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Carolyn Chun
*Brunel University London*

Dillon Mayhew
*Victoria University of Wellington*

James Oxley
*Louisiana State University*

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# Towards a splitter theorem for internally 4-connected binary matroids V [☆]

Carolyn Chun [a], Dillon Mayhew [b], James Oxley [c,*]

[a] *School of Mathematical Sciences, Brunel University, London, England, United Kingdom*
[b] *School of Mathematics, Statistics and Operations Research, Victoria University, Wellington, New Zealand*
[c] *Department of Mathematics, Louisiana State University, Baton Rouge, LA, USA*

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ABSTRACT

Let $M$ be an internally 4-connected binary matroid and $N$ be an internally 4-connected proper minor of $M$. In our search for a splitter theorem for internally 4-connected binary matroids, we proved in the third paper in this series that, except when $M$ or its dual is a cubic Möbius or planar ladder or a certain coextension thereof, either $M$ has a proper internally 4-connected minor $M'$ with an $N$-minor such that $|E(M) - E(M')| \leqslant 3$, or, up to duality, $M$ has a triangle $T$ and an element $e$ of $T$ such that $M \backslash e$ has an $N$-minor and has the property that one side of every 3-separation is a fan with at most four elements. The fourth paper in the series proved that, when we cannot find such a proper internally 4-connected minor $M'$ of $M$, we can incorporate the triangle $T$ into one of two substructures of $M$, a good bowtie or a good augmented 4-wheel. The goal of this paper is essentially to eliminate the need to consider good augmented 4-wheels by showing that, when $M$ contains such a substructure, either it also contains a good bowtie, or, in an easily described way, we can obtain an internally 4-connected minor of $M$ with an $N$-minor.


## 1. Introduction

For 3-connected matroids, Seymour's Splitter Theorem [10] has proved to have numerous applications, both inductive and constructive. That theorem proves that if a 3-connected matroid $M$ has a

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* Corresponding author.
*E-mail addresses:* chchchun@gmail.com (C. Chun), dillon.mayhew@msor.vuw.ac.nz (D. Mayhew), oxley@math.lsu.edu (J. Oxley).

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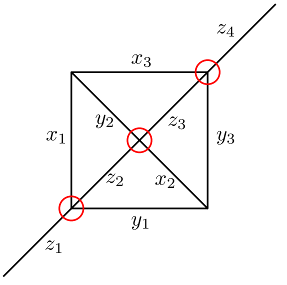


**Fig. 1.** An augmented 4-wheel.

proper 3-connected minor $N$, then $M$ has a proper 3-connected minor $M'$ with an $N$-minor such that $|E(M) - E(M')| = 1$ unless $r(M) \geqslant 3$ and $M$ is a wheel or a whirl. The current paper is the fifth in a series whose aim is to obtain a splitter theorem for binary internally 4-connected matroids. Specifically, we believe we can prove that if $M$ and $N$ are internally 4-connected binary matroids, and $M$ has a proper $N$-minor, then $M$ has a proper minor $M'$ such that $M'$ is internally 4-connected with an $N$-minor, and $M'$ can be produced from $M$ by a small number of simple operations.

In earlier work [1], we found it useful to consider weaker variants of internal 4-connectivity. The only 3-separations allowed in an internally 4-connected matroid have a triangle or a triad on one side. A 3-connected matroid $M$ is $(4,4,S)$*-connected* if, for every 3-separation $(X,Y)$ of $M$, one of $X$ and $Y$ is a triangle, a triad, or a 4-element fan, that is, a 4-element set $\{x_1,x_2,x_3,x_4\}$ that can be ordered so that $\{x_1,x_2,x_3\}$ is a triangle and $\{x_2,x_3,x_4\}$ is a triad.

To state our main theorem, we need to define some special structures. Let $M$ be an internally 4-connected binary matroid and $N$ be an internally 4-connected proper minor of $M$. Suppose $M$ has disjoint triangles $T_1$ and $T_2$ and a 4-cocircuit $D^*$ contained in their union. We call this structure a *bowtie* and denote it by $(T_1,T_2,D^*)$. If $D^*$ has an element $d$ such that $M\backslash d$ has an $N$-minor and $M\backslash d$ is $(4,4,S)$-connected, then $(T_1,T_2,D^*)$ is a *good bowtie*. An *augmented* 4*-wheel* consists of a 4-wheel restriction of $M$ with triangles $\{z_2,x_1,y_2\}$, $\{y_2,x_3,z_3\}$, $\{z_3,y_3,x_2\}$, $\{x_2,y_1,z_2\}$ along with two additional distinct elements $z_1$ and $z_4$ such that $M$ has $\{x_1,y_1,z_1,z_2\}$, $\{x_2,y_2,z_2,z_3\}$, and $\{x_3,y_3,z_3,z_4\}$ as cocircuits. We say that an augmented 4-wheel labeled in this way is *good* if $M\backslash y_1$ is $(4,4,S)$-connected having an $N$-minor, while $M\backslash y_2$ has an $N$-minor. A diagrammatic representation of an augmented 4-wheel is shown in Fig. 1. Although the matroid $M$ we are dealing with need not be graphic, we follow the convention begun in [1] of using a modified graph diagram to keep track of some of the circuits and cocircuits in $M$. By that convention, the cycles in the graph diagram correspond to circuits of the matroid while a circled vertex indicates a known cocircuit of $M$. We call $\{x_2,y_2,z_2,z_3\}$ the *central cocircuit* of the augmented 4-wheel. We require one further special graph. A *terrahawk* is the graph that is obtained from a cube by adjoining one new vertex and adding edges from the new vertex to each of the four vertices that bound some fixed face of the cube.

The purpose of this paper is to prove the following result.

**Theorem 1.1.** *Let $M$ and $N$ be internally 4-connected binary matroids such that $N$ is a proper minor of $M$. Suppose that $|E(M)| \geqslant 16$ and $|E(N)| \geqslant 6$. If $M$ contains a good augmented 4-wheel, then*

(i) *$M$ has an internally 4-connected minor $M'$ that has an $N$-minor such that either $1 \leqslant |E(M) - E(M')| \leqslant 3$; or $|E(M) - E(M')| = 4$ and $M'$ is obtained from $M$ by deleting the central cocircuit of the good augmented 4-wheel*; *or*
(ii) *$M$ has a good bowtie*; *or*
(iii) *$M$ is the cycle matroid of a terrahawk*; *or*
(iv) *$M$ contains a configuration of the form shown in Fig. 2 where all the elements shown are distinct*; *for some $n \geqslant 5$, there are $n$ dashed elements*; *and the deletion of all of the dashed elements produces an internally 4-connected matroid having an $N$-minor.*

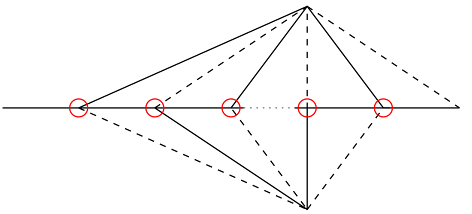

**Fig. 2.** Delete all the dashed edges.

We now describe the role that this theorem plays in obtaining the desired splitter theorem. Johnson and Thomas [6] showed that, even for graphs, a splitter theorem in the internally 4-connected case must take account of some special examples. For $n \geqslant 3$, let $G_{2n+2}$ be the *biwheel* with $2n+2$ vertices, that is, $G$ consists of a $2n$-cycle $v_1, v_2, \ldots, v_{2n}, v_1$, the *rim*, and two additional vertices, $u$ and $w$, both of which are adjacent to every $v_i$. Thus the dual of $G_{2n+2}$ is a cubic planar ladder. Let $M$ be the cycle matroid of $G_{2n+2}$ for some $n \geqslant 3$ and let $N$ be the cycle matroid of the graph that is obtained by proceeding around the rim of $G_{2n+2}$ and alternately deleting the edges from the rim vertex to $u$ and to $w$. Both $M$ and $N$ are internally 4-connected but there is no internally 4-connected proper minor of $M$ that has a proper $N$-minor. We can modify $M$ slightly and still see the same phenomenon. Let $G^+_{n+2}$ be obtained from $G_{n+2}$ by adding a new edge $a$ joining the hubs $u$ and $w$. Let $\Delta_{n+1}$ be the binary matroid that is obtained from $M(G^+_{n+2})$ by deleting the edge $v_n v_1$ and adding the third element on the line spanned by $wv_n$ and $uv_{n-1}$. This new element is also on the line spanned by $uv_n$ and $wv_{n-1}$. For $r \geqslant 3$, Mayhew, Royle, and Whittle [7] call $\Delta_r$ the *rank-r triangular Möbius matroid* and note that $\Delta_r \backslash a$ is the dual of the cycle matroid of a cubic Möbius ladder. The following is the main result of [3, Theorem 1.2].

**Theorem 1.2.** *Let $M$ be an internally 4-connected binary matroid with an internally 4-connected proper minor $N$ such that $|E(M)| \geqslant 15$ and $|E(N)| \geqslant 6$. Then*

(i) *$M$ has a proper minor $M'$ such that $|E(M) - E(M')| \leqslant 3$ and $M'$ is internally 4-connected with an $N$-minor; or*
(ii) *for some $(M_0, N_0)$ in $\{(M, N), (M^*, N^*)\}$, the matroid $M_0$ has a triangle $T$ that contains an element $e$ such that $M_0 \backslash e$ is $(4, 4, S)$-connected having an $N$-minor; or*
(iii) *$M$ is isomorphic to $M(G^+_{r+1})$, $M(G_{r+1})$, $\Delta_r$, or $\Delta_r \backslash z$ for some $r \geqslant 5$.*

That theorem prompted us to consider those matroids for which the second condition in the last theorem holds. The next theorem, the main result of [4, Theorem 1.3], identifies some more specific outcomes that occur when (ii) above holds.

**Theorem 1.3.** *Let $M$ and $N$ be internally 4-connected binary matroids such that $|E(M)| \geqslant 16$ and $|E(N)| \geqslant 6$. Suppose that $M$ has a triangle $T$ containing an element $e$ for which $M \backslash e$ is $(4, 4, S)$-connected having an $N$-minor. Then one of the following holds.*

(i) *$M$ has an internally 4-connected minor $M'$ that has an $N$-minor such that $1 \leqslant |E(M) - E(M')| \leqslant 3$; or*
(ii) *$M$ or $M^*$ has a good bowtie; or*
(iii) *$M$ or $M^*$ has a good augmented 4-wheel; or*
(iv) *$N \cong M(K_4)$ and $M$ is the cycle matroid of a terrahawk.*

By combining this theorem with Theorem 1.1, we obtain the following result.

**Corollary 1.4.** *Let $M$ and $N$ be internally 4-connected binary matroids such that $|E(M)| \geqslant 16$ and $|E(N)| \geqslant 6$. Suppose that $M$ has a triangle $T$ containing an element $e$ for which $M \backslash e$ is $(4, 4, S)$-connected having an $N$-minor. Then one of the following holds.*

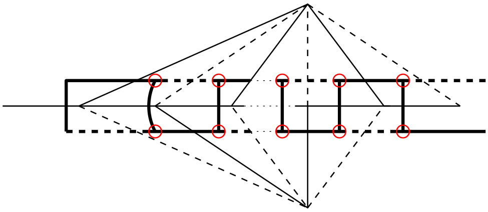

**Fig. 3.** Turning a quartic ladder segment into a cubic one.

(i) *$M$ has an internally 4-connected minor $M'$ that has an $N$-minor such that either $1 \leqslant |E(M) - E(M')| \leqslant 3$; or $|E(M) - E(M')| = 4$ and, for some $(M_1, M_2)$ in $\{(M, M'), (M^*, (M')^*)\}$, the matroid $M_2$ is obtained from $M_1$ by deleting the central cocircuit of an augmented 4-wheel*; *or*
(ii) *$M$ or $M^*$ has a good bowtie*; *or*
(iii) *$M$ is the cycle matroid of a terrahawk*; *or*
(iv) *for some $(M_0, N_0)$ in $\{(M, N), (M^*, N^*)\}$, the matroid $M_0$ contains the configuration shown in Fig. 2 where the deletion of all of the dashed elements is an internally 4-connected matroid having an $N_0$-minor.*

This points us towards examining internally 4-connected matroids having good bowties and, indeed, the remaining papers in this series deal with precisely this situation. As we can see from Fig. 3, the operation of deleting all of the dashed elements in Fig. 2 is dual to the operation of contracting alternate side elements in a quartic ladder segment, turning it into a cubic ladder segment.

## 2. Preliminaries

The matroid terminology used here will follow Oxley [8]. We shall sometimes write $N \preccurlyeq M$ to indicate that $M$ has an *$N$-minor*, that is, a minor isomorphic to the matroid $N$. If $x$ is an element of a matroid $M$ and $Y \subseteq E(M)$, we write $x \in \mathrm{cl}^{(*)}(Y)$ to mean that $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$. The property that a circuit and a cocircuit in a matroid cannot have exactly one common element will be referred to as *orthogonality*. It is well known [8, Theorem 9.1.2] that, in a binary matroid, a circuit and cocircuit must meet in an even number of elements.

