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Relaxations of GF(4)-representable matroids

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Abstract

We consider the GF(4)-representable matroids with a circuit-hyperplane such that the matroid obtained by relaxing the circuit-hyperplane is also GF(4)-representable. We characterize the structure of these matroids as an application of structure theorems for the classes of $U_{2,4}$-fragile and $\{U_{2,5},U_{3,5}\}$-fragile matroids. In addition, we characterize the forbidden submatrices in GF(4)-representations of these matroids.

Mathematics Subject Classifications: 05B35

1 Introduction

Lucas [9] determined the binary matroids that have a circuit-hyperplane whose relaxation yields another binary matroid. Truemper [16], and independently, Oxley and Whittle [13], did the same for ternary matroids. In this paper, we solve the corresponding problem for quaternary matroids. We give both a structural characterization and a characterization in terms of forbidden submatrices.

Truemper [16] used the structure of circuit-hyperplane relaxations of binary and ternary matroids to give new proofs of the excluded-minor characterizations for the classes of binary, ternary, and regular matroids. It is natural to ask if Truemper’s techniques can be extended to give excluded-minor characterizations for classes of quaternary matroids. The main results of this paper can be viewed as a first step towards answering this question.

Our structural characterization can be summarized as follows. A matroid has path width 3 if there is an ordering $(e_1,e_2,\ldots,e_n)$ of its ground set such that $\{e_1,e_2,\ldots,e_t\}$ is a 3-separating set for all $t \in \{1,2,\ldots,n\}$.

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Theorem 1. Let $M$ and $M'$ be GF(4)-representable matroids such that $M'$ is obtained from $M$ by relaxing a circuit-hyperplane. Then $M'$ has path width 3.

In fact, our main result, Theorem 35, describes precisely how the matroids in Theorem 1 of path width 3 can be constructed using the generalized $\Delta$-$Y$ exchange of [12] and the notion of gluing a wheel onto a triangle from [2]. Our description uses the structure of $U_{2,4}$-fragile matroids from [10] and the structure of $\{U_{2,5},U_{3,5}\}$-fragile matroids from [3].

In future work, we hope to obtain a description of these matroids that is independent of the notion of fragility. Specifically, we would like to characterize the representations of these matroids. As a step in this direction, we describe minimal GF(4)-representations of matroids with a circuit-hyperplane whose relaxation is not GF(4)-representable. Note that the proof uses the excluded-minor characterization of the class of GF(4)-representable matroids. The setup for this result is as follows.

Let $M$ be a GF(4)-representable matroid on $E$ with a circuit-hyperplane $X$. Choose $e \in X$ and $f \in E - X$ such that $(X - e) \cup f$ is a basis of $M$. Then $M = M[I|C]$ for a quaternary matrix $C$ of the following block form.

$$C = \begin{bmatrix}
(E-X) - f & e \\
f & \begin{bmatrix}
A & 1 \\
1^T & 0
\end{bmatrix}
\end{bmatrix}.$$

In the above matrix, $A$ is an $(X - e) \times ((E - X) - f)$ matrix, and we have scaled so that every non-zero entry in the row labelled by $f$ and the column labelled by $e$ is 1. Let $M'$ be the matroid obtained from $M$ by relaxing the circuit-hyperplane $X$. We call the matrix $C$ a reduced representation of $M$. If $M'$ is GF(4)-representable, then we can find a reduced representation $C'$ of $M'$ in the following block form.

$$C' = \begin{bmatrix}
(E-X) - f & e \\
f & \begin{bmatrix}
A' & 1 \\
1^T & \omega
\end{bmatrix}
\end{bmatrix}.$$

We have scaled the rows and columns of the matrix such that the entry $C'_{fe} = \omega \in GF(4) - \{0, 1\}$, and the remaining entries in row $f$ and column $e$ are all 1. The following theorem is our characterization in terms of forbidden submatrices.

Theorem 2. Let $M$ and $C$ be constructed as described above. There is a reduced representation $C'$ of the above form for $M'$ if and only if, up to permuting rows and columns, $A$ and $A^T$ have no submatrix in the following list, where $x,y,z$ denote distinct non-zero elements of GF(4).

$$\begin{bmatrix}
x & y & z \\
0 & x & 0
\end{bmatrix}, \begin{bmatrix}
x & y \\
y & x
\end{bmatrix}, \begin{bmatrix}
x & x \\
x & x
\end{bmatrix}, \begin{bmatrix}
x & y \\
y & z
\end{bmatrix}, \begin{bmatrix}
x & x \\
x & 0
\end{bmatrix}, \begin{bmatrix}
x & 0 \\
x & 0
\end{bmatrix}, \begin{bmatrix}
x & 0 \\
0 & x
\end{bmatrix}, \begin{bmatrix}
x & 0 \\
0 & y
\end{bmatrix}, \begin{bmatrix}
x & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
x & 0 \\
0 & x
\end{bmatrix}, \begin{bmatrix}
x & 0 \\
0 & y
\end{bmatrix}.$$
This paper is organized as follows. In the next section, we collect some results on connectivity and circuit-hyperplane relaxation. In Section 3, we prove a fragility theorem. In Section 4, we describe the structure of the \( \{U_{2,5}, U_{3,5}\}\)-fragile matroids. In Section 5, we prove the structural characterization. In Section 6, we reduce the proof of Theorem 2 to a finite computer check. This check, carried out using SageMath, can be found in the Appendix [4].

2 Circuit-hyperplane relaxations and connectivity

We assume the reader is familiar with the fundamentals of matroid theory. Any undefined matroid terminology will follow Oxley [11]. Let \( M \) be a matroid on \( E \), and let \( B(M) \) denote the collection of bases of \( M \). If \( M \) has a circuit-hyperplane \( X \), then \( B(M') = B(M) \cup \{X\} \) is the collection of bases of a matroid \( M' \) on \( E \). We say that \( M' \) is obtained from \( M \) by relaxing the circuit-hyperplane \( X \). We list here a number of useful results on circuit-hyperplane relaxation.

Lemma 3. [11, Proposition 2.1.7] If \( M' \) is obtained from \( M \) by relaxing the circuit-hyperplane \( X \) of \( M \), then \( (M')^* \) is obtained from \( M^* \) by relaxing the circuit-hyperplane \( E(M) - X \) of \( M^* \).

The following elementary results are originally from [8].

Lemma 4. [11, Proposition 3.3.5] Let \( X \) be a circuit-hyperplane of a matroid \( M \), and let \( M' \) be the matroid obtained from \( M \) by relaxing \( X \). When \( e \in E(M) - X \),

(i) \( M/e = M'/e \) and, unless \( M \) has \( e \) as a coloop, \( M\setminus e \) is obtained from \( M\setminus e \) by relaxing the circuit-hyperplane \( X \) of the latter.

Dually, when \( f \in X \),

(ii) \( M\setminus f = M'/f \) and, unless \( M \) has \( f \) as a loop, \( M/f \) is obtained from \( M/f \) by relaxing the circuit-hyperplane \( X - f \) of the latter.

For a set \( N \) of matroids, we say that a matroid \( M \) has an \( N \)-minor if \( M \) has an \( N \)-minor for some \( N \in N \). We say \( M \) is \( N \)-fragile if \( M \) has an \( N \)-minor and, for each element \( e \) of \( M \), at most one matroid in \( \{M\setminus e, M/e\} \) has an \( N \)-minor. We say an element \( e \) of an \( N \)-fragile matroid \( M \) is nondeletable if \( M\setminus e \) has no \( N \)-minor; the element \( e \) is noncontractible if \( M/e \) has no \( N \)-minor.

The following lemma is an immediate consequence of Lemma 4.
Lemma 5. Let $X$ be a circuit-hyperplane of a matroid $M$, and let $M'$ be the matroid obtained from $M$ by relaxing $X$. If $N$ is a set of matroids such that $M'$ has an $N$-minor but $M$ has no $N$-minor, then $M'$ is $N$-fragile. Moreover, $X$ is a basis of $M'$ whose elements are nondeletable such that the elements of the cobasis $E(M') - X$ are noncontractible.

We use the following connectivity result.

Lemma 6. [11, Proposition 8.4.2] Let $M'$ be a matroid that is obtained by relaxing a circuit-hyperplane of a matroid $M$. If $M$ is $n$-connected, then $M'$ is $n$-connected.

Kahn [8] proved the following result on the representability of a circuit-hyperplane relaxation.

Lemma 7. Let $M'$ be a matroid that is obtained by relaxing a circuit-hyperplane of a matroid $M$. If $M$ is connected, then $M'$ is non-binary.

