Towards a splitter theorem for internally 4-connected binary matroids VII

Carolyn Chun  
*US Naval Academy*

James Oxley  
*Louisiana State University*

Follow this and additional works at: [https://repository.lsu.edu/mathematics_pubs](https://repository.lsu.edu/mathematics_pubs)

**Recommended Citation**


This Article is brought to you for free and open access by the Department of Mathematics at LSU Scholarly Repository. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Scholarly Repository. For more information, please contact [ir@lsu.edu](mailto:ir@lsu.edu).
Abstract. Let \( M \) be a 3-connected binary matroid; \( M \) is internally 4-connected if one side of every 3-separation is a triangle or a triad, and \( M \) is \((4,4,S)\)-connected if one side of every 3-separation is a triangle, a triad, or a 4-element fan. Assume \( M \) is internally 4-connected and that neither \( M \) nor its dual is a cubic Möbius or planar ladder or a certain coextension thereof. Let \( N \) be an internally 4-connected proper minor of \( M \). Our aim is to show that \( M \) has a proper internally 4-connected minor with an \( N \)-minor that can be obtained from \( M \) either by removing at most four elements, or by removing elements in an easily described way from a special substructure of \( M \). When this aim cannot be met, the earlier papers in this series showed that, up to duality, \( M \) has a good bowtie, that is, a pair, \( \{ x_1, x_2, x_3 \} \) and \( \{ x_4, x_5, x_6 \} \), of disjoint triangles and a cocircuit, \( \{ x_2, x_3, x_4, x_5 \} \), where \( M \setminus x_3 \) has an \( N \)-minor and is \((4,4,S)\)-connected. We also showed that, when \( M \) has a good bowtie, either \( M \setminus x_3, x_6 \) has an \( N \)-minor and \( M \setminus x_6 \) is \((4,4,S)\)-connected; or \( M \setminus x_3/x_2 \) has an \( N \)-minor and is \((4,4,S)\)-connected. In this paper, we show that, when \( M \setminus x_3, x_6 \) has no \( N \)-minor, \( M \) has an internally 4-connected proper minor with an \( N \)-minor that can be obtained from \( M \) by removing at most three elements, or by removing elements in a well-described way from a special substructure of \( M \). This is the penultimate step towards obtaining a splitter theorem for the class of internally 4-connected binary matroids.

1. Introduction

Seymour’s Splitter Theorem [11] proved that if \( N \) is a 3-connected proper minor of a 3-connected matroid \( M \), then \( M \) has a proper 3-connected minor \( M' \) with an \( N \)-minor such that \( |E(M) - E(M')| \leq 2 \). Furthermore, such an \( M' \) can be found with \( |E(M) - E(M')| = 1 \) unless \( r(M) \geq 3 \) and \( M \) is a wheel or a whirl. This result has been extremely useful in inductive and constructive arguments for 3-connected matroids. In this paper, we prove the penultimate step in obtaining a corresponding result for internally 4-connected binary matroids. Specifically, we will prove that if \( M \) and \( N \) are internally 4-connected binary matroids, and \( M \) has a proper \( N \)-minor, then \( M \) has a proper minor \( M' \) such that \( M' \) is internally 4-connected with an \( N \)-minor, and \( M' \) can be produced from \( M \) by a small number of simple operations.

Any unexplained matroid terminology used here will follow [10]. The only 3-separations allowed in an internally 4-connected matroid have a triangle or a...
triad on one side. A 3-connected matroid $M$ is $(4, 4, S)$-connected if, for every 3-separation $(X, Y)$ of $M$, one of $X$ and $Y$ is a triangle, a triad, or a 4-element fan, that is, a 4-element set $\{x_1, x_2, x_3, x_4\}$ that can be ordered so that $\{x_1, x_2, x_3\}$ is a triangle and $\{x_2, x_3, x_4\}$ is a triad.

To provide a context for the main theorem of this paper, we briefly describe our progress towards obtaining the desired splitter theorem. Johnson and Thomas [7] showed that, even for graphs, a splitter theorem in the internally 4-connected case must take account of some special examples. For $n \geq 3$, let $G_n + 2$ be the biwheel with $n + 2$ vertices, that is, $G_n + 2$ consists of an $n$-cycle $v_1, v_2, \ldots, v_n, v_1$, the rim, and two additional vertices, $u$ and $w$, both of which are adjacent to every $v_i$. Thus the dual of $G_n + 2$ is a cubic planar ladder. Let $M$ be the cycle matroid of $G_2 n + 2$ for some $n \geq 3$ and let $N$ be the cycle matroid of the graph that is obtained by proceeding around the rim of $G_2 n + 2$ and alternately deleting the edges from the rim vertex to $u$ and to $w$. Both $M$ and $N$ are internally 4-connected but there is no internally 4-connected proper minor of $M$ that has a proper $N$-minor. We can modify $M$ slightly and still see the same phenomenon. Let $G_r n + 2$ be obtained from $G_n + 2$ by adding a new edge $z$ joining the hubs $u$ and $w$. Let $\Delta n + 1$ be the binary matroid that is obtained from $M(G_r n + 2)$ by deleting the edge $v_1 v_n$ and adding the third element on the line spanned by $w v_n$ and $u v_1$. This new element is also on the line spanned by $w v_n$ and $w v_1$. For $r \geq 3$, Mayhew, Royle, and Whittle [9] call $\Delta r$ the rank-$r$ triangular Möbius matroid and note that $\Delta r \backslash z$ is the dual of the cycle matroid of a cubic Möbius ladder. The following is the main result of [2, Theorem 1.2].

**Theorem 1.1.** Let $M$ be an internally 4-connected binary matroid with an internally 4-connected proper minor $N$ such that $|E(M)| \geq 15$ and $|E(N)| \geq 6$. Then

(i) $M$ has a proper minor $M'$ such that $|E(M) - E(M')| \leq 3$ and $M'$ is internally 4-connected with an $N$-minor; or

(ii) for some $(M_0, N_0) \in \{(M, N), (M^*, N^*)\}$, the matroid $M_0$ has a triangle $T$ that contains an element $e$ such that $M_0 \backslash e$ is $(4, 4, S)$-connected with an $N_0$-minor; or

(iii) $M$ is isomorphic to $M(G_r n + 2)$, $M(G_r + 1)$, $\Delta r$, or $\Delta r \backslash z$ for some $r \geq 5$.

\[\text{Figure 1. All the elements shown are distinct. There are at least three dashed elements; and all dashed elements are deleted.}\]

That theorem led us to consider those matroids for which the second outcome in the theorem holds. In order to state the next result, we need to define some special structures. Let $M$ be an internally 4-connected binary matroid and $N$ be an internally 4-connected proper minor of $M$. Suppose $M$ has disjoint triangles $T_1$
and $T_2$ and a 4-cocircuit $D^*$ contained in their union. We call this structure a bowtie and denote it by $(T_1, T_2, D^*)$. If $D^*$ has an element $d$ such that $M \setminus d$ has an $N$-minor and $M \setminus d$ is $(4,4,S)$-connected, then $(T_1, T_2, D^*)$ is a good bowtie. Motivated by (ii) of the last theorem, we aim to discover more about the structure of $M$ when it has a triangle containing an element $e$ such that $M \setminus e$ is $(4,4,S)$-connected with an $N$-minor. One possible outcome here is that $M$ has a good bowtie. Indeed, as the next result shows, if that outcome or its dual does not arise, we get a small number of easily described alternatives. We shall need two more definitions. A terrahawk is the graph that is obtained from a cube by adjoining a new vertex and adding edges from the new vertex to each of the four vertices that bound some fixed face of the cube. Figure 1 shows a modified graph diagram, which we will use to keep track of some of the circuits and cocircuits in $M$, even though $M$ need not be graphic. Each of the cycles in such a graph diagram corresponds to a circuit of $M$ while a circled vertex indicates a known cocircuit of $M$. We refer to the structure in Figure 1 as an open rotor chain noting that all of the elements in the figure are distinct and, for some $n \geq 3$, there are $n$ dashed edges. The figure may suggest that $n$ must be even but we impose no such restriction. We will refer to deleting the dashed elements from Figure 1 as trimming an open rotor chain.

![Figure 1. An open rotor chain.](image1)

We need to define another special structure. An augmented 4-wheel consists of a 4-wheel restriction of $M$ with triangles $\{z_2, x_1, y_2\}, \{y_2, x_3, z_3\}, \{z_3, y_3, x_2\}, \{x_2, y_1, z_2\}$ along with two additional distinct elements $z_1$ and $z_4$ such that $M$ has $\{x_1, y_1, z_1, z_2\}, \{x_2, y_2, z_2, z_3\}$, and $\{x_3, y_3, z_3, z_4\}$ as cocircuits. We call $\{x_2, y_2, z_2, z_3\}$ the central cocircuit of the augmented 4-wheel. A diagrammatic representation of an augmented 4-wheel is shown in Figure 2.

![Figure 2. An augmented 4-wheel.](image2)

We refer to the structure in Figure 2 as an open rotor chain noting that all of the elements in the figure are distinct and, for some $n \geq 3$, there are $n$ dashed edges. The figure may suggest that $n$ must be even but we impose no such restriction. We will refer to deleting the dashed elements from Figure 2 as trimming an open rotor chain.

**Theorem 1.2.** Let $M$ and $N$ be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 6$. Suppose that $M$ has a triangle $T$ containing an element $e$ for which $M \setminus e$ is $(4,4,S)$-connected with an $N$-minor. Then one of the following holds.

(i) $M$ has an internally 4-connected minor $M'$ that has an $N$-minor such that $1 \leq |E(M) - E(M')| \leq 3$; or $|E(M) - E(M')| = 4$ and, for some $(M_1, M_2)$
in \(\{M, M', (M^*, (M')^*)\}\), the matroid \(M_2\) is obtained from \(M_1\) by deleting
the central cocircuit of an augmented 4-wheel; or

(ii) \(M\) or \(M^*\) has a good bowtie; or

(iii) \(M\) is the cycle matroid of a terrahawk; or

(iv) for some \((M_0, N_0)\) in \(\{(M, N), (M^*, N^*)\}\), the matroid \(M_0\) contains an
open rotor chain that can be trimmed to obtain an internally 4-connected
matroid with an \(N_0\)-minor.

Note that there is a small error in [4, Theorem 1.1] since it requires at least
five elements to be removed when trimming an open rotor chain. But, as the proof
there makes clear, trimming exactly four elements is a possibility. Trimming exactly
three elements is also possible but that is included under (i) of [4, Theorem 1.1].

This theorem leads us to consider a good bowtie \(\{\{x_1, x_2, x_3\},\{x_4, x_5, x_6\},\{x_2, x_3, x_4, x_5\}\}\) in an internally 4-connected binary matroid
where \(M\setminus x_3\) is \((4,4,S)\)-connected with an \(N\)-minor. In \(M\setminus x_3\), we see that
\(\{x_5, x_4, x_2\}\) is a triad and \(\{x_6, x_5, x_4\}\) is a triangle, so \(\{x_6, x_5, x_4, x_2\}\) is a 4-element
fan. By [3, Lemma 2.5], which is included below as Lemma 3.1, either

(i) \(M\setminus x_3, x_6\) has an \(N\)-minor; or

(ii) \(M\setminus x_3, x_6\) does not have an \(N\)-minor, but \(M\setminus x_3/x_2\) is \((4,4,S)\)-connected
with an \(N\)-minor.

In [5], we considered the case when (i) holds and \(M\setminus x_6\) is not \((4,4,S)\)-connected.
In this paper, we focus on the case when (ii) holds. The next and final paper in this
series will complete the work to obtain the splitter theorem by considering the case
when \(M\setminus x_3, x_6\) has an \(N\)-minor and \(M\setminus x_6\) is \((4,4,S)\)-connected. Before stating
the main result of [5], we define some structures that require special attention.

In a matroid \(M\), a string of bowties is a sequence \(\{a_0, b_0, c_0\},\{b_0, c_0, a_1, b_1\},\{a_1, b_1, c_1\},\{b_1, c_1, a_2, b_2\},\ldots,\{a_n, b_n, c_n\}\) with \(n \geq 1\) such that

(i) \(\{a_i, b_i, c_i\}\) is a triangle for all \(i\) in \(\{0,1,\ldots,n\}\);

(ii) \(\{b_j, c_j, a_{j+1}, b_{j+1}\}\) is a cocircuit for all \(j\) in \(\{0,1,\ldots,n-1\}\); and

(iii) the elements \(a_0, b_0, c_0, a_1, b_1, c_1,\ldots,a_n, b_n, c_n\) are distinct except that
\(a_0\) and \(c_n\) may be equal.

The reader should note that this differs slightly from the definition we gave in [11]
in that here we allow \(a_0\) and \(c_n\) to be equal instead of requiring all of the elements to
be distinct. Figure 3 illustrates a string of bowties, but this diagram may obscure
the potential complexity of such a string. Evidently \(M\setminus c_0\) has \(\{c_1, b_1, a_1, b_0\}\) as a
4-fan. Indeed, \(M\setminus c_0, c_1,\ldots,c_i\) has a 4-fan for all \(i\) in \(\{0,1,\ldots,n-1\}\). We shall
say that the matroid \(M\setminus c_0, c_1,\ldots,c_n\) has been obtained from \(M\) by trimming a string of bowties.
This operation plays a prominent role in our main theorem, and is the underlying operation in trimming an open rotor chain. Before stating

![Figure 3. A string of bowties. All elements are distinct except that \(a_0\) may be the same as \(c_n\).](image-url)
this theorem, we introduce the other operations that incorporate this process of trimming a string of bowties. Such a string can attach to the rest of the matroid in a variety of ways. In most of these cases, the operation of trimming the string will produce an internally 4-connected minor of \( M \) with an \( N \)-minor. But, when the bowtie string is embedded in a modified quartic ladder in certain ways, we need to adjust the trimming process.

Consider the three configurations shown in Figure 4 and Figure 5 where the elements in each configuration are distinct except that \( d_2 \) may equal \( w_k \). We refer to each of these configurations as an \textit{enhanced quartic ladder}. Indeed, in each configuration, we can see a portion of a quartic ladder, which can be thought of

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{In both (a) and (b), the elements shown are distinct, except that \( d_2 \) may be \( w_k \). Furthermore, in (a), \( k \geq 0 \); and in (b), \( k \geq 1 \) and \( \{w_{k-2}, u_{k-1}, v_{k-1}, u_k, v_k\} \) is a cocircuit.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{In this configuration, \( k \geq 2 \) and the elements are all distinct except that \( d_2 \) may be \( w_k \).}
\end{figure}
as two interlocking bowtie strings, one pointing up and one pointing down. In each case, we focus on \( M \setminus c_2, c_1, c_0, v_0, v_1, \ldots, v_k \) saying that this matroid has been obtained from \( M \) by an enhanced-ladder move.

Suppose that \( \{ a_0, b_0, c_0 \}, \{ b_0, c_0, a_1, b_1 \}, \{ a_1, b_1, c_1 \}, \ldots, \{ a_n, b_n, c_n \} \) is a bowtie string for some \( n \geq 2 \). Assume, in addition, that \( \{ b_n, c_n, a_0, b_0 \} \) is a cocircuit. Then the string of bowties has wrapped around on itself as in Figure 7 and we call the resulting structure a ring of bowties. We refer to each of the structures in Figure 6 as a ladder structure and we refer to removing the dashed elements in Figure 7 and Figure 6 as trimming a ring of bowties and trimming a ladder structure, respectively.

In the case that trimming a string of bowties in \( M \) yields an internally 4-connected matroid with an \( N \)-minor, we are able to ensure that the string of bowties belongs to one of the more highly structured objects we have described above. The following theorem is the main result of [5, Theorem 1.3].

**Theorem 1.3.** Let \( M \) and \( N \) be internally 4-connected binary matroids such that \( |E(M)| \geq 13 \) and \( |E(N)| \geq 7 \). Assume that \( M \) has a bowtie \( \{ x_0, y_0, z_0 \}, \{ x_1, y_1, z_1 \}, \{ y_0, z_0, x_1, y_1 \} \), where \( M \setminus z_0 \) is \( (4,4,S) \)-connected.

![Figure 6](image6.png)

**Figure 6.** In (a) and (b), \( n \geq 2 \) and the elements shown are distinct, with the exception that \( d_n \) may be the same as \( \gamma \) in (b). Either \( \{ d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1} \} \) or \( \{ d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n \} \) is a cocircuit in (a) and (b). Either \( \{ b_0, c_0, a_1, b_1 \} \) or \( \{ \beta, a_0, c_0, a_1, b_1 \} \) is also a cocircuit in (b).

![Figure 7](image7.png)

**Figure 7.** A bowtie ring. All elements are distinct. The ring contains at least three triangles.
Figure 8. All of the elements are distinct except that $a$ may be $f$, or \{a, b, c\} may be \{d, e, f\}. There are at least three dashed elements.

$M \backslash z_0, z_1$ has an $N$-minor, and $M \backslash z_1$ is not $(4, 4, S)$-connected. Then one of the following holds.

(i) $M$ has a proper minor $M'$ such that $|E(M)| - |E(M')| \leq 3$ and $M'$ is internally 4-connected with an $N$-minor; or

(ii) $M$ contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an $N$-minor; or

(iii) $M$ contains an enhanced quartic ladder from which an internally 4-connected minor of $M$ with an $N$-minor can be obtained by an enhanced-ladder move.

In Theorem 1.3, not all of the moves that we perform on $M$ to obtain an intermediate internally 4-connected binary matroid with an $N$-minor are bounded in size, but each unbounded move is highly structured. In this paper, we shall require one more such unbounded move. When $M$ contains the structure in Figure 8, where the elements are all distinct except that $a$ may be $f$, or \{a, b, c\} may be \{d, e, f\}, we say that $M$ contains an open quartic ladder. We will refer to deleting the dashed elements and contracting the arrow edge as a mixed ladder move. This is the only unbounded move that uses a contraction as well as a number of deletions. Note that both of the vertices of degree one in the diagram differ from the vertices closest to them.

The following theorem is the main result of this paper.

**Theorem 1.4.** Let $M$ and $N$ be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 7$. Let $M$ have a bowtie ($\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\}$), where $M \backslash A$ is $(4, 4, S)$-connected with an $N$-minor, and $M \backslash 1, 4$ has no $N$-minor. Then, for some $(M_0, N_0)$ in \{(M, N), (M*, N*)\}, one of the following holds.

(i) $M_0$ has a proper internally 4-connected minor $M'$ such that $M'$ has an $N_0$-minor and either $|E(M)| - |E(M')| \leq 3$, or $|E(M) - E(M')| = 4$ and $M'$ is obtained from $M_0$ by deleting the central cocircuit of an augmented 4-wheel; or

(ii) $M_0$ contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an $N_0$-minor; or

(iii) $M_0$ contains an open quartic ladder and an internally 4-connected matroid with an $N_0$-minor can be obtained by a mixed ladder move; or
Let Corollary 1.5. series, which proves a splitter theorem for internally 4-connected binary matroids.

An outline of the proof of this theorem is given in Section 5. That section separates the argument into three subcases and these cases are treated in the three subsequent sections. The results from those three sections are combined in Section 9 to complete the proof of the theorem. Before all of that, Section 2 gives some basic preliminaries while Sections 3 and 4 present some properties of, respectively, bowties and quasi rotors, and bowties and ladders.

The next corollary follows immediately by combining Theorem 1.4 with Theorem 1.3. This corollary provides the context for the next and final paper [6] in this series, which proves a splitter theorem for internally 4-connected binary matroids.

**Corollary 1.5.** Let $M$ and $N$ be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 7$. If $M$ has a bowtie $\{(1, 2, 3), \{4, 5, 6\}, \{2, 3, 4, 5\}\}$, where $M \setminus 4$ is $(4, 4, S)$-connected with an $N$-minor, then either $M \setminus 1, 4$ has an $N$-minor and $M \setminus 1$ is $(4, 4, S)$-connected, or one of the following holds for some $(M_0, N_0)$ in $\{(M, N), (M^*, N^*)\}$.

(i) $M_0$ has a proper internally 4-connected minor $M'$ such that $M'$ has an $N_0$-minor and either $|E(M)| - |E(M')| \leq 3$, or $|E(M) - E(M')| = 4$ and $M'$ is obtained from $M_0$ by deleting the central cocircuit of an augmented 4-wheel; or

(ii) $M_0$ contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an $N_0$-minor; or

(iii) $M_0$ contains an open quartic ladder and an internally 4-connected matroid with an $N_0$-minor can be obtained by a mixed ladder move; or

(iv) $M_0$ contains an enhanced quartic ladder from which an internally 4-connected minor of $M_0$ with an $N_0$-minor can be obtained by an enhanced-ladder move.

**2. Preliminaries**

In this section, we give some basic definitions mainly relating to matroid connectivity. Let $M$ and $N$ be matroids. We shall sometimes write $N \preceq M$ to indicate that $M$ has an $N$-minor, that is, a minor isomorphic to $N$. Now let $E$ be the ground set of $M$ and $r$ be its rank function. The **connectivity function** $\lambda_M$ of $M$ is defined on all subsets $X$ of $E$ by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate $\lambda_M$ as $\lambda$. For a positive integer $k$, a subset $X$ or a partition $(X, E - X)$ of $E$ is **$k$-separating** if $\lambda_M(X) \leq k - 1$. A $k$-separating partition $(X, E - X)$ of $E$ is a **$k$-separation** if $|X|, |E - X| \geq k$. If $n$ is an integer exceeding one, a matroid is **$n$-connected** if it has no $k$-separations for all $k < n$. This definition [12] has the attractive property that a matroid is $n$-connected if and only if its dual is. Moreover, this matroid definition of $n$-connectivity is relatively compatible with the graph notion of $n$-connectivity when $n$ is 2 or 3. For example, when $G$ is a graph with at least four vertices and with no isolated vertices, $M(G)$ is a 3-connected matroid if and only if $G$ is a 3-connected simple graph. But the link between $n$-connectivity for matroids and graphs breaks down for $n \geq 4$. In particular, a 4-connected matroid with at
least six elements cannot have a triangle. Hence, for \( r \geq 3 \), neither \( M(K_{r+1}) \) nor \( PG(r - 1, 2) \) is 4-connected. This motivates the consideration of other types of 4-connectivity in which certain 3-separations are allowed.

A matroid is **internally 4-connected** if it is 3-connected and, whenever \( (X, Y) \) is a 3-separation, either \( |X| = 3 \) or \( |Y| = 3 \). Equivalently, a 3-connected matroid \( M \) is internally 4-connected if and only if, for every 3-separation \( (X, Y) \) of \( M \), either \( X \) or \( Y \) is a triangle or a triad of \( M \). A graph \( G \) without isolated vertices is **internally 4-connected** if \( M(G) \) is internally 4-connected.

Let \( M \) be a matroid. A subset \( S \) of \( E(M) \) is a **fan** in \( M \) if \( |S| \geq 3 \) and there is an ordering \((s_1, s_2, \ldots, s_n)\) of \( S \) such that \( \{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \ldots, \{s_{n-2}, s_{n-1}, s_n\} \) alternate between triangles and triads. We call \((s_1, s_2, \ldots, s_n)\) a fan ordering of \( S \). For convenience, we will often refer to the fan ordering as the fan. We will be mainly concerned with 4-element and 5-element fans. By convention, we shall always view a fan ordering of a 4-element fan as beginning with a triangle and we shall use the term 4-fan to refer to both the 4-element fan and such a fan ordering of it. Moreover, we shall use the terms 5-fan and 5-cofan to refer to the two different types of 5-element fan where the first contains two triangles and the second two triads. Let \((s_1, s_2, \ldots, s_n)\) be a fan ordering of a fan \( S \). When \( M \) is 3-connected and \( n \geq 4 \), every fan ordering of \( S \) has its first and last elements in \( \{s_1, s_n\} \). We call these elements the **ends** of the fan while the elements of \( S - \{s_1, s_n\} \) are called the **internal elements** of the fan. When \((s_1, s_2, s_3, s_4)\) is a 4-fan, our convention is that \( \{s_1, s_2, s_3\} \) is a triangle, and we call \( s_1 \) the **guts element** of the fan and \( s_4 \) the **coguts element** of the fan since \( s_1 \in cl(\{s_2, s_3, s_4\}) \) and \( s_4 \in cl^*(\{s_1, s_2, s_3\}) \).

A set \( U \) in a matroid \( M \) is **fully closed** if it is closed in both \( M \) and \( M^* \). Let \((X, Y)\) be a partition of \( E(M) \). If \((X, Y)\) is \( k \)-separating in \( M \) for some positive integer \( k \), and \( y \) is an element of \( Y \) that is also in \( cl(X) \) or \( cl^*(X) \), then it is well known and easily checked that \((X \cup y, Y - y)\) is \( k \)-separating, and we say that we have moved \( y \) into \( X \). More generally, \((fcl(X), Y - fcl(X))\) is \( k \)-separating in \( M \). Let \( n \) be an integer exceeding one. If \( M \) is \( n \)-connected, an \( n \)-separation \((U, V)\) of \( M \) is **sequential** if \( fcl(U) \) or \( fcl(V) \) is \( E(M) \). In particular, when \( fcl(U) = E(M) \), there is an ordering \((v_1, v_2, \ldots, v_m)\) of the elements of \( V \) such that \( U \cup \{v_m, v_{m-1}, \ldots, v_1\} \) is \( n \)-separating for all \( i \in \{1, 2, \ldots, m\} \). When this occurs, the set \( V \) is called sequential. Moreover, if \( n \leq m \), then \( \{v_1, v_2, \ldots, v_n\} \) is a circuit or a cocircuit of \( M \). A 3-connected matroid is **sequentially 4-connected** if all of its 3-separations are sequential. It is straightforward to check that, when \( M \) is binary, a sequential set with 3, 4, or 5 elements is a fan. Let \((X, Y)\) be a 3-separation of a 3-connected binary matroid \( M \). We shall frequently be interested in 3-separations that indicate that \( M \) is, for example, not internally 4-connected. We call \((X, Y)\) or \( X \) a \((4, 3)\)-violator if \( |Y| \geq |X| \geq 4 \). Similarly, \((X, Y)\) is a \((4, 4, S)\)-violator if, for each \( Z \in \{X, Y\} \), either \( |Z| \geq 5 \), or \( Z \) is non-sequential. We also say that \((X, Y)\) is a \((4, 5, S, +)\)-violator if, for each \( Z \in \{X, Y\} \), either \( |Z| \geq 6 \), or \( Z \) is non-sequential, or \( Z \) is a 5-cofan. A binary matroid that has no \((4, 4, S)\)-violator is \((4, 4, S)\)-connected, as we defined in the introduction, and it is \((4, 5, S, +)\)-connected if its has no \((4, 5, S, +)\)-violator.

