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# A matroid analogue of a theorem of Brooks for graphs



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## ABSTRACT

Brooks proved that the chromatic number of a loopless connected graph  $G$  is at most the maximum degree of  $G$  unless  $G$  is an odd cycle or a clique. This note proves an analogue of this theorem for  $GF(p)$ -representable matroids when  $p$  is prime, thereby verifying a natural generalization of a conjecture of Peter Nelson.

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## 1. Introduction

The matroid terminology and notation used here will follow [8]. For a matroid  $M$  having ground set  $E$  and rank function  $r$ , the *chromatic* or *characteristic polynomial* of  $M$  is defined by

$$p(M; \lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{r(M) - r(X)}.$$

If  $M$  is the cycle matroid of a graph  $G$  and  $G$  has  $\omega(G)$  components, then the chromatic polynomial  $P_G(\lambda)$  of the graph  $G$  is linked to the chromatic polynomial of its cycle matroid  $M(G)$  via the following equation:

$$P_G(\lambda) = \lambda^{\omega(G)} p(M(G); \lambda).$$

Of course, the chromatic number  $\chi(G)$  of  $G$  is the smallest positive integer  $j$  for which  $P_G(j)$  is positive unless  $G$  has a loop, in which case, the chromatic number is  $\infty$ . Let  $M$  be a rank- $r$  simple matroid that is representable over  $GF(q)$  and let  $T$  be a subset of  $PG(r-1, q)$  such that  $M \cong PG(r-1, q)|T$ . Let  $Q$  be a flat of  $PG(r-1, q)$  that avoids  $T$  and has maximum rank. The *critical exponent*  $c(M; q)$  of  $M$  is  $r - r(Q)$ . If  $M$  is loopless but has parallel elements, we define  $c(M; q) = c(\text{si}(M); q)$ . If  $M$  has a loop,

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$c(M; q) = \infty$ . Ostensibly,  $c(M; q)$  depends on the embedding of  $M$  in  $PG(r - 1, q)$  but the following fundamental result of Crapo and Rota [4] establishes that this is not the case.

**Theorem 1.1.** *Let  $M$  be a loopless matroid that is representable over  $GF(q)$ . Then*

$$c(M; q) = \min\{j : p(M; q^j) > 0\}.$$

Evidently, the critical exponent is an analogue of the chromatic number of a graph. Indeed, Geelen and Nelson [5] use the term ‘critical number’ rather than ‘critical exponent’ to highlight this analogy. For a loopless graph  $G$ , it is immediate that, for all prime powers  $q$ ,

$$q^{c(M(G); q)-1} < \chi(G) \leq q^{c(M(G); q)}.$$

Brooks [1] proved the following well-known result. For a graph  $G$ , let  $\Delta(G)$  denote its maximum vertex degree.

**Theorem 1.2.** *Let  $G$  be a loopless connected graph. Then*

$$\chi(G) \leq \Delta(G) + 1.$$

Indeed,  $\chi(G) \leq \Delta(G)$  unless  $G$  is an odd cycle or a complete graph.

The purpose of this note is to prove the following analogue of this result for  $GF(q)$ -representable matroids when  $q$  is prime. This new result was essentially conjectured by Peter Nelson [7]. An alternative analogue of Brooks’ Theorem, one for regular matroids, was proved in [9, Theorem 2.12].

**Theorem 1.3.** *Let  $p$  be a prime and  $M$  be a loopless non-empty  $GF(p)$ -representable matroid whose largest cocircuit has  $c^*$  elements. Then*

$$c(M; p) \leq \lceil \log_p(1 + c^*) \rceil.$$

Indeed, if  $M$  is connected, then  $c(M; p) \leq \lceil \log_p c^* \rceil$  unless  $M$  is a projective geometry or  $M$  is an odd circuit, where the latter only occurs when  $p = 2$ .

The requirement that  $M$  be connected appears in the last part of the theorem only to streamline the statement. It is not difficult to state a result in the absence of that requirement since the critical exponent of a loopless matroid  $M$  is the maximum of the critical exponents of its components while the maximum cocircuit size of  $M$  is the maximum of the maximum cocircuit sizes of its components.

We conjecture that Theorem 1.3 remains true if  $p$  is replaced by an arbitrary prime power  $q$ , but the proof technique used here only works when  $q$  is prime.