We use the following definition of the rank function of the 2-sum from [7]. Let $M_1$ and $M_2$ be matroids with at least two elements such that $E(M_1) \cap E(M_2) = \{p\}$. Then $M = M_1 \oplus_2 M_2$ has rank function $r_M$ defined for all $A_1 \subseteq E(M_1)$ and $A_2 \subseteq E(M_2)$ by

$$r_M(A_1 \cup A_2) = r_{M_1}(A_1) + r_{M_2}(A_2) - \theta(A_1, A_2) + \theta(\emptyset, \emptyset)$$

where $\theta(X, Y) = 1$ if $r_{M_1}(X \cup p) = r_{M_1}(X)$ and $r_{M_2}(Y \cup p) = r_{M_2}(Y)$, and $\theta(X, Y) = 0$ otherwise.

The next three results on 2-sums and minors of 2-sums are well known.

Lemma 8. [11, Proposition 7.1.20] Let $M$ and $N$ be matroids with at least two elements. Let $E(M) \cap E(N) = \{p\}$ and suppose that neither $M$ nor $N$ has $\{p\}$ as a separator. The set of circuits of $M \oplus_2 N$ is

$$C(M \setminus p) \cup C(N \setminus p) \cup \{(C \cup D) - p : p \in C \in C(M) \text{ and } p \in D \in C(N)\}.$$  

Lemma 9. [11, Theorem 8.3.1] A connected matroid $M$ is not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids $M_1$ and $M_2$, each of which has at least three elements and is isomorphic to a proper minor of $M$.

Lemma 10. [11, Proposition 8.3.5] Let $M, N, M_1, M_2$ be matroids such that $M = M_1 \oplus_2 M_2$ and $N$ is 3-connected. If $M$ has an $N$-minor, then $M_1$ or $M_2$ has an $N$-minor.

We can now describe the structure of circuit-hyperplanes in matroids of low connectivity. We omit the straightforward proof of the next lemma.

Lemma 11. Let $M$ be a GF(4)-representable matroid with a circuit-hyperplane $H$. If $M$ is not connected, then $M \cong U_{1,m} \oplus U_{n-1,n}$ for some positive integers $m$ and $n$.

We now work towards a description of the 2-separations of a connected matroid in which the relaxation of some circuit-hyperplane is GF(4)-representable.
Lemma 12. Let $M$ be a matroid with a circuit-hyperplane $X$. If $A$ is a non-trivial parallel class of $M$, then either $A \subseteq E - X$, or $A = X$ and $|A| = 2$.

Proof. If $A \cap X$ and $A \cap (E - X)$ are both non-empty, then there is a circuit $\{x, y\}$ contained in $A$ such that $x \in X$ and $y \in E - X$. But $E - X$ is a cocircuit of $M$, so this is a contradiction to orthogonality. Thus either $A \cap X$ or $A \cap (E - X)$ is empty. In the case that $A \cap (E - X)$ is empty, there is a circuit $\{x, y\}$ contained in $A$ that is also contained in the circuit $X$, so $X = A = \{x, y\}$. □

For the next result, we say that $M$ is 3-connected up to series and parallel classes if $M$ is connected and, for any 2-separation $(X, Y)$ of $M$, either $X$ or $Y$ is a series class or a parallel class.

Lemma 13. Let $M$ be a GF(4)-representable matroid with a circuit-hyperplane $X$ such that the matroid $M'$ obtained from $M$ is also GF(4)-representable. If $M$ is connected but not 3-connected, then $M$ is 3-connected up to series and parallel classes.

Proof. Assume that $M$ has a 2-separation $(S, T)$ where neither side is a series or parallel class. Then $M$ has a 2-sum decomposition of the form $M = N \oplus_2 N'$ for some $N$ and $N'$ with $E(N) \cap E(N') = \{p\}$, where neither $N$ nor $N'$ is a circuit or cocircuit.

First suppose that the circuit $X$ of $M$ has the form $(C \cup C') - p$, where $C$ is a circuit of $N$, and $C'$ is a circuit of $N'$ while $p \in C \cap C'$. Then

\[
\begin{align*}
r(X) &= r(M) - 1, \quad (1) \\
r(N) + r(N') - 1 &= r(M), \quad (2)
\end{align*}
\]

and

\[
r_M(X) = r_N(C) + r_{N'}(C') - 1. \quad (3)
\]

Equation (1) follows from the fact that $X$ is a hyperplane of $M$; Equations (2) and (3) follow from the definition of the rank function of the 2-sum of $N$ and $N'$. Combining (1) and (2), we see that $r(X) = r(N) + r(N') - 2$. Then combining this equation with (3), we see that

\[
r(C) + r(C') = r(N) + r(N') - 1.
\]

We may therefore assume that $C$ is a spanning circuit of $N$, and hence that $E(N) = C$ because the hyperplane $X$ is closed. Therefore $N$ is a circuit, a contradiction.

By symmetry, it remains to consider the case when $X$ is a circuit of $N' \setminus p$. Then $r(X) \leq r(N')$. Since $X$ is a hyperplane of $M$, and $r(M) = r(N) + r(N') - 1$, it follows that $r(N) \leq 2$. Since $N$ is not a cocircuit, we deduce that $r(N) = 2$. Then $r(M) = r(N') + 1$, so $r(X) = r(N') = r(N' \setminus p)$. Since $N$ is not a circuit we deduce that $\si(N) \cong U_{2,m}$ for some $m \geq 4$. Moreover, $p$ is not in a non-trivial parallel class in $N$ otherwise $X$ is not a hyperplane of $M$.

Switching to $M^*$, we see that $r_{M^*}(N') = |X| + r(N) - r(M) = r(N) = 2$. As above, it follows that $\co(N') \cong U_{n-1,n+1}$ for some $n \geq 3$. Moreover, $p$ is not in a non-trivial series class in $N'$. Let $X_1$ consist of one representative of each series class of $N'$, and let
3 A fragility theorem

We will use the following consequence of Geelen, Oxley, Vertigan, and Whittle [6, Theorem 8.4].

Theorem 14. Let $M$ and $M'$ be GF(4)-representable matroids with the properties that $M$ is connected, $M'$ is 3-connected, and $M'$ is obtained from $M$ by relaxing a circuit-hyperplane.

(i) If $M'$ has a $U_{2,4}$-minor but no $\{U_{2,5}, U_{3,5}\}$-minor, then $M$ is binary.

(ii) If $M'$ has a $\{U_{2,5}, U_{3,5}\}$-minor but no $U_{3,6}$-minor, then $M$ has no $\{U_{2,5}, U_{3,5}\}$-minor.

We can now prove the main result of this section.

Theorem 15. Let $M$ and $M'$ be GF(4)-representable matroids such that $M$ is connected, $M'$ is 3-connected, and $M'$ is obtained from $M$ by relaxing a circuit-hyperplane $X$. Then $M'$ is either $U_{2,4}$-fragile or $\{U_{2,5}, U_{3,5}\}$-fragile. Moreover, $X$ is a basis of $M'$ whose elements are nondeletable such that the elements of the cobasis $E(M') - X$ are noncontractible.

Proof. First assume that $M'$ has no $\{U_{2,5}, U_{3,5}\}$-minor. By Lemma 7 and Theorem 14 (i), $M'$ has a $U_{2,4}$-minor and $M$ has no $U_{2,4}$-minor. Then it follows from Lemma 5 that $M'$ is $U_{2,4}$-fragile, and $M'$ has a basis $X$ whose elements are nondeletable such that the elements of the cobasis $E(M') - X$ are noncontractible.

We may now assume that $M'$ has a $\{U_{2,5}, U_{3,5}\}$-minor. Suppose that $M$ also has a $\{U_{2,5}, U_{3,5}\}$-minor, and assume that $M$ is a minor-minimal matroid with respect to the hypotheses; that is, we assume that $M$ has no proper minor $M_0$ such that $M_0$ is connected, $M_0$ has a $\{U_{2,5}, U_{3,5}\}$-minor, and $M_0$ has a circuit-hyperplane whose relaxation $M'_0$ is 3-connected, GF(4)-representable, and has a $\{U_{2,5}, U_{3,5}\}$-minor.

Claim 16. $M$ is $\{U_{2,5}, U_{3,5}\}$-fragile.