Next we note another special structure from [13], which has arisen frequently in our work towards the desired splitter theorem. In an internally 4-connected binary matroid \( M \), we call \( \{(1, 2, 3), \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\}\) a **quasi rotor** with **central triangle** \( \{4, 5, 6\} \) and **central element** 5 if \( \{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\}\).
and \{7, 8, 9\} are disjoint triangles in \(M\) such that \{2, 3, 4, 5\} and \{5, 6, 7, 8\} are cocircuits and \{3, 5, 7\} is a triangle. The next section is dedicated to results concerning bowties and quasi rotors.

Figure 9. A quasi rotor, where \{2, 3, 4, 5\} and \{5, 6, 7, 8\} are cocircuits.

For all non-negative integers \(i\), it will be convenient to adopt the convention throughout the paper of using \(T_i\) and \(D_i\) to denote, respectively, a triangle \(\{a_i, b_i, c_i\}\) and a cocircuit \(\{b_i, c_i, a_{i+1}, b_{i+1}\}\). Let \(M\) have \((T_0, T_1, T_2, D_0, D_1, \{c_0, b_1, a_2\})\) as a quasi rotor. Now \(T_2\) may also be the central triangle of a quasi rotor. In fact, we may have a structure like one of the two depicted in Figure 10. If \(T_0, D_0, T_1, D_1, \ldots, T_n\) is a string of bowties in \(M\), for some \(n \geq 2\), and \(M\) has the additional structure that \(\{c_i, b_{i+1}, a_{i+2}\}\) is a triangle for all \(i\) in \(\{0, 1, \ldots, n-2\}\), then we say that \(((a_0, b_0, c_0), (a_1, b_1, c_1), \ldots, (a_n, b_n, c_n))\) is a rotor chain. Clearly, deleting \(a_0\) from a rotor chain gives an open rotor chain. Note that every three consecutive triangles within a rotor chain have the structure of a quasi rotor; that is, for all \(i\) in \(\{0, 1, \ldots, n-2\}\), the sequence \((T_i, T_{i+1}, T_{i+2}, D_i, D_{i+1}, \{c_i, b_{i+1}, a_{i+2}\})\) is a quasi rotor. Zhou [13] considered a similar structure that he called a double fan of length \(n - 1\); it consists of all of the elements in the rotor chain except for \(a_0, b_0, b_n, \) and \(c_n\).

Figure 10. Right-maximal rotor chain configurations. In the case that \(n\) is even, the rotor chain is depicted on the left. If \(n\) is odd, then the rotor chain has the form on the right.
If a rotor chain \(((a_0, b_0, c_0), (a_1, b_1, c_1), \ldots, (a_n, b_n, c_n))\) cannot be extended to a rotor chain of the form \(((a_0, b_0, c_0), (a_1, b_1, c_1), \ldots, (a_{n+1}, b_{n+1}, c_{n+1}))\), then we call it a right-maximal rotor chain.

In the introduction, we defined a string of bowties. We say that such a string \(T_0, D_0, T_1, D_1, \ldots, T_n\) is a right-maximal bowtic string in \(M\) if \(M\) has no triangle \(\{u, v, w\}\) such that \(T_0, D_0, T_1, D_1, \ldots, T_n, \{x, c_n, u, v\}, \{u, v, w\}\) is a bowtie string for some \(x\) in \(\{a_n, b_n\}\).

For each positive integer \(n \geq 3\), let \(M_n\) be the binary matroid that is obtained from a wheel of rank \(n\) by adding a single element \(\gamma\) such that if \(B\) is the basis of \(M(W_n)\) consisting of the set of spokes of the wheel, then the fundamental circuit \(C(\gamma, B) = B \cup \gamma\). Observe that \(M_3 \cong F_2\) and \(M_4 \cong M^*(K_{3,3})\). Assume that the spokes of \(M(W_n)\), in cyclic order, are \(x_1, x_2, \ldots, x_n\) and that \(\{x_i, y_i, x_{i+1}\}\) is a triangle of \(M(W_n)\) for all \(i\) in \(\{1, 2, \ldots, n\}\) where we interpret all subscripts modulo \(n\). Then, for all \(i\) in \(\{1, 2, \ldots, n\}\), the set \(\{y_{i-1}, x_i, y_i\}\) is a triad of \(M(W_n)\) and \(\{\gamma, y_{i-1}, x_i, y_i\}\) is a cocircuit of \(M_n\). It is straightforward to check that \(M_n\) is internally 4-connected. Kingan and Lemos \([8]\) denote \(M_n\) by \(F_{2n+1}\). When \(n\) is odd, which is the case that will be of most interest to us here, \(M_n\) is isomorphic to what Mayhew, Royle, and Whittle \([9]\) call the rank-(\(n + 1\)) triadic Möbius matroid, \(\Upsilon_{n+1}\).

3. Some results for bowties and quasi rotors

In this section, we gather together a number of results that will be needed to prove the main theorem beginning with Lemma 2.5 from \([3]\) and Lemmas 4.1 and 4.2 from \([5]\). The second of these will often be used implicitly without reference.

**Lemma 3.1.** Let \(M\) and \(N\) be internally 4-connected binary matroids and \(\{e, f, g\}\) be a triangle of \(M\) such that \(N \preceq M \setminus e\) and \(M \setminus e\) is \((4, 4, S)\)-connected. Suppose \(|E(N)| \geq 7\) and \(M \setminus e\) has \((1, 2, 3, 4)\) as a 4-fan. Then either

i. \(N \preceq M \setminus e \setminus 1\); or

ii. \(N \preceq M \setminus e \setminus 4\) and \(M \setminus e \setminus 4\) is \((4, 4, S)\)-connected.

**Lemma 3.2.** Let \(N\) be an internally 4-connected matroid having at least seven elements and \(M\) be a binary matroid with an \(N\)-minor. If \((s_1, s_2, s_3, s_4)\) is a 4-fan in \(M\), then \(M \setminus s_1\) or \(M / s_4\) has an \(N\)-minor. If \((s_1, s_2, s_3, s_4, s_5)\) is a 5-fan in \(M\), then either \(M \setminus s_1, s_5\) has an \(N\)-minor, or both \(M \setminus s_1 / s_2\) and \(M \setminus s_5 / s_4\) have \(N\)-minors.

**Lemma 3.3.** Let \(M\) be an internally 4-connected matroid having at least ten elements. If \((\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})\) is a bowtie in \(M\), then \(\{2, 3, 4, 5\}\) is the unique 4-cocircuit of \(M\) that meets both \(\{1, 2, 3\}\) and \(\{4, 5, 6\}\).

When dealing with bowtie structures, we will repeatedly use the following result from \([5]\) Lemma 4.3, a modification of \([1]\) Lemma 6.3.

**Lemma 3.4.** Let \((\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})\) be a bowtie in an internally 4-connected binary matroid \(M\) with \(|E(M)| \geq 13\). Then \(M \setminus 6\) is \((4, 4, S)\)-connected unless \(\{4, 5, 6\}\) is the central triangle of a quasi rotor \((\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{y, 6, 7, 8\}, \{x, y, 7\})\) for some \(x\) in \(\{2, 3\}\) and some \(y\) in \(\{4, 5\}\). In addition, when \(M \setminus 6\) is \((4, 4, S)\)-connected, one of the following holds.

i. \(M \setminus 6\) is internally 4-connected; or
(ii) $M$ has a triangle $\{7, 8, 9\}$ disjoint from $\{1, 2, 3, 4, 5, 6\}$ such that
$(\{4, 5, 6\}, \{7, 8, 9\}, \{a, 6, 7, 8\})$ is a bowtie for some $a$ in $\{4, 5\}$; or

(iii) every $(4, 3)$-violator of $M\setminus 6$ is a 4-fan of the form $(u, v, w, x)$, where $M$ has
a triangle $\{u, v, w\}$ and a cocircuit $\{v, w, \{x, 6\}\}$ for some $u$ and $v$ in $\{2, 3\}$
and $\{4, 5\}$, respectively, and $|\{1, 2, 3, 4, 5, 6, w, x\}| = 8$; or

(iv) $M\setminus 1$ is internally 4-connected and $M$ has a triangle $\{1, 7, 8\}$ and a cocircuit
$\{a, 6, 7, 8\}$ where $|\{1, 2, 3, 4, 5, 6, 7, 8\}| = 8$ and $a \in \{4, 5\}$.

In Theorem 1.3, we dealt with the case when $M$ has a bowtie
$(\{a_0, b_0, c_0\}, \{a_1, b_1, c_1\}, \{b_0, c_0, a_1, b_1\})$ such that $M\setminus c_0$ is $(4, 4, S)$-connected with
an $N$-minor and $M\setminus c_1$ has an $N$-minor but is not $(4, 4, S)$-connected. We will therefore use the following hypothesis in the next lemma, and throughout this paper.

**Hypothesis VII.** If, for $(M_1, N_1) \in \{(M, N), (M^*, N^*)\}$, the matroid $M_1$ has a bowtie
$(\{a_0, b_0, c_0\}, \{a_1, b_1, c_1\}, \{b_0, c_0, a_1, b_1\})$, where $M_1\setminus c_0$ is $(4, 4, S)$-connected
and $M_1\setminus c_1$ has an $N_1$-minor, then $M_1\setminus c_1$ is $(4, 4, S)$-connected.

The next lemma is related to the previous lemma. We begin with the same
structure in $M$, a bowtie, but we add the additional consideration of preserving an
$N$-minor, and we eliminate one outcome by adding Hypothesis VII.

**Lemma 3.5.** Let $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ be a bowtie in an internally 4-connected
binary matroid $M$ with $|E(M)| \geq 13$. Let $N$ be an internally 4-connected minor of $M$
having at least seven elements. Suppose that $M\setminus 4$ is $(4, 4, S)$-connected, that $N \preceq M\setminus 1, 4$, and that Hypothesis VII holds. Then $M\setminus 1$ is $(4, 4, S)$-connected
with an $N$-minor and

(i) $M\setminus 1$ is internally 4-connected; or

(ii) $M$ has a triangle $\{7, 8, 9\}$ such that $(\{1, 2, 3\}, \{7, 8, 9\}, \{7, 8, 1, s\})$ is a
bowtie for some $s$ in $\{2, 3\}$ and $|\{1, 2, \ldots, 9\}| = 9$; or

(iii) every $(4, 3)$-violator of $M\setminus 1$ is a 4-fan of the form $(4, t, 7, 8)$, for some $t$ in
$\{2, 3\}$ where $|\{1, 2, 3, 4, 5, 6, 7, 8\}| = 8$; or

(iv) $M\setminus 6$ is internally 4-connected with an $N$-minor.

**Proof.** By Hypothesis VII, we may assume that $M\setminus 1$ is $(4, 4, S)$-connected. Now, if part (i)
or (ii) of Lemma 3.4 holds, then (i) or (ii) of the current lemma holds. Moreover, if (iii) of Lemma 3.4 holds, then, since $M\setminus 4$ is $(4, 4, S)$-connected, part (iii) of the current lemma holds. Thus we may assume that (iv) of Lemma 3.4 holds. Then, by symmetry, we may assume that $M$ has a triangle $\{6, 7, 8\}$ and a cocircuit $\{1, 3, 7, 8\}$ where $|\{1, 2, 3, 4, 5, 6, 7, 8\}| = 8$ and $M\setminus 6$ is internally 4-connected. We may also assume that $M\setminus 6$ has no $N$-minor otherwise (iv) holds.

Now $M\setminus 1, 4$ is 3-connected having $\{6, 7, 8, 3\}$ as a 4-fan and $N \preceq M\setminus 1, 4$. As $N \not\preceq M\setminus 6$, it follows, by Lemma 3.2, that $N \preceq M\setminus 1, 4/3$. But $M\setminus 1, 4/3 \cong M\setminus 1, 2, 4/3 \cong M\setminus 2, 4/5 \cong M\setminus 2/5 \setminus 6$. Hence $N \preceq M\setminus 6$; a contradiction. □

The next two results [5, Lemmas 5.2 and 5.7] are helpful when dealing with bowtie strings.
Lemma 3.6. Let $T_0, D_0, T_1, D_1, \ldots, T_n$ be a string of bowties in a matroid $M$. Then

$$M \backslash c_0, c_1, \ldots, c_n/b_n \cong M \backslash a_0, a_1, \ldots, a_n/b_0 \cong M \backslash c_0, c_1, \ldots, c_{k-1}/b_k \backslash a_k, a_{k+1}, \ldots, a_n \cong M \backslash c_0, c_1, \ldots, c_{k-1}/b_{k-1} \backslash a_k, a_{k+1}, \ldots, a_n$$

for all $k$ in $\{1, 2, \ldots, n\}$.

Lemma 3.7. Let $M$ be a binary matroid with an internally 4-connected minor $N$ where $|E(N)| \geq 7$. Let $T_0, D_0, T_1, D_1, \ldots, T_n$ be a string of bowties in $M$. Suppose $M \backslash c_0, c_1$ has an $N$-minor but $M \backslash c_0, c_1/b_1$ does not. Then $M \backslash c_0, c_1, \ldots, c_n$ has an $N$-minor, but $M \backslash c_0, c_1, \ldots, c_j/b_j$ has no $N$-minor for all $i$ in $\{1, 2, \ldots, n\}$, and $M \backslash c_0, c_1, \ldots, c_j/a_j$ has no $N$-minor for all $j$ in $\{2, 3, \ldots, n\}$.

In the following result, we consider a short bowtie string.

Lemma 3.8. Let $T_0, D_0, T_1, D_1, T_2$ be a string of bowties in an internally 4-connected binary matroid $M$. Suppose $M \backslash c_1$ is $(4, 4, S)$-connected. Then $M \backslash c_1/b_1$ is $(4, 5, S, +)$-connected. Moreover, either $M \backslash c_1/b_1$ is internally 4-connected, or $M \backslash c_1/b_1$ has a 4-fan and, whenever $(\alpha, \beta, \gamma, \delta)$ is such a 4-fan, $a_1 \in \{\beta, \gamma, \delta\}$, and $\{\beta, \gamma, \delta, c_1\}$ is a cocircuit of $M$.

Proof. As $M \backslash c_1$ is $(4, 4, S)$-connected having $(c_2, b_2, a_2, b_1)$ as a 4-fan, $M \backslash c_1/b_1$ is 3-connected. Suppose $M \backslash c_1/b_1$ has a $(4, 5, S, +)$-violator $(U, V)$. Then, without loss of generality, $|T_2 \cap U| \geq 2$. It is not difficult to check that $(U \cup T_2 \cup b_1, V - T_2)$ is a $(4, 4, S)$-violator of $M \backslash c_1$; a contradiction. We conclude that $M \backslash c_1/b_1$ is $(4, 5, S, +)$-connected.

Suppose that $(\alpha, \beta, \gamma, \delta)$ is a 4-fan in $M \backslash c_1/b_1$ such that $a_1 \notin \{\beta, \gamma, \delta\}$. Orthogonality with $T_1$ implies that $\{\beta, \gamma, \delta, c_1\}$ is not a cocircuit of $M$. Hence $\{\beta, \gamma, \delta\}$ is a triad of $M$. As $M$ is internally 4-connected, we deduce that $\{\alpha, \beta, \gamma, b_1\}$ is a circuit of $M$, so, by orthogonality, $\{\alpha, \beta, \gamma\}$ meets $\{b_0, c_0, a_1\}$ and $\{a_2, b_2\}$. Thus $\{\beta, \gamma, \delta\}$ meets a triangle of $M$; a contradiction. We conclude that $a_1 \in \{\beta, \gamma, \delta\}$. Since $a_1$ is in a triangle of $M$, we deduce that $\{c_1, \beta, \gamma, \delta\}$ is a 4-cocircuit of $M$. \qed

We continue on this theme with the following lemma.

Lemma 3.9. Let $M$ be an internally 4-connected binary matroid and suppose that $M$ has $T_0, D_0, T_1, D_1, T_2$ as a string of bowties and that $M \backslash c_1$ is $(4, 4, S)$-connected. Then

(i) $M \backslash c_1/b_1$ is internally 4-connected; or
(ii) $M \backslash c_1/b_1$ is $(4, 5, S, +)$-connected and $M$ has a triangle $\{1, 2, 3\}$ that avoids $T_1$ such that $\{2, 3, a_1, c_1\}$ is a cocircuit; or
(iii) $M \backslash c_1/b_1$ is $(4, 5, S, +)$-connected and $M$ has elements $d_0$ and $d_1$ such that $\{d_0, d_1\}$ avoids $T_0 \cup T_1 \cup T_2$ where $\{d_0, a_1, c_1, d_1\}$ is a cocircuit, and $\{d_0, a_1, s\}$ or $\{d_1, c_1, t\}$ is a triangle for some $s$ in $\{b_0, c_0\}$ or $t$ in $\{a_2, b_2\}$.

Proof. Suppose (i) does not hold. By 3.8 $M \backslash c_1/b_1$ has a 4-fan, $(1, 2, 3, 4)$, where $a_1 \in \{2, 3, 4\}$, and $(2, 3, 4, c_1)$ is a cocircuit. Lemma 3.3 implies that $(2, 3, 4)$ avoids $T_0$ and $T_2$. Now $M$ has $\{1, 2, 3\}$ or $\{1, 2, 3, b_1\}$ as a circuit. Suppose that $a_1 = 4$. If $\{1, 2, 3\}$ is a triangle, then (ii) holds, so we assume not. Then $\{1, 2, 3, b_1\}$ is a circuit. Now orthogonality implies that $\{1, 2, 3\}$ meets both $\{b_0, c_0\}$ and $\{a_2, b_2\}$, so $\{2, 3\}$ meets $T_0$ or $T_2$; a contradiction. We deduce that $a_1 \neq 4$. Without loss
of generality, \( a_1 = 3 \). If \( \{1, 2, a_1\} \) is a triangle in \( M \), then orthogonality implies that \( \{1, 2\} \) meets \( \{b_0, c_0\} \). Hence \( 1 \in \{b_0, c_0\} \) and, relabelling \( (1, 2, 4) \) as \( (s, d_0, d_1) \), we see that (iii) holds. If \( \{1, 2, a_1, b_1\} \) is a circuit of \( M \), then orthogonality with \( D_1 \) implies that \( \{1, 2\} \) meets \( \{a_2, b_2\} \). Thus \( 1 \in \{a_2, b_2\} \) and \( \{1, 2, a_1, b_1\} \) \( \Delta T_1 \) is \( \{1, 2, c_1\} \), a triangle, so, relabelling \( (1, 2, 4) \) as \( (t, d_1, d_0) \), (iii) holds.

Next, we prove a stronger version of [1, Lemma 8.4].

**Lemma 3.10.** Let \( M \) be an internally 4-connected binary matroid having \( T_0, D_0, T_1, D_1, T_2 \) as a string of bowties. Then

(i) \( M/T_1 \) is internally 4-connected; or
(ii) \( T_1 \) is the central triangle of a quasi rotor; or
(iii) \( M\setminus c_1/b_1 \) is internally 4-connected; or
(iv) \( M\setminus c_1/b_1 \) is \( (4, 5, S, +) \)-connected and \( M \) has elements \( d_0 \) and \( d_1 \) such that \( \{d_0, d_1\} \) avoids \( T_0 \cup T_1 \cup T_2 \), and \( \{d_0, a_1, c_1\} \) is a cocircuit, and \( \{d_0, a_1, s\} \) or \( \{d_1, c_1, t\} \) is a triangle for some \( s \) in \( \{b_0, c_0\} \) or \( t \) in \( \{a_2, b_2\} \).

**Proof.** Assume the lemma does not hold. By Lemma 3.4 since \( T_1 \) is not the central triangle of a quasi rotor, \( M\setminus c_1 \) is \( (4, 4, S) \)-connected. By Lemma 3.9 \( M \) has a triangle \( \{1, 2, 3\} \) avoiding \( T_1 \) such that \( \{2, 3, a_1, c_1\} \) is a cocircuit. Lemma 3.3 implies that \( \{1, 2, 3\} \) avoids \( D_0 \) and \( D_1 \), and that \( T_0 \) and \( T_2 \) avoid \( \{2, 3\} \). Then (i) or (ii) holds by [1, Lemma 8.3]; a contradiction.

To conclude this section, we recall [5, Lemma 4.5], which is useful for dealing with quasi rotors.

**Lemma 3.11.** Let \( M \) be an internally 4-connected binary matroid having \( (\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\}) \) as a quasi rotor and having at least thirteen elements. Let \( N \) be an internally 4-connected matroid containing at least seven elements such that \( M/e \) has an \( N \)-minor for some \( e \) in \( \{3, 5, 7\} \). Then one of \( M\setminus 1, M\setminus 9, M\setminus 1/2, M\setminus 9/8, \) or \( M\setminus 3, 4/5 \) is internally 4-connected with an \( N \)-minor.

4. Bowties and Ladders

In this section, we consider how bowties can interact with ladders. We begin with a lemma that builds from the configuration in Figure 11.

\[
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\cdots \\
\delta_n \\
b_n \\
\end{array}
\]

**Figure 11.** \( n \geq 1 \) and \( M \) has either \( \{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\} \) or \( \{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\} \) as a cocircuit, where \( d_{-1} = \alpha \) if \( n = 1 \).
Lemma 4.1. Let $M$ be an internally 4-connected binary matroid that has at least thirteen elements. Assume that $M$ contains the configuration shown in Figure 12 where $n \geq 1$, all the elements shown are distinct except that $d_n$ and $\gamma$ may be equal, and, in addition to the cocircuits shown, exactly one of \{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\} and \{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\} is a cocircuit of $M$. Assume also that $M$ is not isomorphic to the cycle matroid of a quartic M"{o}bius ladder and that $M\setminus c_n$ is $(4,4,S)$-connected. Then $M\setminus c_0, c_1, \ldots, c_n, \beta$ is $(4,4,S)$-connected and if it has a $(4,3)$-violator, then one side of that $(4,3)$-violator is a 4-fan $F$ where either

- (i) $F$ is a 4-fan in $M\setminus c_n$ with $b_n$ as its coguts element; or
- (ii) $F$ is a 4-fan in $M\setminus \beta$ with $\alpha$ as its coguts element.

Proof. First we show the following.

4.1.1. When $n = 1$, neither $\{d_0, d_1\}$ nor $\{b_0, b_1\}$ is contained in a triangle of $M$. Moreover, none of $a_1, b_0, b_1, d_0$, nor $d_1$ is in a triangle of $M\setminus c_0, c_1$.

If $\{d_{n-1}, d_n\}$ is in a triangle, then $M\setminus c_n$ has a 5-fan; a contradiction. If $\{b_0, b_1\}$ is in a triangle, then orthogonality implies that the triangle’s third element is in $\{\beta, \gamma\}$, so $\lambda(\alpha, \beta, \gamma, d_0) \cup T_0 \cup \{a_1, b_1\} \leq 2; a$ contradiction. By [5, Lemma 6.1] $a_1$ is not in a triangle of $M\setminus c_0, c_1$. If $b_0$ or $b_1$ is in a triangle of $M\setminus c_0, c_1$, then orthogonality implies that this triangle contains $\{b_0, b_1\}$; a contradiction. Similarly, if $d_0$ or $d_1$ is in a triangle of $M\setminus c_0, c_1$, then orthogonality implies that this triangle contains $\{d_0, d_1\}$; a contradiction. We conclude that 4.1.1 holds.

Next we note that

4.1.2. $M\setminus c_0, c_1, \ldots, c_n$ is $(4,4,S)$-connected.

This follows immediately from [5, Lemma 6.5] when $n > 1$. Moreover, it holds when $n = 1$ by combining 4.1.1 with [5, Lemma 6.1].

The matroid $M\setminus c_0, c_1, \ldots, c_n$ has $(\beta, \alpha, a_0, d_0)$ or $(\beta, \alpha, a_0, a_1)$ as a 4-fan. Thus $M\setminus c_0, c_1, \ldots, c_n, \beta$ is 3-connected. Next we observe that $M\setminus c_0, c_1, \ldots, c_n, \beta$ is sequentially 4-connected. To see this, note that if $M\setminus c_0, c_1, \ldots, c_n, \beta$ has a non-sequence 3-separation $(U, V)$, then, as $\{a_0, \alpha\}$ is in a triad, we may assume that this triad is contained in $U$. Thus $(U \cup \beta, V)$ is a non-sequence 3-separation of $M\setminus c_0, c_1, \ldots, c_n, \beta$; a contradiction.

Now suppose $M\setminus c_0, c_1, \ldots, c_n, \beta$ has a 4-fan $(w_1, w_2, w_3, w_4)$. Then $M$ has a cocircuit $C^*$ such that $\{w_2, w_3, w_4\} \subseteq C^*$ and $\{w_2, w_3, w_4, \beta, c_0, c_1, \ldots, c_n\}$. Then $M\setminus c_0, c_1, \ldots, c_n, \beta$ is 3-connected.

4.1.3. If $(w_1, w_2, w_3, w_4)$ is a 4-fan of $M\setminus c_0, c_1, \ldots, c_n$, then $w_4 = b_n$ and $(b_n, c_n, w_2, w_3)$ is a cocircuit of $M$, so $(w_1, w_2, w_3, w_4)$ is a 4-fan of $M\setminus c_n$.