Let $M$ be a matroid with ground set $E$ and rank function $r$. The *connectivity function* $\lambda_M$ of $M$ is defined on all subsets $X$ of $E$ by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate $\lambda_M$ as $\lambda$. For a positive integer $k$, a subset $X$ or a partition $(X, E - X)$ of $E$ is *$k$-separating* if $\lambda_M(X) \leqslant k - 1$. A $k$-separating partition $(X, E - X)$ is a *$k$-separation* if $|X|, |E - X| \geqslant k$. If $n$ is an integer exceeding one, a matroid is *$n$-connected* if it has no $k$-separations for all $k < n$. This definition [11] has the attractive property that a matroid is $n$-connected if and only if its dual is. Moreover, this matroid definition of $n$-connectivity is relatively compatible with the graph notion of $n$-connectivity when $n$ is 2 or 3. For example, when $G$ is a graph with at least four vertices and with no isolated vertices, $M(G)$ is a 3-connected matroid if and only if $G$ is a 3-connected simple graph. But the link between $n$-connectivity for matroids and graphs breaks down for $n \geqslant 4$. In particular, a 4-connected matroid with at least six elements cannot have a triangle. Hence, for $r \geqslant 3$, neither $M(K_{r+1})$ nor $PG(r-1, 2)$ is 4-connected. This motivates the consideration of other types of 4-connectivity in which certain 3-separations are allowed. In particular, a matroid is *internally 4-connected* if it is 3-connected and, whenever $(X, Y)$ is a 3-separation, either $|X| = 3$ or $|Y| = 3$. Equivalently, a 3-connected matroid is internally 4-connected if and only if, for every 3-separation $(X, Y)$ of $M$, either $X$ or $Y$ is a triangle or a triad of $M$. As Geelen and Zhou [5, p. 539] note, "For binary matroids, internal 4-connectivity is certainly the most natural variant of 4-connectivity". A graph $G$ without isolated vertices is *internally 4-connected* if $M(G)$ is internally 4-connected.

Let $M$ be a matroid. A subset $S$ of $E(M)$ is a *fan* in $M$ if $|S| \geqslant 3$ and there is an ordering $(s_1, s_2, \dots, s_n)$ of $S$ such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \dots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads. We call $(s_1, s_2, \dots, s_n)$ a *fan ordering* of $S$. We will be mainly concerned with 4-element

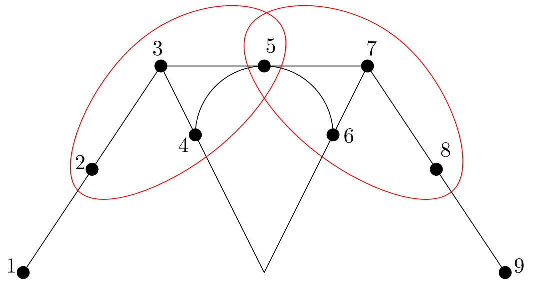


**Fig. 4.** A quasi rotor.

and 5-element fans. For convenience, we shall always view a fan ordering of a 4-element fan as beginning with a triangle and we shall use the term 4-*fan* to refer to both the 4-element fan and such a fan ordering of it. Moreover, we shall use the terms 5-*fan* and 5-*cofan* to refer to the two different types of 5-element fan where the first contains two triangles and the second two triads. Let $(s_1, s_2, \dots, s_n)$ be a fan ordering of a fan $S$. When $n \geqslant 4$, every fan ordering of $S$ has its first and last elements in $\{s_1, s_n\}$. We call these elements the *ends* of the fan while the elements of $S - \{s_1, s_n\}$ are called the *internal elements* of the fan. When $(s_1, s_2, s_3, s_4)$ is a 4-fan, our convention is that $\{s_1, s_2, s_3\}$ is a triangle, and we call $s_1$ the *guts element* of the fan and $s_4$ the *coguts element* of the fan since $s_1 \in \mathrm{cl}(\{s_2, s_3, s_4\})$ and $s_4 \in \mathrm{cl}^*(\{s_1, s_2, s_3\})$.

A set $U$ in a matroid $M$ is *fully closed* if it is closed in both $M$ and $M^*$. The intersection of two fully closed sets is fully closed, and the *full closure* $\mathrm{fcl}(U)$ of $U$ is the intersection of all fully closed sets that contain $U$. Let $(X, Y)$ be a partition of $E(M)$. If $(X, Y)$ is $k$-separating in $M$ for some positive integer $k$, and $y$ is an element of $Y$ that is also in $\mathrm{cl}^{(*)}(X)$, then it is well known and easily checked that $(X \cup y, Y - y)$ is $k$-separating, and we say that we have *moved* $y$ into $X$. More generally, $(\mathrm{fcl}(X), Y - \mathrm{fcl}(X))$ is $k$-separating in $M$. Let $n$ be an integer $n$ exceeding one. If $M$ is $n$-connected, an $n$-separation $(U, V)$ of $M$ is *sequential* if $\mathrm{fcl}(U)$ or $\mathrm{fcl}(V)$ is $E(M)$. In particular, when $\mathrm{fcl}(U) = E(M)$, there is an ordering $(v_1, v_2, \dots, v_m)$ of the elements of $V$ such that $U \cup \{v_m, v_{m-1}, \dots, v_i\}$ is $n$-separating for all $i$ in $\{1, 2, \dots, m\}$. When this occurs, the set $V$ is called *sequential*. Moreover, if $n \leqslant m$, then $\{v_1, v_2, \dots, v_n\}$ is a circuit or a cocircuit of $M$. A 3-connected matroid is *sequentially* 4-*connected* if all of its 3-separations are sequential. It is straightforward to check that, when $M$ is binary, a sequential set with 3, 4, or 5 elements is a fan.

To conclude this section, we note another special structure [12], which has arisen frequently in our work towards the desired splitter theorem and which is also important here. In an internally 4-connected binary matroid $M$, we call $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ a *quasi rotor* with *central triangle* $\{4, 5, 6\}$ and *central element* 5 if $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$ are disjoint triangles in $M$ such that $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ are cocircuits and $\{3, 5, 7\}$ is a triangle (see Fig. 4).

## 3. Some properties of good augmented 4-wheels and terrahawks

This section collects together a number of lemmas that will be needed in the proof of the main theorem. The first is an easy modification of [9, Lemma 6.1] that uses the fact that $F_7$ and $F_7^*$ are the only 7-element internally 4-connected binary matroids; the second is [2, Lemma 2.2].

**Lemma 3.1.** *Let $M$ be an internally* 4*-connected binary matroid with $|E(M)| \geqslant 7$. If $e$ is an element of $M$ that is not in a triad, then $M \backslash e$ is* 3*-connected. In particular, if $f$ is an element of $M$ that is in a triangle, then $M \backslash f$ is* 3*-connected.*

**Lemma 3.2.** *Let* $(1, 2, 3, 4)$ *be a* 4*-element fan in a binary matroid $M$ that has an internally* 4*-connected minor $N$ such that $N$ has at least seven elements. Then $M \backslash 1$ or $M/4$ has an $N$-minor. Also, if* $(1, 2, 3, 4, 5)$ *is*

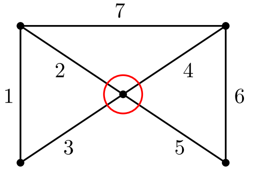


**Fig. 5.** The configuration in Lemma 3.3.

*a 5-fan in $M$, then either $M\backslash 1,5$ has an $N$-minor, or both $M/2\backslash 1$ and $M/4\backslash 5$ have $N$-minors. In particular, both $M\backslash 1$ and $M\backslash 5$ have $N$-minors.*

The following lemma will be used repeatedly in the proof of the main result.

**Lemma 3.3.** *For some $k$ in $\{2,3\}$, let $M$ be a $k$-connected matroid that contains the configuration in Fig. 5. If $(U,V)$ is a non-sequential $k$-separation of $M$, then $\mathrm{fcl}(U)$ or $\mathrm{fcl}(V)$ contains $\{1,2,\dots,7\}$.*

**Proof.** Without loss of generality, we may assume that $\{2,4,7\}\subseteq U$. If $6\in U$, then we can add 5, then 3, then 1 to $U$ to get the required result. Thus we may assume that $6\in V$. Then $5\in V$ otherwise we can move 6 into $U$. By symmetry, $\{3,1\}\subseteq V$, so $\{6,5,3,1\}\subseteq V$. Hence $\{2,4,7\}\subseteq \mathrm{cl}(V)$ and the lemma holds. $\square$

**Lemma 3.4.** *Let $M$ and $N$ be internally 4-connected binary matroids such that $N$ is a proper minor of $M$. Suppose that $|E(N)|\geqslant 7$ and that $M$ contains a good augmented 4-wheel labeled as in Fig. 1. Then $N\preccurlyeq M\backslash y_2,y_1$ and $M\backslash y_2,y_1$ is 3-connected.*

**Proof.** By the definition of a good augmented 4-wheel, as $M\backslash y_2$ has an $N$-minor and has $(y_1,z_2,x_2,z_3,y_3)$ as a 5-fan, it follows by Lemma 3.2 that $N\preccurlyeq M\backslash y_2,y_1$. Now, by [4, Lemma 2.1], $M\backslash y_2,y_1$ is 3-connected unless $M\backslash y_2$ has a triad containing $\{y_1,z_2\}$. In the exceptional case, $M$ has a 4-cocircuit containing $\{y_1,y_2,z_2\}$. Thus, by orthogonality, $x_3$ or $z_3$ is in this cocircuit. Then $\lambda(\{y_1,z_2,x_2,y_2,z_3,x_3\})\leqslant 2$; a contradiction. $\square$

Continuing to assume that $M$ contains a good augmented 4-wheel labeled as in Fig. 1, we have, by the last lemma, that $N\preccurlyeq M\backslash y_2,y_1$. As $M\backslash y_2,y_1$ has $(y_3,x_2,z_3,z_2)$ as a 4-fan, it follows that $N\preccurlyeq M\backslash y_1,y_2,y_3$ or $N\preccurlyeq M\backslash y_1,y_2/z_2$. The first of these two cases is dealt with in the next lemma. The second is treated in Theorem 4.1.

**Lemma 3.5.** *Let $M$ and $N$ be internally 4-connected binary matroids such that $N$ is a proper minor of $M$. Suppose that $|E(N)|\geqslant 7$ and that $M$ contains a good augmented 4-wheel labeled as in Fig. 1. If $N\preccurlyeq M\backslash y_1,y_2,y_3$, then either*

(i) *$M$ has a 4-cocircuit containing $\{y_1,x_2,y_3\}$; or*
(ii) *$M\backslash y_1,y_2,y_3$ is 3-connected.*

**Proof.** As $M\backslash y_1,y_2$ is 3-connected having $(y_3,z_3,x_2,z_2)$ as a 4-fan, $M\backslash y_1,y_2,y_3$ is 3-connected unless $M\backslash y_1,y_2$ has a triad $T^*$ that avoids $z_2$ but contains $\{y_3,z_3\}$ or $\{y_3,x_2\}$. If $\{y_3,z_3\}\subseteq T^*$, then orthogonality implies that $\{x_1,x_3,z_2\}$ meets $T^*$, so $\{y_1,y_2,y_3,z_3,x_1,x_3,z_2\}$ contains a cocircuit of $M$ containing $y_3,z_3\}$, so $\lambda_M(\{x_1,x_2,x_3,y_1,y_2,y_3,z_2,z_3\})\leqslant 2$; a contradiction. Thus $\{y_3,z_3\}\not\subseteq T^*$, so we may assume that $T^*=\{y_3,x_2,t\}$. Then $T^*\cup y_1$, $T^*\cup y_2$, or $T^*\cup\{y_1,y_2\}$ is a cocircuit $C^*$ of $M$. By orthogonality, $y_2\notin C^*$. Thus $\{y_3,x_2,y_2,t\}$ is a cocircuit of $M$. $\square$

We omit the straightforward proof of the following result, which identifies all internally 4-connected minors of the cycle matroid of the terrahawk.

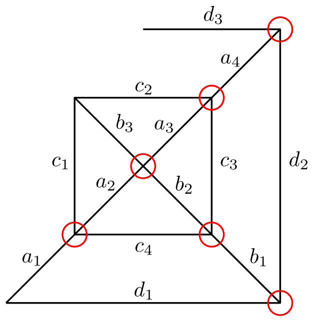


**Fig. 6.** A partial terrahawk.

**Lemma 3.6.** *The only internally 4-connected proper minors of the cycle matroid of the terrahawk are the cycle matroids of the cube, the octahedron, and $K_4$. Moreover, both the cube and the octahedron have $K_4$ as a minor.*

**Lemma 3.7.** *Assume that $M$ is an internally 4-connected binary matroid having at least sixteen elements and containing the configuration in Fig. 6 where all the elements shown are distinct. Then $M$ is isomorphic to the cycle matroid of the terrahawk.*

**Proof.** Let $A$ be the set of fourteen elements shown in the configuration in Fig. 6. Then $r(A - d_3) \leqslant 7$ and $A - d_3$ contains five cocircuits none of which is a symmetric difference of any others. Thus $\lambda(A - d_3) \leqslant 7 - 5 = 2$. Since $|E(M)| \geqslant 16$, we deduce that $\lambda(A - d_3) = 2$ and $|E(M)| = 16$. Moreover, $r(A - d_3) = 7$. Let $E(M) - A = \{b_4, d_4\}$. Clearly $\{b_4, d_4, d_3\}$ is a triangle or a triad of $M$. Since $d_3$ is already in a triad, $\{b_4, d_4, d_3\}$ must be a triad.

