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Abstract

We consider the GF(4)-representable matroids with a circuit-hyperplane such that the matroid obtained by relaxing the circuit-hyperplane is also GF(4)-representable. We characterize the structure of these matroids as an application of structure theorems for the classes of $U_{2,4}$-fragile and $\{U_{2,5}, U_{3,5}\}$-fragile matroids. In addition, we characterize the forbidden submatrices in GF(4)-representations of these matroids.

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1 Introduction

Lucas [9] determined the binary matroids that have a circuit-hyperplane whose relaxation yields another binary matroid. Truemper [16], and independently, Oxley and Whittle [13], did the same for ternary matroids. In this paper, we solve the corresponding problem for quaternary matroids. We give both a structural characterization and a characterization in terms of forbidden submatrices.

Truemper [16] used the structure of circuit-hyperplane relaxations of binary and ternary matroids to give new proofs of the excluded-minor characterizations for the classes of binary, ternary, and regular matroids. It is natural to ask if Truemper's techniques can be extended to give excluded-minor characterizations for classes of quaternary matroids. The main results of this paper can be viewed as a first step towards answering this question.

Our structural characterization can be summarized as follows. A matroid has *path width* 3 if there is an ordering $(e_1, e_2, \ldots, e_n)$ of its ground set such that $\{e_1, e_2, \ldots, e_t\}$ is a 3-separating set for all $t \in \{1, 2, \ldots, n\}$.

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Theorem 1. *Let M and M' be* GF(4)*-representable matroids such that M' is obtained from M by relaxing a circuit-hyperplane. Then M' has path width* 3.

In fact, our main result, Theorem 35, describes precisely how the matroids in Theorem 1 of path width 3 can be constructed using the *generalized Δ-Y exchange* of [12] and the notion of *gluing a wheel onto a triangle* from [2]. Our description uses the structure of $U_{2,4}$-fragile matroids from [10] and the structure of $\{U_{2,5}, U_{3,5}\}$-fragile matroids from [3].

In future work, we hope to obtain a description of these matroids that is independent of the notion of fragility. Specifically, we would like to characterize the representations of these matroids. As a step in this direction, we describe minimal GF(4)-representations of matroids with a circuit-hyperplane whose relaxation is not GF(4)-representable. Note that the proof uses the excluded-minor characterization of the class of GF(4)-representable matroids. The setup for this result is as follows.

Let M be a GF(4)-representable matroid on E with a circuit-hyperplane X. Choose $e \in X$ and $f \in E - X$ such that $(X - e) \cup f$ is a basis of M. Then $M = M[I|C]$ for a quaternary matrix C of the following block form.

$$C = \begin{array}{c c} & \begin{array}{cc} \scriptstyle (E-X)-f & \scriptstyle e \end{array} \\ \begin{array}{c} \scriptstyle X-e \\ \scriptstyle f \end{array} & \begin{bmatrix} A & \underline{1} \\ \underline{1}^T & 0 \end{bmatrix} \end{array}.$$

In the above matrix, A is an $(X - e) \times ((E - X) - f)$ matrix, and we have scaled so that every non-zero entry in the row labelled by f and the column labelled by e is 1. Let M' be the matroid obtained from M by relaxing the circuit-hyperplane X. We call the matrix C a *reduced representation* of M. If M' is GF(4)-representable, then we can find a reduced representation C' of M' in the following block form.

$$C' = \begin{array}{c c} & \begin{array}{cc} \scriptstyle (E-X)-f & \scriptstyle e \end{array} \\ \begin{array}{c} \scriptstyle X-e \\ \scriptstyle f \end{array} & \begin{bmatrix} A' & \underline{1} \\ \underline{1}^T & \omega \end{bmatrix} \end{array}.$$

We have scaled the rows and columns of the matrix such that the entry $C'_{fe} = \omega \in \mathrm{GF}(4) - \{0, 1\}$, and the remaining entries in row f and column e are all 1. The following theorem is our characterization in terms of forbidden submatrices.

Theorem 2. *Let M and C be constructed as described above. There is a reduced representation C' of the above form for M' if and only if, up to permuting rows and columns, A and A^T have no submatrix in the following list, where x, y, z denote distinct non-zero elements of* GF(4).

$$\begin{bmatrix} x & y & z \end{bmatrix}, \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}, \begin{bmatrix} x & y \\ y & x \end{bmatrix}, \begin{bmatrix} x & x \\ y & z \end{bmatrix}, \begin{bmatrix} x & y \\ z & x \end{bmatrix}, \begin{bmatrix} x & x & 0 \\ x & 0 & x \end{bmatrix}, \begin{bmatrix} x & x & 0 \\ x & 0 & y \end{bmatrix},$$

$$\begin{bmatrix} x & x & 0 \\ y & 0 & y \end{bmatrix}, \begin{bmatrix} x & y & 0 \\ x & 0 & y \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \end{bmatrix}, \begin{bmatrix} x & y & 0 \\ x & 0 & z \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix},$$

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & 0 \end{bmatrix}, \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & z \end{bmatrix}.$$

This paper is organized as follows. In the next section, we collect some results on connectivity and circuit-hyperplane relaxation. In Section 3, we prove a fragility theorem. In Section 4, we describe the structure of the $\{U_{2,5}, U_{3,5}\}$-fragile matroids. In Section 5, we prove the structural characterization. In Section 6, we reduce the proof of Theorem 2 to a finite computer check. This check, carried out using SageMath, can be found in the Appendix [4].

2 Circuit-hyperplane relaxations and connectivity

We assume the reader is familiar with the fundamentals of matroid theory. Any undefined matroid terminology will follow Oxley [11]. Let M be a matroid on E, and let $\mathcal{B}(M)$ denote the collection of bases of M. If M has a circuit-hyperplane X, then $\mathcal{B}(M') = \mathcal{B}(M) \cup \{X\}$ is the collection of bases of a matroid M' on E. We say that M' is obtained from M by *relaxing the circuit-hyperplane* X. We list here a number of useful results on circuit-hyperplane relaxation.

Lemma 3. *[11, Proposition 2.1.7] If M' is obtained from M by relaxing the circuit-hyperplane X of M, then $(M')^*$ is obtained from M^* by relaxing the circuit-hyperplane $E(M) - X$ of M^*.*

The following elementary results are originally from [8].

Lemma 4. *[11, Proposition 3.3.5] Let X be a circuit-hyperplane of a matroid M, and let M' be the matroid obtained from M by relaxing X. When $e \in E(M) - X$,*

(i) $M/e = M'/e$ and, unless M has e as a coloop, $M'\backslash e$ is obtained from $M\backslash e$ by relaxing the circuit-hyperplane X of the latter.

Dually, when $f \in X$,

(ii) $M\backslash f = M'\backslash f$ and, unless M has f as a loop, M'/f is obtained from M/f by relaxing the circuit-hyperplane $X - f$ of the latter.

For a set $\mathcal{N}$ of matroids, we say that a matroid M has an *$\mathcal{N}$-minor* if M has an N-minor for some $N \in \mathcal{N}$. We say M is *$\mathcal{N}$-fragile* if M has an $\mathcal{N}$-minor and, for each element e of M, at most one matroid in $\{M\backslash e, M/e\}$ has an $\mathcal{N}$-minor. We say an element e of an $\mathcal{N}$-fragile matroid M is *nondeletable* if $M\backslash e$ has no $\mathcal{N}$-minor; the element e is *noncontractible* if M/e has no $\mathcal{N}$-minor.

The following lemma is an immediate consequence of Lemma 4.

Lemma 5. *Let X be a circuit-hyperplane of a matroid M, and let M' be the matroid obtained from M by relaxing X. If $\mathcal{N}$ is a set of matroids such that M' has an $\mathcal{N}$-minor but M has no $\mathcal{N}$-minor, then M' is $\mathcal{N}$-fragile. Moreover, X is a basis of M' whose elements are nondeletable such that the elements of the cobasis $E(M') - X$ are noncontractible.*

We use the following connectivity result.