Proof of 16. Suppose that $M$ has an element $e \in E(M) - X$ such that $M \setminus e$ has a $\{U_{2,5}, U_{3,5}\}$-minor. If $M \setminus e$ is 3-connected, then we have a contradiction to the minimality of $M$. Therefore, by Lemma 13, $M \setminus e$ is 3-connected up to series and parallel pairs. Suppose that $A$ is a non-trivial parallel class of $M \setminus e$. Suppose $A \subseteq X$. Then $A = X$ and $|A| = 2$ by Lemma 12, so we deduce that $M \setminus e$ is a parallel extension of $U_{2,5}$ and hence
that $M'\setminus e$ has a $U_{2,5}$-minor, a contradiction to the fact that the matroid $M'$ obtained from $M$ by relaxing $X$ is $GF(4)$-representable. Thus $A \subseteq E(M\setminus e) - X$ by Lemma 12. By duality, any non-trivial series class of $M\setminus e$ must be contained in $X$. Then, by Lemma 10, the matroid $M_0$ obtained from $M\setminus e$ by deleting all but one element of every non-trivial parallel class and contracting all but one element of every non-trivial series class has a $\{U_{2,5}, U_{3,5}\}$-minor. We deduce from Lemma 13 that $M_0$ is $3$-connected. Then $M_0$ contradicts the minimality of $M$. Therefore $M\setminus e$ has no $\{U_{2,5}, U_{3,5}\}$-minor for all $e \in E(M) - X$, and, by duality, $M/e$ has no $\{U_{2,5}, U_{3,5}\}$-minor for all $e \in X$, so $M$ is $\{U_{2,5}, U_{3,5}\}$-fragile. This completes the proof of 16.

Since $M$ has a $\{U_{2,5}, U_{3,5}\}$-minor, it follows from Theorem 14 (ii) that $M'$ has a $U_{3,6}$-minor, that is, $M'/C\setminus D \cong U_{3,6}$ for some subsets $C$ and $D$. If $C \subseteq X$ and $D \subseteq E(M') - X$, then it follows from Lemma 4 that $U_{3,6}$ can be obtained from $M/C\setminus D$ by relaxing the circuit-hyperplane $X - C$. Hence $M/C\setminus D \cong P_6$, a contradiction because $M/C\setminus D$ is $GF(4)$-representable but $P_6$ is not. Therefore $C \cap (E(M') - X)$ or $D \cap X$ is nonempty, so $M/C\setminus D = M'/C\setminus D \cong U_{3,6}$ by Lemma 4. This is a contradiction to 16 because any minor of $M$ must also be $\{U_{2,5}, U_{3,5}\}$-fragile, but for any $e$, both $U_{3,6}\setminus e$ and $U_{3,6}/e$ have a $\{U_{2,5}, U_{3,5}\}$-minor. We conclude that $M$ has no $\{U_{2,5}, U_{3,5}\}$-minor. It now follows from Lemma 5 that $M'$ is $\{U_{2,5}, U_{3,5}\}$-fragile, and that $M'$ has a basis $X$ whose elements are nondeletable such that the elements of the cobasis $E(M') - X$ are noncontractible. 

4 The structure of $\{U_{2,5}, U_{3,5}\}$-fragile matroids

4.1 Partial Fields and Constructions

We briefly state the necessary material on partial fields. For a more thorough treatment, we refer the reader to [14].

A partial field is a pair $\mathbb{P} = (R, G)$, where $R$ is a commutative ring with unity, and $G$ is a subgroup of the units of $R$ with $-1 \in G$. A matrix with entries in $G$ is a $\mathbb{P}$-matrix if $\det(D) \in G \cup \{0\}$ for any square submatrix $D$ of $A$. We use $\langle X \rangle$ to denote the multiplicative subgroup of $R$ generated by the subset $X$.

A rank-$r$ matroid $M$ on the ground set $E$ is $\mathbb{P}$-representable if there is an $r \times |E|$ $\mathbb{P}$-matrix $A$ such that, for each $r \times r$ submatrix $D$, the determinant of $D$ is nonzero if and only if the corresponding subset of $E$ is a basis of $M$. When this occurs, we write $M = M[A]$.

The 2-regular partial field is defined as follows.

$$\mathbb{U}_2 = \langle \mathbb{Q}(\alpha, \beta), \langle -1, \alpha, \beta, 1 - \alpha, 1 - \beta, \alpha - \beta \rangle \rangle,$$

where $\alpha, \beta$ are indeterminates.

It is well-known that any $\mathbb{U}_2$-representable matroid is $GF(4)$-representable [12]. On the other hand, there are $GF(4)$-representable matroids that are not $\mathbb{U}_2$-representable. We now define three such matroids. The matroid $P_8$ has a unique pair of disjoint circuit-hyperplanes; we let $P_8^+$ denote the unique matroid obtained by relaxing one of these
circuit-hyperplanes. We denote by $F_7^-$ the matroid obtained from the non-Fano matroid $F_7^-$ by relaxing a circuit-hyperplane. The GF(4)-representable matroids $P_8^-, F_7^-$, $(F_7^-)^*$ are not $U_2$-representable. We note that this can be deduced from [1] since $P_8^-, F_7^-, (F_7^-)^*$ are $\{U_{2,5}, U_{3,5}\}$-fragile matroids. Since these matroids are not $U_2$-representable, we have the following lemma.

**Lemma 17.** The class of $U_2$-representable matroids is contained in the class of GF(4)-representable matroids with no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor.

To describe the structure of $\{U_{2,5}, U_{3,5}\}$-fragile matroids as in [3], we need two constructions: the generalized $\Delta$-$Y$ exchange, and gluing on wheels. For a more thorough treatment of these constructions, we refer the reader to [12] and [2].

Loosely speaking, the operations of generalized $\Delta$-$Y$ exchange and gluing on wheels both involve gluing matroids together along a common restriction. Let $M_1$ and $M_2$ be matroids with a common restriction $A$, where $A$ is a modular flat of $M_1$. The generalized parallel connection of $M_1$ and $M_2$ along $A$, denoted $P_A(M_1, M_2)$, is the matroid obtained by gluing $M_1$ and $M_2$ along $A$. It has ground set $E(M_1) \cup E(M_2)$, and a set $F$ is a flat of $P_A(M_1, M_2)$ if and only if $F \cap E(M_i)$ is a flat of $M_i$ for each $i$ (see [11, Section 11.4]).

A subset $S$ of $E(M)$ is a segment of $M$ if every three-element subset of $S$ is a triangle of $M$. Let $M$ be a matroid with a $k$-element segment $A$. Intuitively, a generalized $\Delta$-$Y$ exchange on $A$ turns the segment $A$ into a $k$-element cosegment. To define the generalized $\Delta$-$Y$ exchange formally, we first recall the following definition of a family of matroids $\Theta_k$ from [12]. For $k \geq 3$, fix a basis $B = \{b_1, b_2, \ldots, b_k\}$ of the rank-$k$ projective geometry $PG(k - 1, \mathbb{R})$, and choose a line $L$ of $PG(k - 1, \mathbb{R})$ that is freely placed relative to $B$. If follows from modularity that, for each $i$, the hyperplane spanned by $B - \{b_i\}$ meets $L$; we let $a_i$ be the point of intersection. Let $A = \{a_1, a_2, \ldots, a_k\}$, and let $\Theta_k$ be the matroid obtained by restricting $PG(k - 1, \mathbb{R})$ to the set $A \cup B$. Note that the matroid $\Theta_k$ has $A$ as a modular $k$-point segment $A$, so the generalized parallel connection of $\Theta_k$ and $M$ along $A$ is well-defined. If the $k$-element segment $A$ is coindependent in $M$, then we define the matroid $\Delta_A(M)$ to be the matroid obtained from $P_A(\Theta_k, M) \backslash A$ by relabeling the elements of $E(\Theta_k) - A$ by $A$ in the natural way, and we say that $\Delta_A(M)$ is obtained from $M$ by performing a generalized $\Delta$-$Y$ exchange on $A$. For a matroid $M$ with an independent cosegment $A$, a generalized $Y$-$\Delta$ exchange on $A$, denoted by $\nabla_A(M)$, is defined to be the matroid $(\Delta_A(M^*))^*$.

We use the following results on representability and the minor operations.

**Lemma 18.** [12, Lemma 3.7] Let $\mathbb{P}$ be a partial field. Then $M$ is $\mathbb{P}$-representable if and only if $\Delta_A(M)$ is $\mathbb{P}$-representable.

**Lemma 19.** [12, Lemma 2.13] Suppose that $\Delta_A(M)$ is defined. If $x \in A$ and $|A| \geq 3$, then $\Delta_{A-x}(M \backslash x)$ is also defined, and $\Delta_A(M)/x = \Delta_{A-x}(M \backslash x)$.

