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\textbf{A B S T R A C T}

Let \(M\) and \(N\) be internally 4-connected binary matroids such that \(M\) has a proper \(N\)-minor, and \(|E(N)| \geq 7\). As part of our project to develop a splitter theorem for internally 4-connected binary matroids, we prove the following result: if \(M\setminus e\) has no \(N\)-minor whenever \(e\) is in a triangle of \(M\), and \(M/e\) has no \(N\)-minor whenever \(e\) is in a triad of \(M\), then \(M\) has a minor, \(M'\), such that \(M'\) is internally 4-connected with an \(N\)-minor, and \(1 \leq |E(M)| - |E(M')| \leq 2\).

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1. Introduction

It would be useful for structural matroid theory if we could make the following statement: there exists an integer, \(k\), such that whenever \(M\) and \(N\) are internally 4-connected binary matroids and \(M\) has a proper \(N\)-minor, then \(M\) has an internally 4-connected minor, \(M'\), such that \(M'\) has an \(N\)-minor, and \(1 \leq |E(M)| - |E(M')| \leq k\). However this statement is false; no such \(k\) exists. To see this, we let \(M\) be the cycle matroid of a quartic planar ladder on \(n\) vertices, and we let \(N\) be the cycle matroid of the cubic planar ladder on the same number of vertices. Then \(M\) and \(N\) are internally 4-connected, and \(M\) has a proper minor isomorphic to \(N\). Moreover, \(|E(M)| = 2n\), and \(|E(N)| = 3n/2\). However, the only proper minor of \(M\) that is internally 4-connected with an \(N\)-minor is itself isomorphic to \(N\).

In light of this example, we concentrate on a different goal. To aid brevity, let us introduce some notation. Say that \(\delta\) is the set of all ordered pairs, \((M, N)\) where \(M\) and \(N\) are internally 4-connected binary matroids, and \(M\) has a proper \(N\)-minor. We will let \(\delta_k\) be the subset of \(\delta\) for which there is an internally 4-connected minor, \(M'\), of \(M\) that has an \(N\)-minor and satisfies \(1 \leq |E(M)| - |E(M')| \leq k\). The discussion in the previous paragraph shows that we cannot find a \(k\) so that \(\delta \subseteq \delta_k\). Instead, we want to show that, for any \((M, N) \in \delta\), either \((M, N) \in \delta_k\), for some small value of \(k\), or there is some easily described operation we can perform on \(M\) to produce an internally 4-connected minor...
that has an $N$-minor. To this end, we are trying to identify as many pairs as possible that belong to $\delta_k$, for small values of $k$. For example, our first step [1] was to show that if $M$ is 4-connected, then $(M, N)$ is in $\delta_2$. In fact, in almost every case, $(M, N)$ belongs to $\delta_1$.

**Theorem 1.1.** Let $M$ and $N$ be binary matroids such that $M$ has a proper $N$-minor, and $|E(N)| \geq 7$. If $M$ is 4-connected and $N$ is internally 4-connected, then $M$ has an internally 4-connected minor $M'$ with an $N$-minor such that $1 \leq |E(M)| - |E(M')| \leq 2$. Moreover, unless $M$ is isomorphic to a specific 16-element self-dual matroid, such an $M'$ exists with $|E(M)| - |E(M')| = 1$.

An internally 4-connected binary matroid is 4-connected if and only if it has no triangles and triads. Therefore we have shown that if $M$ has no triangles or triads (and $|E(N)| \geq 7$), then $(M, N) \in \delta_2$. Hence we now assume that $M$ does contain a triangle or triad. In this chapter of the series, we consider the case that all triangles and triads of $M$ must be contained in the ground set of every $N$-minor. In other words, deleting an element from a triangle of $M$, or contracting an element from a triad, destroys all $N$-minors. We show that under these circumstances, $(M, N)$ is in $\delta_2$.

**Theorem 1.2.** Let $M$ and $N$ be internally 4-connected binary matroids, such that $|E(N)| \geq 7$, and $N$ is isomorphic to a proper minor of $M$. Assume that if $T$ is a triangle of $M$ and $e \in T$, then $M \setminus e$ does not have an $N$-minor. Dually, assume that if $T$ is a triad of $M$ and $e \in T$, then $M \setminus e$ does not have an $N$-minor. Then $M$ has an internally 4-connected minor $M'$, such that $M'$ has an $N$-minor, and $1 \leq |E(M)| - |E(M')| \leq 2$.

With this result in hand, in the next chapter [2] we will be able to assume that (up to duality) $M$ has a triangle $T$ and an element $e \in T$ such that $M \setminus e$ has an $N$-minor.

We note that Theorem 1.2 is not strictly a strengthening of Theorem 1.1 as, in the earlier theorem, we completely characterized when $(M, N)$ was in $\delta_2 - \delta_1$. We make no attempt to obtain the corresponding characterization in Theorem 1.2, as we believe that $\delta_2 - \delta_1$ will contain many more pairs when we relax the constraint that $M$ is 4-connected. For example, let $N$ be obtained from a binary projective geometry by performing a $\Delta$-$\gamma$ exchange on a triangle $T$. Let $T'$ be a triangle that is disjoint from $T$. We obtain $M$ from $N$ by coextending by the element $x$ so that it is in a triad with two elements from $T'$, and then extending by $y$ so that it is in a circuit with $x$ and two elements from $T$. It is not difficult to confirm that the hypotheses of Theorem 1.2 hold, but $M$ has no internally 4-connected single-element deletion or contraction with an $N$-minor. Clearly this technique could be applied to create even more diverse examples.

## 2. Preliminaries

We assume familiarity with standard matroid notions and notations, as presented in [3]. We make frequent, and sometimes implicit, use of the following well-known facts. If $M$ is $n$-connected, and $|E(M)| \geq 2(n - 1)$, then $M$ has no circuit or cocircuit with fewer than $n$ elements [3, Proposition 8.2.1]. In a binary matroid, a circuit and a cocircuit must meet in a set of even cardinality [3, Theorem 9.1.2(ii)].

The symmetric difference, $C \triangle C'$, of two circuits in a binary matroid is a disjoint union of circuits [3, Theorem 9.1.2(iv)].

We use 'by orthogonality' as shorthand for the statement 'by the fact that a circuit and a cocircuit cannot intersect in a set of cardinality one' [3, Proposition 2.1.11]. A **triangle** is a 3-element circuit, and a **triad** is a 3-element cocircuit. We use $\lambda_M$ or $\lambda$ to denote the connectivity function of the matroid $M$. If $M$ and $N$ are matroids, an $N$-**minor** of $M$ is a minor of $M$ that is isomorphic to $N$.

Let $M$ be a matroid. A subset $S$ of $E(M)$ is a **fan** in $M$ if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of $S$ such that

$$
\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \ldots, \{s_{n-2}, s_{n-1}, s_n\}
$$

is an alternating sequence of triangles and triads. We call $(s_1, s_2, \ldots, s_n)$ a **fan ordering** of $S$. Sometimes we blur the distinction between a fan and an ordering of that fan. Most of the fans we encounter have four or five elements. We adopt the following convention: if $(s_1, s_2, s_3, s_4)$ is a fan ordering of a 4-element fan, then $(s_1, s_2, s_3)$ is a triangle. We call such a fan ordering a 4-***fan***. We distinguish between the two different types of 5-element fan by using 5-***fan*** to refer to a 5-element fan containing two triangles, and using 5-***cofan*** to refer to a 5-element fan containing two triads.
The next proposition is proved by induction on $n$, using the fact that $s_n$ is contained in either the closure or the coclosure of $\{s_1, \ldots, s_{n-1}\}$.

**Proposition 2.1.** Let $(s_1, \ldots, s_n)$ be a fan ordering in a matroid $M$. Then
\[
\lambda_M(\{s_1, \ldots, s_n\}) \leq 2.
\]

**Lemma 2.2.** Let $M$ be a binary matroid that has an internally 4-connected minor, $N$, satisfying $|E(N)| \geq 8$. If $(s_1, s_2, s_3, s_4)$ is a 4-fan of $M$, then $M \setminus s_1$ or $M / s_4$ has an $N$-minor. If $(s_1, s_2, s_3, s_4, s_5)$ is a 5-fan in $M$, then either $M \setminus s_1 \setminus s_2$ has an $N$-minor, or both $M \setminus s_1 / s_2$ and $M / s_4 \setminus s_3$ have $N$-minors. In particular, both $M \setminus s_1$ and $M \setminus s_5$ have $N$-minors.

**Proof.** Let $(s_1, s_2, s_3, s_4)$ be a 4-fan. Since $\{s_1, s_2, s_3, s_4\}$ contains a circuit and a cocircuit, $\lambda_{N_0}(\{s_1, s_2, s_3, s_4\}) \leq 2$ for any minor, $N_0$, of $E(M)$ that contains $\{s_1, s_2, s_3, s_4\}$ in its ground set. As $N$ is internally 4-connected and $|E(N)| \geq 8$, we deduce that $N$ is obtained from $M$ by removing at least one element of $\{s_1, s_2, s_3, s_4\}$. Let $x$ be an element in $\{s_1, s_2, s_3, s_4\} - E(N)$. If $M \setminus x$ has an $N$-minor, then either $x = s_1$, as desired; or $(s_2, s_3, s_4) - x$ is a 2-cocircuit in $M \setminus x$. In the latter case, as $N$ is internally 4-connected, either $x \in \{s_2, s_3\}$, and $M / s_4$ has an $N$-minor, as desired; or $x = s_4$, and $M / s_2$ has an $N$-minor. But $(s_1, s_2)$ is a 2-circuit of the last matroid, so $M \setminus s_1$ has an $N$-minor, and the lemma holds. We may now suppose that deleting any element of $\{s_1, s_2, s_3, s_4\}$ from $M$ yields a matroid with no $N$-minor. Then $N$ is a minor of $M / x$ for some $x \in \{s_1, s_2, s_3, s_4\}$. But $x$ is not in $\{s_1, s_2, s_3\}$, or else $(s_1, s_2, s_3) - x$ is a 2-circuit in $M / x$, and we may delete one of its elements while keeping an $N$-minor. Thus $x = s_4$, and the lemma holds.

