 9\}$. Again we get a 4-cocircuit contained in $\{1, 2, \dots, 9\}$ other than $\{2, 3, 4, 5\}$ or $\{5, 6, 7, 8\}$. Hence 3.5.2 holds.

Next we show the following.

**3.5.3.** *M has triangles* $\{2, 4, 11\}$ *and* $\{6, 8, 10\}$ *such that* $|\{1, 2, \dots, 11\}| = 11$.

By 3.5.1, we may assume that $x_3 = 4$ and $y_1 = 6$. Then the cocircuit $\{x_2, x_3, a, b\}$ is $\{x_2, 4, a, 5\}$. By orthogonality, this cocircuit contains 3 or 7. In the latter case, by orthogonality again, it also contains 8 or 9, and we have a 4-cocircuit contained in $\{1, 2, \dots, 9\}$ other than $\{2, 3, 4, 5\}$ or $\{5, 6, 7, 8\}$. We deduce that $(x_2, x_3, a, b)$ is $(2, 4, 3, 5)$ or $(3, 4, 2, 5)$. But $a \in \{3, 5, 7\}$, so $(x_2, x_3, a, b) = (2, 4, 3, 5)$. Thus $c = 7$, so $(b, c, y_1, y_2) = (5, 7, 6, 8)$. Hence $M$ has disjoint triangles $\{2, 4, 11\}$ and $\{6, 8, 10\}$, neither of which meets $\{3, 5, 7\}$. Thus $|\{1, 2, \dots, 11\}| = 11$ unless $\{10, 11\}$ meets $\{1, 9\}$. As $\{2, 4, 11\}$ and $\{1, 2, 3\}$ are triangles, $11 \neq 1$. By symmetry, it suffices to show that $11 \neq 9$. If $11 = 9$, then $\{1, 2, \dots, 9\}$ is spanned by $\{2, 3, 4, 5\}$, and so $\lambda(\{1, 2, \dots, 9\}) \leqslant 2$; a contradiction. We conclude that 3.5.3 holds.

**3.5.4.** *Both* $M\backslash 1/2$ *and* $M\backslash 3, 4/5$ *have N-minors.*

Assume first that $M/4$ has no $N$-minor. As $M\backslash 5$ has an $N$-minor having $(1, 3, 2, 4, 11)$ as a 5-fan, and $M/4$ has no $N$-minor, $M\backslash 5/4$ has no $N$-minor, so, by Lemma 3.3, $M\backslash 5\backslash\{1, 11\}$ has an $N$-minor. Now we may assume that $M\backslash 1$ is not internally 4-connected otherwise (i) holds. Thus, by [2, Lemma 6.5], $M$ has a 4-cocircuit $C^*$ meeting $\{1, 2, \dots, 9\}$ in $\{1, 2\}$. The triangle $\{2, 4, 11\}$ implies that $11 \in C^*$. Thus $M\backslash 5\backslash\{1, 11\}$ has a 2-cocircuit $\{2, z\}$ where $C^* = \{1, 2, 11, z\}$. Hence $M\backslash\{5, 1, 11\}/2$ has an $N$-minor and therefore so do each of $M\backslash 1/2$ and $M/2$; hence $M/2\backslash 3, 4$ does also. But $M\backslash 3, 4$ has $\{2, 5\}$ as a cocircuit. Thus $M\backslash 3, 4/5$ has an $N$-minor. Hence 3.5.4 holds when $M/4$ has no $N$-minor.

Now suppose that $M/4$ does have an $N$-minor. Clearly 2 and 5 are in distinct parallel classes of $M/4$. Hence $M/4\backslash 2, 5$ has an $N$-minor. But $M/4\backslash 2, 5$ has $(3, 6, 7, 8, 9)$ as a 5-fan, so, by Lemma 3.3, $M/4\backslash 2, 5, 3$ has an $N$-minor. Thus $M\backslash 2, 3$ has an $N$-minor and, as $\{4, 5\}$ is a cocircuit in this matroid, $M\backslash 2, 3/5$ has an $N$-minor. Thus so do $M/5$ and $M/5\backslash 3, 4$. As $M\backslash 3, 4/5 \cong M\backslash 3, 4/2$, we deduce that $M/2$ has an $N$-minor and so does $M/2\backslash 1$. We conclude that 3.5.4 holds.

By [2, Theorem 6.1], one of $M\backslash 1$, $M\backslash 9$, $M\backslash 1/2$, $M\backslash 9/8$, or $M\backslash 3, 4/5$ is internally 4-connected. By 3.5.4 and symmetry, each of these five matroids has an $N$-minor. Thus the theorem holds. □

The first part of the next lemma is in [13, Lemma 6.1], so we omit the proof. The second part will be used repeatedly throughout the rest of the paper. In particular, we shall need the two corollaries of the lemma that are proved following it.

**Lemma 3.6.** *Let $\{e, f, g\}$ be a triangle in an internally 4-connected binary matroid $M$ having at least eight elements. Then*

(i) *$M\backslash e$ is 3-connected; and*
(ii) *if $(X, Y)$ is a 2-separation of $M\backslash e, f$ with $g$ in $Y$, then $|X| \leqslant 3$; in particular, $M\backslash e, f$ is $(3, 3)$-connected.*

**Proof.** Let $(X, Y)$ be a 2-separation of $M\backslash e, f$ with $g$ in $Y$ and $|X| \geqslant 4$. Then $(X, Y \cup f)$ is a 3-separation of $M\backslash e$. Hence $(X, Y \cup \{e, f\})$ is a 3-separation of $M$, contradicting the fact that $M$ is internally 4-connected. We conclude that $|X| \leqslant 3$ and (ii) holds. □

**Corollary 3.7.** *Let $\{e, f, g\}$ be a triangle in an internally 4-connected binary matroid $M$ having at least eight elements. Let $X$ be a 2-separating set of $M\backslash e, f$ such that $|X| \geqslant 2$ and $g \notin X$. Then either $X$ is a 2-cocircuit of $M\backslash e, f$ and $X \cup \{e, f\}$ is a cocircuit of $M$, or $X$ is a triangle and, for some $\{x_1, x_2\} \subseteq X$, the set $\{x_1, x_2\} \cup \{e, f\}$ is a cocircuit of $M$.*

**Proof.** By Lemma 3.6, $|X| \leqslant 3$. If $|X| = 2$, then, as $M$ is simple, $X$ is a 2-cocircuit of $M\backslash e, f$; and, as $M$ is internally 4-connected, $X \cup \{e, f\}$ is a cocircuit of $M$.

Now let $|X| = 3$. Then $r(X) + r^*_{M\backslash e,f}(X) - 3 = 1$. If $r(X) = 3$, then $r^*_{M\backslash e,f}(X) = 1$. But $M\backslash e$ is 3-connected and binary, so $M\backslash e, f$ has no series classes of size more than two. Thus $r(X) = 2$, so $r^*_{M\backslash e,f}(X) = 2$. Hence $X$ is a triangle. Since $M$ is binary, $X$ is not a triad. Thus $X$ contains a 2-cocircuit, and the corollary holds. □

**Corollary 3.8.** *Let $\{e, f, g\}$ be a triangle in an internally 4-connected binary matroid $M$ having at least eight elements. Then $\mathrm{si}(\mathrm{co}(M\backslash e, f))$ is 3-connected and no parallel class of $\mathrm{co}(M\backslash e, f)$ has more than two elements.*

**Proof.** The fact that $\mathrm{si}(\mathrm{co}(M\backslash e, f))$ is 3-connected is an immediate consequence of the last corollary. Now assume that $\mathrm{co}(M\backslash e, f)$ has a parallel class of size at least three. Then $M\backslash e, f$ has triangles $\{1, 2, 3\}$ and $\{1, 4, 5\}$ such that $\{2, 3\}$ and $\{4, 5\}$ are cocircuits. Then $\{2, 3, e, f\}$ and $\{4, 5, e, f\}$ are cocircuits of $M$. Hence so is $\{2, 3, 4, 5\}$. But $\{2, 3, 4, 5\}$ is also a circuit of $M$, so $M$ has a quad, contradicting the fact that $M$ is internally 4-connected. □

The next lemma will be used frequently.

**Lemma 3.9.** *Let $e$ be an element of an internally 4-connected matroid $M$.*

(i) *If $(U, V)$ is a $(4, k)$-violator of $M\backslash e$ for some $k \geqslant 3$ and $C$ is a circuit of $M$ containing $e$, then $C$ meets both $U$ and $V$.*
(ii) *If $(U, V)$ is a $(4, 4)$-violator or a $(4, 4, S)$-violator of $M\backslash e$ and $Z$ is a circuit or a cocircuit of $M\backslash e$ such that $V \cup Z$ spans $e$, then $|Z \cap U| \geqslant 2$.*

**Proof.** For (i), suppose $C - e \subseteq U$. Then $e \in \mathrm{cl}(U)$ and $(U \cup e, V)$ is a $(4, k)$-violator of $M$; a contradiction. Thus (i) holds. For (ii), observe first that $U$ must meet $Z$ otherwise $(U, V \cup e)$ is a $(4, 3)$-violator of $M$; a contradiction. Now either $U$ is a quad, or $|U| \geqslant 5$. If $U$ is a quad, then, by orthogonality, $|Z \cap U| \geqslant 2$ as desired. Thus we may assume that $|U| \geqslant 5$. Suppose that $U$ contains a single element, say $z$, of $Z$. Then $z$ is in the closure or coclosure of $V$ in $M\backslash e$. Hence $(U - z, V \cup z \cup e)$ is a $(4, 3)$-violator of $M$; a contradiction. We conclude that $|Z \cap U| \geqslant 2$. □

Next we prove a lemma that extracts some common features from two of the longer proofs in the paper, those of Lemma 4.3 and Theorem 2.1.

**Lemma 3.10.** *Let $\{e, f, g\}$ be a triangle of an internally 4-connected matroid $M$. Let $(X_e, Y_e)$ and $(X_f, Y_f)$ be 3-separations of $M\backslash e$ and $M\backslash f$, respectively, where $f \in X_e$ and $e \in X_f$. Suppose that $\min\{|X_e|, |Y_e|, |X_f|, |Y_f|\} \geqslant 4$. Then the following hold:*

(i) $g \in Y_e \cap Y_f$;
(ii) *either* $e \in \mathrm{cl}(X_f - e)$, *or* $X_f$ *is a* 4*-element fan of* $M\backslash f$;
(iii) *either* $f \in \mathrm{cl}(X_e - f)$, *or* $X_e$ *is a* 4*-element fan of* $M\backslash e$;
(iv) *if* $X_f$ *is not a* 4*-element fan of* $M\backslash f$ *and* $X_e$ *is not a* 4*-element fan of* $M\backslash e$, *then*
   (a) $e \in \mathrm{cl}(X_f - e)$ *and* $f \in \mathrm{cl}(X_e - f)$;
   (b) $\lambda_M(X_e \cap Y_f) + \lambda_M(X_f \cap Y_e) \leqslant 4$;
   (c) $\lambda_{M\backslash e,f}(X_e \cap X_f) + \lambda_M(Y_e \cap Y_f) \leqslant 4$; *and*
   (d) $|Y_e \cap Y_f| \geqslant 2$ *unless* $Y_e$ *is a* 4*-element fan of* $M\backslash e$ *and* $Y_f$ *is a* 4*-element fan of* $M\backslash f$.

**Proof.** Part (i) is an immediate consequence of Lemma 3.9(i). To prove (ii), assume that $e \notin \mathrm{cl}(X_f - e)$. Then $(X_f - e, Y_f \cup e)$ is a 3-separation of $M\backslash f$. But $f \in \mathrm{cl}(Y_f \cup e)$, so $(X_f - e, Y_f \cup e \cup f)$ is a 3-separation of $M$. As $M$ is internally 4-connected, it follows that $|X_f - e| = 3$. Hence $X_f$ is a 4-element sequential 3-separating set so $X_f$ is a 4-element fan. Thus (ii) holds. Hence, by symmetry, so does (iii).

Part (iv)(a) is immediate from (ii) and (iii). Now

$$2 = \lambda_{M\backslash f}(X_f) = \lambda_{M\backslash f,e}(X_f - e). \tag{1}$$

Likewise, $\lambda_{M\backslash e,f}(Y_e) = 2$. Thus

$$2 + 2 = \lambda_{M\backslash e,f}(X_f - e) + \lambda_{M\backslash e,f}(Y_e) \geqslant \lambda_{M\backslash e,f}\big((X_f - e) \cup Y_e\big) + \lambda_{M\backslash e,f}\big((X_f - e) \cap Y_e\big).$$

Hence

$$4 \geqslant \lambda_{M\backslash e,f}(Y_f \cap X_e) + \lambda_{M\backslash e,f}(X_f \cap Y_e). \tag{2}$$

As $Y_f \cap X_e$ avoids $X_f - e$, and $e \in \mathrm{cl}(X_f - e)$, we have

$$\lambda_{M\backslash e,f}(Y_f \cap X_e) = \lambda_{M\backslash f}(Y_f \cap X_e) = \lambda_M(Y_f \cap X_e)$$

where the last step follows as $\{e, g\} \subseteq E(M\backslash f) - (Y_f \cap X_e)$. Therefore, by (2) and symmetry,

$$4 \geqslant \lambda_M(Y_f \cap X_e) + \lambda_M(X_f \cap Y_e) = \lambda_{M\backslash f}(Y_f \cap X_e) + \lambda_{M\backslash e}(X_f \cap Y_e). \tag{3}$$

Hence (iv)(b) holds.

To prove (iv)(d), suppose $|Y_e \cap Y_f| < 2$. Then, by (i), $Y_e \cap Y_f = \{g\}$. Thus $\min\{|X_e \cap Y_f|, |X_f \cap Y_e|\} \geqslant 3$. Hence, by (3), $\lambda_{M\backslash f}(Y_f \cap X_e) = 2 = \lambda_{M\backslash e}(X_f \cap Y_e)$, so $\lambda_{M\backslash f}(Y_f \cap X_e) = \lambda_{M\backslash f}(Y_f)$. But $Y_f - (Y_f \cap X_e) = \{g\}$. Thus $g \in \mathrm{cl}(Y_f \cap X_e)$ or $g \in \mathrm{cl}^*{}_{M\backslash f}(Y_f \cap X_e)$. The first possibility gives the contradiction that $e \in \mathrm{cl}(X_e)$ since $f \in X_e$. Thus $g \in \mathrm{cl}^*{}_{M\backslash f}(Y_f \cap X_e)$ so $g \in \mathrm{cl}^*{}_M((X_e \cap Y_f) \cup f)$. Hence $g \in \mathrm{cl}^*{}_M(X_e)$ so $g \in \mathrm{cl}^*{}_{M\backslash e}(X_e)$. Thus $(X_e \cup g, Y_e - g)$ is a 3-separation of $M\backslash e$, so $(X_e \cup g \cup e, Y_e - g)$ is a 3-separation of $M$. Hence $|Y_e - g| = 3$. Thus $Y_e - g$ is a triangle or a triad of $M$ and hence of $M\backslash e$. As $g \in \mathrm{cl}^*{}_{M\backslash e}(Y_e - g)$ and $M$ is binary, we deduce that $Y_e - g$ is a triangle of $M\backslash e$, and $Y_e$ is a 4-element fan of $M\backslash e$. By symmetry, $Y_f$ is a 4-element fan of $M\backslash f$, and (iv)(d) holds.

Next, we note that, by (1),

$$\begin{aligned} 2 + 2 &= \lambda_{M\backslash e,f}(X_e - f) + \lambda_{M\backslash e,f}(X_f - e) \\ &\geqslant \lambda_{M\backslash e,f}(X_e \cap X_f) + \lambda_{M\backslash e,f}\big((X_e - f) \cup (X_f - e)\big) \\ &= \lambda_{M\backslash e,f}(X_e \cap X_f) + \lambda_M(X_e \cup X_f). \end{aligned}$$

Thus (iv)(c) holds. $\square$

In the next lemma, we will assume the following.

**Hypothesis I.** *No triangle of $M$ contains two elements $e$ and $f$ such that $M\backslash e$ and $M\backslash f$ each have an $N$-minor, and no triad of $M$ contains two elements $e'$ and $f'$ such that $M/e'$ and $M/f'$ each have an $N$-minor.*

Recall that the matroid $M(C_3^2)$ is obtained from a triangle by adding a new element in parallel to each existing element.

**Lemma 3.11.** *Let $M$ be a binary internally 4-connected matroid and $N$ be an internally 4-connected proper minor of $M$ with at least eight elements. If Hypothesis I holds, then either*

(i) *$M$ has a triangle $T$ such that $M\backslash e$ is $(4, 5, S, +)$-connected with an $N$-minor for some $e \in T$; or*
(ii) *$M^*$ has a triangle $T$ such that $M^*\backslash e$ is $(4, 5, S, +)$-connected with an $N^*$-minor for some $e \in T$; or*
(iii) *$M$ has an internally 4-connected minor $M'$ having an $N$-minor such that $|E(M) - E(M')| \leqslant 3$.*

**Proof.** By Theorem 1.1 and duality, we may assume that $M$ has a triangle $\{e, f, g\}$ such that $M\backslash e$ has an $N$-minor. We may also assume that $M\backslash e$ is not $(4, 5, S, +)$-connected, so $M\backslash e$ has a $(4, 5, S, +)$-violator $(X, Y)$. Then $|X|, |Y| \geqslant 4$, and neither $X$ nor $Y$ is a 4-fan or a 5-fan, although either may be a 5-cofan. Since $e$ is in neither $\mathrm{cl}(X)$ nor $\mathrm{cl}(Y)$, we may assume that $f \in X$ and $g \in Y$. Let $\{a, b, c\}$ be the guts line of $(X, Y)$. Then $M\backslash e = P_{\{a,b,c\}}(M_X, M_Y)|E(M\backslash e)$ where $M_X = PG(r-1, 2)|(X \cup \{a, b, c\})$ and $M_Y = PG(r-1, 2)|(Y \cup \{a, b, c\})$. Note that $f \in E(M_X) - \{a, b, c\}$ and $g \in E(M_Y) - \{a, b, c\}$. As $(E(N) \cap X, E(N) \cap Y)$ is not a $(4, 3)$-violator of $N$, we may also assume that $|E(N) \cap X| \leqslant 3$. Thus, by Lemma 3.2, $M_Y$ or $\Delta_{\{a,b,c\}}(M_Y)$ has an $N$-minor. As $M$ is internally 4-connected, it is easily shown [15, (4.3)] that $M_X$ is 3-connected.

We show first that

**3.11.1.** *$M_X$ is graphic.*

Suppose $M_X$ is not graphic. Then, by Asano, Nishizeki, and Seymour [1], $M_X$ has a minor $M'_X$ isomorphic to $F_7$ or $M^*(K_{3,3})$ that uses $\{a, b, c\}$. Suppose $f \in E(M'_X)$. Then $M'_X/f$ has an $M(C_3^2)$-minor using the triangle $\{a, b, c\}$. Moreover, $M'_X\backslash f$ has an $M(K_4)$-minor using the triangle $\{a, b, c\}$. Since $M_Y$ or $\Delta_{\{a,b,c\}}(M_Y)$ has an $N$-minor, we deduce that $M\backslash e\backslash f$ or $M\backslash e/f$ has an $N$-minor. Indeed, this assertion holds in general since it also holds when $f \notin E(M'_X)$. Thus the triangle $\{e, f, g\}$ of $M$ contains distinct elements $x$ and $y$ such that $M\backslash x$ and $M\backslash y$ have $N$-minors. This contradiction to Hypothesis I completes the proof of 3.11.1.

Next we show that

**3.11.2.** $\mathrm{co}(M_X\backslash f)$ *is 3-connected up to parallel classes of size 2. Moreover, $\{a, b, c\}$ is a triangle in* $\mathrm{co}(M_X\backslash f)$.

Let $(U, V)$ be a 2-separation of $M_X\backslash f$. We may assume that $|U \cap \{a, b, c\}| \geqslant 2$. Then $(U \cup \{a, b, c\}, V - \{a, b, c\})$ is a 2-separation of $M_X\backslash f$. It follows easily that $((U \cup E(M_Y)) \cap E(M), V - \{a, b, c\})$ is a 2-separation in $M\backslash e, f$. Since $g \notin V - \{a, b, c\}$, Corollary 3.7 shows that either $V - \{a, b, c\}$ is a 2-cocircuit in $M\backslash e, f$, or $V - \{a, b, c\}$ is a triangle that contains a 2-cocircuit. Using this, it is not difficult to show that one side of every 2-separation of $M_X\backslash f$ is either a 2-cocircuit, or a triangle that contains a 2-cocircuit. This implies that $\mathrm{co}(M_X\backslash f)$ is 3-connected up to parallel classes of size 2.