## 2. The proof

The proof of the main result will use three lemmas, the first of which is [9, Theorem 3.5]. For a matroid  $M$ , let  $\mathcal{R}(M)$  be the set of simple restrictions of  $M$ , and let  $\mathcal{C}^*(M)$  be the set of cocircuits of  $M$ .

**Lemma 2.1.** *Let  $M$  be a  $GF(q)$ -representable matroid having no loops. Then*

$$c(M; q) \leq \lceil \log_q(1 + \max_{N \in \mathcal{R}(M)} (\min_{C^* \in \mathcal{C}^*(N)} |C^*|)) \rceil.$$

Murty [6] considered the class of matroids in which all circuits have the same cardinality. His main result, which can be stated as follows, determined all binary matroids with this property.

**Lemma 2.2.** *Let  $M$  be a connected binary matroid with at least two elements. Then every cocircuit of  $M$  has the same cardinality if and only if, for some positive integer  $t$ , the matroid  $M$  can be obtained by adding  $t - 1$  elements in parallel to each element of one of the following:*

- (i)  $U_{r, r+1}$  for some  $r \geq 2$ ;
- (ii)  $PG(r - 1, 2)$  for some  $r \geq 1$ ; or
- (iii)  $AG(r - 1, 2)$  for some  $r \geq 2$ .

The proof of this result relies heavily on a property that characterizes binary matroids, namely, that the symmetric difference of two circuits is a disjoint union of circuits. Although the last result has not been generalized to all  $GF(q)$ -representable matroids, the next result is a partial generalization of it that treats the case when  $q$  is prime and the cardinality of all cocircuits is a power of  $q$ . Recall that a *point* in a matroid is a rank-1 flat in the matroid.

**Lemma 2.3.** *Let  $p$  be a prime exceeding two and  $M$  be a loopless non-empty  $GF(p)$ -representable matroid in which all cocircuits have  $p^k$  elements for some non-negative integer  $k$ . If  $M'$  is a component of  $M$ , then  $\text{si}(M')$  is a projective geometry and every parallel class of  $M'$  has the same size, this size being a power of  $p$ .*

**Proof.** For each  $e \in E(M)$ , we write  $d(e)$  for  $|\text{cl}(\{e\})|$ . We argue by induction on  $r(M)$  noting that the result is immediate if  $r(M) = 1$ . If  $r(M) = 2$ , then either  $M \cong U_{1,p^k} \oplus U_{1,p^k}$ , or  $M$  is a line with  $n$  points, for some  $n \geq 3$ . As the lemma holds in the former case, we consider the latter case, letting  $\{e_1, e_2, \dots, e_n\}$  be a transversal of the set of points of  $M$ . Then  $|E(M)| - d(e_i) = p^k$  for all  $i$ . Thus  $d(e_i) = d(e_j)$  for all distinct  $i$  and  $j$ . Now  $(n - 1)d(e_1) = p^k$ . As  $p$  is prime, we deduce that  $n = p + 1$  and  $d(e_1) = p^{k-1}$ . Hence  $M$  is obtained from  $PG(1, p)$  by replacing each element by  $p^{k-1}$  elements in parallel. Thus the lemma holds when  $r(M) = 2$ .

Assume the lemma holds when  $r(M) < r$  and let  $r(M) = r \geq 3$ . If  $M$  is disconnected, the result follows by the induction assumption. Thus we may assume that  $M$  is connected.

We show next that

**2.3.1.**  $M/\text{cl}(\{e\})$  is connected for all  $e \in E(M)$ .

Assume that  $M/\text{cl}(\{e\})$  is disconnected for some  $e \in E(M)$ . Then every cocircuit of  $M/\text{cl}(\{e\})$  has  $p^k$  elements and, by the induction assumption, each component is a projective geometry in which every parallel class has the same size, this size being a power of  $p$ . As  $M/\text{cl}(\{e\})$  is disconnected, by a result of Brylawski [2],  $M$  can be written as the parallel connection, with basepoint  $e$ , of some set,  $S$ , of connected matroids. Let  $M_1, M_2, \dots, M_s$  be the matroids in  $S$  that have rank at least two. The only other possible member of  $S$  is  $M/\text{cl}(\{e\})$  and it is present if and only if  $d(e) > 1$ . For each  $i$  in  $\{1, 2, \dots, s\}$ , take  $N_i = M/[E(M_i) \cup \text{cl}(\{e\})]$ . Then  $N_i$  is certainly connected.