Next we assume that $(s_1, s_2, s_3, s_4, s_5)$ is a 5-fan in $M$. First we show that $M \setminus s_1 / s_2$ has an $N$-minor if and only if $M / s_4 \setminus s_5$ has an $N$-minor. As $(s_1, s_3)$ is a 2-circuit of $M / s_2$, it follows that if $M \setminus s_1 / s_2$ has an $N$-minor, so does $M / s_3$. As $(s_4, s_5)$ is a 2-cocircuit of the last matroid, this implies that $M / s_4$ has an $N$-minor. Hence so does $M / s_4 \setminus s_5$. Thus $M / s_4 \setminus s_5$ has an $N$-minor if $M / s_1 / s_2$ does. The converse statement yields to a symmetrical argument.

Now $(s_1, s_2, s_3, s_4, s_5)$ is a 4-fan of $M$. By applying the first statement of the lemma, we see that $M \setminus s_1$ or $M / s_4$ has an $N$-minor. In the latter case, $M / s_4 \setminus s_5$ has an $N$-minor, and we are done. Therefore we assume that $M \setminus s_1$ has an $N$-minor. There is a cocircuit of $M \setminus s_1$ that contains $s_2$, and is contained in $(s_2, s_3, s_4)$. If this cocircuit is not a triad, then $M / s_1 / s_2$ has an $N$-minor, and we are done. Therefore we assume that $(s_5, s_4, s_3, s_2)$ is a 4-fan of $M \setminus s_1$. We apply the first statement of the lemma, and deduce that either $M \setminus s_1 / s_2$ or $M \setminus s_1 / s_5$ has an $N$-minor. In either case the proof is complete. $\Box$

A quad is a 4-element circuit–cocircuit. It is clear that if $Q$ is a quad, then $\lambda(Q) \leq 2$. The next result is easy to verify.

**Proposition 2.3.** Let $(X, Y)$ be a 3-separation of a 3-connected binary matroid with $|X| = 4$. Then $X$ is a quad or a 4-fan.

The next result is Lemma 2.2 in [1].

**Lemma 2.4.** Let $Q$ be a quad in a binary matroid $M$. If $x$ and $y$ are in $Q$, then $M \setminus x$ and $M \setminus y$ are isomorphic.

A matroid is $(4, k)$-connected if it is 3-connected, and, whenever $(X, Y)$ is a 3-separation, either $|X| \leq k$ or $|Y| \leq k$. A matroid is internally 4-connected precisely when it is $(4, 3)$-connected. If a matroid is 3-connected, but not $(4, k)$-connected, then it contains a 3-separation, $(X, Y)$, such that $|X|, |Y| > k$. We will call such a 3-separation a $(4, k)$-violator.

For $n \geq 3$, we let $G_{n+2}$ denote the biwheel graph with $n + 2$ vertices. Thus $G_{n+2}$ consists of a cycle $v_1, v_2, \ldots, v_n$, and two additional vertices, $u$ and $v$, each of which is adjacent to every vertex in $\{v_1, v_2, \ldots, v_n\}$. The planar dual of a biwheel is a cubic planar ladder. We construct $G_{n+2}^+$ by adding an edge between $u$ and $v$. It is easy to see that $M(G_{n+2}^+)$ is represented over GF(2) by the following matrix:

\[
\begin{bmatrix}
I_{n+1} & 1 & 0 \\
I_n & A_n
\end{bmatrix}
\]
where $A_n$ is the $n \times n$ matrix
\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 1 \\
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]
and $1$ and $0$ are $1 \times n$ vectors with all entries equal to $1$ or $0$ respectively. Thus $M(G^+_{n+2})$ is precisely equal to the matroid $D_n$, as defined by Zhou [4], and the element $f_1$ of $E(D_n)$ is the edge $uv$.

For $n \geq 2$ let $\Delta_{n+1}$ be the rank-$(n+1)$ binary matroid represented by the following matrix.
\[
\begin{bmatrix}
I_{n+1} & 1 & e_n \\
I_n & A_n
\end{bmatrix}.
\]
In this case, $e_n$ is the standard basis vector with a one in position $n$. Then $\Delta_{n+1}$ is a triangular Möbius matroid (see [5]). In [4], the notation $D^n$ is used for the matroid $\Delta_{n+1}$, and $f_1$ denotes the element represented by the first column in the matrix. We use $z$ to denote the same element. We observe that $\Delta_{n+1} \setminus z$ is the bond matroid of a Möbius cubic ladder.

The next result is a consequence of a theorem due to Zhou [4].

**Theorem 2.5.** Let $M$ and $N$ be internally $4$-connected binary matroids such that $N$ is a proper minor of $M$ satisfying $|E(N)| \geq 7$. Then either

(i) $M \setminus e$ or $M / e$ is $(4, 4)$-connected with an $N$-minor, for some element $e \in E(M)$, or

(ii) $M$ or $M^*$ is isomorphic to either $M(G^+_{n+2})$, $M(G^+_{n+2} \setminus z)$, or $\Delta_{n+1}$, or $\Delta_{n+1} \setminus z$, for some $n \geq 4$.

Note that the theorem in [4] is stated with the weaker hypothesis that $|E(N)| \geq 10$. However, Zhou explains that by using results from [6,7] and performing a relatively simple case-analysis, we can strengthen the theorem so that it holds under the condition that $|E(N)| \geq 7$.

### 3. Proof of the main theorem

In this section we prove Theorem 1.2. Throughout the section, we assume that the theorem is false. This means that there exist internally $4$-connected binary matroids, $\tilde{M}$ and $\tilde{N}$, with the following properties:

(i) $\tilde{M}$ has a proper $\tilde{N}$-minor,

(ii) if $e$ is in a triangle of $\tilde{M}$, then $\tilde{M} \setminus e$ has no $\tilde{N}$-minor,

(iii) if $e$ is in a triad of $\tilde{M}$, then $\tilde{M} / e$ has no $\tilde{N}$-minor,

(iv) there is no internally $4$-connected minor, $M'$, of $\tilde{M}$ such that $M'$ has an $\tilde{N}$-minor and $1 \leq |E(\tilde{M} / e)| - |E(M')| \leq 2$, and

(v) $|E(\tilde{N})| \geq 7$.

Note that $(\tilde{M}^*, \tilde{N}^*)$ also provides a counterexample to Theorem 1.2. We start by showing that we can assume $|E(\tilde{N})| \geq 8$. If $|E(\tilde{N})| = 7$, then $\tilde{N}$ is isomorphic to $F_7$ or $F_7^*$. Then $\tilde{M}$ is non-regular, and contains one of the five internally $4$-connected non-regular matroids $N_{10}$, $K_5$, $\tilde{K}_5$, $T_{12} / e$ or $T_{12} / e$ as a minor [7, Corollary 1.2]. But $N_{10}$ contains an element in a triangle whose deletion is non-regular, so $\tilde{M}$ is not isomorphic to $N_{10}$. The same statement applies to $K_5$ and $T_{12} / e$, so $\tilde{M}$ is not isomorphic to these matroids, or their duals, $\tilde{K}_5^*$ and $T_{12} / e$. Thus $\tilde{M}$ has a proper internally $4$-connected minor, $\tilde{N}'$, isomorphic to one of the five matroids listed above. Therefore we can relabel $N'$ as $\tilde{N}$. As each of the five matroids has more than seven elements, we are justified in assuming that $|E(\tilde{N})| \geq 8$. As $(\tilde{M}, \tilde{N})$ provides a counterexample to Theorem 1.2, it follows that $|E(\tilde{M})| \geq 11$.

**Lemma 3.1.** Let $(M, N)$ be $(\tilde{M}, \tilde{N})$ or $(\tilde{M}^*, \tilde{N}^*)$. Let $N_0$ be an arbitrary $N$-minor of $M$. If $T$ is a triangle or a triad of $M$, then $T \subseteq E(N_0)$. 
Proof. By duality, we can assume that \( \{e, f, g\} \) is a triangle of \( M \). Assume that \( e \not\in E(N_0) \). Since \( M \setminus e \) has no minor isomorphic to \( N \), it follows that \( N_0 \) is a minor of \( M/e \). As \( \{f, g\} \) is a 2-circuit in \( M/e \), it follows that \( N_0 \) is a minor of either \( M/e \setminus f \) or \( M/e \setminus g \), and hence of \( M \setminus f \) or \( M \setminus g \). But neither of these matroids has an \( N \)-minor, so we have a contradiction. \( \Box \)

Lemma 3.2. There is an element \( e \in E(\tilde{M}) \) such that either \( \tilde{M} \setminus e \) or \( \tilde{M} / e \) is \((4, 4)\)-connected with an \( \tilde{N} \)-minor.

Proof. If the lemma fails, then by Theorem 2.5, either \( \tilde{M} \) or its dual is isomorphic to one of \( M(G_{n+2}) \), \( M(G_{n+2}^+) \), \( \Delta_{n+1} \), or \( \Delta_{n+1} \setminus z \), for some \( n \geq 4 \). In these cases it is easy to verify that every element of \( E(\tilde{M}) \) is contained in a triangle or a triad. Therefore Lemma 3.1 implies that \( E(\tilde{M}) = E(\tilde{N}) \), contradicting the fact that \( \tilde{N} \) is a proper minor of \( M \). \( \Box \)

If \( e \) is an element such that \( \tilde{M} \setminus e \) is \((4, 4)\)-connected with an \( \tilde{N} \)-minor, then \( \tilde{M} \setminus e \) has a quad or a 4-fan, for otherwise it follows from Proposition 2.3 that \( M \setminus e \) is internally 4-connected, contradicting the fact that \( M \) is a counterexample to Theorem 1.2. We will make frequent use of the following fact.

Proposition 3.3. Let \((M, N)\) be either \((\tilde{M}, \tilde{N})\) or \((\tilde{M}^*, \tilde{N}^*)\). If \( M \setminus e \) is 3-connected and has an \( N \)-minor, and \((X, Y)\) is a 3-separation of \( M \setminus e \) such that \(|Y| = 5\), then \( Y \) is a 5-cofan of \( M \setminus e \).