Now $M \backslash \{b_4, d_4, d_3\}$ has $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3\}$ as a basis. Thus adjoining $d_3$ to this set gives a basis $B$ for $M$. For each element $\alpha$ of $\{b_4, d_4\}$, the triad $\{b_4, d_4, d_3\}$ of $M$ implies, by orthogonality, that the fundamental circuit $C(\alpha, B)$ must contain $d_3$. It follows by using the other known cocircuits of $M$ that $C(\alpha, B)$ must also contain $a_4$ and $a_3$, and must avoid $b_1$ and $b_2$. Moreover, $C(\alpha, B)$ contains exactly one of $\{b_3\}$ and $\{a_2, a_1\}$. As $|C(b_4, B) \triangle C(d_4, B)| \neq 2$, it follows that we may assume that $C(b_4, B) = \{b_4, b_3, d_3, a_3, a_4\}$ and $C(d_4, B) = \{d_4, d_3, a_4, a_3, a_2, a_1\}$. Using this, we are now able to construct a binary representation for the binary matroid $M$. But that representation is the same as the one we get for the cycle matroid of the terrahawk that is obtained by adding a new vertex to the configuration in Fig. 6 where this new vertex is incident with $d_3$, $b_4$, and $d_4$, while $b_4$ and $d_4$ have their other ends meeting the vertices incident with $\{c_1, c_2\}$ and $\{a_1, d_1\}$, respectively. We conclude that $M$ is isomorphic to the cycle matroid of the terrahawk. □

## 4. A four-element win

The purpose of this section is to prove the following theorem, which will be a major step in the proof of Theorem 1.1. Although this theorem involves an augmented 4-wheel, we are not assuming that this augmented 4-wheel is good.

**Theorem 4.1.** *Let $M$ and $N$ be internally 4-connected binary matroids such that $|E(M)| \geqslant 16$ and $|E(N)| \geqslant 7$. Assume that $M$ contains an augmented 4-wheel labeled as in Fig. 1. Suppose that $M \backslash x_2, y_2/z_2$ has an $N$-minor. Then*

(i) *$M$ has an internally 4-connected minor $M'$ such that $M'$ has an $N$-minor; and either $1 \leqslant |E(M) - E(M')| \leqslant 3$ or $M'$ is obtained from $M$ by deleting the central cocircuit of the augmented 4-wheel; or*
(ii) *$M$ is the cycle matroid of a terrahawk.*

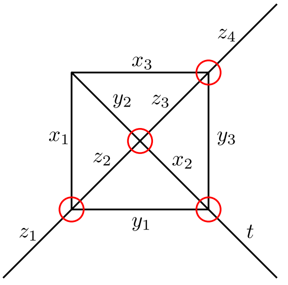


**Fig. 7.** The configuration in Lemma 4.2.

Before proving this, we establish a number of preliminary lemmas. Throughout this section and the next, we shall denote by $Z$ the set of ten elements shown in Fig. 1, that is, $Z = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, z_4\}$.

**Lemma 4.2.** *Assume that $M$ is an internally 4-connected binary matroid that contains the configuration shown in Fig. 7 where $|Z \cup t| = 11$ and $|E(M)| \geqslant 15$. If $T$ is a triangle of $M$ that meets $Z \cup t$, then either $T$ is one of $\{z_2, y_2, x_1\}$, $\{y_2, z_3, x_3\}$, $\{z_3, x_2, y_3\}$, and $\{x_2, z_2, y_1\}$, or $T \cap (Z \cup t)$ is $\{x_1, z_1\}$ or $\{x_3, z_4\}$.*

**Proof.** Suppose first that $T$ meets $\{z_2, y_2, x_2, z_3\}$ but that $T$ is not one of the four triangles shown in Fig. 7. Then, by orthogonality, $T$ contains exactly two elements of the set $\{z_2, y_2, x_2, z_3\}$. By orthogonality again, the third element of $T$ must also be in $Z \cup t$. If this third element avoids $\{z_1, t, z_4\}$, then we get a contradiction to orthogonality. Thus the third element meets $\{z_1, t, z_4\}$, so $r(Z \cup t) \leqslant 6$ and $\lambda(Z \cup t) \leqslant 2$; a contradiction.

We may now assume that $T$ avoids $\{z_2, y_2, x_2, z_3\}$. If $t \in T$, then, by orthogonality and symmetry, we may assume that $y_3 \in T$, so $T$ is $\{t, y_3, z_4\}$ or $\{t, y_3, x_3\}$. Thus, again, $r(Z \cup t) \leqslant 6$ and so we obtain the contradiction that $\lambda(Z \cup t) \leqslant 2$. We conclude that $t \notin T$. Thus $T$ avoids $\{y_1, y_3\}$ otherwise $T$ must contain $\{y_1, y_3\}$ and meet both $\{x_3, z_4\}$ and $\{x_1, z_1\}$; a contradiction. We conclude that $T$ avoids $\{z_2, y_2, x_2, z_3, t, y_1, y_3\}$. Thus $T \cap (Z \cup t) \subseteq \{x_1, z_1, x_3, z_4\}$. Hence, by orthogonality, $T \cap (Z \cup t)$ is $\{x_1, z_1\}$ or $\{x_3, z_4\}$, and the lemma holds. □

**Lemma 4.3.** *Assume that $M$ is an internally 4-connected binary matroid that contains the configuration shown in Fig. 7 where $|Z \cup t| = 11$ and $|E(M)| \geqslant 16$. Then*

(i) *$M/t$ is internally 4-connected; or*
(ii) *$M$ has distinct elements $u$ and $v$ not in $Z \cup t$ such that*
  (a) *$\{y_1, t, u, z_1\}$ is a circuit and $\{z_1, u, v\}$ is a triad; or*
  (b) *$\{y_3, t, u, z_4\}$ is a circuit and $\{z_4, u, v\}$ is a triad.*

**Proof.** We assume that $M/t$ is not internally 4-connected otherwise the lemma holds. Evidently $M/t$ is cosimple. We show next that

**4.3.1.** *$M/t$ is simple.*

Suppose that $M/t$ is not simple. Then $M$ has a triangle $T$ containing $t$. By orthogonality, $T$ must contain $y_1$, $x_2$, or $y_3$. But each of $y_1, x_2$, and $y_3$ is contained in a 4-cocircuit contained in $Z \cup t$ other than $\{t, y_1, x_2, y_3\}$. Thus $T - t \subseteq Z$. Hence $\lambda(Z \cup t) \leqslant 2$. As $|Z \cup t| \leqslant 11$, this is a contradiction. Thus 4.3.1 holds.

Next we show that

**4.3.2.** *$M/t$ is 3-connected and has a 4-fan.*

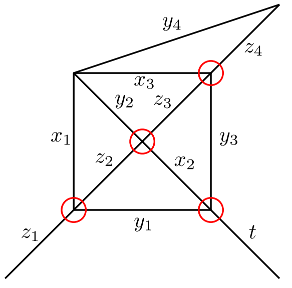


**Fig. 8.** The initial configuration in Lemma 4.4.

Since $M/t$ is simple and cosimple, if it is not 3-connected, it has a non-sequential 2-separation. If $M/t$ is 3-connected but has no 4-fan, then, as $M/t$ is not internally 4-connected, it has a non-sequential 3-separation. Thus to prove 4.3.2, it suffices to show the following.

**4.3.3.** *For each $k$ in $\{2,3\}$, the matroid $M/t$ has no non-sequential $k$-separations.*

Suppose $M/t$ has a non-sequential $k$-separation $(U,V)$ for some $k$ in $\{2,3\}$. Then $M/t$ contains the configuration in Fig. 5 where the elements of this configuration are labeled $\{x_1,x_2,y_1,y_2,y_3,z_2,z_3\}$. By Lemma 3.3, we may assume that $U$ contains the last set. Then we can adjoin $t$ to $U$ to get a non-sequential $k$-separation of $M$; a contradiction. We conclude that 4.3.3 holds. Hence so does 4.3.2.

We now know that $M/t$ has a 4-fan $(s_1,s_2,s_3,s_4)$, so $\{s_1,s_2,s_3,t\}$ is a circuit of $M$. By orthogonality, $\{s_1,s_2,s_3\}$ meets $\{y_1,x_2,y_3\}$. But each of $y_1$, $x_2$, and $y_3$ is in a triangle of $M$ so none is in the triad $\{s_2,s_3,s_4\}$. Thus $s_1 \in \{y_1,x_2,y_3\}$. If $s_1 = x_2$, then, by orthogonality, $\{s_2,s_3\}$ meets $\{z_2,y_2,z_3\}$, so $M$ has a 4-fan; a contradiction. Thus, by symmetry, we may assume that $s_1 = y_1$. By orthogonality, $\{s_2,s_3\}$ meets $\{z_1,x_1,z_2\}$, so, as neither $s_2$ nor $s_3$ is in a triangle, it follows, by symmetry, that we may assume that $z_1 = s_2$. Then (ii)(a) holds provided $\{s_3,s_4\}$ avoids $Z \cup t$. Certainly $\{s_3,s_4\} \cap \{z_1,y_1,t\} = \emptyset$. As $\{s_3,s_4\}$ is contained in a triad and hence meets no triangle, $\{s_3,s_4\}$ can only meet $Z$ in $z_4$. By orthogonality between the circuit $\{t,y_1,z_1,s_3\}$ and the cocircuit $\{z_3,x_3,y_3,z_4\}$, we deduce that $z_4 \neq s_3$. If $z_4 = s_4$, then $\lambda(Z \cup t \cup s_3) \leqslant 2$. This is a contradiction as $|Z \cup t \cup s_3| = 12$ but $|E(M)| \geqslant 16$. □

**Lemma 4.4.** *Let $M$ be an internally 4-connected binary matroid with at least sixteen elements. Suppose that $M$ contains the configuration shown in Fig. 8 where all the elements shown are distinct. Then $M \backslash y_4$ or $M/z_4 \backslash x_3$ is internally 4-connected.*

**Proof.** We show first that

**4.4.1.** *$M/z_4 \backslash x_3$ is both simple and cosimple.*

As $x_3$ is in a triangle of $M$, it is not in a triad. Hence $M/z_4 \backslash x_3$ is cosimple. Now suppose $M/z_4 \backslash x_3$ has a 2-circuit $C$. Then $C \cup z_4$ is a triangle of $M$. Lemma 4.2 implies that $x_3 \in C$; a contradiction. We conclude that 4.4.1 holds.

**4.4.2.** *For each $k$ in $\{2,3\}$, the matroid $M/z_4 \backslash x_3$ has no non-sequential $k$-separations.*

Assume that $M/z_4 \backslash x_3$ has a non-sequential $k$-separation $(U,V)$ for some $k$ in $\{2,3\}$. Then, by Lemma 3.3, we may assume that $\{x_1,x_2,y_1,y_2,y_3,z_2,z_3\} \subseteq U$. Thus we can adjoin $x_3$ and then $z_4$ to $U$ to get a non-sequential $k$-separation of $M$; a contradiction. Therefore 4.4.2 holds.

It follows immediately from 4.4.2 and Lemma 4.2 that $M/z_4 \backslash x_3$ is 3-connected. If it is internally 4-connected, then the lemma holds. Thus, we may assume, by 4.4.2 again, that $M/z_4 \backslash x_3$ has a 4-fan $(s_1,s_2,s_3,s_4)$.

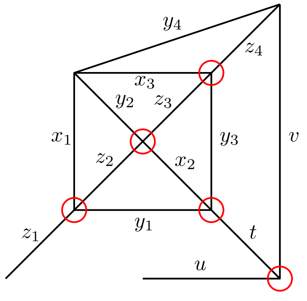


**Fig. 9.** The configuration that results from 4.4.3.

**4.4.3.** *$M$ has distinct elements $u$ and $v$ that are not in $Z \cup t$ such that $\{u, v, t\}$ is a triad and $\{v, t, y_3, z_4\}$ is a circuit.*

Suppose first that $\{x_3, s_2, s_3, s_4\}$ is a cocircuit of $M$. By orthogonality with the circuit $\{x_3, z_4, y_4\}$, it follows, since $z_4 \notin \{s_2, s_3, s_4\}$, that $y_4 \in \{s_2, s_3, s_4\}$. By orthogonality between $\{x_3, s_2, s_3, s_4\}$ and $\{x_3, y_2, z_3\}$, it follows that $\{y_2, z_3\}$ meets $\{s_2, s_3, s_4\}$. If $y_2 \in \{s_2, s_3, s_4\}$, then so is $x_1$ or $z_2$. If $z_3 \in \{s_2, s_3, s_4\}$, then so is $x_2$ or $y_3$. It follows that $\{x_3, s_2, s_3, s_4\} \subseteq (Z - z_1) \cup y_4$, so $\lambda((Z - z_1) \cup y_4) \leqslant 2$; a contradiction. We deduce that $\{x_3, s_2, s_3, s_4\}$ is not a cocircuit of $M$. Thus $\{s_2, s_3, s_4\}$ is a cocircuit of $M$. Hence $\{s_1, s_2, s_3, z_4\}$ is a circuit of $M$.

By orthogonality, $\{s_1, s_2, s_3\}$ meets $\{x_3, y_3, z_3\}$. But $x_3 \notin \{s_1, s_2, s_3\}$. As $M$ has no 4-fans, we deduce that $s_1 \in \{y_3, z_3\}$. If $s_1 = z_3$, then $\{s_2, s_3\}$ meets $\{x_2, y_2, z_2\}$; a contradiction. If $s_1 = y_3$, then $\{s_2, s_3\}$ meets $\{x_2, y_1, t\}$, so, by symmetry, we may assume that $s_2 = t$. Then, letting $(s_4, s_3) = (u, v)$, we get that 4.4.3 holds unless $\{u, v\}$ meets $Z \cup t$. Consider the exceptional case. As $\{u, v, t\}$ is a triad, none of its elements is in a triangle. Thus $z_1 \in \{u, v\}$. If $z_1 = v$, then $\{z_1, t, y_3, z_4\}$ is a circuit, so $\lambda(Z \cup t \cup z_4) \leqslant 2$; a contradiction as $|E(M)| \geqslant 16$. Hence $z_1 = u$. Then $\lambda(Z \cup t \cup v) \leqslant 2$ and again we get a contradiction. We deduce that 4.4.3 holds.