Lemma 6. *[11, Proposition 8.4.2] Let M' be a matroid that is obtained by relaxing a circuit-hyperplane of a matroid M. If M is n-connected, then M' is n-connected.*

Kahn [8] proved the following result on the representability of a circuit-hyperplane relaxation.

Lemma 7. *Let M' be a matroid that is obtained by relaxing a circuit-hyperplane of a matroid M. If M is connected, then M' is non-binary.*

We use the following definition of the rank function of the 2-sum from [7]. Let M_1 and M_2 be matroids with at least two elements such that $E(M_1) \cap E(M_2) = \{p\}$. Then $M = M_1 \oplus_2 M_2$ has rank function r_M defined for all $A_1 \subseteq E(M_1)$ and $A_2 \subseteq E(M_2)$ by

$$r_M(A_1 \cup A_2) = r_{M_1}(A_1) + r_{M_2}(A_2) - \theta(A_1, A_2) + \theta(\emptyset, \emptyset)$$

where $\theta(X, Y) = 1$ if $r_{M_1}(X \cup p) = r_{M_1}(X)$ and $r_{M_2}(Y \cup p) = r_{M_2}(Y)$, and $\theta(X, Y) = 0$ otherwise.

The next three results on 2-sums and minors of 2-sums are well known.

Lemma 8. *[11, Proposition 7.1.20] Let M and N be matroids with at least two elements. Let $E(M) \cap E(N) = \{p\}$ and suppose that neither M nor N has $\{p\}$ as a separator. The set of circuits of $M \oplus_2 N$ is*

$$\mathcal{C}(M\backslash p) \cup \mathcal{C}(N\backslash p) \cup \{(C \cup D) - p : p \in C \in \mathcal{C}(M) \text{ and } p \in D \in \mathcal{C}(N)\}.$$

Lemma 9. *[11, Theorem 8.3.1] A connected matroid M is not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2, each of which has at least three elements and is isomorphic to a proper minor of M.*

Lemma 10. *[11, Proposition 8.3.5] Let M, N, M_1, M_2 be matroids such that $M = M_1 \oplus_2 M_2$ and N is 3-connected. If M has an N-minor, then M_1 or M_2 has an N-minor.*

We can now describe the structure of circuit-hyperplanes in matroids of low connectivity. We omit the straightforward proof of the next lemma.

Lemma 11. *Let M be a* GF(4)*-representable matroid with a circuit-hyperplane H. If M is not connected, then $M \cong U_{1,m} \oplus U_{n-1,n}$ for some positive integers m and n.*

We now work towards a description of the 2-separations of a connected matroid in which the relaxation of some circuit-hyperplane is GF(4)-representable.

Lemma 12. *Let M be a matroid with a circuit-hyperplane X. If A is a non-trivial parallel class of M, then either $A \subseteq E - X$, or $A = X$ and $|A| = 2$.*

Proof. If $A \cap X$ and $A \cap (E - X)$ are both non-empty, then there is a circuit $\{x, y\}$ contained in A such that $x \in X$ and $y \in E - X$. But $E - X$ is a cocircuit of M, so this is a contradiction to orthogonality. Thus either $A \cap X$ or $A \cap (E - X)$ is empty. In the case that $A \cap (E - X)$ is empty, there is a circuit $\{x, y\}$ contained in A that is also contained in the circuit X, so $X = A = \{x, y\}$. $\square$

For the next result, we say that M is 3-*connected up to series and parallel classes* if M is connected and, for any 2-separation (X, Y) of M, either X or Y is a series class or a parallel class.

Lemma 13. *Let M be a* GF(4)*-representable matroid with a circuit-hyperplane X such that the matroid M' obtained from M is also* GF(4)*-representable. If M is connected but not* 3*-connected, then M is* 3*-connected up to series and parallel classes.*

Proof. Assume that M has a 2-separation (S, T) where neither side is a series or parallel class. Then M has a 2-sum decomposition of the form $M = N \oplus_2 N'$ for some N and N' with $E(N) \cap E(N') = \{p\}$, where neither N nor N' is a circuit or cocircuit.

First suppose that the circuit X of M has the form $(C \cup C') - p$, where C is a circuit of N, and C' is a circuit of N' while $p \in C \cap C'$. Then

$$r(X) = r(M) - 1, \tag{1}$$

$$r(N) + r(N') - 1 = r(M), \tag{2}$$

and

$$r_M(X) = r_N(C) + r_{N'}(C') - 1. \tag{3}$$

Equation (1) follows from the fact that X is a hyperplane of M; Equations (2) and (3) follow from the definition of the rank function of the 2-sum of N and N'. Combining (1) and (2), we see that $r(X) = r(N) + r(N') - 2$. Then combining this equation with (3), we see that

$$r(C) + r(C') = r(N) + r(N') - 1.$$

We may therefore assume that C is a spanning circuit of N, and hence that $E(N) = C$ because the hyperplane X is closed. Therefore N is a circuit, a contradiction.

By symmetry, it remains to consider the case when X is a circuit of $N' \backslash p$. Then $r(X) \leqslant r(N')$. Since X is a hyperplane of M, and $r(M) = r(N) + r(N') - 1$, it follows that $r(N) \leqslant 2$. Since N is not a cocircuit, we deduce that $r(N) = 2$. Then $r(M) = r(N') + 1$, so $r(X) = r(N') = r(N' \backslash p)$. Since N is not a circuit we deduce that $\mathrm{si}(N) \cong U_{2,m}$ for some $m \geqslant 4$. Moreover, p is not in a non-trivial parallel class in N otherwise X is not a hyperplane of M.

Switching to M^*, we see that $r_{M^*}(N') = |X| + r(N) - r(M) = r(N) = 2$. As above, it follows that $\mathrm{co}(N') \cong U_{n-1,n+1}$ for some $n \geqslant 3$. Moreover, p is not in a non-trivial series class in N'. Let X_1 consist of one representative of each series class of N', and let

Y_1 consist of one representative of each parallel class of N. By contracting elements of $X - X_1$ and deleting elements of $(E(M) - X) - Y_1$, we obtain $U_{n-1,n+1} \oplus_2 U_{2,m}$ as a minor of M for some $n \geqslant 3$ and $m \geqslant 4$. Moreover, by Lemma 4, X_1 is a circuit-hyperplane of this minor whose relaxation is GF(4)-representable. Thus $X_1 \subseteq E(U_{n-1,n+1})$. Contract $n-3$ elements from X_1 and delete $m-4$ elements from Y_1 to get $U_{2,4} \oplus_2 U_{2,4}$. Relaxing a circuit-hyperplane of this minor gives P_6 which is not GF(4)-representable (see [11, Proposition 6.5.8]), a contradiction. $\square$

3 A fragility theorem

We will use the following consequence of Geelen, Oxley, Vertigan, and Whittle [6, Theorem 8.4].

Theorem 14. *Let M and M' be* GF(4)*-representable matroids with the properties that M is connected, M' is* 3*-connected, and M' is obtained from M by relaxing a circuit-hyperplane.*

(i) If M' has a $U_{2,4}$-minor but no $\{U_{2,5}, U_{3,5}\}$-minor, then M is binary.

(ii) If M' has a $\{U_{2,5}, U_{3,5}\}$-minor but no $U_{3,6}$-minor, then M has no $\{U_{2,5}, U_{3,5}\}$-minor.

We can now prove the main result of this section.

Theorem 15. *Let M and M' be* GF(4)*-representable matroids such that M is connected, M' is* 3*-connected, and M' is obtained from M by relaxing a circuit-hyperplane X. Then M' is either $U_{2,4}$-fragile or $\{U_{2,5}, U_{3,5}\}$-fragile. Moreover, X is a basis of M' whose elements are nondeletable such that the elements of the cobasis $E(M') - X$ are noncontractible.*

Proof. First assume that M' has no $\{U_{2,5}, U_{3,5}\}$-minor. By Lemma 7 and Theorem 14 (i), M' has a $U_{2,4}$-minor and M has no $U_{2,4}$-minor. Then it follows from Lemma 5 that M' is $U_{2,4}$-fragile, and M' has a basis X whose elements are nondeletable such that the elements of the cobasis $E(M') - X$ are noncontractible.