**Lemma 20.** [12, Lemma 2.16] Suppose that $\Delta_A(M)$ is defined.

(i) If $x \in E(M) - A$ and $A$ is coindependent in $M \backslash x$, then $\Delta_A(M \backslash x)$ is defined and $\Delta_A(M \backslash x) = \Delta_A(M \backslash x)$. 

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(ii) If \( x \in E(M) - \text{cl}(A) \), then \( \Delta_A(M/x) \) is defined and \( \Delta_A(M)/x = \Delta_A(M/x) \).

**Lemma 21.** [12, Lemma 2.15] Suppose that \( x \in \text{cl}(A) - A \) and let \( a \) be an arbitrary element of the \( k \)-element segment \( A \). Then \( \Delta_A(M)/x \) equals the 2-sum, with basepoint \( p \), of a copy of \( U_{k-1,k+1} \) with groundset \( A \cup p \) and the matroid obtained from \( M/x \backslash (A - a) \) by relabeling \( a \) as \( p \).

The next result implies that every \( \{U_{2,5},U_{3,5}\} \)-fragile matroid is 3-connected up to series and parallel classes.

**Lemma 22.** [10, Proposition 4.3] Let \( M \) be a matroid with a 2-separation \((A,B)\), and let \( N \) be a 3-connected minor of \( M \). Assume \( |E(N) \cap A| \geq |E(N) \cap B| \). Then \( |E(N) \cap B| \leq 1 \). Moreover, unless \( B \) is a parallel or series class, there is an element \( x \in B \) such that both \( M \backslash x \) and \( M/x \) have a minor isomorphic to \( N \).

The following is an easy consequence of the property that \( \{U_{2,5},U_{3,5}\} \)-fragile matroids are 3-connected up to parallel and series classes.

**Lemma 23.** Let \( M \) be a \( \{U_{2,5},U_{3,5}\} \)-fragile matroid with at least 8 elements. If \( S \) is a triangle or 4-element segment of \( M \) such that \( E(M) - S \) is not a series or parallel class of \( M \), then \( S \) is coinddependent in \( M \). If \( C \) is a triad or 4-element cosegment of \( M \) such that \( E(M) - C \) is not a series or parallel class of \( M \), then \( C \) is independent.

Let \( M \) be a \( \{U_{2,5},U_{3,5}\} \)-fragile matroid. A segment \( S \) of \( M \) is **allowable** if \( S \) is coinddependent and some element of \( S \) is nondeletable. A cosegment \( C \) of \( M \) is **allowable** if the segment \( C \) of \( M^* \) is allowable. In [3], it was shown that we can obtain a new \( \{U_{2,5},U_{3,5}\} \)-fragile \( U_2 \)-representable matroid from an old \( \{U_{2,5},U_{3,5}\} \)-fragile \( U_2 \)-representable matroid by performing a generalized \( \Delta-Y \) exchange on an allowable segment. We will prove an analogous result for \( \{U_{2,5},U_{3,5}\} \)-fragile GF(4)-representable matroids with no \( \{P_8^-,F_7^-, (F_7^*)^\} \)-minor.

Let \( \mathcal{U} \) be the class of GF(4)-representable matroids with no \( \{U_{2,5},U_{3,5}\} \)-minor. The class of **sixth-root-of-unity** matroids is the class of matroids that are representable over both GF(3) and GF(4). Semple and Whittle [15, Theorem 5.2] showed that \( \mathcal{U} \) is the class of matroids that can be obtained by taking direct sums and 2-sums of binary and sixth-root-of-unity matroids.

**Lemma 24.** Let \( M \) be a matroid in the class \( \mathcal{U} \). If \( M' \) is obtained from \( M \) by performing a generalized \( \Delta-Y \) exchange or a generalized \( Y-\Delta \) exchange, then \( M' \in \mathcal{U} \).

**Proof.** Suppose that there exists a matroid \( M \in \mathcal{U} \) with a coinddependent segment \( A \) such that \( \Delta_A(M) \notin \mathcal{U} \). Among all counterexamples, suppose that \( M \) has been chosen so that \( |E(M)| \) is as small as possible. Suppose \( M \) is 3-connected. Since any 3-connected member of \( \mathcal{U} \) is either a binary or sixth-root-of-unity matroid, this also holds for \( \Delta_A(M) \) by Lemma 18. Hence \( \Delta_A(M) \in \mathcal{U} \), contradicting the assumption that \( M \) is a counterexample. Therefore \( M \) is not 3-connected.

Now either \( M = M_1 \oplus M_2 \) or \( M = M_1 \oplus_2 M_2 \) for some \( M_1, M_2 \in \mathcal{U} \) with \( |E(M_i)| < |E(M)| \) for each \( i \in \{1,2\} \). Moreover, we may assume that \( M_1 \) and \( M_2 \) have been chosen.
so that the segment $A$ of $M$ is contained in $E(M_1)$. Now either $\Delta_A(M) = \Delta_A(M_1) \oplus M_2$ or $\Delta_A(M) = \Delta_A(M_1) \oplus_2 M_2$. Since $|E(M_1)| < |E(M)|$, it follows that $\Delta_A(M_1) \in \mathcal{U}$. Hence $\Delta_A(M) \in \mathcal{U}$. Since $\mathcal{U}$ is closed under duality, the result follows. \hfill \Box

**Lemma 25.** Let $M$ be a $\{U_{2,5}, U_{3,5}\}$-fragile $GF(4)$-representable matroid with no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor. If $A$ is an allowable segment of $M$ with $|A| \in \{3, 4\}$, then $\Delta_A(M)$ is a $\{U_{2,5}, U_{3,5}\}$-fragile $GF(4)$-representable matroid with no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor. Moreover, $A$ is an allowable cosegment of $\Delta_A(M)$.

**Proof.** The proof that $\Delta_A(M)$ is a $\{U_{2,5}, U_{3,5}\}$-fragile $GF(4)$-representable matroid where $A$ is an allowable cosegment of $\Delta_A(M)$ closely follows the proof of [3, Lemma 2.21]. The only difference is where the proof of [3, Lemma 2.21] uses the fact that a $U_2$-representable matroid with no $\{U_{2,5}, U_{3,5}\}$-minor is near-regular and the class of near-regular matroids is closed under the generalized $\Delta-Y$ exchange, we instead use Lemma 24.

We must also show that $\Delta_A(M)$ has no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor. This follows for $|E(M)| \leq 9$ from the generation of the 3-connected $\{U_{2,5}, U_{3,5}\}$-fragile $GF(4)$-representable matroids with no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor on at most 9 elements (see the Appendix [4]), since all such matroids are $U_2$-representable. Suppose that $M$ is a minimum-sized counterexample, so $\Delta_A(M)$ has a $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor and $\Delta_A(M)$ has at least ten elements. Then $\Delta_A(M)$ has a minor $N$, obtained by deleting or contracting an element $x$ say, that also has a $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor. Since $\Delta_A(M)$ is a $\{U_{2,5}, U_{3,5}\}$-fragile it follows that the minor $N$ is also $\{U_{2,5}, U_{3,5}\}$-fragile. Suppose that $N = \Delta_A(M)/x$. Suppose that $x \notin A$. Then $\Delta_A(M)/x = \Delta_A(M \setminus x)$ by Lemma 19, a contradiction since $M$ is a minimum-sized counterexample. Next suppose that $x \in cl(A) - A$. Since $N$ is a $\{U_{2,5}, U_{3,5}\}$-fragile it follows from Lemma 21 and Proposition 22 that $|A| = 4$ and $M/x \setminus (A - a) \cong U_{1,n}$ for some $n \geq 2$. Hence $\Delta_A(M)$ has no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor, a contradiction. We may now assume $x \in E(M) - cl(A)$. Then $\Delta_A(M)/x = \Delta_A(M/x)$ by Lemma 20, a contradiction since $M$ is a minimum-sized counterexample. We deduce that $N = \Delta_A(M \setminus x)$, and we may assume that any minor obtained from $\Delta_A(M)$ by contracting an element has no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor. Now if $x \in A$, then $A - x$ is a series class of $\Delta_A(M)$, so there is some $y \in A$ such that $\Delta_A(M)/y$ has a $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor, a contradiction. Therefore $x \notin A$. If $A$ is not coindependent in $M \setminus x$, then it follows from Lemma 23 that $\Delta_A(M)$ has no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor, a contradiction. Therefore $A$ is coindependent in $M \setminus x$, so $\Delta_A(M)/x = \Delta_A(M \setminus x)$ by Lemma 20, a contradiction since $M$ is a minimum-sized counterexample. \hfill \Box