Proof. If \( Y \) is not a fan, then \( Y \) contains a quad (see [4, Lemma 2.14]). As in the proof of [4, Lemma 2.15], we can show that in \( M \), there is either a triangle or a triad of \( M \) that is contained in \( Y \) and which contains two elements from the quad. In the first case, the triangle contains an element we can delete to keep an \( N \)-minor. In the second case, the triad contains an element we can contract and keep an \( N \)-minor. In either case, we have a contradiction to Lemma 3.1. Therefore \( Y \) is a 5-element fan. If \( Y \) is a 5-fan, then by Lemma 2.2, we can delete an element from a triangle in \( M \setminus e \) and preserve an \( N \)-minor. This contradicts Lemma 3.1, so \( Y \) is a 5-cofan. \( \Box \)

At this point, we give a quick summary of the lemmas that follow. Lemma 3.4 considers the matroid produced by contracting the last element of a 4-fan in \( M \setminus e \). Lemma 3.5 deals with deleting an element from a quad in \( M \setminus e \). In Lemma 3.6 we show that whenever we delete such an element, we destroy all \( N \)-minors. We exploit this information in Lemma 3.7, and show that \( M \setminus e \) has no 4-fans. The only case left to consider is one in which we contract an element from a quad in \( M \setminus e \). This case is covered in Lemma 3.8. After this lemma, there is only a small amount of work to be done before we obtain a final contradiction and complete the proof.

Lemma 3.4. Let \((M, N)\) be either \((\tilde{M}, \tilde{N})\) or \((\tilde{M}^*, \tilde{N}^*)\). Assume that \( e \) is an element of \( M \) such that \( M \setminus e \) is \((4, 4)\)-connected with an \( N \)-minor, and that \((a, b, c, d)\) is a 4-fan of \( M \setminus e \). Then \( M \setminus e / d \) is 3-connected with an \( N \)-minor, and \( M \setminus d \) is \((4, 4)\)-connected. Moreover, if \((X, Y)\) is a \((4, 3)\)-violer of \( M \setminus d \) such that \(|X \cap \{a, b, c\}| \geq 2\), then \( Y \) is a quad of \( M \setminus d \), and \( Y \cap \{a, b, c, e\} = \{e\} \).

Proof. From Proposition 2.1 and the fact that \( M \) is internally 4-connected with at least eleven elements, it follows that \((a, b, c, d)\) is not a 4-fan of \( M \). Therefore \((b, c, d, e)\) is a cocircuit in \( M \).

3.4.1. \( M \setminus e / d \) and \( M \setminus d \) are 3-connected.

Proof. We start by showing that \( M \setminus e / d \) is 3-connected. Because \( M \setminus e \) is \((4, 4)\)-connected, it is also 3-connected. Assume that \( M \setminus e / d \) is not 3-connected. As \( M \setminus e / c \) contains the parallel pair \( \{a, b\} \), it too is not 3-connected. As \( \{b, c, d\} \) is a triad in \( M \setminus e \), we can apply the dual of [3, Lemma 8.7.7], and see that there is a triangle of \( M \setminus e \) containing \( d \), and exactly one of \( b \) and \( c \). Let \( z \) be the third element of this triangle. Then \( z \neq a \), or else \((a, b, c, d)\) is a \( U_{2,4} \)-restriction of \( M \setminus e \) and in this case \((b, c, d)\) is both a triangle and a triad. This leads to a contradiction to the 3-connectivity of \( M \setminus e \). Therefore \((a, b, c, d, z)\) is a 5-fan in \( M \setminus e \). Since \(|E(M \setminus e)| \geq 10\), this means that \( M \setminus e \) is not \((4, 4)\)-connected, and we have a contradiction. Therefore \( M \setminus e / d \) is 3-connected.

If \( M \setminus d \) is not 3-connected, then it follows easily (see the dual of [8, Lemma 2.6]) that \( \{e, d\} \) is contained in a triangle of \( M \). However, \( N \) is a minor of \( M \setminus e \), so we have a contradiction to Lemma 3.1. \( \Box \)
3.4.2. $M \setminus e/d$, and hence $M/d$, has an $N$-minor.

**Proof.** Let $N_0$ be an $N$-minor of $M$. Then $\{a, b, c\} \subseteq E(N_0)$, by Lemma 3.1. As $(a, b, c, d)$ is a 4-fan of $M \setminus e$, it now follows by Lemma 2.2 that $M \setminus e/d$ has an $N$-minor. \hfill\(\diamondsuit\)

3.4.3. Let $(X, Y)$ be a $(4, 3)$-violator of $M/d$, and assume that $|X \cap \{a, b, c\}| \geq 2$. Then $|Y| = 4$, and $e \in Y$. Moreover, $Y \cap \{a, b, c\} = \emptyset$.

**Proof.** Assume that the result fails.

3.4.3.1. $|Y| \geq 5$.

**Proof.** Assume otherwise. Then $|Y| = 4$. Assume that $e \in X$. If $\{b, c\} \subseteq X$, then $d \in \text{cl}^e_m(X)$, as $\{b, c, d, e\}$ is a cocircuit. It follows from [3, Corollary 8.2.6(iii)] that $\lambda_M(X \cup d) = \lambda_{M/d}(X)$, and therefore $(X \cup d, Y)$ is a $(4, 3)$-violator of $M$, an impossibility. Hence either $b$ or $c$ is contained in $Y$, so $|X \cap \{a, b, c\}| \geq 2$ implies $a$ is in $X$.

Proposition 2.3 implies that $Y$ is either a quad or a 4-fan of $M/d$. As $(a, b, c)$ is a triangle of $M/d$ that meets $Y$ in a single element, $Y$ is not a cocircuit, and hence not a quad of $M/d$. Thus $Y = \{y_1, y_2, y_3, y_4\}$, where $(y_1, y_2, y_3, y_4)$ is a 4-fan of $M/d$. Since the triangle $(a, b, c)$ cannot meet the triad $\{y_2, y_3, y_4\}$ in a single element, it follows that $y_1$ is equal to $b$ or $c$. Let $N_0$ be an $N$-minor of $M/d$. Since $\{y_2, y_3, y_4\}$ is a triad of $M$, it follows from Lemma 3.1 that $\{y_2, y_3, y_4\} \subseteq E(N_0)$. But $(a, b, c)$ is a triangle of $M$, so $(a, b, c) \subseteq E(N_0)$. Therefore $\{y_1, y_2, y_3, y_4\} \subseteq E(N_0)$, and this contradicts Lemma 2.2. From this contradiction we conclude that $e \in Y$.

Since 3.4.3 fails, yet $|Y| = 4$ and $e \in Y$, we deduce that $Y$ contains exactly one element of the triangle $(a, b, c)$. Thus $Y$ is not a quad of $M/d$, so $Y$ is a 4-fan, $(y_1, y_2, y_3, y_4)$, of $M/d$. Since $(a, b, c)$ is a triangle of $M/d$, and $(y_2, y_3, y_4)$ is a triad, orthogonality requires that the single element in $Y \cap \{a, b, c\}$ is $y_1$. Therefore $e$ is contained in the triad $(y_2, y_3, y_4)$. But this means that $M \setminus e$ contains a 2-cocircuit, a contradiction as it is 3-connected. \hfill\(\diamondsuit\)

Let $T = \{a, b, c\}$. As $Y$ contains at most one element of $T$, it follows from 3.4.3.1 that $|Y - T| \geq 4$. Furthermore, $X$ spans $T$. The next fact follows from these observations and from 3.4.1.

3.4.3.2. $(X \cup T, Y - T)$ is a 3-separation in $M/d$.

3.4.3.3. $e \in Y$.

**Proof.** Assume that $e \in X$. Then 3.4.3.2 and the cocircuit $(b, c, d, e)$ imply that $(X \cup T \cup d, Y - T)$ is 3-separation of $M$. Since $|Y - T| \geq 4$, it follows that $M$ has a $(4, 3)$-violator, which is impossible. \hfill\(\diamondsuit\)

3.4.3.4. $|Y - T| \leq 5$.

**Proof.** By 3.4.3.2 and 3.4.3.3, we see that $(X \cup T, Y - (T \cup e))$ is a 3-separation in $M/d \setminus e$. As $\{b, c, d\}$ is a triad in $M \setminus e$, it follows that $d \in \text{cl}^e_M(T)$, so

$$(X \cup T \cup d, Y - (T \cup e))$$

is a 3-separation of $M \setminus e$. Since $|X \cup T \cup d| > 4$, and $M \setminus e$ is $(4, 4)$-connected, it follows that $|Y - (T \cup e)| \leq 4$, so $|Y - T| \leq 5$. \hfill\(\diamondsuit\)

3.4.3.5. $|Y - T| = 4$.

**Proof.** We have observed that $|Y - T| \geq 4$, so if 3.4.3.5 is false, it follows from 3.4.3.4 that $|Y - T| = 5$. From 3.4.3.2 and the dual of Proposition 3.3, we see that $Y - T$ is a 5-fan of $M/d$. Let $(y_1, \ldots, y_5)$ be a fan ordering of $Y - T$. Since $M \setminus e$ is 3-connected, $e$ is contained in no triads of $M$, so $e = y_1$ or $e = y_5$. By reversing the fan ordering as necessary, we can assume that the first case holds. As $(y_2, y_3, y_4)$ is a triad of $M$, it follows that $(y_2, y_3, y_4, y_5)$ is not a triangle, or else $M$ has a 4-fan. Therefore $(y_3, y_4, y_5, d)$ is a circuit of $M$ that is contained in $(Y - T) \cup d$. It meets the cocircuit $(b, c, d, e)$ in a single element, violating orthogonality. \hfill\(\diamondsuit\)
As \(|Y| \geq 5\), and \(|Y - T| = 4\), it follows that \(|Y| = 5\) and \(|Y \cap T| = 1\). From Proposition 3.3, we see that \(Y\) is a 5-fan of \(M/d\). Let \((y_1, \ldots, y_5)\) be a fan ordering of \(Y\) in \(M/d\). As \(M\backslash e\) is 3-connected, \(e\) is in no triad in \(M\), and hence in \(M/d\), so \(e = y_1\) or \(e = y_5\). By reversing the fan ordering as necessary, we assume \(e = y_1\). Since \((y_2, y_3, y_4)\) is a triad of \(M\), it follows that \((y_2, y_4, y_5)\) is not a triangle, or else \(M\) has a 4-fan. Therefore \([y_2, y_4, y_5] = \{d\}\) is a circuit of \(M\). This circuit cannot meet the cocircuit \([b, c, d, e]\) in the single element \(d\). Therefore the single element in \(T \cap Y\) is \([y_3, y_4, y_5]\). Call this element \(y\). As the triangle \(T\) cannot meet the triad \([y_2, y_3, y_4]\) in a single element, it follows that \(e = y\). Since \((y_5, y_4, y_3, y_2)\) is a 4-fan of \(M/d\), and \([y_2, y_3, y_4]\) is a triad of \(M\), it follows from Lemmas 3.1 and 2.2 that \(M/d\backslash y_5\), and hence \(M\backslash y_5\) has an \(N\)-minor. This contradicts the fact that \(y_5\) is in the triangle \(T\). Thus we have completed the proof of 3.4.3.