Suppose that $\{a, b, c\}$ is not a triangle in $\mathrm{co}(M_X\backslash f)$. Then it contains a 2-cocircuit in $M_X\backslash f$. Thus $M_X$ has a triad that contains $f$ and two elements of $\{a, b, c\}$. By possibly relabelling, we may assume that $\{a, b, f\}$ is a triad in $M_X$. As $M_Y$ is connected, it has a cocircuit that contains $\{a, b\}$, and therefore avoids $c$. Hence $M_Y$ has a hyperplane that meets $\{a, b, c\}$ in $\{c\}$. The union of this hyperplane with $E(M_X) - \{a, b, f\}$ is a hyperplane of $P_{\{a,b,c\}}(M_X, M_Y)$. Thus there is a cocircuit of $P_{\{a,b,c\}}(M_X, M_Y)$ contained in $E(M_Y) \cup f$ that contains $\{a, b, f\}$. Assume that $|E(M_X) - \{a, b, c\}| \geqslant 5$. Then $(E(M_X) - \{a, b, c, f\}, E(M_Y) \cup f)$ is a 3-separation of $P_{\{a,b,c\}}(M_X, M_Y)$, and $(X - \{a, b, c, f\}, Y \cup (E(M) \cap \{a, b, c, f\}))$ is a 3-separation in $M\backslash e$. As $Y$ contains $g$, it follows

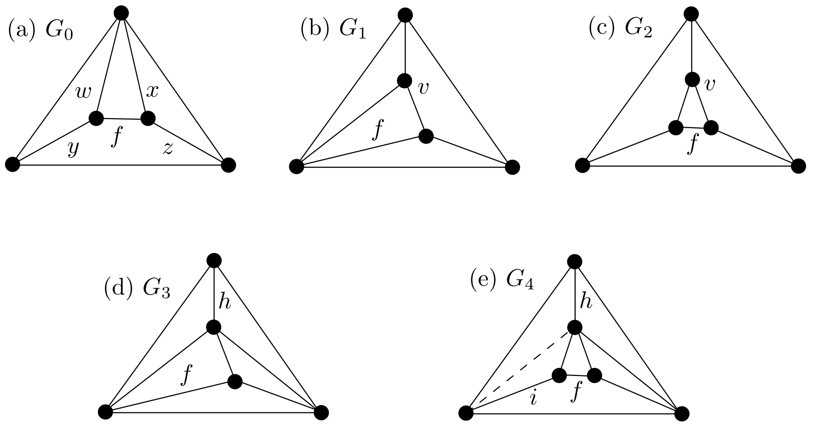


**Fig. 2.** Five graphs used in the proof of 3.11.3.

that $(X - \{a, b, c, f\}, Y \cup (E(M) \cap \{a, b, c, e, f\}))$ is a (4, 3)-violator of $M$. This contradiction shows that $|E(M_X) - \{a, b, c\}| \leqslant 4$. Hence $|E(M_X)| \leqslant 7$, so $M_X$ is isomorphic to $F_7$ or $M(K_4)$. But $M_X$ is graphic. Thus $M_X$ is isomorphic to $M(K_4)$, and $X = E(M_X)$, otherwise $(X, Y)$ is not a $(4, 5, S, +)$-violator in $M\backslash e$. If $\Delta_{\{a,b,c\}}(M_Y)$ has an $N$-minor, then deleting any element of $\{a, b, c\}$ from $M$ produces a matroid with an $N$-minor, contradicting Hypothesis I. Therefore $N \preccurlyeq M_Y$, so $N \preccurlyeq M/f$. Hence both $M\backslash e$ and $M\backslash g$ have $N$-minors. This contradiction shows that 3.11.2 holds.

Let $G$ be a graph such that $M_X = M(G)$. We show next that $M$ has elements $x, y, w$, and $z$ such that if $G_0$ is the graph shown in Fig. 2(a), where the edges of the outside face are labelled $a$, $b$, and $c$, then

**3.11.3.** $M_X = M(G_0)$.

Let $H$ denote the graph obtained from $G\backslash f$ by suppressing degree-2 vertices. Thus $M(H) = \text{co}(M_X\backslash f)$. By 3.11.2, the parallel classes in $H$ have at most two edges. Moreover, $H$ has at most two non-trivial parallel classes as $G$ is simple.

As $\text{co}(M_X\backslash f)$ contains the triangle $\{a, b, c\}$, its rank is at least two. Suppose first that $r(\text{co}(M_X\backslash f)) \geqslant 3$. By 3.11.2, $\text{si}(\text{co}(M_X\backslash f))$ is 3-connected. Since the last matroid has rank at least three, it also has corank at least three. From [10, Corollary 3.7], we see that $\text{si}(\text{co}(M_X\backslash f))$, and hence $\text{co}(M_X\backslash f)$, has an $M(K_4)$-minor using the triangle $\{a, b, c\}$. By Hypothesis I, $M\backslash f$ has no $N$-minor, so $\Delta_{\{a,b,c\}}(M_Y)$ has no $N$-minor. Thus $N \preccurlyeq M_Y$.

Suppose that $\text{si}(\text{co}(M_X\backslash f))$ has at least seven elements. As $\text{si}(\text{co}(M_X\backslash f))$ is 3-connected, [15, (4.1)] implies that $\text{si}(\text{co}(M_X\backslash f))$, and hence $M_X\backslash f$, has an $M(C_3^2)$-minor in which $\{a, b, c\}$ is a triangle. Now it follows easily that $M\backslash e, f$ has a minor isomorphic to $M_Y$. Thus $M\backslash f$ has an $N$-minor, which contradicts Hypothesis I. Therefore $\text{si}(\text{co}(M_X\backslash f))$ is a 3-connected binary matroid with rank at least three, and at most six elements. Hence $\text{si}(\text{co}(M_X\backslash f))$ is isomorphic to $M(K_4)$. Thus the graph $H$ is obtained from $K_4$ by possibly adding parallel edges.

Assume that $a$, $b$, or $c$ is in a non-trivial parallel class in $H$. Then $\text{co}(M_X\backslash f)$, and hence $M_X\backslash f$, has an $M(C_3^2)$-minor in which $\{a, b, c\}$ is a triangle. This implies that $M\backslash e, f$ has a minor isomorphic to $M_Y$, and so has a minor isomorphic to $N$. As this violates Hypothesis I, we deduce that none of $a$, $b$, and $c$ is in a non-trivial parallel class in $H$. If $H$ is simple, then $G$ is one of the graphs $G_1$ or $G_2$, shown in Fig. 2 where, as with $G_0$, the edges in the outside face are labelled with $a$, $b$, and $c$. As $H$ has at most two non-trivial parallel classes, if $H$ is non-simple, then $G$ is either $G_3$ or $G_4$, where the dashed edge in $G_4$ may be either present or absent.

If $G$ is $G_1$ or $G_2$, the set $T^*$ of edges that are incident with the vertex $v$ is a triad of $M\backslash e$ that meets a triangle of $M\backslash e$. As this triad does not contain $f$ or $g$, it follows, by orthogonality, that $T^*$ is a triad in $M$. Thus $M$ has a 4-element fan; a contradiction. Now suppose that $H$ is isomorphic to

$G_3$ or $G_4$. Then each of $G_3/h/f$ and $G_4/h/i/f$ has $\{a,b,c\}$ as a triangle and has edges $a', b'$, and $c'$ parallel to $a, b$, and $c$, respectively. Then $M\backslash e/f$ has a minor isomorphic to $M_Y$, so $N \preccurlyeq M/f$. Thus $N \preccurlyeq M\backslash e$ and $N \preccurlyeq M\backslash g$, contradicting Hypothesis I. We conclude that $r(\mathrm{co}(M_X\backslash f)) \ngeq 3$.

We now know that $r(\mathrm{co}(M_X\backslash f)) = 2$. Then, as $M(H) = \mathrm{co}(M_X\backslash f)$, it follows by 3.11.2 that $H$ is obtained from the triangle $\{a,b,c\}$ by adding parallel edges, and no vertex of $H$ has degree two. It follows easily that $G = G_0$, that is, 3.11.3 holds.

Next we show the following.

**3.11.4.** *$N \preccurlyeq M_Y$, no element of $\{a,b,c\}$ is in $E(M)$, and all of $\{f,w,y,e\}$, $\{f,x,z,e\}$, and $\{w,x,y,z\}$ are cocircuits of $M$. Moreover, both $M\backslash e/y$ and $M/y$ are 3-connected.*

Since $M_X = M(G_0)$, if $N \preccurlyeq \Delta_{\{a,b,c\}}(M_Y)$, then, as $K_4 \preccurlyeq G_0/f$, we have that $N \preccurlyeq M/f$, so $N \preccurlyeq M\backslash g$; a contradiction. Thus $N \preccurlyeq M_Y$. Suppose $E(M)$ meets $\{a,b,c\}$. Then $M_X/f$ has a rank-2 minor that uses the triangle $\{a,b,c\}$ and contains a triangle all of whose elements are in $E(M) - \{e,f\}$. Thus $N \preccurlyeq M/f$, so $M\backslash g$ has an $N$-minor; a contradiction. It follows that $E(M)$ avoids $\{a,b,c\}$. Since $\{f,x,w\}$ is not in a 4-element fan of $M$, we deduce that $\{f,w,y,e\}$ and $\{f,x,z,e\}$ are cocircuits of $M$. Hence $\{w,x,y,z\}$ is a cocircuit of $M$.

As $(y,w,f,x,z)$ is a 5-cofan in $M\backslash e$, the dual of Lemma 3.3 implies that $M/y$ has an $N$-minor. Now $(y,w,f,x,z)$ is a maximal fan in $M\backslash e$ as $E(M) \cap \{a,b,c\} = \emptyset$. Hence $M\backslash e/y$ is 3-connected. Thus $M/y$ is 3-connected, otherwise $\{e,y\}$ is in a triangle of $M$ and we contradict Hypothesis I since $N \preccurlyeq M/y$. Hence 3.11.4 holds.

We shall assume that $M/y$ is not internally 4-connected, otherwise (iii) holds. We now show that

**3.11.5.** *$M/y$ has a triad $\{i,j,k\}$ such that $\{w,y,i,j\}$ is a circuit in $M$.*

Assume that 3.11.5 fails. Suppose that $M/y$ has a 4-fan, $(x_1,x_2,x_3,x_4)$. Then $\{y,x_1,x_2,x_3\}$ is a circuit of $M$. By orthogonality with $\{e,f,w,y\}$, we see that $\{x_1,x_2,x_3\}$ and $\{e,f,w\}$ intersect in exactly one element. In fact, as $e$, $f$, and $w$ are all in triangles of $M$, none is contained in any triads of $M$. Therefore $x_1$ is in $\{e,f,w\}$. Assume that $x_1 \in \{e,f\}$. Then orthogonality between $\{y,x_1,x_2,x_3\}$ and $\{w,x,y,z\}$ requires that $x_2$ or $x_3$ is equal to $x$ or $z$. As $x$ is in a triangle of $M$, and is therefore in no triad, it follows that $z \in \{x_2,x_3\}$. By symmetry, we may assume that $z = x_2$. Note that $M_X/y,z$ is isomorphic to $M(C_3^2)$, and $\{a,b,c\}$ is a triangle in this minor. As $N \preccurlyeq M_Y$, it follows that $M\backslash e/y,z$, and hence $M/y,z$, has an $N$-minor. As $\{x_1,x_3\}$ is a circuit of $M/y,z$, we see that $M/y,z\backslash x_3$, and hence $M\backslash x_3$, has an $N$-minor. But $\{x_2,x_4\}$ is a 2-cocircuit in $M\backslash x_3$. Hence $\{x_2,x_3,x_4\}$ is a triad of $M$ that contains two elements whose contractions have $N$-minors. This contradiction to Hypothesis I shows that $x_1 = w$. But now $\{x_2,x_3,x_4\}$ is a triad of $M/y$, and $\{y,w,x_2,x_3\}$ is a circuit of $M$. This contradicts our assumption that 3.11.5 fails. Thus we may assume that $M/y$ has no 4-fans.

Using Lemma 2.4, we deduce that $M/y$ has a $(4,3)$-violator $(U,V)$ such that $\{f,w,x\} \subseteq U$. If $e \in U$, then $y \in \mathrm{cl}^*_M(U)$, because of the cocircuit $\{e,f,w,y\}$. This implies $(U \cup y, V)$ is a $(4,3)$-violator of $M$, so $e$ is in $V$. Similarly, as $\{w,x,y,z\}$ is a cocircuit in $M$, and $w,x \in U$, it follows that $z \in V$. If $g$ is in $U$, then, by using Lemma 2.4 again, we see that $(U \cup e, V - e)$ is a $(4,3)$-violator of $M/y$, and $(U \cup \{e,y\}, V - e)$ is a $(4,3)$-violator of $M$. Thus $g \in V$. Now $(U - f, V \cup f)$ is a $(4,3)$-violator of $M/y$ as $\{e,f,g\}$ is a triangle and $e,g \in V$. The cocircuit $\{e,f,x,z\}$ shows that $x$ is in $\mathrm{cl}^*_M(V \cup f)$. Hence $(U - \{f,x\}, V \cup \{f,x\})$ is a $(4,3)$-violator of $M/y$. Now $w$ is in $\mathrm{cl}_M(V \cup \{f,x\})$, so $(U - \{f,w,x\}, V \cup \{f,w,x\})$ is a $(4,3)$-violator in $M/y$, and the cocircuit $\{w,x,y,z\}$ shows that $(U - \{f,w,x\}, V \cup \{f,w,x,y\})$ is a $(4,3)$-violator in $M$. This contradiction completes the proof of 3.11.5.

**3.11.6.** $\{i,j,k\} \cap \{e,f,w,x,y,z\} = \emptyset$.

To prove this, note that, as each of $e$, $f$, $w$, and $x$ is in a triangle, none can be in the triad $\{i,j,k\}$. As $\{i,j,k\}$ remains a triad in $M/y$, we see that if 3.11.6 fails, then $z \in \{i,j,k\}$. In this case, by orthogonality between $\{i,j,w,y\}$ and $\{e,f,x,z\}$, we see that $z = k$. Then the 3-connected matroid $M\backslash e/y$ has $\{x,f,z\}$ and $\{i,j,z\}$ as cocircuits and has $\{x,f,w\}$ and $\{i,j,w\}$ as circuits. Thus $M\backslash e/y$ has

$\{x, f, i, j\}$ as a quad containing $f$. Lemma 3.4 now implies that $M\backslash f$ or $M/f$ has an $N$-minor. In each case, we get a contradiction to Hypothesis I. Thus 3.11.6 holds.

We show next that

**3.11.7.** *$M/k$ has an $N$-minor.*

Since $M\backslash e$ is 3-connected, $\{i, j, k\}$ is a triad in $M\backslash e$. Hence $(w, i, j, k)$ is a 4-fan of $M\backslash e/y$. By applying Lemma 3.3 and 3.11.4, we see that $N$ is a minor of $M\backslash e/y/k$, or of $M\backslash e/y\backslash w$. In the latter case, $N \preccurlyeq M\backslash e\backslash w$. As $\{f, y\}$ is a 2-cocircuit in $M\backslash e\backslash w$, this implies $N \preccurlyeq M/f$, and this leads to a contradiction to Hypothesis I. Thus $N \preccurlyeq M\backslash e/y/k \preccurlyeq M/k$, so 3.11.7 holds.

Since $M^*\backslash k$ has an $N^*$-minor, we complete the proof of Lemma 3.11 by showing that $M^*\backslash k$ is $(4, 5, S, +)$-connected. First note that, by Lemma 3.6, $M/k$ is 3-connected since $k$ is in a triad of $M$. Assume that $M^*\backslash k$ is not $(4, 5, S, +)$-connected. Then $M/k$ has a 3-separation $(U, V)$ where $|U|, |V| \geqslant 4$ and neither $U$ nor $V$ is a 4-fan or a 5-cofan.

**3.11.8.** *If $(U, V)$ is a $(4, 5, S, +)$-violator of $M^*\backslash k$, then neither $U$ nor $V$ contains $\{i, j\}$.*

This is immediate, otherwise $(U \cup k, V)$ or $(U, V \cup k)$ is a $(4, 3)$-violator in $M$.

**3.11.9.** *If $(U, V)$ is a $(4, 5, S, +)$-violator of $M^*\backslash k$ such that $i \in U$ and $j \in V$, then neither $U$ nor $V$ contains $\{w, y\}$.*

To prove this, assume that $\{w, y\} \subseteq P$ where $\{P, Q\} = \{U, V\}$. Let $\{p, q\} = \{i, j\}$, where $p \in P$ and $q \in Q$. Because $\{p, q, w, y\}$ is the circuit $\{i, j, w, y\}$ of $M$, it follows that $(P \cup q, Q - q)$ is a 3-separation in $M^*\backslash k$. Now $k \in \mathrm{cl}^*_M(P \cup q)$, because $\{k, p, q\}$ is a triad of $M$, so $(P \cup \{p, k\}, Q - q)$ is a 3-separation in $M^*$. Thus $|Q - q| \leqslant 3$. Since $(P, Q)$ is a $(4, 5, S, +)$-violator in $M^*\backslash k$, this means that $|Q| = 4$, so $Q$ is a quad in $M^*\backslash k$. However, this is impossible, as $q$ is in the coclosure of $P$ in $M^*\backslash k$. Hence 3.11.9 holds.

**3.11.10.** *There is a $(4, 5, S, +)$-violator, $(U, V)$, of $M^*\backslash k$ such that $\{e, f, g\} \subseteq U$.*

Let $(U, V)$ be a $(4, 5, S, +)$-violator of $M^*\backslash k$, and assume that $|U \cap \{e, f, g\}| \geqslant 2$. If $(U \cup \{e, f, g\}, V - \{e, f, g\})$ is a $(4, 5, S, +)$-violator of $M^*\backslash k$, there is nothing left to prove. Therefore we assume that $V$ contains a single element, $\alpha$, of $\{e, f, g\}$, and that $(U \cup \alpha, V - \alpha)$ is not a $(4, 5, S, +)$-violator of $M^*\backslash k$. This means that $|V - \alpha| \leqslant 5$, so $|V| \leqslant 6$. If $V$ contains a quad in $M/k$, then $\alpha$ is not in this quad, by orthogonality with $\{e, f, g\}$, so, in this case, $V - \alpha$ contains a quad of $M/k$, and $(U \cup \alpha, V - \alpha)$ is a $(4, 5, S, +)$-violator of $M^*\backslash k$. Hence $V$ does not contain a quad of $M/k$, so $|V| > 4$. Thus $|V| \in \{5, 6\}$.

Suppose $|V| = 5$. Then $V$ is a 5-element fan of $M/k$. It must contain two triangles in $M/k$, otherwise it is a 5-fan of $M^*\backslash k$, which contradicts the fact that $(U, V)$ is a $(4, 5, S, +)$-violator. Let $(v'_1, v'_2, \dots, v'_5)$ be a fan ordering of $V$ in $M/k$, where $\{v'_1, v'_2, v'_3\}$ is a triangle. Since $\alpha$ is in a triangle of $M$, it is not in the triad $\{v'_2, v'_3, v'_4\}$. Therefore, by replacing $(v'_1, v'_2, \dots, v'_5)$ with $(v'_5, v'_4, \dots, v'_1)$ as necessary, we may assume that $\alpha = v'_1$.

Next assume that $|V| = 6$. Then $V - \alpha$ is a 5-element fan in $M/k$ with two triads, otherwise $V - \alpha$ is a 5-cofan in $M^*\backslash k$, which is impossible as $(U \cup \alpha, V - \alpha)$ is not a $(4, 5, S, +)$-violator. Let $(v_1, v_2, \dots, v_5)$ be a fan ordering of $V - \alpha$ in $M/k$, where $\{v_1, v_2, v_3\}$ is a triad. Now it is easy to see by orthogonality that $\{v_1, v_2, v_4, v_5\}$ is independent in $M/k$ and spans $V - \alpha$. It must also span $\alpha$, for otherwise $(U \cup \alpha, V - \alpha)$ is a 2-separation in $M/k$. Orthogonality shows that one of $\{v_1, v_2, \alpha\}$, $\{v_4, v_5, \alpha\}$, or $\{v_1, v_2, v_4, v_5, \alpha\}$ is a circuit in $M/k$. In the second case, we can reverse the fan $(v_1, v_2, \dots, v_5)$, and assume that $\{v_1, v_2, \alpha\}$ is a circuit.

Let us assume that either $|V| = 5$, or $|V| = 6$ and $\{v_1, v_2, \alpha\}$ is a circuit. Next we shall eliminate these two cases. In the first case, $(\alpha, v'_2, v'_3, v'_4, v'_5)$ is a fan with two triangles in $M/k$, and, in the second case, $(\alpha, v_1, v_2, v_3, v_4)$ is. In both cases, $M/k$ has a 5-fan $(\alpha, w_1, w_2, w_3, w_4)$. Thus

$\{\alpha, w_1, w_2, k\}$ and $\{w_2, w_3, w_4, k\}$ are circuits of $M$ and $\{w_1, w_2, w_3\}$ is a triad. Orthogonality with the triad $\{i, j, k\}$, and the fact that $V$ contains only one element of $\{i, j\}$, means that $w_2 \in \{i, j\}$. Thus $\{w_1, w_2, w_3\}$ is a triad of $M$ that meets the circuit $\{i, j, w, y\}$. Since $|V \cap \{i, j\}| = 1$, and $w$ is in no triads of $M$, we deduce that $y \in \{w_1, w_3\}$. Therefore $\{\alpha, w_1, w_2, k\}$ or $\{w_2, w_3, w_4, k\}$ is a 4-element circuit, $C$, of $M$ that contains $k$, $y$, and a single element from $\{i, j\}$. Thus, by orthogonality between $C$ and the cocircuit $\{e, f, w, y\}$, it follows by 3.11.6 and 3.11.9 that $\alpha \in \{e, f\}$ and $C$ is $\{k, y, \alpha, i\}$ or $\{k, y, \alpha, j\}$. Then $C$ meets the cocircuit $\{w, x, y, z\}$ in a single element. This contradiction to orthogonality eliminates the two targeted cases. We deduce that $|V| = 6$, and $\{v_1, v_2, v_4, v_5, \alpha\}$ is a circuit in $M/k$. By taking the symmetric difference of this circuit with $\{v_2, v_3, v_4\}$, and using the fact that $M/k$ is simple, we see that $\{v_1, v_3, v_5, \alpha\}$ is a circuit in $M/k$.