Suppose that this fails. If $n > 1$, then, by (iii) of [5, Lemma 6.5], $w_4 = d_0$ and $a_0 \in \{w_2, w_3\}$. Moreover, $(w_1, w_2, w_3, w_4)$ is a 4-fan of $M\setminus c_0$. Thus $(w_2, w_3, c_0, d_0)$ is a 4-cocircuit of $M$ containing $\{a_0, c_0, d_0\}$. By orthogonality, this 4-cocircuit contains $\alpha$ or $\beta$. Hence it is $\{\alpha, a_0, c_0, d_0\}$. Thus $\{w_1, w_2, w_3\}$ contains $\{\alpha, a_0\}$ and so is $\{\alpha, a_0\};$ a contradiction. We conclude that 4.1.3 holds if $n > 1$.

Now let $n = 1$. By 4.1.3 neither $\{b_0, b_1\}$ nor $\{d_0, d_1\}$ is contained in a triangle of $M$. It follows by [5, Lemma 6.1] that $w_4 = b_1$. Now 4.1.3 holds if $(w_2, w_3, b_1, c_1)$ is a cocircuit, so we assume that $\{w_2, w_3, b_1, c_0\}$ or $\{w_2, w_3, b_1, c_0, c_1\}$ is a cocircuit. Orthogonality implies that $\{w_2, w_3\}$ meets $\{d_0, a_1\};$ a contradiction to 4.1.1. Thus 4.1.3 holds.

We may now assume that $(w_1, w_2, w_3, w_4)$ is not a 4-fan of $M\setminus c_0, c_1, \ldots, c_n$. Then $\beta \in C^*$. Thus $\{\alpha, a_0\}$ meets $\{w_2, w_3, w_4\}$. Next we show that
4.1.4. \( a_0 \not\in \{ w_1, w_2, w_3, w_4 \} \).

First we show that \( a_0 \not\in \{ w_1, w_2, w_3 \} \). Assume the contrary. Let \( n = 1 \). Then, by orthogonality, \( \{ w_1, w_2, w_3 \} \) meets \( \{ \alpha, d_0 \} \) or \( \{ \alpha, a_1 \} \). Thus \( d_0 \) or \( a_1 \) is in a triangle of \( M \setminus c_0, c_1 \); a contradiction to 4.1.1. Hence we may assume that \( n \geq 2 \). Then, by [5, Lemma 6.3], the triangle \( \{ w_1, w_2, w_3 \} \) of \( M \setminus c_0, c_1, \ldots, c_n \) meets \( \{ a_0, b_0, d_0, a_1, b_1, d_1, \ldots, a_n, b_n, d_n \} \) in \( \{ a_0 \} \) or \( \{ a_0, d_{n-1}, d_n \} \).

By orthogonality, \( \{ a_0, d_{n-1}, d_n \} \) is not a triangle. Thus \( \{ w_1, w_2, w_3 \} \) avoids \( \{ b_0, d_0, a_1, b_1, d_1, \ldots, a_n, b_n, d_n \} \). By orthogonality between \( \{ w_1, w_2, w_3 \} \) and both \( \{ \beta, \gamma, a_0, b_0 \} \) and \( \{ \alpha, a_0, c_0, d_0 \} \), we find that \( \{ w_1, w_2, w_3 \} = \{ a_0, \alpha, \gamma \} \). But \( \{ a_0, \alpha, \beta \} \) is a triangle; a contradiction. Hence \( a_0 \not\in \{ w_1, w_2, w_3 \} \).

Suppose now that \( a_0 = w_4 \). Orthogonality between \( C^* \) and the circuit \( \{ a_0, b_0, d_0, a_1 \} \) implies that \( \{ w_2, w_3 \} \) meets \( \{ b_0, d_0, a_1 \} \). Thus, by [5, Lemma 6.3], \( n = 1 \). But now we have a contradiction to 4.1.1. Thus 4.1.4 holds.

We now know that \( \alpha \in \{ w_2, w_3, w_4 \} \). Suppose \( \alpha \in \{ w_2, w_3 \} \). If \( \{ \alpha, a_0, c_0, d_0 \} \) is a cocircuit, then, by orthogonality, \( \{ a_0, d_0 \} \) meets \( \{ w_1, w_2, w_3 \} \). By [5, Lemma 6.3] and 4.1.4, \( n = 1 \) and \( d_0 \in \{ w_1, w_2, w_3 \} \); a contradiction to 4.1.1. We deduce that \( \{ \alpha, a_0, c_0, a_1, c_1 \} \) is a cocircuit of \( M \), so \( n = 1 \). Then orthogonality implies that \( \{ w_1, w_2, w_3 \} \) meets \( \{ a_0, a_1 \} \). Thus, by 4.1.4 and 4.1.1 we have a contradiction. Hence \( \alpha = w_4 \).

Now suppose that \( c_i \in C^* \) for some \( i \) in \( \{ 0, 1, \ldots, n \} \). Then \( \{ w_2, w_3 \} \) meets \( \{ a_i, b_i \} \). Thus, by [5, Lemma 6.3], if \( n = 0 \) and \( a_0 \in \{ w_2, w_3 \} \); a contradiction to 4.1.4. Moreover, if \( n = 1 \), then one of \( a_0, b_0, a_1, \) or \( b_1 \) is in \( \{ w_2, w_3 \} \); a contradiction to 4.1.4 or 4.1.1. We conclude that \( C^* \) avoids \( \{ c_0, c_1, \ldots, c_n \} \), so \( C^* = \{ w_2, w_3, \alpha, \beta \} \), and \( \{ w_1, w_2, w_3, \alpha \} \) is a 4-fan of \( M \setminus \beta \).

\[\begin{array}{c}
\alpha \\
b_0 \\
c_0 \\
a_0 \\
\vdots \\
\alpha, d_0, d_1, d_2, \ldots, d_{n-1}, d_n \\
a_{n-1}, a_n, c_n, b_n \\
\end{array}\]

FIGURE 12. Either \( \{ d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1} \} \) or \( \{ d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n \} \) is a cocircuit, where \( d_{n-1} = \alpha \).

Beginning with the next lemma and for the rest of the paper, we shall start abbreviating how we refer to the following four outcomes in the main theorem.

(i) \( M \) has a proper minor \( M' \) such that \( |E(M)| - |E(M')| \leq 3 \) and \( M' \) is internally 4-connected with an \( N \)-minor; or

(ii) \( M \) contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an \( N \)-minor;

(iii) \( M \) contains an open quartic ladder from which an internally 4-connected minor of \( M \) with an \( N \)-minor can be obtained by a mixed ladder move;
(iv) \( M \) contains an enhanced quartic ladder from which an internally 4-connected minor of \( M \) with an \( N \)-minor can be obtained by an enhanced-ladder move.

When (i) or (iv) holds, we say, respectively, that \( M \) has a quick win or an enhanced-ladder win. When trimming an open rotor chain, a ladder structure, or a ring of bowties in \( M \) produces an internally 4-connected matroid with an \( N \)-minor, we say, respectively, that \( M \) has an open-rotor-chain win, a ladder win, or a bowtie-ring win. When (iii) holds, we say that \( M \) has a mixed ladder win.

5. AN OUTLINE OF THE PROOF OF THE MAIN THEOREM

Since the proof of the main theorem is long, we give an outline of it in this section. By hypothesis, \( M \) and \( N \) are internally 4-connected binary matroids and \( M \) has a bowtie \( (\{1,2,3\}, \{4,5,6\}, \{2,3,4,5\}) \) where \( M\setminus 4 \) is \( (4,4,S) \)-connected with an \( N \)-minor, and \( M\setminus 1,4 \) has no \( N \)-minor. We may assume that \( M\setminus 6 \) is \( (4,4,S) \)-connected otherwise the theorem holds by Theorem 1.3. The one result in this section, Lemma 5.1, shows that either we get a quick win, or \( M \) contains one of configurations (A), (B), and C in Figure 13. In Section 6, we treat the case when \( M \) contains configuration (C) noting first that, by Lemma 3.1, \( M\setminus 4/5 \) is \( (4,4,S) \)-connected with an \( N \)-minor and with \( (a,b,c,6) \) as a 4-fan. Thus \( M\setminus 4/5 \setminus a \) has an \( N \)-minor. These two possibilities are dealt with in Lemmas 6.1 and 6.4, respectively.

In Section 7, we deal with the case when \( M \) contains configuration (A). First we prove a technical lemma detailing the possible structures surrounding a right-maximal bowtie chain in \( M \) that is also a right-maximal bowtie chain in \( M' \), a minor of \( M \). Then we show in Lemma 7.3 that we obtain our result.

The results of Sections 6 and 7 mean that we can assume that \( M \) contains neither of configurations (C) or (A). It remains to consider when \( M \) contains configuration (B) from Figure 13. This is done in Section 8. Finally, in Section 9, the parts already proved are combined to complete the proof of the main theorem.

![Figure 13](image-url)

FIGURE 13. In each structure, we view the labels on 2 and 3 as being interchangeable. The elements in each part are distinct except that \( a \) may equal 1 in (B) and (C).

We now show that \( M \) does indeed contain one of the three structures in Figure 13.

**Lemma 5.1.** Let \( (\{1,2,3\}, \{4,5,6\}, \{2,3,4,5\}) \) be a bowtie in an internally 4-connected binary matroid \( M \) with \( |E(M)| \geq 13 \). Let \( N \) be an internally 4-connected binary matroid having at least seven elements. Suppose that \( M\setminus 4 \) is \( (4,4,S) \)-connected with an \( N \)-minor and that \( N \not\preceq M\setminus 1,4 \). Then either \( M \) has an internally
4-connected minor $M'$ with an $N$-minor such that $1 \leq |E(M) - E(M')| \leq 2$, or, up to switching the labels on the elements 2 and 3, the matroid $M$ contains one of the configurations shown in Figure 13, the deletion $M\setminus 6$ is $(4, 4, S)$-connected, and \{4, 5, 6\} is the only triangle in $M$ containing 5. Moreover, in each of (A), (B), and (C), the elements shown are distinct except that, in (B) and (C), it is possible that $a = 1$.

**Proof.** Since $N \not\leq M\setminus 1, 4$, it follows by Lemma 3.1 that $M\setminus 4/5$ is $(4, 4, S)$-connected with an $N$-minor. Thus $M\setminus 6/5$ is $(4, 4, S)$-connected with an $N$-minor. We may assume that $M\setminus 6/5$ is not internally 4-connected otherwise the lemma holds. If 5 is in a triangle $T$ other than \{4, 5, 6\}, then $M\setminus 4/5$ has $T - 5$ as a circuit; a contradiction. Thus \{4, 5, 6\} is the only triangle in $M$ containing 5.

With a view to using Lemma 3.4, we now consider $M\setminus 6$. First suppose that \{4, 5, 6\} is the central triangle of a quasi rotor (\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{y, 6, 7, 8\}, \{x, y, 7\}) for some $x$ in \{2, 3\} and some $y$ in \{4, 5\}. Since $M\setminus 4$ is $(4, 4, S)$-connected, we deduce that $y = 4$. Then $M\setminus 4/5$ has $(x, 6, 7, 8, 9)$ as a 5-fan, a contradiction. We conclude that \{4, 5, 6\} is not the central triangle of such a quasi rotor. Then Lemma 3.4 implies that either $M\setminus 6$ is internally 4-connected, and the lemma holds; or $M\setminus 6$ is $(4, 4, S)$-connected but not internally 4-connected. Moreover, in the latter case, one of the following holds.

(i) $M$ has \{a, b, c\} as a triangle and \{b, c, d, 6\} as a cocircuit for some $a$ in \{2, 3\} and $b$ in \{4, 5\}, where $|\{1, 2, \ldots, 6, c, d\}| = 8$; or

(ii) $M$ contains the structure in Figure 13(C) where the elements are all distinct except that $a$ may be the same as 1; or

(iii) $M$ has a triangle \{7, 8, 9\} and a cocircuit \{5, 6, 7, 8\} where the elements are all distinct except that 1 may be the same as 9.

To see this, observe first that (i) above occurs when (iii) of Lemma 3.4 holds. On the other hand, outcomes (ii) and (iv) of Lemma 3.4 have been combined into (ii) and (iii) above with the separation between the latter being determined by the relative placement of the elements 4 and 5.

If (i) holds, then we know that $b = 4$, otherwise \{a, 5, c\} is a triangle in $M$ that contains 5 but is not \{4, 5, 6\}; a contradiction. Thus, up to switching the labels on 2 and 3, if (i) holds, then $M$ contains the structure in Figure 13(A) where all of the elements shown are distinct.

Since (ii) yields (C) in Figure 13, we may now assume that (iii) holds. Recall that $M\setminus 6/5$ is $(4, 4, S)$-connected but not internally 4-connected. Thus $M\setminus 6/5$ has a 4-fan \{a, b, c, d\}. Lemma 3.8 implies that $4 \in \{b, c, d\}$, and \{b, c, d, 6\} is a cocircuit of $M$. By symmetry, we may assume that $4 = b$ or $4 = d$.

Assume first that $4 = b$. Then $M$ has \{4, c, d, 6\} as a cocircuit, and Lemma 3.3 implies that \{c, d\} avoids \{1, 2, 3\} so $|\{1, 2, \ldots, 6, c, d\}| = 8$, and $M$ has \{a, 4, c\} or \{5, a, 4, c\} as a circuit. In the first case, orthogonality with \{2, 3, 4, 5\} implies that $a \in \{2, 3\}$, so, up to switching the labels of 2 and 3, the structure (A) in Figure 13 occurs, where all of the elements are distinct. If \{5, a, 4, c\} is a circuit of $M$, then $M$ also has \{a, c, 6\} as a circuit, so $M$ contains the structure in Figure 13(B) where all of the elements are distinct except that $a$ may be a repeated element. Certainly $a \not\in \{5, 6, b, c, d\}$. Hence, by orthogonality, either $a$ is distinct from the other elements in (B), or $a = 1$.

We may now assume that $4 = d$. Then \{b, c, 4, 6\} is a cocircuit of $M$. Lemma 3.3 implies that \{b, c\} avoids \{1, 2, 3, 7, 8, 9\}. Either \{a, b, c\} or \{a, b, c, 5\} is a circuit.
of $M$. In the first case, (C) in Figure 13 occurs and orthogonality implies that all of the elements are distinct except that $a$ may be 1. Now suppose that $M$ has $\{a, b, c, 5\}$ as a circuit. Orthogonality between this circuit and $\{2, 3, 4, 5\}$ implies that $\{a, b, c\}$ meets $\{2, 3\}$. By symmetry between 2 and 3, we may assume that $3 \in \{a, b, c\}$. As $\{b, c\}$ avoids $\{1, 2, 3, 7, 8, 9\}$, we deduce that $3 = a$. In (iii), we know that $\{5, 6, 7, 8\}$ is a cocircuit. Orthogonality between this cocircuit and the circuit $\{3, b, c, 5\}$ implies that $\{3, b, c\}$ meets $\{6, 7, 8\}$: a contradiction. 

We reiterate here that, in each of configurations (A), (B), and (C) in Figure 13 while the labels of 1, 4, 5, and 6 are fixed, we allow the labels on 2 and 3 to be interchanged without regarding the resulting structures as being different.

6. Configuration (C)

In this section, we treat the case when $M$ contains the configuration in Figure 13(C). We arrived at this configuration by assuming that $N \not\leq M \setminus 1, 4$. Thus, by Lemma 3.1, $M \setminus 4/5$ is $(4, 4, S)$-connected with an $N$-minor. Since $M \setminus 4/5$ has $(a, b, c, 6)$ as a 4-fan, by Lemma 3.2, $N \leq M \setminus 4/5/6$ or $N \not\leq M \setminus 4/5/5/a$. The next lemma deals with the first of these cases.

**Lemma 6.1.** Let $M$ and $N$ be binary internally 4-connected matroids such that $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Suppose that $M$ contains structure (C) in Figure 13, where $M \setminus 4$ is $(4, 4, S)$-connected with an $N$-minor and $N \not\leq M \setminus 1, 4$. If $M \setminus 4/5/6$ has an $N$-minor, then $M$ has a quick win.

**Proof.** We observe that $M \setminus 4/5/6 \cong M \setminus 4/5/6$. Now apply Lemma 3.10. If (i) or (iii) of that lemma holds, then the required result is immediate. If (ii) holds, then, as $M/e$ has an $N$-minor for all $e$ in $\{4, 5, 6\}$, it follows by [5, Lemma 4.5] that the lemma holds. Finally, suppose that (iv) holds. Then either $M$ has a triangle containing 5 and a member of $\{2, 3\}$; or $M$ has a triangle containing 6 and a member of $\{b, c\}$. In each case, we obtain a contradiction to the fact that $M \setminus 4$ is $(4, 4, S)$-connected.

The next lemma concerns the structure in Figure 14 which arises in Lemma 6.3.

**Lemma 6.2.** Suppose $M$ is an internally 4-connected binary matroid. If $M$ contains the structure in Figure 14, where all of the elements are distinct, then $M$ is the cycle matroid of a 16-element quartic planar ladder having $\{a_1, c_0, d_0\}$ and $\{a_0, c_3, d_3\}$ as triangles.

**Proof.** Clearly $|E(M)| \geq 16$. Moreover, $T_0 \cup T_1 \cup T_2 \cup T_3 \cup \{d_1, d_2\}$ has rank at most seven and contains at least five cocircuits, none of which is the symmetric difference of any others. Thus this 14-element set is 3-separating, so $|E(M)| \leq 17$, and has rank equal to seven. Suppose $|E(M)| = 17$. Then $M$ has $\{d_0, d_3, e\}$ as a triad for some element $e$ that is not shown in Figure 14. Then the symmetric difference of all of the vertex cocircuits and $\{d_0, d_1, e\}$ is $\{e, a_0, c_0\}$, so $M$ is not internally 4-connected, a contradiction. We conclude that $|E(M)| = 16$. As $r(M) = 7$, we see that $\{a_0, c_0, a_1, b_1, a_2, b_2, c_3\}$ is a basis $B$ of $M$. By orthogonality with the vertex cocircuits in Figure 14, we deduce that the fundamental circuit $C(d_0, B)$ for $d_0$ is $\{d_0, a_1, c_0\}$. Similarly, $C(d_3, B) = \{d_3, a_0, c_3\}$. The lemma now follows because $M$ is binary and so is determined by its fundamental circuits with respect to $B$. 

Next we deal with a structure that leads to a mixed ladder win.
Lemma 6.3. Let $M$ be an internally 4-connected binary matroid having at least fifteen elements. Suppose that $M$ contains the structure in Figure 15, where $k \geq 2$ and all of the elements are distinct except that $a_0$ may be $g$, or $T_0$ may be $\{e, f, g\}$, or $d_0$ may be $d_k$.

(i) The set $\{b_0, c_0\} \neq \{e, f\}$ and $d_0 \neq d_k$.
(ii) If $M \setminus c_i$ is $(4, 4, S)$-connected for all $i$ in $\{1, 2, \ldots, k\}$, then
(a) $\{d_0, a_1\}$ is contained in a triangle; or
(b) $\{c_k, d_k\}$ is contained in a triangle; or
(c) $M \setminus c_1, c_2, \ldots, c_k/b_k$ is internally 4-connected.

Proof. Let $X = T_1 \cup T_2 \cup \cdots \cup T_k \cup \{d_1, d_2, \ldots, d_{k-1}\}$. Then $E(M) - X$ contains $T_0 \cup d_0$. To establish (i), it suffices to show that $\lambda(X) \leq 2$ if $\{b_0, c_0\} = \{e, f\}$ or if $d_0 = d_k$. In the first case, $D_0 \triangle \{b_k, c_k, e, f\} = \{a_1, b_1, a_k, c_k\}$, which is a cocircuit contained in $X$. In the second case, $\{d_0, a_1, c_1, d_1\} \triangle \{d_{k-1}, a_k, c_k, d_k\}$ is also a cocircuit contained in $X$. In each case, one easily checks that $\lambda(X) \leq 2$, so (i) holds.

Next we note that, by [5, Lemma 5.3], $M \setminus c_1, c_2, \ldots, c_k$ is 3-connected unless this matroid has a 1- or 2-element cocircuit $D^*$ that contains $a_j$ or $b_j$ for some $j$ in
Thus (ii)(b) holds; a contradiction. We may now assume that $e \in C$ and $\beta \in C \setminus \{a_0, a_1\}$. The symmetric difference of $C$ with $T_k$ is a triangle containing $\{t_k, d_k\}$. Thus (ii)(b) holds; a contradiction. We may now assume that $a_k \notin C$. Then $b_{k-1} \in C$ and orthogonality implies that $\{b_{k-2}, b_{k-2}, a_{k-1}\}$ meets $C$. Moreover, orthogonality with $\{d_{k-2}, a_{k-1}, c_{k-1}, d_{k-1}\}$ implies that $a_{k-1}$ is not in $C$. Thus 6.3.2 holds.

Suppose that $k \geq 3$. Then, by orthogonality, $c_{k-2} \notin C$. Moreover, without loss of generality, we may assume that $e \in C$. Thus $C = \{e, b_{k-2}, b_{k-1}, b_k\}$. Then, by orthogonality between $C$ and $\{b_{k-3}, c_{k-2}, a_{k-2}, b_{k-2}\}$, we deduce that $e \in \{b_{k-3}, c_{k-3}\}$. Thus $k = 3$ and $T_0 = \{e, f, g\}$. Without loss of generality, $e = b_0$. Since (i) holds, $c_0 = g$, and $M$ contains the structure in Figure 14 with all the elements in that figure being distinct. Then Lemma 6.2 implies that (ii)(a) holds.

We may now assume that $k = 2$. Recall that $C = \{a, b, c, d, e\}$ where $a, b, c, d, e$ is a 4-fan of $M \setminus C$. Now $M$ has a cocircuit $C^*$ such that $C^* \subseteq \{b_0, b_1, c_0\}$. Since $\{b, c\}$ meets $\{b_0, b_1, c_0\}$, it follows that $|C^*| \neq 3$.

Next we show the following.

6.3.3. If $\{e, f, g\} = T_0$, then $C = \{b_0, b_1, b_2, y\}$ for some element $y$ that is not in $T_0 \cup T_1 \cup T_2 \cup \{d_0, d_1, d_2\}$. If $\{e, f, g\} \neq T_0$, then, without loss of generality, $e \in C$ and $C$ is $\{b_0, b_1, b_2, e\}$ or $\{c_0, b_1, b_2, e\}$.

Suppose first that $\{e, f, g\} = T_0$. By (a), we may assume that $(a_0, b_0, c_0) = (e, f, g)$. If $M$ has a triangle that meets $T_0, T_1$, and $T_2$, then $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$; a contradiction. Thus we may assume that $M$ has no such triangle. Now $C$ contains
If \( C \) meets each of \( T_0 \), \( T_1 \), and \( T_2 \).

If \( C \) contains two elements of one of these triangles, say \( T_i \), then \( C \triangle T_i \) is a triangle that meets each of \( T_0 \), \( T_1 \), and \( T_2 \); a contradiction. Thus \( C = \{b_0, b_1, b_2, y\} \) for some element \( y \) that avoids \( T_0 \cup T_1 \cup T_2 \). By orthogonality, \( y \) also avoids \( \{d_0, d_1, d_2\} \).

Thus the first part of 6.3.2 holds. The second part is an immediate consequence of 6.3.2.

6.3.4. If \( \{e, f, g\} = T_0 \), then \( \{\beta, \gamma\} = \{b_1, y\} \); otherwise \( \{\beta, \gamma\} = \{b_1, e\} \).

To see this, note that, by 6.3.3, \( \{\alpha, \beta, \gamma\} = \{b_0, b_1, y\}, \{b_0, b_1, e\}, \) or \( \{c_0, b_1, e\} \). To prove it suffices to show that \( T_0 \) avoids \( \{\beta, \gamma\} \). Assume the contrary. Then orthogonality between \( C^* \) and \( T_0 \) implies that \( \delta \notin T_0 \). Suppose \( c_1 \in C^* \). Then orthogonality implies that \( \{\beta, \gamma\} - T_0 = \{b_1\} \) and \( c_2 \notin C^* \). Thus \( C^* \) is a 4-cocircuit that meets both \( T_0 \) and \( T_1 \) but is not \( \{b_0, c_0, a_1, b_1\} \); a contradiction to Lemma 6.3. Thus \( c_1 \notin C^* \). Hence \( C^* \) is in \( C \) and we contradict orthogonality with \( T_2 \). We conclude that 6.3.4 holds.

We now know that \( b_1 \in C^* \). As \( C^* \cap T_1 \) is even, either \( c_1 \in C^* \), or \( c_1 \notin C^* \) and \( C^* \) is \( \{b_1, y, a_1, c_2\} \) or \( \{b_1, e, a_1, c_2\} \). Thus \( c_1 \in C^* \) otherwise orthogonality between \( C^* \) and the circuit \( \{c_1, b_2, c_2, d_1\} \) gives a contradiction. Suppose \( c_2 \in C^* \). Then, by orthogonality, \( \delta = a_2 \), so \( C^* \) is a 5-cocircuit containing \( \{b_1, c_1, a_2\} \). The symmetric difference of this cocircuit with the cocircuit \( \{b_1, c_1, a_2, b_2\} \) is a triad that contains \( c_2 \); a contradiction. Hence \( c_2 \notin C^* \). Thus \( C^* \) is \( \{b_1, y, \delta, c_1\} \) or \( \{b_1, e, \delta, c_1\} \). By orthogonality, with the circuit \( \{c_1, c_2, d_1, b_2\} \), we deduce that \( \delta = d_1 \). If \( \{e, f, g\} \neq \{a_0, b_0, c_0\} \), then we have a contradiction to orthogonality between \( C^* \) and \( \{e, f, g\} \). If \( \{e, f, g\} = \{a_0, b_0, c_0\} \), then \( \lambda(T_0 \cup T_1 \cup T_2 \cup \{d_1, y\}) \leq 2 \); a contradiction as \( |E(M)| \geq 15 \). This completes the proof of 6.3.1.

Now assume that part (ii) of the lemma fails. Next we show the following.