We may now assume that M' has a $\{U_{2,5}, U_{3,5}\}$-minor. Suppose that M also has a $\{U_{2,5}, U_{3,5}\}$-minor, and assume that M is a minor-minimal matroid with respect to the hypotheses; that is, we assume that M has no proper minor M_0 such that M_0 is connected, M_0 has a $\{U_{2,5}, U_{3,5}\}$-minor, and M_0 has a circuit-hyperplane whose relaxation M_0' is 3-connected, GF(4)-representable, and has a $\{U_{2,5}, U_{3,5}\}$-minor.

Claim 16. *M is $\{U_{2,5}, U_{3,5}\}$-fragile.*

Proof of 16. Suppose that M has an element $e \in E(M) - X$ such that $M \backslash e$ has a $\{U_{2,5}, U_{3,5}\}$-minor. If $M \backslash e$ is 3-connected, then we have a contradiction to the minimality of M. Therefore, by Lemma 13, $M \backslash e$ is 3-connected up to series and parallel pairs. Suppose that A is a non-trivial parallel class of $M \backslash e$. Suppose $A \subseteq X$. Then $A = X$ and $|A| = 2$ by Lemma 12, so we deduce that $M \backslash e$ is a parallel extension of $U_{2,5}$ and hence that $M'\backslash e$ has a $U_{2,6}$-minor, a contradiction to the fact that the matroid M' obtained from M by relaxing X is GF(4)-representable. Thus $A \subseteq E(M\backslash e) - X$ by Lemma 12. By duality, any non-trivial series class of $M\backslash e$ must be contained in X. Then, by Lemma 10, the matroid M_0 obtained from $M\backslash e$ by deleting all but one element of every non-trivial parallel class and contracting all but one element of every non-trivial series class has a $\{U_{2,5}, U_{3,5}\}$-minor. We deduce from Lemma 13 that M_0 is 3-connected. Then M_0 contradicts the minimality of M. Therefore $M\backslash e$ has no $\{U_{2,5}, U_{3,5}\}$-minor for all $e \in E(M) - X$, and, by duality, M/e has no $\{U_{2,5}, U_{3,5}\}$-minor for all $e \in X$, so M is $\{U_{2,5}, U_{3,5}\}$-fragile. This completes the proof of 16.

Since M has a $\{U_{2,5}, U_{3,5}\}$-minor, it follows from Theorem 14 (ii) that M' has a $U_{3,6}$-minor, that is, $M'/C\backslash D \cong U_{3,6}$ for some subsets C and D. If $C \subseteq X$ and $D \subseteq E(M') - X$, then it follows from Lemma 4 that $U_{3,6}$ can be obtained from $M/C\backslash D$ by relaxing the circuit-hyperplane $X - C$. Hence $M/C\backslash D \cong P_6$, a contradiction because $M/C\backslash D$ is GF(4)-representable but P_6 is not. Therefore $C \cap (E(M') - X)$ or $D \cap X$ is nonempty, so $M/C\backslash D = M'/C\backslash D \cong U_{3,6}$ by Lemma 4. This is a contradiction to 16 because any minor of M must also be $\{U_{2,5}, U_{3,5}\}$-fragile, but for any e, both $U_{3,6}\backslash e$ and $U_{3,6}/e$ have a $\{U_{2,5}, U_{3,5}\}$-minor. We conclude that M has no $\{U_{2,5}, U_{3,5}\}$-minor. It now follows from Lemma 5 that M' is $\{U_{2,5}, U_{3,5}\}$-fragile, and that M' has a basis X whose elements are nondeletable such that the elements of the cobasis $E(M') - X$ are noncontractible. $\square$

4 The structure of $\{U_{2,5}, U_{3,5}\}$-fragile matroids

4.1 Partial Fields and Constructions

We briefly state the necessary material on partial fields. For a more thorough treatment, we refer the reader to [14].

A *partial field* is a pair $\mathbb{P} = (R, G)$, where R is a commutative ring with unity, and G is a subgroup of the units of R with $-1 \in G$. A matrix with entries in G is a $\mathbb{P}$-*matrix* if $\det(D) \in G \cup \{0\}$ for any square submatrix D of A. We use $\langle X \rangle$ to denote the multiplicative subgroup of R generated by the subset X.

A rank-r matroid M on the ground set E is $\mathbb{P}$-*representable* if there is an $r \times |E|$ $\mathbb{P}$-matrix A such that, for each $r \times r$ submatrix D, the determinant of D is nonzero if and only if the corresponding subset of E is a basis of M. When this occurs, we write $M = M[A]$.

The 2-*regular* partial field is defined as follows.

$$\mathbb{U}_2 = (\mathbb{Q}(\alpha, \beta), \langle -1, \alpha, \beta, 1-\alpha, 1-\beta, \alpha-\beta \rangle),$$

where α,β are indeterminates.

It is well-known that any $\mathbb{U}_2$-representable matroid is GF(4)-representable [12]. On the other hand, there are GF(4)-representable matroids that are not $\mathbb{U}_2$-representable. We now define three such matroids. The matroid P_8 has a unique pair of disjoint circuit-hyperplanes; we let P_8^- denote the unique matroid obtained by relaxing one of these circuit-hyperplanes. We denote by $F_7^{=}$ the matroid obtained from the non-Fano matroid F_7^- by relaxing a circuit-hyperplane. The GF(4)-representable matroids P_8^-,$F_7^{=}$, $(F_7^{=})^*$ are not $\mathbb{U}_2$-representable. We note that this can be deduced from [1] since P_8^-,$F_7^{=}$, $(F_7^{=})^*$ are $\{U_{2,5}, U_{3,5}\}$-fragile matroids. Since these matroids are not $\mathbb{U}_2$-representable, we have the following lemma.

Lemma 17. *The class of* $\mathbb{U}_2$*-representable matroids is contained in the class of* GF(4)*-representable matroids with no* $\{P_8^-, F_7^{=}, (F_7^{=})^*\}$*-minor.*

To describe the structure of $\{U_{2,5}, U_{3,5}\}$-fragile matroids as in [3], we need two constructions: the generalized Δ-Y exchange, and gluing on wheels. For a more thorough treatment of these constructions, we refer the reader to [12] and [2].

Loosely speaking, the operations of generalized Δ-Y exchange and gluing on wheels both involve gluing matroids together along a common restriction. Let M_1 and M_2 be matroids with a common restriction A, where A is a modular flat of M_1. The *generalized parallel connection* of M_1 and M_2 along A, denoted $P_A(M_1, M_2)$, is the matroid obtained by gluing M_1 and M_2 along A. It has ground set $E(M_1) \cup E(M_2)$, and a set F is a flat of $P_A(M_1, M_2)$ if and only if $F \cap E(M_i)$ is a flat of M_i for each i (see [11, Section 11.4]).

A subset S of $E(M)$ is a *segment* of M if every three-element subset of S is a triangle of M. Let M be a matroid with a k-element segment A. Intuitively, a generalized Δ-Y exchange on A turns the segment A into a k-element cosegment. To define the generalized Δ-Y exchange formally, we first recall the following definition of a family of matroids Θ_k from [12]. For $k \geqslant 3$, fix a basis $B = \{b_1, b_2, \dots, b_k\}$ of the rank-k projective geometry $PG(k-1, \mathbb{R})$, and choose a line L of $PG(k-1, \mathbb{R})$ that is freely placed relative to B. If follows from modularity that, for each i, the hyperplane spanned by $B - \{b_i\}$ meets L; we let a_i be the point of intersection. Let $A = \{a_1, a_2, \dots, a_k\}$, and let Θ_k be the matroid obtained by restricting $PG(k-1, \mathbb{R})$ to the set $A \cup B$. Note that the matroid Θ_k has A as a modular k-point segment A, so the generalized parallel connection of Θ_k and M along A is well-defined. If the k-element segment A is coindependent in M, then we define the matroid $\Delta_A(M)$ to be the matroid obtained from $P_A(\Theta_k, M)\backslash A$ by relabeling the elements of $E(\Theta_k) - A$ by A in the natural way, and we say that $\Delta_A(M)$ is obtained from M by performing a *generalized* Δ-Y *exchange* on A. For a matroid M with an independent cosegment A, a *generalized* Y-Δ *exchange on* A, denoted by $\nabla_A(M)$, is defined to be the matroid $(\Delta_A(M^*))^*$.

