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Towards a splitter theorem for internally 4-connected binary matroids II

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ABSTRACT

Let M and N be internally 4-connected binary matroids such that M has a proper N-minor, and $|E(N)| \geq 7$. As part of our project to develop a splitter theorem for internally 4-connected binary matroids, we prove the following result: if $M\backslash e$ has no N-minor whenever e is in a triangle of M, and M/e has no N-minor whenever e is in a triad of M, then M has a minor, M', such that M' is internally 4-connected with an N- minor, and $1 \leq |E(M)| - |E(M')| \leq 2$.

1. Introduction

It would be useful for structural matroid theory if we could make the following statement: there exists an integer, k, such that whenever M and N are internally 4-connected binary matroids and M has a proper N-minor, then M has an internally 4-connected minor, M', such that M' has an N-minor, and $1 \leq |E(M)| - |E(M')| \leq k$. However this statement is false; no such k exists. To see this, we let M be the cycle matroid of a quartic planar ladder on n vertices, and we let N be the cycle matroid of the cubic planar ladder on the same number of vertices. Then M and N are internally 4-connected, and M has a proper minor isomorphic to N. Moreover, $|E(M)| = 2n$, and $|E(N)| = 3n/2$. However, the only proper minor of M that is internally 4-connected with an N-minor is itself isomorphic to N.

In light of this example, we concentrate on a different goal. To aid brevity, let us introduce some notation. Say that $\mathcal{S}$ is the set of all ordered pairs, (M, N) where M and N are internally 4-connected binary matroids, and M has a proper N-minor. We will let $\mathcal{S}_k$ be the subset of $\mathcal{S}$ for which there is an internally 4-connected minor, M', of M that has an N-minor and satisfies $1 \leq |E(M)| - |E(M')| \leq k$. The discussion in the previous paragraph shows that we cannot find a k so that $\mathcal{S} \subseteq \mathcal{S}_k$. Instead, we want to show that, for any $(M, N) \in \mathcal{S}$, either $(M, N) \in \mathcal{S}_k$, for some small value of k, or there is some easily described operation we can perform on M to produce an internally 4-connected minor

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that has an N-minor. To this end, we are trying to identify as many pairs as possible that belong to $\mathcal{S}_k$, for small values of k. For example, our first step [1] was to show that if M is 4-connected, then (M, N) is in $\mathcal{S}_2$. In fact, in almost every case, (M, N) belongs to $\mathcal{S}_1$.

Theorem 1.1. *Let M and N be binary matroids such that M has a proper N-minor, and $|E(N)| \geq 7$. If M is 4-connected and N is internally 4-connected, then M has an internally 4-connected minor M' with an N-minor such that $1 \leq |E(M)| - |E(M')| \leq 2$. Moreover, unless M is isomorphic to a specific 16-element self-dual matroid, such an M' exists with $|E(M)| - |E(M')| = 1$.*

An internally 4-connected binary matroid is 4-connected if and only if it has no triangles and triads. Therefore we have shown that if M has no triangles or triads (and $|E(N)| \geq 7$), then $(M, N) \in \mathcal{S}_2$. Hence we now assume that M does contain a triangle or triad. In this chapter of the series, we consider the case that all triangles and triads of M must be contained in the ground set of every N-minor. In other words, deleting an element from a triangle of M, or contracting an element from a triad, destroys all N-minors. We show that under these circumstances, (M, N) is in $\mathcal{S}_2$.

Theorem 1.2. *Let M and N be internally 4-connected binary matroids, such that $|E(N)| \geq 7$, and N is isomorphic to a proper minor of M. Assume that if T is a triangle of M and $e \in T$, then $M\backslash e$ does not have an N-minor. Dually, assume that if T is a triad of M and $e \in T$, then M/e does not have an N-minor. Then M has an internally 4-connected minor, M', such that M' has an N-minor, and $1 \leq |E(M)| - |E(M')| \leq 2$.*

With this result in hand, in the next chapter [2] we will be able to assume that (up to duality) M has a triangle T and an element $e \in T$ such that $M\backslash e$ has an N-minor.

We note that Theorem 1.2 is not strictly a strengthening of Theorem 1.1 as, in the earlier theorem, we completely characterized when (M, N) was in $\mathcal{S}_2 - \mathcal{S}_1$. We make no attempt to obtain the corresponding characterization in Theorem 1.2, as we believe that $\mathcal{S}_2 - \mathcal{S}_1$ will contain many more pairs when we relax the constraint that M is 4-connected. For example, let N be obtained from a binary projective geometry by performing a Δ-Y exchange on a triangle T. Let T' be a triangle that is disjoint from T. We obtain M from N by coextending by the element x so that it is in a triad with two elements from T', and then extending by y so that it is in a circuit with x and two elements from T. It is not difficult to confirm that the hypotheses of Theorem 1.2 hold, but M has no internally 4-connected single-element deletion or contraction with an N-minor. Clearly this technique could be applied to create even more diverse examples.

2. Preliminaries

We assume familiarity with standard matroid notions and notations, as presented in [3]. We make frequent, and sometimes implicit, use of the following well-known facts. If M is n-connected, and $|E(M)| \geq 2(n-1)$, then M has no circuit or cocircuit with fewer than n elements [3, Proposition 8.2.1]. In a binary matroid, a circuit and a cocircuit must meet in a set of even cardinality [3, Theorem 9.1.2(ii)]. The symmetric difference, $C \triangle C'$, of two circuits in a binary matroid is a disjoint union of circuits [3, Theorem 9.1.2(iv)].

We use 'by orthogonality' as shorthand for the statement 'by the fact that a circuit and a cocircuit cannot intersect in a set of cardinality one' [3, Proposition 2.1.11]. A *triangle* is a 3-element circuit, and a *triad* is a 3-element cocircuit. We use λ_M or λ to denote the connectivity function of the matroid M. If M and N are matroids, an N-*minor* of M is a minor of M that is isomorphic to N.

Let M be a matroid. A subset S of $E(M)$ is a *fan* in M if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of S such that

$$\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \ldots, \{s_{n-2}, s_{n-1}, s_n\}$$

is an alternating sequence of triangles and triads. We call $(s_1, s_2, \ldots, s_n)$ a *fan ordering* of S. Sometimes we blur the distinction between a fan and an ordering of that fan. Most of the fans we encounter have four or five elements. We adopt the following convention: if (s_1, s_2, s_3, s_4) is a fan ordering of a 4-element fan, then $\{s_1, s_2, s_3\}$ is a triangle. We call such a fan ordering a 4-*fan*. We distinguish between the two different types of 5-element fan by using 5-*fan* to refer to a 5-element fan containing two triangles, and using 5-*cofan* to refer to a 5-element fan containing two triads.

The next proposition is proved by induction on n, using the fact that s_n is contained in either the closure or the coclosure of $\{s_1, \ldots, s_{n-1}\}$.

Proposition 2.1. *Let* $(s_1, \ldots, s_n)$ *be a fan ordering in a matroid* M. *Then*

$$\lambda_M(\{s_1, \ldots, s_n\}) \leq 2.$$

Lemma 2.2. *Let* M *be a binary matroid that has an internally* 4*-connected minor,* N*, satisfying* $|E(N)| \geq 8$. *If* (s_1, s_2, s_3, s_4) *is a* 4*-fan of* M*, then* $M\backslash s_1$ *or* M/s_4 *has an* N*-minor. If* $(s_1, s_2, s_3, s_4, s_5)$ *is a* 5*-fan in* M*, then either* $M\backslash s_1\backslash s_5$ *has an* N*-minor, or both* $M\backslash s_1/s_2$ *and* $M/s_4\backslash s_5$ *have* N*-minors. In particular, both* $M\backslash s_1$ *and* $M\backslash s_5$ *have* N*-minors.*

Proof. Let (s_1, s_2, s_3, s_4) be a 4-fan. Since $\{s_1, s_2, s_3, s_4\}$ contains a circuit and a cocircuit, $\lambda_{N_0}(\{s_1, s_2, s_3, s_4\}) \leq 2$ for any minor, N_0, of $E(M)$ that contains $\{s_1, s_2, s_3, s_4\}$ in its ground set. As N is internally 4-connected and $|E(N)| \geq 8$, we deduce that N is obtained from M by removing at least one element of $\{s_1, s_2, s_3, s_4\}$. Let x be an element in $\{s_1, s_2, s_3, s_4\} - E(N)$. If $M\backslash x$ has an N-minor, then either $x = s_1$, as desired; or $\{s_2, s_3, s_4\} - x$ is a 2-cocircuit in $M\backslash x$. In the latter case, as N is internally 4-connected, either $x \in \{s_2, s_3\}$, and M/s_4 has an N-minor, as desired; or $x = s_4$, and M/s_2 has an N-minor. But $\{s_1, s_3\}$ is a 2-circuit of the last matroid, so $M\backslash s_1$ has an N-minor, and the lemma holds. We may now suppose that deleting any element of $\{s_1, s_2, s_3, s_4\}$ from M yields a matroid with no N-minor. Then N is a minor of M/x for some $x \in \{s_1, s_2, s_3, s_4\}$. But x is not in $\{s_1, s_2, s_3\}$, or else $\{s_1, s_2, s_3\} - x$ is a 2-circuit in M/x, and we may delete one of its elements while keeping an N-minor. Thus $x = s_4$, and the lemma holds.