By the dual of Lemma 3.3, either $M/\{k, v_1, v_5\}$ or $M/k\backslash v_2/v_1$ has an $N$-minor. Assume that $N \preccurlyeq M/\{k, v_1, v_5\}$. As $\{\alpha, v_3\}$ is a 2-circuit in $M/\{k, v_1, v_5\}$, it follows that $N \preccurlyeq M\backslash v_3$. Now $\{v_1, v_2, v_3\}$ is a triad in $M/k$, and hence in $M$, so $\{v_1, v_2\}$ is a 2-cocircuit in $M\backslash v_3$. Thus $M/v_1$ and $M/v_2$ have $N$-minors, and Hypothesis I is contradicted. It follows that $N \preccurlyeq M/k\backslash v_2/v_1 \preccurlyeq M\backslash v_2$. But $\{v_1, v_3\}$ is a 2-cocircuit in $M\backslash v_2$, and we get exactly the same contradiction. Hence 3.11.10 holds.

Now we let $(U, V)$ be a $(4, 5, S, +)$-violator of $M^*\backslash k$, where $\{e, f, g\} \subseteq U$. By 3.11.8 and 3.11.9, we can let $\{u, v\} = \{i, j\}$ and $\{u', v'\} = \{w, y\}$, where $u, u' \in U$ and $v, v' \in V$. Because $\{e, f, u', v'\}$ is a cocircuit in $M/k$, it follows that $(U \cup v', V - v')$ is 3-separating in $M/k$. Now $\{u, v, u', v'\}$ is a circuit in $M/k$, so $(U \cup \{v, v'\}, V - \{v, v'\})$ is 3-separating in $M/k$. As $\{k, u, v\}$ is a triad, $(U \cup \{k, v, v'\}, V - \{v, v'\})$ is 3-separating in $M$, so $|V - \{v, v'\}| \leqslant 3$. Thus $|V| \leqslant 5$ and $V$ is sequential.

As $(U, V)$ is a $(4, 5, S, +)$-violator of $M^*\backslash k$, we see that $V$ is a 5-fan in $M/k$. This gives a contradiction to orthogonality between the cocircuit $\{e, f, u', v'\}$ and one of the triangles of $M/k$ contained in $V$. This completes the proof of Lemma 3.11. □

The next result is helpful in identifying $N$-minors of $M$.

**Lemma 3.12.** *Let $(X, Y)$ be a 3-separation of a 3-connected binary matroid $M$ with $|X| = 6$ and $|Y| \geqslant 6$. Let $N$ be an internally 4-connected minor of $M$ having at least seven elements. Suppose $r(X) = 3$ and $|X - \mathrm{cl}(Y)| = 3$. Let $T = X \cap \mathrm{cl}(Y)$. Then $M|X \cong M(K_4)$ and has $T$ as a triangle and $M = P_T(M|X, M|\mathrm{cl}(Y))$. Moreover, either*

(i) *for all $x$ in $X \cap \mathrm{cl}(Y)$, the matroid $M\backslash x$ has an $N$-minor*; *or*
(ii) *for all $y$ in $X - \mathrm{cl}(Y)$, both $M\backslash y$ and $M/y$ have an $N$-minor.*

**Proof.** Since $|X - \mathrm{cl}(Y)| = 3$, the set $X - \mathrm{cl}(Y)$ is a triad of $M$. As $r(X) = 3$, it follows that $X \cap \mathrm{cl}(Y)$ is the guts line of the 3-separation $(X, Y)$ of $M$. Thus $X \cap \mathrm{cl}(Y)$ is a triangle of $M$ and $M|X \cong M(K_4)$. Now $N$ is internally 4-connected with at least seven elements and so has no 4-element fans. Thus either $M\backslash T$ has an $N$-minor or $M|\mathrm{cl}(Y)$ has an $N$-minor, and so (i) or (ii) holds. □

## 4. A big step

The goal of this section is to prove Lemma 4.3. When that result is combined with the next two lemmas, it proves Theorem 2.1 when we replace $(4, 5, S, +)$-connectivity by $(4, 5)$-connectivity. We begin the section with a result of Zhou [18, Lemma 2.15].

**Lemma 4.1.** *Let $N$ be an internally 4-connected proper minor of an internally 4-connected binary matroid $M$ with $|E(N)| \geqslant 7$. Suppose that $M\backslash e$ has an $N$-minor and a 5-element 3-separating set $A$. If $A$ is not a 5-fan or a 5-cofan, then either $M$ has a triangle $T$ such that $M\backslash x$ has an $N$-minor for all $x$ in $T$, or $M$ has a triad $T^*$ such that $M/y$ has an $N$-minor for all $y$ in $T^*$.*

The next lemma follows from Lemma 4.1, Theorem 3.5, and duality.

**Lemma 4.2.** *Let $N$ be an internally 4-connected proper minor of an internally 4-connected binary matroid $M$ such that $|E(N)| \geqslant 7$ and $|E(M)| \geqslant 13$. Suppose $e \in E(M)$ and $\{M\backslash e, M/e\}$ contains a member that has an*

*$N$-minor and a 5-element 3-separating set $A$. Then either $A$ is a 5-fan or a 5-cofan, or one of the following holds.*

(i) *$M$ has an internally 4-connected proper minor that has an $N$-minor and has at least $|E(M)|-3$ elements*; *or*
(ii) *for some $a$ in a triangle of $M$, the matroid $M\backslash a$ is $(4,4,S)$-connected having an $N$-minor*; *or*
(iii) *for some $z$ in a triad of $M$, the matroid $M/z$ is $(4,4,S)$-connected having an $N$-minor.*

The following is the main result of this section.

**Lemma 4.3.** *Let $M$ be an internally 4-connected binary matroid with $|E(M)| \geqslant 15$ and let $\{e,f,g\}$ be a triangle of $M$. Let $N$ be an internally 4-connected matroid with $|E(N)| \geqslant 6$. Suppose that both $M\backslash e$ and $M\backslash f$ have $N$-minors and have $(4,5)$-violators. Then*

(i) *$M$ has an internally 4-connected proper minor $M'$ with $|E(M)-E(M')| \leqslant 3$ such that $M'$ has an $N$-minor*; *or*
(ii) *$M$ has a triangle $T$ such that $M\backslash z$ has an $N$-minor for all $z$ in $T$*; *or*
(iii) *$M$ has a triad $T^*$ such that $M/z$ has an $N$-minor for all $z$ in $T^*$.*

**Proof.** In view of Lemma 2.3, we may assume that $|E(N)| \geqslant 10$. Let $(X_e, Y_e)$ and $(X_f, Y_f)$ be $(4,5)$-violators of $M\backslash e$ and $M\backslash f$, respectively, where $f \in X_e$ and $e \in X_f$. Then $\min\{|X_e|, |Y_e|, |X_f|, |Y_f|\} \geqslant 6$. By Lemma 3.10,

**4.3.1.** *$g \in Y_e \cap Y_f$ and $e \in \mathrm{cl}(X_f - e)$ and $f \in \mathrm{cl}(X_e - f)$.*

The following are immediate consequences of Lemma 3.10(iv)(b).

**4.3.2.**

(a) *If $|X_e \cap Y_f| \geqslant 4$, then $|X_f \cap Y_e| \leqslant 1$.*
(b) *If $|X_e \cap Y_f| \in \{2,3\}$, then $|X_f \cap Y_e| \leqslant 3$.*
(c) *If $|X_f \cap Y_e| \geqslant 4$, then $|X_e \cap Y_f| \leqslant 1$.*
(d) *If $|X_f \cap Y_e| \in \{2,3\}$, then $|X_e \cap Y_f| \leqslant 3$.*

Since $M$ is internally 4-connected and $M\backslash e, f$ is $(3,3)$-connected, the following is immediate from Lemma 3.10(iv)(c).

**4.3.3.** *If $|X_e \cap X_f| \geqslant 4$, then $|Y_e \cap Y_f| \leqslant 3$.*

Next we show the following.

**4.3.4.** *Either $|X_e| \leqslant |Y_e|$ and $|X_f| \leqslant |Y_f|$; or $|X_e| \geqslant |Y_e|$ and $|X_f| \geqslant |Y_f|$.*

By symmetry, suppose that $|Y_e| < |X_e|$ and $|X_f| < |Y_f|$. As $|E(M)| \geqslant 15$, we deduce that $|X_e| \geqslant 8$ and $|Y_f| \geqslant 8$. Suppose $|X_f \cap Y_e| \geqslant 4$. Then, by 4.3.2(c), $|X_e \cap Y_f| \leqslant 1$. Since $|X_e| \geqslant 8$, it follows that $|X_e \cap X_f| \geqslant 6$. Thus, by 4.3.3, $|Y_e \cap Y_f| \leqslant 3$. Hence $|Y_f| \leqslant 4$; a contradiction. We deduce that $|X_f \cap Y_e| \leqslant 3$.

Suppose $|X_f \cap Y_e| \in \{2,3\}$. Then, by 4.3.2(d), $|X_e \cap Y_f| \leqslant 3$. As $|Y_f| \geqslant 8$, it follows that $|Y_e \cap Y_f| \geqslant 5$. Thus, by 4.3.3, $|X_e \cap X_f| \leqslant 3$. Hence $|X_e| \leqslant 7$; a contradiction. Finally, suppose that $|X_f \cap Y_e| \leqslant 1$. Then, as $|Y_e|, |X_f| \geqslant 6$, we have $|Y_e \cap Y_f| \geqslant 5$ and $|X_f \cap X_e| \geqslant 4$. This contradicts 4.3.3, so 4.3.4 holds.

**4.3.5.** *$|X_e| \leqslant |Y_e|$ and $|X_f| \leqslant |Y_f|$.*

Assume that this fails. Then, by 4.3.4, $|Y_e| \leqslant |X_e|$ and $|Y_f| \leqslant |X_f|$. Moreover, we may assume that equality does not hold for both of these, so, without loss of generality, $|Y_e| < |X_e|$. Suppose $|X_e \cap X_f| \leqslant 3$. As $|X_e|, |X_f| \geqslant 6$, it follows that $|X_e \cap Y_f|, |X_f \cap Y_e| \geqslant 2$. Thus, by 4.3.2, $|X_f \cap Y_e|$, $|X_e \cap Y_f| \leqslant 3$. Hence $|X_e| \leqslant 7$. As $|Y_e| < |X_e|$, it follows that $|E(M)| \leqslant 14$; a contradiction.

We may now assume that $|X_e \cap X_f| \geqslant 4$. Then, by 4.3.3, $|Y_e \cap Y_f| \leqslant 3$. As $|Y_e| \geqslant 6$, if $|Y_e \cap Y_f| \leqslant 2$, then $|X_e \cap Y_f|, |X_f \cap Y_e| \geqslant 4$ and we contradict 4.3.2. Thus $|Y_e \cap Y_f| = 3$ and $|X_e \cap Y_f|, |X_f \cap Y_e| \geqslant 3$. Hence, by 4.3.2, $|X_e \cap Y_f| = 3 = |X_f \cap Y_e|$. By Lemma 3.10(iv)(c), if $\lambda_M(Y_e \cap Y_f) = 3$, then $\lambda_{M\backslash e,f}(X_e \cap X_f) \leqslant 1$, so, as $M\backslash e, f$ is (3, 3)-connected, $|X_e \cap X_f| \leqslant 3$; a contradiction. Hence $\lambda_M(Y_e \cap Y_f) = 2$, so $Y_e \cap Y_f$ is a triangle or a triad of $M$. But $g \in Y_e \cap Y_f$, so, by orthogonality, $Y_e \cap Y_f$ is a triangle.

As $|X_e \cap Y_f| = 3 = |X_f \cap Y_e|$, it follows by Lemma 3.10(iv)(b) that each of $X_e \cap Y_f$ and $X_f \cap Y_e$ is a triangle or a triad of $M$. If one of these is a triangle, then, by Lemma 2.2, $M\backslash x$ has an $N$-minor for each element $x$ in this triangle and (ii) holds. Thus we may assume that $X_f \cap Y_e$ and $X_e \cap Y_f$ are triads of $M$.

Applying Lemma 2.2 to $M\backslash e$ taking $Y_e \cap Y_f$ to be $T$, we deduce that either (ii) holds, or $\sqcap_{M\backslash e}(Y_e \cap Y_f, Y_e \cap X_f) \neq 2$. In the latter case, as $r(Y_e \cap Y_f) = 2$, it follows that $r(Y_e \cap X_f) \neq r(Y_e)$, so

$$r(Y_e) \geqslant 4. \tag{4}$$

Similarly, applying the dual of Lemma 2.2 to $(M\backslash e)^*$ taking $X_f \cap Y_e$ to be $T$, we deduce that either (iii) holds, or $\sqcap_{(M\backslash e)^*}(X_f \cap Y_e, Y_e \cap Y_f) \neq 2$. In the latter case, as $r_{(M\backslash e)^*}(X_f \cap Y_e) = 2$, it follows that

$$r^*_{M\backslash e}(Y_e) \geqslant 4. \tag{5}$$

We may now assume that (4) and (5) hold. As $Y_e$ is 3-separating in $M\backslash e$, we have that $r(Y_e) + r^*_{M\backslash e}(Y_e) = |Y_e| + 2 = 8$, so

$$r(Y_e) = r^*_{M\backslash e}(Y_e) = 4. \tag{6}$$

Let $Y_e \cap Y_f = \{1, 2, 3\}$ and $X_f \cap Y_e = \{4, 5, 6\}$. Now $\{4, 5, 6\}$ is a triad of $M$ and $\{1, 2, 3\}$ is a triangle. Clearly $\{4, 5, 6\}$ is contained in a basis $B$ of $Y_e$. As $|B| = 4$, we may assume that $3 \in B$. Let $C$ be the fundamental circuit $C_{M|Y_e}(1, B)$. This circuit does not contain the triad $\{4, 5, 6\}$. Moreover, $|C| > 3$, as $M$ has no 4-element fans. As $|B| = 4$, we deduce that $|C| = 4$. Hence $|C \cap \{1, 2, 3\}| = 2$, so $C \triangle \{1, 2, 3\}$ is a triangle of $M$ that meets the triad $\{4, 5, 6\}$; a contradiction to the fact that $M$ is internally 4-connected. We conclude that 4.3.5 holds.

Since $|X_e| \leqslant |Y_e|$ and $|X_f| \leqslant |Y_f|$, we have that

$$|X_e \cap X_f| + |X_e \cap Y_f| + 1 \leqslant |X_f \cap Y_e| + |Y_f \cap Y_e|$$

and

$$|X_f \cap X_e| + |X_f \cap Y_e| + 1 \leqslant |X_e \cap Y_f| + |Y_e \cap Y_f|.$$

Adding these two inequalities and simplifying, we get

$$|X_e \cap X_f| + 1 \leqslant |Y_e \cap Y_f|. \tag{7}$$

From this and 4.3.3, it follows that $|X_e \cap X_f| \leqslant 3$. If $|X_e \cap X_f| \leqslant 1$, then $|X_e \cap Y_f|, |X_f \cap Y_e| \geqslant 4$, which contradicts 4.3.2(a). Hence

$$|X_e \cap X_f| \in \{2, 3\}. \tag{8}$$

As $|X_e|, |X_f| \geqslant 6$, we have

$$\min\{|X_e \cap Y_f|, |X_f \cap Y_e|\} \geqslant 6 - (1 + |X_e \cap X_f|) \geqslant 2. \tag{9}$$

Thus, by 4.3.2,

$$\max\{|X_e \cap Y_f|, |X_f \cap Y_e|\} \leqslant 3. \tag{10}$$

As $|E(M)| \geqslant 15$, we have $|Y_e \cap Y_f| \geqslant 4$. Hence, by Lemma 3.10(iv)(c),

$$\lambda_{M\backslash e,f}(X_e \cap X_f) = 1. \tag{11}$$

Next we show the following.

**4.3.6.** *The lemma holds when $|X_e \cap X_f| = 2$.*

Assume that $|X_e \cap X_f| = 2$. Then $X_e \cap X_f$ is a 2-element cocircuit $\{1, 2\}$ of $M\backslash e, f$ so $\{1, 2, e, f\}$ is a cocircuit of $M$. By (10) and the first inequality in (9), $|X_f \cap Y_e| = 3 = |X_e \cap Y_f|$. Let $X_f \cap Y_e = \{3, 4, 5\}$ and $X_e \cap Y_f = \{6, 7, 8\}$. Then each of $\{3, 4, 5\}$ and $\{6, 7, 8\}$ is a triangle or a triad of $M$ by Lemma 3.10(iv)(b).

Suppose $\{3, 4, 5\}$ is a triad of $M$. Then $\{3, 4, 5\}$ and $\{1, 2, e\}$ are triads of $M\backslash f$. By applying the dual of Lemma 2.2 to $M\backslash f$, we deduce that (iii) holds. Thus we may assume that both $\{3, 4, 5\}$ and $\{6, 7, 8\}$ are triangles. If $r_{M\backslash f}(X_f) = 3$ or $r_{M\backslash e}(X_e) = 3$, then, by applying Lemma 2.2 to $M\backslash f$ or $M\backslash e$, we get that (ii) holds. By duality, if $r^*_{M\backslash f}(X_f) = 3$ or $r^*_{M\backslash e}(X_e) = 3$, then (iii) holds. Hence we may assume, since $r_{M\backslash e}(X_e) + r^*_{M\backslash e}(X_e) = 8$ and $r_{M\backslash f}(X_f) + r^*_{M\backslash f}(X_f) = 8$, that $r_{M\backslash e}(X_e) = r^*_{M\backslash e}(X_e) = r_{M\backslash f}(X_f) = r^*_{M\backslash f}(X_f) = 4$.

As $|E(N)| \geqslant 10$, we have that $|X_f \cap E(N)| \leqslant 3$. Let $L$ be the guts line of the 3-separation $(X_f, Y_f)$ of $M\backslash f$. Let $M_X$ be the matroid obtained by extending $M|X_f$ by the elements of $L - X_f$. Then $M_X$ is a 3-connected matroid of rank 4. Moreover, as $M\backslash f$ has $\{1, 2, e\}$ as a cocircuit, so does $M_X$. Thus $M_X\backslash\{1, 2, e\}$ is a plane $P$ that contains $L$ and $\{3, 4, 5\}$. As $M$ is binary, $L \cap \{3, 4, 5\}$ is non-empty, so we may assume that $L \cap \{3, 4, 5\} = \{3\}$. Let $L'$ be the guts line of the triad $\{1, 2, e\}$ of $M_X$. Then, viewing $M$ as a restriction of a binary projective geometry, we see that $L'$ lies in the plane of the projective geometry that is spanned by $P$. As $r(X_f) = 4$, the lines $L'$ and $\{3, 4, 5\}$ are distinct. Moreover, $L \neq L'$ otherwise $\{4, 5\}$ is a 2-cocircuit of $M_X$; a contradiction. Thus, letting $t$ be the point where $L'$ meets $\{3, 4, 5\}$, we have that $M_X$ has $(t, u, v, w)$ as a 4-fan where $\{u, v, w\} = \{1, 2, e\}$. Moreover, we may assume that $t$ is 3 or 5. Thus $M_X$ is isomorphic to $S_8$ or $M(\mathcal{W}_4)$. In the first case, $M_X$ is the rank-4 tipped cotipped binary spike with 3 as the tip and $w$ as the cotip. Hence $\{w, 4, 5\}$ is a cocircuit of $M_X$ and so of $M\backslash f$. As $M$ has no 4-element fans, it follows that $\{f, w, 4, 5\}$ is a cocircuit of $M$ so, by orthogonality with the circuit $\{e, f, g\}$, we deduce that $w = e$. Thus, when $M_X \cong S_8$, we have that $M_X/e \cong F_7$. Consider the second case, when $M_X \cong M(\mathcal{W}_4)$. Then $M_X$ has $\{u, 4, 5\}$ as a cocircuit, so $M$ has $\{f, u, 4, 5\}$ as a cocircuit and, by orthogonality, $u = e$. In this case, $\mathrm{si}(M_X/e)$ is isomorphic to $M(K_4)$ and uses the line $L$.

Now $N$ is isomorphic to a minor of the matroid that is obtained by replacing $X_f$ by a triangle or a triad on the guts line $L$ of $(X_f, Y_f)$. For both of the choices of $M_X$, we see that $M\backslash f/e$ has an $N$-minor when we need to replace $X_f$ by a triangle on $L$ and when we need to replace it by a triad. We conclude that $M/e\backslash g$ has an $N$-minor and, therefore, so does $M\backslash g$. Thus the lemma holds and so we have proved 4.3.6.

By 4.3.6 and (8), we may now assume that $|X_e \cap X_f| = 3$. By (11) and Corollary 3.7, $X_e \cap X_f$ is a triangle $\{0, 1, 2\}$ containing a 2-element cocircuit, say $\{1, 2\}$, of $M\backslash e, f$. Moreover, as $|X_f \cap Y_e| \in \{2, 3\}$ by (9) and (10), it follows that $|X_f| \in \{6, 7\}$. By symmetry, $|X_e| \in \{6, 7\}$.