6.3.5. \( k \geq 3 \)

Assume that \( k = 2 \). Then applying Lemma 6.1 of [5], we see that part (i) of that lemma does not hold. Moreover, part (v) of that lemma does not hold by 6.3.1. If (ii) of [5] Lemma 6.1 holds, that is, \( \{d_1, d_2\} \) is in a triangle of \( M \), then this triangle together with \( c_1 \) and \( a_2 \) forms a 5-fan in \( M \setminus c_2 \); a contradiction. If (iv) of [5] Lemma 6.1 holds, that is, \( a_1 \) is in a triangle that avoids \( \{b_1, c_1, d_1\} \), then orthogonality with \( \{d_0, a_1, c_1, d_1\} \) implies that this triangle contains \( d_0 \), so (ii)(a) of the current lemma holds; a contradiction. Thus (iii) of [5] Lemma 6.1 holds, that is, \( \{b_1, b_2\} \) is in a triangle \( T \) of \( M \). By orthogonality, \( T \) meets both \( \{b_0, c_0\} \) and \( \{e, f\} \). Thus \( T_0 = \{e, f, g\} \), so \( \{b_1, b_2, b_0\} \) or \( \{b_1, b_2, c_0\} \) is a triangle. Then \( \lambda(T_0 \cup T_1 \cup T_2) \leq 2 \); a contradiction. We conclude that 6.3.5 holds.

We now apply Lemma 6.5 of [5] to the configuration induced by \( T_1 \cup T_2 \cup \cdots \cup T_k \cup \{d_1, d_2, \ldots, d_k\} \). Neither (i) nor (ii) of that lemma holds, and if (iv) holds, then \( \{d_k, c_0\} \) is in a triangle, that is, (ii)(b) of the current lemma holds; a contradiction. We deduce that (iii) of [5] Lemma 6.5 holds. Thus \( M \setminus \{c_1, c_2, \ldots, c_k\} \) is \( (4, 4, 4) \)-connected and every \( (4, 3) \)-violator of it is a 4-fan \( \{u_1, u_2, u_3, u_4\} \) where either \( u_4 = d_1 \) and \( a_1 \) is in \( \{u_2, u_3\} \); or \( u_4 = b_k \). Suppose that \( \{u_1, u_2, a_1, d_1\} \) is a 4-fan in \( M \setminus \{c_1, c_2, \ldots, c_k\} \). By orthogonality, \( \{u_1, u_2\} \) meets \( \{d_0, d_1\} \). Hence \( \{d_0, a_1\} \) is contained in a triangle and (ii)(a) holds; a contradiction. We conclude that every 4-fan of \( M \setminus \{c_1, c_2, \ldots, c_k\} \) has \( b_k \) as its coguts element. This contradiction to 6.3.1 completes the proof of the lemma.
Proof. Suppose that $3 = \lambda(N)$. We conclude that 6.4.1 holds.

Lemma 6.1. We deduce that 4 has an open-rotor-chain win or a ladder win; or $M$ has an enhanced-ladder win.

Proof. Suppose that $M$ has no quick win. As $M\setminus 6\setminus 5\setminus a \cong M\setminus 4\setminus 5\setminus a$, each of these matroids has an $N$-minor.

6.4.1. Neither $M\setminus 4, a/c$ nor $M\setminus 4, a/b$ has an $N$-minor.

Assume that $M\setminus 4, a/c$ or $M\setminus 4, a/b$ has an $N$-minor, we deduce that $M\setminus 4\setminus 6\setminus 5\setminus a$ has an $N$-minor. As $M\setminus 4\setminus 6\setminus 5\setminus a$ has (1,2,3,5) as a 4-fan, but $M\setminus 4, 1$ has no $N$-minor, it follows that $M\setminus 4\setminus 6\setminus 5\setminus a$ has an $N$-minor. Therefore, by Lemma 6.1, we conclude that 6.4.1 holds.

Next we relabel letting $(5, 6, 4) = (a_0, b_0, c_0)$ and $(b, c, a) = (a_1, b_1, c_1)$. Take $T_0, T_1, T_2, \ldots, T_n$ to be a right-maximal bowtie string in $M$. Now $M\setminus c_0, c_1$ has an $N$-minor but none of $M\setminus c_0, c_1/b_1, M\setminus c_0, c_1/a_1$, and $M\setminus c_0, 1$ has an $N$-minor.

Suppose that $\{a_0, b_0, x, c_n\}$ is not a cocircuit for all $x$ in $\{a_n, b_n\}$. Now, by Lemma 5.1, $a_0$ is in a unique triangle of $M$. Therefore, by [5, Lemma 10.1], the lemma holds.

We may now assume that $\{a_0, b_0, x, c_n\}$ is a cocircuit for some $x$ in $\{a_n, b_n\}$. Then $a_0 \neq c_n$, so all of the elements in the bowtie string are distinct. Up to relabelling $a_n$ and $b_n$, we may assume that $x = b_n$. Lemma 3.3 implies that $n > 1$. If $n = 2$, then $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$; a contradiction. Therefore $n \geq 3$. By Lemma 3.7 and the observations at the end of the second-last paragraph, we have that

6.4.2. $M\setminus c_0, c_1, \ldots, c_n$ has an $N$-minor, but $M\setminus c_0, c_1, \ldots, c_n/a_i$ has no $N$-minor for all $i$ in $\{1, 2, \ldots, n\}$.

Next we show that

6.4.3. $\{1, 2, 3\}$ avoids $\{c_0, c_1, \ldots, c_n\} \cup \{a_n, b_n\}$.

Suppose first that $\{1, 2, 3\}$ meets $\{c_0, c_1, \ldots, c_n\}$. Then $\{2, 3\}$ meets the last set, since $M\setminus c_0, 1$ has no $N$-minor. Up to switching the labels on 2 and 3, we may assume that $3 = c_i$ for some $i$ in $\{0, 1, \ldots, n\}$. Then 2 is in a 1- or 2-cocircuit of $M\setminus c_0, c_i$, so $M\setminus c_0, c_i/2$ has an $N$-minor. Hence so does $M\setminus c_0, 2/1$; a contradiction. We deduce that $\{1, 2, 3\}$ avoids $\{c_0, c_1, \ldots, c_n\}$.

Now suppose that $\{1, 2, 3\}$ meets $\{a_n, b_n\}$. As $\{1, 2, 3\} \neq T_n$, orthogonality between $\{1, 2, 3\}$ and $D_{n-1}$ implies that $b_{n-1} \in \{1, 2, 3\}$. If $n = 2$, then $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$; a contradiction. If $n \geq 3$, then orthogonality with $D_{n-2}$ implies, since $\{1, 2, 3\} \neq T_{n-1}$, that $b_{n-2} \in \{1, 2, 3\}$. Then orthogonality between $\{1, 2, 3\}$ and $D_{n-3}$ gives a contradiction. Thus 6.4.3 holds.

Evidently $\{1, 2, 3, a_0\}$ is a 4-fan in $M\setminus c_0, c_1, \ldots, c_n$. Since deleting 1 from the last matroid destroys all $N$-minors, Lemma 3.2 implies that $M\setminus c_0, c_1, \ldots, c_n/a_0$ has an $N$-minor. Now
\[ M \setminus c_0, c_1, \ldots, c_{n-1}, c_n/a_0 \cong M \setminus b_0, c_1, \ldots, c_{n-1}, c_n/a_0 \]
\[ \cong M \setminus b_0, c_1, \ldots, c_{n-1}, c_n/b_n \]
\[ \cong M \setminus b_0, c_1, \ldots, c_{n-1}, a_n/b_n \]
\[ \cong M \setminus b_0, c_1, \ldots, c_{n-1}, a_n/b_n \]
\[ \vdots \]
\[ \cong M \setminus b_0, c_1, a_2, \ldots, a_{n-1}, a_n/b_1 \]
\[ \cong M \setminus b_0, a_1, \ldots, a_{n-1}, a_n/c_0. \]

Therefore \( M/b_i \setminus c_i \) has an \( N \)-minor for all \( i \) in \( \{1, 2, \ldots, n\} \). Hypothesis VII implies that \( M \setminus c_1 \) is \((4, 4, S)\)-connected and, indeed, that \( M \setminus c_i \) is \((4, 4, S)\)-connected for all \( i \) in \( \{1, 2, \ldots, n\} \).

Consider \( M/b_n \setminus c_n \), which has an \( N \)-minor. Lemma 3.9 implies that \( M/b_n \setminus c_n \) is \((4, 5, S, +)\)-connected and either

(I) \( M \) has a triangle \( \{x, y, z\} \) such that \( \{y, z, a_n, c_n\} \) is a cocircuit; or

(II) \( M \) has elements \( d_{n-1} \) and \( d_n \) such that \( \{d_{n-1}, d_n\} \) avoids \( T_{n-1} \cup T_n \cup T_0 \)

where \( \{d_{n-1}, a_n, c_n\} \) is a cocircuit, and \( \{d_{n-1}, a_n, s\} \) or \( \{d_n, c_n, t\} \) is a triangle for some \( s \) in \( \{b_{n-1}, c_{n-1}\} \) or \( t \) in \( \{a_0, b_0\} \).

**6.4.4.** Part (I) does not hold.

Suppose that (I) does hold. Since we have a right-maximal bowtie string, we know that \( \{x, y, z\} \) meets \( T_0 \cup T_1 \cup \cdots \cup T_n \). By Lemma 5.1 \( T_0 \) is the only triangle containing \( a_0 \). Thus, by 5.1 Lemma 5.4, \( \{x, y, z\} = T_i \) for some \( i \) in \( \{0, 1, \ldots, n-2\} \). Moreover, by Lemma 3.3 \( i \neq 0 \). If \( c_i \in \{y, z\} \), then \( M \setminus c_0, c_1, \ldots, c_n \) has \( a_n \) in a cocircuit of size at most two, so we can contract \( a_n \) from the last matroid keeping an \( N \)-minor; a contradiction to 6.4.2. Therefore \( c_i = x \), so \( \{a_i, b_i\} = \{y, z\} \). Now \( D_{n-1} \triangle \{y, z, a_n, c_n\} \) is \( \{b_{n-1}, c_{n-1}, a_n, c_n\} \), which must be a cocircuit. Again \( a_n \) is in a cocircuit in \( M \setminus c_0, c_1, \ldots, c_n \) of size at most two, so contracting \( a_n \) from the last matroid retains an \( N \)-minor; a contradiction to 6.4.2. We conclude that 6.4.4 holds.

We may now assume that (II) holds. Next we show the following.

**6.4.5.** \( M \) has no triangle containing \( \{d_n, c_n, t\} \).

Suppose \( M \) has a triangle \( T \) containing \( \{d_n, c_n\} \). By orthogonality with the cocircuit \( \{b_n, c_n, a_0, b_0\} \), we deduce that \( a_0 \) or \( b_0 \) is in \( T \). As \( T_0 \) is the only triangle containing \( a_0 \), it follows that \( T = \{d_n, c_n, b_0\} \). Orthogonality implies that \( d_n \in \{c_0, a_1, b_1\} \) and hence that \( \{d_{n-1}, d_n\} \subseteq T_1 \). Then (I) holds so we have a contradiction to 6.4.4 that completes the proof of 6.4.5.

We now know that \( \{d_{n-1}, a_n, s\} \) is a triangle for some \( s \) in \( \{b_{n-1}, c_{n-1}\} \). If \( s = b_{n-1} \), then orthogonality implies that \( d_{n-1} \in \{b_{n-2}, c_{n-2}\} \). Hence orthogonality implies that \( \{d_{n-1}, d_n\} \subseteq T_{n-2} \), and \( \lambda(T_{n-2} \cup T_{n-1} \cup T_n) \leq 2 \); a contradiction. Thus \( s = c_{n-1} \). By assumption, \( \{d_{n-1}, d_n\} \) avoids \( T_{n-1} \cup T_n \cup T_0 \). If \( \{d_{n-1}, d_n\} \) meets \( T_0 \cup T_1 \cup \cdots \cup T_n \), then, by orthogonality between \( \{d_{n-1}, a_n, b_n, d_n\} \) and each \( T_i \), we see that \( \{d_{n-1}, d_n\} \subseteq T_i \) for some \( i \notin \{n-1, n, 0\} \), and (I) holds; a contradiction. We deduce that the elements of \( T_0 \cup T_1 \cup \cdots \cup T_n \cup \{d_{n-1}, d_n\} \) are distinct.
By taking \( j = n - 1 \), we see that \( M \) contains the structure in Figure 16 where all of the elements shown are distinct except those with the same labels. Next we show the following.

**6.4.6.** Suppose \( M \) contains the structure in Figure 16 for some \( j \) with \( 1 \leq j \leq n - 1 \) where all of the elements are distinct except those with the same label. Then either \( M \) has a mixed ladder win, or there is an element \( d_{j-1} \) that is not in \( T_0 \cup T_1 \cup \cdots \cup T_n \cup \{d_j, d_{j+1}, \ldots, d_n\} \) such that \( \{c_{j-1}, d_{j-1}, a_j\} \) is a triangle and \( \{d_{j-1}, a_j, c_j, d_j\} \) is a cocircuit.

We apply Lemma 6.3 to the bowtie string \( T_{j-1}, D_{j-1}, T_j, D_j, T_{j+1} \), noting that \( M/b_j \cap c_j \) has an \( N \)-minor. As \( M \) has no quick win, outcome (i) of that lemma does not hold. Thus (ii) or (iii) of Lemma 6.3 holds, so \( \{a_j, c_j\} \) is contained in a 4-cocircuit \( D^* \). Lemma 3.3 implies that \( D^* \) avoids \( T_{j+1} \). By orthogonality with the circuit \( \{c_j, d_j, a_{j+1}\} \), we see that \( D^* \) contains \( d_j \). Let \( d_{j-1} \) be the fourth element of \( D^* \). The structure induced on \( T_{j-1} \cup T_j \cup \cdots \cup T_n \cup T_0 \cup \{d_{j-1}, d_j, \ldots, d_n\} \) has the form of the one shown in Figure 16.

By orthogonality using the cocircuit \( \{d_{j-1}, a_j, c_j, d_j\} \) and the triangles in Figure 16, we see that \( d_{j-1} \) avoids \( T_0 \cup T_1 \cup \cdots \cup T_n \), and \( d_{j-1} \) avoids \( \{d_j, d_{j+1}, \ldots, d_{n-1}\} \). We now apply Lemma 6.3 to the structure on \( T_{j-1} \cup T_j \cup \cdots \cup T_n \cup T_0 \cup \{d_{j-1}, d_{j+1}, \ldots, d_n\} \). If (ii) or (iii) of that lemma holds, then \( M \) has a mixed ladder win. Part (ii)(b) does not hold by 6.4.5 so part (ii)(a) holds; that is, \( \{d_{j-1}, a_j\} \) is contained in a triangle \( T \). By orthogonality between \( T \) and the cocircuits \( \{b_{j-1}, c_{j-1}, a_j, b_j\} \) and \( \{b_{j-2}, c_{j-2}, a_j, b_{j-1}\} \), we deduce that \( T = \{d_{j-1}, a_j, c_{j-1}\} \). We conclude that 6.4.6 holds.

By repeatedly applying 6.4.6, we find that either \( M \) has a mixed ladder win or \( M \) has \( \{c_0, d_0, a_1\} \) as a triangle. Hence we may assume the latter. But recall that we began with a triangle \( \{1, 2, 3\} \) and a cocircuit that, after relabelling, became \( \{2, 3, a_0, c_0\} \). Now the elements in \( \{1, 2, 3, a_0, b_0, c_0, a_1, b_1, c_1\} \) are distinct except that \( 1 \) and \( c_1 \) may be equal. By orthogonality between \( \{c_0, d_0, a_1\} \) and \( \{2, 3, a_0, c_0\} \), we see that \( \{d_0, a_1\} \) meets \( \{2, 3, b_0\} \) so \( d_0 \in \{2, 3\} \). Suppose \( 1 = c_1 \). Then the circuit \( \{1, 2, 3\} \) is \( \{d_0, c_1, a_2\} \) or \( \{d_0, c_1, b_2\} \). The first possibility contradicts the fact that \( \{d_1, c_1, a_2\} \) is a circuit; the second violates orthogonality. We deduce that \( 1 \neq c_1 \), so \( \{1, 2, 3\} \) avoids \( \{a_0, b_0, c_0, a_1, b_1, c_1\} \). Now orthogonality between \( \{1, 2, 3\} \) and \( \{d_0, a_1, c_1, d_1\} \) implies that \( \{d_0, d_1\} \subseteq \{1, 2, 3\} \). Then \( \lambda(\{1, 2, 3\} \cup T_0 \cup T_1) \leq 2 \); a contradiction.

The results in this section enable us to conclude that \( M \) does not contain the structure in Figure 13 (C).
7. Configuration (A).

In this section, we deal with the case when \( M \) contains configuration (A) from Figure 13, where \( M/4 \) has an \( N \)-minor and \( M/4 \) has no \( N \)-minor. We begin with a straightforward lemma that will aid our efforts in this case.

**Lemma 7.1.** Suppose that \((\{1, 2, 3\}, \{4, 5, 6\}), \{2, 3, 4, 5\})\) is a bowtie in an internally 4-connected binary matroid \( M \) and that \( M \) has \( \{2, 4, 7\} \) as a triangle. Let \( N \) be an internally 4-connected matroid with at least seven elements such that \( N \) is a minor of \( M/\{x, y\} \) for some pair \( \{x, y\} \) of elements of \( \{4, 5, 6\} \). Then \( N \preceq M \setminus 1, 4 \).

**Proof.** As \( \{4, 5, 6\} \) is a triangle of \( M \), clearly \( N \preceq M/\{4, 5, 6\} \), so \( N \preceq M/5, 6 \setminus 4 \). Since \( M/5, 6 \setminus 4 \) has \( \{2, 7\} \) as a circuit, it follows that \( N \preceq M/4, 2 \), so \( N \preceq M/4, 2/3 \). Hence \( N \preceq M/1, 4 \). \( \square \)

**Figure 17.** The elements in both structures are all distinct, and we view the labels on \( a_1 \) and \( b_1 \) as being interchangeable.

**Figure 18.** The elements in this structure are all distinct, and we view the labels on \( a_1 \) and \( b_1 \) as being interchangeable. Furthermore, \( d_1 \in Y \), and either \( d_0 \in X \) or \( M \setminus c_0, c_1, d_0 \) has an \( N \)-minor.

In the following lemma, we consider a matroid that is not necessarily internally 4-connected, or even 3-connected. In the case that a binary matroid has \( \{a, b, c\} \) as a disjoint union of circuits and \( \{b, c, d\} \) as a disjoint union of cocircuits, we say that \( (a, b, c, d) \) is a loose 4-fan. If \( (a, b, c, d) \) and \( (e, d, c, b) \) are loose 4-fans, then we say that \( (a, b, c, d, e) \) is a loose 5-fan in \( M \) and a loose 5-cofan in \( M^* \). It is easy to see, by modifying the proof of Lemma 3.2, that if \( M \) has a loose 4-fan and has an internally 4-connected minor \( N \) having at least seven elements, then either deleting the first element or contracting the last element of the loose 4-fan retains the \( N \)-minor. We will use this fact within the proof of the next lemma.
Lemma 7.2. Let $M$ and $N$ be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 7$. Suppose that Hypothesis VII holds and that $M$ is not the cycle matroid of a terrahawk or a quartic Möbius ladder and is not the dual of a triadic Möbius matroid. Let $M' = M\setminus X/Y$ and let $M$ have $T_0, D_0, T_1, D_1, \ldots, T_n$ as a right-maximal bowtie string that is also a bowtie string in $M'$. Suppose that $M'\setminus c_0, c_1, \ldots, c_n$ has an $N$-minor. Then one of the following holds.

(i) $M$ has a quick win; or

(ii) $M$ has an open-rotor-chain win, a ladder win, or an enhanced-ladder win; or

(iii) $M$ has $\{a_0, b_0, z, c_n\}$ as a 4-cocircuit for some $z$ in $\{a_n, b_n\}$; or

(iv) the structure in Figure 12 is contained in $M$, up to switching the labels on $a_n$ and $b_n$, and either $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit, where $d_{-1} = a$; or

(v) $M'\setminus c_0, c_1/b_1$ has an $N$-minor, or $n = 1$ and $M'\setminus c_0, c_1/a_1$ has an $N$-minor; or

(vi) $n = 1$ and $M$ contains one of the structures in Figure 17 or Figure 18, where all of the elements are distinct and the labels $a_1$ and $b_1$ are viewed as being interchangeable. Moreover, if $M$ contains the structure in Figure 18, then $d_1 \in Y$, and either $d_0 \in X$ or $M'\setminus c_0, c_1, d_0$ has an $N$-minor; or

(vii) deleting the central cocircuit of some augmented 4-wheel in $M$ gives an internally 4-connected matroid with an $N$-minor.

Furthermore, if neither (iii) nor (v) holds, then $M$ has no triangle $T_{n+1}$ such that $\{x, c_n, a_{n+1}, b_{n+1}\}$ is a 4-cocircuit for any $x$ in $\{a_n, b_n\}$.

Proof. Suppose that neither (iii) nor (v) holds. By Lemma 3.6 we get the following.

7.2.1. If $i \in \{1, 2, \ldots, n\}$, then $M'\setminus c_0, c_1, \ldots, c_i/b_i$ has no $N$-minor. If $j \in \{1, 2, \ldots, n-1\}$, then $M'\setminus c_0, c_1, \ldots, c_j/a_{j+1}$ has no $N$-minor.

7.2.2. If $M$ has a triangle $T_{n+1}$ where $\{x, c_n, a_{n+1}, b_{n+1}\}$ is a cocircuit for some $x$ in $\{a_n, b_n\}$, then $a_0 \neq c_n$.

To show this, suppose that $a_0 = c_n$. Then $n \geq 2$. Without loss of generality, we assume that $x = b_n$. By orthogonality with $T_0$, the cocircuit $D_n$ meets $\{b_0, c_0\}$. Up to switching the labels on $a_{n+1}$ and $b_{n+1}$, we may assume that $b_{n+1} \in \{b_0, c_0\}$. Then orthogonality with $D_0$ implies that $T_{n+1}$ meets $\{a_1, b_1\}$. As $a_0 \in D_n$, Lemma 3.3 implies that $a_{n+1} \notin T_1$. Hence $c_{n+1} \in \{a_1, b_1\}$.

Suppose $c_{n+1} = b_1$. Then orthogonality between $T_{n+1}$ and $D_1$ implies that $a_{n+1} \in \{a_2, b_2\}$. Moreover, orthogonality with $D_n$ implies that $T_2$ meets $\{b_n, a_0, b_{n+1}\}$. It follows that $n = 2$, so $c_2 = a_0$ and $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$; a contradiction. Thus $c_{n+1} = a_1$.

As the next step towards 7.2.2 we now show that

7.2.3. $\{b_{n+1}, c_{n+1}\} \subseteq E(M')$ and $a_{n+1} \in Y$.

Since $b_{n+1} \in \{b_0, c_0\}$ and $c_{n+1} = a_1$, we know that $\{b_{n+1}, c_{n+1}\} \subseteq E(M')$. If $a_{n+1} \in X$, then $M'\setminus c_0, c_1, \ldots, c_n$ has $b_n$ in a 1- or 2-cocircuit; a contradiction to 7.2.1 Suppose $a_{n+1} \in E(M')$. If $b_{n+1} = c_0$, then $M'\setminus c_0, c_1, \ldots, c_n$ has $\{b_n, a_{n+1}\}$ as a disjoint union of cocircuits; a contradiction to 7.2.1. Thus $b_{n+1} = b_0$ and $\{a_1, b_0, a_{n+1}, b_n\}$ is a loose 4-fan in $M'\setminus c_0, c_1, \ldots, c_n$. By 7.2.1 we see that $M'\setminus c_0, c_1, \ldots, c_n, a_1$ has an $N$-minor. The last matroid has $\{b_0, b_1\}$ as a disjoint union of cocircuits; a contradiction to 7.2.1. Hence 7.2.3 holds.
Still aiming to show 7.2.2 we show next that

7.2.4. \( b_{n+1} = c_0 \).

Suppose that \( b_{n+1} \neq c_0 \). Then \( M' \setminus c_0, c_1, \ldots, c_n \) has \( \{b_{n+1}, a_1\} \) as a disjoint union of circuits, so \( M' \setminus c_0, c_1, \ldots, c_n, a_1 \) has an \( N \)-minor. But the last matroid has \( \{b_0, b_1\} \) as a disjoint union of cocircuits; a contradiction to 7.2.1. Thus 7.2.4 holds.

We now know that \( M \) contains the structure in Figure 19 and that \( M \setminus c_0, c_1, \ldots, c_n, b_n/b_m \) has an \( N \)-minor for all \( i \) in \( \{0, 1, \ldots, n\} \).

As the next step towards 7.2.2 we now show that

7.2.5. \( M \) has a 4-cocircuit containing \( \{a_i, c_i\} \) for all \( i \) in \( \{1, 2, \ldots, n\} \).

Since \( M \) has no quick win, this follows by Lemma 3.8 and Hypothesis VII unless \( M \setminus c_1 \) is not \( (4, 4, S) \)-connected. Consider the exceptional case. By Lemma 3.4, \( M \) has a quasi rotor \( (T_0, T_1, \{7, 8, 9\}, D_0, \{v, c_1, 7, 8\}, \{u, v, 7\}) \) for some \( u \) in \( \{b_0, c_0\} \) and \( v \) in \( \{a_1, b_1\} \). Since \( M/b_1 \) has an \( N \)-minor, Lemma 3.11 implies that \( b_1 \neq v \). Thus \( v = a_1 \). By orthogonality between \( \{c_0, a_1, a_{n+1}\} \) and the cocircuit \( \{a_1, c_1, 7, 8\} \), it follows that \( \{7, 8\} \) meets \( \{c_0, a_{n+1}\} \). Since the triangles \( T_0, T_1 \), and \( \{7, 8, 9\} \) are disjoint, \( c_0 \notin \{a_1, c_1, 7, 8\} \). Hence \( a_{n+1} \in \{7, 8\} \). By orthogonality between \( \{7, 8, 9\} \) and \( \{a_0, c_0, b_n, a_{n+1}\} \), it follows that \( b_n \in \{7, 8, 9\} \). Then orthogonality between \( \{7, 8, 9\} \) and \( D_{n-1} \) implies that \( \{7, 8, 9\} \) meets \( \{b_{n-1}, c_{n-1}\} \). Then \( M \) has \( (T_{n-1}, T_n, \{0, a_1, a_{n+1}\}, D_{n-1}, \{a_0, c_0, b_n, a_{n+1}\}, \{7, 8, 9\}) \) as a quasi rotor in which \( b_n \) is in two triangles. As \( M/b_n \) has an \( N \)-minor, Lemma 3.11 gives a contradiction. Hence 7.2.5 holds.

\[Figure 19.\]

\[Figure 20.\] The elements depicted are all distinct, \( \ell \leq n \), where we let \( d_n = b_0 \), and \( m \geq 0 \). Furthermore, \( m \leq \ell - 2 \).