Next we assume that $(s_1, s_2, s_3, s_4, s_5)$ is a 5-fan in M. First we show that $M\backslash s_1/s_2$ has an N-minor if and only if $M/s_4\backslash s_5$ has an N-minor. As $\{s_1, s_3\}$ is a 2-circuit of M/s_2, it follows that if $M\backslash s_1/s_2$ has an N-minor, so does $M\backslash s_3$. As $\{s_4, s_5\}$ is a 2-cocircuit of the last matroid, this implies that M/s_4 has an N-minor. Hence so does $M/s_4\backslash s_5$. Thus $M/s_4\backslash s_5$ has an N-minor if $M\backslash s_1/s_2$ does. The converse statement yields to a symmetrical argument.

Now (s_1, s_2, s_3, s_4) is a 4-fan of M. By applying the first statement of the lemma, we see that $M\backslash s_1$ or M/s_4 has an N-minor. In the latter case, $M/s_4\backslash s_5$ has an N-minor, and we are done. Therefore we assume that $M\backslash s_1$ has an N-minor. There is a cocircuit of $M\backslash s_1$ that contains s_2 and is contained in $\{s_2, s_3, s_4\}$. If this cocircuit is not a triad, then $M\backslash s_1/s_2$ has an N-minor, and we are done. Therefore we assume that (s_5, s_4, s_3, s_2) is a 4-fan of $M\backslash s_1$. We apply the first statement of the lemma, and deduce that either $M\backslash s_1/s_2$ or $M\backslash s_1\backslash s_5$ has an N-minor. In either case the proof is complete. $\square$

A *quad* is a 4-element circuit–cocircuit. It is clear that if Q is a quad, then $\lambda(Q) \leq 2$. The next result is easy to verify.

Proposition 2.3. *Let* (X, Y) *be a* 3*-separation of a* 3*-connected binary matroid with* $|X| = 4$. *Then* X *is a quad or a* 4*-fan.*

The next result is Lemma 2.2 in [1].

Lemma 2.4. *Let* Q *be a quad in a binary matroid* M. *If* x *and* y *are in* Q*, then* $M\backslash x$ *and* $M\backslash y$ *are isomorphic.*

A matroid is $(4, k)$*-connected* if it is 3-connected, and, whenever (X, Y) is a 3-separation, either $|X| \leq k$ or $|Y| \leq k$. A matroid is *internally* 4*-connected* precisely when it is $(4, 3)$-connected. If a matroid is 3-connected, but not $(4, k)$-connected, then it contains a 3-separation, (X, Y), such that $|X|, |Y| > k$. We will call such a 3-separation a $(4, k)$*-violator*.

For $n \geq 3$, we let G_{n+2} denote the *biwheel* graph with $n + 2$ vertices. Thus G_{n+2} consists of a cycle $v_1, v_2, \ldots, v_n$, and two additional vertices, u and v, each of which is adjacent to every vertex in $\{v_1, v_2, \ldots, v_n\}$. The planar dual of a biwheel is a *cubic planar ladder*. We construct G^+_{n+2} by adding an edge between u and v. It is easy to see that $M(G^+_{n+2})$ is represented over GF(2) by the following matrix

$$\left[\begin{array}{c|cc} I_{n+1} & \mathbf{1} & \mathbf{0} \\ & I_n & A_n \end{array}\right]$$

where A_n is the $n \times n$ matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and $\mathbf{1}$ and $\mathbf{0}$ are $1 \times n$ vectors with all entries equal to 1 or 0 respectively. Thus $M(G^+_{n+2})$ is precisely equal to the matroid D_n, as defined by Zhou [4], and the element f_1 of $E(D_n)$ is the edge uv.

For $n \geq 2$ let Δ_{n+1} be the rank-$(n+1)$ binary matroid represented by the following matrix.

$$\left[\begin{array}{c|cc} I_{n+1} & \begin{matrix} \mathbf{1} \\ I_n \end{matrix} & \begin{matrix} e_n \\ A_n \end{matrix} \end{array} \right].$$

In this case, e_n is the standard basis vector with a one in position n. Then Δ_{n+1} is a *triangular Möbius matroid* (see [5]). In [4], the notation D^n is used for the matroid Δ_{n+1}, and f_1 denotes the element represented by the first column in the matrix. We use z to denote the same element. We observe that $\Delta_{n+1} \backslash z$ is the bond matroid of a *Möbius cubic ladder*.

The next result is a consequence of a theorem due to Zhou [4].

Theorem 2.5. *Let M and N be internally 4-connected binary matroids such that N is a proper minor of M satisfying $|E(N)| \geq 7$. Then either*

(i) *$M \backslash e$ or M/e is $(4, 4)$-connected with an N-minor, for some element $e \in E(M)$, or*
(ii) *M or M^* is isomorphic to either $M(G_{n+2})$, $M(G^+_{n+2})$, Δ_{n+1}, or $\Delta_{n+1} \backslash z$, for some $n \geq 4$.*

Note that the theorem in [4] is stated with the weaker hypothesis that $|E(N)| \geq 10$. However, Zhou explains that by using results from [6,7] and performing a relatively simple case-analysis, we can strengthen the theorem so that it holds under the condition that $|E(N)| \geq 7$.

3. Proof of the main theorem

In this section we prove Theorem 1.2. Throughout the section, we assume that the theorem is false. This means that there exist internally 4-connected binary matroids, $\bar{M}$ and $\bar{N}$, with the following properties:

(i) $\bar{M}$ has a proper $\bar{N}$-minor,
(ii) if e is in a triangle of $\bar{M}$, then $\bar{M} \backslash e$ has no $\bar{N}$-minor,
(iii) if e is in a triad of $\bar{M}$, then $\bar{M}/e$ has no $\bar{N}$-minor,
(iv) there is no internally 4-connected minor, M', of $\bar{M}$ such that M' has an $\bar{N}$-minor and $1 \leq |E(\bar{M})| - |E(M')| \leq 2$, and
(v) $|E(\bar{N})| \geq 7$.

Note that $(\bar{M}^*, \bar{N}^*)$ also provides a counterexample to Theorem 1.2. We start by showing that we can assume $|E(\bar{N})| \geq 8$. If $|E(\bar{N})| = 7$, then $\bar{N}$ is isomorphic to F_7 or F_7^*. Then $\bar{M}$ is non-regular, and contains one of the five internally 4-connected non-regular matroids N_{10}, $\widetilde{K}_5$, $\widetilde{K}_5^*$, $T_{12} \backslash e$, or T_{12}/e as a minor [7, Corollary 1.2]. But N_{10} contains an element in a triangle whose deletion is non-regular, so $\bar{M}$ is not isomorphic to N_{10}. The same statement applies to $\widetilde{K}_5$ and T_{12}/e, so $\bar{M}$ is not isomorphic to these matroids, or their duals, $\widetilde{K}_5^*$ and $T_{12} \backslash e$. Thus $\bar{M}$ has a proper internally 4-connected minor, N', isomorphic to one of the five matroids listed above. Therefore we can relabel N' as $\bar{N}$. As each of the five matroids has more than seven elements, we are justified in assuming that $|E(\bar{N})| \geq 8$. As $(\bar{M}, \bar{N})$ provides a counterexample to Theorem 1.2, it follows that $|E(\bar{M})| \geq 11$.

Lemma 3.1. *Let (M, N) be $(\bar{M}, \bar{N})$ or $(\bar{M}^*, \bar{N}^*)$. Let N_0 be an arbitrary N-minor of M. If T is a triangle or a triad of M, then $T \subseteq E(N_0)$.*

Proof. By duality, we can assume that $\{e, f, g\}$ is a triangle of M. Assume that $e \notin E(N_0)$. Since $M\backslash e$ has no minor isomorphic to N, it follows that N_0 is a minor of M/e. As $\{f, g\}$ is a 2-circuit in M/e, it follows that N_0 is a minor of either $M/e\backslash f$ or $M/e\backslash g$, and hence of $M\backslash f$ or $M\backslash g$. But neither of these matroids has an N-minor, so we have a contradiction. □

Lemma 3.2. *There is an element $e \in E(\bar{M})$ such that either $\bar{M}\backslash e$ or $\bar{M}/e$ is $(4, 4)$-connected with an $\bar{N}$-minor.*

Proof. If the lemma fails, then by Theorem 2.5, either $\bar{M}$ or its dual is isomorphic to one of $M(G_{n+2})$, $M(G^+_{n+2})$, Δ_{n+1}, or $\Delta_{n+1}\backslash z$, for some $n \geq 4$. In these cases it is easy to verify that every element of $E(\bar{M})$ is contained in a triangle or a triad. Therefore Lemma 3.1 implies that $E(\bar{M}) = E(\bar{N})$, contradicting the fact that $\bar{N}$ is a proper minor of $\bar{M}$. □

If e is an element such that $\bar{M}\backslash e$ is $(4, 4)$-connected with an $\bar{N}$-minor, then $\bar{M}\backslash e$ has a quad or a 4-fan, for otherwise it follows from Proposition 2.3 that $\bar{M}\backslash e$ is internally 4-connected, contradicting the fact that $\bar{M}$ is a counterexample to Theorem 1.2. We will make frequent use of the following fact.