We now know that $M$ contains the configuration in Fig. 9. Since $y_4$ is in a triangle, it follows that $M \backslash y_4$ is 3-connected. We show next that

**4.4.4.** *$M \backslash y_4$ has no non-sequential 3-separations.*

Suppose that $M \backslash y_4$ has a non-sequential 3-separation $(U, V)$. Then, by Lemma 3.3, we may assume that $\{x_1, x_2, x_3, y_2, y_3, z_2, z_3\} \subseteq U$. Thus we may assume that $z_4 \in U$, so we can adjoin $y_4$ to $U$ to get a non-sequential 3-separation of $M$; a contradiction. We deduce that 4.4.4 holds.

We may now assume that $M \backslash y_4$ has a 4-fan $(t_1, t_2, t_3, t_4)$ otherwise $M \backslash y_4$ is internally 4-connected and the lemma holds. Then $\{t_2, t_3, t_4, y_4\}$ is a cocircuit of $M$. Thus $\{t_2, t_3, t_4\}$ meets $\{x_3, z_4\}$ in a single element. Suppose $x_3 \in \{t_2, t_3, t_4\}$. Then $\{t_2, t_3, t_4\}$ also contains either $z_3$ and some element of $\{x_2, y_3\}$, or $y_2$ and some element of $\{x_1, z_2\}$. In both cases, $\lambda((Z - z_1) \cup y_4) \leqslant 2$; a contradiction. We deduce that $z_4 \in \{t_2, t_3, t_4\}$. Thus, by orthogonality, $\{v, y_3, t\}$ also meets $\{t_2, t_3, t_4\}$. Suppose $z_4 = t_4$. Then $\{v, y_3, t\}$ meets the triangle $\{t_1, t_2, t_3\}$. But no triangle contains $v$ or $t$. Thus, without loss of generality, $y_3 = t_3$. Hence, by orthogonality, $t_2$ is either $x_2$ or $z_3$, and we violate orthogonality with one of the triangles $\{x_2, y_1, z_2\}$ and $\{z_3, y_2, x_3\}$. We may now assume that $z_4 = t_3$. Then, by Lemma 4.2, $\{t_1, t_2, t_3\} = \{z_4, x_3, y_4\}$; a contradiction. Thus Lemma 4.4 holds. □

**Lemma 4.5.** *Let $M$ be an internally 4-connected binary matroid with at least sixteen elements. Suppose that $M$ contains the configuration shown in Fig. 10 where all the elements shown are distinct. If $M \backslash x_1 / z_1$ is not internally 4-connected, then there is an element $w$ not in $Z \cup \{t, u, v\}$ such that either $\{v, t, w\}$ is a triad and $\{w, t, y_1, z_1\}$ is a circuit, or $\{x_1, y_2, x_3, w\}$ is a cocircuit.*

**Proof.** We show first that

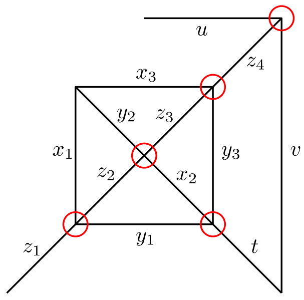


**Fig. 10.** The configuration in Lemma 4.5.

**4.5.1.** *$M\backslash x_1/z_1$ is 3-connected.*

As $x_1$ is in a triangle, $M\backslash x_1$ is 3-connected. Moreover, the last matroid has $(x_2, z_2, y_1, z_1)$ as a 4-fan. Thus we may assume that $z_1$ is in a triangle of $M$ that also contains $z_2$ or $y_1$ otherwise 4.5.1 holds. If $\{z_1, z_2\}$ is contained in a triangle, then that triangle also contains $z_3$ and we contradict orthogonality. If $\{z_1, y_1\}$ is contained in a triangle, then orthogonality implies that triangle must also contain $x_2$, $y_3$, or $t$. The first two possibilities yield contradictions. If the third holds, then $\lambda((Z - z_4) \cup t) \leqslant 2$; a contradiction. We conclude that 4.5.1 holds.

**4.5.2.** *$M\backslash x_1/z_1$ has no non-sequential 3-separations.*

Suppose that $M\backslash x_1/z_1$ has a non-sequential 3-separation $(U, V)$. By Lemma 3.3, we may assume that $\{x_2, x_3, y_1, y_2, y_3, z_2, z_3\} \subseteq U$. Thus we can adjoin $x_1$ and then $z_1$ to $U$ to obtain a non-sequential 3-separation of $M$; a contradiction. We deduce that 4.5.2 holds.

Since $M\backslash x_1/z_1$ is not internally 4-connected, we now know that this matroid has a 4-fan $(s_1, s_2, s_3, s_4)$. Assume first that $\{x_1, s_2, s_3, s_4\}$ is a cocircuit of $M$. Then, by orthogonality, $\{s_2, s_3, s_4\}$ contains either $z_2$ and a member of $\{y_1, x_2\}$, or $y_2$ and a member of $\{x_3, z_3\}$. If $\{s_2, s_3, s_4\}$ contains $\{z_2, x_2\}$, then it also contains $y_3$ or $z_3$ and we obtain the contradiction that $\lambda(\{x_1, y_2, z_2, x_2, z_3, y_3\}) \leqslant 2$. If $\{s_2, s_3, s_4\}$ contains $\{z_2, y_1\}$, then we obtain the contradiction that $\{x_1, s_2, s_3, s_4\}$ is $\{x_1, z_2, y_1, z_1\}$. If $\{s_2, s_3, s_4\}$ contains $\{y_2, z_3\}$, then it also contains $y_3$ or $x_2$, and $\lambda(\{x_1, y_2, z_2, x_2, z_3, y_3\}) \leqslant 2$; a contradiction. Finally, if $\{s_2, s_3, s_4\}$ contains $\{y_2, x_3\}$, then $M$ has a 4-cocircuit containing $\{x_1, y_2, x_3\}$. Let $w$ be the fourth element of this cocircuit. Then the lemma holds, unless $w \in Z \cup \{t, u, v\}$. Consider the exceptional case. If $w \in Z \cup t$, then $\lambda(Z \cup t) \leqslant 2$. If $w = v$, then we contradict orthogonality with the circuit $\{y_3, z_4, v, t\}$; and if $w = u$, then $\lambda((Z - z_1) \cup \{t, u, v\}) \leqslant 2$; a contradiction.

We may now assume that $\{s_2, s_3, s_4\}$ is a cocircuit of $M$. Then $\{s_1, s_2, s_3, z_1\}$ is a circuit of $M$. By orthogonality, $\{s_1, s_2, s_3\}$ must meet $\{z_2, y_1\}$ so $s_1 \in \{z_2, y_1\}$ otherwise $M$ has a 4-fan. If $s_1 = z_2$, then $\{s_2, s_3\}$ meets $\{y_2, z_3, x_2\}$; a contradiction. Thus $s_1 = y_1$. Then $\{s_2, s_3\}$ meets $\{x_2, y_3, t\}$ and so contains $t$. By symmetry, we may assume that $s_2 = t$. By orthogonality, $\{s_2, s_3, s_4\}$ must meet $\{y_3, v, z_4\}$. But $y_3$ is not in a triad. Moreover, $s_3 \notin \{v, z_4\}$ by orthogonality with the cocircuit $\{u, v, z_4\}$. Thus $s_4 \in \{v, z_4\}$. Suppose $s_4 = z_4$. Then $\{y_3, z_3, z_4, x_3\} \triangle \{t, s_3, z_4\}$ is a cocircuit $\{y_3, z_3, x_3, t, s_3\}$ of $M$. Thus $\lambda(\{x_1, x_2, x_3, y_1, y_2, y_3, z_2, z_3, t, s_3\}) \leqslant 2$; a contradiction. We conclude that $s_4 = v$. Then letting $s_3 = w$, we have the required result unless $w$ is in $Z \cup \{t, u, v\}$. Consider the exceptional case. As $w$ is not in a triangle and $w \notin \{z_1, t, v\}$, it follows that $w \in \{z_4, u\}$. Since $\{z_4, u, v\}$ and $\{t, w, v\}$ are distinct triads, $w \notin \{z_4, u\}$ and the lemma follows. □

With these preliminary results, we are now ready to prove the main theorem of this section, the proof of which is quite long.

**Proof of Theorem 4.1.** Assume that the theorem fails. Then $M$ is not isomorphic to the cycle matroid of the terrahawk. First we show the following.

**4.6.1.** *$M\backslash x_2, y_2/z_2$ is connected, simple, and cosimple.*

Since $M$ is 3-connected, $M/z_2$ is certainly connected. Thus $M/z_2\backslash x_2, y_2$ is connected since $x_2$ and $y_2$ are in disjoint 2-circuits of $M/z_2$. Moreover, $M/z_2\backslash x_2, y_2$ is simple otherwise $M$ has a triangle containing $z_2$ but avoiding $\{x_1, x_2, y_1, y_2\}$. By orthogonality, such a triangle must also contain $\{z_1, z_3\}$ and some element of $\{x_3, y_3, z_4\}$; a contradiction. Suppose $M/z_2\backslash x_2, y_2$ is not cosimple. Then it has a 2-cocircuit $C^*$. Hence $C^* \cup \{x_2, y_2\}$ is a cocircuit of $M$. By orthogonality, $C^*$ meets each of $\{z_3, x_3\}$, $\{z_2, x_1\}$, $\{z_2, y_1\}$, and $\{z_3, y_3\}$. Thus $C^* = \{z_2, z_3\}$; a contradiction. We conclude that $M/z_2\backslash x_2, y_2$ is cosimple. Hence 4.6.1 holds.

**4.6.2.** *For each $k$ in $\{2, 3\}$, the matroid $M\backslash x_2, y_2/z_2$ has no non-sequential $k$-separations.*

Assume that $M\backslash x_2, y_2/z_2$ has a non-sequential $k$-separation $(U, V)$ for some $k$ in $\{2, 3\}$. By symmetry, we may assume that $\{y_1, y_3, z_3\} \subseteq U$. Now $x_3 \in V$ otherwise $(U \cup \{x_2, y_2, z_2\}, V)$ is a non-sequential $k$-separation of $M$; a contradiction. It follows that $x_1 \in V$ otherwise we can move $x_3$ into $U$. Then $z_1 \in V$ otherwise we can move $x_1$ into $U$. But now we can move $z_3$, $y_1$, and $y_3$ into $V$ and we obtain the contradiction that $(U - \{y_1, z_3, y_3\}, V \cup \{y_1, z_3, y_3\} \cup \{y_2, x_2, z_2\})$ is a non-sequential $k$-separation of $M$; a contradiction. We conclude that 4.6.2 holds.

By combining 4.6.1 and 4.6.2, we deduce that $M\backslash x_2, y_2/z_2$ is 3-connected. Moreover, we may assume that $M\backslash x_2, y_2/z_2$ has a 4-fan $(s_1, s_2, s_3, s_4)$ otherwise, by 4.6.2, $M\backslash x_2, y_2/z_2$ is internally 4-connected and the theorem holds.

Next we show that

**4.6.3.** *$M$ has a 4-cocircuit that contains $\{x_1, y_2, x_3\}$ or $\{y_1, x_2, y_3\}$ and also contains an element, $s$, that is not in $Z$.*

Suppose first that $\{s_2, s_3, s_4\}$ is a triad of $M$. Then $\{s_1, s_2, s_3, z_2\}$ is a circuit of $M$. Thus, by orthogonality, $\{s_1, s_2, s_3\}$ meets $\{y_2, x_2, z_3\}$. But $\{x_2, y_2\}$ avoids $\{s_1, s_2, s_3\}$, so $z_3 \in \{s_1, s_2, s_3\}$. As $M$ has no 4-fans, we see that $z_3 = s_1$. Then $\{z_3, s_2, s_3, z_2\}$ is a circuit of $M$. By orthogonality, $\{s_2, s_3\}$ meets each of $\{x_1, y_1, z_1\}$ and $\{x_3, y_3, z_4\}$. As none of $x_1, y_1, x_3$, and $y_3$ is in a triad, we deduce that $\{s_2, s_3\} = \{z_1, z_4\}$. Then $M$ has $\{z_1, z_2, z_3, z_4\}$ as a circuit, so $\lambda(Z) \leqslant 2$; a contradiction.

We may now assume that $\{s_2, s_3, s_4\}$ is not a triad of $M$. Then $M$ has a cocircuit $C^*$ such that $\{s_2, s_3, s_4\} \subsetneqq C^* \subseteq \{s_2, s_3, s_4, x_2, y_2\}$. Suppose $C^*$ contains $\{x_2, y_2\}$. Then, since $z_2 \notin C^*$, it follows, as $M$ is binary, that $C^*$ avoids $z_3$. Then, by orthogonality, $C^*$ contains $\{x_1, x_3, y_1, y_3\}$; a contradiction. We deduce that $C^*$ does not contain $\{x_2, y_2\}$, so $C^*$ contains exactly one of $x_2$ and $y_2$. We shall assume that $x_2 \in C^*$. A symmetric argument will cover the case when $y_2 \in C^*$.

By orthogonality with $\{x_2, z_2, y_1\}$, we see that $y_1 \in C^*$. As $x_2 \in C^*$, orthogonality implies that $z_3$ or $y_3$ is in $C^*$. Suppose $z_3 \in C^*$. Then $x_3 \in C^*$, so $C^* = \{x_2, y_1, z_3, x_3\}$ and therefore $\lambda(\{y_1, z_2, x_2, y_2, z_3, x_3\}) \leqslant 2$; a contradiction. We deduce that $z_3 \notin C^*$, so $y_3 \in C^*$. Hence $C^*$ is a 4-cocircuit containing $\{x_2, y_1, y_3\}$. Let the fourth element of this cocircuit be $s$. If $s \in Z$, then $\lambda(Z) \leqslant 2$; a contradiction as $|E(M)| \geqslant 16$. Hence $s \notin Z$ and 4.6.3 holds.