We use the following results on representability and the minor operations.

Lemma 18. *[12, Lemma 3.7] Let $\mathbb{P}$ be a partial field. Then M is $\mathbb{P}$-representable if and only if $\Delta_A(M)$ is $\mathbb{P}$-representable.*

Lemma 19. *[12, Lemma 2.13] Suppose that $\Delta_A(M)$ is defined. If $x \in A$ and $|A| \geqslant 3$, then $\Delta_{A-x}(M\backslash x)$ is also defined, and $\Delta_A(M)/x = \Delta_{A-x}(M\backslash x)$.*

Lemma 20. *[12, Lemma 2.16] Suppose that $\Delta_A(M)$ is defined.*

(i) If $x \in E(M) - A$ and A is coindependent in $M\backslash x$, then $\Delta_A(M\backslash x)$ is defined and $\Delta_A(M)\backslash x = \Delta_A(M\backslash x)$.

(ii) If $x \in E(M) - \mathrm{cl}(A)$, then $\Delta_A(M/x)$ is defined and $\Delta_A(M)/x = \Delta_A(M/x)$.

Lemma 21. *[12, Lemma 2.15] Suppose that $x \in \mathrm{cl}(A) - A$ and let a be an arbitrary element of the k-element segment A. Then $\Delta_A(M)/x$ equals the 2-sum, with basepoint p, of a copy of $U_{k-1,k+1}$ with groundset $A \cup p$ and the matroid obtained from $M/x\backslash(A - a)$ by relabeling a as p.*

The next result implies that every $\{U_{2,5}, U_{3,5}\}$-fragile matroid is 3-connected up to series and parallel classes.

Lemma 22. *[10, Proposition 4.3] Let M be a matroid with a 2-separation (A, B), and let N be a 3-connected minor of M. Assume $|E(N)\cap A| \geqslant |E(N)\cap B|$. Then $|E(N)\cap B| \leqslant 1$. Moreover, unless B is a parallel or series class, there is an element $x \in B$ such that both $M\backslash x$ and M/x have a minor isomorphic to N.*

The following is an easy consequence of the property that $\{U_{2,5}, U_{3,5}\}$-fragile matroids are 3-connected up to parallel and series classes.

Lemma 23. *Let M be a $\{U_{2,5}, U_{3,5}\}$-fragile matroid with at least 8 elements. If S is a triangle or 4-element segment of M such that $E(M) - S$ is not a series or parallel class of M, then S is coindependent in M. If C is a triad or 4-element cosegment of M such that $E(M) - C$ is not a series or parallel class of M, then C is independent.*

Let M be a $\{U_{2,5}, U_{3,5}\}$-fragile matroid. A segment S of M is *allowable* if S is coindependent and some element of S is nondeletable. A cosegment C of M is *allowable* if the segment C of M^* is allowable. In [3], it was shown that we can obtain a new $\{U_{2,5}, U_{3,5}\}$-fragile $\mathbb{U}_2$-representable matroid from an old $\{U_{2,5}, U_{3,5}\}$-fragile $\mathbb{U}_2$-representable matroid by performing a generalized Δ-Y exchange on an allowable segment. We will prove an analogous result for $\{U_{2,5}, U_{3,5}\}$-fragile GF(4)-representable matroids with no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor.

Let $\mathcal{U}$ be the class of GF(4)-representable matroids with no $\{U_{2,5}, U_{3,5}\}$-minor. The class of *sixth-root-of-unity* matroids is the class of matroids that are representable over both GF(3) and GF(4). Semple and Whittle [15, Theorem 5.2] showed that $\mathcal{U}$ is the class of matroids that can be obtained by taking direct sums and 2-sums of binary and sixth-root-of-unity matroids.

Lemma 24. *Let M be a matroid in the class $\mathcal{U}$. If M' is obtained from M by performing a generalized Δ-Y exchange or a generalized Y-Δ exchange, then $M' \in \mathcal{U}$.*

Proof. Suppose that there exists a matroid $M \in \mathcal{U}$ with a coindependent segment A such that $\Delta_A(M) \notin \mathcal{U}$. Among all counterexamples, suppose that M has been chosen so that $|E(M)|$ is as small as possible. Suppose M is 3-connected. Since any 3-connected member of $\mathcal{U}$ is either a binary or sixth-root-of-unity matroid, this also holds for $\Delta_A(M)$ by Lemma 18. Hence $\Delta_A(M) \in \mathcal{U}$, contradicting the assumption that M is a counterexample. Therefore M is not 3-connected.

Now either $M = M_1 \oplus M_2$ or $M = M_1 \oplus_2 M_2$ for some $M_1, M_2 \in \mathcal{U}$ with $|E(M_i)| < |E(M)|$ for each $i \in \{1, 2\}$. Moreover, we may assume that M_1 and M_2 have been chosen so that the segment A of M is contained in $E(M_1)$. Now either $\Delta_A(M) = \Delta_A(M_1) \oplus M_2$ or $\Delta_A(M) = \Delta_A(M_1) \oplus_2 M_2$. Since $|E(M_1)| < |E(M)|$, it follows that $\Delta_A(M_1) \in \mathcal{U}$. Hence $\Delta_A(M) \in \mathcal{U}$. Since $\mathcal{U}$ is closed under duality, the result follows. $\square$

Lemma 25. *Let M be a $\{U_{2,5}, U_{3,5}\}$-fragile* GF(4)*-representable matroid with no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. If A is an allowable segment of M with $|A| \in \{3, 4\}$, then $\Delta_A(M)$ is a $\{U_{2,5}, U_{3,5}\}$-fragile* GF(4)*-representable matroid with no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. Moreover, A is an allowable cosegment of $\Delta_A(M)$.*

Proof. The proof that $\Delta_A(M)$ is a $\{U_{2,5}, U_{3,5}\}$-fragile GF(4)-representable matroid where A is an allowable cosegment of $\Delta_A(M)$ closely follows the proof of [3, Lemma 2.21]. The only difference is where the proof of [3, Lemma 2.21] uses the fact that a $\mathbb{U}_2$-representable matroid with no $\{U_{2,5}, U_{3,5}\}$-minor is near-regular and the class of near-regular matroids is closed under the generalized Δ-Y exchange, we instead use Lemma 24.

We must also show that $\Delta_A(M)$ has no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. This follows for $|E(M)| \leqslant 9$ from the generation of the 3-connected $\{U_{2,5}, U_{3,5}\}$-fragile GF(4)-representable matroids with no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor on at most 9 elements (see the Appendix [4]), since all such matroids are $\mathbb{U}_2$-representable. Suppose that M is a minimum-sized counterexample, so $\Delta_A(M)$ has a $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor and $\Delta_A(M)$ has at least ten elements. Then $\Delta_A(M)$ has a minor N, obtained by deleting or contracting an element x say, that also has a $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. Since $\Delta_A(M)$ is $\{U_{2,5}, U_{3,5}\}$-fragile it follows that the minor N is also $\{U_{2,5}, U_{3,5}\}$-fragile. Suppose that $N = \Delta_A(M)/x$. Suppose that $x \in A$. Then $\Delta_A(M)/x = \Delta_{A-x}(M\backslash x)$ by Lemma 19, a contradiction since M is a minimum-sized counterexample. Next suppose that $x \in \mathrm{cl}(A) - A$. Since N is $\{U_{2,5}, U_{3,5}\}$-fragile it follows from Lemma 21 and Proposition 22 that $|A| = 4$ and $M/x\backslash(A-a) \cong U_{1,n}$ for some $n \geqslant 2$. Hence $\Delta_A(M)$ has no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor, a contradiction. We may now assume $x \in E(M) - \mathrm{cl}(A)$. Then $\Delta_A(M)/x = \Delta_A(M/x)$ by Lemma 20, a contradiction since M is a minimum-sized counterexample. We deduce that $N = \Delta_A(M)\backslash x$, and we may assume that any minor obtained from $\Delta_A(M)$ by contracting an element has no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. Now if $x \in A$, then $A - x$ is a series class of $\Delta_A(M)\backslash x$, so there is some $y \in A$ such that $\Delta_A(M)/y$ has a $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor, a contradiction. Therefore $x \notin A$. If A is not coindependent in $M\backslash x$, then it follows from Lemma 23 that $\Delta_A(M)$ has no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor, a contradiction. Therefore A is coindependent in $M\backslash x$, so $\Delta_A(M)\backslash x = \Delta_A(M\backslash x)$ by Lemma 20, a contradiction since M is a minimum-sized counterexample. $\square$