Proposition 3.3. *Let (M, N) be either $(\bar{M}, \bar{N})$ or $(\bar{M}^*, \bar{N}^*)$. If $M\backslash e$ is 3-connected and has an N-minor, and (X, Y) is a 3-separation of $M\backslash e$ such that $|Y| = 5$, then Y is a 5-cofan of $M\backslash e$.*

Proof. If Y is not a fan, then Y contains a quad (see [4, Lemma 2.14]). As in the proof of [4, Lemma 2.15], we can show that in M, there is either a triangle or a triad of M that is contained in Y and which contains two elements from the quad. In the first case, the triangle contains an element we can delete to keep an N-minor. In the second case, the triad contains an element we can contract and keep an N-minor. In either case, we have a contradiction to Lemma 3.1. Therefore Y is a 5-element fan. If Y is a 5-fan, then by Lemma 2.2, we can delete an element from a triangle in $M\backslash e$ and preserve an N-minor. This contradicts Lemma 3.1, so Y is a 5-cofan. □

At this point, we give a quick summary of the lemmas that follow. Lemma 3.4 considers the matroid produced by contracting the last element of a 4-fan in $M\backslash e$. Lemma 3.5 deals with deleting an element from a quad in $M\backslash e$. In Lemma 3.6 we show that whenever we delete such an element, we destroy all N-minors. We exploit this information in Lemma 3.7, and show that $M\backslash e$ has no 4-fans. The only case left to consider is one in which we contract an element from a quad in $M\backslash e$. This case is covered in Lemma 3.8. After this lemma, there is only a small amount of work to be done before we obtain a final contradiction and complete the proof.

Lemma 3.4. *Let (M, N) be either $(\bar{M}, \bar{N})$ or $(\bar{M}^*, \bar{N}^*)$. Assume that e is an element of M such that $M\backslash e$ is $(4, 4)$-connected with an N-minor, and that (a, b, c, d) is a 4-fan of $M\backslash e$. Then $M \backslash e/d$ is 3-connected with an N-minor, and M/d is $(4, 4)$-connected. Moreover, if (X, Y) is a $(4, 3)$-violator of M/d such that $|X \cap \{a, b, c\}| \geq 2$, then Y is a quad of M/d, and $Y \cap \{a, b, c, e\} = \{e\}$.*

Proof. From Proposition 2.1 and the fact that M is internally 4-connected with at least eleven elements, it follows that (a, b, c, d) is not a 4-fan of M. Therefore $\{b, c, d, e\}$ is a cocircuit in M.

3.4.1. *$M\backslash e/d$ and M/d are 3-connected.*

Proof. We start by showing that $M\backslash e/d$ is 3-connected. Because $M\backslash e$ is $(4, 4)$-connected, it is also 3-connected. Assume that $M\backslash e/d$ is not 3-connected. As $M\backslash e/c$ contains the parallel pair $\{a, b\}$, it too is not 3-connected. As $\{b, c, d\}$ is a triad in $M\backslash e$, we can apply the dual of [3, Lemma 8.7.7], and see that there is a triangle of $M\backslash e$ containing d, and exactly one of b and c. Let z be the third element of this triangle. Then $z \neq a$, or else $\{a, b, c, d\}$ is a $U_{2,4}$-restriction of $M\backslash e$ and in this case $\{b, c, d\}$ is both a triangle and a triad. This leads to a contradiction to the 3-connectivity of $M\backslash e$. Therefore (a, b, c, d, z) is a 5-fan in $M\backslash e$. Since $|E(M\backslash e)| \geq 10$, this means that $M\backslash e$ is not $(4, 4)$-connected, and we have a contradiction. Therefore $M\backslash e/d$ is 3-connected.

If M/d is not 3-connected, then it follows easily (see the dual of [8, Lemma 2.6]) that $\{e, d\}$ is contained in a triangle of M. However, N is a minor of $M\backslash e$, so we have a contradiction to Lemma 3.1. ◇

3.4.2. *$M\backslash e/d$, and hence M/d, has an N-minor.*

Proof. Let N_0 be an N-minor of M. Then $\{a, b, c\} \subseteq E(N_0)$, by Lemma 3.1. As (a, b, c, d) is a 4-fan of $M\backslash e$, it now follows by Lemma 2.2 that $M\backslash e/d$ has an N-minor. ◇

3.4.3. *Let (X, Y) be a $(4, 3)$-violator of M/d, and assume that $|X \cap \{a, b, c\}| \geq 2$. Then $|Y| = 4$, and $e \in Y$. Moreover, $Y \cap \{a, b, c\} = \emptyset$.*

Proof. Assume that the result fails.

3.4.3.1. $|Y| \geq 5$.

Proof. Assume otherwise. Then $|Y| = 4$. Assume that $e \in X$. If $\{b, c\} \subseteq X$, then $d \in \mathrm{cl}^*_M(X)$, as $\{b, c, d, e\}$ is a cocircuit. It follows from [3, Corollary 8.2.6(iii)] that $\lambda_M(X \cup d) = \lambda_{M/d}(X)$, and therefore $(X \cup d, Y)$ is a $(4, 3)$-violator of M, an impossibility. Hence either b or c is contained in Y, so $|X \cap \{a, b, c\}| \geq 2$ implies a is in X.

Proposition 2.3 implies that Y is either a quad or a 4-fan of M/d. As $\{a, b, c\}$ is a triangle of M/d that meets Y in a single element, Y is not a cocircuit, and hence not a quad of M/d. Thus $Y = \{y_1, y_2, y_3, y_4\}$, where (y_1, y_2, y_3, y_4) is a 4-fan of M/d. Since the triangle $\{a, b, c\}$ cannot meet the triad $\{y_2, y_3, y_4\}$ in a single element, it follows that y_1 is equal to b or c. Let N_0 be an N-minor of M/d. Since $\{y_2, y_3, y_4\}$ is a triad of M, it follows from Lemma 3.1 that $\{y_2, y_3, y_4\} \subseteq E(N_0)$. But $\{a, b, c\}$ is a triangle of M, so $\{a, b, c\} \subseteq E(N_0)$. Therefore $\{y_1, y_2, y_3, y_4\} \subseteq E(N_0)$, and this contradicts Lemma 2.2. From this contradiction we conclude that $e \in Y$.

Since 3.4.3 fails, yet $|Y| = 4$ and $e \in Y$, we deduce that Y contains exactly one element of the triangle $\{a, b, c\}$. Thus Y is not a quad of M/d, so Y is a 4-fan, (y_1, y_2, y_3, y_4), of M/d. Since $\{a, b, c\}$ is a triangle of M/d, and $\{y_2, y_3, y_4\}$ is a triad, orthogonality requires that the single element in $Y \cap \{a, b, c\}$ is y_1. Therefore e is contained in the triad $\{y_2, y_3, y_4\}$. But this means that $M\backslash e$ contains a 2-cocircuit, a contradiction as it is 3-connected. ◇

Let $T = \{a, b, c\}$. As Y contains at most one element of T, it follows from 3.4.3.1 that $|Y - T| \geq 4$. Furthermore, X spans T. The next fact follows from these observations and from 3.4.1.

3.4.3.2. $(X \cup T, Y - T)$ *is a 3-separation in M/d.*

3.4.3.3. $e \in Y$.

Proof. Assume that $e \in X$. Then 3.4.3.2 and the cocircuit $\{b, c, d, e\}$ imply that $(X \cup T \cup d, Y - T)$ is 3-separation of M. Since $|Y - T| \geq 4$, it follows that M has a $(4, 3)$-violator, which is impossible. ◇

3.4.3.4. $|Y - T| \leq 5$.

Proof. By 3.4.3.2 and 3.4.3.3, we see that $(X \cup T, Y - (T \cup e))$ is a 3-separation in $M/d\backslash e$. As $\{b, c, d\}$ is a triad in $M\backslash e$, it follows that $d \in \mathrm{cl}^*_{M\backslash e}(T)$, so

$$(X \cup T \cup d, Y - (T \cup e))$$

is a 3-separation of $M\backslash e$. Since $|X \cup T \cup d| > 4$, and $M\backslash e$ is $(4, 4)$-connected, it follows that $|Y - (T \cup e)| \leq 4$, so $|Y - T| \leq 5$. ◇

3.4.3.5. $|Y - T| = 4$.

Proof. We have observed that $|Y - T| \geq 4$, so if 3.4.3.5 is false, it follows from 3.4.3.4 that $|Y - T| = 5$. From 3.4.3.2 and the dual of Proposition 3.3, we see that $Y - T$ is a 5-fan of M/d. Let $(y_1, \ldots, y_5)$ be a fan ordering of $Y - T$. Since $M\backslash e$ is 3-connected, e is contained in no triads of M, so $e = y_1$ or $e = y_5$. By reversing the fan ordering as necessary, we can assume that the first case holds. As $\{y_2, y_3, y_4\}$ is a triad of M, it follows that $\{y_3, y_4, y_5\}$ is not a triangle, or else M has a 4-fan. Therefore $\{y_3, y_4, y_5, d\}$ is a circuit of M that is contained in $(Y - T) \cup d$. It meets the cocircuit $\{b, c, d, e\}$ in a single element, violating orthogonality. ◇

As $|Y| \geq 5$, and $|Y - T| = 4$, it follows that $|Y| = 5$ and $|Y \cap T| = 1$. From Proposition 3.3, we see that Y is a 5-fan of M/d. Let $(y_1, \ldots, y_5)$ be a fan ordering of Y in M/d. As $M\backslash e$ is 3-connected, e is in no triad in M, and hence in M/d, so $e = y_1$ or $e = y_5$. By reversing the fan ordering as necessary, we assume $e = y_1$. Since $\{y_2, y_3, y_4\}$ is a triad of M, it follows that $\{y_3, y_4, y_5\}$ is not a triangle, or else M has a 4-fan. Therefore $\{y_3, y_4, y_5, d\}$ is a circuit of M. This circuit cannot meet the cocircuit $\{b, c, d, e\}$ in the single element d. Therefore the single element in $T \cap Y$ is in $\{y_3, y_4, y_5\}$. Call this element y. As the triangle T cannot meet the triad $\{y_2, y_3, y_4\}$ in a single element, it follows that $y = y_5$. Since (y_5, y_4, y_3, y_2) is a 4-fan of M/d, and $\{y_2, y_3, y_4\}$ is a triad of M, it follows from Lemmas 3.1 and 2.2 that $M/d\backslash y_5$, and hence $M\backslash y_5$ has an N-minor. This contradicts the fact that y_5 is in the triangle T. Thus we have completed the proof of 3.4.3. $\diamond$

From 3.4.3 we know that M/d is $(4, 4)$-connected. Next we must eliminate the possibility that M/d has a 4-fan.