Let $M$ be a matroid, and $(a, b, c)$ an ordered subset of $E(M)$ such that $T = \{a, b, c\}$ is a triangle. Let $r \geq 3$ be a positive integer, and, when $r = 3$, we fix a vertex of $\mathcal{W}_3$ to be the center, so we can refer to rim and spoke elements of $M(\mathcal{W}_3)$. Let $N$ be obtained from $M(\mathcal{W}_r)$ by relabeling some triangle as $\{a, b, c\}$, where $a, c$ are spoke elements, and let $X \subseteq \{a, b, c\}$ such that $b \in X$. We say the matroid $M' := P_r(M, N)|X$ is obtained from $M$ by gluing an $r$-wheel onto $(a, b, c)$. We also say that $M'$ is obtained from $N^*$ by gluing a wheel onto the triad $T$. Suppose that $T_1, T_2, \ldots, T_n$ are ordered triples whose underlying sets are triangles of $M$. We say $M'$ can be obtained from $M$ by gluing wheels

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onto \( T_1, T_2, \ldots, T_n \) if, for some subset \( J \) of \( \{1,2,\ldots,n\} \), \( M' \) can be obtained from \( M \) by a sequence of moves, where each move consists of gluing an \( r_j \)-wheel onto \( T_j \) for \( j \in J \). Note that the spoke elements of a triangle in this sequence may only be deleted as part of the gluing operation when they do not appear in any subsequent triangle in the sequence.

Lemma 26. Let \( M \) be a \( \{U_{2,5},U_{3,5}\} \)-fragile GF(4)-representable matroid with no \( \{P_8^-,F_7^-, (F_7^-)^*\}\)-minor. Let \( A = \{a,b,c\} \) be an allowable triangle of \( M \), where \( b \) is nondeletable. If \( M' \) is obtained from \( M \) by gluing an \( r \)-wheel onto \( (a,b,c) \), where \( X \subseteq \{a,b,c\} \) is such that \( b \in X \), then \( M' \) is a \( \{U_{2,5},U_{3,5}\} \)-fragile GF(4)-representable matroid with no \( \{P_8^-,F_7^-, (F_7^-)^*\}\)-minor. Moreover, \( F = E(W_r) - X \) is the set of elements of a fan, the spoke elements of \( F \) are noncontractible in \( M' \), and the rim elements of \( F \) are nondeletable in \( M' \).

Proof. The proof is the same as [3, Lemma 2.22] except that we use Lemma 25 instead of [3, Lemma 2.21].

4.2 Path sequences

We can now describe a family of \( \{U_{2,5},U_{3,5}\} \)-fragile GF(4)-representable matroids with no \( \{P_8^-,F_7^-, (F_7^-)^*\}\)-minor obtained by performing generalized \( \Delta-Y \) exchanges and gluing on wheels. In fact, the matroids in this family are \( \bar{U}_2 \)-representable and were first described in [3]. Each matroid in this family has a \( \{X_8,Y_8,Y_8^*\}\)-minor, and an associated path of 3-separations that we need to describe in order to define the family.

We call the set \( X \subseteq E(M) \) fully closed if \( X \) is closed in both \( M^* \) and \( M \). The full closure of \( X \), denoted \( \text{fcl}_M(X) \), is the intersection of all fully closed sets containing \( X \). The full closure of \( X \) can be obtained from \( X \) by repeatedly taking closure and coclosure until no new elements are added. We call \( X \) a path-generating set if \( X \) is a 3-separating set of \( M \) such that \( \text{fcl}_M(X) = E(M) \). A path-generating set \( X \) thus gives rise to a natural path of 3-separating sets \( (P_1, \ldots, P_m) \), where \( P_1 = X \) and each step \( P_i \) is either the closure or coclosure of the 3-separating set \( P_i \cup \cdots \cup P_{i-1} \).

Let \( X \) be an allowable cosegment of the \( \{U_{2,5},U_{3,5}\} \)-fragile matroid \( M \). A matroid \( Q \) is an allowable series extension of \( M \) along \( X \) if \( M = Q/Z \) and, for every element \( z \) of \( Z \), there is some element \( x \) of \( X \) such that \( x \) is \( \{U_{2,5},U_{3,5}\} \)-contractible in \( M \) and \( z \) is in series with \( x \) in \( Q \). We also say that \( Q^* \) is an allowable parallel extension of \( M^* \) along \( X \).

Let \( N \) be a matroid with a path-generating allowable segment or cosegment \( A \). We say that \( M \) is obtained from \( N \) by a \( \Delta-\nabla \)-step along \( A \) if, up to duality, \( M \) is obtained from \( N \) by performing a non-empty allowable parallel extension along \( A \), followed by a generalized \( \Delta-Y \) exchange on \( A \).

Let \( X_8 \) be the matroid obtained from \( U_{2,5} \) by choosing a 4-element segment \( C \), adding a point in parallel with each of three distinct points of \( C \), then performing a generalized \( \Delta-Y \)-exchange on \( C \) (see Figure 1). In what follows, \( S \) will be the elements of the 4-element segment of \( X_8 \), and \( C \) the elements of the 4-element cosegment of \( X_8 \), so \( E(X_8) = S \cup C \).

We will build matroids from \( X_8 \) by performing a sequence of \( \Delta-\nabla \)-steps along \( A \in \{S,C\} \). Note that, in such matroids, each of \( S \) and \( C \) can be either a segment or a cosegment.
A sequence of matroids $M_1,\ldots,M_n$ is called a path sequence if the following conditions hold:

(P1) $M_1 = X_8$; and

(P2) For each $i \in \{1,\ldots,n-1\}$, there is some 4-element path-generating segment or cosegment $A \in \{S,C\}$ of $M_i$ such that either:

(a) $M_{i+1}$ is obtained from $M_i$ by a $\Delta$-$\nabla$-step along $A$; or

(b) $M_{i+1}$ is obtained from $M_i$ by gluing a wheel onto an allowable subset $A'$ of $A$.

Note in (P2) that each $\Delta$-$\nabla$-step described in (a) increases the number of elements by at least one, and that the wheels in (b) are only glued onto allowable subsets of 4-element segments or cosegments.

We say that a path sequence $M_1,\ldots,M_n$ describes a matroid $M$ if $M_n \cong M$. We also say that $M$ is a matroid described by $M_1,\ldots,M_n$ that describes $M$. Let $\mathcal{P}$ denote the class of matroids such that $M \in \mathcal{P}$ if and only if there is some path sequence $M_1,\ldots,M_n$ that describes a matroid $M'$ such that $M$ can be obtained from $M'$ by some, possibly empty, sequence of allowable parallel and series extensions. Since $X_8$ is self-dual, it is easy to see that the sequence of dual matroids $M_1^*,\ldots,M_n^*$ of a path sequence $M_1,\ldots,M_n$ is also a path sequence. Thus the class $\mathcal{P}$ is closed under duality.

We denote by $Y_8$ the unique matroid obtained from $X_8$ by performing a $Y$-$\Delta$-exchange on an allowable triad (see Figure 1). We will prove the following result.

**Theorem 27.** If $M$ is a 3-connected $\{U_{2,5},U_{3,5}\}$-fragile GF(4)-representable matroid that has an $\{X_8,Y_8,Y_8^*\}$-minor but no $\{P_8^-,F_7^-,(F_7^-)^*\}$-minor, then there is some path sequence that describes $M$.

The proof of Theorem 27 closely follows the proof of [3, Corollary 4.3]. The strategy is to show that a minor-minimal counterexample has at most 12 elements. Let $M$ be a GF(4)-representable $\{U_{2,5},U_{3,5}\}$-fragile matroid $M$ with an $\{X_8,Y_8,Y_8^*\}$-minor but no $\{P_8^-,F_7^-,(F_7^-)^*\}$-minor. Suppose that $M$ is a minimum-sized matroid that is not in the class $\mathcal{P}$. Then $M$ is 3-connected because $\mathcal{P}$ is closed under series and parallel extensions. Moreover, the dual $M^*$ is also not in $\mathcal{P}$ because $\mathcal{P}$ is closed under duality. Thus, by the Splitter Theorem and duality, we may assume there is some element $x$ of $M$ such that $M \setminus x$
is also a 3-connected GF(4)-representable \{U_{2,5}, U_{3,5}\}-fragile matroid with an \{X_8, Y_8, Y_8^*\}-minor but no \{P_{8}^-, F_{7}^-, (F_{7}^-)^*\}-minor. By the assumption that \(M\) is minimum-sized with respect to being outside the class \(\mathcal{P}\), it follows that \(M\setminus x \in \mathcal{P}\). Thus \(M\setminus x\) is described by a path sequence \(M_1, \ldots, M_n\). The next lemma [3, Lemma 6.3] identifies the three possibilities for the position of \(x\) in \(M\) relative to the path of 3-separations associated with \(M_1, \ldots, M_n\).