We continue the proof of 7.2.2 by showing that \( M \) contains the structure in Figure 20. We will construct the left side of the figure first. Take \( d_n = b_0 \). Let \( \ell = n \) if \( \{c_{n-1}, a_n\} \) is not contained in a triangle of \( M \). If \( \{c_{n-1}, a_n\} \) is contained in a triangle, then take \( \ell \) minimal in \( \{1, 2, \ldots, n - 1\} \) such that \( \{c_i, a_{i+1}, d_i\} \) is a
triangle of $M$ for all $i$ in $\{\ell, \ell + 1, \ldots, n - 1\}$, where $d_i$ is an element in $E(M)$. By \cite[7.2.5]{T} $M$ has a 4-cocircuit containing $\{a_n, a_0\}$. By orthogonality with $T_0$ and Lemma 3.3 applied to $(T_n, \{c_0, a_1, a_{n+1}\}, \{a_0, c_0, b_n, a_{n+1}\})$, this cocircuit contains $b_0$. If $\ell < n$, then orthogonality and Lemma 3.3 imply that the fourth element of this cocircuit is $d_{n-1}$. By repeated application of this argument, we see that $M$ has $\{d_{i-1}, a_i, c_i, d_i\}$ as a cocircuit for all $i$ in $\{\ell + 1, \ell + 2, \ldots, n - 1\}$. Moreover, $M$ has $\{e, a_\ell, c_\ell, d_\ell\}$ as a cocircuit for some element $e$ where we note that this includes the case when $\ell = n$.

We continue to construct the structure in Figure 20 by showing that

\begin{equation}
7.2.6. \ell \geq 2.
\end{equation}

Suppose instead that $\ell = 1$. Then by orthogonality between $\{e, a_1, c_1, d_1\}$ and the triangle $\{c_0, a_1, a_{n+1}\}$ and Lemma \ref{lem:3.3}, we deduce that $e = a_{n+1}$. By \cite[Lemma 6.4]{T}, the elements in $T_1 \cup T_2 \cup \cdots \cup T_n \cup \{c_0, a_{n+1}\} \cup \{d_1, d_2, \ldots, d_{n-1}, b_0\}$ are distinct. Now it is easy to see that $M$ is the cycle matroid of a quartic Möbius ladder or is the dual of a triadic Möbius ladder; a contradiction. Thus \cite[7.2.6]{T} holds.

We now consider the right side of Figure 20. It is convenient to let $a_{n+1} = d_0$. Take $m$ maximal in $\{0, 1, \ldots, \ell - 1\}$ such that, for all $j$ in $\{0, 1, \ldots, m\}$, there is an element $d_j$ such that $\{c_j, d_j, a_{j+1}\}$ is a triangle. By the definition of $\ell$, we know that $\{c_{\ell-1}, a_\ell\}$ is not contained in a triangle of $M$. Hence

\begin{equation}
7.2.7. m \leq \ell - 2.
\end{equation}

As before, by orthogonality and Lemma \ref{lem:3.3} we know that $\{d_{j-1}, a_j, c_j, d_j\}$ is a cocircuit for all $j$ in $\{1, 2, \ldots, m\}$. Moreover, $\{d_m, a_m, c_m, d_{m+1}\}$ is a cocircuit for some element $d_{m+1}$. We deduce that $M$ contains the structure in Figure 20 where $\ell \leq n$, and $m \geq 0$.

Next we show that

\begin{equation}
7.2.8. the elements in Figure 20 are distinct.
\end{equation}

By \cite[Lemma 6.4]{T}, since $M$ is neither the dual of the triadic Möbius matroid nor the cycle matroid of a quartic Möbius ladder, the elements in Figure 20 other than $\{e, c_{m+1}, d_{m+1}\}$ are distinct except that $(b_m, b_{m+1})$ may equal $(a_\ell, b_\ell)$. As $\ell \neq m + 1$, we deduce that the elements in Figure 20 are distinct unless $e, c_{m+1}$, or $d_{m+1}$ is equal to one of the other elements.

Suppose $c_{m+1}$ is equal to another element in Figure 20. Then $c_{m+1} \in \{e, d_0, d_1, \ldots, d_m, d_\ell, d_{\ell+1}, \ldots, d_{n-1}\}$. By orthogonality between $T_{m+1}$ and the cocircuits in Figure 20 it follows that $e \in \{a_{m+1}, b_{m+1}\}$ and $c_{m+1} = d_\ell$. Then orthogonality with the cocircuits in Figure 20 implies that $(\ell, m) = (n, 0)$, and $T_1$ is $\{e, b_0, b_1\}$ or $\{e, b_0, a_1\}$. In either case, $\lambda(T_n \cup \{c_0, d_0, a_1, e, b_0, b_1\}) \leq 2$; a contradiction. Thus $c_{m+1}$ is not equal to any other element in Figure 20.

By orthogonality between the triangles in Figure 20 and the cocircuit $\{d_m, a_{m+1}, c_{m+1}, d_{m+1}\}$, we know that $d_{m+1}$ avoids all of the triangles in that figure. Similarly, orthogonality between these triangles and the cocircuit $\{e, a_\ell, c_\ell, d_\ell\}$ implies that $e$ avoids all these triangles. We deduce that \cite[7.2.8]{T} holds unless $d_{m+1} = e$. In the exceptional case, the set $\{e, a_\ell, c_\ell, d_\ell\} \triangle \{d_m, a_{m+1}, c_{m+1}, d_{m+1}\}$ contains a cocircuit, and the set $Z$ of elements in Figure 20 other than $\{e, d_{m+1}\}$ is 3-separating. Then $Z$ meets $T_{\ell-1} \cup e$ otherwise we contradict the fact that $M$ is internally 4-connected. But now we get a contradiction to orthogonality between $T_{\ell-1}$ and the cocircuits in the figure. Thus \cite[7.2.8]{T} holds.
Continuing the proof of 7.2.2 we show next that

7.2.9. \( M/d_{m+1} \) has an \( N \)-minor.

Recall that \( M \setminus c_0, c_1, \ldots, c_{n-1}, c_n/b_n \) has an \( N \)-minor. By Lemma 3.6, \( M \setminus a_1, a_2, \ldots, a_{m+1}, c_{m+1}, c_{m+2}, \ldots, c_{n-1}, c_n/d_m \) has an \( N \)-minor. The last matroid is isomorphic to \( M \setminus a_1, a_2, \ldots, a_{m+1}, c_{m+1}, c_{m+2}, \ldots, c_{n-1}, a_0/d_{m+1} \). Hence 7.2.9 holds.

Next we show that

7.2.10. \( d_{m+1} \) is in no triangle of \( M \).

Suppose \( M \) has \( d_{m+1} \) in a triangle, \( T \). By orthogonality between \( T \) and the 4-cocircuits in Figure 20, we see that \( \{c_{m+1}, d_{m+1}\} \subseteq T \). By the maximality of \( m \), we know that \( a_{m+2} \notin T \). Thus orthogonality with \( D_{m+1} \) implies that \( b_{m+2} \in T \); a contradiction to orthogonality with \( D_{m+2} \). Thus 7.2.10 holds.

As \( M \) has no quick win, we know that \( M/d_{m+1} \) is not internally 4-connected. We complete the proof of 7.2.2 by giving a contradiction to this fact.

Suppose \( (U, V) \) is a non-sequential 2- or 3-separation of \( M/d_{m+1} \). By 4 Lemma 3.3, we may assume that \( T_m \cup T_{m+1} \cup d_m \subseteq U \). Then \( (U \cup d_{m+1}, V) \) is a non-sequential 2- or 3-separation of \( M \); a contradiction. By 7.2.10 \( M/d_{m+1} \) has no sequential 2-separation. It follows that \( M/d_{m+1} \) is 3-connected having a 4-fan, \( \{u, v, w, x\} \). Hence \( M \) has \( \{v, w, x\} \) as a cocircuit and \( \{d_{m+1}, u, v, w\} \) as a circuit. By orthogonality with the cocircuit \( \{d_m, a_{m+1}, c_{m+1}, d_{m+1}\} \), the set \( \{u, v, w\} \) meets \( \{d_m, a_{m+1}, c_{m+1}\} \). If \( \{u, v, w\} \) meets \( D_m \) or \( D_{m+1} \), then orthogonality implies that two elements of \( \{u, v, w\} \) is in one of these cocircuits. Then \( \{v, w, x\} \) meets \( T_m, T_{m+1} \) or \( T_{m+2} \); a contradiction. Thus \( \{u, v, w\} \) avoids \( \{a_{m+1}, c_{m+1}\} \), so \( d_m \in \{u, v, w\} \). Either \( \{d_{m-1}, a_m, c_m, d_m\} \) is a cocircuit, or \( m = 0 \) and \( \{b_n, a_0, c_0, d_0\} \) is a cocircuit. Hence \( \{u, v, w\} \) contains two elements from one of these cocircuits, so the triad \( \{v, w, x\} \) meets a triangle in Figure 20 a contradiction. Thus \( M/d_{m+1} \) has no 4-fan; a contradiction. We conclude that \( a_0 \neq c_n \). Thus 7.2.2 holds.

Next we show that

7.2.11. \( M \) does not have a triangle \( T_{n+1} \) such that \( (T_n, T_{n+1}, \{x, c_n, a_{n+1}, b_{n+1}\}) \) is a bowtie for some \( x \) in \( \{a_n, b_n\} \).

Suppose \( M \) has such a bowtie. Then 5 Lemma 5.4] implies that \( T_{n+1} = T_j \) for some \( j \) in \( \{0, 1, \ldots, n-2\} \), so \( n \geq 2 \). If \( c_j \in \{a_{n+1}, b_{n+1}\} \), then \( M' \setminus c_0, c_1, \ldots, c_n \) has \( x \) in a 1- or 2-cocircuit, so \( M' \setminus c_0, c_1, \ldots, c_n/x \) has an \( N \)-minor; a contradiction to 7.2.1. Thus \( c_j = c_{n+1} \) so \( \{a_j, b_j\} = \{a_{n+1}, b_{n+1}\} \). If \( j = 0 \), then (iii) holds; a contradiction. Thus \( j \geq 1 \), and \( D_{j-1} \triangle \{x, c_n, a_{n+1}, b_{n+1}\} = \{b_{j-1}, c_{j-1}, x, c_n\} \), a disjoint union of cocircuits in \( M' \). Again \( M' \setminus c_0, c_1, \ldots, c_n \) has \( x \) in a 1- or 2-cocircuit; a contradiction. Thus 7.2.11 holds, as does the last assertion of the lemma.

We now assume that the lemma fails. We show next that

7.2.12. \( M \setminus c_n \) is \( (4, 4, S) \)-connected and every 4-fan in \( M \setminus c_n \) has the form \( (u, v, d_{n-1}, d_n) \) for some \( u \) and \( v \) in \( \{b_{n-1}, c_{n-1}\} \) and \( \{a_n, b_n\} \), respectively, where \( |T_{n-1} \cup T_n \cup \{d_{n-2}, d_n\}| = 8 \).

As (i) does not hold, \( M \setminus c_n \) is not internally 4-connected. Observe that, if \( T_n \) is the central triangle of a quasi rotor with \( a_n \) or \( b_n \) as its central element, then
we have a contradiction to 7.2.11. Thus, applying Lemma 3.3 to the bowtie $(T_{n-1}, T_n, D_{n-1})$, we deduce using 7.2.11 that outcome (iii) of that lemma holds; that is, 7.2.12 holds.

Without loss of generality, we may now assume that $M\setminus c_n$ has a 4-fan of the form $(u, a_n, d_{n-1}, d_n)$ for some $u$ in $\{b_{n-1}, c_{n-1}\}$.

7.2.13. $u \neq b_{n-1}$.

To show this, we assume the contrary. Suppose that $n > 1$. Then orthogonality between $\{b_{n-1}, a_n, d_{n-1}\}$ and $D_{n-2}$ implies that $d_{n-1} \in \{b_{n-2}, c_{n-2}\}$. Then orthogonality between $\{a_n, d_{n-1}, d_n, c_n\}$ and $T_{n-2}$ implies that $\{a_n, c_n, d_n\}$ meets $T_{n-2}$. If $a_{n-2} = c_n$, then $n = 2$ and, since $d_1 \in \{b_0, c_0\}$, the triangle $T_0$ is in the closure of $T_1 \cup T_2$, so $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$; a contradiction. Thus $a_{n-2} \neq c_n$, so $d_n \in T_{n-2}$, a contradiction to 7.2.11. We deduce that $n = 1$ and $M$ contains the structure in Figure 18.

Suppose $\{d_0, d_1\}$ avoids $Y$. If $\{d_0, d_1\}$ meets $X$, then $M\setminus c_0, c_1$ has a 1- or 2-cocircuit, so (v) holds; a contradiction. Thus $\{d_0, d_1\}$ avoids $X$, and $(b_1, b_0, a_1, d_0, d_1)$ is a loose 5-cofan of $M\setminus c_0, c_1$, so (v) holds; a contradiction. We deduce that $\{d_0, d_1\}$ meets $X$. If $d_0 \in Y$, then $\{b_0, a_1\}$ is a disjoint union of circuits in $M\setminus c_0, c_1$, so $M\setminus c_0, c_1$ has an $N$-minor. As this matroid has $\{a_1, b_1\}$ as a disjoint union of cocircuits, (v) holds; a contradiction. Thus $d_0 \notin Y$, so $d_1 \in Y$. If $d_0 \in X$, then (vi) holds; a contradiction. Thus $d_0 \notin X$, so $(d_0, b_0, a_1, b_1)$ is a loose 4-fan in $M\setminus c_0, c_1$, and (v) or (vi) holds; a contradiction. We conclude that 7.2.13 holds.

![Figure 21](https://example.com/figure21.png)

**Figure 21.** We view the labels $a_n$ and $b_n$ as being interchangeable. These elements are all distinct with the possible exception that $(a_0, b_0)$ may be $(c_n, d_n)$ in the case that $n > 1$. Furthermore, if $n > 1$, then $d_{n-1} \notin Y$.

We show next that 7.2.14. $M$ contains the configuration in Figure 21 where the elements $a_n$ and $b_n$ are viewed as being interchangeable. Moreover, the elements in this figure are distinct with the possible exception that $(a_0, b_0)$ may equal $(c_n, d_n)$ when $n > 1$. Also if $n > 1$, then $d_{n-1} \notin Y$.

By 7.2.13 $u = c_{n-1}$. Thus, $M$ contains the configuration in Figure 21. Moreover, by 7.2.12 when $n = 1$, the elements in the figure are all distinct. Hence we may suppose that $n \geq 2$. By 7.2.11 the pair $\{d_{n-1}, d_n\}$ is not contained in a triangle of $M$. Hence, by orthogonality, $\{d_{n-1}, a_n, c_n, d_n\}$ avoids $T_1 \cup T_2 \cup \cdots \cup T_{n-2}$. Now $d_{n-1} \notin Y$ otherwise $(b_{n-2}, a_{n-1}, b_{n-1}, a_n, b_n)$ is a loose 5-cofan in $M\setminus c_0, c_1, \ldots, c_n$, so $M\setminus c_0, c_1, \ldots, c_n/b_n$ has an $N$-minor; a contradiction to 7.2.1.
Suppose $T_0$ meets $\{d_{n-1}, d_n\}$. By orthogonality with $\{d_{n-1}, a_n, c_n, d_n\}$, it follows that $a_0 = c_n$, and $\{b_0, c_0\}$ meets $\{d_{n-1}, d_n\}$. Orthogonality between $D_0$ and $\{c_{n-1}, d_{n-1}, a_n\}$ implies that $d_{n-1} \not\in \{b_0, c_0\}$. Hence $a_n \in \{b_0, c_0\}$. Suppose $d_n = c_0$. As $d_{n-1} \in E(M') \cup X$, we see that $a_n$ is in a 1- or 2-cocircuit of $M' \setminus c_0, c_1, \ldots, c_n$; a contradiction to 7.2.1. We conclude that 7.2.14 holds.

7.2.16. If $n = 1$, then $M$ has $\{b_0, b_1, q\}$ as a triangle for some element $q$ that is not in $T_0 \cup T_1 \cup \{d_0, d_1\}$.

As $n = 1$, 7.2.14 implies that the elements in Figure 21 are distinct. We now apply [5] Lemma 6.1] assuming first that (v) of that lemma does not hold, that is, that $M$ has no triangle containing $\{b_0, b_1\}$. As the current lemma fails, it follows using 7.2.11 that none of (i), (ii), or (iv) of [5] Lemma 6.1] holds. Thus (iii) of that lemma holds, so $M \setminus c_0, c_1$ has a 4-fan of the form $\{1, 2, 3, b_1\}$. By 7.2.11 $M$ does not have $\{2, 3, b_1, c_1\}$ as a cocircuit. Thus $M$ has $\{2, 3, b_1, c_0, c_1\}$ or $\{2, 3, b_1, c_0\}$ as a cocircuit. Since $T_0 \cup T_1 \cup d_0$ is not 3-separating, this set contains no cocircuit of $M$ other than $D_0$. By orthogonality, $\{2, 3\}$ meets both $\{a_0, b_0\}$ and $\{d_0, a_1\}$. Hence $\{2, 3\} = \{b_0, a_1\}$, so $a_1$ is in a triangle of $M \setminus c_0, c_1$; a contradiction to [5] Lemma 6.1]. We conclude that (v) of [5] Lemma 6.1] holds, that is, $M$ has a triangle of the form $\{b_0, b_1, q\}$. Clearly $q \not\equiv a_0$, and orthogonality with the cocircuits $D_0$ and $\{d_0, a_1, c_1, d_1\}$ implies that $q \not\in T_0 \cup T_1 \cup \{d_0, d_1\}$. Thus 7.2.16 holds.

We continue our proof of 7.2.15 by showing the following.

7.2.17. Either $q \in X$, or $M' \setminus c_0, c_1, q$ has an $N$-minor.

To prove this, suppose first that $q \in Y$. Then $\{b_0, b_1\}$ is a disjoint union of circuits in $M' \setminus c_0, c_1$, so $M' \setminus c_0, c_1, b_0$ has an $N$-minor. As this matroid has $\{a_1, b_1\}$ as a disjoint union of cocircuits, (v) holds; a contradiction. Suppose $q \in E(M')$. Then $(q, b_0, b_1, a_1)$ is a loose 4-fan in $M' \setminus c_0, c_1$, and, as (v) does not hold, $M' \setminus c_0, c_1, q$ has an $N$-minor. Thus 7.2.17 holds.

Consider $M \setminus q$. Since this matroid is not internally 4-connected, by applying Lemma 3.4 to the bowtie $\{\{d_0, c_0, a_1\}, \{b_0, b_1, q\}, \{a_1, c_0, b_0, b_1\}\}$, we deduce that $\{q, b_0\}$ or $\{q, b_1\}$ is contained in a 4-cocircuit of $M$. By orthogonality with the circuits $T_0$ and $T_1$, it follows using Lemma 3.3 that $\{q, a_0, b_0\}$ or $\{q, b_1, c_1\}$ is contained in a 4-cocircuit of $M$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure22.png}
\caption{We view the labels $a_1$ and $b_1$ as being interchangeable. The elements shown are distinct.}
\end{figure}

Next we show that
7.2.18. \( M \) has no 4-cocircuit containing \( \{q, b_1, c_1\} \).

Suppose that \( \{q, b_1, c_1, s\} \) is a 4-cocircuit in \( M \), for some element \( s \). By orthogonality between this cocircuit and \( T_0, T_1, \) and \( \{c_0, d_0, a_1\} \), we see that either \( s \) is a new element, or \( s = d_1 \). The latter gives the contradiction that \( \lambda(T_0 \cup T_1 \cup \{d_0, q, s\}) \leq 2 \). Thus \( s \) is a new element and \( M \) contains the structure in Figure 22. Since \( M \setminus \{c_0, c_1, q\} \) has \( \{b_1, s\} \) as a disjoint union of cocircuits, \( M \setminus \{c_0, c_1, q/b_1\} \) has an \( N \)-minor. This matroid is isomorphic to \( M \setminus \{a_0, a_1, q/b_0\} \) by Lemma 3.6. By Lemma 3.4 as \( M \) has no quick win, \( a_0 \) is in a 4-cocircuit \( D \) of \( M \). By orthogonality, \( D \) meets \( \{b_0, c_0\} \). Lemma 3.3 implies that \( D \) avoids \( T_1 \), so orthogonality with \( \{c_0, d_0, a_1\} \) and \( \{b_0, b_1, q\} \) implies that \( D \) contains \( \{a_0, c_0, d_0\} \) or \( \{q, a_0, b_0\} \). If the fourth element of \( D \) is \( d_1 \) or \( s \), then the symmetric difference of \( D \) with the 4-cocircuit in Figure 22 containing \( d_1 \) or \( s \), respectively, is a new cocircuit contained in \( T_0 \cup T_1 \cup \{d_0, q\} \). Thus the last set is 3-separating; a contradiction. Using \( D \) together with the structure in Figure 22, we see that \( M \) contains a good augmented 4-wheel. Since \( M \setminus \{c_0, c_1, q/b_1\} \) has an \( N \)-minor and is isomorphic to \( M \setminus \{c_0, a_1, b_0, b_1\} \) and hence to \( M \setminus \{c_0, a_1, b_0, b_1\} \), we see that \( M \) contains an augmented 4-wheel such that removing the central cocircuit preserves an \( N \)-minor. Then \([4, \text{Theorem 4.1}]\) implies that (i) or (vii) holds; a contradiction. Thus 7.2.18 holds.

We now continue the proof of 7.2.15 knowing that \( \{a_0, b_0, q, p\} \) is a cocircuit in \( M \), for some element \( p \). By orthogonality between this cocircuit and the circuits \( T_0, T_1, \) and \( \{c_0, d_0, a_1\} \), we see that either \( p \) is a new element, or \( p = d_1 \). If \( p = d_1 \), then \( \lambda(T_0 \cup T_1 \cup \{d_0, q, d_1\}) \leq 2 \); a contradiction. We now apply \([5, \text{Lemma 6.2}]\). Clearly (i) of that lemma does not hold. Also 7.2.11 and 7.2.18 imply that neither (ii) nor (v) holds. If (iv) holds, then \( M \) has a triangle \( \{a_0, p, t\} \) for some element \( t \) which is new unless it equals \( s \). Orthogonality excludes the exceptional case. Thus \( M \) contains structure (a) in Figure 17 and (vi) of the current lemma holds; a contradiction. We deduce that (iii) of \([5, \text{Lemma 6.2}]\) holds, that is, \( M \) has a triangle \( \{s_1, s_2, s_3\} \) and a cocircuit \( \{q, c_1, b_1, s_2, s_3\} \) where \( \{s_1, s_2, s_3\} \) avoids \( \{b_0, c_0, q, a_1, b_1, c_1\} \). Orthogonality and 7.2.11 imply that \( \{s_1, s_2, s_3\} \) avoids \( \{d_0, a_1\} \). If \( \{s_1, s_2, s_3\} \) meets \( \{p, a_0\} \), then orthogonality implies that \( \{p, a_0\} \subseteq \{s_1, s_2, s_3\} \), and \( \lambda(\{b_0, c_0\} \cup T_1 \cup \{s_1, s_2, s_3, q\}) \leq 2 \); a contradiction. We conclude that \( \{s_1, s_2, s_3\} \) avoids \( p \) as well as all of the elements in Figure 22. Thus \( M \) contains structure (a) in Figure 17 and (vi) holds; a contradiction. Therefore 7.2.15 holds.

Take \( m \) minimal such that, for all \( i \) in \( \{m, m+1, \ldots, n-1\} \), the matroid \( M \) has \( \{c_i, d_i, a_{i+1}\} \) as a triangle and has \( \{d_i, a_{i+1}, c_{i+1}, d_{i+1}\} \) or \( \{d_i, a_{i+1}, c_{i+1}, a_{i+2}, c_{i+2}\} \) as a cocircuit, where the 5-cocircuit option is only possible when \( i = n-2 \).

7.2.19. The elements in \( T_m \cup T_{m+1} \cup \cdots \cup T_n \cup \{d_m, d_{m+1}, \ldots, d_n\} \) are all distinct and \( d_j \notin Y \) if max\{1, m\} \leq j \leq n - 1.\]

To see this, suppose first that the elements in this set are not all distinct. By 7.2.11 \( \{d_{n-1}, d_n\} \) is not contained in a triangle. By \([5, \text{Lemma 6.4}]\), we deduce that \( (a_m, b_m) = (c_n, d_n) \), and \( M \) has \( \{b_n, c_n, d_m\} \) as a cocircuit. Then \( m < n - 2 \). But \( \{c_m, d_m, a_{m+1}\} \) is a triangle of \( M \), so we get a contradiction to 7.2.11. Thus the elements in \( T_m \cup T_{m+1} \cup \cdots \cup T_n \cup \{d_m, d_{m+1}, \ldots, d_n\} \) are all distinct.
Suppose that \( d_j \in Y \) for some \( j \) with \( \max\{1, m\} \leq j \leq n - 1 \). Then \( (b_{j-1}, a_j, b_j, a_{j+1}, b_{j+1}) \) is a loose 5-cofan in \( M \setminus c_0, c_1, \ldots, c_n \). Then \( M \setminus c_0, c_1, \ldots, c_n/b_j+1 \) has an \( N \)-minor; a contradiction to 7.2.1 Thus 7.2.19 holds.

Next we show that

7.2.20. \( m = n - 1 \).