3.4.4. *Let (X, Y) be a $(4, 3)$-violator of M/d, where $|X \cap \{a, b, c\}| \geq 2$. Then Y is not a 4-fan of M/d.*

Proof. Assume that Y is a 4-fan, (y_1, y_2, y_3, y_4). Thus $\{y_2, y_3, y_4\}$ is a triad in M/d, and hence in M. It follows from 3.4.3 that $e \in Y$. But e is not in a triad of M, so $e = y_1$. Since M has no 4-fan, it follows that $\{e, y_2, y_3\}$ is not a triangle of M, so $\{e, d, y_2, y_3\}$ is a circuit.

From 3.4.1, we see that $M/d\backslash e$ is 3-connected. We shall show that it is internally 4-connected. Once we prove this assertion, we will have shown that (M, N) is not a counterexample to Theorem 1.2, since $M/d\backslash e$ has an N-minor by 3.4.2. This contradiction will complete the proof of 3.4.4.

3.4.4.1. *If (U, V) is a $(4, 3)$-violator of $M/d\backslash e$, then $\{b, c\} \not\subseteq U$ and $\{b, c\} \not\subseteq V$.*

Proof. If the result fails, then by symmetry we can assume that (U, V) is a $(4, 3)$-violator of $M/d\backslash e$ such that $b, c \in U$. Then $d \in \mathrm{cl}^*_{M\backslash e}(U)$, because of the triad $\{b, c, d\}$, so $(U \cup d, V)$ is a $(4, 3)$-violator in $M\backslash e$. As $|U \cup d| > 4$, and $M\backslash e$ is $(4, 4)$-connected, we deduce that $|V| = 4$. Assume that V is a quad of $M\backslash e$. Then $V \cup e$ is a cocircuit of M, which cannot meet the circuit $\{e, d, y_2, y_3\}$ in a single element. Hence y_2 or y_3 is in V. However, V is a circuit in $M\backslash e$, and $\{y_2, y_3, y_4\}$ is a cocircuit in $M\backslash e$, as it is a triad of M, and $M\backslash e$ is 3-connected. Orthogonality requires that $|V \cap \{y_2, y_3, y_4\}| = 2$. This means that $\{y_2, y_3, y_4\} \subseteq \mathrm{cl}^*_{M\backslash e}(V)$, so $V \cup \{y_2, y_3, y_4\}$ is a 5-element 3-separating set in $M\backslash e$. As $M\backslash e$ is $(4, 4)$-connected, it follows that $M\backslash e$ has at most nine elements, contradicting our earlier assumption that $|E(M)| \geq 11$. Thus V is not a quad of $M\backslash e$, and Proposition 2.3 implies that V is a 4-fan in $M\backslash e$.

Let T^* be the triad of $M\backslash e$ that is contained in V. As M has no 4-fans, $T^* \cup e$ is a cocircuit of M. It cannot meet the circuit $\{e, d, y_2, y_3\}$ in the single element e. Let y be an element in $\{y_2, y_3\} \cap T^*$. As $\{y_2, y_3, y_4\}$ is a triad in M/d, and hence in M, it does not contain any element that is in a triangle of M, or else M has a 4-fan. Therefore y is not in the triangle of $M\backslash e$ that is contained in V, so $V - y$ is a triangle of M. Thus $V - y$ is contained in the ground set of every N-minor of M, so Lemma 2.2 implies that $M\backslash e/y$, and hence M/y has an N-minor. However, since y is contained in the triad $\{y_2, y_3, y_4\}$ of M, this contradicts Lemma 3.1. $\diamond$

Let (U, V) be a $(4, 3)$-violator of $M/d\backslash e$, and assume that $a \in U$. By 3.4.4.1, we may assume that $x \in U$ and $y \in V$, where $\{x, y\} = \{b, c\}$. Then $y \in \mathrm{cl}_{M/d\backslash e}(U)$, as $\{a, x, y\}$ is a triangle, so $(U \cup y, V - y)$ is a 3-separation of $M/d\backslash e$. It follows from 3.4.4.1 that it is not a $(4, 3)$-violator, so $|V| = 4$. Since V contains an element that is in $\mathrm{cl}_{M/d\backslash e}(U)$, it cannot be a quad of $M/d\backslash e$, so it is a 4-fan. Moreover, as $y \in \mathrm{cl}_{M/d\backslash e}(U)$, it follows that y is not in the triad of $M/d\backslash e$ that is contained in V. Therefore $V - y$ is a triad of $M/d\backslash e$. If $V - y$ is a triad of M, then $V - y$ is contained in every N-minor of M. Because $\{a, x, y\} = \{a, b, c\}$ is a triangle, it follows that y is contained in every N-minor. Thus V is in every N-minor of M. This implies that N has a 4-element 3-separating set, which is impossible as $|E(N)| \geq 8$. Therefore $V - y$ is not a triad of M, so $(V - y) \cup e$ is a cocircuit. It cannot meet the circuit $\{e, d, y_2, y_3\}$ in the single element e, so either y_2 or y_3 is in the triad $V - y$ of $M\backslash e$.

Note that $V - y$ is not equal to $\{y_2, y_3, y_4\}$, as one set is a triad of M and the other is not. They are both triads of $M\backslash e$, and they have at least one element in common. Hence they have exactly one element in common, as $M\backslash e$ is 3-connected, and therefore does not contain a series pair. Let z be the unique element in $(V - y) \cap \{y_2, y_3\}$. If T' is the triangle of $M/d\backslash e$ that is contained in V, then z is not in T', as otherwise the triad $\{y_2, y_3, y_4\}$ in M/d meets the triangle T' in a single element, z. Therefore $V - z$ is a triangle in $M/d\backslash e$, and $V - y$ is a triad.

Note that y is in every N-minor of M, because of the triangle $\{a, x, y\} = \{a, b, c\}$, and z is in every N-minor of M because of the triad $\{y_2, y_3, y_4\}$. But V cannot be contained in an N-minor of $M/d\backslash e$, as it is a 3-separating set. Therefore there is some element $w \in V - \{y, z\}$ such that N is a minor of $M/d\backslash e\backslash w$ or $M/d\backslash e/w$. But z is in the 2-cocircuit $V - \{y, w\}$ of the first matroid, and y is in the 2-circuit $V - \{z, w\}$ of the second. This leads to a contradiction, as y and z are in the ground set of every N-minor of M.

We conclude that there can be no $(4, 3)$-violator in $M/d\backslash e$, and therefore $M/d\backslash e$ is internally 4-connected and has an N-minor. This contradicts our assumption that M is a counterexample to Theorem 1.2. Thus 3.4.4 holds. $\diamond$

Now we can complete the proof of Lemma 3.4. If (a, b, c, d) is a 4-fan of $M\backslash e$, where this matroid is $(4, 4)$-connected with an N-minor, then 3.4.2 implies that M/d has an N-minor. It follows from 3.4.1 that $M\backslash e/d$ and M/d are 3-connected, and 3.4.3 implies that M/d is $(4, 4)$-connected. Moreover, if (X, Y) is a $(4, 3)$-violator of M/d where X contains at least two elements of $\{a, b, c\}$, then 3.4.3 also implies that $|Y| = 4$ and $Y \cap \{a, b, c, e\} = \{e\}$. As 3.4.4 implies that Y cannot be a 4-fan in M/d, Proposition 2.3 implies that Y is a quad. Thus Lemma 3.4 is proved. $\square$

Lemma 3.5. *Let (M, N) be either $(\bar{M}, \bar{N})$ or $(\bar{M}^*, \bar{N}^*)$. Assume that the element e is such that $M\backslash e$ is $(4, 4)$-connected with an N-minor, and that Q is a quad of $M\backslash e$. If $x \in Q$ and $M\backslash e\backslash x$ has an N-minor, then $M\backslash x$ is $(4, 4)$-connected. In particular, if (X, Y) is a $(4, 3)$-violator of $M\backslash x$ such that $|X \cap (Q - x)| \geq 2$, then Y is a quad of $M\backslash x$ such that $|Y \cap Q| = 1$, and $e \in Y$.*

Proof. As M has no quads, we deduce that $Q \cup e$ is a cocircuit in M.