Consider  $\text{si}(N_i)$  for some  $i$ . By [10, Lemma 2.3] (see also [8, Lemma 4.3.10]), each cocircuit of  $\text{si}(N_i)$  has an element  $f_i$  such that  $N_i/\text{cl}(\{f_i\})$  is connected. Choose a cocircuit of  $N_i$  avoiding  $e$ . Then  $M/\text{cl}(\{f_i\})$  is connected and so its simplification is a projective geometry and all the parallel classes of  $M/\text{cl}(\{f_i\})$  have the same size. It follows that  $s = 2$  and  $r(N_1) = r(N_2) = 2$  otherwise  $M/\text{cl}(\{f_1\})$  and  $M/\text{cl}(\{f_2\})$  cannot be matroids whose simplifications are projective geometries. We deduce that  $r(M) = 3$ . As  $M/\text{cl}(\{f_i\})$  is connected,  $\text{si}(N_i) \cong U_{2,p+1}$  for each  $i$ . Thus  $|\text{cl}_M(\{f\})| = p^{k-1}$  for all  $f \neq e$ . But, in  $M/\text{cl}(\{f_i\})$ , the parallel class containing  $e$  has more than  $(p - 1)p^{k-1}$  elements whereas every other parallel class has exactly  $p^{k-1}$  elements; a contradiction. Thus 2.3.1 holds.

By the induction assumption,  $M/\text{cl}(\{e\})$  is obtained from  $PG(r - 2, p)$  by replacing each element by the same number of parallel elements. Every cocircuit of  $M/\text{cl}(\{e\})$  contains  $p^k$  elements and contains  $p^{r-2}$  parallel classes. Thus, in  $M/\text{cl}(\{e\})$ , each parallel class has size  $p^{k-r+2}$ . This number will be the same irrespective of the choice of  $e$ .

Next we show the following.

**2.3.2.** *Every parallel class of  $M$  has the same size,  $n_1$ ; and every line of  $M$  contains the same number,  $n_2$ , of points. Moreover,  $(n_1, n_2)$  is  $(p^{k-r+2}, 2)$  or  $(p^{k-r+1}, p + 1)$ .*

Consider a line  $L$  of  $M$ . Let  $\{e_1, e_2, \dots, e_n\}$  be a transversal of the set of points of  $L$ . Then, in  $M/\text{cl}(\{e_i\})$ , we have a parallel class having exactly  $[d(e_1) + d(e_2) + \dots + d(e_n)] - d(e_i)$  elements. Thus, for all  $i$ ,

$$p^{k-r+2} + d(e_i) = \sum_{j=1}^n d(e_j).$$

Therefore  $d(e_1) = d(e_2) = \dots = d(e_n)$ . Hence

$$(n - 1)d(e_1) = p^{k-r+2}. \tag{2.1}$$

Now let  $f$  be an element not in  $\text{cl}(\{e_1\})$ . Since  $L$  was chosen arbitrarily, every point on the line  $\text{cl}(\{e_1, f\})$  has the same size, so  $d(f) = d(e_1)$ . Thus every parallel class of  $M$  has the same size. Moreover, if some line containing  $f$  has  $n'$  points, then

$$(n' - 1)d(f) = p^{k-r+2}. \tag{2.2}$$

From (2.1) and (2.2), we see that  $n' = n$ , so every line of  $M$  contains the same number of points. It follows since  $p$  is prime that  $(n_1, n_2)$  is  $(p^{k-r+2}, 2)$  or  $(p^{k-r+1}, p + 1)$ , that is, 2.3.2 holds.

Suppose every line of  $M$  contains exactly two points. Take a 3-element independent set,  $\{e, f, g\}$ , in  $\text{si}(M)$ . Then, in  $\text{si}(M)/e$ , the line through  $f$  and  $g$  must have  $p + 1$  elements. Thus, the plane in  $\text{si}(M)$  spanned by  $\{e, f, g\}$  has  $p + 2$  elements. As  $p$  is odd,  $M$  does not have  $U_{3,p+2}$  as a minor (see, for example, [8, Table 6.1]). This contradiction implies that every line of  $M$  contains exactly  $p + 1$  points. Hence  $M$  is a projective geometry in which every point has  $p^{k-r+1}$  elements. Thus the lemma follows by induction.