Suppose \( m < n - 1 \). We consider \( M \setminus c_m, c_{m+1}, \ldots, c_n \). By [5, Lemma 6.5] and 7.2.11 since \( M \) has no ladder win, \( M \setminus c_m, c_{m+1}, \ldots, c_n \) is \( (4,4,S) \)-connected and every 4-fan of this matroid is either a 4-fan of \( M \setminus c_m \) with \( b_n \) as its coguts element or a 4-fan of \( M \setminus c_m \) with \( d_m \) as its coguts element and with \( a_m \) as an interior element. By 7.2.11 \( M \setminus c_m \) does not have a 4-fan with \( b_n \) as its coguts element. Therefore \( M \setminus c_m \) has a 4-fan of the form \( (\beta, \alpha, a_m, d_m) \), so \( \{\alpha, a_m, d_m, c_m\} \) is a cocircuit of \( M \). As (iv) does not occur, we know that \( m \geq 1 \). Orthogonality implies that \( \{\alpha, \beta\} \) meets \( \{b_{m-1}, c_{m-1}, b_m\} \). As \( \{\alpha, \beta, a_m\} \notin T_{m-1} \), we know that \( b_m \notin \{\alpha, \beta\} \). Furthermore, Lemma 3.3 implies that \( \alpha \notin T_{m-1} \). Hence \( \beta \in \{b_{m-1}, c_{m-1}\} \). By the minimality of \( m \), we deduce that \( b = b_{m-1} \).

Suppose \( m > 1 \). Then orthogonality between \( \{b_{m-1}, \alpha, a_m\} \) and \( D_{m-2} \), implies that \( \alpha \in \{b_{m-2}, c_{m-2}, d_{m-1}\} \). As \( \alpha \notin T_{m-1} \), the cocircuit \( \{\alpha, a_m, c_m, d_m\} \) meets \( T_{m-2} \). Orthogonality implies that this 4-cocircuit is contained in \( T_{m-2} \cup T_m \). Hence \( \lambda(T_{m-2} \cup T_{m-1} \cup T_m) \leq 2 \); a contradiction. We conclude that \( m = 1 \).

By 7.2.19 \( d_1 \notin Y \). Thus \( d_1 \in X \cup E(M') \). If \( d_1 \in X \), then \( a_2 \) is in a 1- or 2-cocircuit of \( M \setminus c_0, c_1, \ldots, c_n \), so \( M \setminus c_0, c_1/ a_2 \) has an \( N \)-minor; a contradiction to 7.2.1. Hence \( d_1 \in E(M') \). If \( \alpha \in E(M') \), then \( (b_1, b_0, a_1, \alpha, d_1) \) is a loose 5-cofan in \( M \setminus c_0, c_1 \), so \( M \setminus c_0, c_1/b_1 \) has an \( N \)-minor; a contradiction to 7.2.1. Thus \( \alpha \notin E(M') \). Then \( \alpha \in Y \) or \( \alpha \in X \), so in \( M \setminus c_0, c_1, \ldots, c_n \), either \( \{b_0, a_1\} \) is a disjoint union of circuits, or \( \{a_1, d_1\} \) is a disjoint union of cocircuits. The former implies that \( M \setminus c_0, c_1, \ldots, c_n, a_1 \) has an \( N \)-minor. As the last matroid also has \( \{b_0, c_1\} \) as a disjoint union of cocircuits, we get a contradiction to 7.2.1. Therefore \( \{a_1, d_1\} \) is a disjoint union of cocircuits in \( M \setminus c_0, c_1, \ldots, c_n \), so \( M \setminus c_0, c_1/d_1 \) has an \( N \)-minor. As the last matroid has \( \{b_0, a_1, b_1, a_2, b_2\} \) as a loose 5-cofan, \( M \setminus c_0, c_1/b_2 \) has an \( N \)-minor. This contradiction to 7.2.1 completes the proof of 7.2.20.

We now apply [5, Lemma 6.1] to \( M \setminus c_{n-1}, c_n \). By assumption and 7.2.11 neither (i) nor (ii) of that lemma holds.

We show next that

7.2.21. \( b_n \) is not the coguts element of any 4-fan of \( M \setminus c_{n-1}, c_n \).

Suppose \( (\alpha, \beta, \gamma, b_n) \) is a 4-fan in \( M \setminus c_{n-1}, c_n \). By 7.2.11 we know that \( \{\beta, \gamma, b_n, c_n\} \) is not a cocircuit of \( M \). Hence either \( \{\beta, \gamma, b_n, c_{n-1}\} \) or \( \{\beta, \gamma, b_n, c_{n-1}\} \) is a cocircuit of \( M \). By orthogonality between \( D \) and the triangles \( T_{n-1} \) and \( \{c_{n-1}, d_{n-1}, a_n\} \), we see that \( \{\beta, \gamma\} \subseteq \{a_{n-1}, b_{n-1}, d_{n-1}, a_n\} \). Now \( T_{n-1} \cup T_n \cup d_n \) is not 3-separating in \( M \), and so \( D = D_{n-1} \). Thus \( \{\beta, \gamma\} \in \{a_{n-1}, a_n\} \), so \( a_n \) is in a triangle of \( M \setminus c_{n-1}, c_n \); a contradiction to 5, Lemma 6.1]. Hence 7.2.21 holds.

By 7.2.21 part (v) of [5, Lemma 6.1] does not hold. Now suppose that (iv) of 5, Lemma 6.1] holds. Then \( M \) has a triangle \( \{\alpha, a_{n-1}\} \) that differs from \( T_{n-1} \), and \( M \) has \( \{\alpha, a_{n-1}, c_{n-1}, d_{n-1}\} \) or \( \{\alpha, a_{n-1}, c_{n-1}, a_n, c_n\} \) as a cocircuit. By the minimality of \( m \), we deduce that \( \beta \neq c_{n-2} \). By 7.2.15 \( n \geq 2 \). Orthogonality between \( \{\alpha, \beta, a_{n-1}\} \) and \( D_{n-2} \) implies that \( \{\alpha, \beta\} \) meets \( \{b_{n-2}, c_{n-2}, b_{n-1}\} \), so \( b_{n-2} = \beta \), or \( \alpha \in \{b_{n-2}, c_{n-2}\} \).
7.2.22. \( b_{n-2} \neq \beta \).

Suppose that \( b_{n-2} = \beta \). We begin by locating the element \( \alpha \). Suppose first that \( \alpha \in Y \). Then \( \{b_{n-2}, a_{n-1}\} \) is a disjoint union of circuits in \( M \backslash c_0, c_1, \ldots, c_n \), so \( M \backslash c_0, c_1, \ldots, c_n, a_{n-1} \) has an \( N \)-minor. The last matroid has \( \{b_{n-2}, b_{n-1}\} \) as a disjoint union of cocircuits, so we get a contradiction to 7.2.1 Thus \( \alpha \notin Y \).

Suppose next that \( \alpha \in X \). Then, since \( \{\alpha, a_{n-1}, c_{n-1}, d_{n-1}\} \) or \( \{\alpha, a_{n-1}, c_{n-1}, a_n, c_n\} \) is a cocircuit in \( M \), either \( M \backslash c_0, c_1, \ldots, c_n, a_{n-1} \) or \( M \backslash c_0, c_1, \ldots, c_n, a_n \) has an \( N \)-minor. The second option contradicts 7.2.1If \( M \backslash c_0, c_1, \ldots, c_n, d_{n-1} \) has an \( N \)-minor, then, as the last matroid has \( \{b_{n-2}, a_{n-1}, b_{n-1}, a_n, b_n\} \) as a loose 5-cofan, \( M \backslash c_0, c_1, \ldots, c_n, d_{n-1}, b_n \) has an \( N \)-minor. This contradiction to 7.2.1 establishes that \( \alpha \notin X \).

We now know that \( \alpha \in E(M') \). Then \( \{b_{n-1}, b_{n-2}, a_{n-1}, \alpha, x\} \) is a loose 5-cofan in \( M \backslash c_0, c_1, \ldots, c_n \) for some \( x \in \{d_{n-1}, a_n\} \), so \( M \backslash c_0, c_1, \ldots, c_n, b_{n-1} \) has an \( N \)-minor. This contradiction to 7.2.1 completes the proof of 7.2.22.

From the above, if (iv) of \([5, \text{Lemma } 6.1]\) holds, then \( \alpha \notin \{b_{n-2}, c_{n-2}\} \). Thus, by Lemma 3.3 \( \{\alpha, a_{n-1}, c_{n-1}, d_{n-1}\} \) is not a cocircuit of \( M \), so \( \{\alpha, a_{n-1}, c_{n-1}, a_n, c_n\} \) is a cocircuit. Orthogonality with \( T_{n-2} \) implies that \( n = 2 \) and \( a_0 = c_2 \). By orthogonality with \( \{d_1, a_2, c_2, d_2\} \), the triangle \( T_0 \) meets \( \{d_1, d_2\} \). Thus \( T_0 \) is \( \{\alpha, c_2, d_1\} \) or \( \{\alpha, c_2, d_2\} \), so \( \lambda(T_1 \cup T_2 \cup \{\alpha, d_1, d_2\}) \leq 2 \); a contradiction. We conclude that (iv) of \([5, \text{Lemma } 6.1]\) does not hold.

It now follows that (iii) of \([5, \text{Lemma } 6.1]\) holds, that is, \( \{b_{n-1}, b_n\} \) is contained in a triangle. Recall that \( n \geq 2 \) by 7.2.15 Orthogonality with \( D_{n-2} \) implies that the third element of this triangle is in \( \{b_{n-2}, c_{n-2}\} \). If \( \{b_{n-2}, b_{n-1}, b_n\} \) is a triangle, then \( M \backslash c_0, c_1, \ldots, c_n \) has \( (a_n, b_n, b_{n-1}, a_{n-2}, a_{n-1}) \) as a loose 5-cofan, so \( M \backslash c_0, c_1, \ldots, c_n, a_n \) has an \( N \)-minor; a contradiction to 7.2.1 We conclude that

7.2.23. \( \{c_{n-2}, b_{n-1}, b_n\} \) is a triangle of \( M \).

Next we show that

7.2.24. \( \{c_n, d_n\} \) is not contained in a triangle.

Suppose \( \{c_n, d_n\} \) is contained in a triangle \( T \). Then \( (T, \{a_n, d_{n-1}, c_{n-1}\}, \{d_n, c_n, a_{n-1}, d_{n-1}\}) \) is a bowtie of \( M \). By 7.2.12 \( M \backslash c_n \) is \((4, 4, S)\)-connected and \( M \backslash a_n, c_n, c_{n-1} \) has an \( N \)-minor. Thus, by applying Hypothesis VII to the last bowtie, we get that \( M \backslash c_{n-1} \) is \((4, 4, S)\)-connected. This is a contradiction as the last matroid has a 5-fan. Hence 7.2.24 holds.

Consider the rotor chain \( ((c_n, a_n, b_n), (c_{n-1}, b_{n-1}, a_{n-1}), (c_{n-2}, b_{n-2}, a_{n-2})) \). If this rotor chain can be extended to the right by adjoining \( (x, y, z) \), then \( \{b_{n-2}, a_{n-2}, x, y\} \) is a cocircuit and \( \{a_{n-1}, b_{n-2}, x\} \) is a circuit. By orthogonality, \( x \in \{b_{n-3}, c_{n-3}\} \). Indeed, by orthogonality with \( D_{n-4} \), it follows that \( x = c_{n-3} \) unless \( n = 3 \). Thus \( \{x, y\} = \{b_{n-3}, c_{n-3}\} \), so \( z = a_{n-3} \). Continuing in this way, we see that a right-maximal bowtie chain that begins as above has one of the following forms:

(a) \( (c_n, a_n, b_n), (c_{n-1}, b_{n-1}, a_{n-1}), \ldots, (c_{\ell}, b_{\ell}, a_{\ell}) \) for some \( \ell \), which may be negative; or
(b) \( ((c_n, a_n, b_n), (c_{n-1}, b_{n-1}, a_{n-1}), \ldots, (c_1, b_1, a_1), (b_0, c_0, a_0), (c_{-1}, b_{-1}, a_{-1}), \ldots, (c_{\ell}, b_{\ell}, a_{\ell}) \) for some \( \ell \leq 0 \).
To eliminate the second possibility, assume that it arises. Recall that $M \setminus c_0, c_1, \ldots, c_n$ has an $N$-minor. The last matroid has $(b_0, b_1, a_2, b_2)$ as a 4-fan. Then $M \setminus c_0, c_1, \ldots, c_n \setminus b_0$ or $M \setminus c_0, c_1, \ldots, c_n \setminus b_2$ has an $N$-minor. In both cases, $M/b_1$ has an $N$-minor. To see this, observe in the second case that $M \setminus c_0, c_1, \ldots, c_n \setminus b_2 \cong M \setminus c_0, c_1, c_3, c_4, \ldots, c_n \setminus a_2/b_2 \cong M \setminus c_0, c_1, c_3, c_4, \ldots, c_n \setminus a_2/b_1$; in the first case, note that $M \setminus c_0, c_1, \ldots, c_n \setminus b_0$ has $b_1$ in a cocircuit of size at most two. Thus $M/b_1$ does indeed have an $N$-minor, and it follows by Lemma 3.11 that $M$ has a quick win; a contradiction. We conclude that (a) holds.

If $\ell > 0$, then $M \setminus c_0, c_1, \ldots, c_\ell$ has an $N$-minor. Now suppose that $\ell \leq 0$. Let $k$ be a non-negative integer such that $-k \geq \ell$. We argue by induction on $k$ that $M \setminus c_0, c_1, \ldots, c_\ell, c_{-1}, \ldots, c_{-k}$ has an $N$-minor. This is certainly true if $k = 0$. Hence we may assume that $k \geq 1$. Now $M \setminus c_0, c_{n-1}, \ldots, c_0, c_{-1}, \ldots, c_{-k+1}$ has $(c_{-k}, b_{-k+1}, a_{-k+2}, b_{-k+2})$ as a loose 4-fan. By Lemma 3.11, $M/b_{-k+2}$ does not have an $N$-minor. Thus $M \setminus c_0, c_1, \ldots, c_{-k}$ has an $N$-minor.

When $\ell \neq 0$, we will now adjust the notation to make $\ell = 0$ irrespective of whether it was originally positive or negative. This will change $n$ but we will continue to use the same symbol for this quantity noting that we still know that its value is at least two and that $M \setminus c_n, c_{n-1}, \ldots, c_0$ has an $N$-minor.

We show next that the elements in the rotor chain $((c_n, a_n, b_n), (c_{n-1}, b_{n-1}, a_{n-1}), \ldots, (c_0, b_0, a_0))$ are distinct. Suppose not. Then $a_0 = c_n$. Orthogonality between $T_0$ and $\{d_{n-1}, a_n, c_n, d_n\}$ implies that $T_0$ meets $\{d_{n-1}, d_n\}$. By [7.2.24] we see that $d_{n-1} \in T_0$ but $d_{n-1} \neq a_0$. Then $\{c_{n-1}, d_{n-1}, a_n\}$ meets $D_0$ in a single element; a contradiction to orthogonality. We conclude that the elements in the rotor chain are distinct.

Suppose $\{d_{n-1}, d_n\}$ meets a triangle in this rotor chain. Then orthogonality with the cocircuit $\{d_{n-1}, a_n, c_n, d_n\}$ implies that $\{d_{n-1}, d_n\}$ is contained in this triangle; a contradiction to [7.2.11]. Thus $M$ contains the structure in Figure 23 where all of the elements shown are distinct.

Since $N \preceq M \setminus c_1, c_0$ and $M \setminus c_1$ is not $(4, 4, S)$-connected, it follows by Hypothesis VII that $M \setminus c_0$ is not $(4, 4, S)$-connected. Now $M$ has a bowtie of the form $(T, \{z, b_1, c_0\}, D_1)$ where $z$ is $b_2$ when $n = 2$, and $z$ is $c_2$ otherwise. Applying Lemma 3.4 to this bowtie gives that $M$ has a quasi rotor having $\{z, b_1, c_0\}$ as its

**Figure 23.** These elements are all distinct.
central triangle. Moreover, \( M \setminus c_0 \) has a 5-fan \((p, q, s, t, u)\) whose elements are contained in this quasi rotor. Then \( \{q, s, t, c_0\} \) is a cocircuit \( D \) of \( M \). By orthogonality with \( \{z, b_1, c_0\} \), the cocircuit \( D \) meets either \( T_1 \) or \( T_2 \). If \( D \) meets \( T_2 \), then orthogonality implies that it is contained in \( T_0 \cup T_2 \), and \( \lambda(T_0 \cup T_1 \cup T_2) \leq 2 \); a contradiction. Thus \( D \) meets \( T_1 \). Lemma \ref{lem:orthogonality} implies that \( D = D_0 \). Since \( D_0 \cap D_1 = \{b_1\} \), it follows that \( b_1 \) is the central element of the quasi rotor. Thus \( \{b_0, a_1, g\} \) is a triangle for some element \( g \). By orthogonality with the cocircuits shown in Figure \ref{fig:7.2.23} we know that \( g \) is a new element.

Now \( M \setminus \{c_0, c_1, \ldots, c_n\} \) has an \( N \)-minor and has \( \{g, b_0, a_1, b_1\} \) as a loose 4-fan. Since, by Lemma \ref{lem:orthogonality} \( M/b_1 \) has no \( N \)-minor, we deduce that

7.2.25. \( M \setminus \{c_0, c_1, \ldots, c_n, g\} \) has an \( N \)-minor.

Now \( M \) has \( \{\{z, b_1, c_0\}, \{b_0, a_1, g\}, D_0\} \) as a bowtie where \( z = b_2 \) when \( n = 2 \) and is \( a_2 \) otherwise. As \( M \setminus g \) is not internally 4-connected, Lemma \ref{lem:orthogonality} implies that \( M \) has a 4-cocircuit of the form \( \{v, w, x, g\} \). By orthogonality with \( \{g, b_0, a_1\}, T_0 \), and \( T_1 \), we know that \( \{v, w, x\} \) contains two elements of \( T_0 \) or two elements of \( T_1 \). By Lemma \ref{lem:orthogonality} \( \{v, w, x\} \) avoids \( \{z, b_1, c_0\} \), so \( \{v, w, x, g\} \) contains \( \{a_0, b_0, g\} \) or \( \{a_1, c_1, g\} \). If the latter holds, then, since \( M \setminus \{c_0, c_1, g\} \) has an \( N \)-minor, we deduce that \( M \setminus \{c_0, c_1, g\}/a_1 \) has an \( N \)-minor. This gives a contradiction to Lemma \ref{lem:orthogonality} because \( a_1 \) is in two triangles of a quasi rotor of \( M \). Thus \( \{v, w, x, g\} = \{a_0, b_0, f, g\} \) for some element \( f \). By orthogonality with the triangles in Figure \ref{fig:7.2.23} we deduce that either \( f = d_n \), or \( f \) differs from all the elements in Figure \ref{fig:7.2.23}.

We show next that

7.2.26. \( \{f, g\} \) is not contained in a triangle of \( M \).

Suppose \( \{e, f, g\} \) is a triangle of \( M \). Since we constructed a right-maximal rotor chain, we know that \( \{e, f, g\} \) meets the set of elements in Figure \ref{fig:7.2.23}. But \( g \) avoids this set of elements, and so does \( f \) unless \( f = d_n \). By orthogonality with the cocircuits shown in Figure \ref{fig:7.2.23} it follows that \( f = d_n \) and \( e \in \{d_{n-1}, a_n, c_n\} \). Furthermore, orthogonality implies that \( e \notin D_{n-1} \), so \( e \neq a_n \). By \ref{lem:orthogonality} we deduce that \( e = c_n \), so \( \{c_n, d_n\} \) is contained in a triangle; a contradiction to \ref{lem:orthogonality}. Thus 7.2.26 holds.

![Figure 24](image)

Figure 24. These elements are all distinct except that \( f \) may be \( d_n \). No set in \( \{\{f, g\}, \{c_n, d_n\}, \{d_{n-1}, d_n\}\} \) is contained in a triangle. Deleting the dashed elements preserves an \( N \)-minor.
Now $M$ contains the structure in Figure 24. Moreover, by 7.2.11 and 6.3.3, none of $\{d_{n-1},d_n\}$, $\{c_n,d_n\}$, and $\{f,g\}$ is contained in a triangle. Next we show that

7.2.27. neither $M/a_1$ nor $M/b_0$ has an $N$-minor, and $\{f,g,a_0,b_0\}$ is the only 4-cocircuit in $M$ containing $a_0$.

Suppose $M/b_0$ has an $N$-minor. Then, since $M/b_0$ has $\{a_0,c_0\}$ and $\{g,a_1\}$ as circuits, $M/b_0\setminus a_1,c_0$ has an $N$-minor. This matroid is isomorphic to $M/b_1\setminus a_1,c_0$, so $M/b_1$ has an $N$-minor; a contradiction to Lemma 3.11. Thus $M/b_0$ has no $N$-minor. Moreover, an immediate consequence of Lemma 3.11 is that $M/a_1$ has no $N$-minor.

Now suppose that $a_0$ is in a 4-cocircuit $D$ other than $\{f,g,a_0,b_0\}$. Lemma 3.3 implies that $D$ avoids $T_1$. Orthogonality with $T_0$ implies that $D$ meets $\{b_0,c_0\}$, so orthogonality with $\{g,b_0,a_1\}$ and either $\{a_2,b_1,c_0\}$ or $\{b_2,b_1,c_0\}$ implies that $D$ contains $\{a_0,b_0,g\}$ or $\{a_0,c_0,z\}$, for some element $z$ in $\{t_2,b_2\}$. Since $D$ is not $\{f,g,a_0,b_0\}$, the latter holds. Then orthogonality implies that it is contained in $T_0\cup T_2$, and $\lambda(T_0\cup T_1\cup T_2) \leq 2$; a contradiction. Thus 7.2.27 holds.

7.2.28. $M\setminus c_0,c_1,\ldots,c_n, g$ is sequentially 4-connected.

To see this, first observe that, by Lemma 7.1, $M\setminus c_0,c_1,\ldots,c_n$ is sequentially 4-connected. The last matroid has $g$ in a triangle, so either it has $g$ in a triad, or $M\setminus c_0,c_1,\ldots,c_n,g$ is 3-connected. The former implies, by orthogonality with $\{g,b_0,a_1\}$, that $\{b_0,g\}$ or $\{a_1,g\}$ is contained in a triad of $M\setminus c_0,c_1,\ldots,c_n$. Then 7.2.25 implies that $M\setminus g,c_0,c_1,\ldots,c_n/x$ has an $N$-minor for some $x \in \{b_0,a_1\}$; a contradiction to 7.2.27. Thus $M\setminus c_0,c_1,\ldots,c_n,g$ is 3-connected. Suppose this matroid has a non-sequential 3-separation. Without loss of generality, we may assume that the triad $\{b_0,a_1,b_1\}$ is contained on one side of the 3-separation, and we can add $g$ to that side to get a non-sequential 3-separation of $M\setminus c_0,c_1,\ldots,c_n$; a contradiction. Thus 7.2.28 holds.

Next we show that

7.2.29. $M$ has an element $h$ such that $\{a_0,f,h\}$ is a triangle and $M\setminus g,h$ has an $N$-minor.

Since $M$ has no open-rotor-chain win, $M\setminus c_0,c_1,\ldots,c_n,g$ has a 4-fan $(1,2,3,4)$, so $M$ has a cocircuit $C^*$ such that $\{2,3,4\} \subset C^* \subseteq \{2,3,4,c_0,c_1,\ldots,c_n\}$. By orthogonality with the cocircuits in Figure 24, we deduce that $\{1,2,3\}$ can only meet the set of elements in that figure if it contains $\{d_{n-1},d_n\},\{c_n,d_n\},\{f,g\}$, or $\{a_0,f\}$. The first three possibilities have been excluded, so either $\{1,2,3\}$ avoids the set of elements in Figure 24 or $\{a_0,f\} \subseteq \{1,2,3\}$. Suppose the latter occurs. Then $M$ has a triangle of the form $\{a_0,f,h\}$, so $M\setminus g$ has $(h,f,a_0,b_0)$ as a 4-fan. By 7.2.27, $M\setminus g/b_0$ does not have an $N$-minor. Thus $M\setminus g,h$ has an $N$-minor, so 7.2.29 holds. We may now assume that $\{1,2,3\}$ avoids the set of elements in Figure 24. Suppose $c_i \in C^*$ for some $i \in \{0,1,\ldots,n-1\}$. Then, since $c_i$ is in two triangles in Figure 24, orthogonality with these triangles implies that $\{2,3,4\}$ contains two elements in this figure. Hence $\{2,3\}$ contains an element in the figure; a contradiction. Thus $C^*$ avoids $\{c_0,c_1,\ldots,c_{n-1}\}$, so $C^* \subseteq \{2,3,4,c_n\}$. Furthermore, orthogonality with $\{g,b_0,a_1\}$ and $T_n$ implies that either $C^* = \{2,3,4,g\}$ and $4 \in \{b_0,a_1\}$, or $C^* = \{2,3,4,c_n\}$ and $4 \in \{a_n,b_n\}$. Thus $C^*$ meets one of the triangles $T_0,T_1,\{c_{n-1},d_{n-1},a_n\}$, or $\{c_{n-2},b_{n-1},b_n\}$ in a single element; a contradiction to orthogonality. Thus 7.2.29 holds.
By orthogonality with the cocircuits in Figure 24, either \(h\) differs from all the elements in that figure, or \(d_n = f\) and \(h \in \{c_n, d_{n-1}\}\). By 7.2.11 and 7.2.24 we deduce that \(h\) is a new element.

We now show that

**7.2.30.** \(M' \setminus h\) is \((4, 4, S)\)-connected, and every 4-fan of this matroid has \(f\) as its coguts element.