From 3.4.3 we know that \(M/d\) is \((4, 4)\)-connected. Next we must eliminate the possibility that \(M/d\) has a 4-fan.

3.4.4. Let \((X, Y)\) be a \((4, 3)\)-violer of \(M/d\), where \(|X \cap \{a, b, c\}| \geq 2\). Then \(Y\) is not a 4-fan of \(M/d\).

**Proof.** Assume that \(Y\) is a 4-fan, \((y_1, y_2, y_3, y_4)\). Thus \([y_2, y_3, y_4]\) is a triad in \(M/d\), and hence in \(M\). It follows from 3.4.3 that \(e \in Y\). But \(e\) is not in a triad of \(M\), so \(e = y_1\). Since \(M\) has no 4-fan, it follows that \([e, y_2, y_3]\) is not a triangle of \(M\), so \([e, d, y_2, y_3]\) is a circuit.

From 3.4.1, we see that \(M/d\backslash e\) is 3-connected. We shall show that it is internally 4-connected. Once we prove this assertion, we will have shown that \((M, N)\) is not a counterexample to Theorem 1.2, since \(M/d\backslash e\) has an \(N\)-minor by 3.4.2. This contradiction will complete the proof of 3.4.4.

3.4.4.1. If \((U, V)\) is a \((4, 3)\)-violer of \(M/d\backslash e\), then \([b, c] \not\subseteq U\) and \([b, c] \not\subseteq V\).

**Proof.** If the result fails, then by symmetry we can assume that \((U, V)\) is a \((4, 3)\)-violer of \(M/d\backslash e\) such that \(b, c \in U\). Then \(d \in \text{cl}_{M\backslash e}(U)\), because of the triad \([b, c, d]\), so \((U \cup d, V)\) is a \((4, 3)\)-violer in \(M\backslash e\).

As \(|U \cup d| > 4\), and \(M\backslash e\) is \((4, 4)\)-connected, we deduce that \(|V| = 4\). Assume that \(V\) is a quad of \(M\backslash e\). Then \(V \cup e\) is a cocircuit of \(M\), which cannot meet the circuit \([e, d, y_2, y_3]\) in a single element. Hence \(y_2\) or \(y_3\) is in \(V\). However, \(V\) is a circuit in \(M\backslash e\), and \([y_2, y_3, y_4]\) is a cocircuit in \(M\backslash e\), as it is a triad of \(M\), and \(M\backslash e\) is 3-connected. Orthogonality requires that \(|V \cap [y_2, y_3, y_4]| = 2\). This means that \([y_2, y_3, y_4] \subseteq \text{cl}_{M\backslash e}(V)\), so \(V \cup [y_2, y_3, y_4]\) is a 5-element 3-separating set in \(M\backslash e\). As \(M\backslash e\) is \((4, 4)\)-connected, it follows that \(M\backslash e\) has at most nine elements, contradicting our earlier assumption that \(|E(M)| \geq 11\). Thus \(V\) is not a quad of \(M\backslash e\), and Proposition 2.3 implies that \(V\) is a 4-fan in \(M\backslash e\).

Let \(T^*\) be the triad of \(M\backslash e\) that is contained in \(V\). As \(M\) has no 4-fans, \(T^* \cup e\) is a cocircuit of \(M\). It cannot meet the circuit \([e, d, y_2, y_3]\) in the single element \(e\). Let \(y_1\) be an element in \([y_2, y_3]\) \(\cap T^*\). As \([y_2, y_3, y_4]\) is a triad in \(M/d\), and hence in \(M\), it does not contain any element that is in a triad of \(M\), or else \(M\) has a 4-fan. Therefore \(y\) is not in the triangle of \(M\backslash e\) that is contained in \(V\), so \(V \backslash y\) is a triangle of \(M\). Thus \(V \backslash y\) is contained in the ground set of every \(N\)-minor of \(M\), so Lemma 2.2 implies that \(M\backslash e/y\), and hence \(M/y\) has an \(N\)-minor. However, since \(y\) is contained in the triad \([y_2, y_3, y_4]\) of \(M\), this contradicts Lemma 3.1.

Let \((U, V)\) be a \((4, 3)\)-violer of \(M/d\backslash e\), and assume that \(a \in U\). By 3.4.4.1, we may assume that \(x \in U\) and \(y \in V\), where \([x, y] = \{b, c\}\). Then \(y \in \text{cl}_{M/d\backslash e}(U)\), as \([x, y, z]\) is a triangle, so \((U \cup y, V - y)\) is a 3-separating set of \(M/d\backslash e\). It follows from 3.4.4.1 that it is not a \((4, 3)\)-violer, so \(|V| = 4\). Since \(V\) contains an element that is in \(\text{cl}_{M/d\backslash e}(U)\), it cannot be a quad of \(M/d\backslash e\), so it is a 4-fan. Moreover, as \(y \in \text{cl}_{M/d\backslash e}(U)\), it follows that \(y\) is not in the triad of \(M/d\backslash e\) that is contained in \(V\). Therefore \(V \backslash y\) is a triad of \(M/d\backslash e\). If \(V \backslash y\) is a triad of \(M\), then \(V \backslash y\) is contained in every \(N\)-minor of \(M\). Because \([a, x] = \{a, b, c\}\) is a triangle, it follows that \(y\) is contained in every \(N\)-minor. Thus \(V\) is in every \(N\)-minor of \(M\). This implies that \(N\) has a 4-element 3-separating set, which is impossible as \(|E(N)| \geq 8\). Therefore \(V \backslash y\) is not a triad of \(M\), so \((V \backslash y) \cup e\) is a cocircuit. It cannot meet the circuit \([e, d, y_2, y_3]\) in the single element \(e\), so either \(y_2\) or \(y_3\) is in the triad \(V \backslash y\) of \(M\backslash e\).

Note that \(V \backslash y\) is not equal to \([y_2, y_3, y_4]\), as one set is a triad of \(M\) and the other is not. They are both triads of \(M\backslash e\), and they have at least one element in common. Hence they have exactly one element in common, as \(M\backslash e\) is 3-connected, and therefore does not contain a series pair. Let \(z\) be the unique element in \((V \backslash y) \cap [y_2, y_3]\). If \(T^*\) is the triangle of \(M/d\backslash e\) that is contained in \(V\), then \(z\) is not
Let $M$ be $3$-connected. Since $M$ is $3$-connected, it follows easily that $M$ has no $2$-cocircuit of $x$. Therefore there is some element $w$ such that $N$ is a minor of $M$, and $w$ is a cocircuit of the second. This leads to a contradiction, as $y$ and $z$ are in the ground set of every $N$-minor of $M$.

We conclude that there can be no $(4,3)$-violator in $M$, and therefore $M$ is internally $4$-connected and has an $N$-minor. This contradicts our assumption that $M$ is a counterexample to Theorem 1.2. Thus 3.4.4 holds.

Now we can prove the following Lemma 3.4. If $(a, b, c, d)$ is a $4$-fan of $M$, then $M$ is a $4$-fan in $M$. If $x = e$ and $M$ is a $4$-fan in $M$, then $3.4.4$ also holds. Moreover, if $(X, Y)$ is a $(4,3)$-violator of $M$, then $3.4.3$ also holds.

**Lemma 3.5.** Let $(M, N)$ be either $(\tilde{M}, \tilde{N})$ or $(\tilde{M}^*, \tilde{N}^*)$. Assume that the element $e$ is such that $M$ is a $4$-fan in $M$, and that $Q$ is a quad of $M$. If $x = e$ and $M$ is a $4$-fan in $M$, then $3.4.4$ also holds. Moreover, if $(X, Y)$ is a $(4,3)$-violator of $M$, then $3.4.3$ also holds.

**Proof.** As $M$ has no quads, we deduce that $Q$ is a cocircuit in $M$.

3.5.1. $M\backslash x\backslash e$ and $M\backslash x$ are $3$-connected.

**Proof.** Let $(U, V)$ be a $2$-separation in $M\backslash x\backslash e$. By relabeling as necessary, we assume that $|U \cap (Q - x)| \geq 2$. If $U$ contains $Q - x$, then $(U \cup x, V)$ is a $2$-separation in $M$. This is impossible, so $V$ contains a single element of $Q - x$. Then

$$\lambda_{M\backslash x\backslash e}(V - (Q - x)) \leq \lambda_{M\backslash x\backslash e}(V) \leq 1,$$

as $Q - x$ is a triad of $M\backslash x\backslash e$. Now $x \in cl_{M\backslash x\backslash e}(U \cup (Q - x))$, so

$$\lambda_{M\backslash x\backslash e}(V - (Q - x)) = \lambda_{M\backslash x\backslash e}(V - (Q - x)) \leq 1.$$

But $M\backslash e$ is $3$-connected, so this means that $|V - (Q - x)| \leq 1$. Thus $|V| = 2$, and $V$ must be a $2$-cocircuit of $M\backslash x\backslash e$. This means that $x$ is in a triad of $M\backslash e$. This triad must meet $Q$ in two elements, by orthogonality. Thus $|cl_{M\backslash x\backslash e}(Q)| \geq 5$, and $M\backslash e$ contains a $5$-element $3$-separating set. This is a contradiction as $M\backslash e$ is $4$-connected with at least ten elements. Thus $M\backslash x\backslash e$ is $3$-connected, and it follows easily that $M\backslash x$ is $3$-connected.