We show next that we may assume, by possibly interchanging $e$ and $f$, that

**4.3.7.** $|X_f \cap E(N)| \leqslant 3$.

Since $|E(N)| \geqslant 10$, this is immediate if $|X_f| = 6$. Now suppose that $|X_f| = 7$, but that 4.3.7 fails. Then $|Y_f \cap E(N)| \leqslant 3$, so $|E(N)| \leqslant 10$. Thus $|X_f \cap E(N)| = 7$ and $|E(N)| = 10$. Now $|X_e \cap E(N)| \geqslant 4$ otherwise we can interchange $e$ and $f$ to get that 4.3.7 holds. Thus $|Y_e \cap E(N)| \leqslant 3$. Hence $|X_e \cap E(N)| = 7$. Therefore $E(N) \supseteq X_e \cup X_f$ so $|E(N)| \geqslant 11$. This contradiction completes the proof of 4.3.7.

Suppose $r(X_f) = 4$. Let $L$ be the guts line of the 3-separation $(X_f, Y_f)$ of $M\backslash f$. Let $M_X$ be the matroid obtained by extending $M|X_f$ by the elements of $L - X_f$. As $M\backslash f$ is 3-connected, so is $M_X$. Now $\{1, 2, e\}$ is a cocircuit of $M\backslash f$ and so is a cocircuit of $M_X$. As $\mathrm{co}(M_X\backslash e)$ is not simple, Bixby's Lemma implies that $\mathrm{si}(M_X/e)$ is 3-connected. Since $\{e, 1, 2\}$ is a cocircuit of $M_X$, it follows by orthogonality that $e$ is not on the line $L$. The matroid $M_X/e$ has rank 3 and both $L$ and $\{0, 1, 2\}$ span lines of $M_X/e$. These lines either coincide or meet in a single point. In each case, $\mathrm{si}(M_X/e)$ has $M(K_4)$ as a restriction. This means that, in $M/e\backslash f$, we can put a triangle or a triad, as desired, on $L$. Thus, by Lemma 3.2, $M/e\backslash f$, and so $M/e\backslash g$, has an $N$-minor. Therefore $M\backslash g$ has an $N$-minor. Hence the lemma holds if $r(X_f) = 4$.

Next suppose that $r(X_f) \in \{5, 6\}$ and $|X_f| = r(X_f) + 1$. Then $r^*_{M\backslash f}(X_f) = 3$, so $(M\backslash f)^*|X_f$ is isomorphic to $M(K_4)$ or $F_7$ and has $\{1, 2, e\}$ as a triangle. Moreover, $(M\backslash f)^*|X_f$ has a triangle, $T_1$, that avoids $e$ and contains 1. Thus $T_1$ or $T_1 \cup f$ is a cocircuit of $M$. In the first case, as $1 \in T_1$ and $\{0, 1, 2\}$ is a triangle of $M$, we obtain the contradiction that $M$ has a 4-element fan. Thus $T_1 \cup f$ is a cocircuit of $M$ avoiding $e$. By orthogonality, $g \in T_1$ so $g \in X_f$; a contradiction.

Now assume that $r(X_f) = 5$ and $|X_f| = 7$. Then $r^*_{M\backslash f}(X_f) = 4$ and $|X_f \cap Y_e| = 3$. Let $X_f \cap Y_e = \{3, 4, 5\}$. Then $\{3, 4, 5\}$ is a triangle or triad of $M$. Assume it is a triangle. Then, as $M|X_f$ has rank 5 and contains $\{0, 1, 2\}$ and $\{3, 4, 5\}$ as disjoint triangles, it has $e$ as a coloop; a contradiction to 4.3.1. Thus $\{3, 4, 5\}$ is a triad of $M$. Since $r^*_{M\backslash f}(X_f) = 4$, the set $X_f$ contains at least three cocircuits of $M\backslash f$. Two of these are $\{1, 2, e\}$ and $\{3, 4, 5\}$. Let $C^*$ be a third such cocircuit. If $C^* \subseteq \{0, 1, 2, e\}$, then $\lambda_{M\backslash f}(\{0, 1, 2, 3\}) \leqslant 1$; a contradiction. We deduce that $C^*$ meets $\{3, 4, 5\}$. If $|C^* \cap \{3, 4, 5\}| = 2$, then $C^* \triangle \{3, 4, 5\}$ is a cocircuit meeting $\{3, 4, 5\}$ in just one element. Hence we may assume that $C^* \cap \{3, 4, 5\} = \{3\}$. Thus $3 \in C^* \subseteq \{3, 0, 1, 2, e\}$. As $\{1, 2, e\}$ is a cocircuit of $M\backslash f$, by replacing $C^*$ by $C^* \triangle \{1, 2, e\}$ if necessary, we may assume that $|C^* \cap \{1, 2, e\}| = 1$. Since $|C^*| \geqslant 3$, it follows that $|C^*| = 3$ and $\{0, 3\} \subseteq C^*$. By orthogonality with the circuit $\{0, 1, 2\}$, we deduce that $C^*$ is $\{0, 1, 3\}$ or $\{0, 2, 3\}$. As $\{e, f, g\}$ is a triangle of $M$, it follows, by orthogonality again, that $C^*$ is a cocircuit of $M$, so $M$ has a 4-element fan; a contradiction.

It remains to consider the case when $r(X_f) = 3$. Since $\{1, 2, e\}$ is a cocircuit of $M\backslash f$, it follows that $|X_f| = 6$ and $X_f - \{1, 2, e\}$ is a triangle $T$ that equals the guts line $L$ of the 3-separation $(X_f, Y_f)$ of $M\backslash f$. Then, by Lemma 3.12, either $M\backslash f/e$ has an $N$-minor, so $M\backslash g$ has an $N$-minor and the lemma holds; or, for all $z$ in $T$, the matroid $M\backslash z$ has an $N$-minor and again the lemma holds. This completes the proof of Lemma 4.3. □

## 5. Proof of Theorem 2.1

The purpose of this section is to prove the last major step towards the proof of the main result of the paper.

**Proof of Theorem 2.1.** By Lemma 2.3, we may assume that $|E(N)| \geqslant 10$. Suppose that the theorem does not hold. By Lemma 3.11, Hypothesis I does not hold for $M$ and we may assume, up to duality, that $M$ has a triangle $\{e, f, g\}$ such that $N \preccurlyeq M\backslash e$ and $N \preccurlyeq M\backslash f$. Then $M\backslash e$ and $M\backslash f$ have $(4, 5, S, +)$-violators $(X_e, Y_e)$ and $(X_f, Y_f)$, respectively, such that $e \in X_f$ and $f \in X_e$. By Lemma 3.10(i), $g \in Y_e \cap Y_f$. Without loss of generality, we may assume that

$$\min\{|X_e|, |Y_e|\} \leqslant \min\{|X_f|, |Y_f|\}. \tag{12}$$

We now show that this implies that

**5.1.1.** $(X_e, Y_e)$ *is not a* $(4,5)$*-violator.*

Assume the contrary. Then, by Lemma 4.3 and Theorem 3.5, $(X_f, Y_f)$ is not a $(4,5)$-violator otherwise the theorem holds. Hence $\min\{|X_f|, |Y_f|\} \leqslant 5$. It follows by (12) that $\min\{|X_e|, |Y_e|\} \leqslant 5$; a contradiction.

Next we observe the following.

**5.1.2.** *Either* $X_e$ *or* $Y_e$ *is a* 5*-cofan or a quad in* $M\backslash e$*. Moreover, if* $Z \in \{X_h, Y_h\}$ *for some* $h$ *in* $\{e, f\}$ *and* $|Z| \leqslant 5$*, then* $Z$ *is a* 5*-cofan or a quad in* $M\backslash h$*. In particular, if* $Z$ *is a* 5*-cofan* $(v, w, x, y, z)$ *in* $M\backslash h$*, then* $\{v, w, x, y, z\} \cap \{e, f, g\} = \{x\}$ *and* $\{h, x, v, w\}$ *and* $\{h, x, y, z\}$ *are cocircuits of* $M$*.*

Since $(X_e, Y_e)$ is not a $(4,5)$-violator but is a $(4,5,S,+)$-violator, Lemma 4.2 implies that $X_e$ or $Y_e$ is a 5-cofan or a quad in $M\backslash e$. Using the same lemma, we also see that if $|Z| = 5$ for some $Z$ in $\{X_e, Y_e, X_f, Y_f\}$, then $Z$ is a 5-cofan. The rest of 5.1.2 follows by using orthogonality and the fact that $M$ has no 4-fans.

By Theorem 3.5,

**5.1.3.** $N$ *is not a minor of* $M\backslash g$*.*

By Lemma 3.10,

**5.1.4.** $e \in \mathrm{cl}_{M\backslash f}(X_f - e)$ *and* $f \in \mathrm{cl}_{M\backslash e}(X_e - f)$*.*

Next we show that

**5.1.5.** $\emptyset \notin \{X_e \cap X_f, X_e \cap Y_f, X_f \cap Y_e\}$*.*

Suppose $X_e \cap X_f = \emptyset$. Then $X_e - f \subseteq Y_f$, so $f \in \mathrm{cl}(Y_f)$. Hence $(X_f, Y_f \cup f)$ is a 3-separation of $M$; a contradiction. Thus $X_e \cap X_f \neq \emptyset$. Similar arguments establish that $X_e \cap Y_f$ and $X_f \cap Y_e$ are non-empty. Thus 5.1.5 holds.

By Lemma 3.10(iv)(c),

**5.1.6.** $4 \geqslant \lambda_M(Y_e \cap Y_f) + \lambda_{M\backslash e,f}(X_e \cap X_f)$*.*

Next we show the following.

**5.1.7.** *If* $|X_e| \geqslant |Y_e|$*, then* $|Y_e \cap Y_f| \leqslant 3$*.*

Assume that $|Y_e \cap Y_f| \geqslant 4$. As $|Y_e| \leqslant 5$, it follows, by 5.1.5, that $|X_f \cap Y_e| = 1$ and $|Y_e| = 5$. Since $\min\{|X_e|, |Y_e|\} \leqslant \min\{|X_f|, |Y_f|\}$, we deduce that $|X_f| \geqslant 5$. Now $\lambda_M(Y_e \cap Y_f) \geqslant 3$ otherwise $M$ has a $(4,3)$-violator. Thus by 5.1.6, $\lambda_{M\backslash e,f}(X_e \cap X_f) \leqslant 1$. Hence, by Lemma 3.6, $|X_e \cap X_f| \leqslant 3$, so $|X_f| = 5$ and $|X_e \cap X_f| = 3$. Therefore, by 5.1.2, $X_f$ is a 5-cofan of $M\backslash f$. Moreover, by Corollary 3.7, $X_e \cap X_f$ is a triangle. But, by 5.1.4, $e \in \mathrm{cl}(X_f - e)$. This contradicts the fact that $X_f$ is a 5-cofan of $M\backslash f$. Thus 5.1.7 holds.

We break up the rest of the proof into the following four cases.

(I) $|X_e| \geqslant |Y_e|$ and $|X_f| \geqslant |Y_f|$.
(II) $|X_e| \geqslant |Y_e|$ and $|X_f| < |Y_f|$.
(III) $|X_e| < |Y_e|$ and $|X_f| \geqslant |Y_f|$.
(IV) $|X_e| < |Y_e|$ and $|X_f| < |Y_f|$.

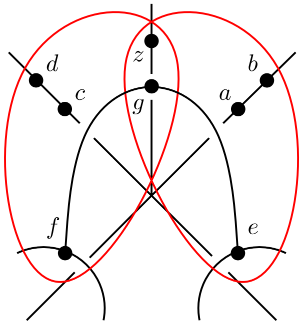


**Fig. 3.** By 5.1.9, $M$ has the structure shown here.

The first of these cases is the most difficult.

**Case I.** $|X_e| \geqslant |Y_e|$ and $|X_f| \geqslant |Y_f|$.

In this case, we first observe that 5.1.7 immediately gives that

**5.1.8.** $|Y_e \cap Y_f| \leqslant 3$.

We now know, by Lemma 3.10(iv)(d), that $|Y_e \cap Y_f|$ is 2 or 3. Each of these cases will require a very detailed analysis. We begin with the following case.

(I)(A) $Y_e \cap Y_f = \{z, g\}$.

First we show that

**5.1.9.** *$Y_e$ is a quad of $M \backslash e$ and $Y_f$ is a quad of $M \backslash f$.*

Suppose that $Y_e$ is not a quad of $M \backslash e$. By 5.1.1, $|Y_e| = 5$, so $|X_f \cap Y_e| = 3$. Then, by 5.1.2, $Y_e$ is a 5-cofan with $g$ as its central element. Hence $g$ is in the coclosure of $X_f \cap Y_e$ in $M \backslash e$, so $g$ is in the coclosure of $X_f$ in $M \backslash f$. Thus $(X_f \cup g, Y_f - g)$ is a 3-separation of $M \backslash f$, so $(X_f \cup g \cup f, Y_f - g)$ is a 3-separation of $M$. Thus $Y_f$ is a 4-element sequential 3-separating set in $M \backslash f$, so $Y_f$ is a 4-fan in $M \backslash f$; a contradiction. We deduce that $Y_e$ is a quad in $M \backslash e$, so $Y_e$ is a circuit of $M$, and $Y_e \cup e$ is a cocircuit of $M$. Since $\{z, g\} \subseteq Y_f$ and $\{e, f, g\}$ is a triangle, it follows that

$$2 = \lambda_{M \backslash f}(X_f) \geqslant \lambda_{M \backslash f}\big(X_f \cup \{z, g\}\big) \geqslant \lambda_M\big(X_f \cup \{z, g, f\}\big).$$

Thus $Y_f - \{z, g\}$ is a 3-separating set in $M$. Hence $|Y_f| \leqslant 5$ so, by 5.1.2, $Y_f$ is a quad or a 5-cofan in $M \backslash f$. Suppose $Y_f$ is a 5-cofan of $M \backslash f$. Then $g$ is its central element, so $g$ is in two triads of $M \backslash f$ that are contained in $Y_f$. By orthogonality with the circuit $Y_e$, each of these triads contains $z$. Hence $M \backslash f$ has a 2-cocircuit, so $M$ has a triad containing $f$ and therefore has a 4-fan; a contradiction. We conclude that $Y_f$ is a quad of $M \backslash f$, so 5.1.9 holds.

Now $M$ has the structure shown in Fig. 3, where $Y_e = \{a, b, g, z\}$ and $Y_f = \{c, d, g, z\}$ while the 5-element cocircuits $Y_e \cup e$ and $Y_f \cup f$ have been circled. Observe that $\{a, b, c, d\}$, the symmetric difference of the circuits $Y_e$ and $Y_f$, is itself a circuit of $M$. We now show that

**5.1.10.** *$M$ has no triad meeting $\{a, b, c, d, e, f, g, z\}$.*

As $M$ is internally 4-connected, no triad contains $e$, $f$, or $g$. Suppose $x$ is in a triad $T^*$ for some $x$ in $\{a,b,c,d,z\}$. Without loss of generality, we may assume that $x \in \{a,b,z\}$. By orthogonality with the circuit $\{a,b,g,z\}$, we know that two elements of $T^*$ are in $\{a,b,g,z\}$. Thus $T^* \subseteq \mathrm{cl}^*_M(\{a,b,g,z\})$. As $e$ is not in a triad of $M$, we know that $e \notin T^*$. Thus $T^* \subseteq \mathrm{cl}^*_{M\backslash e}(\{a,b,g,z\})$. Hence $\{a,b,g,z\} \cup T^*$ is a 5-element 3-separating set of $M\backslash e$ that is not a fan, and the result follows by Lemma 4.2. We conclude that 5.1.10 holds.

Next we show that

**5.1.11.** *$M/z$ is 3-connected having an $N$-minor.*

We know that $N$ is internally 4-connected, $N \preccurlyeq M\backslash e$, and $g$ is in a quad of $M\backslash e$. Thus, by Lemma 3.4, $N \preccurlyeq M\backslash e, g$ or $N \preccurlyeq M\backslash e/g$. The first option contradicts 5.1.3, so the second occurs. Since $g$ is in a quad with $z$ in $M\backslash e$, it follows, by Lemma 3.4, that $N \preccurlyeq M/z$.

As $M\backslash e$ is 3-connected having $\{a,b,g,z\}$ as a quad, as noted in [12, Lemma 2.9], it is routine to check that $\mathrm{si}(M\backslash e/z)$ is 3-connected. It follows that $M/z$ is 3-connected unless $z$ is in a triangle $T$ of $M$. In the exceptional case, since $\{a,b,e,g,z\}$ and $\{c,d,f,g,z\}$ are cocircuits of $M$, it follows by orthogonality that $T$ meets each of $\{a,b,e,g\}$ and $\{c,d,f,g\}$ in a single element. Suppose $g \notin T$. Then it is straightforward to check that $\lambda(\{a,b,c,d,e,f,g,z\}) \leqslant 2$. This is a contradiction since $|E(M)| \geqslant 15$. We deduce that $g \in T$. Let $h$ be the element of $T - \{z,g\}$. Then $(X_f - h, Y_f \cup h)$ is a $(4,5,S,+)$-violator for $M\backslash f$ with $|Y_f \cup h| = 5$. But $Y_f \cup h$ is not a 5-cofan, so 5.1.2 fails; a contradiction. We conclude that 5.1.11 holds.

We may assume that $M/z$ is not internally 4-connected otherwise the theorem holds. We show next that

**5.1.12.** *$M$ has a circuit $\{z,g,v_1,v_2\}$ and a triad $\{v_1,v_2,v_3\}$ such that $\{a,b,c,d,e,f,g,z\} \cap \{v_1,v_2,v_3\} = \emptyset$.*

As $M/z$ is not internally 4-connected, it has a $(4,3)$-violator $(U_z,V_z)$ such that $|U_z \cap \{e,f,g\}| \geqslant 2$. We show next that we may assume that

**5.1.13.** $\{e,f,g\} \subseteq U_z$.

Assume not. Then $(U_z \cup \{e,f,g\}, V_z - \{e,f,g\})$ is not a $(4,3)$-violator of $M/z$. Thus $V_z - \{e,f,g\}$ is a triad $\{v_1,v_2,v_3\}$ of $M/z$ and hence of $M$. Hence, by 5.1.10, $\{a,b,c,d\} \subseteq U_z$. Now $\{e,g,a,b,z\}$ and $\{f,g,c,d,z\}$ are cocircuits of $M$. Thus, as $|U_z \cap \{e,f,g\}| \geqslant 2$, it follows that $\{e,f\} \subseteq U_z$ and $g \in V_z$ otherwise $(U_z \cup z, V_z)$ is a $(4,3)$-violator of $M$; a contradiction. Then $V_z$ is a 4-fan $(g,v_1,v_2,v_3)$ in $M/z$, so $\{z,g,v_1,v_2\}$ is a circuit of $M$ and 5.1.12 holds. We deduce that 5.1.13 holds.

If $\{a,b\}$ or $\{c,d\}$ is contained in $U_z$, then $(U_z \cup z, V_z)$ is a $(4,3)$-violator of $M$; a contradiction. Thus, without loss of generality, we may assume that $\{b,d\} \subseteq V_z$. Suppose $a \in U_z$. Then $b \in \mathrm{cl}_{M/z}(U_z)$, so $(U_z \cup b \cup z, V_z - b)$ is a 3-separation of $M$. Since $d \in V_z - b$, by 5.1.10, $V_z - b$ is not a triad of $M$. Thus $(U_z \cup b \cup z, V_z - b)$ is a $(4,3)$-violator of $M$; a contradiction. We deduce that $a \notin U_z$, so $a \in V_z$. The circuit $\{a,b,c,d\}$ implies that $c \in \mathrm{cl}(V_z)$, so $(U_z - c, V_z \cup c)$ is a $(4,3)$-violator of $M/z$ unless $U_z - c$ is a triad containing $\{e,f,g\}$; a contradiction. Thus we may assume that $\{a,c\} \subseteq V_z$.

Now $\lambda_{M/z}(V_z) \geqslant \lambda_{M/z}(V_z \cup g) \geqslant \lambda_{M/z}(V_z \cup g \cup \{e,f\})$, where the second inequality holds since $\{e,f,g\}$ is a circuit of $M/z$ and $\{e,f,a,b,c,d\}$ is a cocircuit of $M/z$. Thus $|U_z - \{e,f,g\}| \leqslant 3$ otherwise $(U_z - \{e,f,g\}, V_z \cup \{e,f,g\} \cup z)$ is a $(4,3)$-violator of $M$; a contradiction. Suppose $|U_z - \{e,f,g\}| = 3$. Then, as $U_z - g$ is a 5-element 3-separating set in $M/z$, the theorem follows by Lemma 4.2 unless $U_z - g$ is a 5-fan or a 5-cofan. In the exceptional case, by 5.1.10, $(e,u_1,u_2,u_3,f)$ is a 5-fan of $M/z$ where $\{e,u_1,u_2\}$ is a triangle of $M/z$ that meets the cocircuit $\{a,b,c,d,e,f\}$ of $M/z$ in a single element; a contradiction. Suppose next that $|U_z - \{e,f,g\}| = 2$. Then $U_z$ is a 5-element 3-separating set in $M/z$ and we again apply Lemma 4.2 to obtain the desired result because $U_z$ is neither a 5-fan nor a 5-cofan otherwise $e$, $f$, or $g$ is in a triad of $M$; a contradiction to 5.1.10. We may now assume that $|U_z - \{e,f,g\}| = 1$. Then $U_z$ is a 4-fan in $M/z$ containing the triangle $\{e,f,g\}$ of $M$. Thus $M$ has a 4-fan; a contradiction. We deduce that 5.1.12 holds.