Let M be a matroid, and (a, b, c) an ordered subset of $E(M)$ such that $T = \{a, b, c\}$ is a triangle. Let $r \geqslant 3$ be a positive integer, and, when $r = 3$, we fix a vertex of $\mathcal{W}_3$ to be the center, so we can refer to rim and spoke elements of $M(\mathcal{W}_3)$. Let N be obtained from $M(\mathcal{W}_r)$ by relabeling some triangle as $\{a, b, c\}$, where a, c are spoke elements, and let $X \subseteq \{a, b, c\}$ such that $b \in X$. We say the matroid $M' := P_T(M, N)\backslash X$ is obtained from M by *gluing an r-wheel onto* (a, b, c). We also say that M^* is obtained from N^* by gluing a wheel onto the triad T. Suppose that $T_1, T_2, \ldots, T_n$ are ordered triples whose underlying sets are triangles of M. We say M' can be obtained from M by *gluing wheels* *onto* $T_1, T_2, \ldots, T_n$ if, for some subset J of $\{1, 2, \ldots, n\}$, M' can be obtained from M by a sequence of moves, where each move consists of gluing an r_j-wheel onto T_j for $j \in J$. Note that the spoke elements of a triangle in this sequence may only be deleted as part of the gluing operation when they do not appear in any subsequent triangle in the sequence.

Lemma 26. *Let M be a $\{U_{2,5}, U_{3,5}\}$-fragile* GF(4)*-representable matroid with no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. Let $A = \{a, b, c\}$ be an allowable triangle of M, where b is nondeletable. If M' is obtained from M by gluing an r-wheel onto (a, b, c), where $X \subseteq \{a, b, c\}$ is such that $b \in X$, then M' is a $\{U_{2,5}, U_{3,5}\}$-fragile* GF(4)*-representable matroid with no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. Moreover, $F = E(\mathcal{W}_r) - X$ is the set of elements of a fan, the spoke elements of F are noncontractible in M', and the rim elements of F are nondeletable in M'.*

Proof. The proof is the same as [3, Lemma 2.22] except that we use Lemma 25 instead of [3, Lemma 2.21]. □

4.2 Path sequences

We can now describe a family of $\{U_{2,5}, U_{3,5}\}$-fragile GF(4)-representable matroids with no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor obtained by performing generalized Δ-Y exchanges and gluing on wheels. In fact, the matroids in this family are $\mathbb{U}_2$-representable and were first described in [3]. Each matroid in this family has a $\{X_8, Y_8, Y_8^*\}$-minor, and an associated path of 3-separations that we need to describe in order to define the family.

We call the set $X \subseteq E(M)$ *fully closed* if X is closed in both M^* and M. The *full closure* of X, denoted $\mathrm{fcl}_M(X)$, is the intersection of all fully closed sets containing X. The full closure of X can be obtained from X by repeatedly taking closure and coclosure until no new elements are added. We call X a *path-generating* set if X is a 3-separating set of M such that $\mathrm{fcl}_M(X) = E(M)$. A path-generating set X thus gives rise to a natural path of 3-separating sets $(P_1, \ldots, P_m)$, where $P_1 = X$ and each step P_i is either the closure or coclosure of the 3-separating set $P_1 \cup \cdots \cup P_{i-1}$.

Let X be an allowable cosegment of the $\{U_{2,5}, U_{3,5}\}$-fragile matroid M. A matroid Q is an *allowable series extension of M along X* if $M = Q/Z$ and, for every element z of Z, there is some element x of X such that x is $\{U_{2,5}, U_{3,5}\}$-contractible in M and z is in series with x in Q. We also say that Q^* is an *allowable parallel extension of M^* along X*.

Let N be a matroid with a path-generating allowable segment or cosegment A. We say that M is obtained from N by a Δ-∇-*step* along A if, up to duality, M is obtained from N by performing a non-empty allowable parallel extension along A, followed by a generalized Δ-Y exchange on A.

Let X_8 be the matroid obtained from $U_{2,5}$ by choosing a 4-element segment C, adding a point in parallel with each of three distinct points of C, then performing a generalized Δ-Y-exchange on C (see Figure 1). In what follows, S will be the elements of the 4-element segment of X_8, and C the elements of the 4-element cosegment of X_8, so $E(X_8) = S \cup C$. We will build matroids from X_8 by performing a sequence of Δ-∇-steps along $A \in \{S, C\}$. Note that, in such matroids, each of S and C can be either a segment or a cosegment.

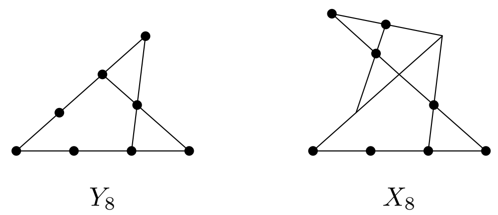

Figure 1: The matroids Y_8 and X_8.

A sequence of matroids $M_1, \ldots, M_n$ is called a *path sequence* if the following conditions hold:

(P1) $M_1 = X_8$; and

(P2) For each $i \in \{1, \ldots, n-1\}$, there is some 4-element path-generating segment or cosegment $A \in \{S, C\}$ of M_i such that either:

  (a) M_{i+1} is obtained from M_i by a Δ-∇-step along A; or

  (b) M_{i+1} is obtained from M_i by gluing a wheel onto an allowable subset A' of A.

Note in (P2) that each Δ-∇-step described in (a) increases the number of elements by at least one, and that the wheels in (b) are only glued onto allowable subsets of 4-element segments or cosegments.

We say that a path sequence $M_1, \ldots, M_n$ *describes* a matroid M if $M_n \cong M$. We also say that M is a matroid *described by* a path sequence if there is some path sequence that describes M. Let $\mathcal{P}$ denote the class of matroids such that $M \in \mathcal{P}$ if and only if there is some path sequence $M_1, \ldots, M_n$ that describes a matroid M' such that M can be obtained from M' by some, possibly empty, sequence of allowable parallel and series extensions. Since X_8 is self-dual, it is easy to see that the sequence of dual matroids $M_1^*, \ldots, M_n^*$ of a path sequence $M_1, \ldots, M_n$ is also a path sequence. Thus the class $\mathcal{P}$ is closed under duality.

We denote by Y_8 the unique matroid obtained from X_8 by performing a Y-Δ-exchange on an allowable triad (see Figure 1). We will prove the following result.