By 4.6.3 and symmetry, we may assume that $M$ has a 4-cocircuit $\{y_1, x_2, y_3, t\}$ where $t \notin Z$, so $M$ contains the configuration in Fig. 7. Since $M\backslash x_2, y_2/z_2$ is 3-connected having an $N$-minor and having $(z_3, y_1, y_3, t)$ as a 4-fan, either $M\backslash x_2, y_2/z_2/t$ or $M\backslash x_2, y_2/z_2\backslash z_3$ has an $N$-minor. Next we establish the following.

**4.6.4.** *If $N \preccurlyeq M\backslash x_2, y_2, z_3/z_2$, then there is an element $u$ that is not in $Z \cup t$ such that $\{z_1, x_1, u\}$ or $\{x_3, z_4, u\}$ is a circuit of $M$.*

Because the theorem fails, we know that $M\backslash x_2, y_2, z_3/z_2$ is not internally 4-connected. Now $M\backslash x_2, y_2/z_2\backslash z_3$ is 3-connected unless $z_3$ is in a triad $T^*$ of $M\backslash x_2, y_2/z_2$. Consider the exceptional case. Then $T^*$ meets $\{x_1, x_3\}$ and $\{y_1, y_3\}$, so $\lambda(\{x_1, x_2, x_3, y_1, y_2, y_3, z_2, z_3\}) \leqslant 2$; a contradiction. Thus $M\backslash x_2, y_2, z_3/z_2$ is indeed 3-connected.

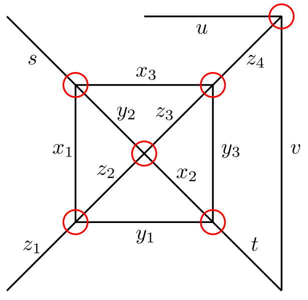


**Fig. 11.** A configuration obtained by Lemma 4.5.

Now assume that $M\backslash x_2, y_2, z_3/z_2$ has a non-sequential 3-separation $(U, V)$. We may also assume that $\{y_1, y_3, t\} \subseteq U$. Suppose $x_3$ or $x_1$ is in $U$. Then we may assume that $\{x_3, x_1\} \subseteq U$ and hence that $\{z_4, z_1\} \subseteq U$. Thus $(U \cup \{z_2, x_2, y_2, z_3\}, V)$ is a non-sequential 3-separation of $M$. It follows that we may assume that $\{x_3, x_1\} \subseteq V$. Then $\{z_4, z_1\} \subseteq V$ otherwise we can move $x_3$ or $x_1$ into $U$. But now we can move $y_1$ and $y_3$ into $V$ to get that $\{U - \{y_1, y_3\}, V \cup \{z_2, x_2, y_2, z_3\} \cup \{y_1, y_3\})$ is a non-sequential 3-separation of $M$; a contradiction. We conclude that $M\backslash x_2, y_2, z_3/z_2$ is sequentially 4-connected, so it must have a 4-fan $(t_1, t_2, t_3, t_4)$.

First observe that $\{t_1, t_2, t_3\}$ is a triangle of $M$ otherwise $\{t_1, t_2, t_3, z_2\}$ is a circuit of $M$ that violates orthogonality with $\{x_2, y_2, z_2, z_3\}$. As $M$ has no 4-fans, it has a cocircuit $C^*$ such that $\{t_2, t_3, t_4\} \subsetneq C^* \subseteq \{t_2, t_3, t_4, x_2, y_2, z_3\}$. Moreover, $z_2 \notin C^*$.

Now $\{t_1, t_2, t_3, t_4\}$ avoids $\{x_2, y_2, z_2, z_3\}$. Thus, by Lemma 4.2, either $\{t_1, t_2, t_3\}$ meets $Z \cup t$ in $\{x_1, z_1\}$ or $\{x_3, z_4\}$; or $\{t_1, t_2, t_3\}$ avoids $Z \cup t$. In the former case, 4.6.4 holds. Now consider the latter case. Then $C^*$ meets $\{x_2, y_2, z_3\}$ as $|C^*| \geqslant 4$. Suppose $z_3 \in C^*$. Then the circuits $\{z_3, x_3, x_1, z_2\}$ and $\{z_3, y_3, y_1, z_2\}$ imply, by orthogonality, that $t_4 \in \{x_1, x_3\} \cap \{y_1, y_3\}$. This contradiction implies that $z_3 \notin C^*$. If $x_2 \in C^*$, then, as $z_3 \notin C^*$, orthogonality implies that $t_4 \in \{z_2, y_1\} \cap \{y_3\}$; a contradiction. Thus $x_2 \notin C^*$ and, similarly, $y_2 \notin C^*$. Hence $C^*$ avoids $\{x_2, y_2, z_3\}$; a contradiction. We conclude that 4.6.4 holds.

Now observe that

$$M\backslash x_2, y_2/z_2 \cong M\backslash x_2, y_2/z_3 \cong M/z_3\backslash x_3, y_3 \cong M\backslash x_3, y_3/z_4.$$

Thus, as $N \preccurlyeq M\backslash x_2, y_2/z_2$, we see that

**4.6.5.** $N \preccurlyeq M/z_4$.

Next we show that

**4.6.6.** *$N \not\preccurlyeq M\backslash x_2, y_2, z_3/z_2$, so $N \preccurlyeq M\backslash x_2, y_2/z_2, t$.*

Assume that $N \preccurlyeq M\backslash x_2, y_2, z_3/z_2$. Then, by 4.6.4 and symmetry, $M$ contains the configuration shown in Fig. 8. As $N \preccurlyeq M/z_4$, we see that $N \preccurlyeq M/z_4\backslash y_4$ and $N \preccurlyeq M/z_4\backslash x_3$. Thus $N \preccurlyeq M\backslash y_4$ and $N \preccurlyeq M/z_4\backslash x_3$. By Lemma 4.4, $M\backslash y_4$ or $M/z_4\backslash x_3$ is internally 4-connected; a contradiction. We conclude that 4.6.6 holds.

We now know that $N \preccurlyeq M/t$. By Lemma 4.3 and symmetry, we may assume that $M$ contains the configuration in Fig. 10.

As $N \preccurlyeq M\backslash x_2, y_2/z_2$ and $M\backslash x_2, y_2/z_2 \cong M/z_2\backslash x_1, y_1 \cong M\backslash x_1, y_1/z_1$, we see that $N \preccurlyeq M\backslash x_1/z_1$. As $M\backslash x_1/z_1$ is not internally 4-connected, Lemma 4.5 implies that there is an element $s$ not in $Z \cup \{t, u, v\}$ such that either $\{v, t, s\}$ is a triad and $\{s, t, y_1, z_1\}$ is a circuit, or $\{x_1, y_2, x_3, s\}$ is a cocircuit. If the first option occurs, then $M$ contains a relabeled form of the configuration shown in Fig. 6, so

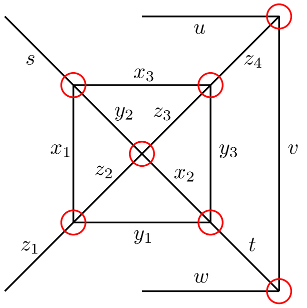


**Fig. 12.** The configuration that is used from 4.6.7 onwards.

Lemma 3.7 implies that $M$ is the cycle matroid of the terrahawk; a contradiction. We conclude that $\{x_1, y_2, x_3, s\}$ is a cocircuit, and that $M$ contains the configuration in Fig. 11, where all of the elements shown are distinct.

Now rotate the configuration in Fig. 11 a quarter turn clockwise and compare the resulting configuration with that shown in Fig. 7. By 4.6.5, $N \preccurlyeq M/z_4$ so $M/z_4$ is not internally 4-connected. Thus, by Lemma 4.3, $M$ has distinct elements $q$ and $w$ not in $(Z - z_1) \cup \{s, t\}$ such that either $\{y_3, z_4, q, t\}$ is a circuit and $\{t, q, w\}$ is a triad, or $\{x_3, z_4, q, s\}$ is a circuit and $\{s, q, w\}$ is a triad. Suppose $\{x_3, z_4, q, s\}$ is a circuit. Orthogonality with the cocircuit $\{u, v, z_4\}$ implies that $q$ is in $\{u, v\}$. If $q = v$, then $\lambda((Z - z_1) \cup \{t, v, s\}) \leqslant 2$; a contradiction. Thus $q = u$. Again rotating the configuration in Fig. 11 a quarter turn clockwise and comparing the resulting configuration with that shown in Fig. 6, we deduce, by Lemma 3.7, that $M$ is the cycle matroid of the terrahawk; a contradiction. We conclude that $\{y_3, z_4, q, t\}$ is a circuit and $\{t, q, w\}$ is a triad. Thus $q = v$ and $M$ contains the configuration shown in Fig. 12.

**4.6.7.** $w \notin Z \cup \{s, t, u, v\}$.

By construction, $w \notin (Z - z_1) \cup \{s, t\}$, and clearly $w \neq v$. If $w \in \{z_1, u\}$, then $\lambda((Z - z_1) \cup \{t, v, w\}) \leqslant 2$; a contradiction. Hence 4.6.7 holds.

Now $M\backslash x_2, y_2/z_2, t \cong M\backslash x_2, y_2/z_3, t \cong M/z_3\backslash x_3, y_3/t \cong M\backslash x_3, y_3/z_4, t$. Since $N \preccurlyeq M\backslash x_2, y_2/z_2, t$, we deduce that $N \preccurlyeq M\backslash y_3/z_4, t$.

As $y_3$ is in a triangle, $M\backslash y_3$ is 3-connected. Since $M\backslash y_3$ has $(y_2, x_3, z_3, z_4)$ as a 4-fan and has no triangle containing $z_4$, it follows that $M\backslash y_3/z_4$ is 3-connected. We deduce that $M\backslash y_3/z_4, t$ is 2-connected and cosimple. It is also simple since, by orthogonality, $M$ has no 4-circuit that contains $\{z_4, t\}$ but avoids $y_3$.

**4.6.8.** *For each $k$ in $\{2, 3\}$, the matroid $M\backslash y_3/z_4, t$ has no non-sequential $k$-separations.*

Suppose $M\backslash y_3/z_4, t$ has a non-sequential $k$-separation $(U, V)$ for some $k$ in $\{2, 3\}$. Then, by Lemma 3.3, we may assume that $\{x_1, x_2, x_3, y_1, y_2, z_2, z_3\} \subseteq U$. Thus we can adjoin $y_3$, then $t$ and $z_4$ to $U$ to get a non-sequential $k$-separation of $M$; a contradiction. We conclude that 4.6.8 holds.

By 4.6.8, we deduce that $M\backslash y_3/z_4, t$ is 3-connected. Since it is not internally 4-connected, it must have a 4-fan $(s_1, s_2, s_3, s_4)$.

**4.6.9.** $\{s_2, s_3, s_4\}$ *is a triad of $M$ avoiding $v$.*

Suppose $\{s_2, s_3, s_4\}$ is not a triad of $M$. Then $\{s_2, s_3, s_4, y_3\}$ is a cocircuit of $M$. By orthogonality, $\{s_2, s_3, s_4\}$ meets each of $\{z_4, v, t\}$, $\{z_3, x_2\}$, and $\{x_3, x_1, y_1\}$. But $\{s_1, s_2, s_3, s_4\}$ avoids $\{z_4, t\}$. Thus $v \in \{s_2, s_3, s_4\}$. As $\{s_2, s_3, s_4\} \subseteq (Z - \{z_1, z_4\}) \cup v$, it follows that $\lambda(Z \cup \{t, v\}) \leqslant 2$; a contradiction. We conclude that $\{s_2, s_3, s_4\}$ is a triad of $M$. This triad avoids $v$ since the circuit $\{t, y_3, z_4, v\}$ implies that the only triads containing $v$ contain $z_4$ or $t$, but $\{z_4, t\}$ avoids $\{s_1, s_2, s_3, s_4\}$. Thus 4.6.9 holds.

From 4.6.9, it follows that $M$ has a circuit $C$ such that $\{s_1, s_2, s_3\} \subsetneq C \subseteq \{s_1, s_2, s_3, z_4, t\}$. If $z_4 \in C$, then, by orthogonality, $\{s_1, s_2, s_3\}$ meets both $\{u, v\}$ and $\{x_3, z_3\}$; so $s_1 \in \{x_3, z_3\}$ and $u \in \{s_2, s_3\}$. If $t \in C$, then, by orthogonality, $\{s_1, s_2, s_3\}$ meets both $\{v, w\}$ and $\{y_1, x_2\}$; so $s_1 \in \{y_1, x_2\}$ and $w \in \{s_2, s_3\}$. By combining the last two sentences, we immediately obtain that $C$ contains exactly one of $z_4$ and $t$.

Suppose $t \in C$. Then $z_4 \notin C$. Moreover, as noted above, $s_1 \in \{y_1, x_2\}$ and $w \in \{s_2, s_3\}$. By orthogonality, $s_1 \neq x_2$ otherwise $\{s_2, s_3\}$ meets $\{y_2, z_2, z_3\}$; a contradiction. Therefore $s_1 = y_1$. Hence, by orthogonality with the cocircuit $\{y_1, x_1, z_2, z_1\}$, we deduce that $z_1 \in \{s_2, s_3\}$. Thus $C = \{y_1, z_1, w, t\}$. Hence $M$ contains the configuration in Fig. 6, so $M$ is the cycle matroid of the terrahawk; a contradiction.