We now show that

**5.1.14.** $(M/v_3)^*$ *is* $(4, 5, S, +)$*-connected with an* $N^*$*-minor.*

Now $M/z$ has an $N$-minor, and $M/z$ has $(g, v_1, v_2, v_3)$ as a 4-fan. By Lemma 3.3, $M\backslash g$ or $M/v_3$ has an $N$-minor. The first possibility contradicts 5.1.3, so $N \preccurlyeq M/v_3$.

Next we show that $M/v_3$ is 3-connected. This is certainly true if $M/z/v_3$ is 3-connected so assume it is not. Then we have a 5-fan $(g, v_i, v_j, v_3, v_4)$ in $M/z$ where $\{i, j\} = \{1, 2\}$. Thus $\{z, v_j, v_3, v_4\}$ is a circuit of $M$. By orthogonality with the cocircuits $\{a, b, g, z, e\}$ and $\{c, d, g, z, f\}$ of $M$, we deduce from 5.1.10 that $v_4 = g$; a contradiction. Hence $M/v_3$ is indeed 3-connected.

Let $(U_{v_3}, V_{v_3})$ be a $(4, 5, S, +)$-violator of $M^*\backslash v_3$. We may assume that $v_1 \in U_{v_3}$ and $v_2 \in V_{v_3}$. Now $v_1 \notin \mathrm{cl}_{M/v_3}(V_{v_3})$, otherwise $U_{v_3} - v_1$ is a $(4, 3)$-violator of $M$; a contradiction. Likewise, $v_2 \notin \mathrm{cl}_{M/v_3}(U_{v_3})$. As each of $\{z, g, v_1, v_2\}$, $\{a, b, v_1, v_2\}$, and $\{c, d, v_1, v_2\}$ is a circuit of $M$, we may assume that $\{a, c, g\} \subseteq U_{v_3}$ and $\{b, d, z\} \subseteq V_{v_3}$ because of the symmetry between $a$ and $b$, and between $c$ and $d$.

Suppose that $\{e, f\} \subseteq V_{v_3}$. Then we obtain the contradiction that $(U_{v_3} - g - v_1, V_{v_3} \cup g \cup v_1 \cup v_3)$ is a $(4, 3)$-violator of $M$ unless $|U_{v_3} - g - v_1| \leqslant 3$. But, in the exceptional case, $U_{v_3}$ is a sequential 3-separating set that is contained in $\mathrm{cl}_{M/v_3}(V_{v_3})$ and again we have a contradiction. We deduce that $\{e, f\} \nsubseteq V_{v_3}$.

Since we have maintained symmetry between $e$ and $f$, we may assume that $e \in U_{v_3}$. Now $(U_{v_3} \cup f, V_{v_3} - f)$ is a $(4, 5, S, +)$-violator in $M^*\backslash v_3$ because $V_{v_3} - f$ is not a 4-fan or a 5-fan of $M^*\backslash v_3$, otherwise $b, z$, or $d$ is in a triad of $M$; a contradiction to 5.1.10. Therefore we may assume that $f \in U_{v_3}$. Then, because $M$ has $\{a, b, c, d\}$ as a circuit and $\{a, b, c, d, e, f\}$ as a cocircuit, and $U_{v_3}$ contains $\{a, c, e, f\}$, it follows that $\lambda_{M/v_3}(U_{v_3} \cup \{b, d\}) \leqslant 2$. Observe that $U_{u_3} \cup \{b, d\}$ spans $z$ in $M/v_3$. Thus, by 5.1.12, $U_{u_3} \cup \{b, d\}$ spans $v_2$ in $M/v_3$. It follows that $(U_{v_3} \cup \{b, d\} \cup v_2 \cup v_3, V_{v_3} - \{b, d\} - v_2)$ is 3-separating in $M$. Thus $|V_{v_3}| \leqslant 6$. Suppose $|V_{v_3}| = 4$. Then $V_{v_3}$ is a quad in $M/v_3$, so $\{b, d, z, v_2\}$ and $\{b, d, z, v_2, v_3\}$ are a cocircuit and a circuit, respectively, of $M$. Thus, letting $Z = \{a, b, c, d, e, f, g, z, v_1, v_2, v_3\}$, we have that $\lambda_M(Z) = r(Z) + r^*(Z) - |Z| \leqslant 6 + 7 - 11 = 2$. Hence $|E(M)| \leqslant 14$; a contradiction. Next suppose $|V_{v_3}| = 5$. Then, by Lemma 4.2, $V_{v_3}$ is a 5-fan or a 5-cofan of $M/v_3$, so $b, d$, or $z$ is in a triad of $M$; a contradiction to 5.1.10. We conclude that $|V_{v_3}| = 6$ and $V_{v_3} - \{b, d\}$ is a 4-fan of $M/v_3$ with $z$ in its triad; a contradiction to 5.1.10. This completes the proof in Case I(A).

To complete the proof of Case I, it remains to consider the following.

(I)(B) $Y_e \cap Y_f = \{y, z, g\}$.

As $|E(M)| \geqslant 15$ and $|X_f| \geqslant |Y_f|$, we must have that $|X_f| \geqslant 7$. By 5.1.1, $|Y_e| \in \{4, 5\}$, so $|Y_e \cap X_f| \leqslant 2$. Hence $|X_e \cap X_f| \geqslant 4$. Thus, by Lemma 3.6, $\lambda_{M\backslash e,f}(X_e \cap X_f) \geqslant 2$. Hence, by 5.1.6, $\lambda_M(Y_e \cap Y_f) \leqslant 2$, so $\lambda_M(Y_e \cap Y_f) = 2$ and $\{y, z, g\}$ is a triangle or a triad of $M$. As $g$ is in a triangle of $M$, we know that $\{y, z, g\}$ is not a triad, so it is a triangle. By 5.1.2, $Y_e$ is a 5-cofan $(x_1, y, g, z, x_2)$ in $M\backslash e$ and $\{e, g, y, x_1\}$ and $\{e, g, z, x_2\}$ are cocircuits of $M$. We have $\lambda_{M\backslash f}(X_f) = 2$. Thus $\lambda_{M\backslash f}(X_f \cup z) \leqslant 3$, so $\lambda_{M\backslash f}(X_f \cup z \cup g) \leqslant 3$ as $g \in \mathrm{cl}^*_{M\backslash f}(\{e, x_2, z\})$. Hence $\lambda_{M\backslash f}(X_f \cup \{y, z, g\}) \leqslant 2$, as $y \in \mathrm{cl}_{M\backslash f}(\{z, g\})$ and $y \in \mathrm{cl}^*_{M\backslash f}(\{e, g, x_1\})$. Thus $\lambda_M(X_f \cup \{y, z, f, g\}) \leqslant 2$, so $|Y_f \cap X_e| \leqslant 3$. Moreover, by 5.1.2, as $Y_e \cap Y_f$ is a triangle, $|Y_f| \geqslant 5$, so $|Y_f \cap X_e| \in \{2, 3\}$.

We show next that

**5.1.15.** $|Y_f \cap X_e| = 2$.

Suppose $|Y_f \cap X_e| = 3$. Then $Y_f \cap X_e$ is a triangle or a triad of $M$. If $Y_f \cap X_e$ is a triangle, then, by Lemma 2.2, $M\backslash f$ has a triangle $T$ such that $M\backslash f\backslash t$, and hence $M\backslash t$, has an $N$-minor for each element $t$ in $T$; and we can apply Theorem 3.5 to obtain the desired result. We deduce that $Y_f \cap X_e$ is a triad. By Lemma 2.2, the result follows if $r(Y_f) = 3$. Thus we may assume that $r(Y_f) \geqslant 4$. If $r(Y_f) = 5$, then $r^*_{M\backslash f}(Y_f) = 3$, so $(M\backslash f)^*|Y_f \cong M(K_4)$. Hence, in $M\backslash f$, we know that $Y_f$ is as depicted in Fig. 4,

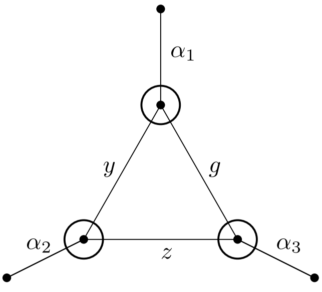


**Fig. 4.** The structure of $Y_f$ in $M\backslash f$.

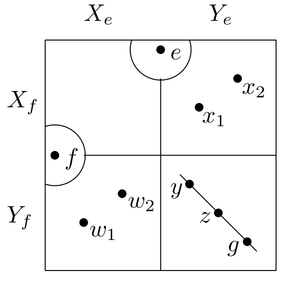


**Fig. 5.** $|X_e \cap Y_f| = 2 = |X_f \cap Y_e|$ and $Y_e \cap Y_f$ is a triangle.

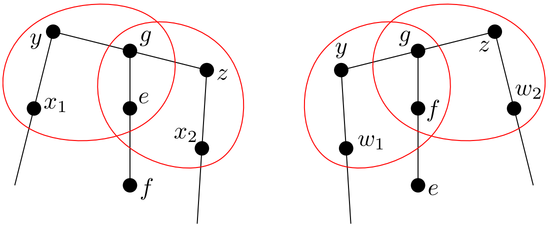


**Fig. 6.** $M$ contains both of these rank-5 structures.

where circled vertices correspond to cocircuits of $M\backslash f$. By orthogonality, $\{z, y, \alpha_2\}$ is a triad of $M$, so $M$ has a 4-fan; a contradiction. We deduce that $r(Y_f) = 4$.

Now $Y_f \cap X_e$ is a triad of $M$ and it spans the guts line $L'$ of the 3-separation $(Y_f \cap X_e, E(M) - (Y_f \cap X_e))$ of $M$. As $r(Y_f) = 4$, we must have that $L'$ meets $\{y, z, g\}$. Since $M$ is binary, we deduce that the triad $Y_f \cap X_e$ is in a 4-fan with an element of $\{y, z, g\} \cap L'$; a contradiction. Hence 5.1.15 holds.

We may now assume that we are dealing with the situation shown in Fig. 5. By 5.1.2, we may also assume that $(x_1, y, g, z, x_2)$ is a 5-cofan in $M\backslash e$ and $(w_1, y, g, z, w_2)$ is a 5-cofan in $M\backslash f$. Thus $\{e, g, y, x_1\}, \{e, g, z, x_2\}, \{f, g, y, w_1\}$, and $\{f, g, z, w_2\}$ are cocircuits of $M$. Therefore $M$ contains the two rank-5 structures shown in Fig. 6, where some elements are common to the two structures and each circled set is a cocircuit of $M$. Observe that $\{y, z, x_1, x_2\}, \{y, z, w_1, w_2\}$, and their symmetric difference, $\{x_1, x_2, w_1, w_2\}$, are also cocircuits of $M$.

By Lemma 3.3, $M\backslash e/x_1$ has an $N$-minor. Thus $N \preccurlyeq M/x_1$. Likewise, $M/x_2$, $M/w_1$, and $M/w_2$ has an $N$-minor. We will show next that

**5.1.16.** $(M/x_1)^*, (M/x_2)^*, (M/w_1)^*$, *and* $(M/w_2)^*$ *are* $(4, 5, S, +)$*-connected.*

By symmetry, it suffices to prove that $(M/x_1)^*$ is $(4,5,S,+)$-connected. First we show that this matroid is 3-connected. Since $M\backslash e$ has $(x_1,y,g,z,x_2)$ as a 5-cofan, by Bixby's Lemma, as $\mathrm{co}(M\backslash e\backslash x_1)$ is not 3-connected, either $\{x_1,y\}$ is contained in a triangle of $M\backslash e$, or $M\backslash e/x_1$ is 3-connected. Suppose first that $\{x_1,y\}$ is contained in a triangle with some element $t$. Then $\{x_1,y,t\}$ meets both of the cocircuits $\{y,w_1,g,f\}$ and $\{y,w_1,z,w_2\}$, so $t=w_1$. Then, letting $Z=\{e,f,g,y,z,x_1,x_2,w_1,w_2\}$, we have $\lambda_M(Z)=r(Z)+r^*(Z)-|Z|\leqslant 6+5-9=2$; a contradiction. Hence $\{x_1,y\}$ is not in a triangle.

As we now know that $M\backslash e/x_1$ is 3-connected, it follows that $M/x_1$ is 3-connected unless $\{x_1,e\}$ is contained in a triangle $T$ of $M$. In the exceptional case, by orthogonality, $T$ must meet each of $\{g,y,w_1\}$ and $\{g,z,w_2\}$, so $T=\{x_1,e,g\}$ contradicting the fact that $\{e,f,g\}$ is a triangle. We conclude that $M/x_1$ is 3-connected.

Suppose that $(M/x_1)^*$ has a $(4,5,S,+)$-violator $(U,V)$. We may assume that

$$\big|U\cap\{y,z,g\}\big|\geqslant 2. \tag{13}$$

Thus $\{y,z,g\}\subseteq \mathrm{cl}_{M/x_1}(U)$. Then $(U\cup\{y,z,g\},V-\{y,z,g\})$ is 3-separating in $(M/x_1)^*$, and $|V-\{y,z,g\}|\geqslant 4$. If $e$ or $x_2$ is in $U$, then, as $\{e,g,y,x_1\}$ and $\{y,z,x_1,x_2\}$ are cocircuits of $M$, it follows that $(U\cup\{y,z,g\}\cup x_1,V-\{y,z,g\})$ is a $(4,3)$-violator of $M$; a contradiction. Thus $\{e,x_2\}\subseteq V$.

We show next that

**5.1.17.** $\{e,x_2,f\}\subseteq V$.

From above, we need only show that $f\in V$. Suppose that $f\in U$. Then $(U\cup\{y,z,g\}\cup e,V-\{y,z,g,e\})$ is 3-separating in $M/x_1$. Thus $|V-\{y,z,g,e\}|\leqslant 3$ otherwise we obtain the contradiction that $(U\cup\{y,z,g\}\cup e\cup x_1,V-\{y,z,g,e\})$ is a $(4,3)$-violator of $M$. Hence $V$ is a 5-fan $(v_1,v_2,v_3,v_4,e)$ of $M/x_1$ with $v_1$ in $\{y,z,g\}$ and $x_2$ in $\{v_2,v_3,v_4\}$. Since $M$ is internally 4-connected, neither $\{v_1,v_2,v_3\}$ nor $\{v_3,v_4,e\}$ is a circuit of $M$. Hence both $\{v_1,v_2,v_3,x_1\}$ and $\{v_3,v_4,e,x_1\}$ are circuits. By orthogonality between the last circuit and the cocircuit $\{y,z,x_1,x_2\}$, we deduce that $x_2\in\{v_3,v_4\}$.

Suppose $x_2=v_4$. Then, by orthogonality, $\{v_1,v_2,v_3\}$ meets both $\{y,z\}$ and $\{g,y\}$. As $\{v_2,v_3\}\cap\{y,z,g\}=\emptyset$, we deduce that $v_1=y$. The cocircuit $\{f,g,y,w_1\}$ and the circuits $\{y,v_2,v_3,x_1\}$ and $\{e,x_2,v_3,x_1\}$ imply that $w_1\in\{v_2,v_3\}$ and $w_1\neq v_3$, so $w_1=v_2$. Now $\lambda_{M\backslash e,f}(\{y,z,g\})\leqslant 2$ and $(x_1,y,g,z,x_2)$ and $(w_1,y,g,z,w_2)$ are 5-cofans in $M\backslash e$ and $M\backslash f$, respectively. Thus $\{x_1,x_2,w_1,w_2\}\subseteq \mathrm{cl}^*{}_{M\backslash e,f}(\{y,z,g\})$, so $\lambda_{M\backslash e,f}(\{y,z,g,x_1,x_2,w_1,w_2\})\leqslant 2$. Hence $\lambda_{M\backslash e}(\{y,z,g,x_1,x_2,w_1,w_2,f\})\leqslant 3$, so $\lambda_M(\{y,z,g,x_1,x_2,w_1,w_2,f,e\})=3$. But $v_3\in \mathrm{cl}^*(\{w_1,x_2\})$ and $v_3\in\mathrm{cl}(\{y,w_1,x_1\})$, so we obtain the contradiction that $\lambda_M(\{y,z,g,x_1,x_2,w_1,w_2,f,e,v_3\})=2$ unless $v_3\in\{y,z,g,x_1,x_2,w_1,w_2,f,e\}$. In the exceptional case, $v_3=w_2$ so $\{y,w_1,w_2,x_1\}$ is a circuit meeting the cocircuit $\{x_1,x_2,w_1,w_2\}$ in exactly three elements; a contradiction. We deduce that $x_2\neq v_4$.

We may now assume that $x_2=v_3$. Then $\{v_1,v_2,x_1,x_2\}$ and $\{x_1,x_2,v_4,e\}$ are circuits of $M$. Recall that $\{y,g,z\}\cap\{v_1,v_2,v_3,v_4,e\}=\{v_1\}$. By orthogonality between the circuit $\{v_1,v_2,x_1,x_2\}$ and the cocircuits $\{x_1,y,g,e\}$ and $\{x_1,y,z,x_2\}$, we deduce that $v_1\in\{y,g\}$ but $v_1\neq y$. Hence $v_1=g$. The symmetric difference of the circuits $\{g,v_2,x_1,x_2\}$, $\{x_1,x_2,v_4,e\}$, and $\{e,f,g\}$ is $\{v_2,v_4,f\}$, which must be a triangle of $M$. As this triangle meets the triad $\{v_2,v_3,v_4\}$, we have a contradiction. Hence 5.1.17 holds.

As $\{x_1,x_2,w_1,w_2\}$ is a cocircuit of $M$, it follows that $\{w_1,w_2\}\nsubseteq V$ otherwise $(U,V\cup x_1)$ is a $(4,3)$-violator of $M$; a contradiction. Therefore $w_1$ or $w_2$ is in $U$. Thus $f\in \mathrm{cl}^*{}_{M/x_1}(U\cup\{y,z,g\})$ and $e\in\mathrm{cl}_{M/x_1}(U\cup\{y,z,g,f\})$, so $(U\cup\{y,z,g,f,e\}\cup x_1,V-\{y,z,g,f,e\})$ is 3-separating in $M$. Hence

$$|V|\leqslant 6. \tag{14}$$

Now, in the 3-separation $(U,V)$ of $M^*\backslash x_1$, we know that $V$ contains $\{e,f,x_2\}$. Thus the triangle $\{e,f,g\}$, the cocircuit $\{e,g,x_2,z\}$, and the triangle $\{y,g,z\}$ of $M$ imply that $(U-g-z-y,V\cup g\cup z\cup y)$ is 3-separating in $M^*\backslash x_1$. As $M^*$ has $\{y,x_1,z,x_2\}$ as a circuit, $(U-\{g,z,y\},V\cup\{g,z,y\}\cup x_1)$ is 3-separating in $M$. Thus $|U|\leqslant 6$. But, by (14), $|V|\leqslant 6$, so $|E(M)|\leqslant 13$; a contradiction. We conclude that $(M/x_1)^*$ is $(4,5,S,+)$-connected, so 5.1.16 holds.

We will now show that $x_1$, $x_2$, $w_1$, or $w_2$ is in a triad of $M$, and conclude that (ii) of the theorem holds, or that contracting one of these elements in $M$ yields an internally 4-connected matroid, in which case (i) holds. Assume neither of these occurs. Suppose $(J, K)$ is a $(4,3)$-violator of $M/x_1$. Then, without loss of generality, $|J \cap \{y,z,g\}| \geqslant 2$. Then $(J \cup \{y,z,g\}, K - \{y,z,g\})$ is a $(4,3)$-violator of $M/x_1$ unless $K - \{y,z,g\}$ is a triad of $M$. In the exceptional case, $\{w_1, w_2, x_2\} \subseteq J$. But $\{w_1, w_2, x_1, x_2\}$ is a cocircuit of $M$, so $(J \cup x_1, K)$ is a $(4,3)$-violator of $M$; a contradiction. We conclude that we may assume that $\{y,z,g\}$ is contained in $J$. As no triad of $M$ meets this triangle, it follows, by 5.1.16, that $K$ is a 4-fan or a 5-cofan of $M/x_1$. Now $\{e,g,y,x_1\}$ and $\{y,z,x_1,x_2\}$ are cocircuits of $M$. Hence if $e$ or $x_2$ is in $J$, then $(J \cup x_1, K)$ is a $(4,3)$-violator of $M$; a contradiction. Thus $\{e, x_2\} \subseteq K$. But, at most one element of $K$ is not in a triad of $M/x_1$, and we have assumed that $x_2$ is not in a triad of $M$. Thus $e$ is in a triad of $M$. This contradiction completes the proof of Case I.