Theorem 27. *If M is a 3-connected $\{U_{2,5}, U_{3,5}\}$-fragile* GF(4)*-representable matroid that has an $\{X_8, Y_8, Y_8^*\}$-minor but no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor, then there is some path sequence that describes M.*

The proof of Theorem 27 closely follows the proof of [3, Corollary 4.3]. The strategy is to show that a minor-minimal counterexample has at most 12 elements. Let M be a GF(4)-representable $\{U_{2,5}, U_{3,5}\}$-fragile matroid M with an $\{X_8, Y_8, Y_8^*\}$-minor but no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. Suppose that M is a minimum-sized matroid that is not in the class $\mathcal{P}$. Then M is 3-connected because $\mathcal{P}$ is closed under series and parallel extensions. Moreover, the dual M^* is also not in $\mathcal{P}$ because $\mathcal{P}$ is closed under duality. Thus, by the Splitter Theorem and duality, we may assume there is some element x of M such that $M\backslash x$

is also a 3-connected GF(4)-representable $\{U_{2,5}, U_{3,5}\}$-fragile matroid with an $\{X_8, Y_8, Y_8^*\}$-minor but no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. By the assumption that M is minimum-sized with respect to being outside the class $\mathcal{P}$, it follows that $M\backslash x \in \mathcal{P}$. Thus $M\backslash x$ is described by a path sequence $M_1, \dots, M_n$. The next lemma [3, Lemma 6.3] identifies the three possibilities for the position of x in M relative to the path of 3-separations associated with $M_1, \dots, M_n$.

Lemma 28. *Let M and $M\backslash x$ be 3-connected $\{U_{2,5}, U_{3,5}\}$-fragile matroids. If $M\backslash x$ is described by a path sequence with associated path of 3-separations $\mathbf{P}$, then either:*

(i) there is some 3-separation (X, Y) displayed by $\mathbf{P}$ such that $x \in \mathrm{cl}(X)$ and $x \in \mathrm{cl}(Y)$; or

(ii) there is some 3-separation (X, Y) displayed by $\mathbf{P}$ such that $x \notin \mathrm{cl}(X)$ and $x \notin \mathrm{cl}(Y)$; or

(iii) for each 3-separation (R, G) of M displayed by $\mathbf{P}$, there is some $X \in \{R, G\}$ such that $x \in \mathrm{cl}_M(X)$ and $x \in \mathrm{cl}^*_M(X)$.

The proofs of the next three lemmas follow the proofs of [3, Lemma 7.4], [3, Lemma 8.7], and [3, Lemma 9.7] but use Lemma 25 above instead of [3, Lemma 2.21].

Lemma 29. *Lemma 28 (i) does not hold.*

Lemma 30. *If Lemma 28 (ii) holds, then $|E(M\backslash x)| \leqslant 10$.*

Lemma 31. *If Lemma 28 (iii) holds, then $|E(M\backslash x)| \leqslant 11$.*

Proof of Theorem 27. In view of the last three lemmas, it suffices to verify that $\mathcal{P}$ contains each 3-connected $\{U_{2,5}, U_{3,5}\}$-fragile GF(4)-representable matroid with an $\{X_8, Y_8, Y_8^*\}$-minor and no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor having at most 12 elements. This is done in the Appendix [4]. $\square$

4.3 Fan extensions

The following theorem describes the structure of the matroids with no $\{X_8, Y_8, Y_8^*\}$-minor. Note that $M_{9,9}$ is the rank-4 matroid on 9 elements in Figure 2. The matroid $M_{7,1}$ is the 7-element matroid that is obtained from Y_8 by deleting the unique point that is contained in the two 4-element segments of Y_8. We label the points of a triangle of $M_{7,1}$ by $\{1, 2, 3\}$ as in Figure 2.

Theorem 32. *Let M' be a 3-connected $\{U_{2,5}, U_{3,5}\}$-fragile GF(4)-representable matroid with no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. Then M' is isomorphic to a matroid M for which at least one of the following holds:*

(i) M has an $\{X_8, Y_8, Y_8^*\}$-minor;

(ii) $M \in \{M_{9,9}, M_{9,9}^*\}$;

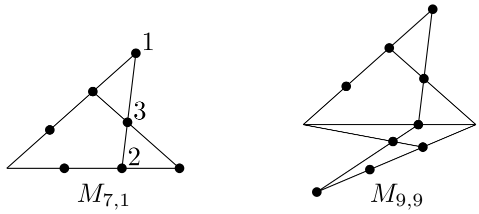

Figure 2: The matroids $M_{7,1}$ and $M_{9,9}$.

(iii) *M or M^* can be obtained from $U_{2,5}$ (with ground set $\{a, b, c, d, e\}$) by gluing wheels to (a, c, b),(a, d, b), (a, e, b);*

(iv) *M or M^* can be obtained from $U_{2,5}$ (with ground set $\{a, b, c, d, e\}$) by gluing wheels to (a, b, c), (c, d, e);*

(v) *M or M^* can be obtained from $M_{7,1}$ by gluing a wheel to $(1, 3, 2)$.*

Proof. Assume M has no $\{X_8, Y_8, Y_8^*\}$-minor. For (ii), we show in Lemma 1 of the Appendix [4] that the matroids $M_{9,9}$ and $M_{9,9}^*$ are splitters for the class of 3-connected $\{U_{2,5}, U_{3,5}\}$-fragile GF(4)-representable matroids with no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor.

We may therefore assume M has no $\{M_{9,9}, M_{9,9}^*, X_8, Y_8, Y_8^*\}$-minor. To show that (iii), (iv), or (v) holds, we use the main result of [2] called the "Fan Lemma", which reduces the proof to showing that extensions and coextensions of the 9-element matroids with this structure also have this structure. These verifications are completed in Lemmas 2 through 7 of the Appendix [4]. $\square$

5 From fragility to relaxations

We use the following result of Mayhew, Whittle, and Van Zwam [10, Lemma 8.2].

Lemma 33. *Let M be a 3-connected $U_{2,4}$-fragile matroid that has no $\{U_{2,6}, U_{4,6}\}$-minor. Then exactly one of the following holds.*

(i) *M has rank or corank two;*

(ii) *M has an $\{F_7^-, (F_7^-)^*\}$-minor;*

(iii) *M has rank and corank at least 3 and is a whirl.*

We show next that $P_8^-, F_7^=, (F_7^=)^*$ do not arise from circuit-hyperplane relaxation of a GF(4)-representable matroid.

Lemma 34. *Let M and M' be GF(4)-representable matroids such that M is connected, M' is 3-connected, and M' is obtained from M by relaxing a circuit-hyperplane X. Then M' has no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor.*

Proof. Assume that M' has a $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. Since M' is obtained from M by relaxing X, it follows from Theorem 15 and Lemma 33 that M' is $\{U_{2,5}, U_{3,5}\}$-fragile. Each of the matroids in $\{P_8^-, F_7^=, (F_7^=)^*\}$ has a $\{U_{2,5}, U_{3,5}\}$-minor, so if C and D are such that $M'/C\backslash D \cong N'$ for some $N' \in \{P_8^-, F_7^=, (F_7^=)^*\}$, then $C \subseteq X$ and $D \subseteq E(M') - X$ since the elements of X are nondeletable and the elements of $E(M') - X$ are noncontractible by Theorem 15. But then it follows from Lemma 4 that N' can be obtained from $M/C\backslash D$ by relaxing the circuit-hyperplane $X - C$. It follows that $M/C\backslash D \cong N$ for some $N \in \{P_8, F_7^-, (F_7^-)^*\}$, a contradiction because M is GF(4)-representable. □

We can now describe the structure of the GF(4)-representable matroids that are circuit-hyperplane relaxations of GF(4)-representable matroids.

Theorem 35. *Let M and M' be* GF(4)*-representable matroids such that M is connected, M' is* 3*-connected, and M' is obtained from M by relaxing a circuit-hyperplane. Then at least one of the following holds.*

(a) M' is a whirl;

(b) $M' \in \{M_{9,9}, M_{9,9}^*\}$;

(c) M' or $(M')^*$ can be obtained from $U_{2,5}$ (with groundset $\{a,b,c,d,e\}$) by gluing wheels to (a,c,b),(a,d,b);

(d) M' or $(M')^*$ can be obtained from $U_{2,5}$ (with groundset $\{a,b,c,d,e\}$) by gluing wheels to (a,b,c),(c,d,e);

(e) M' or $(M')^*$ can be obtained from $M_{7,1}$ by gluing a wheel to $(1,3,2)$;

(f) there is some path sequence that describes M'.