We are now ready to prove the main result of the paper.

**Proof of Theorem 1.3.** By Lemma 2.1,  $M$  has a simple restriction  $N$  such that

$$c(M; p) \leq \lceil \log_p(1 + \min_{C^* \in \mathcal{C}^*(N)} |C^*|) \rceil.$$

Every cocircuit of  $N$  is a subset of a cocircuit of  $M$ , so

$$c(M; p) \leq \lceil \log_p(1 + \max_{C^* \in \mathcal{C}^*(M)} |C^*|) \rceil. \tag{2.3}$$

Hence the first part of the theorem holds.

Now suppose that  $M$  is connected and that

$$c(M; p) > \lceil \log_p(\max_{C^* \in \mathcal{C}^*(M)} |C^*|) \rceil. \tag{2.4}$$

By combining (2.3) and (2.4), we deduce that  $\max_{C^* \in \mathcal{C}^*(M)} |C^*| = p^k$  for some positive integer  $k$ . Then  $c(M; p) = k + 1$ , so  $k + 1 \leq \lceil \log_p(1 + \min_{C^* \in \mathcal{C}^*(N)} |C^*|) \rceil$ . Hence  $\min_{C^* \in \mathcal{C}^*(N)} |C^*| \geq p^k$ . Thus every cocircuit of  $N$  has exactly  $p^k$  elements.

Let  $N = M \setminus T$ . The cocircuits of  $N$  are the minimal non-empty sets in  $\{C^* - T : C^* \in \mathcal{C}^*(M)\}$ . If  $M$  has a cocircuit  $D^*$  that meets both  $T$  and  $E - T$ , then  $|D^*| > p^k$ ; a contradiction. Thus  $T$  is a (possibly empty) union of components of  $M$ . But  $M$  is connected, so  $T = \emptyset$ , and  $N = M$ . By Lemmas 2.2 and 2.3,  $M$  is isomorphic to one of  $PG(r - 1, p)$ ,  $U_{r,r+1}$ , and  $AG(r - 1, 2)$  with the last two possibilities only arising when  $p = 2$ . But  $c(U_{r,r+1}; 2) = 1$  when  $r$  is odd, while  $c(AG(r - 1, 2); 2) = 1$  for all choices of  $r$ , so the theorem follows.  $\square$

One may hope to be able to eliminate the ceiling function in Theorem 1.3 but this is not possible. To see this, observe that  $c(M(K_5); 2) = 3$  but  $\max_{C^* \in \mathcal{C}^*(M)} |C^*| = 6$ . This example is far from the only exception one would need to add. To show this, we shall use the following result of Brylawski [3, Theorem 7.8].

**Lemma 2.4.** *Let  $M_1$  and  $M_2$  be matroids such that  $E(M_1) \cap E(M_2) = X$  and  $M_1|X = M_2|X$ . Let  $X$  be a modular flat in  $M_1$  and let  $M$  be the generalized parallel connection of  $M_1$  and  $M_2$  across  $X$ . Then*

$$p(M; \lambda) = \frac{p(M_1; \lambda)p(M_2; \lambda)}{p(M_1|X; \lambda)}.$$

As is well known and follows easily from Theorem 1.1,

$$p(PG(r - 1, q); \lambda) = (\lambda - 1)(\lambda - q)(\lambda - q^2) \dots (\lambda - q^{r-1}).$$

Combining this with the last lemma, it is straightforward to check that, for  $s \leq r$ , the generalized parallel connection  $M$  of  $PG(r - 1, q)$  and  $PG(s - 1, q)$  across a  $PG(t - 1, q)$  for  $1 \leq t < s < r$  has

critical exponent  $r$ . Now a largest cocircuit of  $PG(r - 1, 2)$  has  $q^{r-1}$  elements, and one easily checks that a largest cocircuit of  $M$  has fewer than  $q^r$  elements. Hence  $c(M; q) > \log_q(\max_{C^* \in \mathcal{C}^*(M)} |C^*|)$ . One can delete elements from  $PG(s - 1, q)$  that are not in the common  $PG(t - 1, q)$  to obtain numerous other examples that exhibit this behaviour.

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