**Case II.** $|X_e| \geqslant |Y_e|$ and $|X_f| < |Y_f|$.

Since $|E(M\backslash f)| \geqslant 14$, it follows that

$$|Y_f| \geqslant 8. \tag{15}$$

Suppose that $|X_f \cap Y_e| \geqslant 2$. Then $\lambda_M(X_f \cap Y_e) \geqslant 2$. Thus, by Lemma 3.10(iv)(b), $\lambda_M(X_e \cap Y_f) \leqslant 2$, so $|X_e \cap Y_f| \leqslant 3$. This is a contradiction as $|Y_f| \geqslant 8$ yet, by 5.1.7, $|Y_e \cap Y_f| \leqslant 3$. We deduce that $|X_f \cap Y_e| \leqslant 1$. Hence $|X_f \cap Y_e| = 1$ and $|Y_e \cap Y_f| = 3$, so $Y_e$ is a quad of $M\backslash e$. Thus $Y_e \cap Y_f$ is neither a triangle nor a triad in $M$, so $\lambda_M(Y_e \cap Y_f) \geqslant 3$. Thus, by Lemma 3.10(iv)(c), $\lambda_{M\backslash e,f}(X_e \cap X_f) = 1$, so, by Lemma 3.6, $|X_e \cap X_f| \leqslant 3$. Therefore $|X_f| \leqslant 5$ so, by 5.1.2, $X_f$ is a quad or a 5-cofan of $M\backslash f$. Thus the unique element of $X_f \cap Y_e$ is in a cocircuit of $M\backslash f$ that is contained in $X_f$. This cocircuit meets the circuit $Y_e$ in a single element; a contradiction. We conclude that Case II does not arise.

**Case III.** $|X_e| < |Y_e|$ and $|X_f| \geqslant |Y_f|$.

First we show that

**5.1.18.** $|X_e| = 5$.

Assume that 5.1.18 fails. Then, by 5.1.2, $X_e$ is a quad of $M\backslash e$. Thus $|Y_e| \geqslant 10$. By 5.1.5, $|X_e \cap Y_f| \geqslant 1$. Since $|X_f| \geqslant |Y_f|$, we deduce that $|Y_e \cap X_f| \geqslant 4$. Thus $\lambda_M(Y_e \cap X_f) \geqslant 3$, so, by Lemma 3.10(iv)(c), $\lambda_M(X_e \cap Y_f) \leqslant 1$. Hence, by 5.1.5, $|X_e \cap Y_f| = 1$ so $|X_e \cap X_f| = 2$. Suppose $|Y_e \cap Y_f| \geqslant 4$. Then $\lambda_M(Y_e \cap Y_f) \geqslant 3$, so, by Lemma 3.10(iv)(c), $\lambda_{M\backslash e,f}(X_e \cap X_f) \leqslant 1$. Thus $X_e \cap X_f$ is a 2-cocircuit $\{a,b\}$ in $M\backslash e,f$. Then $\{a,b,f\}$ is a cocircuit of $M\backslash e$ properly contained in the quad $X_e$; a contradiction. We conclude that $|Y_e \cap Y_f| \leqslant 3$, so $|Y_f| = 4$. Thus $Y_f$ is a quad of $M\backslash f$. Then $Y_f$ is a circuit of $M$ that meets cocircuit $X_e \cup e$ in a single element; a contradiction to orthogonality. We conclude that 5.1.18 holds.

By 5.1.18, $|Y_e| \geqslant 9$. Moreover, by (12), $|Y_f| \geqslant |X_e|$. It follows that $|Y_f| \geqslant 5$. By 5.1.2, $X_e$ is a 5-cofan with $f$ as its central element. For each triad $T^*$ of $M\backslash e$ contained in $X_e$, the set $T^* \cup e$ is a cocircuit of $M$. The symmetric difference of the two such cocircuits, which is $X_e - f$, is a cocircuit of $M$. Suppose $X_e \cap Y_f$ contains a single element, say $z$. Then, since $X_e \cap X_f = X_e - \{f, z\}$, we deduce that $z \in \mathrm{cl}^*_{M\backslash f}(X_e \cap X_f)$, so $(X_f \cup z, Y_f - z)$ is a 3-separation of $M\backslash f$. Hence $(X_f \cup z \cup f, Y_f - z)$ is a $(4,3)$-violator of $M$; a contradiction. Thus $|X_e \cap Y_f| \geqslant 2$. It follows that $|X_e \cap X_f| \leqslant 2$ and $\lambda_M(X_e \cap Y_f) \geqslant 2$. Thus, by Lemma 3.10(iv)(b), $\lambda_M(X_f \cap Y_e) \leqslant 2$, so $|X_f \cap Y_e| \leqslant 3$. But $|X_f| \geqslant |Y_f|$, so $|X_f| \geqslant \frac{|E(M\backslash f)|}{2} \geqslant 7$. Hence $|X_f \cap Y_e| \geqslant 4$; a contradiction. We conclude that Case III does not arise.

**Case IV.** $|X_e| < |Y_e|$ and $|X_f| < |Y_f|$.

Since $|E(M\backslash f)| \geqslant 14$, we deduce that

$$|Y_f| \geqslant 8. \tag{16}$$

By 5.1.2, since $|X_e| \in \{4,5\}$, either $X_e$ is a quad of $M\backslash e$, or $X_e$ is 5-cofan of $M\backslash e$ with $f$ as its central element. By 5.1.5, $X_e \cap X_f \neq \emptyset$. Hence $|X_e \cap Y_f| \leqslant 3$. As $|Y_f| \geqslant 8$, it follows that $|Y_e \cap Y_f| \geqslant 5$, so $\lambda_M(Y_e \cap Y_f) \geqslant 3$. Hence, by Lemma 3.10(iv)(c),

$$\lambda_{M\backslash e,f}(X_e \cap X_f) \leqslant 1. \tag{17}$$

We show next that

**5.1.19.** $|X_e \cap Y_f| \geqslant 2$ *and* $|X_f \cap Y_e| \leqslant 3$.

Suppose $|X_e \cap Y_f| \leqslant 1$. Then, by 5.1.5, we may assume that $X_e \cap Y_f$ contains a single element, say $z$. Suppose $X_e$ is a quad of $M\backslash e$. Then $|X_e \cap X_f| = 2$ so, by (17) and Corollary 3.7, $X_e \cap X_f$ is a 2-cocircuit of $M\backslash e, f$. Thus $(X_e \cap X_f) \cup \{e, f\}$ is a cocircuit of $M$ that is properly contained in the cocircuit $X_e \cup e$ of $M$. Hence $X_e$ is not a quad of $M\backslash e$. Thus $X_e$ is a 5-cofan of $M\backslash e$. It follows, by arguing as in Case III, that $(X_f \cup z, Y_f - z)$ is a 3-separation of $M\backslash f$, so $(X_f \cup z \cup f, Y_f - z)$ is a $(4,3)$-violator of $M$; a contradiction. Thus $|X_e \cap Y_f| \geqslant 2$. Hence $\lambda_M(X_e \cap Y_f) \geqslant 2$, so, by Lemma 3.10(iv)(b), $\lambda_M(X_e \cap Y_f) \leqslant 2$. Thus $|X_f \cap Y_e| \leqslant 3$ and 5.1.19 holds.

Next we show that

**5.1.20.** $|X_e \cap X_f| = 1$.

Suppose that $|X_e \cap X_f| \geqslant 2$. Then, as $|X_e| \leqslant 5$, we deduce, by 5.1.19, that $|X_e \cap X_f| = 2$ and $X_e$ is a 5-cofan of $M\backslash e$ having $f$ as its central element. Since, by (17), $\lambda_{M\backslash e,f}(X_e \cap X_f) \leqslant 1$, it follows that $X_e \cap X_f$ is a 2-cocircuit $\{a, b\}$ of $M\backslash e, f$. Thus $\{a, b, f\}$ is a cocircuit of $M\backslash e$. Hence we may assume that $X_e = (a, b, f, y, z)$ where $\{y, z\} = X_e \cap Y_f$. Thus $\{a, b, f, e\}$ and $\{y, z, f, e\}$ are cocircuits of $M$. Therefore $e \in \mathrm{cl}^*{}_{M\backslash f}(Y_f)$. Thus, as $|X_f| \geqslant |X_e| = 5$, we deduce that $(X_f - e, Y_f \cup e \cup f)$ is a $(4,3)$-violator of $M$; a contradiction. Hence 5.1.20 holds.

Let $X_e \cap X_f = \{a\}$. We show next that

**5.1.21.** *$X_e$ is a quad of $M\backslash e$.*

Assume 5.1.21 is false. Then $X_e$ is a 5-cofan $(x_1, x_2, f, x_3, x_4)$ in $M\backslash e$ and $\{x_1, x_2, f, e\}$ and $\{x_3, x_4, f, e\}$ are cocircuits of $M$. Thus the symmetric difference, $\{x_1, x_2, x_3, x_4\}$, of these two cocircuits is a cocircuit of $M$ containing $a$. Hence $(X_f - a, Y_f \cup a)$ is a 3-separation of $M\backslash f$, and $X_f$ is not a quad of $M\backslash f$, so $|X_f| \geqslant 5$. The circuit $\{x_2, f, x_3\}$ of $M$ implies that $(X_f - a, Y_f \cup a \cup f)$ is a $(4,3)$-violator of $M$; a contradiction. Hence 5.1.21 holds.

We now show that

**5.1.22.** *$X_f$ is a quad of $M\backslash f$.*

Assume that this is false. Then, by 5.1.19 and 5.1.20, $|X_f| = 5$, so $X_f$ is a 5-cofan $(x_1, x_2, e, x_3, x_4)$ in $M\backslash f$. Thus $\{x_1, x_2, e, f\}$, $\{x_3, x_4, e, f\}$, and hence $\{x_1, x_2, x_3, x_4\}$ are cocircuits of $M$. By 5.1.20, the last of these meets the circuit $X_e$ of $M$ in a single element. This contradiction to orthogonality establishes 5.1.22.

We may now assume that $X_e \cap Y_f = \{y, z\}$ and $Y_e \cap X_f = \{b, c\}$, as shown in Fig. 7. Then $X_e \cup X_f = \{a, b, c, e, f, y, z\}$. Note that $\{a, b, c, e, f\}$ and $\{a, y, z, e, f\}$ are cocircuits of $M$. Hence so is their symmetric difference, $\{b, c, y, z\}$. Next we show that

**5.1.23.** *$M$ has no triangle other than $\{e, f, g\}$ meeting $\{a, b, c, e, f, y, z\}$.*

Suppose 5.1.23 does not hold. Then, without loss of generality, some element of $\{a, b, c, f\}$ is in a triangle $T$ of $M$, where $T \neq \{e, f, g\}$. By orthogonality with the cocircuit $\{a, b, c, e, f\}$, we know

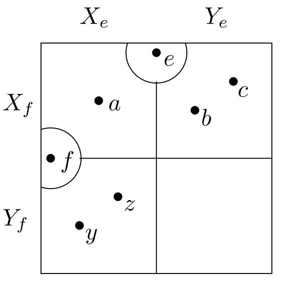


**Fig. 7.** $|X_e \cap Y_f| = 2 = |X_f \cap Y_e|$ and $|X_e \cap X_f| = 1$.

that $T$ contains exactly two elements in $\{a, b, c, e, f\}$. If $T \subseteq X_e \cup X_f$, then $\lambda_M(X_e \cup X_f) = r(X_e \cup X_f) + r^*(X_e \cup X_f) - |X_e \cup X_f| \leqslant 4 + 5 - 7 = 2$, contradicting the fact that $M$ is internally 4-connected. Hence $|T \cap (X_e \cup X_f)| = 2$. By orthogonality with the cocircuit $\{b, c, y, z\}$, either $T$ contains $\{b, c\}$, or $T$ contains two elements of $\{a, e, f\}$. As $T \neq \{e, f, g\}$, we deduce that $T$ contains $\{b, c\}$, $\{a, e\}$, or $\{a, f\}$. Thus $T$ is contained in $\mathrm{cl}_{M\backslash e}(X_e)$ or $\mathrm{cl}_{M\backslash f}(X_f)$, so $M\backslash e$ or $M\backslash f$ has a 5-element 3-separating set that is not a 5-fan or a 5-cofan. By Lemma 4.2, we have a contradiction. Thus 5.1.23 holds.

As $M\backslash e$ has an $N$-minor and a quad $\{a, f, z, y\}$, by Lemma 3.4, $M\backslash e, f$ or $M\backslash e/f$ has an $N$-minor. If $M/f$ has an $N$-minor, then $M/f\backslash g$ has an $N$-minor, as $\{e, g\}$ is a circuit in $M/f$; a contradiction to 5.1.3. Thus $N \preccurlyeq M\backslash e, f$, and so, as $\{a, f, z, y\}$ is a quad of $M\backslash e$, it follows, by Lemma 3.4, that $N \preccurlyeq M\backslash e, a$. Hence $N \preccurlyeq M\backslash a$.

We show next that

**5.1.24.** *$M\backslash a$ is 3-connected.*

First we show that $M\backslash a$ has no 2-cocircuits. Assume $M\backslash a$ has a 2-cocircuit $S$. Then $S \cup a$ is a triad of $M$ which, by orthogonality, must be contained in $X_e \cup X_f$. Since $X_e \cup X_f$ also contains two 5-element cocircuits, $r^*(X_e \cup X_f) \leqslant 4$ and hence $\lambda_M(X_e \cup X_f) \leqslant 2$; a contradiction. Thus $M\backslash a$ has no 2-cocircuits. Hence every 2-separation of $M\backslash a$ is non-minimal. Let $(X, Y)$ be such a 2-separation. Then we may assume that $|X \cap \{e, f, g\}| \geqslant 2$, so $(X \cup \{e, f, g\}, Y - \{e, f, g\})$ is a 2-separation of $M\backslash a$. As $M\backslash a$ has $\{e, f, b, c\}$ and $\{e, f, y, z\}$ as cocircuits, and $M$ has $\{e, b, c, a\}$ and $\{f, y, z, a\}$ as circuits, it is not difficult to see that $\{b, c, y, z\} \subseteq Y$. Thus $(X \cup \{e, f, g\} \cup a, Y - \{e, f, g\})$ is a 3-separation and hence is a $(4, 3)$-violator of $M$; a contradiction. We conclude that 5.1.24 holds.

If $M\backslash a$ is internally 4-connected, then the theorem holds. Thus we may assume that $M\backslash a$ has a $(4, 3)$-violator $(U, V)$ with $|\{e, f, g\} \cap U| \geqslant 2$. Then $(U \cup \{e, f, g\}, V - \{e, f, g\})$ is a $(4, 3)$-violator of $M\backslash a$ unless $V$ is a 4-fan $(x, v_1, v_2, v_3)$ having $\{x, v_1, v_2\}$ as a triangle and $\{v_1, v_2, v_3\}$ as a triad and with $x \in \{e, f, g\}$. In the exceptional case, by 5.1.23, we deduce that $x = g$. As $M$ is internally 4-connected, $\{v_1, v_2, v_3, a\}$ is a cocircuit of $M$. By orthogonality with the circuits $\{a, f, y, z\}$ and $\{a, b, c, e\}$, we get that two elements in $\{y, z, b, c\}$ are in $\{v_1, v_2, v_3\}$; a contradiction to 5.1.23 as $\{g, v_1, v_2\}$ is a triangle other than $\{e, f, g\}$.

We may now assume that $\{e, f, g\} \subseteq U$. If $U$ contains $\{b, c\}$ or $\{y, z\}$, then $(U \cup a, V)$ is a $(4, 3)$-violator of $M$; a contradiction. Without loss of generality, we may assume that $\{c, z\} \subseteq V$. If $b \in U$, then $(U \cup c \cup a, V - c)$ is a $(4, 3)$-violator of $M$ unless $V - c$ is a triangle of $M$ containing $z$; a contradiction to 5.1.23. Thus $b \in V$ and, likewise, $y \in V$. Therefore

**5.1.25.** *$\{e, f, g\} \subseteq U$ and $\{b, c, y, z\} \subseteq V$.*

The symmetric difference $\{a, b, c, e\} \triangle \{a, f, y, z\} \triangle \{e, f, g\}$, which equals $\{b, c, y, z, g\}$, is a circuit of $M$, so

$$2 = \lambda_{M\backslash a}(V) \geqslant \lambda_{M\backslash a}(V \cup g) \geqslant \lambda_{M\backslash a}\big(V \cup g \cup \{e, f\}\big) \geqslant \lambda_M\big(V \cup g \cup \{e, f\} \cup a\big).$$

Thus $4 \leqslant |U| \leqslant 6$.

Suppose that $|U| = 6$. Then $|U - g|$ is a 5-element 3-separating set in $M\backslash a$. By Lemma 4.2, this 3-separating set is a 5-fan or a 5-cofan of $M\backslash a$. If $U - g$ is a 5-fan, then $e$ or $f$ is in a triangle of $M$ that is not $\{e, f, g\}$; a contradiction to 5.1.23. Thus $U - g$ is 5-cofan $(e, u_1, u_2, u_3, f)$ of $M\backslash a$ and $\{a, e, u_1, u_2\}$ is a cocircuit of $M$; a contradiction to orthogonality with $\{e, f, g\}$. Thus $|U| \neq 6$.

Suppose next that $|U| = 5$. By Lemma 4.2, this 3-separating set is a 5-fan or a 5-cofan of $M\backslash a$. By 5.1.23, as before, we deduce that $U$ is a 5-cofan $(u_1, u_2, u_3, u_4, u_5)$ in $M\backslash a$ where $\{u_2, u_3, u_4\} = \{e, f, g\}$. Moreover, $\{a, u_1, u_2, u_3\}$ and $\{u_3, u_4, u_5, a\}$ are cocircuits of $M$. By orthogonality with the circuits $\{a, f, y, z\}$ and $\{a, e, b, c\}$, we deduce that $\{u_1, u_2, u_3\}$ meets $\{f, y, z\}$ and $\{e, b, c\}$. But $\{u_1, u_2, u_3\}$ avoids $\{y, z, b, c\}$, so $\{u_1, u_2, u_3\}$ contains $\{f, e\}$. By symmetry, $\{u_3, u_4, u_5\}$ contains $\{f, e\}$; a contradiction. Hence $|U| \neq 5$.

We may now assume that $|U| = 4$. Then $U - g$ is a triad of $M\backslash a$ containing $\{e, f\}$ but avoiding $g$. Hence $M$ has a 4-cocircuit $C^*$ containing $\{e, f, a\}$. Thus $M\backslash e, f$ has a 2-cocircuit, $\{a, x\}$ say. Moreover, $x \in Y_e \cap Y_f$ otherwise $\lambda_M(X_e \cup X_f) \leqslant 2$; a contradiction. Clearly $(X_e \cup x, Y_e - x)$ is a 3-separation of $M\backslash e$. Thus, by Lemma 4.2, $X_e \cup x$ is a 5-fan or a 5-cofan of $M\backslash e$ contradicting 5.1.21. This completes the proof of the theorem. □

## 6. Proof of the main theorem

In this section, we prove the main result of the paper, Theorem 1.2. To do this, we shall use two more lemmas.

**Lemma 6.1.** *Let $M$ and $N$ be internally 4-connected binary matroids with $|E(M)| \geqslant 15$. Let $\{e, f, g\}$ be a triangle of $M$ such that $M\backslash e$ has an $N$-minor and is $(4, 5, S, +)$-connected. Let $(1, 2, 3, 4, 5)$ be a 5-fan in $M\backslash e$. If $M\backslash 5$ has a quad, then $4 \in \{f, g\}$ and either*

(i) *$M$ has a proper minor $M'$ such that $|E(M) - E(M')| \leqslant 3$ and $M'$ is internally 4-connected with an $N$-minor; or*
(ii) *$M\backslash 1$ is $(4, 4, S)$-connected having an $N$-minor.*

**Proof.** Assume that the lemma fails. By Lemma 3.3, $N \preccurlyeq M\backslash e\backslash 1$, so $N \preccurlyeq M\backslash 1$. Thus $M\backslash 1$ is not $(4, 4, S)$-connected. As $M$ is binary, $\{2, 3, 4\}$ contains a single element of $f$ and $g$, say $f$. Let $Q$ be a quad of $M\backslash 5$. Then $Q \cup 5$ is a cocircuit of $M$ as $M$ has no quads. Hence, by orthogonality with the triangle $\{3, 4, 5\}$, we deduce that $Q$ contains 3 or 4. Indeed, since $M$ is binary, $Q$ contains exactly one of 3 and 4.

Now $M$ has $\{2, 3, 4, e\}$ as a cocircuit. This cocircuit meets the circuit $Q$ in two or four elements. But $\{3, 4\} \nsubseteq Q$, so $|Q \cap \{2, 3, 4, e\}| = 2$. Next we show the following.

**6.1.1.** *If $e \notin Q$, then $Q = \{2, 3, q_3, q_4\}$ for some $q_3, q_4$ not in $\{1, 2, 3, 4, 5, e\}$. Moreover, $4 = f$.*

Assume $e \notin Q$. Then $Q$ meets the cocircuit $\{2, 3, 4, e\}$ in $\{2, 4\}$ or $\{2, 3\}$. First suppose that $Q \cap \{2, 3, 4, e\} = \{2, 4\}$. Then, as $Q \cup 5$ is a cocircuit of $M$ and $\{1, 2, 3\}$ is a circuit, it follows, by orthogonality, that $1 \in Q$. Thus $Q$ is $\{1, 2, 4, z\}$. Hence $\{1, 2, 3, 4, 5, z\}$ is 3-separating in $M\backslash e$; a contradiction to the fact that this matroid is $(4, 5, S, +)$-connected. We deduce that $Q \cap \{2, 3, 4, e\} = \{2, 3\}$.