We may now assume that $z_4 \in C$, so $C = \{z_4, s_1, s_2, s_3\}$ with $s_1 \in \{x_3, z_3\}$ and $u \in \{s_2, s_3\}$. If $s_1 = z_3$, then orthogonality implies that $\{s_2, s_3\}$ meets $\{x_2, y_2, z_2\}$; a contradiction. Thus $s_1 = x_3$. Then orthogonality implies that $C = \{z_4, x_3, s, u\}$. Adding this circuit to the configuration in Fig. 12 and then inverting the resulting configuration, we see that $M$ contains the configuration in Fig. 6. Thus, by Lemma 3.7, $M$ is the cycle matroid of the terrahawk; a contradiction. □

## 5. Proof of the main theorem

In this section, we complete the proof of the main result of the paper.

**Proof of Theorem 1.1.** We shall assume that the theorem fails. First assume that $|E(N)| = 6$. Then $N \cong M(K_4)$. As $|E(M)| \geqslant 16$, it follows by [1, Theorem 1.3] that $M$ has an internally 4-connected minor $M'$ with $1 \leqslant |E(M) - E(M')| \leqslant 3$ unless $M$ or $M^*$ is the cycle matroid of a planar or Möbius quartic ladder. If such an $M'$ exists, then, since every 3-connected binary matroid with at least six elements has an $M(K_4)$-minor, $M'$ has an $N$-minor; a contradiction. We deduce that the exceptional case arises. But then we obtain the contradiction that $M$ does not contain an augmented 4-wheel. We may now assume that $|E(N)| \geqslant 7$. Then, by Lemma 3.4, $N \preccurlyeq M\backslash y_2, y_1$ and $M\backslash y_2, y_1$ is 3-connected. Since $M\backslash y_2, y_1$ has $(y_3, x_2, z_3, z_2)$ as a 4-fan, we know that $N \preccurlyeq M\backslash y_1, y_2/z_2$ or $N \preccurlyeq M\backslash y_1, y_2, y_3$. By Theorem 4.1, $N \not\preccurlyeq M\backslash y_1, y_2/z_2$. Thus $N \preccurlyeq M\backslash y_1, y_2, y_3$. By Lemma 3.5, either $M\backslash y_1, y_2, y_3$ is 3-connected, or $M$ has $\{y_1, x_2, y_3, t\}$ as a cocircuit for some element $t$. Assume the latter holds. As $N \preccurlyeq M\backslash y_1, y_2, y_3$, we have that $N \preccurlyeq M\backslash y_1, y_3\backslash y_2/x_2$, so $N \preccurlyeq M\backslash y_2/x_2\backslash z_2, z_3$. Hence $N \preccurlyeq M\backslash x_2, y_2, z_2, z_3$, so $N \preccurlyeq M/z_2\backslash y_1, y_2$; a contradiction. Thus $M\backslash y_1, y_2, y_3$ is 3-connected. Since the theorem fails, $M\backslash y_1, y_2, y_3$ is not internally 4-connected. Next we show the following.

**Lemma 5.1.** *$M\backslash y_1, y_2, y_3$ is sequentially 4-connected.*

**Proof.** Assume that $M\backslash y_1, y_2, y_3$ has a non-sequential 3-separation $(U, V)$. Without loss of generality, $\{z_2, z_3, x_2\} \subseteq U$. If $x_1$ or $x_3$ is in $U$, then $(U \cup \{y_1, y_2, y_3\}, V)$ is a non-sequential 3-separation of $M$; a contradiction. Thus $\{x_1, x_3\} \subseteq V$. It follows that $z_1 \in V$, otherwise we can move $x_1$ into $U$. By symmetry, we deduce that $z_4 \in V$. Then $\{z_2, z_3\} \subseteq \mathrm{cl}^*_{M\backslash y_1, y_2, y_3}(V)$, so $(U - \{z_2, z_3, x_2\}, V \cup \{z_2, z_3, x_2\})$ is a non-sequential 3-separation of $M\backslash y_1, y_2, y_3$. Hence $(U - \{z_2, z_3, x_2\}, V \cup \{z_2, z_3, x_2\} \cup \{y_1, y_2, y_3\})$ is a non-sequential 3-separation of $M$; a contradiction. □

It follows immediately from the last lemma that $M\backslash y_1, y_2, y_3$ has a 4-fan and this fact will play a key role in the proof of the next lemma. Recall that $Z = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, z_4\}$.

**Lemma 5.2.** *The matroid $M$ has a triangle $\{x_3, z_4, y_4\}$ for some element $y_4$ not in $Z$.*

**Proof.** If $M$ has a triangle containing $\{x_3, z_4\}$, then the third element of this triangle is not in $Z$, otherwise $r(Z) \leqslant 5$ and we obtain the contradiction that $\lambda(Z) \leqslant 2$. We assume now that $M$ has no triangle containing $\{x_3, z_4\}$. Let $(a, b, c, d)$ be a 4-fan in $M\backslash y_1, y_2, y_3$. Since $M$ has no 4-fans, $M$ has a cocircuit $C^*$ such that $\{b, c, d\} \subsetneq C^* \subseteq \{b, c, d, y_1, y_2, y_3\}$.

**5.2.1.** *$\{a, b, c\} \cap \{x_2, z_2, z_3\} = \emptyset$ and $\{a, b, c\}$ contains neither $\{x_1, x_3\}$ nor $\{z_1, z_4\}$.*

Suppose $\{a, b, c\}$ meets $\{x_2, z_2, z_3\}$. Then, by orthogonality, $\{a, b, c\}$ contains two elements of $\{x_2, z_2, z_3\}$. As $\{a, b, c\}$ avoids $\{y_1, y_3\}$, it follows that $\{z_2, z_3\} \subseteq \{a, b, c\}$. This yields a contradiction to orthogonality. If $\{a, b, c\}$ contains $\{x_1, x_3\}$ or $\{z_1, z_4\}$, then we also get a contradiction to orthogonality. Thus 5.2.1 holds.

Next we show that

**5.2.2.** *$y_2 \notin C^*$.*

Suppose $y_2 \in C^*$. Then $C^*$ meet both $\{x_1, z_2\}$ and $\{x_3, z_3\}$. By 5.2.1, $\{b, c\}$ avoids $\{z_2, z_3\}$, and $\{b, c\} \neq \{x_1, x_3\}$. Thus $d \in \{x_1, z_2, x_3, z_3\}$, and $\{x_1, x_3\}$ meets $\{b, c\}$ in a single element. Hence $x_1$ or $x_3$ is in the triangle $\{a, b, c\}$. But $M\backslash y_1$ is $(4, 4, S)$-connected, so $x_1$ is not in a triangle avoiding $\{y_1, y_2\}$. Hence $x_3 \in \{b, c\}$, so the triangle $\{a, b, c\}$ must contain $z_4$; a contradiction. Thus 5.2.2 holds.

It follows immediately from 5.2.2 that $C^*$ meets $\{y_1, y_2, y_3\}$ in $\{y_1\}, \{y_3\}$, or $\{y_1, y_3\}$.

**5.2.3.** *If $C^* \cap \{y_1, y_2, y_3\} = \{y_3\}$, then $d = z_3$ and $\{b, c\} = \{x_3, z_4\}$.*

By orthogonality, $x_2$ or $z_3$ is in $\{b, c, d\}$. Now 5.2.1 implies that neither $x_2$ nor $z_3$ is in $\{a, b, c\}$. Thus $d$ is $x_2$ or $z_3$. Assume first that $d = x_2$. Then $\{y_3, x_2, b, c\}$ is a cocircuit, so $z_2 \in \{b, c\}$; a contradiction to 5.2.1. We deduce that $d = z_3$, so $\{y_3, z_3, b, c\}$ is a cocircuit of $M$. Hence, by orthogonality, $x_3 \in \{b, c\}$. Thus $\{y_3, z_3, b, c\}$ is the cocircuit $\{y_3, z_3, x_3, z_4\}$ so $\{b, c\} = \{x_3, z_4\}$. We conclude that 5.2.3 holds.

Since 5.2.3 implies that the lemma holds, we deduce that $C^* \cap \{y_1, y_2, y_3\} \neq \{y_3\}$.

**5.2.4.** *$C^* \cap \{y_1, y_2, y_3\} \neq \{y_1\}$.*

By symmetry from 5.2.3, we see that if $C^* \cap \{y_1, y_2, y_3\} = \{y_1\}$, then $d = z_2$ and $\{b, c\} = \{x_1, z_1\}$. But this means that $M\backslash y_1$ has a triangle containing $\{x_1, z_1\}$ and so it is not $(4, 4, S)$-connected; a contradiction. Hence 5.2.4 holds.

We now know that $C^* \cap \{y_1, y_2, y_3\} = \{y_1, y_3\}$. Next we show that

**5.2.5.** *$d = x_2$ and $\{a, b, c\} \cap Z = \emptyset$, so $C^* = \{b, c, x_2, y_1, y_3\}$.*

As $\{y_1, y_3\} \subseteq C^*$, either $x_2 \in \{b, c, d\}$, or $\{z_2, z_3\} \subseteq \{b, c, d\}$. By 5.2.1, $\{b, c\} \cap \{x_2, z_2, z_3\} = \emptyset$. Thus $x_2 = d$, so $C^* = \{b, c, x_2, y_1, y_3\}$. We now show that $\{a, b, c\} \cap Z = \emptyset$. Clearly $\{a, b, c\}$ avoids $\{y_1, y_2, y_3\}$ and, by 5.2.1, $\{a, b, c\}$ avoids $\{z_2, z_3, x_2\}$. Now $x_1 \notin \{b, c\}$ otherwise, by orthogonality, $z_2 \in \{b, c\}$. Similarly, $x_3 \notin \{b, c\}$. If $x_1 = a$, then $z_1 \in \{b, c\}$ so $M\backslash y_1$ has a 5-fan; a contradiction. If $x_3 = a$, then $z_4 \in \{b, c\}$; so $M$ has a triangle containing $\{x_3, z_4\}$; a contradiction. Finally, suppose $z_1 \in \{a, b, c\}$. Then $x_1$ or $z_2$ is in $\{a, b, c\}$; a contradiction. Hence $z_1$ avoids $\{a, b, c\}$ and, by symmetry, so does $z_4$. We conclude that 5.2.5 holds.

Since $M\backslash y_1, y_2, y_3$ has $(a, b, c, x_2)$ as a 4-fan and has an $N$-minor, either

(i) $N \preccurlyeq M\backslash y_1, y_2, y_3/x_2$; or
(ii) $N \preccurlyeq M\backslash y_1, y_2, y_3\backslash a$.

In the first case, $N \preccurlyeq M/x_2\backslash y_2, z_2, z_3$, so $N \preccurlyeq M\backslash y_2, z_2, z_3, x_2$. Hence $N \preccurlyeq M/z_2\backslash x_2, y_2$ and we obtain a contradiction by Theorem 4.1. Thus we may assume that (ii) holds. Then $N \preccurlyeq M\backslash a$.

As $a$ is in a triangle, $M\backslash a$ is 3-connected. Next we show that

**5.2.6.** *$M\backslash a$ is sequentially 4-connected.*

Assume that $M\backslash a$ has a non-sequential 3-separation $(U, V)$. Then we may assume that $b \in U$ and $c \in V$. Moreover, by Lemma 3.3, we may assume that $\{x_2, x_3, y_1, y_2, y_3, z_2, z_3\} \subseteq U$. Now, since

$C^* = \{b, c, x_2, y_1, y_3\}$, we can move $c$ into $U$ and then adjoin $a$ to $U$ to get a non-sequential 3-separation of $M\backslash a$; a contradiction. Hence 5.2.6 holds.

We now know, since $M\backslash a$ is not internally 4-connected, that it has a 4-fan.

**5.2.7.** *If $(\alpha, \beta, \gamma, \delta)$ is a 4-fan in $M\backslash a$, then $\delta \in \{b, c\}$ and $\{\alpha, \beta, \gamma\} \cap \{a, b, c\} = \emptyset$.*

Since $\{\beta, \gamma, \delta, a\}$ is a cocircuit of $M$, it follows that $\{b, c\}$ meets $\{\beta, \gamma, \delta\}$ in a single element. Suppose $\delta \notin \{b, c\}$. Then $\{b, c\}$ meets $\{\beta, \gamma\}$ so, by symmetry, we may assume that $\gamma = c$. Since $\{\alpha, \beta, \gamma\} \neq \{a, b, c\}$, it follows that $b \notin \{\alpha, \beta\}$. By orthogonality between $\{\alpha, \beta, c\}$ and the cocircuit $\{b, c, x_2, y_1, y_3\}$, we deduce that $\{\alpha, \beta\}$ meets $\{x_2, y_1, y_3\}$ in a single element.

Suppose $x_2 \in \{\alpha, \beta\}$. Then $\{z_2, z_3\}$ avoids $\{\alpha, \beta, c\}$ otherwise the triangle $\{\alpha, \beta, c\}$ contains $y_1$ or $y_3$; a contradiction. Thus, by orthogonality, $y_2 \in \{\alpha, \beta\}$. Hence $\{x_2, y_2, c\}$ is a triangle of $M$. By taking the symmetric difference of this triangle with each of the circuits $\{x_1, y_1, x_2, y_2\}$ and $\{x_2, y_2, x_3, y_3\}$, separately, we deduce that $M$ has $\{c, x_1, y_1\}$ and $\{c, x_2, y_2\}$ as circuits. The cocircuit $\{a, c, \beta, \delta\}$ must have $\{\beta, \delta\}$ meeting each of $\{x_2, y_2\}$, $\{x_1, y_1\}$, and $\{x_3, y_3\}$; a contradiction. We deduce that $x_2 \notin \{\alpha, \beta\}$.