**Lemma 28.** Let \(M\) and \(M\setminus x\) be 3-connected \{U_{2,5}, U_{3,5}\}-fragile matroids. If \(M\setminus x\) is described by a path sequence with associated path of 3-separations \(P\), then either:

(i) there is some 3-separation \((X,Y)\) displayed by \(P\) such that \(x \in \text{cl}(X)\) and \(x \in \text{cl}(Y)\); or

(ii) there is some 3-separation \((X,Y)\) displayed by \(P\) such that \(x \notin \text{cl}(X)\) and \(x \notin \text{cl}(Y)\); or

(iii) for each 3-separation \((R,G)\) of \(M\) displayed by \(P\), there is some \(X \in \{R,G\}\) such that \(x \in \text{cl}_M(X)\) and \(x \in \text{cl}_M^*(X)\).

The proofs of the next three lemmas follow the proofs of [3, Lemma 7.4], [3, Lemma 8.7], and [3, Lemma 9.7] but use Lemma 25 above instead of [3, Lemma 2.21].

**Lemma 29.** Lemma 28 (i) does not hold.

**Lemma 30.** If Lemma 28 (ii) holds, then \(|E(M\setminus x)| \leq 10\).

**Lemma 31.** If Lemma 28 (iii) holds, then \(|E(M\setminus x)| \leq 11\).

**Proof of Theorem 27.** In view of the last three lemmas, it suffices to verify that \(\mathcal{P}\) contains each 3-connected \{U_{2,5}, U_{3,5}\}-fragile GF(4)-representable matroid with an \{X_8, Y_8, Y_8^*\}-minor and no \{P_{8}^-, F_{7}^-, (F_{7}^-)^*\}-minor having at most 12 elements. This is done in the Appendix [4].

**4.3 Fan extensions**

The following theorem describes the structure of the matroids with no \{X_8, Y_8, Y_8^*\}-minor. Note that \(M_{9,9}\) is the rank-4 matroid on 9 elements in Figure 2. The matroid \(M_{7,1}\) is the 7-element matroid that is obtained from \(Y_8\) by deleting the unique point that is contained in the two 4-element segments of \(Y_8\). We label the points of a triangle of \(M_{7,1}\) by \{1,2,3\} as in Figure 2.

**Theorem 32.** Let \(M'\) be a 3-connected \{U_{2,5}, U_{3,5}\}-fragile GF(4)-representable matroid with no \{P_{8}^-, F_{7}^-, (F_{7}^-)^*\}-minor. Then \(M'\) is isomorphic to a matroid \(M\) for which at least one of the following holds:

(i) \(M\) has an \{X_8, Y_8, Y_8^*\}-minor;

(ii) \(M \in \{M_{9,9}, M_{9,9}^*\}\);
Figure 2: The matroids $M_{7,1}$ and $M_{9,9}$.

(iii) $M$ or $M^*$ can be obtained from $U_{2,5}$ (with ground set $\{a, b, c, d, e\}$) by gluing wheels to $(a, c, b), (a, d, b), (a, e, b)$;

(iv) $M$ or $M^*$ can be obtained from $U_{2,5}$ (with ground set $\{a, b, c, d, e\}$) by gluing wheels to $(a, b, c), (c, d, e)$;

(v) $M$ or $M^*$ can be obtained from $M_{7,1}$ by gluing a wheel to $(1, 3, 2)$.

Proof. Assume $M$ has no $\{X_8, Y_8, Y^*_8\}$-minor. For (ii), we show in Lemma 1 of the Appendix [4] that the matroids $M_{9,9}$ and $M_{9,9}^*$ are splitters for the class of 3-connected $\{U_{2,5}, U_{3,5}\}$-fragile GF(4)-representable matroids with no $\{P_7^-, F_7^-, (F_7^*)^\}$-minor.

We may therefore assume $M$ has no $\{M_{9,9}, M_{9,9}^*, X_8, Y_8, Y^*_8\}$-minor. To show that (iii), (iv), or (v) holds, we use the main result of [2] called the “Fan Lemma”, which reduces the proof to showing that extensions and coextensions of the 9-element matroids with this structure also have this structure. These verifications are completed in Lemmas 2 through 7 of the Appendix [4].

5 From fragility to relaxations

We use the following result of Mayhew, Whittle, and Van Zwam [10, Lemma 8.2].

**Lemma 33.** Let $M$ be a 3-connected $U_{2,4}$-fragile matroid that has no $\{U_{2,6}, U_{4,6}\}$-minor. Then exactly one of the following holds.

(i) $M$ has rank or corank two;

(ii) $M$ has an $\{F_7^-, (F_7^-)^*\}$-minor;

(iii) $M$ has rank and corank at least 3 and is a whirl.

We show next that $P_8^-, F_7^-, (F_7^-)^*$ do not arise from circuit-hyperplane relaxation of a GF(4)-representable matroid.

**Lemma 34.** Let $M$ and $M'$ be GF(4)-representable matroids such that $M$ is connected, $M'$ is 3-connected, and $M'$ is obtained from $M$ by relaxing a circuit-hyperplane $X$. Then $M'$ has no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor.
Proof. Assume that $M'$ has a $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor. Since $M'$ is obtained from $M$ by relaxing $X$, it follows from Theorem 15 and Lemma 33 that $M'$ is $\{U_{2,5}, U_{3,5}\}$-fragile. Each of the matroids in $\{P_8^-, F_7^-, (F_7^-)^*\}$ has a $\{U_{2,5}, U_{3,5}\}$-minor, so if $C$ and $D$ are such that $M'/C \setminus D \cong N'$ for some $N' \in \{P_8^-, F_7^-, (F_7^-)^*\}$, then $C \subseteq X$ and $D \subseteq E(M') - X$ since the elements of $X$ are nondeletable and the elements of $E(M') - X$ are noncontractible by Theorem 15. But then it follows from Lemma 4 that $N'$ can be obtained from $M/C \setminus D$ by relaxing the circuit-hyperplane $X - C$. It follows that $M/C \setminus D \cong N$ for some $N \in \{P_8, F_7^-, (F_7^-)^*\}$, a contradiction because $M$ is GF(4)-representable.

We can now describe the structure of the GF(4)-representable matroids that are circuit-hyperplane relaxations of GF(4)-representable matroids.

**Theorem 35.** Let $M$ and $M'$ be GF(4)-representable matroids such that $M$ is connected, $M'$ is 3-connected, and $M'$ is obtained from $M$ by relaxing a circuit-hyperplane. Then at least one of the following holds.

(a) $M'$ is a whirl;

(b) $M' \in \{M_{9,9}, M_{9,10}\}$;

(c) $M'$ or $(M')^*$ can be obtained from $U_{2,5}$ (with groundset $\{a, b, c, d, e\}$) by gluing wheels to $(a, c, b), (a, d, b)$;

(d) $M'$ or $(M')^*$ can be obtained from $U_{2,5}$ (with groundset $\{a, b, c, d, e\}$) by gluing wheels to $(a, b, c), (c, d, e)$;

(e) $M'$ or $(M')^*$ can be obtained from $M_{7,1}$ by gluing a wheel to $(1, 3, 2)$;

(f) there is some path sequence that describes $M'$.

Proof. It follows from Theorem 15 that $M'$ is either $U_{2,4}$-fragile or $\{U_{2,5}, U_{3,5}\}$-fragile. If $M'$ is $U_{2,4}$-fragile, then it follows from Lemma 33 that $M'$ is a whirl. We may therefore assume that $M'$ is $\{U_{2,5}, U_{3,5}\}$-fragile. It follows from Lemma 34 that $M'$ has no $\{P_8^-, F_7^-, (F_7^-)^*\}$-minor. Then, by Theorem 32 and Theorem 15, one of (b) through (e) holds or else $M'$ has an $\{X_8, Y_8, Y_8^*\}$-minor. Note that outcome (iii) of Theorem 32 corresponds to outcome (c) here, since a matroid or its dual that is obtained from $U_{2,5}$ (with groundset $\{a, b, c, d, e\}$) by gluing wheels onto all three of the triangles $(a, c, b), (a, d, b), (a, e, b)$ does not have a basis of nondeletable elements and a cobasis of noncontractible elements, and therefore cannot be obtained by relaxing a circuit-hyperplane. We can see this by the following counting argument. Observe that the rank of a matroid obtained from $U_{2,5}$ (with groundset $\{a, b, c, d, e\}$) by gluing wheels $A, B$ and $C$ onto the triangles $(a, c, b), (a, d, b), (a, e, b)$ is $r(A) + r(B) + r(C) - 4$. But the nondeletable elements of this matroid are precisely the rim elements of the wheels of which there are $r(A) + r(B) + r(C) - 3$. Hence any cobasis must contain a nondeletable element $e$. Since this matroid has $M_{9,18}$ as a minor (see Appendix [4, Lemma 2]), $M$ has no essential elements, which implies that $e$ must be contractible.