Proof. It follows from Theorem 15 that M' is either $U_{2,4}$-fragile or $\{U_{2,5}, U_{3,5}\}$-fragile. If M' is $U_{2,4}$-fragile, then it follows from Lemma 33 that M' is a whirl. We may therefore assume that M' is $\{U_{2,5}, U_{3,5}\}$-fragile. It follows from Lemma 34 that M' has no $\{P_8^-, F_7^=, (F_7^=)^*\}$-minor. Then, by Theorem 32 and Theorem 15, one of (b) through (e) holds or else M' has an $\{X_8, Y_8, Y_8^*\}$-minor. Note that outcome (iii) of Theorem 32 corresponds to outcome (c) here, since a matroid or its dual that is obtained from $U_{2,5}$ (with groundset $\{a,b,c,d,e\}$) by gluing wheels onto all three of the triangles (a,c,b),(a,d,b),(a,e,b) does not have a basis of nondeletable elements and a cobasis of noncontractible elements, and therefore cannot be obtained by relaxing a circuit-hyperplane. We can see this by the following counting argument. Observe that the rank of a matroid obtained from $U_{2,5}$ (with groundset $\{a,b,c,d,e\}$) by gluing wheels A, B and C onto the triangles (a,c,b),(a,d,b),(a,e,b) is $r(A) + r(B) + r(C) - 4$. But the nondeletable elements of this matroid are precisely the rim elements of the wheels of which there are $r(A) + r(B) + r(C) - 3$. Hence any cobasis must contain a nondeletable element e. Since this matroid has $M_{9,18}$ as a minor (see Appendix [4, Lemma 2]), M has no essential elements, which implies that e must be contractible.

Finally, if M' has an $\{X_8, Y_8, Y_8^*\}$-minor, then it follows from Theorem 27 that (f) holds. □

We can now show that if M and M' are GF(4)-representable matroids such that M' is obtained from M by relaxing a circuit-hyperplane, then M' has path width 3.

Proof of Theorem 1. If M is not connected, then it follows from Lemma 11 that M' has path width 3. We may therefore assume that M is connected. Then, by Lemma 13, M' can be obtained from a matroid in Theorem 35 (a) - (f) by performing some, possibly empty, sequence of series or parallel extensions. The result now follows from the fact that all the matroids in Theorem 35 (a) - (f) have path width 3. $\square$

6 Forbidden submatrices

In this section, we will prove our second characterization, Theorem 2. Let M be a GF(4)-representable matroid with a circuit-hyperplane X. Choose $e \in X$ and $f \in E - X$ such that $B = (X - e) \cup f$ is a basis of M. Then we can find a reduced GF(4)-representation of M in block form,

$$C = \begin{array}{c} \\ X-e \\ f \end{array} \begin{array}{cc} (E-X)-f & e \\ \left[\begin{array}{c} A \\ \underline{1}^T \end{array} \right. & \left. \begin{array}{c} \underline{1} \\ 0 \end{array} \right] \end{array}.$$

Here A is an $(X - e) \times ((E - X) - f)$ matrix over GF(4), and we have scaled so that every non-zero entry in the row labelled by f and the column labelled by e is 1. We denote by A_{ij} the entry in row i and column j of A.

Let M' be the matroid obtained from M by relaxing the circuit-hyperplane X. If M' is GF(4)-representable, then we can find a reduced representation of M' in block form,

$$C' = \begin{array}{c} \\ X-e \\ f \end{array} \begin{array}{cc} (E-X)-f & e \\ \left[\begin{array}{c} A' \\ \underline{1}^T \end{array} \right. & \left. \begin{array}{c} \underline{1} \\ \omega \end{array} \right] \end{array}.$$

We have scaled the rows and columns of the matrix such that the entry in the row labelled by f and column labelled by e is $\omega \in \mathrm{GF}(4) - \{0, 1\}$, and every remaining entry in row e and column f is a 1.

We omit the straightforward proof of the following lemma.

Lemma 36. $A_{ij} = 0$ *if and only if* $A'_{ij} = 0$.

Next we show that the only non-zero entries of A' are 1 and ω.

Lemma 37. $A'_{ij} \neq \omega + 1$.

Proof. Suppose $A'_{ij} = \omega + 1$. Then C' has a submatrix

$$C'[\{i, f\}, \{e, j\}] = \begin{array}{c} \\ i \\ f \end{array} \begin{array}{c} \begin{array}{cc} j & e \end{array} \\ \left[\begin{array}{cc} \omega + 1 & 1 \\ 1 & \omega \end{array} \right] \end{array},$$

which has determinant zero. Therefore $B\triangle\{e,f,i,j\}$ is not a basis of the matroid $M[I|C']$. But the corresponding submatrix of C is

$$C[\{i,f\},\{e,j\}] = \begin{array}{c c} & \begin{array}{cc} j & e \end{array} \\ \begin{array}{c} i \\ f \end{array} & \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \end{array},$$

for some non-zero x. Since $C[\{i,f\},\{e,j\}]$ has non-zero determinant, $B\triangle\{e,f,i,j\}$ is a basis of M, and hence of M'. Therefore $M' \neq M[I|C']$. □

Lemma 38. *$A_{ij} = A_{ik}$ if and only if $A'_{ij} = A'_{ik}$. Similarly, $A_{ij} = A_{kj}$ if and only if $A'_{ij} = A'_{kj}$*

Proof. We show that $A_{ij} = A_{ik}$ implies that $A'_{ij} = A'_{ik}$. The proof of the converse, and the proof of the second statement proceed by similar easy arguments. Suppose that $A_{ij} = A_{ik}$. Then C has a submatrix

$$C[\{i,f\},\{j,k\}] = \begin{array}{c c} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} i \\ f \end{array} & \begin{bmatrix} x & x \\ 1 & 1 \end{bmatrix} \end{array},$$

for some non-zero x. Since $C[\{i,f\},\{j,k\}]$ has zero determinant, $B\triangle\{f,i,j,k\}$ is not a basis of M, and hence not a basis of $M' = M[I|C']$. Therefore $\det(C'[\{i,f\},\{j,k\}]) = 0$, so it follows that $A'_{ij} = A'_{ik}$. □

The following lemma on diagonal submatrices will be used frequently.

Lemma 39. *Let*

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

be corresponding submatrices of A and A' respectively, where x, y, a, b are non-zero entries. Then $x = y$ if and only if $a \neq b$.

Proof. Adjoining e and f to the specified 2×2 submatrices, we get the 3×3 submatrices

$$\begin{bmatrix} x & 0 & 1 \\ 0 & y & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 1 & 1 & \omega \end{bmatrix}.$$

These matrices have determinants $x + y$ and $ab\omega + a + b$. Thus if $x = y$, then $a \neq b$. Conversely, if $a \neq b$, then $\{a,b\} = \{1,\omega\}$ by Lemma 37 so $ab\omega + a + b = \omega^2 + \omega + 1 = 0$. Hence $x = y$. □

We can now identify all of the forbidden submatrices. We use Lemma 38 to identify the first such matrix in the following lemma.

Lemma 40. *Neither A nor A^T has a submatrix of the form*

$$\begin{bmatrix} x & y & z \end{bmatrix},$$

where x, y, z are distinct non-zero entries.

Proof. By Lemma 38, the corresponding submatrix of A' must have the form

$$\begin{bmatrix} a & b & c \end{bmatrix},$$

where a, b, c are distinct non-zero entries, which is a contradiction to Lemma 37. $\square$

We now use Lemma 38 and Lemma 39 to find several more forbidden submatrices.

Lemma 41. *A has no submatrices of the following forms, where x, y, and z are distinct non-zero entries.*

$$(i) \begin{bmatrix} x & x & 0 \\ x & 0 & x \end{bmatrix}; \ (ii) \begin{bmatrix} x & x & 0 \\ x & 0 & y \end{bmatrix}; \ (iii) \begin{bmatrix} x & x & 0 \\ y & 0 & y \end{bmatrix}; \ (iv) \begin{bmatrix} x & y & 0 \\ x & 0 & y \end{bmatrix};$$

$$(v) \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \end{bmatrix}; \ (vi) \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}; \ (vii) \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}; \ (viii) \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}.$$

Proof. Suppose A has the submatrix (i). By applying Lemma 38 to the rows and the first column, we deduce that the corresponding submatrix of A' has the form

$$\begin{bmatrix} a & a & 0 \\ a & 0 & a \end{bmatrix},$$

where a is a non-zero entry, a contradiction of Lemma 39.