3.5.1. *$M\backslash x\backslash e$ and $M\backslash x$ are 3-connected.*

Proof. Let (U, V) be a 2-separation in $M\backslash x\backslash e$. By relabeling as necessary, we assume that $|U \cap (Q - x)| \geq 2$. If U contains $Q - x$, then $(U \cup x, V)$ is a 2-separation of $M\backslash e$. This is impossible, so V contains a single element of $Q - x$. Then

$$\lambda_{M\backslash x\backslash e}(V - (Q - x)) \leq \lambda_{M\backslash x\backslash e}(V) \leq 1,$$

as $Q - x$ is a triad of $M\backslash x\backslash e$. Now $x \in \mathrm{cl}_{M\backslash e}(U \cup (Q - x))$, so

$$\lambda_{M\backslash e}(V - (Q - x)) = \lambda_{M\backslash x\backslash e}(V - (Q - x)) \leq 1.$$

But $M\backslash e$ is 3-connected, so this means that $|V - (Q - x)| \leq 1$. Thus $|V| = 2$, and V must be a 2-cocircuit of $M\backslash x\backslash e$. This means that x is in a triad of $M\backslash e$. This triad must meet Q in two elements, by orthogonality. Thus $|\,\mathrm{cl}^*_{M\backslash e}(Q)| \geq 5$, and $M\backslash e$ contains a 5-element 3-separating set. This is a contradiction as $M\backslash e$ is $(4, 4)$-connected with at least ten elements. Thus $M\backslash x\backslash e$ is 3-connected, and it follows easily that $M\backslash x$ is 3-connected. $\diamond$

Let (X, Y) be a $(4, 3)$-violator of $M\backslash x$, and assume that X contains at least two elements of $Q - x$. If $Q - x \subseteq X$, then $x \in \mathrm{cl}_M(X)$, as Q is a circuit of M. This implies that $(X \cup x, Y)$ is a $(4, 3)$-violator of M, which is impossible. Therefore Y contains exactly one element of $Q - x$. Let us call this element y.

3.5.2. *$(X - e, Y - e)$ is a 3-separation of $M\backslash x\backslash e$ and $y \in \mathrm{cl}^*_{M\backslash x\backslash e}(Y - \{e, y\})$.*

Proof. The fact that $(X - e, Y - e)$ is a 3-separation of $M\backslash x\backslash e$ follows because $|X|, |Y| \geq 4$, and $M\backslash x\backslash e$ is 3-connected. Since $Q - x$ is a triad of $M\backslash x\backslash e$, and $Q - \{x, y\} \subseteq X - e$, we deduce that $y \in \mathrm{cl}^*_{M\backslash x\backslash e}(X - e)$. This means that $y \in \mathrm{cl}^*_{M\backslash x\backslash e}(Y - \{e, y\})$, as otherwise $((X - e) \cup y, Y - \{e, y\})$ is a 2-separation of $M\backslash x\backslash e$. $\diamond$

3.5.3. $\lambda_{M\backslash e}(Y - \{e, y\}) \leq 2$.

Proof. Since $(X - e, Y - e)$ is a 3-separation of $M\backslash x\backslash e$ and $y \in \mathrm{cl}^*_{M\backslash x\backslash e}(X - e)$, it follows that

$$\lambda_{M\backslash x\backslash e}(Y - \{e, y\}) \leq \lambda_{M\backslash x\backslash e}(Y - e) = 2.$$

Since $(X - e) \cup \{x, y\}$ contains Q, it follows that $x \in \mathrm{cl}_{M\backslash e}((X - e) \cup y)$. This means that

$$\lambda_{M\backslash e}(Y - \{e, y\}) = \lambda_{M\backslash x\backslash e}(Y - \{e, y\}) \le 2,$$

as desired. $\diamond$

3.5.4. $|Y| \le 6$.

Proof. As $M\backslash e$ is (4, 4)-connected, $|Y - \{e, y\}| \le 4$ by 3.5.3. Thus $|Y| \le 6$. $\diamond$

3.5.5. $|Y| \ne 6$.

Proof. Assume that $|Y| = 6$. If $e \notin Y$, then 3.5.3 implies that $((X - e) \cup \{x, y\}, Y - y)$ is a 3-separation of $M\backslash e$. As $|Y - y| = 5$ and $|(X - e) \cup \{x, y\}| = |X| + 1 \ge 5$, this contradicts the fact that $M\backslash e$ is (4, 4)-connected. Therefore $e \in Y$, and $Y - \{e, y\}$ is a 4-element 3-separating set in $M\backslash e$. Thus $Y - \{e, y\}$ is either a quad or a 4-fan of $M\backslash e$. The next two assertions show that both these cases are impossible, thereby finishing the proof of 3.5.5.

3.5.5.1. *$Y - \{e, y\}$ is not a quad of $M\backslash e$.*

Proof. Assume that $Y - \{e, y\}$ is a quad of $M\backslash e$. Thus it is a circuit of M, and $Y - y$ is a cocircuit of M. If $Y - y$ is not a cocircuit of $M\backslash x$, then x is in the coclosure of $Y - y$ in M. This leads to a contradiction to orthogonality with the circuit Q of M. Thus $Y - y$ is a cocircuit of $M\backslash x$, so $Y - \{e, y\}$ is a quad in both $M\backslash e$ and $M\backslash x\backslash e$. As $Y - y$ is a cocircuit of $M\backslash x$ and $|Y| = 6$, we see that $r^*_{M\backslash x}(Y) \ge 4$, so

$$r_{M\backslash x}(Y) = \lambda_{M\backslash x}(Y) - r^*_{M\backslash x}(Y) + |Y| \le 2 - 4 + 6 = 4.$$

By 3.5.2, there is a cocircuit of $M\backslash x\backslash e$ contained in $Y - e$ that contains y. The symmetric difference of this cocircuit with $Y - \{e, y\}$ is a disjoint union of cocircuits. As $M\backslash x\backslash e$ contains no cocircuit with fewer than three elements, it follows that there are two triads, T_1^* and T_2^*, of $M\backslash x\backslash e$, such that $T_1^* \cap T_2^* = \{y\}$, and $T_1^* \cup T_2^* = Y - e$. If both $T_1^* \cup e$ and $T_2^* \cup e$ are cocircuits of $M\backslash x$, then we can take the symmetric difference of these cocircuits, and deduce that $Y - \{e, y\}$ is a cocircuit of $M\backslash x$ that is properly contained in the cocircuit $Y - y$. Since this is impossible, we deduce that we can relabel as necessary, and assume that T_1^* is a triad of $M\backslash x$.

Let z be an arbitrary element of $T_2^* - y$. Then $Y - \{e, y, z\}$ is independent in $M\backslash x$. Orthogonality with the cocircuit $(Q - x) \cup e$ means that y cannot be in the closure of $Y - \{e, y, z\}$ in $M\backslash x$. Thus $Y - \{e, z\}$ is independent. Since $r_{M\backslash x}(Y) \le 4$, it follows that $Y - \{e, z\}$ spans Y in $M\backslash x$. Let C be a circuit of $M\backslash x$ such that $\{e\} \subseteq C \subseteq Y - z$. If $y \notin C$, then C and the cocircuit $(Q - x) \cup e$ meet in $\{e\}$. Therefore $y \in C$. This implies that C contains exactly one element of $T_1^* - y$. Now C cannot be a triangle, as $M\backslash e$ has an N-minor. Therefore C also contains the single element in $T_2^* - \{y, z\}$. But now the circuit C meets the cocircuit $Y - y$ of $M\backslash x$ in three elements: e, and a single element from each of $T_1^* - y$ and $T_2^* - y$. This contradiction proves 3.5.5.1. $\diamond$

3.5.5.2. *$Y - \{e, y\}$ is not a 4-fan of $M\backslash e$.*

Proof. Assume that (y_1, y_2, y_3, y_4) is a fan ordering of $Y - \{e, y\}$ in $M\backslash e$. Then $\{y_2, y_3, y_4, e\}$ is a cocircuit of M.

As $M\backslash x\backslash e$ has no cocircuits with fewer than three elements, $\{y_2, y_3, y_4\}$ is a triad of $M\backslash x\backslash e$. Thus (y_1, y_2, y_3, y_4) is a 4-fan of $M\backslash x\backslash e$. By 3.5.2, there is a cocircuit C^* of $M\backslash x\backslash e$ such that $\{y\} \subseteq C^* \subseteq Y - e$. This cocircuit must meet the triangle $\{y_1, y_2, y_3\}$ in exactly two elements. If $\{y_2, y_3\} \subseteq C^*$, then the symmetric difference of $\{y_2, y_3, y_4\}$ and C^* is $\{y, y_4\}$, as $\{y_2, y_3, y_4\}$ is not properly contained in C^*. Since $M\backslash x\backslash e$ has no 2-cocircuit, we deduce that $y_1 \in C^*$. Either C^*, or its symmetric difference with $\{y_2, y_3, y_4\}$, is a triad of $M\backslash x\backslash e$ that contains y, y_1, and a single element from $\{y_2, y_3\}$. We can swap the labels on y_2 and y_3 if necessary, so we can assume that $\{y, y_1, y_2\}$ is a triad. Thus (y, y_1, y_2, y_3, y_4) is a 5-cofan of $M\backslash x\backslash e$. The dual of Lemma 2.2 implies that $M\backslash x\backslash e/y$ and $M\backslash x\backslash e/y_4$ have N-minors.

Recall that $\{y_2, y_3, y_4, e\}$ is a cocircuit of M. It is also a cocircuit of $M\backslash x$, as otherwise $x \in \mathrm{cl}^*_M(\{y_2, y_3, y_4, e\})$, and this contradicts orthogonality with the circuit Q. Therefore $\{y_2, y_3, y_4\}$ is coindependent in $M\backslash x$.