Next suppose that $y_1 \in \{\alpha, \beta\}$. Then, by orthogonality, $\{\alpha, \beta\}$ also meets $\{z_1, x_1, z_2\}$. As $\{\alpha, \beta, c\} \neq \{y_1, z_2, x_2\}$, we see that $\{\alpha, \beta\}$ meets $\{z_1, x_1\}$. If $x_1 \in \{\alpha, \beta\}$, then $M$ has $\{x_1, y_1, c\}$ as a circuit and so has $\{x_2, y_2, c\}$ and $\{x_3, y_3, c\}$ as circuits. This gives a contradiction as above. We conclude that $\{\alpha, \beta\} = \{y_1, z_1\}$. Moreover, $c \notin Z$ by 5.2.5. We deduce that $M$ has $(\{y_1, z_1, c\}, \{x_1, z_2, y_2\}, \{y_1, z_1, x_1, z_2\})$ as a bowtie. Moreover, this bowtie is good as $M\backslash y_1$ is $(4, 4, S)$-connected having an $N$-minor; a contradiction. Hence $y_1 \notin \{\alpha, \beta\}$.

Finally, suppose that $y_3 \in \{\alpha, \beta\}$. Then, by a symmetric argument to that given in the last paragraph, we get that $M$ has $(\{y_3, z_4, c\}, \{x_3, z_3, y_2\}, \{y_3, z_4, x_3, z_3\})$ as a bowtie and $c \notin Z$. Since $N \preccurlyeq M\backslash y_3$ and this bowtie is not good, we deduce that $M\backslash y_3$ is not $(4, 4, S)$-connected. We now switch attention to the bowtie $(\{x_1, y_2, z_2\}, \{x_2, z_3, y_3\}, \{y_2, z_2, x_2, z_3\})$ and use [4, Lemma 2.6]. By that result, $\{x_2, z_3, y_3\}$ is the central triangle of a quasi rotor whose other triangles include $\{x_1, y_2, z_2\}$ and whose 4-cocircuits include $\{y_2, z_2, x_2, z_3\}$.

Suppose $z_3$ is the central element of this quasi rotor. By orthogonality, $M$ has no triangle containing $\{z_2, z_3\}$. Thus the second triangle of the quasi rotor containing $z_3$ also contains $y_2$ and so is $\{z_3, y_2, x_3\}$. Hence the second 4-cocircuit of the quasi rotor apart from $\{y_2, z_2, x_2, z_3\}$ must contain $\{z_3, y_3, x_3\}$ and so is $\{z_3, y_3, x_3, z_4\}$. Thus the quasi rotor contains a triangle containing $\{x_3, z_4\}$; a contradiction.

We may now assume that $x_2$ is the central element of this quasi rotor. Then the 4-cocircuit, $D^*$, of the quasi rotor other than $\{y_2, z_2, x_2, z_3\}$ must contain $\{x_2, y_3\}$. By orthogonality, $D^*$ must meet $\{y_1, z_2\}$ and $\{z_4, c\}$. If $z_2 \in D^*$, then $D^*$ also meets $\{x_1, y_2\}$ so $|D^*| \geqslant 5$; a contradiction. Thus $y_1 \in D^*$, so $D^*$ contains $\{y_1, x_2, y_3\}$. Then $D^* \triangle \{b, c, x_2, y_1, y_3\}$ is a triad that must meet $\{b, c\}$, so $M$ has a 4-fan. This contradiction completes the proof that $\delta \in \{b, c\}$. Certainly $\{\alpha, \beta, \gamma\}$ avoids $a$; if it meets $\{b, c\}$, then $M$ is not internally 4-connected; a contradiction. We conclude that 5.2.7 holds.

We show next that

**5.2.8.** *$M\backslash a$ is $(4, 4, S)$-connected.*

Assume that this fails. Then, by 5.2.6, $M\backslash a$ has a sequential 3-separating set with at least five elements and so has a sequential 3-separating set with exactly five elements. Thus $M\backslash a$ has a 5-fan or a 5-cofan. First suppose that $M\backslash a$ has a 5-fan $(\alpha, \beta, \gamma, \delta, \varepsilon)$. Then, by 5.2.7, $\delta \in \{b, c\}$ and $\beta \in \{b, c\}$. Thus $a \in \mathrm{cl}(\{\alpha, \beta, \gamma, \delta, \varepsilon\})$, so $M$ is not internally 4-connected; a contradiction. We deduce that $M\backslash a$ has a 5-cofan $(\omega, \alpha, \beta, \gamma, \delta)$. Then $M\backslash a$ has $(\alpha, \beta, \gamma, \delta)$ and $(\gamma, \beta, \alpha, \omega)$ as 4-fans. Thus $\delta \in \{b, c\}$ and $\omega \in \{b, c\}$, so $a \in \mathrm{cl}(\{\omega, \alpha, \beta, \gamma, \delta\})$. Hence $M$ is not internally 4-connected; a contradiction. We conclude that 5.2.8 holds.

Now let $(\alpha, \beta, \gamma, \delta)$ be a 4-fan in $M\backslash a$. Then, by 5.2.7, $\delta \in \{b, c\}$. By symmetry, we may assume that $\delta = b$ and so $(\{\alpha, \beta, \gamma\}, \{a, b, c\}, \{a, b, \beta, \gamma\})$ is a good bowtie in $M$ since $M\backslash a$ is $(4, 4, S)$-connected having an $N$-minor. This contradiction completes the proof of Lemma 5.2. □

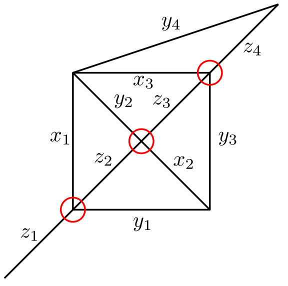


**Fig. 13.** The configuration that arises from Lemma 5.2.

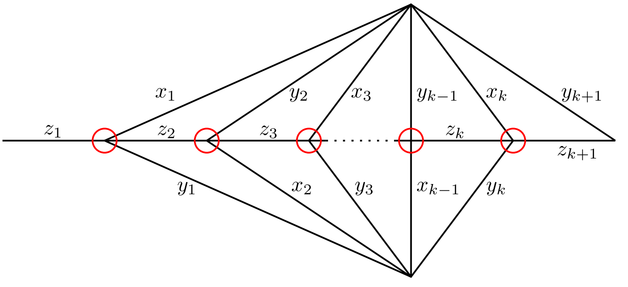


**Fig. 14.** $N \preccurlyeq M\backslash y_1, y_2, \ldots, y_k$, which is sequentially 4-connected.

By Lemma 5.2, $M$ contains the structure shown in Fig. 13. This means that, for some $k \geqslant 3$, we have a configuration of the form shown in Fig. 14 where the following hold:

(i) $M\backslash y_1$ is $(4, 4, S)$-connected;
(ii) $N \not\preccurlyeq M\backslash x_1, y_1/z_2$;
(iii) all the elements shown are distinct; and
(iv) $M\backslash y_1, y_2, \ldots, y_k$ is sequentially 4-connected having an $N$-minor.

We shall assume that $k$ is chosen to be maximal subject to these conditions. The particular form of Fig. 14 may suggest to the reader that we are requiring $k$ to be odd, but no such assumption is being made.

For the rest of the proof of Theorem 1.1, we shall be arguing in terms of the configuration shown in Fig. 14 and assuming that conditions (i)–(iv) above hold.

**Lemma 5.3.** *If $N \preccurlyeq M\backslash x_j, y_j/z_j$ for some $j$ with $2 \leqslant j \leqslant k$, then $N \preccurlyeq M\backslash x_2, y_2/z_2$ and $N \preccurlyeq M\backslash x_1, y_1/z_2$.*

**Proof.** In $M/z_j$, we have $\{x_j, y_{j-1}\}$ and $\{y_j, x_{j-1}\}$ as 2-circuits. Thus $M/z_j\backslash x_j, y_j \cong M/z_j\backslash x_{j-1}, y_{j-1} \cong M\backslash x_{j-1}, y_{j-1}/z_{j-1}$. By repeatedly applying this observation, we obtain the required result. □

Now consider $M\backslash y_1, y_2, \ldots, y_k$. It is 3-connected having an $N$-minor and having $(y_{k+1}, z_{k+1}, x_k, z_k)$ as a 4-fan. Thus either

(i) $N \preccurlyeq M\backslash y_1, y_2, \ldots, y_k\backslash y_{k+1}$; or
(ii) $N \preccurlyeq M\backslash y_1, y_2, \ldots, y_k/z_k$.

If (ii) holds, then $N \preccurlyeq M/z_k\backslash y_k, y_{k-1}$, so $N \preccurlyeq M/z_k\backslash y_k, x_k$. Thus, by Lemma 5.3, $N \preccurlyeq M/z_2\backslash y_1, x_1$; a contradiction.

**Lemma 5.4.** *$M\backslash y_1, y_2, \ldots, y_k, y_{k+1}$ is 3-connected.*

**Proof.** Assume that the lemma fails. Then $M\backslash y_1, y_2, \ldots, y_k$ has a triad containing $y_{k+1}$ and one of $x_k$ and $z_{k+1}$. Thus either $M\backslash y_1, y_2, \ldots, y_k, y_{k+1}/x_k$ or $M\backslash y_1, y_2, \ldots, y_k, y_{k+1}/z_{k+1}$ has an $N$-minor. In the first case, $N \preccurlyeq M\backslash y_1, y_2, \ldots, y_k/x_k\backslash z_{k+1}$, so $N \preccurlyeq M\backslash y_1, y_2, \ldots, y_k\backslash z_{k+1}/z_k$. Hence $N \preccurlyeq M\backslash x_k, y_k/z_k$. We obtain the same conclusion in the second case, as $N \preccurlyeq M/z_{k+1}\backslash x_k, y_k$. Thus, in both cases, we may apply Lemma 5.3 to get that $N \preccurlyeq M/z_2\backslash x_1, y_1$; a contradiction. □

**Lemma 5.5.** *$M\backslash y_1, y_2, \ldots, y_k, y_{k+1}$ is sequentially 4-connected.*

**Proof.** Assume that $M\backslash y_1, y_2, \ldots, y_k, y_{k+1}$ has a non-sequential 3-separation $(U, V)$. Then, without loss of generality, we may assume that $\{z_k, x_k, z_{k+1}\} \subseteq U$. Thus $(U \cup y_{k+1}, V)$ is a non-sequential 3-separation of $M\backslash y_1, y_2, \ldots, y_k$; a contradiction. □

We observe that $M$ has $(\{y_{k+1}, z_{k+1}, x_k\}, \{z_k, y_k, x_{k-1}\}, \{z_{k+1}, x_k, z_k, y_k\})$ as a bowtie. The following is an immediate consequence of [4, Lemma 2.6].

**Lemma 5.6.** *Exactly one of the following holds.*

(i) *$M\backslash y_{k+1}$ is internally 4-connected; or*
(ii) *$M\backslash y_{k+1}$ is $(4, 4, S)$-connected but not internally 4-connected; or*
(iii) *$\{y_{k+1}, z_{k+1}, x_k\}$ is the central triangle of a quasi rotor in $M$.*

Because $M\backslash y_{k+1}$ has an $N$-minor but the theorem fails, (i) of the last lemma cannot occur.

**Lemma 5.7.** *When $M\backslash y_{k+1}$ is $(4, 4, S)$-connected but not internally 4-connected, $M\backslash y_{k+1}$ has a unique 4-fan. Moreover, this 4-fan has $y_k$ as its guts element and has $z_{k+1}$ as an internal element.*

**Proof.** Let $(u_1, u_2, u_3, u_4)$ be a 4-fan in $M\backslash y_{k+1}$. Then $M$ has $\{u_2, u_3, u_4, y_{k+1}\}$ as a cocircuit. Thus, by orthogonality and symmetry, $\{u_3, u_4\}$ meets $\{x_k, z_{k+1}\}$. Now $u_4 \notin \{x_k, z_{k+1}\}$ otherwise $(\{u_1, u_2, u_3\}, \{x_k, z_{k+1}, y_{k+1}\}, \{u_2, u_3, u_4, y_{k+1}\})$ is a good bowtie in $M$ since $N \preccurlyeq M\backslash y_{k+1}$. Thus we may assume that $u_3 \in \{x_k, z_{k+1}\}$.

Suppose $u_3 = x_k$. By orthogonality, the cocircuit $\{u_2, u_4, x_k, y_{k+1}\}$ must meet both $\{z_k, y_{k-1}\}$ and $\{y_k, y_{k-2}, x_{k-2}\}$. Thus $\{u_2, u_4, x_k, y_{k+1}\} \subseteq \{x_{k-2}, y_{k-2}, x_{k-1}, y_{k-1}, z_{k-1}, x_k, y_k, z_k, y_{k+1}, z_{k+1}\}$. Letting the last set be $Z'$, we see that $\lambda(Z') \leqslant 2$; a contradiction. We deduce that $u_3 = z_{k+1}$.