Finally, if $M'$ has an $\{X_8, Y_8, Y_8^*\}$-minor, then it follows from Theorem 27 that (f) holds.

\[\square\]
We can now show that if \( M \) and \( M' \) are GF(4)-representable matroids such that \( M' \) is obtained from \( M \) by relaxing a circuit-hyperplane, then \( M' \) has path width 3.

**Proof of Theorem 1.** If \( M \) is not connected, then it follows from Lemma 11 that \( M' \) has path width 3. We may therefore assume that \( M \) is connected. Then, by Lemma 13, \( M' \) can be obtained from a matroid in Theorem 35 (a) - (f) by performing some, possibly empty, sequence of series or parallel extensions. The result now follows from the fact that all the matroids in Theorem 35 (a) - (f) have path width 3. \( \square \)

6 Forbidden submatrices

In this section, we will prove our second characterization, Theorem 2. Let \( M \) be a GF(4)-representable matroid with a circuit-hyperplane \( X \). Choose \( e \in X \) and \( f \in E - X \) such that \( B = (X - e) \cup f \) is a basis of \( M \). Then we can find a reduced GF(4)-representation of \( M \) in block form,

\[
C = \begin{bmatrix}
X - e \\
\omega \\
\end{bmatrix} \begin{bmatrix}
\omega & 1 \\
1 & \omega \\
\end{bmatrix}.
\]

Here \( A \) is an \((X - e) \times (E - X) \) matrix over GF(4), and we have scaled so that every non-zero entry in the row labelled by \( f \) and the column labelled by \( e \) is 1. We denote by \( A_{ij} \) the entry in row \( i \) and column \( j \) of \( A \).

Let \( M' \) be the matroid obtained from \( M \) by relaxing the circuit-hyperplane \( X \). If \( M' \) is GF(4)-representable, then we can find a reduced representation of \( M' \) in block form,

\[
C' = \begin{bmatrix}
X - e \\
\omega \\
\end{bmatrix} \begin{bmatrix}
\omega & 1 \\
1 & \omega \\
\end{bmatrix}.
\]

We have scaled the rows and columns of the matrix such that the entry in the row labelled by \( f \) and column labelled by \( e \) is \( \omega \in GF(4) - \{0, 1\} \), and every remaining entry in row \( e \) and column \( f \) is a 1.

We omit the straightforward proof of the following lemma.

**Lemma 36.** \( A_{ij} = 0 \) if and only if \( A'_{ij} = 0 \).

Next we show that the only non-zero entries of \( A' \) are 1 and \( \omega \).

**Lemma 37.** \( A'_{ij} \neq \omega + 1 \).

**Proof.** Suppose \( A'_{ij} = \omega + 1 \). Then \( C' \) has a submatrix

\[
C'[\{i, f\}, \{e, j\}] = \begin{bmatrix}
\omega + 1 \\
1 \\
\end{bmatrix}.
\]
which has determinant zero. Therefore $B \triangle \{e, f, i, j\}$ is not a basis of the matroid $M[I|C']$. But the corresponding submatrix of $C$ is

$$C[\{i, f\}, \{e, j\}] = \begin{bmatrix} j & e \\ i & 1 \\ f & 0 \end{bmatrix},$$

for some non-zero $x$. Since $C[\{i, f\}, \{e, j\}]$ has non-zero determinant, $B \triangle \{e, f, i, j\}$ is a basis of $M$, and hence of $M'$. Therefore $M' \neq M[I|C']$.

Lemma 38. $A_{ij} = A_{ik}$ if and only if $A'_{ij} = A'_{ik}$. Similarly, $A_{ij} = A_{kj}$ if and only if $A'_{ij} = A'_{kj}$.

Proof. We show that $A_{ij} = A_{ik}$ implies that $A'_{ij} = A'_{ik}$. The proof of the converse, and the proof of the second statement proceed by similar easy arguments. Suppose that $A_{ij} = A_{ik}$. Then $C$ has a submatrix

$$C[\{i, f\}, \{j, k\}] = \begin{bmatrix} i & f \\ j & x \\ 1 & 1 \end{bmatrix},$$

for some non-zero $x$. Since $C[\{i, f\}, \{j, k\}]$ has zero determinant, $B \triangle \{f, i, j, k\}$ is not a basis of $M$, and hence not a basis of $M' = M[I|C']$. Therefore $\det(C'[\{i, f\}, \{j, k\}]) = 0$, so it follows that $A'_{ij} = A'_{ik}$.

The following lemma on diagonal submatrices will be used frequently.

Lemma 39. Let

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \text{ and } \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

be corresponding submatrices of $A$ and $A'$ respectively, where $x, y, a, b$ are non-zero entries. Then $x = y$ if and only if $a \neq b$.

Proof. Adjoining $e$ and $f$ to the specified $2 \times 2$ submatrices, we get the $3 \times 3$ submatrices

$$\begin{bmatrix} x & 0 & 1 \\ 0 & y & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 1 & 1 & \omega \end{bmatrix}.$$  

These matrices have determinants $x + y$ and $ab\omega + a + b$. Thus if $x = y$, then $a \neq b$. Conversely, if $a \neq b$, then $\{a, b\} = \{1, \omega\}$ by Lemma 37 so $ab\omega + a + b = \omega^2 + \omega + 1 = 0$. Hence $x = y$.

We can now identify all of the forbidden submatrices. We use Lemma 38 to identify the first such matrix in the following lemma.
Lemma 40. Neither $A$ nor $A^T$ has a submatrix of the form
\[
\begin{bmatrix}
x & y & z
\end{bmatrix},
\]
where $x, y, z$ are distinct non-zero entries.

Proof. By Lemma 38, the corresponding submatrix of $A'$ must have the form
\[
\begin{bmatrix}
a & b & c
\end{bmatrix},
\]
where $a, b, c$ are distinct non-zero entries, which is a contradiction to Lemma 37.

We now use Lemma 38 and Lemma 39 to find several more forbidden submatrices.

Lemma 41. $A$ has no submatrices of the following forms, where $x, y, z$ are distinct non-zero entries.

\begin{itemize}
  \item[(i)] \begin{bmatrix} x & x & 0 \\
        x & 0 & x \\
        0 & x & 0 
\end{bmatrix};
  \item[(ii)] \begin{bmatrix} x & x & 0 \\
        x & 0 & y \\
        y & 0 & y 
\end{bmatrix};
  \item[(iii)] \begin{bmatrix} x & x & 0 \\
        y & 0 & y \\
        0 & 0 & y 
\end{bmatrix};
  \item[(iv)] \begin{bmatrix} x & y & 0 \\
        x & 0 & y \\
        x & 0 & y 
\end{bmatrix};
  \item[(v)] \begin{bmatrix} x & 0 & 0 \\
        0 & y & z \\
        0 & 0 & y 
\end{bmatrix};
  \item[(vi)] \begin{bmatrix} x & 0 & 0 \\
        0 & x & 0 \\
        0 & 0 & x 
\end{bmatrix};
  \item[(vii)] \begin{bmatrix} x & 0 & 0 \\
        0 & x & 0 \\
        0 & 0 & 0 
\end{bmatrix};
  \item[(viii)] \begin{bmatrix} x & 0 & 0 \\
        0 & x & 0 \\
        0 & 0 & z 
\end{bmatrix}.
\end{itemize}

Proof. Suppose $A$ has the submatrix (i). By applying Lemma 38 to the rows and the first column, we deduce that the corresponding submatrix of $A'$ has the form
\[
\begin{bmatrix}
a & a & 0 \\
        a & 0 & a 
\end{bmatrix},
\]
where $a$ is a non-zero entry, a contradiction of Lemma 39.

Suppose $A$ has the submatrix (ii). By applying Lemma 38 to the rows and the first column, and since $A'$ has at most two distinct non-zero entries by Lemma 37, we deduce that the corresponding submatrix of $A'$ has the form
\[
\begin{bmatrix}
a & a & 0 \\
        a & 0 & b 
\end{bmatrix},
\]
where $a$ and $b$ are the two non-zero entries of $A'$, a contradiction to Lemma 39.