Assume $y_1 \in \mathrm{cl}^*_{M\backslash x}(\{y_2, y_3, y_4\})$, so y_1 is in $\mathrm{cl}^*_{M\backslash x\backslash e}(\{y_2, y_3, y_4\})$. As it is also in $\mathrm{cl}_{M\backslash x\backslash e}(\{y_2, y_3, y_4\})$, it follows that $\lambda_{M\backslash x\backslash e}(\{y_1, y_2, y_3, y_4\}) \leq 1$. This leads to a contradiction to the fact that $M\backslash x\backslash e$ is 3-connected. Therefore $y_1 \notin \mathrm{cl}^*_{M\backslash x}(\{y_2, y_3, y_4\})$. Thus $\{y_1, y_2, y_3, y_4\}$ is a coindependent set in $M\backslash x$, so $r^*_{M\backslash x}(Y) \geq 4$. Now we see that

$$r_{M\backslash x}(Y) = \lambda_{M\backslash x}(Y) - r^*_{M\backslash x}(Y) + |Y| \leq 2 - 4 + 6 = 4.$$

If $\{y, y_1, y_3, y_4\}$ is dependent in $M\backslash x\backslash e$, then it is a circuit, by orthogonality with the triads $\{y, y_1, y_2\}$ and $\{y_2, y_3, y_4\}$, and the fact that $M\backslash x\backslash e$ has no 2-circuits. In this case, $\{y, y_1, y_3, y_4\}$ is a circuit of M, and $Q \cup e$ is a cocircuit that meets it in the single element y. Therefore $\{y, y_1, y_3, y_4\}$ is independent in $M\backslash x\backslash e$, and hence in $M\backslash x$. Therefore $\{y, y_1, y_3, y_4\}$ spans Y in $M\backslash x$. Let C be a circuit of $M\backslash x$ such that $\{e\} \subseteq C \subseteq \{e, y, y_1, y_3, y_4\}$.

First observe that $y \in C$, as otherwise C and $Q \cup e$ are a circuit and a cocircuit of M that meet in $\{e\}$. We have noted that $\{e, y_2, y_3, y_4\}$ is a cocircuit of $M\backslash x$. Therefore orthogonality implies that C contains exactly one element of $\{y_3, y_4\}$. Since $M\backslash e$ has an N-minor, it follows that e is in no triangles of M. Therefore y_1 must be in C. Hence C is either $\{e, y, y_1, y_3\}$ or $\{e, y, y_1, y_4\}$. In the first case, we take the symmetric difference of C with the triangle $\{y_1, y_2, y_3\}$, and discover that e is in the triangle $\{e, y, y_2\}$, a contradiction. Therefore $C = \{e, y, y_1, y_4\}$.

We noted earlier that $M\backslash x\backslash e/y$, and hence M/y, has an N-minor. The symmetric difference of C with the triangle $\{y_1, y_2, y_3\}$ is $\{e, y, y_2, y_3, y_4\}$, which must therefore be a circuit of M. Thus $\{e, y_2, y_3, y_4\}$ is a circuit of M/y. It is also a cocircuit, as it is a cocircuit in M. Thus M/y contains a quad that contains e. Since $M/y\backslash e$ has an N-minor, it follows from Lemma 2.4 that if z is an arbitrary member of the quad $\{e, y_2, y_3, y_4\}$, then $M/y\backslash z$ has an N-minor. In particular, $M/y\backslash y_2$, and hence $M\backslash y_2$ has an N-minor. This is contradictory, as y_2 is contained in the triangle $\{y_1, y_2, y_3\}$ of M. This completes the proof of 3.5.5.2. $\diamond$

The proof of 3.5.5 now follows immediately from 3.5.5.1 and 3.5.5.2. $\diamond$

3.5.6. $|Y| \neq 5$.

Proof. Assume that $|Y| = 5$. If $e \in X$, then $y \in \mathrm{cl}^*_{M\backslash x}(X)$, since $(Q - x) \cup e$ is a cocircuit of $M\backslash x$ that is contained in $X \cup y$. This means that $(X \cup y, Y - y)$ is a 3-separation of $M\backslash x$. As Q is a circuit of M, and $Q - x \subseteq X \cup y$, it follows that $x \in \mathrm{cl}_M(X \cup y)$. Therefore $(X \cup \{x, y\}, Y - y)$ is a 3-separation of M, and as $|X \cup \{x, y\}|, |Y - y| \geq 4$, we have violated the internal 4-connectivity of M. Therefore e is in Y.

Proposition 3.3 implies that Y is a 5-cofan of $M\backslash x$. Let $(y_1, y_2, y_3, y_4, y_5)$ be a fan ordering of Y in $M\backslash x$. Since e is contained in no triangle of M by Lemma 3.1, we can assume that $e = y_1$. The element y cannot be contained in $\{y_2, y_3, y_4\}$, or else this triangle meets the cocircuit $Q \cup e$ of M in the single element y. Now $\{y_1, y_2, y_3\}$ is a triad of M, or else $\{x, y_1, y_2, y_3\}$ is a cocircuit, and it meets the circuit Q in the single element x. However, $\{y_1, y_2, y_3\}$ cannot be a triad, as $e = y_1$ and $M\backslash e$ is 3-connected. $\diamond$

We can now complete the proof of Lemma 3.5. Recall that (X, Y) is a (4, 3)-violator of $M\backslash x$, where x is contained in the quad Q of $M\backslash e$, and $|X \cap (Q - x)| \geq 2$. By combining 3.5.4–3.5.6, we deduce that $|Y| = 4$. Therefore $M\backslash x$ is (4, 4)-connected with an N-minor, and Y is either a quad or a 4-fan of $M\backslash x$.

Assume that e is not in Y. If Y is a quad of $M\backslash x$, then it is a circuit of M that meets the cocircuit $Q \cup e$ in the single element y. Therefore Y must be a 4-fan of $M\backslash x$. Certainly y is not contained in the triangle of Y, by orthogonality with $Q \cup e$. Therefore $Y = (y_1, y_2, y_3, y)$ is a 4-fan. We can apply Lemma 3.4 to $M\backslash x$, and deduce that M/y is (4, 4)-connected with an N-minor. Since M/y is not internally 4-connected, Lemma 3.4 also implies that M/y contains a quad and that this quad contains x. However, $Q - y$ is a triangle in M/y, so M/y contains a quad, and a triangle that contains an element of this quad. It follows that the triangle and the quad meet in two elements, and their union is a 5-element 3-separating set of M/y. As M/y is (4, 4)-connected, this leads to a contradiction, so now we know that e is in Y.

If Y is a 4-fan of $M\backslash x$, then e is not contained in the triangle of this fan, as $M\backslash e$ has an N-minor. Therefore y is contained in a triangle of $M\backslash x$ that is contained in $Y - e$. This triangle violates orthogonality with the cocircuit $Q \cup e$ in M. Hence Y is a quad of $M\backslash x$, and Lemma 3.5 holds. $\square$

The next proof is essentially the same as an argument used in [1].

Lemma 3.6. *Let (M, N) be either $(\bar{M}, \bar{N})$ or $(\bar{M}^*, \bar{N}^*)$. Assume that the element e is such that $M\backslash e$ is $(4, 4)$-connected with an N-minor, and that Q is a quad of $M\backslash e$. Then N is not a minor of $M\backslash e\backslash x$, for any element $x \in Q$.*

Proof. Assume that N is a minor of $M\backslash e\backslash x$ for some element x of Q. By Lemma 2.4, deleting any element of Q from $M\backslash e$ produces a matroid with an N-minor. Lemma 3.5 implies that $M\backslash x$ is $(4, 4)$-connected and contains a quad, Q_x, such that $e \in Q_x$, and $|Q \cap Q_x| = 1$.

3.6.1. *Assume that x_1 and x_2 are elements of Q, and that $M\backslash x_1$ contains a quad, Q_1, such that $e \in Q_1$, and $Q \cap Q_1 = \{x_2\}$. Then $M\backslash x_2$ contains a quad, Q_2, such that $Q \cap Q_2 = \{x_1\}$ and $Q_1 \cap Q_2 = \{e\}$.*

Proof. Since $M\backslash e\backslash x_2$ has an N-minor, we can apply Lemma 3.5, and deduce that $M\backslash x_2$ contains a quad Q_2 such that $e \in Q_2$ and $|Q \cap Q_2| = 1$. On the other hand, $M\backslash x_1$ is $(4, 4)$-connected, and contains a quad, Q_1. Moreover, $M\backslash x_1\backslash x_2$ is isomorphic to $M\backslash x_1\backslash e$, by Lemma 2.4 and the fact that e and x_2 are both in Q_1, so $M\backslash x_1\backslash x_2$ has an N-minor. Hence we can apply Lemma 3.5 again, and deduce that $M\backslash x_2$ contains a quad Q_2' such that $x_1 \in Q_2'$ and $|Q_1 \cap Q_2'| = 1$.

We will show that $Q_2 = Q_2'$. Assume this is not the case. As Q_2 and Q_2' are both quads of $M\backslash x_2$, orthogonality demands that they are disjoint, or they meet in two elements. In the latter case, $Q_2 \triangle Q_2'$ is a circuit of M, and $(Q_2 \cup x_2) \triangle (Q_2' \cup x_2) = Q_2 \triangle Q_2'$ must be a cocircuit of M, so M has a quad. As this is impossible, we deduce that Q_2 and Q_2' are disjoint. Therefore $e \notin Q_2'$, as e is in Q_2. This means that $|Q \cap Q_2'| = 2$, as otherwise the circuit Q_2' and the cocircuit $Q \cup e$ meet in $\{x_1\}$. But $Q_2' \cup x_2$ is a cocircuit, and Q is a circuit, and they meet in three elements: x_2 and the two elements of $Q \cap Q_2'$. This contradiction shows that $Q_2 = Q_2'$, so $x_1 \in Q_2$. Furthermore, $Q \cap Q_2 = \{x_1\}$ and $Q_1 \cap Q_2 = \{e\}$. $\diamond$

Now we return to the proof of Lemma 3.6. Let y be the single element in $Q \cap Q_x$. By 3.6.1, we see that $M\backslash y$ has a quad Q_y such that $Q \cap Q_y = \{x\}$ and $Q_x \cap Q_y = \{e\}$.