Now let $Q = \{2, 3, q_3, q_4\}$. Then $\{1, q_3, q_4\}$, which is $Q \triangle \{1, 2, 3\}$, is a circuit of $M$. Hence, as $M\backslash e$ is $(4, 5, S, +)$-connected, $\{q_3, q_4\} \cap \{1, 2, 3, 4, 5, e\} = \emptyset$. Suppose $f \in \{2, 3\}$. Then $g \in \{q_3, q_4\}$. Let $Z = \{1, 2, 3, 4, 5, e, q_3, q_4\}$. Then $r(Z) \leqslant 4$. Moreover, $r^*(Z) \leqslant |Z| - 2$ as $Z$ contains the cocircuits $\{2, 3, 4, e\}$ and $\{2, 3, q_3, q_4, 5\}$. Hence $\lambda(Z) \leqslant 2$; a contradiction as $|E(M)| \geqslant 15$. Thus $f = 4$, so 6.1.1 holds.

**6.1.2.** *If $e \notin Q$, then $M\backslash 1$ is $(4, 4)$-connected.*

Let $(U, V)$ be a $(4, 4)$-violator for $M\backslash 1$. Then $|U|, |V| \geqslant 5$. The triangles $\{1, 2, 3\}$ and $\{1, q_3, q_4\}$ imply that we may assume that $\{2, q_3\} \subseteq U$ and $\{3, q_4\} \subseteq V$. Observe that both $U \cup \{2, 3, 4, e\}$ and

$V \cup \{2,3,4,e\}$ span 1 in $M$. By Lemma 3.9(ii), $|\{2,3,4,e\} \cap V| \geqslant 2$ and $|\{2,3,4,e\} \cap U| \geqslant 2$. Therefore $\{4,e\} \nsubseteq U$ and $\{4,e\} \nsubseteq V$.

Suppose $4 \in U$ and $e \in V$. Then $5 \notin U$ otherwise $1 \in \mathrm{cl}(U)$. Thus $5 \in V$. Then $4 \in \mathrm{cl}(V)$ and $2 \in \mathrm{cl}^*{}_{M\backslash 1}(V \cup 4)$. Hence $(U - 4 - 2, V \cup 4 \cup 2 \cup 1)$ is a 3-separation of $M$. Thus $|U| = 5$ and $U - 4 - 2$ is a triangle of $M$. It follows that $\{2,4\}$ is contained in a triangle of $M$. But this is a contradiction as $M\backslash e$ is $(4,5,S,+)$-connected.

We may now assume that $e \in U$ and $4 \in V$. Suppose $5 \in V$. Then $(U - 2, V)$ is a 3-separation of $M\backslash 1,2$. But $q_3 \in U - 2$ and $\{3,q_4,5\} \subseteq V$. Hence, by 6.1.1, $q_3 \in \mathrm{cl}^*{}_{M\backslash 1,2}(V)$, so $(U - 2 - q_3, V \cup q_3)$ is 3-separating in $M\backslash 1,2$. But $\{1,2\} \subseteq \mathrm{cl}(V \cup q_3)$. Thus $(U - 2 - q_3, V \cup q_3 \cup \{1,2\})$ is 3-separating in $M$, so $|U - 2 - q_3| = 3$. As $e$ is in the 3-separating set $U - 2 - q_3$, this set is a triangle of $M$ that meets the cocircuit $\{2,3,4,e\}$ in $\{e\}$; a contradiction. We conclude that $5 \in U$. Then $5 \in \mathrm{cl}(V)$, so $(U - 5, V \cup 5)$ is a 3-separation of $M\backslash 1$. Hence $|U| = 5$ otherwise $|U - 5| \geqslant 5$ and we can replace $(U,V)$ by $(U - 5, V \cup 5)$ to obtain a contradiction as above. Now $q_4 \in \mathrm{cl}^*{}_{M\backslash 1,3}(U)$, so $(U \cup q_4, V - 3 - q_4)$ is 3-separating in $M\backslash 1,3$. But $1 \in \mathrm{cl}(U \cup q_4)$ and $3 \in \mathrm{cl}(U \cup q_4 \cup 1)$. Hence $(U \cup q_4 \cup 1 \cup 3, V - 3 - q_4)$ is 3-separating in $M$. Thus $|V| = 5$, so $|E(M)| = 11$; a contradiction. Hence 6.1.2 holds.

We now strengthen 6.1.2 to show the following.

**6.1.3.** $e \in Q$.

Assume $e \notin Q$. Since $M\backslash 1$ is not $(4,4,S)$-connected, by 6.1.2, the assertion holds unless $M\backslash 1$ has a quad $Q'$. Consider the exceptional case. Then $Q' \cup 1$ is a cocircuit of $M$, so exactly one of $q_3$ and $q_4$ is in $Q'$ since $\{1,q_3,q_4\}$ is a triangle of $M$. Without loss of generality, we may assume that $q_3 \in Q'$.

**6.1.4.** $3 \notin Q'$ *and* $2 \in Q'$.

Clearly $|Q' \cap \{2,3\}| = 1$. Suppose that $3 \in Q'$. Then 4 or 5 is in $Q'$, so $Q' \cup \{4,5\}$ is a $(4,4)$-violator of $M\backslash 1$; a contradiction to 6.1.2. We conclude that 6.1.4 holds.

Next we show that

**6.1.5.** $4 \notin Q'$ *and* $e \in Q'$.

Since $2 \in Q'$ and $\{2,3,4,e\}$ is a cocircuit, exactly one of 3, 4 and $e$ is in $Q'$. By 6.1.4, $3 \notin Q'$. Suppose $4 \in Q'$. Then $Q' = \{2,q_3,4,v\}$ for some element $v$, so $\{2,q_3,4,v,1\}$ is a cocircuit of $M$. As $3 \notin Q'$ and $\{3,4,5\}$ is a circuit, it follows that $5 \in Q'$ so $v = 5$. Thus $Q' = \{2,q_3,4,5\}$. Taking the symmetric difference with $\{1,2,4,5\}$, we see that $\{q_3,1\}$ contains a circuit of $M$; a contradiction. We deduce that $4 \notin Q'$. Therefore $e \in Q'$. Thus 6.1.5 holds.

We may now assume that $Q' = \{2,q_3,e,w\}$ for some element $w$ that is not in $\{1,2,3,4\}$ and so is not in $\{1,2,3,4,5\}$ by orthogonality between $Q' \cup 1$ and the triangle $\{3,4,5\}$. Let $Z'' = \{1,2,3,4,5,q_3,q_4,e,w\}$. Then $r(Z'') \leqslant 5$ and $r^*(Z'') \leqslant |Z''| - 3$ as $Z''$ contains the cocircuits $\{2,3,4,e\}$, $Q \cup 5$, and $Q' \cup 1$. Hence $\lambda(Z'') \leqslant 2$; a contradiction, so 6.1.3 holds.

**6.1.6.** $Q = \{e,4,q_3,q_4\}$ *for some* $q_3, q_4$ *not in* $\{1,2,3,4,5,e,g\}$. *Moreover,* $f = 4$ *and* $\{g,q_3,q_4\}$ *is a circuit of* $M$.

As $e \in Q$, exactly one of 2, 3, and 4 is in $Q$. Suppose that $Q \cap \{2,3\} \neq \emptyset$. Then, by orthogonality, $1 \in Q$, so $\{1,2,3,4,5\} \cup Q$ is a 7-element 3-separating set in $M$; a contradiction. Thus $Q \cap \{2,3\} = \emptyset$, so $4 \in Q$. We now know that $Q = \{e,4,q_3,q_4\}$ for some elements $q_3, q_4$ not in $\{1,2,3,4,5\}$. Suppose $f \in \{2,3\}$. Then, as $|\{e,4,q_3,q_4,5\} \cap \{e,f,g\}| = 2$, we may assume that $q_3 = g$. Thus $Q \triangle \{e,f,g\} = \{4,f,q_4\}$. Then $q_4 \in \mathrm{cl}(\{1,2,3,4,5\})$, so $\{1,2,3,4,5,q_4\}$ is 3-separating in $M\backslash e$; a contradiction. We conclude that $f \notin \{2,3\}$. Thus $f = 4$ and so $g \notin \{q_3,q_4\}$. Moreover, $Q \triangle \{e,f,g\}$, which is $\{g,q_3,q_4\}$, is a circuit of $M$. Hence 6.1.6 holds.

Since $N \preccurlyeq M\backslash e$ and $(1,2,3,4,5)$ is a maximal fan of $M\backslash e$, it follows that $M\backslash e\backslash 5$ is 3-connected. Moreover, $N \preccurlyeq M\backslash e\backslash 5$. Now $M\backslash e\backslash 5$ has $\{4,q_3,q_4\}$ as a triad and has $\{g,q_3,q_4\}$ as a triangle, so has

$(g, q_3, q_4, 4)$ as a fan. Thus $N \preccurlyeq M\backslash e\backslash 5\backslash g$ or $N \preccurlyeq M\backslash e\backslash 5/4$. In the latter case, as $4 = f$ and $\{e, g\}$ is a circuit of $M/4$, we deduce that $N \preccurlyeq M\backslash g$. This also holds in the former case.

As the quad $Q$ of $M\backslash 5$ is $\{4, e, q_3, q_4\}$ and $M\backslash 5\backslash e$ has an $N$-minor, Lemma 3.4 implies that $N \preccurlyeq M\backslash 5\backslash 4$. But $f = 4$. Hence $N \preccurlyeq M\backslash f$. We now know that, for all $t$ in the triangle $\{e, f, g\}$, the matroid $M\backslash t$ has an $N$-minor. Then, by Theorem 3.5, since we have assumed that (i) of the lemma does not hold, we deduce that

**6.1.7.** *$M\backslash f$ or $M\backslash g$ is $(4, 4, S)$-connected.*

Recall that $M\backslash 1$ is not $(4, 4, S)$-connected and that $4 = f$. Next we show the following.

**6.1.8.** *Let $(U, V)$ be a $(4, 4, S)$-violator of $M\backslash 1$ with 4 in $U$. Then $g \in U$, and $V$ is a 5-fan $(e, 2, s_1, s_2, s_3)$ where $\{s_1, s_2, s_3\} \cap \{1, 2, 3, 4, e, g\} = \emptyset$. Moreover, $M\backslash g$ is $(4, 4, S)$-connected.*

We may assume that $|U|, |V| \geqslant 5$, or $U$ or $V$ is a quad of $M\backslash 1$. Neither $U$ nor $V$ spans 1, so we may also assume that $a \in U$ and $b \in V$ where $\{a, b\} = \{2, 3\}$. Suppose $e \in U$. Then $(U \cup b \cup 1, V - b)$ is a 3-separation of $M$. Therefore $|V - b| \leqslant 3$, so $|V| = 4$. Hence $V$ is a quad of $M\backslash 1$. But $V$ is sequential in $M\backslash 1$; a contradiction. We conclude that $e \in V$. Then $(U \cup b, V - e - b)$ is 3-separating in $M\backslash 1\backslash e$. Hence $(U \cup b \cup 1, V - e - b)$ is 3-separating in $M\backslash e$. Thus $|V| \leqslant 7$.

Suppose $g \in V$. Then $f \in \mathrm{cl}(V)$. But, by 6.1.6, $f = 4$, so $4 \in \mathrm{cl}(V)$ and $a \in \mathrm{cl}^*_{M\backslash 1}(V \cup 4)$. Thus $(U - 4 - a, V \cup 4 \cup a \cup 1)$ is 3-separating in $M$, so $|U| \leqslant 5$. Hence $|E(M)| \leqslant 13$; a contradiction. We deduce that $g \in U$.

Now $e \in \mathrm{cl}(U)$ and $b \in \mathrm{cl}^*_{M\backslash 1}(U \cup e)$. Thus $(U \cup e \cup b \cup 1, V - e - b)$ is 3-separating in $M$. Hence $V$ is a 5-fan $(e, b, s_1, s_2, s_3)$ in $M\backslash 1$, and $\{s_1, s_2, s_3\} \cap \{1, 2, 3, 4, e, g\} = \emptyset$. Thus $\{1, b, s_1, s_2\}$ is a cocircuit of $M$. Suppose that $b = 3$. Then the circuit $\{3, 4, 5\}$ implies, by orthogonality, that $5 \in \{s_1, s_2\}$. But $\{e, 3, 5\}$ is not a circuit of $M$, so $5 = s_2$. Taking the symmetric difference of the circuits $\{e, 3, s_1\}$, $\{s_1, 5, s_3\}$, and $\{3, 4, 5\}$, we deduce that $\{e, s_3, 4\}$ is a circuit. Thus $s_3 = g$; a contradiction. We conclude that $b \neq 3$, so $(a, b) = (3, 2)$.

Finally, suppose that $M\backslash g$ is not $(4, 4, S)$-connected and recall, from 6.1.6, that $f = 4$. Then, by 6.1.7, $M\backslash f$ is $(4, 4, S)$-connected. As $(U - 4 - 3, V \cup 3)$ is 3-separating in $M\backslash 1\backslash 4$, we deduce that $(U - 4 - 3, V \cup 3 \cup 1)$ is 3-separating in $M\backslash 4$. Hence $|U| \leqslant 6$, so $|E(M)| \leqslant 12$; a contradiction. Thus $M\backslash g$ is $(4, 4, S)$-connected. We conclude that 6.1.8 holds.

As $M\backslash g$ is not internally 4-connected, it has a 4-fan $(t_1, t_2, t_3, t_4)$. Then $\{t_2, t_3, t_4, g\}$ is a cocircuit of $M$. By orthogonality, $\{t_2, t_3, t_4\}$ contains exactly one element of $\{e, f\}$ and contains exactly one element of $\{q_3, q_4\}$, say $q_3$.

Suppose $f \in \{t_2, t_3, t_4\}$. Then, by orthogonality, 3 or 5 is in $\{t_2, t_3, t_4\}$. If $3 \in \{t_2, t_3, t_4\}$, then $\{t_2, t_3, t_4\}$ meets $\{1, 2\}$; a contradiction as $q_3 \in \{t_2, t_3, t_4\}$. Thus $5 \in \{t_2, t_3, t_4\}$, so $\{f, 5, q_3, g\}$ is a cocircuit of $M$. Let $Z = \{1, 2, 3, 4, 5, e, g, q_3, q_4\}$. Then $r(Z) \leqslant 5$ and $r^*(Z) \leqslant |Z| - 3$ as $Z$ contains the cocircuits $\{2, 3, 4, e\}$, $\{f, 5, q_3, g\}$, and $\{e, 4, q_3, q_4, 5\}$. Thus $\lambda(Z) \leqslant 2$; a contradiction.

We may now assume that $e \in \{t_2, t_3, t_4\}$. First suppose that $e = t_4$. As $q_3 \in \{t_2, t_3\}$, we may assume that $t_3 = q_3$. Orthogonality between the circuit $\{t_1, t_2, q_3\}$ and the cocircuit $\{e, 4, q_3, q_4, 5\}$ implies that $4, 5$, or $q_4$ is in $\{t_1, t_2\}$. If $q_4 \in \{t_1, t_2\}$, then $\{t_1, t_2, t_3\} = \{q_3, q_4, g\}$; a contradiction. If $4 \in \{t_1, t_2\}$, then 2 or 3 is in $\{t_1, t_2\}$ by orthogonality with $\{2, 3, 4, e\}$. Thus $\{2, 4\}$ or $\{3, 4\}$ is in a triangle of $M$ other than $\{3, 4, 5\}$; a contradiction. We deduce that $5 \in \{t_1, t_2\}$. Let $Z' = \{1, 2, 3, 4, 5, e, g, q_3, q_4, t_1, t_2\}$. Then $r(Z') \leqslant 5$ and $r^*(Z') \leqslant |Z'| - 3$ as $Z'$ contains the cocircuits $\{2, 3, 4, e\}$, $\{t_2, q_3, e, g\}$, and $\{e, 4, q_3, q_4, 5\}$. Thus $\lambda(Z') \leqslant 2$; a contradiction. We conclude that $e \neq t_4$. Thus $e \in \{t_2, t_3\}$. Also $q_3 \in \{t_3, t_4\}$.

Next we show that

**6.1.9.** $q_3 \neq t_3$.

Assume the contrary. Then $e = t_2$. Orthogonality between $\{t_1, e, q_3\}$ and $\{2, 3, 4, e\}$ implies that $t_1 \in \{2, 3, 4\}$. By 6.1.1, $4 = f$, so $t_1 \neq 4$. Hence $t_1 \in \{2, 3\}$. Then $\{t_1, e, q_3\} \triangle \{g, q_3, q_4\} \triangle \{e, f, g\} =$

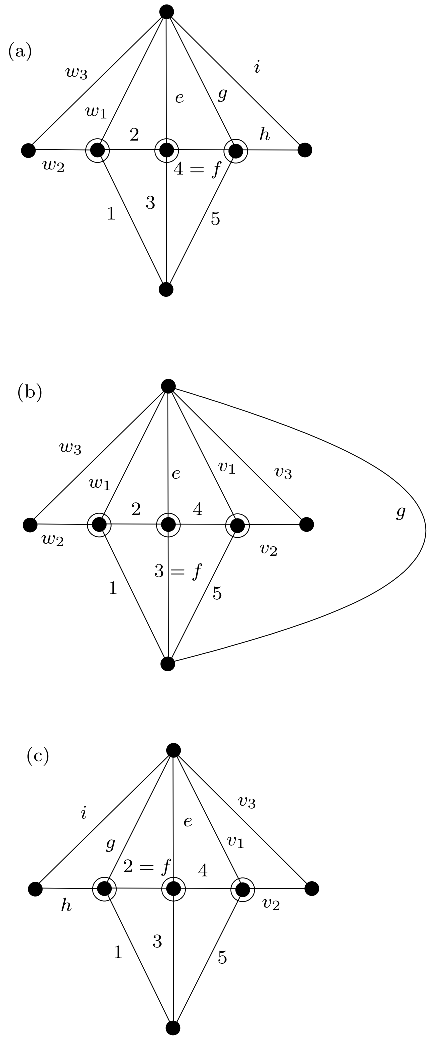


**Fig. 8.** Circled vertices correspond to known cocircuits.

$\{t_1, f, q_4\}$. Thus $\{t_1, f, q_4\}$ is a circuit, so $q_4 \in \text{cl}(\{1, 2, 3, 4, 5\})$; a contradiction. We conclude that 6.1.9 holds.

We now know that $q_3 = t_4$. By symmetry, we may assume that $e = t_2$. Thus $(t_1, e, t_3, q_3)$ is a 4-fan in $M\backslash g$. From 6.1.8, $M$ has $\{e, 2, s_1\}$ as a triangle and $g \neq s_1$. It follows by orthogonality that $\{2, s_1\}$ must meet $\{t_3, q_3, g\}$. If $q_3 \in \{2, s_1\}$, then $\{t_1, e, t_3, q_3\} \cup \{2, s_1\}$ is a 5-element 3-separating set in $M\backslash g$. This is a contradiction as $M\backslash g$ is $(4, 4, S)$-connected. Therefore $\{e, 2, s_1\} = \{e, t_1, t_3\}$. Thus 2 is $t_3$ or $t_1$.

If $2 = t_3$, then orthogonality between the cocircuit $\{e, 2, q_3, g\}$ and the circuit $\{1, 2, 3\}$ implies that $q_3 \in \{1, 3\}$, contradicting 6.1.6. Thus $2 = t_1$, so $t_3 = s_1$. The cocircuit $\{e, s_1, q_3, g\}$ meets the circuit $\{s_1, s_2, s_3\}$, so $\{s_2, s_3\}$ meets $\{e, q_3, g\}$. But, by 6.1.8, $\{s_2, s_3\}$ avoids $\{e, g\}$, so $q_3 \in \{s_2, s_3\}$. Then the 4-fan $(t_1, e, s_1, q_3)$ in $M\backslash g$ can be extended to a 5-fan; a contradiction. Thus Lemma 6.1 holds. □

Although the matroid $M$ we are dealing with need not be graphic, we follow [2] in using a modified graph diagram to keep track of some of the circuits and cocircuits in $M$ (see Fig. 8). By convention, the cycles in the graph correspond to circuits of the matroid while a circled vertex indicates a known cocircuit of $M$.

**Lemma 6.2.** *Let $M$ and $N$ be internally 4-connected matroids with $|E(M)| \geqslant 15$. Let $\{e, f, g\}$ be a triangle of $M$ such that $M\backslash e$ has an $N$-minor and is $(4, 5, S, +)$-connected. Let $(1, 2, 3, 4, 5)$ be a 5-fan in $M\backslash e$. Then each of $M\backslash e, 1$ and $M\backslash e, 5$ has an $N$-minor. Moreover,*

- (i) *$M$ has a proper minor $M'$ such that $|E(M) - E(M')| \leqslant 3$ and $M'$ is internally 4-connected with an $N$-minor*; *or*
- (ii) *$M\backslash 1$ or $M\backslash 5$ is $(4, 4, S)$-connected having an $N$-minor*; *or*
- (iii) *both $M\backslash 1$ and $M\backslash 5$ are $(4, 5, S, +)$-connected and $M$ contains one of the configurations shown in Fig. 8 where all the indicated elements are distinct unless $f = 3$, $|E(M)| = 15$, and $v_3 = w_3$.*

**Proof.** Certainly both $M\backslash e, 1$ and $M\backslash e, 5$ have $N$-minors. Moreover, as $(1, 2, 3, 4, 5)$ is a maximal fan in $M\backslash e$, by [14, Lemma 1.5], each of $M\backslash e, 1$ and $M\backslash e, 5$ is 3-connected. Assume that neither (i) nor (ii) holds. Then, by Lemma 6.1 and symmetry, we may assume that neither $M\backslash 1$ nor $M\backslash 5$ has a quad. Let $(U_5, V_5)$ be a $(4, 4, S)$-violator of $M\backslash 5$. Then $|U_5|, |V_5| \geqslant 5$. Without loss of generality, we may assume that $3 \in U_5$ and $4 \in V_5$. By Lemma 3.9(ii), each of $U_5$ and $V_5$ must contain at least two elements of $\{2, 3, 4, e\}$, so $\{2, e\} \nsubseteq U_5$ and $\{2, e\} \nsubseteq V_5$.