Let $(X, Y)$ be a $(4,3)$-violator of $M\backslash x$, and assume that $X$ contains at least two elements of $Q - x$. If $Q - x \subseteq X$, then $x \in cl_M(X)$, as $Q$ is a circuit of $M$. This implies that $(X \cup x, Y)$ is a $(4,3)$-violator of $M$, which is impossible. Therefore $Y$ contains exactly one element of $Q - x$. Let us call this element $y$.

3.5.2. $(X - e, Y - e)$ is a $3$-separation of $M\backslash x\backslash e$ and $y \in cl^{*}_{M\backslash x\backslash e}(Y - (e, y))$.

**Proof.** The fact that $(X - e, Y - e)$ is a $3$-separation of $M\backslash x\backslash e$ follows because $|X|, |Y| \geq 4$, and $M\backslash x\backslash e$ is $3$-connected. Since $Q - x$ is a triad of $M\backslash x\backslash e$, and $Q - (x, y) \subseteq X - e$, we deduce that $y \in cl^{*}_{M\backslash x\backslash e}(X - e)$. This means that $y \in cl^{*}_{M\backslash x\backslash e}(Y - (e, y))$, as otherwise $((X - e) \cup y, Y - (e, y))$ is a $2$-separation of $M\backslash x\backslash e$.

3.5.3. $\lambda_{M\backslash x\backslash e}(Y - (e, y)) \leq 2$.

**Proof.** Since $(X - e, Y - e)$ is a $3$-separation of $M\backslash x\backslash e$ and $y \in cl^{*}_{M\backslash x\backslash e}(X - e)$, it follows that

$$\lambda_{M\backslash x\backslash e}(Y - (e, y)) \leq \lambda_{M\backslash x\backslash e}(Y - e) = 2.$$
Since \((X - e) \cup \{x, y\}\) contains \(Q\), it follows that \(x \in \text{cl}_{M\setminus e}((X - e) \cup y)\). This means that
\[
\lambda_{M\setminus e}(Y - \{e, y\}) = \lambda_{M\setminus x\setminus e}(Y - \{e, y\}) \leq 2,
\]
as desired. \(\diamondsuit\)

**3.5.4.** \(|Y| \leq 6.** Proof.** As \(M\setminus e\) is \((4, 4)\)-connected, \(|Y - \{e, y\}| \leq 4\) by 3.5.3. Thus \(|Y| \leq 6.** \(\diamondsuit\)

**3.5.5.** \(|Y| \neq 6.** Proof.** Assume that \(|Y| = 6. If e \not\in Y\), then 3.5.3 implies that \(((X - e) \cup \{x, y\}, Y - y)\) is a 3-separation of \(M\setminus e\). As \(|Y - y| = 5 \text{ and } |(X - e) \cup \{x, y\}| = |X| + 1 \geq 5\), this contradicts the fact that \(M\setminus e\) is \((4, 4)\)-connected. Therefore \(e \in Y\), and \(Y - \{e, y\}\) is a 4-element 3-separating set in \(M\setminus e\). Thus \(Y - \{e, y\}\) is either a quad or a 4-fan of \(M\setminus e\). The next two assertions show that both these cases are impossible, thereby finishing the proof of 3.5.5.

**3.5.5.1.** \(Y - \{e, y\}\) is not a quad of \(M\setminus e\).** Proof.** Assume that \(Y - \{e, y\}\) is a quad of \(M\setminus e\). Thus it is a circuit of \(M\), and \(Y - y\) is a cocircuit of \(M\).

If \(Y - y\) is not a cocircuit of \(M\setminus x\), then \(x\) is in the closure of \(Y - y\) in \(M\). This leads to a contradiction to orthogonality with the circuit \(Q\) of \(M\). Thus \(Y - y\) is a cocircuit of \(M\setminus x\), so \(Y - \{e, y\}\) is a quad in both \(M\setminus e\) and \(M\setminus x\). As \(Y - y\) is a cocircuit of \(M\) and \(|Y| = 6\), we see that \(r_{M\setminus x}(Y) \geq 4\), so
\[
|Y| = 4 - 2 + 6 = 4.
\]

By 3.5.2, there is a cocircuit of \(M\setminus x\) contained in \(Y - e\) that contains \(y\). The symmetric difference of this cocircuit with \(Y - \{e, y, z\}\) is a disjoint union of cocircuits. As \(M\setminus x\) contains no cocircuit with fewer than three elements, it follows that there are two triads, \(T^+_1\) and \(T^+_2\), of \(M\setminus x\), such that \(T^+_1 \cap T^+_2 = \{y\}\), and \(T^+_1 \cup T^+_2 = Y - e\). If both \(T^+_1 \cup e\) and \(T^+_2 \cup e\) are cocircuits of \(M\setminus x\), then we can take the symmetric difference of these cocircuits, and deduce that \(Y - \{e, y\}\) is a cocircuit of \(M\setminus x\) that is properly contained in the cocircuit \(Y - y\). Since this is impossible, we deduce that we can relabel as necessary, and assume that \(T^+_1\) is a triad of \(M\setminus x\).

Let \(z\) be an arbitrary element of \(T^+_2 - y\). Then \(Y - \{e, y, z\}\) is independent in \(M\setminus x\). Orthogonality with the cocircuit \((Q - x) \cup e\) means that \(y\) cannot be in the closure of \(Y - \{e, y, z\}\) in \(M\setminus x\). Thus \(Y - \{e, z\}\) is independent. Since \(r_{M\setminus x}(Y) \leq 4\), it follows that \(Y - \{e, z\}\) spans \(Y\) in \(M\setminus x\). Let \(C\) be a circuit of \(M\setminus x\) such that \(\{e\} \subseteq C \subseteq Y - y\). If \(y \not\in C\), then \(C\) and the cocircuit \((Q - x) \cup e\) meet in \(e\). Therefore \(y \in C\).

This implies that \(C\) contains exactly one element of \(T^+_1\). Now \(C\) cannot be a triangle, as \(M\setminus e\) has an \(N\)-minor. Therefore \(C\) also contains the single element in \(T^+_1 - \{y, z\}\). But now the circuit \(C\) meets the cocircuit \(Y - y\) of \(M\setminus x\) in three elements: \(e\), and a single element from each of \(T^+_1 - y\) and \(T^+_2 - y\). This contradiction proves 3.5.5.1. \(\diamondsuit\)

**3.5.5.2.** \(Y - \{e, y\}\) is not a 4-fan of \(M\).** Proof.** Assume that \((y_1, y_2, y_3, y_4)\) is a fan ordering of \(Y - \{e, y\}\) in \(M\). Then \(y_2, y_3, y_4, e\) is a cocircuit of \(M\).

As \(M\setminus x\) has no cocircuits with fewer than three elements, \(y_2, y_3, y_4\) is a triad of \(M\setminus x\). Thus \((y_1, y_2, y_3, y_4)\) is a 4-fan of \(M\setminus x\). By 3.5.2, there is a cocircuit \(C^*\) of \(M\setminus x\) such that \(|y| \subseteq C^*\). This cocircuit must meet the triangle \((y_1, y_2, y_3)\) in exactly two elements. If \(y_2, y_3 \subseteq C^*\), then the symmetric difference of \(y_2, y_3, y_4\) and \(C^*\) is \(y, y_4\), as \(y_2, y_3, y_4\) is not properly contained in \(C^*\). Since \(M\setminus x\) has no 2-cocircuit, we deduce that \(y_1 \in C^*. Either C^* or its symmetric difference with \(y_2, y_3, y_4\), is a triad of \(M\setminus x\) that contains \(y, y_1\), and a single element from \(y_2, y_3\). We can swap the labels on \(y_2\) and \(y_3\) if necessary, so we can assume that \(y_1, y_2, y_3\) is a triad. Thus \((y_1, y_1, y_2, y_3, y_4)\) is a 5-cofan of \(M\). The dual of Lemma 2.2 implies that \(M\setminus x\) and \(M\setminus x\) have \(N\)-minors.

Recall that \((y_2, y_3, y_4, e)\) is a cocircuit of \(M\). It is also a cocircuit of \(M\setminus x\), as otherwise \(x \in \text{cl}_{M\setminus e}((y_2, y_3, y_4, e))\), and this contradicts orthogonality with the circuit \(Q\). Therefore \((y_2, y_3, y_4)\) is coindependent in \(M\setminus x\).
Assume \( y_1 \in \text{cl}_{M \setminus x}^e(\{y_2, y_3, y_4\}) \), so \( y_1 \) is in \( \text{cl}_{M \setminus x}^e(\{y_2, y_3, y_4\}) \). As it is also in \( \text{cl}_{M \setminus x}^e(\{y_2, y_3, y_4\}) \), it follows that \( \lambda_{M \setminus x}^e(\{y_1, y_2, y_3, y_4\}) \leq 1 \). This leads to a contradiction to the fact that \( M \setminus x \) is 3-connected. Therefore \( y_1 \notin \text{cl}_{M \setminus x}^e(\{y_2, y_3, y_4\}) \). Thus \( \{y_1, y_2, y_3, y_4\} \) is a co-independent set in \( M \setminus x \), so \( r_{M \setminus x}^e(Y) \geq 4 \). Now we see that

\[
r_{M \setminus x}(Y) = \lambda_{M \setminus x}^e(Y) - r_{M \setminus x}^e(Y) + |Y| \leq 2 - 4 + 6 = 4.
\]

If \( \{y, y_1, y_3, y_4\} \) is dependent in \( M \setminus x \), then it is a circuit, by orthogonality with the triads \( \{y, y_1, y_2\} \) and \( \{y_2, y_3, y_4\} \), and the fact that \( M \setminus x \) has no 2-circuits. In this case, \( \{y, y_1, y_3, y_4\} \) is a circuit of \( M \), and \( Q \cup e \) is a cocircuit that meets it in the single element \( y \). Therefore \( \{y, y_1, y_3, y_4\} \) is independent in \( M \setminus x \), and hence in \( M \setminus x \). Therefore \( \{y, y_1, y_3, y_4\} \) spans \( Y \) in \( M \setminus x \). Let \( C \) be a circuit of \( M \setminus x \) such that \( \{y\} \subseteq C \subseteq \{e, y, y_1, y_3, y_4\} \).