Suppose A has the submatrix (ii). By applying Lemma 38 to the rows and the first column, and since A' has at most two distinct non-zero entries by Lemma 37, we deduce that the corresponding submatrix of A' has the form

$$\begin{bmatrix} a & a & 0 \\ a & 0 & b \end{bmatrix},$$

where a and b are the two non-zero entries of A', a contradiction to Lemma 39.

The proofs for (iii) and (iv) are similar to that for (ii). We omit the details.

Suppose A has the submatrix (v). Then, by two applications of Lemma 39, the corresponding submatrix of A' must have the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & a \end{bmatrix},$$

for some non-zero entry a. This is a contradiction to Lemma 38.

Suppose A has the submatrix (vi). By Lemma 39, the corresponding submatrix of A' must be a diagonal matrix with distinct non-zero entries, a contradiction to Lemma 37.

Suppose A has the submatrix (vii). Applying Lemma 39 to the two submatrices the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix},$$

it follows that the corresponding submatrix of A' is

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix},$$

for some a, which is a contradiction to Lemma 39.

Suppose A has the submatrix (viii). Then the corresponding submatrix of A' is

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix},$$

for some a. Adjoining e and f, we have a submatrix of C,

$$\begin{bmatrix} x & 0 & 0 & 1 \\ 0 & y & 0 & 1 \\ 0 & 0 & z & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

which has zero determinant, while the corresponding submatrix of C',

$$\begin{bmatrix} a & 0 & 0 & 1 \\ 0 & a & 0 & 1 \\ 0 & 0 & a & 1 \\ 1 & 1 & 1 & \omega \end{bmatrix},$$

has non-zero determinant, a contradiction. □

Lemma 42. *A has no submatrices of the following forms, where x, y, and z are distinct non-zero entries:*

$$(i) \begin{bmatrix} x & y \\ 0 & x \end{bmatrix};\ (ii) \begin{bmatrix} x & y \\ y & x \end{bmatrix};\ (iii) \begin{bmatrix} x & x \\ y & z \end{bmatrix};\ (iv) \begin{bmatrix} x & y \\ z & x \end{bmatrix};\ (v) \begin{bmatrix} x & y & 0 \\ x & 0 & z \end{bmatrix}.$$

Proof. Suppose A has the submatrix (i). Then, adjoining e and f, we see that C has the following submatrix with non-zero determinant.

$$\begin{bmatrix} x & y & 1 \\ 0 & x & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

But then, by Lemma 38, the corresponding submatrix of C' must have the following form.

$$\begin{bmatrix} a & b & 1 \\ 0 & a & 1 \\ 1 & 1 & \omega \end{bmatrix},$$

where $\{a, b\} = \{1, \omega\}$ by Lemma 37. This gives a contradiction because this submatrix of C' has zero determinant. A similar proof handles (ii).

Suppose A has the submatrix (iii). Then, by Lemma 38, in the corresponding submatrix of A', the entries in the first row are the same and the entries in the second row are different. But, by Lemma 37, there are only two distinct non-zero entries in A', so the entries are the same in one of the columns of A', which is a contradiction to Lemma 38.

Suppose A has the submatrix (iv). Note that this submatrix has zero determinant. By Lemma 38, the corresponding submatrix of A' must have the following form.

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

where $\{a, b\} = \{1, \omega\}$ by Lemma 37. But this submatrix of A' has non-zero determinant, a contradiction.

Suppose A has the submatrix (v). Then C contains the following submatrix, which does not use its last column:

$$\begin{bmatrix} x & y & 0 \\ x & 0 & z \\ 1 & 1 & 1 \end{bmatrix}.$$

This matrix has determinant 0. By Lemmas 37, 38, and 39, the corresponding submatrix of C' is

$$\begin{bmatrix} a & b & 0 \\ a & 0 & b \\ 1 & 1 & 1 \end{bmatrix},$$

where $\{a, b\} = \{1, \omega\}$. This matrix has non-zero determinant, a contradiction. □

Finally, we find two more 3×3 forbidden submatrices of A.

Lemma 43. *A has no submatrices of the following forms, where x, y, and z are distinct non-zero entries:*

$$(i) \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & 0 \end{bmatrix}; \ (ii) \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & z \end{bmatrix}.$$

Proof. Suppose that A has the submatrix (i). Then, adjoining e and f, we see that C has the submatrix

$$\begin{bmatrix} x & y & x & 1 \\ y & y & 0 & 1 \\ x & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

which has zero determinant. The corresponding submatrix of C' is

$$\begin{bmatrix} a & b & a & 1 \\ b & b & 0 & 1 \\ a & 0 & 0 & 1 \\ 1 & 1 & 1 & \omega \end{bmatrix},$$

for distinct $a, b \in \{1, \omega\}$. This submatrix of C has non-zero determinant, a contradiction.

Suppose that A has the submatrix (ii). Note that the determinant of this submatrix is not zero. By Lemma 37 and Lemma 38, the corresponding submatrix of A' is

$$\begin{bmatrix} a & b & a \\ b & b & 0 \\ a & 0 & b \end{bmatrix},$$

for distinct $a, b \in \{1, \omega\}$. This submatrix of A' has zero determinant, which is a contradiction. □

To prove the main theorem of this section, we need the following theorem [5, Theorem 5.1].

Theorem 44. *Minor-minimal non-*GF(4)*-representable matroids have rank and corank at most 4.*

We can now prove the main theorem, which we repeat for convenience.

Theorem 45. *There is some matrix C' representing M' if and only if, up to permuting rows and columns, A and A^T have no submatrix in the following set, where x, y, z are distinct non-zero elements of* GF(4)*:*

$$\begin{bmatrix} x & y & z \end{bmatrix}, \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}, \begin{bmatrix} x & y \\ y & x \end{bmatrix}, \begin{bmatrix} x & x \\ y & z \end{bmatrix}, \begin{bmatrix} x & y \\ z & x \end{bmatrix}, \begin{bmatrix} x & x & 0 \\ x & 0 & x \end{bmatrix}, \begin{bmatrix} x & x & 0 \\ x & 0 & y \end{bmatrix},$$

$$\begin{bmatrix} x & x & 0 \\ y & 0 & y \end{bmatrix}, \begin{bmatrix} x & y & 0 \\ x & 0 & y \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \end{bmatrix}, \begin{bmatrix} x & y & 0 \\ x & 0 & z \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix},$$

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & 0 \end{bmatrix}, \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & z \end{bmatrix}.$$

Proof. It follows from Lemmas 40, 41, 42, and 43 that both A and A^T have no submatrix on the above list.

Conversely, suppose that the GF(4)-representable matroid M is chosen to be minimal subject to the property that the relaxation M' is not GF(4)-representable. Then M' has a minor N isomorphic to one of the excluded minors for the class of GF(4)-representable matroids. Assume that $N = M'/C\backslash D$ for some subsets C and D. If there is an element

g in both D and the circuit-hyperplane X of M, then $M\backslash g = M'\backslash g$ by Lemma 4, so M also has an N-minor, contradicting the fact that M is GF(4)-representable. We deduce that $D \subseteq E(M) - X$, and dually, $C \subseteq X$. Now if $|D| \geqslant 2$, then there is some element g in both D and $E(M') - (X \cup f)$, so relaxing the circuit-hyperplane X of $M\backslash g$ gives $M'\backslash g$ that is not GF(4)-representable, which contradicts the minimality of M. Therefore $|D| \leqslant 1$, and by a dual argument, there is no element g in both C and $X - e$, so $|C| \leqslant 1$. Since we know, by Theorem 44, that $|E(N)| \leqslant 8$, it now follows that $|E(M')| \leqslant 10$. The computations in the Appendix [4] show that M' must have a submatrix from the above list. □

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