Let $\{z, w\} = Q - \{x, y\}$. We can again apply Lemma 3.5 and deduce the existence of Q_z, a quad of $M\backslash z$ that contains e and a single element of Q. Note that $Q_z \neq Q_y$, or else we can take the symmetric difference of $Q_y \cup y$ and $Q_z \cup z$ and deduce that $\{y, z\}$ is a series pair of M. Assume that $y \in Q_z$. As the cocircuit $Q_z \cup z$ and the circuit Q_y both contain e, and x is not the single element in $Q \cap Q_z$, it follows that one of the elements in $Q_y - \{x, e\}$ is in Q_z. Then the cocircuit $Q_y \cup y$ and the circuit Q_z meet in three elements: e, y, and an element in $Q_y - \{x, e\}$. This contradiction shows that the single element in $Q \cap Q_z$ is not y nor z. Therefore it is x or w.

First we assume that $Q \cap Q_z = \{w\}$. Then 3.6.1 implies that $M\backslash w$ has a quad Q_w such that $Q \cap Q_w = \{z\}$ and $Q_z \cap Q_w = \{e\}$. The cocircuit $Q_w \cup w$ and the circuit Q_x both contain the element e. Moreover, $y \notin Q_w$, so there is an element α in $(Q_x - \{e, y\}) \cap Q_w$. Let β be the unique element in $Q_x - \{e, y, \alpha\}$. Similarly, the cocircuit $Q_w \cup w$ and the circuit Q_y have e in common, but $x \notin Q_w$, so there is an element γ in $(Q_y - \{e, x\}) \cap Q_w$. Thus $Q_w = \{e, z, \alpha, \gamma\}$. Consider the set $X = \{x, y, z, \alpha, \beta\}$. It spans: w because of the circuit Q; e because of the circuit Q_x; γ because it spans the circuit Q_w; and Q_y because it spans x, e, and γ. This shows that $Q \cup Q_x \cup Q_y$ is a 9-element set satisfying $r(Q \cup Q_x \cup Q_y) \leq 5$. Moreover, X cospans e because of the cocircuit $Q_x \cup x$. It cospans w because it cospans e, and $Q \cup e$ is a cocircuit. Now it cospans: γ as $Q_w \cup w$ is a cocircuit; and Q_y as $Q_y \cup y$ is a cocircuit. Thus X spans and cospans $Q \cup Q_x \cup Q_y$, so

$$\lambda_M(Q \cup Q_x \cup Q_y) \leq 5 + 5 - 9 = 1.$$

As M is 3-connected, this means that there is at most 1 element not in $Q \cup Q_x \cup Q_y$. This is a contradiction as $|E(M)| \geq 11$. Hence we conclude that $Q \cap Q_z = \{x\}$.

Now the cocircuit $Q_z \cup z$ and the circuit Q_x both contain e, so $|Q_z \cap (Q_x - \{e, y\})| = 1$. But this means that the circuit Q_z and the cocircuit $Q_x \cup x$ have three elements in common: e, x, and the element in $Q_z \cap (Q_x - \{e, y\})$. This contradiction completes the proof of Lemma 3.6. $\square$

Lemma 3.7. *Let (M, N) be either $(\bar{M}, \bar{N})$ or $(\bar{M}^*, \bar{N}^*)$. Assume that the element e is such that $M\backslash e$ is $(4, 4)$-connected with an N-minor. Then $M\backslash e$ has no 4-fans.*

Proof. Assume that (a, b, c, d) is a 4-fan of $M\backslash e$. It follows from Lemma 3.4 that M/d is $(4, 4)$-connected with an N-minor. Since it is not internally 4-connected, it contains a quad Q such that $Q \cap \{a, b, c, e\} = \{e\}$. We will show that $M\backslash e/d$ is internally 4-connected, and this will contradict the fact that M and N provide a counterexample to Theorem 1.2, thereby proving Lemma 3.7. Note that Lemma 3.4 states that $M\backslash e/d$ is 3-connected with an N-minor.

3.7.1. *Let (X', Y') be a $(4, 3)$-violator of $M\backslash e/d$. Then neither X' nor Y' contains $Q - e$.*

Proof. Assume that $Q - e \subseteq X'$. As Q is a circuit of M/d, this means that $e \in \mathrm{cl}_{M/d}(X')$. Thus $(X' \cup e, Y')$ is a $(4, 3)$-violator of M/d. Lemma 3.4 says that one side of this $(4, 3)$-violator is a quad that contains e. But this is impossible as $|X' \cup e| > 4$, and $e \notin Y'$. $\diamond$

Let (X, Y) be a $(4, 3)$-violator of $M\backslash e/d$, and assume that $|X \cap (Q - e)| \geq 2$. By 3.7.1 we see that there is a single element in $Y \cap (Q - e)$. Let this element be y. Since $Q - e$ is a triad in $M\backslash e/d$, it follows that $y \in \mathrm{cl}^*_{M\backslash e/d}(X)$. Therefore $(X \cup y, Y - y)$ is a 3-separation in $M\backslash e/d$, but 3.7.1 implies that it is not a $(4, 3)$-violator. Hence $|Y| = 4$. Orthogonality with the triad $Q - e$ implies that Y is not a quad of $M\backslash e/d$. Thus $Y = \{y_1, y_2, y_3, y_4\}$, where (y_1, y_2, y_3, y_4) is a 4-fan in $M\backslash e/d$. Orthogonality also implies that $y = y_4$.

Assume that $M\backslash e/d/y$ has an N-minor. Then $M^*\backslash d\backslash y$ has an N^*-minor. As $M^*\backslash d$ is $(4, 4)$-connected, and y is in the quad Q of this matroid, we now have a contradiction to Lemma 3.6. Therefore $M\backslash e/d/y$ has no N-minor. Lemma 2.2 implies that $M\backslash e/d\backslash y_1$, and hence $M\backslash y_1$, has an N-minor. From this, we deduce that $\{y_1, y_2, y_3\}$ is not a triangle of M, so $\{d, y_1, y_2, y_3\}$ is a circuit. Since $\{b, c, d, e\}$ is a cocircuit, this implies that exactly one of b or c is in $\{y_1, y_2, y_3\}$. Let α be the single element in $\{b, c\} \cap \{y_1, y_2, y_3\}$. Then $\alpha \neq y_1$, as $M\backslash e/d\backslash y_1$ has an N-minor, and b and c are contained in a triangle of M.

Both $\{y_2, y_3, y\}$ and $\{a, b, c\}$ contain the element α. As $\{y_2, y_3, y\}$ is a triad in $M\backslash e/d$, and hence in $M\backslash e$, and $\{a, b, c\}$ is a triangle of $M\backslash e$, it follows that $\{y_2, y_3\} = \{\alpha, a\}$, since $y \in Q$ and $Q \cap \{a, b, c\} = \emptyset$. Hence either (y, a, b, c, d) or (y, a, c, b, d) is a 5-cofan of $M\backslash e$, depending on whether $\alpha = b$ or $\alpha = c$. In either case, from Proposition 2.1, and the fact that $M\backslash e$ is $(4, 4)$-connected, we deduce that $|E(M\backslash e)| \leq 9$, a contradiction. Thus $M\backslash e/d$ has no $(4, 3)$-violator, and is therefore internally 4-connected. This contradiction completes the proof of Lemma 3.7. $\square$

By Lemma 3.2, we know we can choose (M, N) to be $(\bar{M}, \bar{N})$ or $(\bar{M}^*, \bar{N}^*)$ in such a way that $M\backslash e$ is $(4, 4)$-connected with an N-minor for some element $e \in E(M)$. From Lemma 3.7, we deduce that $M\backslash e$ has no 4-fans, and therefore contains at least one quad. Moreover, deleting any element from this quad destroys all N-minors, by Lemma 3.6. Therefore we next consider contracting an element from a quad in $M\backslash e$.

Lemma 3.8. *Let (M, N) be either $(\bar{M}, \bar{N})$ or $(\bar{M}^*, \bar{N}^*)$. Assume that the element e is such that $M\backslash e$ is $(4, 4)$-connected with an N-minor, and that Q is a quad of $M\backslash e$. If $x \in Q$, then $M\backslash e/x$ is 3-connected, and M/x is $(4, 4)$-connected with an N-minor. In particular, if (X, Y) is a $(4, 3)$-violator of M/x such that $|X \cap (Q - x)| \geq 2$, then Y is a quad of M/x, and $Y \cap (Q \cup e) = \{e\}$.*

Proof. To see that M/x has an N-minor, we note that Q is not contained in the ground set of any N-minor of $M\backslash e$. By Lemma 3.6, we cannot delete any element of Q in $M\backslash e$ and preserve an N-minor. Therefore we must contract an element of Q. By the dual of Lemma 2.4, we can contract any element. Thus $M\backslash e/x$, and hence M/x, has an N-minor.

3.8.1. *$M\backslash e/x$ and M/x are 3-connected.*

Proof. Assume that (U, V) is a 2-separation of $M\backslash e/x$ such that $|U \cap (Q - x)| \geq 2$. If $Q - x \subseteq U$, then $(U \cup x, V)$ is a 2-separation in $M\backslash e$, as Q is a cocircuit in this matroid. Since $M\backslash e$ is 3-connected, this is not true, so V contains a single element, y, of $Q - x$. Then $y \in \mathrm{cl}_{M\backslash e/x}(U)$. However $(U \cup y, V - y)$ is not a 2-separation of $M\backslash e/x$, or else $(U \cup \{x, y\}, V - y)$ is a 2-separation of $M\backslash e$. Thus V is either a 2-circuit or a 2-cocircuit in $M\backslash e/x$. Orthogonality with $Q - x$ tells us that the latter case is impossible. Therefore x is in a triangle in $M\backslash e$ that contains two elements of Q. The union of Q with this triangle is a 5-element 3-separating set in $M\backslash e$, contradicting the fact that $M\backslash e$ is $(4, 4)$-connected. Therefore

$M\backslash e/x$ is 3-connected. If M/x is not, then e must be in a triangle with x in M, and this is impossible by Lemma 3.1. $\diamond$

We will prove that M/x is $(4, 4)$-connected. Assume otherwise, and let (X, Y) be a $(4, 4)$-violator of M/x, so that $|X|, |Y| \geq 5$. We can assume that $|X \cap (Q - x)| \geq 2$.