First observe that \( y \in C \), as otherwise \( C \) and \( Q \cup e \) are a circuit and a cocircuit of \( M \) that meet in \( \{e\} \). We have noted that \( \{y, y_1, y_3, y_4\} \) is a cocircuit of \( M \setminus x \). Therefore orthogonality implies that \( C \) contains exactly one element of \( \{y, y_1, y_3, y_4\} \). Since \( M \setminus x \) has an \( N \)-minor, it follows that \( e \) is not in any triples of \( M \). Therefore \( y_1 \) must be in \( C \). Hence \( C \) is either \( \{e, y, y_1, y_3, y_4\} \) or \( \{e, y, y_1, y_4\} \). In the first case, we take the symmetric difference of \( C \) with the triangle \( \{y_1, y_2, y_3\} \), and discover that \( e \) is in the triangle \( \{e, y, y_2\} \), a contradiction. Therefore \( C = \{e, y, y_1, y_4\} \).

We noted earlier that \( M \setminus x \) is an \( N \)-minor. The symmetric difference of \( C \) with the triangle \( \{y_1, y_2, y_3\} \) is \( \{e, y, y_2, y_3, y_4\} \), which must therefore be a circuit of \( M \). Thus \( \{e, y, y_2, y_3, y_4\} \) is a circuit of \( M \). Thus \( M \) contains a quad that contains \( e \), and \( M \) contains a quad of \( e \). Since \( M \setminus x \) is an \( N \)-minor, it follows from \emph{Lemma 2.4} that if \( z \) is an arbitrary member of the quad \( \{e, y_2, y_3, y_4\} \), then \( M \setminus x \) is not a quad. Thus \( M \setminus x \) is not a quad, and \( M \setminus x \) is a circuit of \( M \). Therefore \( y \in C \) is a quad of \( M \). Thus \( M \) contains a quad of \( y \). This is a contradiction, as \( y \) is contained in the triangle \( \{y_1, y_2, y_3\} \) of \( M \). This completes the proof of \emph{3.5.5.2}. \( \diamond \)

The proof of 3.5.5 now follows immediately from 3.5.5.1 and 3.5.5.2. \( \diamond \)

\subsection*{3.5.6. \( |Y| \neq 5 \)}

\textbf{Proof.} Assume that \( |Y| = 5 \). If \( e \in X \), then \( y \in \text{cl}_{M \setminus x}^e(X) \), since \( (Q - x) \cup e \) is a cocircuit of \( M \setminus x \) that is contained in \( X \cup y \). This means that \( (X \cup y, Y - y) \) is a 3-separation of \( M \setminus x \). As \( Q \) is a circuit of \( M \), and \( Q - x \subseteq X \cup y \), it follows that \( x \in \text{cl}_M(X \cup y) \). Therefore \( (X \cup \{x, y\}, Y - y) \) is a 3-separation of \( M \), and as \( |X \cup \{x, y\}|, |Y - y| \geq 4 \), we have violated the internal 4-connectivity of \( M \). Therefore \( e \in Y \).

\emph{Proposition 3.3} implies that \( Y \) is a 5-cofan of \( M \). Let \( (y_1, y_2, y_3, y_4, y_5) \) be a fan ordering of \( Y \) in \( M \setminus x \). Since \( e \) is contained in no triangle of \( M \) by \emph{Lemma 3.1}, we can assume that \( e = y_1 \). The element \( y_1 \) cannot be contained in \( \{y_2, y_3, y_4\} \), or else this triangle meets the cocircuit \( Q \cup e \) of \( M \) in the single element \( y \). Now \( \{y_1, y_2, y_3\} \) is a triad of \( M \), or else \( \{x, y_1, y_2, y_3\} \) is a cocircuit, and it meets the circuit \( Q \) in the single element \( x \). However, \( \{y_1, y_2, y_3\} \) cannot be a triad, as \( e = y_1 \) and \( M \setminus x \) is 3-connected. \( \diamond \)

We can now complete the proof of \emph{Lemma 3.5}. Recall that \( (X, Y) \) is a \((4, 3)\)-violator of \( M \setminus x \), where \( x \) is contained in the quad \( Q \) of \( M \setminus x \), and \( |X \cap (Q - x)| \geq 2 \). By combining 3.5.4–3.5.6, we deduce that \( |Y| = 4 \). Therefore \( M \setminus x \) is \((4, 4)\)-connected with an \( N \)-minor, and \( Y \) is either a quad or a 4-fan of \( M \setminus x \).

Assume that \( e \) is not in \( Y \). If \( Y \) is a quad of \( M \setminus x \), then it is a circuit of \( M \) that meets the cocircuit \( Q \cup e \) in the single element \( y \). Therefore \( Y \) must be a 4-fan of \( M \setminus x \). Certainly \( y \) is not contained in the triangle of \( Y \), by orthogonality with \( Q \cup e \). Therefore \( Y = \{y_1, y_2, y_3, y\} \) is a 4-fan. We can apply \emph{Lemma 3.4} to \( M \setminus x \), and deduce that \( M \setminus y \) is \((4, 4)\)-connected with an \( N \)-minor. Since \( M \setminus y \) is not internally 4-connected, \emph{Lemma 3.4} also implies that \( M \setminus y \) contains a quad and that this quad contains \( x \). However, \( Q - y \) is a triangle in \( M \setminus y \), so \( M \setminus y \) contains a quad, and a triangle that contains an element of this quad. It follows that the triangle and the quad meet in two elements, and their union is a 5-element 3-separating set of \( M \setminus y \). As \( M \setminus y \) is \((4, 4)\)-connected, this leads to a contradiction, so now we know that \( e \) is in \( Y \).

If \( Y \) is a 4-fan of \( M \setminus x \), then \( e \) is not contained in the triangle of this fan, as \( M \setminus e \) has an \( N \)-minor. Therefore \( y \) is contained in a triangle of \( M \setminus x \) that is contained in \( Y - e \). This triangle violates orthogonality with the cocircuit \( Q \cup e \) in \( M \). Hence \( Y \) is a quad of \( M \setminus x \), and \emph{Lemma 3.5} holds. \( \Box \)
Lemma 3.6. Let \((M, N)\) be either \((\bar{M}, \bar{N})\) or \((\bar{M}^*, \bar{N}^*)\). Assume that the element \(e\) is such that \(M \setminus e\) is (4, 4)-connected with an \(N\)-minor, and that \(Q\) is a quad of \(M \setminus e\). Then \(N\) is not a minor of \(M \setminus e\) \(x\), for any element \(x \in Q\).

**Proof.** Assume that \(N\) is a minor of \(M \setminus e\) \(x\) for some element \(x\) of \(Q\). By Lemma 2.4, deleting any element of \(Q\) from \(M \setminus e\) produces a matroid with an \(N\)-minor. Lemma 3.5 implies that \(M \setminus x\) is (4, 4)-connected and contains a quad, \(Q_x\), such that \(e \in Q_x\), and \(|Q \cap Q_x| = 1\).

3.6.1. Assume that \(x_1\) and \(x_2\) are elements of \(Q\), and that \(M \setminus x_1\) contains a quad, \(Q_1\), such that \(e \in Q_1\), and \(Q \cap Q_1 = \{x_2\}\). Then \(M \setminus x_2\) contains a quad, \(Q_2\), such that \(Q \cap Q_2 = \{x_1\}\) and \(Q_1 \cap Q_2 = \{e\}\).

**Proof.** Since \(M \setminus e\) \(x_2\) has an \(N\)-minor, we can apply Lemma 3.5, and deduce that \(M \setminus x_2\) contains a quad \(Q_2\) such that \(e \in Q_2\) and \(|Q \cap Q_2| = 1\). On the other hand, \(M \setminus x_1\) is (4, 4)-connected, and contains a quad, \(Q_1\). Moreover, \(M \setminus x_1\) \(x_2\) is isomorphic to \(M \setminus x_1\) \(e\), by Lemma 2.4 and the fact that \(e\) and \(x_2\) are both in \(Q_1\), so \(M \setminus x_1\) \(x_2\) has an \(N\)-minor. Hence we can apply Lemma 3.5 again, and deduce that \(M \setminus x_2\) contains a quad \(Q_2\) such that \(x_1 \in Q_2\) and \(|Q_1 \cap Q_2| = 1\).

We will show that \(Q_2 = Q_1\). Assume this is not the case. As \(Q_2\) and \(Q_5\) are both quads of \(M \setminus x_2\), orthogonality demands that they are disjoint, or they meet in two elements. In the latter case, \(Q_2 \triangle Q_5\) is a circuit of \(M\), and \((Q_2 \cup x_2) \setminus \{Q_2, Q_5\} = Q_2 \cup x_2\) must be a cocircuit of \(M\), so \(M\) has a quad. As this is impossible, we deduce that \(Q_2\) and \(Q_5\) are disjoint. Therefore \(e \notin Q_5\), as \(e\) is in \(Q_2\). This means that \(|Q \cap Q_5| = 2\), as otherwise the circuit \(Q_2\) and the cocircuit \(Q \cup e\) meet in \(\{x_1\}\). But \(Q_5 \cup x_2\) is a cocircuit, and \(Q\) is a circuit, and they meet in three elements: \(x_2\) and the two elements of \(Q \cap Q_5\). This contradiction shows that \(Q_2 = Q_5\), so \(x_1 \in Q_2\). Furthermore, \(Q \cap Q_2 = \{x_1\}\) and \(Q_1 \cap Q_2 = \{e\}\).

Now we return to the proof of Lemma 3.6. Let \(y\) be the single element in \(Q \cap Q_2\). By 3.6.1, we see that \(M \setminus y\) has a quad \(Q_0\) such that \(Q \cap Q_0 = \{x\}\) and \(Q_1 \cap Q_0 = \{e\}\).

Let \((z, w) = Q \setminus \{x, y\}\). We can again apply Lemma 3.5 and deduce the existence of \(Q_2\), a quad of \(M \setminus z\) that contains \(e\) and \(\{x\}\) as a single element of \(Q\). Note that \(Q_2 \neq Q_0\), or else we can take the symmetric difference of \(Q_2 \cup y\) and \(Q_2 \cup z\) and deduce that \(\{y, z\}\) is a series pair of \(M\). Assume that \(y \in Q_0\). As the cocircuit \(Q_2 \cup z\) and the circuit \(Q_0\) both contain \(e\), and \(x\) is not the single element in \(Q \cap Q_2\), it follows that one of the elements in \(Q_2 \setminus \{x, e\}\) is in \(Q_0\). Then the cocircuit \(Q_0 \cup y\) and the circuit \(Q_2\) meet in three elements: \(e, y, y\) and an element in \(Q_0 \setminus \{x, e\}\). This contradiction shows that the single element in \(Q \cap Q_0\) is not \(y\) nor \(z\). Therefore it is \(x\) or \(w\).