Let \((z, y, x, w)\) be a 4-fan in \(M' \setminus h\). Then \(\{w, x, y, h\}\) is a cocircuit of \(M\), and orthogonality with \(\{a_0, f, h\}\) implies that \(\{w, x, y\}\) meets \(\{a_0, f\}\) in a single element. By 7.2.27 \(a_0 \not\in \{w, x, y\}\), so \(f \in \{w, x, y\}\). Suppose \(f \in \{x, y, z\}\). Then orthogonality with \(\{f, g, a_0, b_0\}\) implies that \(\{x, y, z\}\) meets \(\{g, a_0, b_0\}\). By 7.2.26 \(g \not\in \{x, y, z\}\). As \(\{x, y, z\}\) does not contain \(\{e_0, f\}\), it must contain \(\{f, b_0\}\). By orthogonality with \(D_0\), the triangle \(\{x, y, z\}\) is \(\{f, b_0, a_1\}\); a contradiction to orthogonality with \(D_1\). Thus \(f = w\), that is, \(f\) is the coguts element of every 4-fan of \(M' \setminus h\). Thus \(M' \setminus h\) has no 5-fan. It follows by Lemma 3.4 that \(M' \setminus h\) is \((4, 4, S)\)-connected. Thus 7.2.30 holds.

Since \(M\) has no quick win, \(M' \setminus h\) has a 4-fan \((z_0, y_0, x_0, f)\). Thus \((\{a_0, f, h\}, \{x_0, y_0, z_0\}, \{f, h, x_0, y_0\})\) is a bowtie.

**7.2.31.** \(M' \setminus f\) has an \(N\)-minor.

To show this, we assume the contrary. Then \(M' \setminus z_0\) has an \(N\)-minor. Extend the bowtie \((\{a_0, f, h\}, \{x_0, y_0, z_0\}, \{f, h, x_0, y_0\})\) to a right-maximal bowtie string \(\{a_0, f, h\}, \{f, h, x_0, y_0\}, \{x_0, y_0, z_0\}, \ldots, \{x_k, y_k, z_k\}\). Now \(M' \setminus z_0/ y_0 \cong M' \setminus h, x_0/ f\). Thus \(M' \setminus h, z_0/ y_0\) has no \(N\)-minor. Therefore, by Lemma 3.7, \(M' \setminus h, z_0, z_1, \ldots, z_k\) has an \(N\)-minor.

To complete the proof of 7.2.31, we aim to apply [5, Lemma 10.1], but first we need to show that

**7.2.32.** \((\{x_k, y_k, z_k\}, \{a_0, f, h\}, \{\gamma, z_k, a_0, f\})\) is not a bowtie for all \(\gamma \in \{x_k, y_k\}\).

Assume that \(M\) contains such a bowtie. Then, by 7.2.27 \(\{\gamma, z_k\} = \{g, b_0\}\). By Lemma 3.3 \(k > 0\). Suppose \(z_k = b_0\). Then \(M' \setminus b_0\) has an \(N\)-minor. This matroid has \(\{c_1, a_1, b_1, c_0, z\}\) as a 5-fan where \(z = b_2\) unless \(n = 2\), in which case \(z = b_2\). By Lemma 3.2 \(M' \setminus b_0, c_1, z\) or \(M' \setminus b_0, c_1, a_1\) has an \(N\)-minor. The former implies that \(M/b_1\) has an \(N\)-minor, so in both cases we get a contradiction to Lemma 3.11. We conclude that \(z_k \neq b_0\), so \(z_k = g\). Now \(M' \setminus h, z_0, z_1, \ldots, z_k\) has an \(N\)-minor, and, by 7.2.30 \(M' \setminus h\) is \((4, 4, S)\)-connected. By Hypothesis VII, it follows that \(M' \setminus z_0\) is \((4, 4, S)\)-connected, that \(M' \setminus z_i\) is \((4, 4, S)\)-connected for all \(i \in \{1, 2, \ldots, k-1\}\), and that \(M' \setminus g\) is \((4, 4, S)\)-connected. But \(M' \setminus g\) has a 5-fan; a contradiction. Thus 7.2.32 holds.

We can now apply [5, Lemma 10.1]. None of (i), (ii), or (v) of that lemma holds. If (iii) holds, then \(a_0\) is in a 4-cocircuit with \(h\); a contradiction to 7.2.27. Thus (iv) holds, so \(M' \setminus h, z_0/ y_0\) or \(M' \setminus h, z_0/ x_0\) has an \(N\)-minor. Lemma 3.6 implies that \(M' \setminus f\) has an \(N\)-minor; a contradiction. We conclude that 7.2.31 holds.

Since \(M\) has no quick win, we know that \(M' \setminus h/ f\) is not internally 4-connected. We now apply Lemma 3.8 to the bowtie string \(\{a_1, g, b_0\}, \{b_0, g, a_0, f\}, \{a_0, f, h\}, \{f, h, x_0, y_0\}\) to obtain that \(M\) has \(\{a_0, h\}\) contained in a 4-cocircuit; a contradiction to 7.2.27. □
Lemma 7.3. Let $M$ and $N$ be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 7$. Suppose Hypothesis VII holds. Suppose that $M$ contains configuration (A) labelled as in Figure 25 where $M \setminus 4$ is $(4, 4, S)$-connected with an $N$-minor, but $N \not\preceq M \setminus 1, 4$. Then

(i) $M$ has a quick win; or
(ii) $M^*$ has an open-rotor-chain win, a ladder win, or an enhanced-ladder win; or
(iii) deleting the central cocircuit of an augmented 4-wheel in $M^*$ gives an internally 4-connected matroid with an $N^*$-minor.

Proof. Assume the lemma does not hold. First we show that

7.3.1. $M \setminus 6/8$ and $M/8$ are 3-connected, and $M/5 \setminus 6/8$ is 3-connected with an $N$-minor.

Clearly $M/5 \setminus 4 \cong M/5 \setminus 6$. As $N \not\preceq M \setminus 1, 4$, it follows by Lemma 3.1 that $M/5 \setminus 6$ is $(4, 4, S)$-connected having an $N$-minor. Since $M/5 \setminus 6$ has $(2, 4, 7, 8)$ as a 4-fan, both $M/5 \setminus 6/8$ and $M/5 \setminus 6/2$ are 3-connected, and at least one of these matroids has an $N$-minor. As $M/5 \setminus 6, 2 \cong M/5 \setminus 4, 2 \cong M/4, 2/3 \cong M/3 \setminus 4, 1$, and $N \not\preceq M \setminus 1, 4$, we deduce that $N \preceq M/5 \setminus 6/8$.

By Lemma 5.1 we know that $M \setminus 6$ is $(4, 4, S)$-connected. As $M \setminus 6$ has $(2, 4, 7, 8)$ as a 4-fan, $M \setminus 6/8$ is 3-connected. Thus $M/8$ is 3-connected unless $M$ has a triangle.
containing \( \{6, 8\} \). In the exceptional case, \( M \setminus 4/5 \) has a 5-fan containing \( \{2, 6, 7, 8\} \). This contradiction completes the proof of 7.3.1.

Next we note that

7.3.2. \( M/8 \) is sequentially 4-connected.

Suppose that \( M/8 \) has a non-sequential 3-separation \((U, V)\). Then, by Lemma 3.3, we may assume that \( \{1, 2, \ldots, 7\} \subseteq U \). Thus we can adjoin 8 to \( U \) to obtain a non-sequential 3-separation of \( M \); a contradiction. Thus 7.3.2 holds.

Next we show that

7.3.3. \( M \) contains one of the configurations \((F)\) and \((G)\) in Figure 25, where all of the elements shown are distinct, and every 4-fan in \( M/8 \) has its guts element in \( \{6, 7\} \).

To see this, note that, since \( N \leq M/8 \) but \( M \) has no quick win, \( M/8 \) has a 4-fan \((u_1, u_2, u_3, u_4)\). Thus \( M \) has \( \{u_1, u_2, u_3, 8\} \) as a circuit and has \( \{u_2, u_3, u_4\} \) as a triad. By orthogonality, \( \{u_1, u_2, u_3\} \) meets \( \{4, 6, 7\} \), so \( v_1 \in \{4, 6, 7\} \). If \( u_1 = 4 \), then, by orthogonality, 2, 3, or 5 is in \( \{u_2, u_3, u_4\} \), so \( M \) has a 4-fan; a contradiction. Thus \( u_1 \in \{6, 7\} \). By construction, \( 8 \notin \{u_2, u_3, u_4\} \). As \( \{u_2, u_3, u_4\} \) is a triad, this set also avoids \( \{1, 2, \ldots, 7\} \). Hence 7.3.3 holds.

Next we show the following.

7.3.4. If \( M \) contains configuration \((G)\), then \( M \setminus 6/8 \) is sequentially 4-connected.

By 7.3.1 we know that \( M \setminus 6/8 \) is 3-connected. Let \( (U, V) \) be a non-sequential 3-separation of \( M \setminus 6/8 \). Then we may assume that \( \{u_2, u_3, u_4\} \subseteq U \). Thus \( (U \cup 6, V) \) is a non-sequential 3-separation of \( M/8 \); a contradiction to 7.3.2. Hence 7.3.4 holds.

7.3.5. If \( M \) contains configuration \((G)\), then \( M \) contains configuration \((F)\).

To see this, observe, since \( M \setminus 6/8 \) is sequentially 4-connected with an \( N \)-minor, but (i) does not hold, \( M \setminus 6/8 \) has a 4-fan \((v_1, v_2, v_3, v_4)\). Suppose first that \( \{v_2, v_3, v_4\} \) is a triad of \( M \). Then the 4-fan is a fan in \( M \), and 7.3.3 implies that \( v_1 = 7 \). Hence \( M \) contains configuration \((F)\) where \( v_i \) replaces \( u_i \) for each \( i \) in \( \{1, 2, 3\} \).

We may now assume that \( \{v_2, v_3, v_4, 6\} \) is a cocircuit of \( M \). Lemma 3.3 implies that \( \{v_2, v_3, v_4\} \) avoids \( \{1, 2, 3\} \). Orthogonality implies that \( \{4, 5\} \) meets \( \{v_2, v_3, v_4\} \). Suppose \( 4 \in \{v_2, v_3, v_4\} \). Then, by orthogonality, \( \{2, 7\} \) meets \( \{v_2, v_3, v_4\} \). Thus \( \{v_2, v_3, v_4\} \) contains \( \{4, 7\} \), so \( \{6, v_2, v_3, v_4\} = \{6, 4, 7, 8\} \); a contradiction. We conclude that \( 4 \notin \{v_2, v_3, v_4\} \). Thus \( 5 \in \{v_2, v_3, v_4\} \).

By Lemma 5.1 we know that \( \{4, 5, 6\} \) is the only triangle containing 5. Suppose that \( 5 \in \{v_2, v_3\} \). Then, without loss of generality, \( \{v_1, v_2, 5, 8\} \) is a circuit. In this case, orthogonality with \( \{2, 3, 4, 5\} \) and \( \{4, 6, 7, 8\} \) implies that either \( 4 \notin \{v_1, v_2\} \), or \( \{v_1, v_2\} = \{7, z\} \), for some \( z \) in \( \{2, 3\} \). If \( 4 \in \{v_1, v_2\} \), then \( \{v_1, v_2, 5, 8\} \Delta \{4, 5, 8, u_2, u_3\} \) is a triangle meeting \( \{u_2, u_3\} \); a contradiction. Thus \( \{v_1, v_2\} = \{7, z\} \), for some \( z \) in \( \{2, 3\} \), and \( \lambda(\{1, 2, \ldots, 8\}) \leq 2 \); a contradiction. We conclude that \( 5 \notin \{v_2, v_3\} \), so \( 5 = v_4 \). Moreover, by orthogonality between \( \{6, 8, u_2, u_3\} \) and \( \{v_2, v_3, 5, 6\} \), we deduce that \( \{u_2, u_3\} \) meets \( \{v_2, v_3\} \). Thus, we may assume that \( v_3 = u_3 \). As \( u_3 \) is not in a triangle of \( M \), the set \( \{v_1, v_2, u_3, 8\} \) is a circuit. By orthogonality, \( \{4, 7\} \) and \( \{u_2, u_4\} \) meet \( \{v_1, v_2\} \). Hence \( v_2 \in \{4, 7, u_2, u_4\} \). By orthogonality between \( \{v_2, v_3, 5, 6\} \) and the circuits \( \{2, 4, 7\} \) and \( \{6, 8, u_2, u_3\} \), we
know that \( v_2 \notin \{4, 7, u_2\} \). Hence \( v_2 = u_4 \). Then \( \{u_4, u_3, 5, 6\} \Delta \{u_2, u_3, u_4\} \), which is \( \{u_2, 5, 6\} \), is a triad; a contradiction. We conclude that 7.3.5 holds.

We may now assume that \( M \) contains configuration (F). We relabel \( (u_2, u_3, u_4) \) as \( (a_0, b_0, c_0) \), for reasons that will become clear later. We show next that

7.3.6. \( M/5\{6/8\}/c_0 \) has an \( N \)-minor but \( M/5\{6/8\}/7 \) does not.

As \( M/5\{6/8\} \) has an \( N \)-minor and has \( (7, a_0, b_0, c_0) \) as a 4-fan, we deduce that \( N \leq M/5\{6/8\}/7 \) or \( N \leq M/5\{6/8\}/c_0 \). In the first case, as \( M/5\{6/8\}/7 \cong M/5\{6,7/4\} \), we see that \( N \leq M/5,4 \). Thus, by Lemma 7.1, we deduce that \( N \leq M/4,1 \); a contradiction. We conclude that 7.3.6 holds.

From 7.3.6 \( N \leq M/c_0 \). Since \( c_0 \) is in a triad of \( M \), we see that \( M/c_0 \) is 3-connected.

7.3.7. \( M/c_0 \) is sequentially 4-connected.

To show this, suppose that \( (U, V) \) is a non-sequential 3-separation of \( M/c_0 \). Then, by [4, Lemma 3.3], we may assume that \( \{1, 2, \ldots, 7\} \subseteq U \). Hence we may also assume that \( 8 \in U \). Now if \( a_0 \) or \( b_0 \) is in \( U \), then we may assume that both are in \( U \), in which case we can adjoin \( c_0 \) to \( U \) to get a non-sequential 3-separation of \( M \); a contradiction. Hence \( \{a_0, b_0\} \subseteq V \) and we can adjoin \( c_0 \) to \( V \) to get a non-sequential 3-separation of \( M \); a contradiction. Thus 7.3.7 holds.

Next we observe the following.

7.3.8. If \( a_0 \) or \( b_0 \) is the guts element of a 4-fan of \( M/c_0 \), then, up to switching the labels on \( a_0 \) and \( b_0 \), the matroid \( M \) contains structure (J) in Figure 25 where all of the elements shown are distinct.

We need only check that the elements are distinct. Clearly \( \{a_1, b_1, c_1\} \) avoids \( \{1, 2, \ldots, 7, b_0, c_0\} \). If \( \{a_1, b_1, c_1\} \) meets \( \{a_0, 8\} \), then it contains this set. Now \( a_0 \notin \{a_1, b_1\} \) as \( M \) is binary, so \( a_0 = c_1 \). Then \( 8 \in \{a_1, b_1\} \) and we get a contradiction to orthogonality. Hence 7.3.8 holds.

We now show that

7.3.9. \( M \) contains structure (J) in Figure 25 where all of the elements shown are distinct and the labels on \( a_0 \) and \( b_0 \) may be interchanged.

As \( M \) has no quick win, by 7.3.7 \( M/c_0 \) has a 4-fan \( (\alpha, \beta, \gamma, \delta) \). Thus \( M \) has \( \{\alpha, \beta, \gamma, c_0\} \) as a circuit. By orthogonality, \( \{a_0, b_0\} \) meets \( \{\alpha, \beta, \gamma\} \). If \( \alpha \in \{a_0, b_0\} \), then the result follows by 7.3.8. Thus we may assume that \( \gamma = b_0 \). Then \( \{\beta, \gamma, \delta\} \) contains exactly two elements of \( \{a_0, b_0, 7, 8\} \). Now \( 7 \notin \{\beta, \gamma, \delta\} \), and \( \{\beta, \gamma, \delta\} \) does not contain \( \{a_0, b_0\} \), so \( 8 \in \{\beta, \delta\} \).

As the next step towards proving 7.3.9 we now show that

7.3.10. \( M \) does not have \( \{2, 5, a_0, c_0\} \) or \( \{6, 8, b_0, c_0\} \) as a circuit.

Since \( M \) has \( \{a_0, b_0, 2, 5, 6, 8\} \) as a circuit and \( \{a_0, b_0, 2, 5, 6, 8\} \Delta \{2, 5, a_0, c_0\} = \{6, 8, b_0, c_0\} \), it suffices to prove that \( \{2, 5, a_0, c_0\} \) is not a circuit. Assume otherwise. By 7.3.6 we know that \( M/5,8/c_0 \) has an \( N \)-minor. Hence so does \( M/5,8/c_0 \), \( 6 \). But \( M/5,8/c_0 \) \( \cong M/5,8/c_0 \), \( 4 \), \( 2 \) \( \cong M/4,2/3,8/c_0 \) \( \cong M/3,8/c_0 \), \( 4,1 \), so \( N \leq M/4,1 \); a contradiction. Hence 7.3.10 holds.

Recall that \( 8 \in \{\beta, \delta\} \). Suppose that \( 8 = \beta \). Then, by orthogonality between \( \{\alpha, 8, b_0, c_0\} \) and the cocircuits \( \{4, 6, 7, 8\} \) and \( \{2, 3, 4, 5\} \), it follows that \( \alpha \in \{6, 7\} \). As \( \{7, 8, a_0, b_0\} \) is also a circuit, \( \alpha \neq 7 \), so \( \alpha = 6 \) and we have a contradiction to 7.3.10. We deduce that \( 8 \neq \beta \), so \( 8 = \delta \). Hence \( M/c_0 \) has \( (\alpha, \beta, b_0, 8) \) as a 4-fan.
7.3.11. $M$ has $\{\alpha, \beta, b_0, c_0\}$ as a circuit and has $\{\beta, b_0, 8\}$ and $\{a_0, b_0, c_0\}$ as triads, these being its only triads containing $b_0$.

The first part of this is immediate. By orthogonality, a triad containing $b_0$ must contain $a_0$ or 8, so the last part also holds.

By 7.3.6 we see that $N \subseteq M/c_0, 8$. Still aiming at obtaining 7.3.9 we show next that

7.3.12. $M/c_0, 8$ is 3-connected.

Assume the contrary. As $M/c_0$ is 3-connected having $(\alpha, \beta, b_0, 8)$ as a 4-fan, we deduce that $M/c_0$ has $\{8, b_0\}$ or $\{8, \beta\}$ in a triangle. Suppose $\{8, \beta\}$ is in a triangle of $M/c_0$. Then $\{8, \beta, c_0\}$ is contained in a 4-circuit of $M$, which, by orthogonality, must be $\{8, \beta, c_0, a_0\}$ or $\{8, \beta, c_0, b_0\}$. But $M$ has $\{\alpha, \beta, b_0, c_0\}$ as a circuit. As $\alpha \neq 8$, it follows that $\{8, \beta, c_0, a_0\}$ is a circuit. By taking the symmetric difference of the last two circuits, we deduce that $\{\alpha, 8, a_0, b_0\}$ is a circuit, so $\alpha = 7$. Then $\{\beta, b_0, c_0\}$ is a circuit, so $\beta \in \{4, 6\}$; a contradiction to orthogonality. We conclude that $\{8, \beta\}$ is not in a triangle of $M/c_0$. Thus $M/c_0$ has $\{8, b_0\}$ contained in a triangle, so $M$ has $\{8, b_0, c_0\}$ contained in a 4-circuit. By orthogonality, this 4-circuit is $\{8, b_0, c_0, 6\}$; a contradiction to 7.3.10. We conclude that 7.3.12 holds.

By 7.3.7 $M/c_0$ is sequentially 4-connected. It follows that $M/c_0, 8$ is sequentially 4-connected for if $(U, V)$ is a non-sequential 3-separation of the last matroid, then we may assume that $\{\alpha, \beta, b_0\} \subseteq U$, so $(U \cup 8, V)$ is a non-sequential 3-separation of $M/c_0$; a contradiction.

Since $M$ has no quick win, $M/c_0, 8$ has a 4-fan $(s_1, s_2, s_3, s_4)$. Thus $M$ has a circuit $C$ such that $\{s_1, s_2, s_3\} \subseteq C \subseteq \{s_1, s_2, s_3, c_0, 8\}$, and $\{s_2, s_3, s_4\}$ avoids $\{1, 2, \ldots, 7, 8, c_0\}$. Suppose $c_0 \notin C$. Then $C = \{s_1, s_2, s_3, 8\}$. By orthogonality, $s_1 \in \{4, 6, 7\}$. But $s_1 \neq 4$ otherwise $\{s_2, s_3\}$ meets $\{2, 3, 5\}$. Thus $s_1 \in \{6, 7\}$. Now $M$ has $\{\beta, b_0, 8\}$ as a triad, so $\{s_2, s_3\}$ meets $\{\beta, b_0\}$ in a single element. The triad $\{s_2, s_3, s_4\}$ avoids $\{c_0, 8\}$ so differs from both $\{a_0, b_0, c_0\}$ and $\{b_0, 8\}$. Hence, by 7.3.11 $b_0 \notin \{s_2, s_3, s_4\}$, so $\beta \in \{s_2, s_3\}$. Thus, by orthogonality between $\{s_2, s_3, s_4\}$ and $\{\alpha, \beta, b_0, c_0\}$, we deduce that $\alpha \in \{s_2, s_3, s_4\}$. It follows that $M/c_0$ has a 4-fan having $b_0$ as its guts element and $\{s_2, s_3, s_4\}$ as its triad; a contradiction to 7.3.8. We conclude that $c_0 \in C$.

Suppose $C = \{s_1, s_2, s_3, c_0\}$. By orthogonality and 7.3.8 $\{s_2, s_3\}$ meets $\{a_0, b_0\}$. As $\{s_2, s_3, s_4\}$ is not $\{a_0, b_0, c_0\}$, it follows by orthogonality that $\{s_2, s_3, s_4\}$ meets $\{7, 8\}$; a contradiction. We deduce that $C \neq \{s_1, s_2, s_3, c_0\}$, so $C = \{s_1, s_2, s_3, c_0, 8\}$. Then, by orthogonality, $s_1 \in \{4, 6, 7\}$, and $\{s_2, s_3\}$ meets $\{a_0, b_0\}$. By 7.3.11 $b_0 \notin \{s_2, s_3, s_4\}$ so $a_0 \in \{s_2, s_3\}$. Thus, by orthogonality between $\{s_2, s_3, s_4\}$ and $\{a_0, b_0, 7, 8\}$, we obtain a contradiction. We conclude that 7.3.9 holds.

We may now assume that $M^*$ has $(T_0, T_1, D_0)$ as a bowtie. In $M^*$, take a right-maximal bowtie string $T_0, D_0, T_1, D_2, \ldots, T_n$ noting that $n$ may equal 1. Next we show the following.

7.3.13. The elements in $\{1, 2, \ldots, 8\} \cup T_0 \cup T_1 \cup \cdots \cup T_n$ are distinct.

The triangles in this string are triads in $M$, so the elements in the string avoid $\{1, 2, \ldots, 7\}$. Therefore 7.3.13 holds unless either $a_0 = c_n$, or 8 is in the bowtie string. Suppose $8 \in T_i$. Then, since $8 \notin T_0$, orthogonality between $T_i$ and $\{7, 8, a_0, b_0\}$ implies that $a_0 = c_0$ and $i = n$. Then $8 \in \{a_n, b_n\}$. By orthogonality, $\{b_{n-1}, c_{n-1}, a_n, b_n\}$ meets $\{4, 6, 7\}$, so a triangle in $M$ meets a triangle in
$M^*$; a contradiction. We conclude that 8 is not in the bowtie string. By orthogonality between $T_n$ and the cocircuit $\{7, 8, a_0, b_0\}$ in $M^*$, we see that $c_n \neq a_0$. Thus 7.3.13 holds.

We want to apply Lemma 7.2 to $M^*$. Since $M^*$ contains both triangles and triads, it is not isomorphic to the cycle matroid of a quartic Möbius ladder. Moreover, if $M^*$ is isomorphic to the cycle matroid of a terrahawk, then so is $M$, and therefore $M$ has a second triangle containing 5; a contradiction to Lemma 5.1. Let $X = \{5, 8\}$, let $Y = \{6\}$, and let $M' = M^*/X/Y$. In $M'$, each $T_i$ is a disjoint union of circuits while each $D_j$ is a disjoint union of cocircuits. Since $M'$ is 3-connected, it follows that $T_0, D_0, T_1, D_2, \ldots, T_n$ is a bowtie string in $M'$.

7.3.14. $M\setminus c_0, c_1, \ldots, c_n$ has an $N^*$-minor and $M\setminus c_0, c_1, \ldots, c_i/e$ has no $N^*$-minor for all $e$ in $\{a_i, b_i\}$ where $i \in \{1, 2, \ldots, n\}$.

To see this, observe first that, by 7.3.6 $M\setminus c_0$ has an $N^*$-minor but $M'/7$ does not. It follows from this that $M\setminus a_0/b_0$ has no $N^*$-minor since $M\setminus a_0$ has $\{b_0, 7\}$ as a cocircuit, so $M\setminus a_0/b_0 \cong M\setminus a_0/7$. Suppose $M\setminus c_0, c_1, \ldots, c_i$ has an $N^*$-minor for some $i \in \{0, 1, \ldots, n - 1\}$. As this matroid has $(c_{i+1}, a_{i+1}, b_{i+1}, b_i)$ as a 4-fan, either $M\setminus c_0, c_1, \ldots, c_{i+1}$ has an $N^*$-minor or $M\setminus c_0, c_1, \ldots, c_i/b_i$ has an $N^*$-minor. By Lemma 3.6 the latter implies that $M\setminus a_0, a_1, \ldots, a_i/b_0$ has an $N^*$-minor. Hence so does $M\setminus a_0/b_0$; a contradiction. It follows by induction that $M\setminus c_0, c_1, \ldots, c_n$ has an $N^*$-minor. Furthermore, if $M\setminus c_0, c_1, \ldots, c_i/a_i$ has an $N^*$-minor for some $i$ in $\{1, 2, \ldots, n\}$, then Lemma 3.6 implies that $M\setminus a_0, a_1, \ldots, a_{i-1}, b_i/b_0$ has an $N^*$-minor, so $M\setminus a_0/b_0$ has an $N^*$-minor; a contradiction. Thus 7.3.14 holds.