Next we show that

**6.2.1.** $2 \in U_5$ *and* $e \in V_5$.

Assume that this fails. Then $2 \in V_5$ and $e \in U_5$. It follows that $1 \in U_5$ otherwise $V_5$ contains $\{1, 2, 4\}$ and so spans 5, a contradiction. Now $(U_5 \cup 2, V_5 - 2)$ and $(U_5 \cup 2 \cup 4, V_5 - 2 - 4)$ are 3-separations of $M\backslash 5$. Thus $(U_5 \cup 2 \cup 4 \cup 5, V_5 - 2 - 4)$ is 3-separating in $M$. Hence $|V_5 - \{2, 4\}| \leqslant 3$, so $|V_5| \leqslant 5$.

Now $(U_5 - e, V_5)$ is 3-separating in $M\backslash 5\backslash e$. Hence so is $(U_5 - e - 3, V_5 \cup 3)$. Thus $(U_5 - e - 3, V_5 \cup 3 \cup 5)$ is 3-separating in $M\backslash e$. Therefore $|U_5 - e - 3| \leqslant 5$, so $|U_5| \leqslant 7$. But $|V_5| \leqslant 5$. Thus $|E(M)| \leqslant 13$; a contradiction. Thus 6.2.1 holds.

We show next that

**6.2.2.** $|V_5| \leqslant 7$ *and* $|U_5| \geqslant 7$.

We know that $(U_5, V_5 - e)$ is a 3-separation of $M\backslash 5\backslash e$, so $(U_5 \cup 4, V_5 - e - 4)$ is 3-separating in $M\backslash 5\backslash e$. Hence $(U_5 \cup 4 \cup 5, V_5 - e - 4)$ is 3-separating in $M\backslash e$, so $|V_5 - e - 4| \leqslant 5$ and $|V_5| \leqslant 7$. Thus $|U_5| \geqslant 7$ and 6.2.2 holds.

Next we show the following.

**6.2.3.** *If $f = 4$, then $g \in V_5$.*

Assume $g \in U_5$. We have $\lambda_{M\backslash 5}(U_5) = 2$, so $\lambda_{M\backslash 5}(U_5 \cup 4) \leqslant 3$. But $e \in \mathrm{cl}_{M\backslash 5}(U_5 \cup 4) \cap \mathrm{cl}^*_{M\backslash 5}(U_5 \cup 4)$ since $\{f, g\} \subseteq U_5 \cup 4$ and $\{2, 3, 4\} \subseteq U_5 \cup 4$. Thus $\lambda_{M\backslash 5}(U_5 \cup 4 \cup e) \leqslant 2$. Hence $(U_5 \cup \{4, e, 5\}, V_5 - \{4, e\})$ is 3-separating in $M$, so $|V_5 - \{4, e\}| \leqslant 3$. But $|V_5| \geqslant 5$. Thus $V_5 - \{4, e\}$ is a triangle or a triad of $M$.

Now $(U_5, V_5 - e)$ is a 3-separation of $M\backslash 5\backslash e$ and $4 \in \mathrm{cl}^*_{M\backslash 5, e}(U_5)$. Thus $V_5 - \{4, e\}$ is a triangle $\{v_1, v_2, v_3\}$ of $M\backslash 5, e$ where $\{4, v_1, v_2\}$ is a triad of $M\backslash 5, e$. As $4 = f$ and $g \in U_5$, it follows that $\{4, v_1, v_2, e\}$ is a cocircuit of $M\backslash 5$. We know that $M\backslash 5$ has no quads and that $V_5$ is a 5-element 3-separating set in this matroid that contains the triangle $\{v_1, v_2, v_3\}$. Suppose $r(V_5) = 3$. Then $V_5$ is a 5-fan in $M\backslash 5$. Thus $\{4, e\}$ is contained in a triangle contained in $V_5$; a contradiction. We deduce that $r(V_5) = 4$. Thus $V_5$ is a 5-cofan in $M\backslash 5$ in which the triangle is $\{v_1, v_2, v_3\}$ and the other two elements are $e$ and 4. Thus $e$ is in a triad of $M\backslash 5$. But $M\backslash 5, e$ is 3-connected; a contradiction. We conclude that 6.2.3 holds.

**6.2.4.** *If $f \in \{2,3\}$, then $g \in U_5$.*

Suppose $g \in V_5$. First assume that $f = 3$. Then $(U_5 - 3, V_5 \cup 3)$ is a 3-separation of $M\backslash 5$, so $(U_5 - 3, V_5 \cup 3 \cup 5)$ is a 3-separation of $M$; a contradiction as $|U_5 - 3| \geqslant 4$. We may now assume that $f = 2$. Then $(U_5 - 2, V_5 \cup 2)$ and $(U_5 - 2 - 3, V_5 \cup 2 \cup 3)$ are 3-separations of $M\backslash 5$. Hence $(U_5 - 2 - 3, V_5 \cup 2 \cup 3 \cup 5)$ is 3-separating in $M$. Thus $|U_5 - 2 - 3| \leqslant 3$. This contradicts 6.2.2. Hence 6.2.4 holds.

**6.2.5.** *If $f \in \{2,3\}$, then $V_5$ is a 5-fan $(e, 4, v_1, v_2, v_3)$ in $M\backslash 5$ and $\{4, v_1, v_2, 5\}$ is a cocircuit of $M$. Moreover, $\{v_1, v_2, v_3\} \cap \{1,2,3,4,5,e,g\} = \emptyset$.*

First we note that, as $f \in \{2,3\}$, by 6.2.4, $g \in U_5$. Thus each of $(U_5 \cup e, V_5 - e)$ and $(U_5 \cup e \cup 4, V_5 - e - 4)$ is 3-separating in $M\backslash 5$, so $(U_5 \cup e \cup 4 \cup 5, V_5 - e - 4)$ is 3-separating in $M$. Hence $|V_5 - e - 4| \leqslant 3$, so $|V_5| = 5$. As $e \in \mathrm{cl}_{M\backslash 5}(U_5)$ and $4 \in \mathrm{cl}^*{}_{M\backslash 5}(U_5 \cup e)$, it follows that $V_5$ is a 5-fan $(e, 4, v_1, v_2, v_3)$ as asserted. As $M$ has no 4-fans, we deduce that $\{4, v_1, v_2, 5\}$ is a cocircuit of $M$.

Now $\{2,3,4,5,e,g\} \cap \{v_1, v_2, v_3\} = \emptyset$ as $\{2,3,g\} \subseteq U_5$ and $|\{e, 4, v_1, v_2, v_3\}| = 5$. Clearly $1 \notin \{v_1, v_2\}$ by orthogonality between the cocircuit $\{4, v_1, v_2, 5\}$ and the circuit $\{1,2,3\}$. Finally, $v_3 \neq 1$ otherwise $r(\{1,2,3,4,5,e,v_1,v_2\}) \leqslant 4$ and then $\lambda(\{1,2,3,4,5,e,v_1,v_2\}) \leqslant 2$; a contradiction. Thus 6.2.5 holds.

Now let $(U_1, V_1)$ be a $(4,4,S)$-violator of $M\backslash 1$. As $M\backslash 1$ has no quads, $|U_1|, |V_1| \geqslant 5$. Without loss of generality, we may assume that $3 \in U_1$ and $2 \in V_1$. By 6.2.1, 6.2.3, 6.2.5, and symmetry, we get the following.

**6.2.6.**

(i) *$4 \in U_1$ and $e \in V_1$;*
(ii) *if $f = 2$, then $g \in V_1$; and*
(iii) *if $f \in \{3,4\}$, then $V_1$ is a 5-fan $(e, 2, w_1, w_2, w_3)$ in $M\backslash 1$ where $\{2, w_1, w_2, 1\}$ is a cocircuit of $M$ and $\{w_1, w_2, w_3\} \cap \{1,2,3,4,5,e,g\} = \emptyset$.*

We now show the following.

**6.2.7.** *If $f = 4$, then $V_5$ is a 5-fan $(e, 4, g, h, i)$ in $M\backslash 5$ and $\{4, g, h, 5\}$ is a cocircuit of $M$ where $\{h, i\} \cap \{1,2,3,4,5,e,g\} = \emptyset$.*

By 6.2.3, $g \in V_5$. If $w_1 \in V_5$, then $(U_5 - 2, V_5 \cup 2)$ and $(U_5 - 2 - 3, V_5 \cup 2 \cup 3)$ are 3-separating in $M\backslash 5$, so $(U_5 - 2 - 3, V_5 \cup 2 \cup 3 \cup 5)$ is 3-separating in $M$; a contradiction as $|U_5| \geqslant 7$ by 6.2.2. We deduce that $w_1 \in U_5$. Then $e \in \mathrm{cl}(U_5)$ and $4 \in \mathrm{cl}^*{}_{M\backslash 5}(U_5 \cup e)$. Thus $(U_5 \cup e \cup 4 \cup 5, V_5 - e - 4)$ is 3-separating in $M$. Hence $|V_5| = 5$ and $V_5$ is a 5-fan $(e, 4, g, h, i)$ in $M\backslash 5$. It follows that $\{4, g, h, 5\}$ is a cocircuit of $M$. Clearly $\{h, i\} \cap \{2,3,4,5,e,g\} = \emptyset$. Moreover, $h \neq 1$ otherwise the circuit $\{1,2,3\}$ and the cocircuit $\{4, g, h, 5\}$ meet in a single element. Finally, $i \neq 1$ otherwise $\lambda(\{1,2,3,4,5,e,g,h\}) \leqslant 2$; a contradiction. Thus 6.2.7 holds.

The case when $f = 2$ is symmetric to that when $f = 4$. We may now combine the information above to obtain that both $M\backslash 1$ and $M\backslash 5$ are $(4,5,S,+)$-connected, and that $M$ contains one of the configurations shown in Fig. 8. It is not difficult to check that all of the elements shown are distinct unless $|E(M)| = 15$, $f = 3$, and $v_3 = w_3$, otherwise $M$ has a $(4,3)$-violator. $\square$

We are now ready to complete the proof of the main theorem. This will use the following notion, which was motivated by Zhou's [18] definition of a double $k$-fan. For an integer $k \geqslant 3$, we shall say that an internally 4-connected binary matroid $M$ has a *good $k$-configuration* if $M$ has distinct elements $c_1, c_2, \dots, c_k, a_0, a_1, \dots, a_{k-1}, b_0, b_1, \dots, b_{k-1}$ such that

(i) for all $i$ in $[k-1]$, the set $\{c_i, a_i, b_i, c_{i+1}\}$ is a cocircuit and each of $\{a_{i-1}, c_i, a_i\}$ and $\{b_{i-1}, c_i, b_i\}$ is a triangle of $M$; and

(ii) when $k$ is even, $\{a_{k-1}, c_k, a_k\}$ is a triangle for some element $a_k$ not in $\{a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_{k-1}, b_{k-1}, c_{k-1}\}$; and, when $k$ is odd, $\{b_{k-1}, c_k, b_k\}$ is a triangle for some element $b_k$ not in $\{a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_{k-1}, b_{k-1}, c_{k-1}\}$; and

(iii) if $i$ is odd, then $M\backslash b_i$ is $(4, 5, S, +)$-connected and has an $N$-minor, while if $j$ is even, then $M\backslash a_j$ is $(4, 5, S, +)$-connected and has an $N$-minor.

**Proof of Theorem 1.2.** Assume that neither (i) nor (ii) holds. By Theorem 2.1, taking the dual when necessary, we may assume that $M$ has a triangle $T$ that contains an element $e$ such that $M\backslash e$ is $(4, 5, S, +)$-connected. Let $T = \{e, f, g\}$. Since $M\backslash e$ is not $(4, 4, S)$-connected, it has a 5-fan $(1, 2, 3, 4, 5)$ where we may assume that $f \in \{2, 3, 4\}$. By Lemma 6.1, neither $M\backslash 1$ nor $M\backslash 5$ has a quad. Thus neither $M\backslash 1$ nor $M\backslash 5$ is $(4, 4)$-connected. Hence, by Lemma 6.2, both $M\backslash 1$ and $M\backslash 5$ are $(4, 5, S, +)$-connected having $N$-minors, and $M$ contains one of the configurations shown in Fig. 8.

By symmetry, we may assume that $f \in \{3, 4\}$. Now, in $M$, we have distinct elements $c_1, c_2, c_3, a_0, a_1, a_2, b_0, b_1, b_2, b_3$ where $(a_0, a_1, a_2) = (5, 3, 1)$, $(b_1, b_2, b_3) = (e, w_1, w_3)$, $(c_1, c_2, c_3) = (4, 2, w_2)$, and $b_0 = g$ when $f = 4$, while $b_0 = v_1$ when $f = 3$. We also know that each of $M\backslash a_0$, $M\backslash b_1$, and $M\backslash a_2$ is $(4, 5, S, +)$-connected having an $N$-minor. We deduce that $M$ has a good $k$-configuration when $k = 3$. Let $n$ be the largest value of $k$ for which $M$ has a good $k$-configuration. We shall show that $M$ is isomorphic to one of $M(G^+_{n+1})$, $M(G_{n+1})$, $\Delta_n$, or $\Delta_n\backslash a$.

By taking symmetric differences, we see that if $M$ has a good $k$-configuration, then $\{a_0, b_0, a_i, b_i\}$ is a circuit of $M$ for all $i$ in $[k-1]$. The arguments for the cases when $n$ is odd and when $n$ is even are essentially identical. We shall present only the former. In that case, $M\backslash a_{n-1}$ is $(4, 5, S, +)$-connected having an $N$-minor and having $(b_{n-2}, c_{n-1}, b_{n-1}, c_n, b_n)$ as a fan. Applying Lemma 6.2 taking $(e, f, g) = (a_{n-1}, c_{n-1}, a_{n-2})$, we get that $M\backslash b_n$ is $(4, 5, S, +)$-connected having an $N$-minor and a 5-fan $(a_{n-1}, c_n, a_n, c_{n+1}, a_{n+1})$ for some elements $a_n, c_{n+1}, a_{n+1}$ where $\{c_n, a_n, c_{n+1}, b_n\}$ is a cocircuit of $M$. Moreover, $M\backslash a_{n+1}$ is $(4, 5, S, +)$-connected having an $N$-minor. Now, by orthogonality between the circuit $\{a_{n-1}, c_n, a_n\}$ and the cocircuits $\{c_1, a_1, b_1, c_2\}, \{c_2, a_2, b_2, c_3\}, \dots, \{c_{n-2}, a_{n-2}, b_{n-2}, c_{n-1}\}$, we deduce that $a_n \notin \{c_1, b_1, a_1, c_2, b_2, a_2, \dots, c_{n-1}, b_{n-1}, a_{n-1}\}$. Moreover, $a_n \notin \{c_n, b_n\}$. In addition, as $\{a_{n-1}, b_{n-1}, a_n, b_n\}$ and $\{a_0, b_0, a_{n-1}, b_{n-1}\}$ are circuits, their symmetric difference is either a 4-circuit or it is empty. Hence either $\{a_0, b_0\} \cap \{a_n, b_n\} = \emptyset$ and $\{a_0, b_0, a_n, b_n\}$ is a 4-circuit of $M$, or $\{a_n, b_n\} = \{a_0, b_0\}$.

Suppose first that $\{a_0, b_0\} \cap \{a_n, b_n\} = \emptyset$. The cocircuit $\{c_n, a_n, c_{n+1}, b_n\}$ implies, using orthogonality, that $c_{n+1} \notin \{a_0, b_0, a_1, b_1, c_1, \dots, a_n, b_n, c_n\}$. Now $N \preccurlyeq M\backslash a_{n+1}$ and, by Lemma 6.2, $M\backslash a_{n+1}$ is $(4, 5, S, +)$-connected. Moreover, using orthogonality, it follows that $a_{n+1} \notin \{a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_n, b_n, c_n\}$. We conclude that if $\{a_0, b_0\} \cap \{a_n, b_n\} = \emptyset$, then $M$ has a good $(n+1)$-configuration; a contradiction. It follows that $\{a_0, b_0\} = \{a_n, b_n\}$. In that case, either

(i) $(a_0, b_0) = (a_n, b_n)$; or
(ii) $(a_0, b_0) = (b_n, a_n)$.

If (i) holds, the triangles $\{a_0, c_1, a_1\}$ and $\{a_0, c_{n+1}, a_{n+1}\}$ imply that either $\{a_1, a_{n+1}, c_1, c_{n+1}\}$ is a circuit, or $\{a_1, c_1\} = \{a_{n+1}, c_{n+1}\}$. The former contradicts orthogonality with the cocircuit $\{b_0, a_0, c_n, c_{n+1}\}$. Thus $\{a_1, c_1\} = \{a_{n+1}, c_{n+1}\}$ and it is not difficult to check using orthogonality that $(a_1, c_1) = (a_{n+1}, c_{n+1})$. In case (ii), a similar argument establishes that $(b_1, c_1) = (a_{n+1}, c_{n+1})$.

Now let $Z = \{a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_n, b_n, c_n\}$. Then $r(Z) \leqslant n + 1$ and $r^*(Z) \leqslant |Z| - n$. Thus $\lambda(Z) \leqslant 1$. Hence either $Z = E(M)$, or $E(M) - Z$ contains a single element, say $z$. In the latter case, let $Z' = E(M) - \{a_n, b_n, z\}$. Then $r(Z') \leqslant n + 1$ and $r^*(Z') \leqslant |Z'| - n - 1$. It follows that equality holds in each of the last two inequalities and $\lambda(Z') = 2$. Thus $\{a_n, b_n, z\}$ is a triangle of $M$. It is now straightforward to check that, in case (i), $M$ is isomorphic to $M(G^+_{n+1})$ or $M(G_{n+1})$ depending on whether $z$ does or does not exist; and, in case (ii), $M$ is isomorphic to $\Delta_n$ or $\Delta_n\backslash a$ depending on whether $z$ does or does not exist. □


## Acknowledgments

The authors thank the referee for meticulously reading the paper and for helping us to clarify a number of points.

## References


[1] T. Asano, T. Nishizeki, P.D. Seymour, A note on non-graphic matroids, J. Combin. Theory Ser. B 37 (1984) 290–293.
[2] C. Chun, D. Mayhew, J. Oxley, A chain theorem for internally 4-connected binary matroids, J. Combin. Theory Ser. B 101 (2011) 141–189.
[3] C. Chun, D. Mayhew, J. Oxley, Towards a splitter theorem for internally 4-connected binary matroids, J. Combin. Theory Ser. B 102 (2012) 688–700.
[4] C. Chun, D. Mayhew, J. Oxley, Towards a splitter theorem for internally 4-connected binary matroids II, submitted for publication.
[5] C. Chun, D. Mayhew, J. Oxley, Towards a splitter theorem for internally 4-connected binary matroids IV, in preparation.
[6] J. Geelen, X. Zhou, A splitter theorem for internally 4-connected binary matroids, SIAM J. Discrete Math. 20 (2006) 578–587.
[7] J. Geelen, X. Zhou, Generating weakly 4-connected matroids, J. Combin. Theory Ser. B 98 (2008) 538–557.
[8] T. Johnson, R. Thomas, Generating internally four-connected graphs, J. Combin. Theory Ser. B 85 (2002) 21–58.
[9] D. Mayhew, G. Royle, G. Whittle, The binary matroids with no $M(K_{3,3})$-minor, Mem. Amer. Math. Soc. 208 (981) (2010).
[10] J.G. Oxley, On nonbinary 3-connected matroids, Trans. Amer. Math. Soc. 300 (1987) 663–679.
[11] J. Oxley, Matroid Theory, second edition, Oxford University Press, New York, 2011.
[12] J. Oxley, C. Semple, G. Whittle, Wild triangles in 3-connected matroids, J. Combin. Theory Ser. B 98 (2008) 291–323.
[13] J. Oxley, C. Semple, G. Whittle, A chain theorem for matroids, J. Combin. Theory Ser. B 98 (2008) 447–483.
[14] J. Oxley, H. Wu, On the structure of 3-connected matroids and graphs, European J. Combin. 21 (2000) 667–688.
[15] P.D. Seymour, Decomposition of regular matroids, J. Combin. Theory Ser. B 28 (1980) 305–359.
[16] W.T. Tutte, Connectivity in matroids, Canad. J. Math. 18 (1966) 1301–1324.
[17] X. Zhou, On internally 4-connected non-regular binary matroids, J. Combin. Theory Ser. B 91 (2004) 327–343.
[18] X. Zhou, Generating an internally 4-connected matroid from another, Discrete Math. 312 (2012) 2375–2387.