First we assume that \(Q \cap Q_1 = \{w\}\). Then 3.6.1 implies that \(M \setminus w\) has a quad \(Q_w\) such that \(Q \cap Q_w = \{z\}\) and \(Q_x \cap Q_w = \{e\}\). The cocircuit \(Q_0 \cup w\) and the circuit \(Q_0\) both contain the element \(e\). Moreover, \(y \notin Q_w\), so there is an element \(\alpha\) in \((Q_x \setminus \{e, y\}) \cap Q_w\). Let \(\beta\) be the unique element in \(Q_x \setminus \{e, y, \alpha\}\). Similarly, the cocircuit \(Q_0 \cup w\) and the circuit \(Q_0\) have \(e\) in common, but \(x \notin Q_w\), so there is an element \(\gamma\) in \((Q_x \setminus \{e, x\}) \cap Q_w\). Thus \(Q_w = \{e, z, \alpha, \gamma\}\). Consider the set \(X = \{x, y, z, e, \alpha, \beta, \gamma\}\). It spans: \(w\) because of the circuit \(Q_y\) \(e\) because of the circuit \(Q_0\); \(\gamma\) because it spans the circuit \(Q_w\); and \(Q_y\) because it spans \(x, e, \gamma\). This shows that \(Q \cup Q_w \cup Q_y\) is a 9-element set satisfying \(r(Q \cup Q_w \cup Q_y) \leq 5\). Moreover, \(X\) spans \(e\) because of the cocircuit \(Q_0 \cup x\). It spans \(w\) because it spans \(e\), and \(Q \cup e\) is a cocircuit. Now it spans: \(\gamma\) as \(Q_w \cup w\) is a cocircuit; and \(Q_y\) as \(Q_y \cup y\) is a cocircuit. Thus \(X\) spans and \(Q \cup Q_w \cup Q_y\), so

\[
\lambda_M(Q \cup Q_x \cup Q_y) \leq 5 + 5 - 9 = 1.
\]

As \(M\) is 3-connected, this means that there is at most 1 element not in \(Q \cup Q_x \cup Q_y\). This is a contradiction as \(|E(M)| \geq 11\). Hence we conclude that \(Q \cap Q_1 = \{x\}\).

Now the cocircuit \(Q_0 \cup z\) and the circuit \(Q_0\) both contain \(e\), so \(|Q_1 \cap (Q_x \setminus \{e, y\})| = 1\). But this means that the circuit \(Q_x\) and the cocircuit \(Q_0 \cup x\) have three elements in common: \(e, x\), and the element in \(Q_x \cap (Q_x \setminus \{e, y\})\). This contradiction completes the proof of Lemma 3.6. \(\square\)

Lemma 3.7. Let \((M, N)\) be either \((\bar{M}, \bar{N})\) or \((\bar{M}^*, \bar{N}^*)\). Assume that the element \(e\) is such that \(M \setminus e\) is (4, 4)-connected with an \(N\)-minor. Then \(M \setminus e\) has no 4-fans.
Proof. Assume that \((a, b, c, d)\) is a 4-fan of \(M\setminus e\). It follows from Lemma 3.4 that \(M/d\) is \((4, 4)\)-connected with an \(N\)-minor. Since it is not internally 4-connected, it contains a quad \(Q\) such that \(Q \cap \{a, b, c, e\} = \{e\}\). We will show that \(M\setminus e/d\) is internally 4-connected, and this will contradict the fact that \(M\) and \(N\) provide a counterexample to Theorem 1.2, thereby proving Lemma 3.7. Note that Lemma 3.4 states that \(M\setminus e/d\) is 3-connected with an \(N\)-minor.

3.7.1. Let \((X', Y')\) be a \((4, 3)\)-violator of \(M\setminus e/d\). Then neither \(X'\) nor \(Y'\) contains \(Q - e\).

Proof. Assume that \(Q - e \subseteq X'\). As \(Q\) is a circuit of \(M/d\), this means that \(e \in \text{cl}_{M/d}(X')\). Thus \((X' \cup e, Y')\) is a \((4, 3)\)-violator of \(M/d\). Lemma 3.4 says that one side of this \((4, 3)\)-violator is a quad that contains \(e\). But this is impossible as \(|X' \cup e| > 4\), and \(e \not\in Y'\). \(\Box\)

Let \((X, Y)\) be a \((4, 3)\)-violator of \(M\setminus e/d\), and assume that \(|X \cap (Q - e)| \geq 2\). By 3.7.1 we see that there is a single element in \(Y \cap (Q - e)\). Let this element be \(y\). Since \(Q - e\) is a triad in \(M\setminus e/d\), it follows that \(y \in \text{cl}_{M\setminus e/d}(X)\). Therefore \((X \cup y, Y - y)\) is a 3-separation in \(M\setminus e/d\), but 3.7.1 implies that it is not a \((4, 3)\)-violator. Hence \(|Y| = 4\). Orthogonality with the triad \(Q - e\) implies that \(Y\) is not a quad of \(M\setminus e/d\). Thus \(Y = \{y_1, y_2, y_3, y_4\}\), where \(\{y_1, y_2, y_3, y_4\}\) is a 4-fan in \(M\setminus e/d\). Orthogonality also implies that \(y = y_4\).

Assume that \(M\setminus e/d\setminus y\) has an \(N\)-minor. Then \(M^*/d\setminus y\) has an \(N^*\)-minor. As \(M^*/d\) is \((4, 4)\)-connected, and \(y\) is in the quad \(Q\) of this matroid, we now have a contradiction to Lemma 3.6. Therefore \(M\setminus e/d\setminus y\) has no \(N\)-minor. Lemma 2.2 implies that \(M\setminus e/d\setminus y_1\), and hence \(M\setminus y_1\), has an \(N\)-minor. From this, we deduce that \(\{y_1, y_2, y_3\}\) is not a triangle of \(M\), so \([d, y_1, y_2, y_3]\) is a circuit. Since \([b, c, d, e]\) is a cocircuit, this implies that exactly one of \(b\) or \(c\) is in \(\{y_1, y_2, y_3\}\). Let \(\alpha\) be the single element in \([b, c]\)\(\cap\{y_1, y_2, y_3\}\). Then \(\alpha \neq y_1\), as \(M\setminus e/d\setminus y_1\) has an \(N\)-minor, and \(y_1\) and \(y_2\) are contained in a triangle of \(M\).

Both \([y_2, y_3, y]\) and \([a, b, c]\) contain the element \(\alpha\). As \([y_2, y_3, y]\) is a triad in \(M\setminus e/d\), and hence in \(M\setminus e\), and \([a, b, c]\) is a triangle of \(M\setminus e\), it follows that \([y_2, y_3] = \{\alpha, a\}\), since \(y \in Q\) and \(Q \cap \{a, b, c\} = \emptyset\). Hence either \((y, a, b, c, d)\) or \((y, a, c, b, d)\) is a 5-cofan of \(M\setminus e\), depending on whether \(\alpha = b\) or \(\alpha = c\). In either case, from Proposition 2.1, and the fact that \(M\setminus e\) is \((4, 4)\)-connected, we deduce that \(|E(M\setminus e)| \leq 9\), a contradiction. Thus \(M\setminus e/d\) has no \((4, 3)\)-violator, and is therefore internally 4-connected. This contradiction completes the proof of Lemma 3.7. □

By Lemma 3.2, we know we can choose \((M, N)\) to be \((\bar{M}, \bar{N})\) or \((\bar{M}^*, \bar{N}^*)\) in such a way that \(M\setminus e\) is \((4, 4)\)-connected with an \(N\)-minor for some element \(e \in E(M)\). From Lemma 3.7, we deduce that \(M\setminus e\) has no 4-fans, and therefore contains at least one quad. Moreover, deleting any element from this quad destroys all \(N\)-minors, by Lemma 3.6. Therefore we next consider contracting an element from a quad in \(M\setminus e\).

Lemma 3.8. Let \((M, N)\) be either \((\bar{M}, \bar{N})\) or \((\bar{M}^*, \bar{N}^*)\). Assume that the element \(e\) is such that \(M\setminus e\) is \((4, 4)\)-connected with an \(N\)-minor, and that \(Q\) is a quad of \(M\setminus e\). If \(x \in Q\), then \(M\setminus e/x\) is 3-connected, and \(M/x\) is \((4, 4)\)-connected with an \(N\)-minor. In particular, if \((X, Y)\) is a \((4, 3)\)-violator of \(M/x\) such that \(|X \cap (Q - x)| \geq 2\), then \(Y\) is a quad of \(M/x\), and \(Y \cap (Q \cup e) = \{e\}\).

Proof. To see that \(M/x\) has an \(N\)-minor, we note that \(Q\) is not contained in the ground set of any \(N\)-minor of \(M\setminus e\). By Lemma 3.6, we cannot delete any element of \(Q\) in \(M\setminus e\) and preserve an \(N\)-minor. Therefore we must contract an element of \(Q\). By the dual of Lemma 2.4, we can contract any element. Thus \(M\setminus e/x\), and hence \(M/x\), has an \(N\)-minor.

3.8.1. \(M\setminus e/x\) and \(M/x\) are 3-connected.

Proof. Assume that \((U, V)\) is a 2-separation of \(M\setminus e/x\) such that \(|U \cap (Q - x)| \geq 2\). If \(Q - x \subseteq U\), then \((U \cup x, V)\) is a 2-separation in \(M\setminus e\), as \(Q\) is a cocircuit in this matroid. Since \(M\setminus e\) is 3-connected, this is not true, so \(V\) contains a single element, \(y\), of \(Q - x\). Then \(y \in \text{cl}_{M\setminus e/x}(U)\). However \((U \cup y, V - y)\) is not a 2-separation of \(M\setminus e/x\), or else \((U \cup \{x, y\}, V - y)\) is a 2-separation of \(M\setminus e/x\). Thus \(V\) is either a 2-circuit or a 2-cocircuit in \(M\setminus e/x\). Orthogonality with \(Q - x\) tells us that the latter case is impossible. Therefore \(x\) is in a triangle in \(M\setminus e\) that contains two elements of \(Q\). The union of \(Q\) with this triangle is a 5-element 3-separating set in \(M\setminus e\), contradicting the fact that \(M\setminus e\) is \((4, 4)\)-connected. Therefore
$M \setminus e/ x$ is 3-connected. If $M/ x$ is not, then $e$ must be in a triangle with $x$ in $M$, and this is impossible by Lemma 3.1. 