The triangle $\{z_{k+1}, u_1, u_2\}$ meets the cocircuit $\{x_k, z_{k+1}, y_k, z_k\}$. As $\{z_{k+1}, z_k\}$ is not contained in a triangle and $x_k \notin \{u_1, u_2\}$, we deduce that $y_k \in \{u_1, u_2\}$. If $y_k = u_2$, then $\{y_k, z_{k+1}, y_{k+1}, u_4\}$ is a cocircuit, so $u_4 \in \{z_k, x_{k-1}\}$ and hence $\lambda(\{x_{k-1}, y_k, z_k, x_k, y_{k+1}, z_{k+1}\}) \leqslant 2$; a contradiction. It follows that $y_k = u_1$. Thus every 4-fan of $M\backslash y_{k+1}$ has $y_k$ as its guts element and has $z_{k+1}$ as an internal element. It follows by [4, Lemma 2.11] that there is a unique such fan, so the lemma holds. □

**Lemma 5.8.** *When $M\backslash y_{k+1}$ is $(4, 4, S)$-connected but not internally 4-connected, $M\backslash y_1, y_2, \ldots, y_k, y_{k+1}$ is internally 4-connected.*

**Proof.** Assume that this is false. Then, by Lemma 5.7, $M\backslash y_{k+1}$ has a unique 4-fan and this 4-fan has $y_k$ as its guts element and has $z_{k+1}$ as an internal element. Let $x_{k+1}$ be the other internal element of this 4-fan and $z_{k+2}$ be its coguts element. Then $M$ has $\{y_k, z_{k+1}, x_{k+1}\}$ as a triangle and has $\{x_{k+1}, y_{k+1}, z_{k+1}, z_{k+2}\}$ as a cocircuit. Moreover, since $M\backslash y_1, y_2, \ldots, y_k, y_{k+1}$ is sequentially 4-connected, this matroid must have a 4-fan $(w_1, w_2, w_3, w_4)$. Then $M$ has $\{w_1, w_2, w_3\}$ as a triangle and has a cocircuit $C^*$ such that $\{w_2, w_3, w_4\} \subsetneqq C^* \subseteq \{w_2, w_3, w_4, y_1, y_2, \ldots, y_{k+1}\}$.

Next we show that

**5.8.1.** $\{w_1, w_2, w_3\} \cap \{z_1, z_2, \ldots, z_{k+2}, x_1, x_2, \ldots, x_{k+1}\} = \emptyset$.

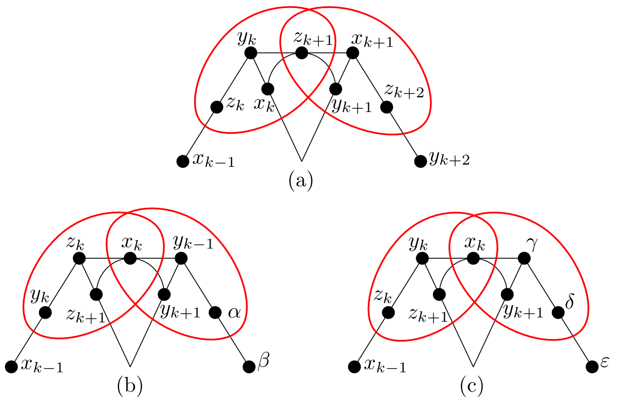


**Fig. 15.** Possible labeled quasi rotors.

First observe that $\{x_1, z_1\}$ is not in a triangle as $M\backslash y_1$ is $(4, 4, S)$-connected. Let $i = \min\{j : z_j \in \{w_1, w_2, w_3\}\}$. If $i = 1$, then, by orthogonality, $\{w_1, w_2, w_3\}$ contains $\{z_1, z_2\}$. By orthogonality again, $\{w_1, w_2, w_3\}$ is $\{z_1, z_2, x_2\}$. This is a contradiction as $\{y_1, z_2, x_2\}$ is a triangle too. Thus $i > 1$. Suppose $i = k + 2$. Then $\{z_{k+2}, x_{k+1}\}$ is contained in a triangle, so $M\backslash y_{k+1}$ is not $(4, 4, S)$-connected; a contradiction. Hence $i < k + 2$. Now, as $z_{i-1} \notin \{w_1, w_2, w_3\}$, we must have that $x_{i-1} \in \{w_1, w_2, w_3\}$. Hence $\{w_1, w_2, w_3\}$ coincides with the triangle $\{z_i, x_{i-1}, y_i\}$; a contradiction. We conclude that $\{w_1, w_2, w_3\}$ avoids $\{z_1, z_2, \dots, z_{k+2}\}$. Then, by orthogonality, $\{w_1, w_2, w_3\}$ avoids $\{x_1, x_2, \dots, x_{k+1}\}$. Hence 5.8.1 holds.

**5.8.2.** *For $j$ in $\{1, 2, \dots, k\}$, if $y_j \in C^*$, then $w_4 \in \{z_{j+1}, x_{j+1}\}$. For $i$ in $\{2, 3, \dots, k+1\}$, if $y_i \in C^*$, then $w_4 \in \{x_{i-1}, z_i\}$.*

To see the first part, observe that $\{y_j, z_{j+1}, x_{j+1}\}$ is a triangle. Then, by orthogonality, $\{z_{j+1}, x_{j+1}\}$ meets $\{w_2, w_3, w_4\}$. The first part follows immediately from 5.8.1. A similar argument establishes the second part. Thus 5.8.2 holds.

Suppose $y_m \in C^*$ for some $m$ with $2 \leqslant m \leqslant k$. Then, by 5.8.2, $w_4 \in \{z_{m+1}, x_{m+1}\} \cap \{x_{m-1}, z_m\}$; a contradiction. We deduce that $C^*$ avoids $\{y_2, y_3, \dots, y_k\}$. Hence $y_1$ or $y_{k+1}$ is in $C^*$. If both $y_1$ and $y_{k+1}$ are in $C^*$, then $w_4 \in \{z_2, x_2\} \cap \{x_k, z_{k+1}\}$; a contradiction as $k \geqslant 3$. Thus $C^*$ contains exactly one of $y_1$ and $y_{k+1}$.

Suppose $y_{k+1} \in C^*$. Then $w_4 \in \{x_k, z_{k+1}\}$. Also $(w_1, w_2, w_3, w_4)$ is a 4-fan of $M\backslash y_{k+1}$. But the unique 4-fan of $M\backslash y_{k+1}$ has $y_k$ as its guts element. This contradiction to the fact that $(w_1, w_2, w_3, w_4)$ is a 4-fan of $M\backslash y_1, y_2, \dots, y_{k+1}$ establishes that $y_{k+1} \notin C^*$.

We now know that $y_1 \in C^*$. Thus $w_4 \in \{x_2, z_2\}$. If $w_4 = x_2$, then, by orthogonality, $\{z_3, y_3\}$ meets $\{w_2, w_3\}$; a contradiction. If $w_4 = z_2$, then, by orthogonality, $\{x_1, y_2\}$ meets $\{w_2, w_3\}$; a contradiction. We conclude that Lemma 5.8 holds. $\square$

This completes the treatment of the case when (ii) of Lemma 5.6 holds. It remains to consider what happens when (iii) of Lemma 5.6 holds.

**Lemma 5.9.** *If $\{y_{k+1}, z_{k+1}, x_k\}$ is the central triangle of a quasi rotor in $M$, then $M$ has elements $x_{k+1}, z_{k+2}$, and $y_{k+2}$ such that this quasi rotor is labeled as in Fig. 15(a).*

**Proof.** By orthogonality, $M$ has no triangle containing $\{z_k, z_{k+1}\}$. Thus the quasi rotor is labeled as in (a), (b), or (c) of Fig. 15. In (b), by orthogonality, the triangle $\{y_{k-1}, z_{k-1}, x_{k-2}\}$ must contain $\alpha$. Then $\lambda(\{x_{k-2}, x_{k-1}, x_k, y_{k-2}, y_{k-1}, y_k, y_{k+1}, z_{k-1}, z_k, z_{k+1}\}) \leqslant 2$; a contradiction. In (c), the other circuits of

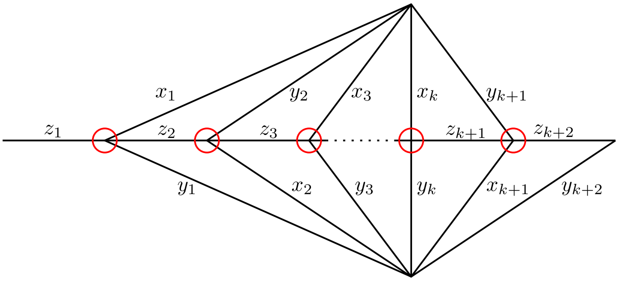


**Fig. 16.** Our basic configuration extended.

$M$ imply that $\{x_j, y_j, \gamma\}$ is a circuit for all $j$ in $\{1, 2, \ldots, k\}$. Thus, by orthogonality, $\delta \in \{x_{k-1}, y_{k-1}\} \cap \{x_{k-2}, y_{k-2}\}$; a contradiction. We conclude that (a) holds. $\square$

By the last lemma, $M$ contains the configuration shown in Fig. 16.

**Lemma 5.10.** $\{x_{k+1}, z_{k+2}, y_{k+2}\} \cap \{x_1, \ldots, x_k, y_1, \ldots, y_{k+1}, z_1, \ldots, z_{k+1}\} = \emptyset$.

**Proof.** First we observe that $\{x_{k+1}, z_{k+2}, y_{k+2}\}$ avoids $\{x_{k-1}, x_k, y_k, y_{k+1}, z_k, z_{k+1}\}$. Moreover, $y_{k-1} \notin \{x_{k+1}, z_{k+2}\}$ by orthogonality; and $y_{k-1} \neq y_{k+2}$ otherwise $\lambda(\{x_{k-1}, x_k, x_{k+1}, y_{k-1}, y_k, y_{k+1}, y_{k+2}, z_k, z_{k+1}, z_{k+2}\}) \leqslant 2$; a contradiction. Now suppose that $\{x_{k+1}, z_{k+2}, y_{k+2}\}$ meets $\{x_1, x_2, \ldots, x_{k-2}, y_1, y_2, \ldots, y_{k-2}, z_1, z_2, \ldots, z_{k-1}\}$. Then, because the last set can be covered by the set of 4-cocircuits of the form $\{z_i, x_i, y_i, z_{i+1}\}$ for $1 \leqslant i \leqslant k-2$, the triangle $\{x_{k+1}, z_{k+2}, y_{k+2}\}$ must contain exactly two elements of one of these 4-cocircuits. Now, by orthogonality, no set of the form $\{z_i, z_{i+1}\}$ with $1 \leqslant i \leqslant k-2$ is contained in a triangle.

**5.10.1.** $\{x_{k+1}, z_{k+2}, y_{k+2}\}$ *contains no set* $\{x_i, y_i\}$ *with* $1 \leqslant i \leqslant k-2$.

Suppose that $\{x_i, y_i\}$ is contained in $\{x_{k+1}, z_{k+2}, y_{k+2}\}$ for some $i$ in $\{1, 2, \ldots, k-2\}$, say $\{x_{k+1}, z_{k+2}, y_{k+2}\} = \{x_i, y_i, t\}$. Since $\{x_i, y_i, x_k, y_k\}$ is a circuit of $M$, so is $\{x_k, y_k, t\}$. Suppose $t = x_{k+1}$. Then the triangles $\{x_k, y_k, t\}$ and $\{y_k, x_{k+1}, z_{k+1}\}$ are equal. Thus $x_k = z_{k+1}$; a contradiction since the members of $\{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k+1}, z_1, z_2, \ldots, z_{k+1}\}$ are distinct. We deduce that $t \in \{z_{k+2}, y_{k+2}\}$. Thus $x_{k+1} \in \{x_i, y_i\}$. Then the triangle $\{z_{k+1}, x_{k+1}, y_k\}$ and the cocircuit $\{x_i, y_i, z_i, z_{i+1}\}$ have a single common element; a contradiction. We conclude that 5.10.1 holds.

We now know that $\{x_{k+1}, z_{k+2}, y_{k+2}\}$ meets $\{x_i, y_i, z_i, z_{i+1}\}$ in $\{x_i, z_i\}$, $\{x_i, z_{i+1}\}$, $\{y_i, z_i\}$, or $\{y_i, z_{i+1}\}$. Thus $\{x_{k+1}, z_{k+2}, y_{k+2}\}$ coincides with $\{x_i, z_i, y_{i-1}\}$, $\{x_i, z_{i+1}, y_{i+1}\}$, $\{y_i, z_i, x_{i-1}\}$, or $\{y_i, z_{i+1}, x_{i+1}\}$ unless $i = 1$ and $\{x_{k+1}, z_{k+2}, y_{k+2}\}$ contains $\{x_1, y_1\}$ or $\{y_1, z_1\}$. Provided the exceptional case does not occur, $x_{k+1}$ is in $\{x_{i-1}, y_{i-1}, x_i, y_i, z_i, x_{i+1}, y_{i+1}, z_{i+1}\}$ so the triangle $\{y_k, z_{k+1}, x_{k+1}\}$ meets some 4-cocircuit $\{x_j, y_j, z_j, z_{j+1}\}$ for $j$ in $\{i-1, i, i+1\}$ in a single element; a contradiction since $i + 1 \leqslant k-1$. It remains to consider the exceptional case. Because $M \backslash y_1$ is $(4, 4, S)$-connected, $\{x_1, z_1\}$ is not contained in a triangle. Hence we may assume that $\{y_1, z_1\} \subseteq \{x_{k+1}, y_{k+2}, z_{k+2}\}$. Letting $x_0$ be the element of $\{x_{k+1}, y_{k+2}, z_{k+2}\} - \{y_1, z_1\}$, we deduce that $M$ has $(\{x_0, y_1, z_1\}, \{x_1, y_2, z_2\}, \{x_1, y_1, z_1, z_2\})$ as a good bowtie; a contradiction. This completes the proof of Lemma 5.10. $\square$

As we now know that all the elements in Fig. 16 are distinct and that $M \backslash y_1, y_2, \ldots, y_{k+1}$ is sequentially 4-connected having an $N$-minor, we have contradicted the choice of $k$, thereby completing the proof of the theorem. $\square$

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