3.8.2. $e \in Y$.

Proof. Assume that $e \in X$. If $Q - x \subseteq X$, then $x \in \mathrm{cl}^*_M(X)$, as $Q \cup e$ is a cocircuit of M. Therefore $(X \cup x, Y)$ is a $(4, 4)$-violator of M, which is impossible. Therefore $Y \cap (Q - x)$ contains a single element, y. Now $y \in \mathrm{cl}_{M/x}(X)$, so $(X \cup y, Y - y)$ is a 3-separation in M/x. As $x \in \mathrm{cl}^*_M(X \cup y)$, and $|Y - y| \geq 4$, it follows that $(X \cup \{x, y\}, Y - y)$ is a $(4, 3)$-violator of M, a contradiction. $\diamond$

3.8.3. $\lambda_{M\backslash e}(Y - (Q \cup e)) \leq 2$.

Proof. As $\lambda_{M/x}(Y) = 2$, and $Q - x \subseteq \mathrm{cl}_{M/x}(X)$, it follows that $\lambda_{M/x}(Y - Q) \leq 2$. Therefore $\lambda_{M\backslash e/x}(Y - (Q \cup e)) \leq 2$. Now x is in the coclosure of the complement of $Y - (Q \cup e)$ in $M\backslash e$, as Q is a cocircuit, so

$$\lambda_{M\backslash e}(Y - (Q \cup e)) = \lambda_{M\backslash e/x}(Y - (Q \cup e)) \leq 2,$$

as desired. $\diamond$

3.8.4. $|Y| \leq 6$.

Proof. Since $M\backslash e$ is $(4, 4)$-connected, 3.8.3 implies that $|Y - (Q \cup e)| \leq 4$. The result follows. $\diamond$

3.8.5. $|Y| \neq 6$.

Proof. Assume that $|Y| = 6$. If $Q - x \subseteq X$, then 3.8.3 implies that $M\backslash e$ has a 5-element 3-separating set, which leads to a contradiction. Therefore $Y \cap (Q - x)$ contains a single element, y. Since $Q - x$ is a triangle in M/x, it follows that $y \in \mathrm{cl}_{M/x}(X)$, so $(X \cup x, Y - y)$ is a 3-separation of M/x. Proposition 3.3 implies that $Y - y$ is a 5-fan of M/x. Let $(y_1, \ldots, y_5)$ be a fan ordering of $Y - y$. As e is contained in no triads of M, we can assume that $e = y_1$. As $\{y_2, y_3, y_4\}$ is a triad of M/x, and hence of M, it cannot be the case that $\{y_3, y_4, y_5\}$ is a triangle, or else M has a 4-fan. Therefore $\{x, y_3, y_4, y_5\}$ is a circuit of M that meets the cocircuit $Q \cup e$ in the single element x. This contradiction completes the proof of 3.8.5. $\diamond$

3.8.6. $|Y| \neq 5$.

Proof. Assume that $|Y| = 5$. First suppose that $Q - x \subseteq X$. Then $(X \cup x, Y - e)$ is a 3-separation of $M\backslash e$, by 3.8.3. Thus $Y - e$ is a quad of $M\backslash e$, by Proposition 2.3 and Lemma 3.7. But Proposition 3.3 implies that Y is a 5-fan of M/x. Thus Y contains a triad of M/x, and hence of M. This means that $Y - e$ contains a cocircuit of size at most three in $M\backslash e$, contradicting the fact that it is a quad. Thus $Y \cap (Q - x)$ contains a single element, y.

By again using Proposition 3.3, we see that Y is a 5-fan of M/x. Let $(y_1, \ldots, y_5)$ be a fan ordering. Orthogonality with $Q - x$ means that y is not contained in a triad of M/x that is contained in Y. Therefore we can assume that $y = y_1$. As e is in no triad of M, it follows that $e \notin \{y_2, y_3, y_4\}$. As $M/x\backslash e$ is 3-connected, by 3.8.1, we deduce that (y_1, y_2, y_3, y_4) is a 4-fan of $M/x\backslash e$. As $\{y_2, y_3, y_4\}$ is a triad of M/x, and hence of M, Lemma 3.1 implies M/y_4 has no N-minor, so neither does $M/x\backslash e/y_4$. Lemma 2.2 now implies that $M/x\backslash e\backslash y_1$, and hence $M\backslash e\backslash y_1$, has an N-minor. As $y_1 = y$ is contained in the quad Q of $M\backslash e$, this means we have a contradiction to Lemma 3.6. $\diamond$

We assumed (X, Y) was a $(4, 4)$-violator of M/x, so we now obtain a contradiction by combining 3.8.4–3.8.6. Therefore M/x is $(4, 4)$-connected. Now assume (X, Y) is a $(4, 3)$-violator of M/x. We can assume that $|X \cap (Q - x)| \geq 2$. Because $M^*\backslash x$ is $(4, 4)$-connected with an N^*-minor, it follows from Lemma 3.7 that either X or Y is a quad of M/x. If X is a quad of M/x, then it does not contain the triangle $Q - x$. Therefore $X \cup (Q - x)$ is a 5-element 3-separating set of M/x. This leads to a contradiction, as $|E(M/x)| \geq 10$ and M/x is $(4, 4)$-connected. Therefore Y is a quad of M/x. Orthogonality shows that Y is disjoint from the triangle $Q - x$. If Y does not contain e, then $x \in \mathrm{cl}^*_M(X)$, and Y is a quad of M, a contradiction. Therefore $Y \cap (Q \cup e) = \{e\}$, and the proof of Lemma 3.8 is complete. $\square$

Finally, we are in a position to prove Theorem 1.2. By Lemma 3.2, we can assume that (M, N) is either $(\bar{M}, \bar{N})$ or $(\bar{M}^*, \bar{N}^*)$, and $M\backslash e$ is (4, 4)-connected with an N-minor, for some element e. Lemma 3.7 implies that $M\backslash e$ has no 4-fans. As it is not internally 4-connected, it contains a quad Q. Deleting any element of Q destroys all N-minors, by Lemma 3.6, so $M\backslash e/x$ has an N-minor, for some element $x \in Q$. Lemma 3.8 says that $M\backslash e/x$ is 3-connected, and M/x is (4, 4)-connected. As M/x is not internally 4-connected, it has a quad, Q_x, such that $(Q \cup e) \cap Q_x = \{e\}$. We will show that $M\backslash e/x$ is internally 4-connected, and this will provide a contradiction that completes the proof of Theorem 1.2.

Assume that (X, Y) is a (4, 3)-violator of $M\backslash e/x$, where $|X \cap (Q_x - e)| \geq 2$. If $Q_x - e \subseteq X$, then $(X \cup e, Y)$ is a (4, 3)-violator of M/x, as Q_x is a circuit in M/x. Then Lemma 3.8 implies that either $X \cup e$ or Y is a quad that contains e. This is impossible, as $|X \cup e| \geq 5$. Therefore $Y \cap (Q_x - e)$ contains a single element, y. In $M\backslash e/x$, the set $Q_x - e$ is a triad, so y is in the coclosure of X. Therefore $(X \cup y, Y - y)$ is a 3-separation. If it is a (4, 3)-violator, then $(X \cup \{y, e\}, Y - y)$ is a (4, 3)-violator of M/x, and this leads to the same contradiction as before, since either $X \cup \{y, e\}$ or $Y - y$ must be a quad of M/x that contains e. Therefore $|Y| = 4$. Orthogonality with $Q_x - e$ shows that Y is not a quad of $M\backslash e/x$. Thus we assume that (y_1, y_2, y_3, y_4) is a 4-fan and a fan ordering of Y in $M\backslash e/x$. Then $y = y_4$, or else we violate orthogonality between $Q_x - e$ and $\{y_1, y_2, y_3\}$.

Since y is in a quad of M/x, Lemma 3.6 implies that $M/x/y$, and hence $M\backslash e/x/y$ has no N-minor. Therefore Lemma 2.2 implies that $M\backslash e/x\backslash y_1$ has an N-minor. As $M\backslash y_1$ has an N-minor, it follows that $\{y_1, y_2, y_3\}$ is not a triangle of M. Therefore $\{x, y_1, y_2, y_3\}$ is a circuit. As $Q \cup e$ is a cocircuit, there is a single element, which we call z, in $(Q - x) \cap \{y_1, y_2, y_3\}$.

Note that $\{y_2, y_3, y\}$ is not a triad of M, by orthogonality with the circuit $Q_x \cup x$. Therefore $\{y_2, y_3, y, e\}$ is a cocircuit. This means that z is not in $\{y_2, y_3\}$, for otherwise $\{y_2, y_3, y, e\}$ meets the circuit Q in the single element z. Therefore $z = y_1$, and $M\backslash e/x\backslash z$ has an N-minor. This means that $M\backslash e\backslash z$ has an N-minor, and as z is in Q, we have contradicted Lemma 3.6. Thus Theorem 1.2 is now proved.

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