The proofs for (iii) and (iv) are similar to that for (ii). We omit the details.

Suppose $A$ has the submatrix (v). Then, by two applications of Lemma 39, the corresponding submatrix of $A'$ must have the form
\[
\begin{bmatrix}
a & 0 & 0 \\
        0 & a & a 
\end{bmatrix},
\]
for some non-zero entry $a$. This is a contradiction to Lemma 38.
Suppose $A$ has the submatrix (vi). By Lemma 39, the corresponding submatrix of $A'$ must be a diagonal matrix with distinct non-zero entries, a contradiction to Lemma 37.

Suppose $A$ has the submatrix (vii). Applying Lemma 39 to the two submatrices the form
\[
\begin{bmatrix}
  x & 0 \\
  0 & y
\end{bmatrix},
\]
it follows that the corresponding submatrix of $A'$ is
\[
\begin{bmatrix}
  a & 0 & 0 \\
  0 & a & 0 \\
  0 & 0 & a
\end{bmatrix},
\]
for some $a$, which is a contradiction to Lemma 39.

Suppose $A$ has the submatrix (viii). Then the corresponding submatrix of $A'$ is
\[
\begin{bmatrix}
  a & 0 & 0 \\
  0 & a & 0 \\
  0 & 0 & a
\end{bmatrix},
\]
for some $a$. Adjoining $e$ and $f$, we have a submatrix of $C$,
\[
\begin{bmatrix}
  x & 0 & 0 & 1 \\
  0 & y & 0 & 1 \\
  0 & 0 & z & 1 \\
  1 & 1 & 1 & 0
\end{bmatrix},
\]
which has zero determinant, while the corresponding submatrix of $C'$,
\[
\begin{bmatrix}
  a & 0 & 0 & 1 \\
  0 & a & 0 & 1 \\
  0 & 0 & a & 1 \\
  1 & 1 & 1 & \omega
\end{bmatrix},
\]
has non-zero determinant, a contradiction. \qed

**Lemma 42.** $A$ has no submatrices of the following forms, where $x$, $y$, and $z$ are distinct non-zero entries:

(i) $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$; (ii) $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$; (iii) $\begin{bmatrix} x & x \\ y & z \end{bmatrix}$; (iv) $\begin{bmatrix} x & y \\ z & x \end{bmatrix}$; (v) $\begin{bmatrix} x & y & 0 \\ x & 0 & z \end{bmatrix}$.

**Proof.** Suppose $A$ has the submatrix (i). Then, adjoining $e$ and $f$, we see that $C$ has the following submatrix with non-zero determinant.
\[
\begin{bmatrix}
  x & y & 1 \\
  0 & x & 1 \\
  1 & 1 & 0
\end{bmatrix}.
\]
But then, by Lemma 38, the corresponding submatrix of $C'$ must have the following form:

$$
\begin{bmatrix}
a & b & 1 \\
0 & a & 1 \\
1 & 1 & \omega
\end{bmatrix},
$$

where $\{a,b\} = \{1,\omega\}$ by Lemma 37. This gives a contradiction because this submatrix of $C'$ has zero determinant. A similar proof handles (ii).

Suppose $A$ has the submatrix (iii). Then, by Lemma 38, in the corresponding submatrix of $A'$, the entries in the first row are the same and the entries in the second row are different. But, by Lemma 37, there are only two distinct non-zero entries in $A'$, so the entries are the same in one of the columns of $A'$, which is a contradiction to Lemma 38.

Suppose $A$ has the submatrix (iv). Note that this submatrix has zero determinant. By Lemma 38, the corresponding submatrix of $A'$ must have the following form:

$$
\begin{bmatrix}
a & b \\
b & a
\end{bmatrix},
$$

where $\{a,b\} = \{1,\omega\}$ by Lemma 37. But this submatrix of $A'$ has non-zero determinant, a contradiction.

Suppose $A$ has the submatrix (v). Then $C$ contains the following submatrix, which does not use its last column:

$$
\begin{bmatrix}
x & y & 0 \\
x & 0 & z \\
1 & 1 & 1
\end{bmatrix}.
$$

This matrix has determinant 0. By Lemmas 37, 38, and 39, the corresponding submatrix of $C'$ is

$$
\begin{bmatrix}
a & b & 0 \\
a & 0 & b \\
1 & 1 & 1
\end{bmatrix},
$$

where $\{a,b\} = \{1,\omega\}$. This matrix has non-zero determinant, a contradiction.

Finally, we find two more $3 \times 3$ forbidden submatrices of $A$.

**Lemma 43.** $A$ has no submatrices of the following forms, where $x, y,$ and $z$ are distinct non-zero entries:

(i) $\begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & 0 \end{bmatrix}$; (ii) $\begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & z \end{bmatrix}$.

**Proof.** Suppose that $A$ has the submatrix (i). Then, adjoining $e$ and $f$, we see that $C$ has the submatrix

$$
\begin{bmatrix}
x & y & x & 1 \\
y & y & 0 & 1 \\
x & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix},
$$
which has zero determinant. The corresponding submatrix of $C'$ is
\[
\begin{bmatrix}
a & b & a & 1 \\
b & b & 0 & 1 \\
a & 0 & 0 & 1 \\
1 & 1 & 1 & \omega \\
\end{bmatrix},
\]
for distinct $a, b \in \{1, \omega\}$. This submatrix of $C$ has non-zero determinant, a contradiction.

Suppose that $A$ has the submatrix (ii). Note that the determinant of this submatrix is not zero. By Lemma 37 and Lemma 38, the corresponding submatrix of $A'$ is
\[
\begin{bmatrix}
a & b & a \\
b & b & 0 \\
a & 0 & b \\
\end{bmatrix},
\]
for distinct $a, b \in \{1, \omega\}$. This submatrix of $A'$ has zero determinant, which is a contradiction. \hfill \Box

To prove the main theorem of this section, we need the following theorem [5, Theorem 5.1].

**Theorem 44.** Minor-minimal non-GF(4)-representable matroids have rank and corank at most 4.

We can now prove the main theorem, which we repeat for convenience.

**Theorem 45.** There is some matrix $C'$ representing $M'$ if and only if, up to permuting rows and columns, $A$ and $A^T$ have no submatrix in the following set, where $x, y, z$ are distinct non-zero elements of GF(4):

\[
\begin{bmatrix}
x & y & z \\
0 & x & y \\
\end{bmatrix}, \begin{bmatrix}
x & y & 0 \\
0 & y & z \\
\end{bmatrix}, \begin{bmatrix}
x & x & 0 \\
y & x & 0 \\
0 & y & z \\
\end{bmatrix}, \begin{bmatrix}
x & x & 0 \\
x & 0 & 0 \\
0 & 0 & x \\
\end{bmatrix}, \begin{bmatrix}
x & y & x \\
0 & 0 & y \\
x & 0 & 0 \\
\end{bmatrix}, \begin{bmatrix}
x & y & 0 \\
0 & y & 0 \\
x & 0 & 0 \\
\end{bmatrix}, \begin{bmatrix}
x & 0 & 0 \\
y & y & 0 \\
x & 0 & 0 \\
0 & z & x \\
\end{bmatrix}, \begin{bmatrix}
x & y & x \\
0 & 0 & z \\
\end{bmatrix}.
\]

**Proof.** It follows from Lemmas 40, 41, 42, and 43 that both $A$ and $A^T$ have no submatrix on the above list.

Conversely, suppose that the GF(4)-representable matroid $M$ is chosen to be minimal subject to the property that the relaxation $M'$ is not GF(4)-representable. Then $M'$ has a minor $N$ isomorphic to one of the excluded minors for the class of GF(4)-representable matroids. Assume that $N = M'/C\setminus D$ for some subsets $C$ and $D$. If there is an element
g in both D and the circuit-hyperplane X of M, then M\g = M'\g by Lemma 4, so M also has an N-minor, contradicting the fact that M is GF(4)-representable. We deduce that D ⊆ E(M) − X, and dually, C ⊆ X. Now if |D| ≥ 2, then there is some element g in both D and E(M') − (X ∪ f), so relaxing the circuit-hyperplane X of M\g gives M'\g that is not GF(4)-representable, which contradicts the minimality of M. Therefore |D| ≤ 1, and by a dual argument, there is no element g in both C and X − e, so |C| ≤ 1. Since we know, by Theorem 44, that |E(N)| ≤ 8, it now follows that |E(M')| ≤ 10. The computations in the Appendix [4] show that M' must have a submatrix from the above list.

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References