We will prove that $M/ x$ is (4, 4)-connected. Assume otherwise, and let $(X, Y)$ be a (4, 4)-violator of $M/ x$, so that $|X|, |Y| \geq 5$. We can assume that $|X \cap (Q - x)| \geq 2$.

3.8.2. $e \in Y$.

**Proof.** Assume that $e \in X$. If $Q - x \subseteq X$, then $x \in cl^*_M (X)$, as $Q \cup e$ is a coocircuit of $M$. Therefore $(X \cup x, Y)$ is a (4, 4)-violator of $M$, which is impossible. Therefore $Y \cap (Q - x)$ contains a single element, $y$. Now $y \in cl^*_M (X)$, so $(X \cup y, Y - y)$ is a 3-separation in $M/ x$. As $x \in cl^*_M (X \cup y)$, and $|Y - y| \geq 4$, it follows that $(X \cup \{ x, y \}, Y - y)$ is a (4, 3)-violator of $M$, a contradiction. 

3.8.3. $\lambda_{M \setminus e} (Y - (Q \cup e)) \leq 2$.

**Proof.** As $\lambda_{M/ x} (Y) = 2$, and $Q - x \subseteq cl^*_M (X)$, it follows that $\lambda_{M/ x} (Y - Q) \leq 2$. Therefore $\lambda_{M \setminus e/ x} (Y - (Q \cup e)) \leq 2$. Now $x$ is in the coclosure of the complement of $Y - (Q \cup e)$ in $M \setminus e$, as $Q$ is a cocircuit, so

$$\lambda_{M \setminus e} (Y - (Q \cup e)) = \lambda_{M \setminus e/ x} (Y - (Q \cup e)) \leq 2,$$

as desired. 

3.8.4. $|Y| \leq 6$.

**Proof.** Since $M \setminus e$ is (4, 4)-connected, 3.8.3 implies that $|Y - (Q \cup e)| \leq 4$. The result follows. 

3.8.5. $|Y| \neq 6$.

**Proof.** Assume that $|Y| = 6$. If $Q - x \subseteq X$, then 3.8.3 implies that $M \setminus e$ has a 5-element 3-separating set, which leads to a contradiction. Therefore $Y \cap (Q - x)$ contains a single element, $y$. Since $Q - x$ is a triangle in $M/ x$, it follows that $y \in cl^*_M (X), \text{so} (X \cup y, Y - y)$ is a 3-separation of $M/ x$. Proposition 3.3 implies that $Y - y$ is a 5-fan of $M/ x$. Let $(y_1, \ldots, y_5)$ be a fan ordering of $Y - y$. As $e$ is contained in no triads of $M$, we can assume that $e = y_1$. As $\{ y_2, y_3, y_4 \}$ is a triad of $M/ x$, and hence of $M$, it cannot be the case that $\{ y_3, y_4, y_5 \}$ is a triangle, or else $M$ has a 4-fan. Therefore $\{ x, y_3, y_4, y_5 \}$ is a circuit of $M$ that meets the cocircuit $Q \cup e$ in the single element $x$. This contradiction completes the proof of 3.8.5. 

3.8.6. $|Y| \neq 5$.

**Proof.** Assume that $|Y| = 5$. First suppose that $Q - x \subseteq X$. Then $(X \cup x, Y - e)$ is a 3-separation of $M \setminus e$, by 3.8.3. Thus $Y - e$ is a quad of $M \setminus e$, by Proposition 2.3 and Lemma 3.7. But Proposition 3.3 implies that $Y$ is a 5-fan of $M/ x$. Thus $Y$ contains a triad of $M/ x$, and hence of $M$. This means that $Y - e$ contains a cocircuit of size at most three in $M \setminus e$, contradicting the fact that it is a quad. Thus $Y \cap (Q - x)$ contains a single element, $y$.

By again using Proposition 3.3, we see that $Y$ is a 5-fan of $M/ x$. Let $(y_1, \ldots, y_5)$ be a fan ordering. Orthogonality with $Q - e$ means that $y_1$ is not contained in a triad of $M/ x$ that is contained in $Y$. Therefore we can assume that $Y = y_1$. As $e$ is in no triad of $M$, it follows that $e \notin \{ y_2, y_3, y_4 \}$. As $M/ x \setminus e$ is 3-connected, by 3.8.1, we deduce that $(y_1, y_2, y_3, y_4)$ is a 4-fan of $M/ x \setminus e$. As $\{ y_2, y_3, y_4 \}$ is a triad of $M/ x$, and hence of $M$, Lemma 3.1 implies $M/ y_4$ has no $N$-minor, so neither does $M/ x \setminus e/ y_4$. Lemma 2.2 now implies that $M/ x \setminus e/ y_1$, and hence $M \setminus e/ y_1$, has an $N$-minor. As $y_1 = y$ is contained in the quad of $M \setminus e$, this means we have a contradiction to Lemma 3.6. 

We assumed $(X, Y)$ was a (4, 4)-violator of $M/ x$, so we now obtain a contradiction by combining 3.8.4–3.8.6. Therefore $M/ x$ is (4, 4)-connected. Now assume $(X, Y)$ is a (4, 3)-violator of $M/ x$. We can assume that $|X \cap (Q - x)| \geq 2$. Because $M^* \setminus x$ is (4, 4)-connected with an $N^*$-minor, it follows from Lemma 3.7 that either $X$ or $Y$ is a quad of $M/ x$. If $X$ is a quad of $M/ x$, then it does not contain the triangle $Q - x$. Therefore $X \cup (Q - x)$ is a 5-element 3-separating set of $M/ x$. This leads to a contradiction, as $|E(M/ x)| \geq 10$ and $M/ x$ is (4, 4)-connected. Therefore $Y$ is a quad of $M/ x$. Orthogonality shows that $Y$ is disjoint from the triangle $Q - x$. If $Y$ does not contain $e$, then $x \in cl^*_M (X)$, and $Y$ is a quad of $M$, a contradiction. Therefore $Y \cap (Q \cup e) = \{ e \}$, and the proof of Lemma 3.8 is complete. 

$\square$
Finally, we are in a position to prove Theorem 1.2. By Lemma 3.2, we can assume that \((M, N)\) is either \((\bar{M}, \bar{N})\) or \((M^*, \bar{N}^*)\), and \(M \setminus e\) is \((4, 4)\)-connected with an \(N\)-minor, for some element \(e\). Lemma 3.7 implies that \(M \setminus e\) has no 4-fans. As it is not internally 4-connected, it contains a quad \(Q\). Deleting any element of \(Q\) destroys all \(N\)-minors, by Lemma 3.6, so \(M \setminus e/x\) has an \(N\)-minor, for some element \(x \in Q\). Lemma 3.8 says that \(M \setminus e/x\) is 3-connected, and \(M/x\) is \((4, 4)\)-connected. As \(M/x\) is not internally 4-connected, it has a quad, \(Q_x\), such that \((Q \cup e) \cap Q_x = \{e\}\). We will show that \(M \setminus e/x\) is internally 4-connected, and this will provide a contradiction that completes the proof of Theorem 1.2.

Assume that \((X, Y)\) is a \((4, 3)\)-violator of \(M \setminus e/x\), where \(|X \cap (Q_x - e)| \geq 2\). If \(Q_x - e \subseteq X\), then \((X \cup e, Y)\) is a \((4, 3)\)-violator of \(M/x\), as \(Q_x\) is a circuit in \(M/x\). Then Lemma 3.8 implies that either \(X \cup e\) or \(Y\) is a quad that contains \(e\). This is impossible, as \(|X \cup e| \geq 5\). Therefore \(Y \cap (Q_x - e)\) contains a single element, \(y\). In \(M \setminus e/x\), the set \(Q_x - e\) is a triad, so \(y\) is in the coclosure of \(X\). Therefore \((X \cup y, Y - y)\) is a \((4, 3)\)-violator of \(M/x\), and this leads to the same contradiction as before, since either \(X \cup \{y, e\}\) or \(Y - y\) must be a quad of \(M/x\) that contains \(e\). Therefore \(|Y| = 4\). Orthogonality with \(Q_x - e\) shows that \(Y\) is not a quad of \(M \setminus e/x\). Thus we assume that \(\{y_1, y_2, y_3, y_4\}\) is a 4-fan and a fan ordering of \(Y\) in \(M \setminus e/x\). Then \(y = y_4\), or else we violate orthogonality between \(Q_x - e\) and \(\{y_1, y_2, y_3\}\).

Since \(y\) is in a quad of \(M/x\), Lemma 3.6 implies that \(M/x/y\), and hence \(M \setminus e/x/y\) has no \(N\)-minor. Therefore Lemma 2.2 implies that \(M \setminus e/x/y_1\) has an \(N\)-minor. As \(M \setminus y_1\) has an \(N\)-minor, it follows that \(\{y_1, y_2, y_3\}\) is not a triangle of \(M\). Therefore \(\{x, y_1, y_2, y_3\}\) is a circuit. As \(Q \cup e\) is a cocircuit, there is a single element, which we call \(z\), in \((Q - x) \cap \{y_1, y_2, y_3\}\).

Note that \(\{y_2, y_3, y\}\) is not a triad of \(M\), by orthogonality with the circuit \(Q_x \cup x\). Therefore \(\{y_2, y_3, y, e\}\) is a cocircuit. This means that \(z\) is not in \(\{y_2, y_3\}\), for otherwise \(\{y_2, y_3, y, e\}\) meets the circuit \(Q\) in the single element \(z\). Therefore \(z = y_1\), and \(M \setminus e/x \setminus z\) has an \(N\)-minor. This means that \(M \setminus e/z\) has an \(N\)-minor, and as \(z\) is in \(Q\), we have contradicted Lemma 3.6. Thus Theorem 1.2 is now proved.

References