Clearly none of (i), (ii), and (vii) of Lemma 7.2 holds in $M^*$.

7.3.15. Neither (iii) nor (v) of Lemma 7.2 holds in $M^*$.

To see this, observe first that, by 7.3.14 it follows that (v) of Lemma 7.2 does not hold. Suppose (iii) of Lemma 7.2 holds in $M^*$. Then $\{a_0, b_0, z, c_0\}$ is a 4-circuit of $M$ for some $z$ in $\{a_n, b_n\}$. Taking the symmetric difference of this circuit with $\{7, 8, a_0, b_0\}$, we see that $\{7, 8, z, c_0\}$ is a circuit of $M$. But $M/5/6/8, c_n$ has an $N$-minor and has 7 in a 2-circuit. Hence $M/5/6/8, c_7$ has an $N$-minor; a contradiction to 7.3.6. We deduce that 7.3.15 holds.

Suppose $M^*$ contains the structure in Figure 18 where $d_1 \in Y$ and either $d_0 \in X$ or $M\setminus c_0, c_1, d_0$ has an $N^*$-minor. Then $d_1 = 6$, and $\{d_0, a_1, c_1, 6\}$ is a circuit in $M$. By orthogonality with $\{4, 6, 7, 8\}$, we know that $d_0 \in \{4, 7, 8\}$. But $\{b_0, a_0, a_1\}$ is a triad of $M$, so it avoids $\{4, 7\}$. Thus $d_0 = 8$. The symmetric difference $\{4, 5, 6\} \triangle \{6, 8, a_1, c_1\}$ is $\{4, 5, 8, a_1, c_1\}$, which must be a circuit in $M$. Now $M/5/6/8/c_0/c_1$ has $\{4, a_1\}$ as a circuit. Hence $M/5/6/8/c_0/c_1/a_1$ has an $N$-minor and (v) of Lemma 7.2 holds; a contradiction. Thus $M^*$ does not contain the structure in Figure 18.

Suppose $M^*$ contains the structure in Figure 17(a). Then $M$ has $\{p, t, a_0\}, \{a_0, b_0, c_0\}$, and $\{b_0, b_1, q\}$ as distinct cocircuits. By orthogonality with the circuit $\{7, 8, a_0, b_0\}$, each of $q$ and $\{p, t\}$ meets $\{7, 8\}$ in a unique element. As 7 is in no triad of $M$, we deduce that $q = 8$ and $p \in \{t\}$; a contradiction. Suppose next that $M^*$ contains the structure in Figure 17(b). Then $\{b_0, b_1, q\}$ and $\{s_1, s_2, s_3\}$ are disjoint cocircuits of $M$ and $\{b_1, c_1, q, s_2, s_3\}$ is a circuit of $M$. By orthogonality with the circuit $\{7, 8, a_0, b_0\}$, it follows that $q = 8$. By orthogonality with the cocircuit $\{4, 6, 7, 8\}$, we see that $\{b_1, c_1, s_2, s_3\}$ meets
{4, 6, 7}; a contradiction. We conclude that $M^*$ contains neither of the structures in Figure 17.

By Lemma 7.2 it follows that $M^*$ contains the structure in Figure 12 and \{d_{n-1}, d_n\} is not contained in a triangle of $M^*$. By [5, Lemma 6.4], as $a_0 \neq c_n$ and \{d_{n-1}, d_n\} is not contained in a triangle, we know that the elements in this figure are all distinct with the possible exception that $\alpha$ and $\beta$ may be repeated elements. Thus $\{\alpha, \beta, a_0\}$ is a triangle of $M^*$ distinct from $T_0$, and $\{\alpha, a_0, c_0, d_0\}$ or $\{\alpha, a_0, c_0, a_1, c_1\}$ is a cocircuit of $M^*$ for some elements $\alpha, \beta$, and $d_0$. Furthermore, $\{c_0, d_0, a_1\}$ is a triangle of $M^*$. Since $M^*$ has \{7, 8, a_0, b_0\} as a cocircuit, orthogonality implies that $\{\alpha, \beta\}$ meets \{7, 8\}. Clearly 7 avoids $\{\alpha, \beta\}$, so $8 \in \{\alpha, \beta\}$.

7.3.16. $\beta = 8$.

To show this, suppose that $\alpha = 8$. Then orthogonality implies that \{4, 6, 7\} meets \{a_0, c_0, d_0\} or \{a_0, c_0, a_1, c_1\}. Thus $M^*$ is not internally 4-connected; a contradiction. Thus 7.3.16 holds.

We relabel 7 as $\gamma$ to see that $M^*$ contains the structure in Figure 11 where the elements are all distinct with the possible exception that $\alpha, \beta$, and $\gamma$ may be repeated elements.

In preparation for applying Lemma 4.1 we now show the following.

7.3.17. The elements in Figure 11 are distinct except that $\gamma$ and $d_n$ may be equal.

As $\gamma$ is in a triad of $M^*$, we know that $\gamma$ avoids all of the other elements in Figure 11 with the possible exception of $d_n$. By orthogonality between the triangles in this figure and the cocircuits $\{\beta, \gamma, a_0, b_0\}$ and $\{\alpha, a_0, c_0, d_0\}$, we deduce that $\beta$ and $\alpha$ avoid all of the elements with the possible exception of $d_n$. Thus the elements are all distinct except that $d_n$ may be in $\{\alpha, \beta, \gamma\}$. But orthogonality between $\{d_{n-1}, a_n, c_n, d_n\}$ and $\{\alpha, \beta, a_0\}$ implies that $d_n \notin \{\alpha, \beta\}$. Thus 7.3.17 holds.

By 7.3.16 and Lemma 7.2 $M^*$ has no triangle $T_{n+1}$ such that $\{x, c_n, a_{n+1}, b_{n+1}\}$ is a 4-cocircuit for any $x$ in $\{a_n, b_n\}$. Then Lemma 3.4 implies that $M^* \setminus c_n$ is (4, 4, *S*)-connected.

By 7.3.14 $M^* \setminus c_0, c_1, \ldots, c_n$ has an $N$-minor. By 7.3.17 we are now in a position to apply Lemma 4.1 to get that $M^* \setminus c_0, c_1, \ldots, c_n$ is (4, 4, *S*)-connected and every 4-fan of this matroid is either a 4-fan in $M^* \setminus c_n$ with $b_n$ as its coguts element, or is a 4-fan in $M^*/\beta$ with $\alpha$ as its coguts element. Let $(u, v, w, x)$ be such a 4-fan. Suppose first that $x = b_n$ and this 4-fan is a 4-fan in $M^* \setminus c_n$. Then $\{v, w, b_n, c_n\}$ is a 4-cocircuit of $M^*$ and, taking $T_{n+1} = \{u, v, w\}$, we get a contradiction to the previous paragraph. It follows that $(u, v, w, \alpha)$ is a 4-fan in $M^* \setminus \beta$. By 7.3.16 we know that $\beta = 8$. Thus $(\alpha, w, v, u)$ is a 4-fan in $M/8$; a contradiction to 7.3.3 We conclude that $M^* \setminus \beta, c_0, c_1, \ldots, c_n$ has no 4-fans and so is internally 4-connected. Thus (ii) holds; a contradiction.

This completes our analysis of the case when $M$ contains configuration (A) in Figure 13.

8. Configuration (B)

In this section, we deal with the case when $M$ contains configuration (B) in Figure 13. The results from the last two sections mean that if we find that $M$
contains configuration (C) or (A) from Figure 13, then we are guaranteed to get
one of the desired outcomes from the main theorem.

![Figure 26: All of the elements are distinct except that u may be 1.](image)

**Lemma 8.1.** Suppose $M$ and $N$ are internally 4-connected binary matroids, $|E(M)| \geq 13$ and $|E(N)| \geq 7$, and $M$ contains structure (B) in Figure 13 where all of the elements are distinct except that 1 may be $a$. Suppose $M \setminus 4$ is $(4, 4, S)$-connected with an $N$-minor but $M \setminus 1, 4$ does not have an $N$-minor. Then

(i) $M$ has a triangle $\{7, 8, 9\}$ where $\{4, 6, 7, 8\}$ is a cocircuit and the elements in $\{1, 2, \ldots, 9\}$ are distinct except that 1 may be 9; or

(ii) $M$ contains the configuration in Figure 26 where all of the elements are distinct except that $u$ may be 1, and $M \setminus 6$ is $(4, 4, S)$-connected with an $N$-minor; or

(iii) $M \setminus 6$ is internally 4-connected with an $N$-minor.

**Proof.** Suppose that the lemma fails. As $N \not\cong M \setminus 4$, it follows by Lemma 3.1 that $N \not\cong M \setminus 4/5$, and $M \setminus 4/5$ is $(4, 4, S)$-connected. Since $M/5 \setminus 4 \cong M/5 \setminus 6$, we deduce that $N \not\cong M \setminus 6$.

8.1.1. $M \setminus 6$ is $(4, 4, S)$-connected with an $N$-minor.

Assume that 8.1.1 fails. Then, by Lemma 3.4, $\{4, 5, 6\}$ is the central triangle of a quasi rotor. Moreover, since $M \setminus 4$ is $(4, 4, S)$-connected, the central element of this quasi rotor is 4, and Lemma 5.1 specifies that this quasi rotor is $\{(1, 2, 3), \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{4, 6, 7, 8\}, \{x, 4, 7\}\}$ for some $x$ in $\{2, 3\}$, so (i) holds; a contradiction. We conclude that 8.1.1 holds.

Since $M \setminus 6$ is not internally 4-connected, it has a 4-fan $\{u, v, w, x\}$. Thus $M$ has $\{v, w, x, 6\}$ as a cocircuit. Hence $\{v, w, x\}$ meets $\{4, 5\}$ and $\{a, c\}$. Clearly $\{u, v, w\} \not= \{4, 5, 6\}$, so Lemma 5.1 implies that 5 $\not\in \{u, v, w\}$. Suppose $4 \in \{u, v, w\}$. Then orthogonality between $\{u, v, w\}$ and $\{4, 6, c, d\}$ implies that $\{c, d\}$ meets $\{u, v, w\}$. If $\{4, c\}$ is in a triangle, then the symmetric difference of this triangle with $\{4, 5, a, c\}$ is a triangle other than $\{4, 5, 6\}$ that contains 5; a contradiction to Lemma 5.1. Thus $\{4, d\} \subseteq \{u, v, w\}$, so $M \setminus 6/5$ has a 5-fan; a contradiction. We conclude that $4 \not\in \{u, v, w\}$, so $x \in \{4, 5\}$ and, without loss of generality, $w \in \{a, c\}$.

8.1.2. Either $\{1, 2, 3\}$ avoids $\{u, v, w\}$, or $\{1, 2, 3\} \cap \{u, v, w\} = \{1\} = \{u\}$.

To see this, observe that, as $\{(u, v, w), \{4, 5, 6\}, \{v, w, x, 6\}\}$ is a bowtie, Lemma 3.3 implies that the cocircuit $\{2, 3, 4, 5\}$ avoids $\{u, v, w\}$. Furthermore, Lemma 3.3 applied to the bowtie $\{(1, 2, 3), \{4, 5, 6\}, \{2, 3, 4, 5\}\}$ implies that $\{v, w, x, 6\}$ avoids $\{1, 2, 3\}$. It follows that 8.1.2 holds.
By 8.1.2 if \( x = 4 \), then (i) of the lemma holds; a contradiction. Thus we may assume that \( x = 5 \). Suppose \( w = c \). Then \( \{c, d\} \) is not in a triangle, as (i) does not hold, so orthogonality implies that \( 4 \in \{u, v\} \), which we have already shown does not occur. We conclude that \( a = w \). Then \( \{v, a, 5, 6\} \) is a cocircuit of \( M \), so \( M \) contains the configuration in Figure 26.

By 8.1.2 since \( a = w \), we see that \( a \neq 1 \). Thus the elements in Figure 13(B) are distinct. Lemma 3.3 implies that \( \{u, v\} \) avoids \( \{2, 3, 4, 5\} \) and \( \{4, 6, c, d\} \). Hence the elements in Figure 26 are distinct with the possible exception that \( 1 \in \{u, v\} \).

By 8.1.2, \( 1 \neq v \). Thus (ii) holds.

If (i) of the last lemma holds, then, possibly after a minor relabelling, we see that \( M \) contains structure (C) from Figure 13. Since this case has already been treated, it remains for us to consider the structure in Figure 26.

**Lemma 8.2.** Suppose \( M \) and \( N \) are internally 4-connected binary matroids, \( |E(M)| \geq 16 \) and \( |E(N)| \geq 7 \), and \( M \) contains the structure in Figure 26 where all of the elements are distinct except that \( u \) may be 1. Suppose Hypothesis VII holds. If \( M \setminus 6 \) is \( (4, 4, S) \)-connected with an \( N \)-minor but \( M \setminus 6, u \) has no \( N \)-minor, then

(i) \( M \) has a quick win; or

(ii) \( M^* \) has an open-rotor-chain win, a ladder win, or an enhanced-ladder win; or

(iii) deleting the central cocircuit of an augmented 4-wheel in \( M^* \) gives an internally 4-connected matroid with a magenta \( N^* \)-minor.

**Proof.** As \( M \setminus 6, u \) has no \( N \)-minor, we relabel the elements 1, 2, 3, 4, 5, 6, a, v, u, c, and d in Figure 26 as \( x, y, z, 6, 5, 4, 2, 3, 1, c \), and d. We then restrict our attention to the configuration in \( M \) that is the same as that in Figure 13(A). Now \( M \setminus 1, 4 \) has no \( N \)-minor. Lemma 7.3 implies that the result holds.

We may now assume that \( M \) contains the structure in Figure 26 and \( M \setminus 6, u \) has an \( N \)-minor. Recall that \( M \setminus 4/5 \) has an \( N \)-minor. As this matroid has \( (a, 6, c, d) \) as a 4-fan, we may delete \( a \) or contract \( d \) keeping an \( N \)-minor. In the next lemma, we consider the second possibility.

**Lemma 8.3.** Suppose \( M \) is an internally 4-connected binary matroid containing at least thirteen elements, and \( M \) has the structure in Figure 26 where all of the elements are distinct except that \( u \) may be 1. Suppose that both \( M \setminus 4 \) and \( M \setminus 6 \) are \( (4, 4, S) \)-connected. Then

(i) \( M/d \) is \( (4, 4, S) \)-connected and every \( (4, 3) \)-violator of \( M/d \) is a 4-fan with \( c \) as its guts element; or

(ii) \( M \) contains the structure in Figure 13(A).

**Proof.** Assume that (ii) does not hold. First we show that

8.3.1. \( M \) has no triangle containing \( d \).

Assume that \( M \) has a triangle \( T \) containing \( d \). Then, by orthogonality, \( T \) contains \( c, 6, \) or 4. In the first two cases, we obtain the contradiction that \( M \setminus 4 \) is not \( (4, 4, S) \)-connected. We conclude that 4 \( \in T \). Then, by orthogonality and symmetry, we may assume that 2 \( \in T \), so (ii) holds. This contradiction implies that 8.3.1 holds.

8.3.2. \( M/d \) is sequentially 4-connected.
By 8.3.1, $M/d$ is 3-connected. Suppose that $M/d$ has a non-sequential 3-separation $(U, V)$. Then, by Lemma 3.3, we may assume that \{4, 5, 6, a, u, v, c\} $\subseteq U$. Then $(U \cup d, V)$ is a non-sequential 3-separation of $M$; a contradiction. Thus 8.3.2 holds.

Now suppose that $M/d$ has a 4-fan $(\alpha, \beta, \gamma, \delta)$. Then $M$ has $\{\alpha, \beta, \gamma\}$ as a circuit. By orthogonality, $\{\alpha, \beta, \gamma\}$ meets $\{4, 6, c\}$. Since $M$ has no 4-fan, $\{\beta, \gamma\}$ avoids the triangles of $M$. Thus $\alpha \in \{4, 6, c\}$. By orthogonality between $\{\alpha, \beta, \gamma, d\}$ and the cocircuits $\{2, 3, 4, 5\}$ and $\{5, 6, a, v\}$, we see that $\alpha \notin \{4, 6\}$. Thus $\alpha = c$.

We now know that every 4-fan in $M/d$ has $c$ as its guts element. It follows easily that $M$ has no 5-fan and no 5-cofan, so $M/d$ is $(4, 4, S)$-connected as required. □

We have not eliminated the case that $M$ contains the structure in Figure 26 and $M/d$ has an $N$-minor, but we have built up more structure, which will assist us in our later analysis.

![Figure 27](image)

**Figure 27.** All of the elements are distinct except that $u_2$ may be the same as 1, or $\{1, 2, 3\}$ may be $\{a_2, u_2, v_2\}$.

**Lemma 8.4.** Suppose $M$ and $N$ are internally 4-connected binary matroids, $|E(M)| \geq 13$ and $|E(N)| \geq 7$, and $M$ contains the structure in Figure 26 where all of the elements are distinct except that $u$ may be 1. Suppose that Hypothesis VII holds. Suppose that $M \setminus 4$ and $M \setminus 6$ are each $(4, 4, S)$-connected with an $N$-minor, and $M \setminus 1, 4$ has no $N$-minor. Then

(i) $\{4, 5, 6\}$ is the only triangle of $M$ containing 4, and $M$ contains the structure in Figure 27 where all of the elements are distinct except that $u_2$ may be the same as 1, or $\{1, 2, 3\}$ may equal $\{a_2, u_2, v_2\}$; or

(ii) $M$ has a quick win; or

(iii) $M$ has a ladder win.

**Proof.** We assume that neither (ii) nor (iii) holds. We show first that

8.4.1. $\{4, 5, 6\}$ is the only triangle that meets $\{4, 5\}$.

Lemma 5.1 implies that $\{4, 5, 6\}$ is the only triangle that contains 5. Let $T$ be a triangle that contains 4 but differs from $\{4, 5, 6\}$. Then orthogonality implies that $T$ meets $\{2, 3, 5\}$ and $\{6, c, d\}$, so, up to switching the labels on 2 and 3, the triangle is $\{2, 4, d\}$ or $\{2, 4, c\}$, so $M \setminus 4/5$ has a 5-fan; a contradiction to Lemma 3.1. We deduce that 8.4.1 holds.

Lemma 8.2 implies that $M \setminus 6, u$ has an $N$-minor. From Lemma 3.5 we know that $M \setminus u$ is $(4, 4, S)$-connected and, as $M \setminus 4$ is not internally 4-connected, either
(I) \( M \) has a triangle \( \{a_2, u_2, v_2\} \) and a cocircuit \( \{x, u, a_2, v_2\} \) where \( x \in \{a, v\} \) and \(|\{4, 5, 6, a, u, v, a_2, v_2\}| = 9\), or
(II) every \((4, 3)\)-violator of \( M \setminus u \) is a 4-fan of the form \((6, y_2, y_3, y_4)\) where \( y_2 \in \{a, v\} \).

Suppose (II) holds. Since \( y_2 \in \{a, v\} \), orthogonality implies that \((y_1, y_2, y_3)\) is \((6, v, d)\) or \((6, a, c)\). If \((6, v, d)\) is a triangle, then \(\lambda(\{4, 5, 6, a, v, u, c, d\}) \leq 2\); a contradiction. Thus \(y_2 = a\) and \(y_3 = c\). We now consider \( M \setminus 6, u \). By [5 Lemma 6.1], since \( M \setminus 6, u \) is not internally 4-connected, one of the following occurs: \(\{c, y_4\}\) is contained in a triangle; or \(\{5, v\}\) is contained in a triangle; or \( M \) has a triangle that contains \(4\) but avoids \(\{5, 6\}\); or \( M \setminus 6, u \) is \((4, 4, S)\)-connected and \( v \) is the coguts of every 4-fan in it. If \(\{c, y_4\}\) is contained in a triangle, then orthogonality implies that the third element of this triangle is in \(\{4, 6, d\}\) and so \(\lambda(\{4, 5, 6, a, u, v, c, d, y_4\}) \leq 2\); a contradiction. By [5.4.1] \(\{5, v\}\) is not contained in a triangle and \( M \) has no triangle that contains \(4\) but avoids \(\{5, 6\}\). We deduce that \( M \setminus 6, u \) has a 4-fan of the form \((z_1, z_2, z_3, v)\). As every 4-fan of \( M \setminus u \) contains \(6\), we see that \((z_1, z_2, z_3, v)\) is not a 4-fan of \( M \setminus u \). Hence \((6, z_2, z_3, v)\) or \((6, u, z_2, z_3, v)\) is a cocircuit of \( M \). Then orthogonality implies that \(\{z_2, z_3\}\) meets \(\{4, 5\}\), a contradiction to [5.4.1]. We conclude that (II) does not hold. Therefore (I) holds.

If the triangle \( \{a_2, u_2, v_2\} \) meets \(\{c, d\}\), then orthogonality with the cocircuit \(\{4, 6, c, d\}\) implies that \(\{c, d\} \subseteq \{a_2, u_2, v_2\}\), so \( M \setminus 4 \) has a 5-fan; a contradiction. Thus the elements in \(\{4, 5, 6, a, u, v, a_2, u_2, v_2, c, d\}\) are distinct. Now consider the cocircuit \(\{x, u, a_2, v_2\}\) recalling that \(x \in \{a, v\}\). If \(x = a\), then orthogonality implies that \(\{u, a_2, v_2\}\) meets \(\{6, c\}\); a contradiction. Thus \(x = v\). By hypothesis, \(\{2, 3\}\) avoids \(\{4, 5, 6, a, c, d, u, v\}\). Suppose \(\{2, 3\}\) meets \(\{a_2, u_2, v_2\}\). Then orthogonality with \(\{2, 3, 4, 5\}\) implies that \(\{2, 3\} \subseteq \{a_2, u_2, v_2\}\), so \(\{1, 2, 3\} = \{a_2, u_2, v_2\}\), and (i) holds. Now suppose that \(\{2, 3\}\) avoids \(\{a_2, u_2, v_2\}\). Then the elements in \(\{2, 3, 4, 5, 6, a, u, v, a_2, u_2, v_2, c, d\}\) are distinct. Finally, orthogonality implies that 1 can only be in the last set if it equals \(u_2\). Thus (i) holds.

When \( M \) contains the configuration in Figure 27 and \( M \setminus 4 \) is \((4, 4, S)\)-connected with an \(N\)-minor but \( M \setminus 1, 4 \) does not have an \(N\)-minor, we know, by Lemma 3.1, that \( M \setminus 6/5 \) is \((4, 4, S)\)-connected with an \(N\)-minor. Since \( M \setminus 6/5 \) has \(\{a, 4, c, d\}\) as a 4-fan, it follows that either
(i) \( N \preceq M \setminus 6/5 \setminus a \); or
(ii) \( N \preceq M \setminus 6/5 \setminus d \).

As we showed in Lemma 8.3 we are able to find a new triad in the case that (ii) holds. In the following lemma, we dispense with the case that (i) holds.

**Lemma 8.5.** Suppose \( M \) and \( N \) are internally 4-connected binary matroids, \(|E(M)| \geq 15\) and \(|E(N)| \geq 7\), and \( M \) contains the structure in Figure 27 where all of the elements are distinct except that \( u_2 \) may be 1, or \(\{1, 2, 3\}\) may equal \(\{a_2, v_3, v_2\}\). Suppose that Hypothesis VII holds and that \(\{4, 5, 6\}\) is the only triangle that contains \(4\). Suppose that \( M \setminus 4 \) and \( M \setminus 6 \) are each \((4, 4, S)\)-connected with an \(N\)-minor and \( M \setminus 6/5 \setminus a \) has an \(N\)-minor but \( M \setminus 1, 4 \) has no \(N\)-minor. Then
(i) \( M \) has a quick win; or
(ii) \( M \) has a ladder win; or
(iii) \( M \) has a mixed ladder win.
Proof. Assume that the lemma does not hold. We relabel the elements 4, 5, 6, a, v, u, d, and c in Figure 27 as \( a_0, v_0, u_0, a_1, v_1, u_1, t_0, \) and \( t_1 \), respectively. Since \( M \backslash u_0 \backslash a_1 \) is isomorphic to \( M \backslash u_1 \backslash t_1 \), the second matroid has an N-minor. We take a right-maximal bowtie string \( \{a_0, v_0, u_0\}, \{v_0, u_0, a_1, v_1\}, \{a_1, v_1, u_1\}, \{v_1, u_1, a_2, v_2\}, \ldots, \{a_n, v_n, u_n\} \). Let \( X \) be the set of elements in this bowtie string.

![Figure 28. All of the elements shown are distinct.](image)

8.5.1. Suppose \( M \) contains the structure in Figure 28 where \( i \geq 2 \), all of the elements are distinct, and \( \{t_0, t_1, \ldots, t_{i-1}\} \) avoids \( X \). Suppose \( M \backslash u_0, u_1, \ldots, u_{i-1}/v_{i-1} \) has an N-minor. Then \( M \) has an element \( t_i \) that is not in \( \{t_0, t_1, \ldots, t_{i-1}\} \cup X \) such that \( t_{i-1}, a_{i-1}, u_{i-1}, t_i \) is a cocircuit, \( \{u_{i-1}, t_i, a_i\} \) is a triangle, and \( M \backslash u_0, u_1, \ldots, u_i/v_i \) has an N-minor. Moreover, if \( \{1, 2, 3\} \) meets \( (\bigcup_{j=0}^{i} \{a_j, u_j, v_j\}) \cup \{0, t_1, \ldots, t_i\} \), then either \( 1 = u_i \), or \( \{1, 2, 3\} = \{a_i, u_i, v_i\} \).

Since \( M/v_{i-1}/u_{i-1} \) has an N-minor, Lemma 3.8 implies that \( \{a_{i-1}, u_{i-1}\} \) is contained in a 4-cocircuit. Orthogonality implies that this cocircuit meets \( \{u_{i-2}, t_{i-1}\} \), and Lemma 3.3 implies that it avoids \( \{a_{i-2}, u_{i-2}, v_{i-2}\} \), hence it contains \( t_{i-1} \). Let \( t_i \) be the fourth element in this cocircuit. Orthogonality implies that \( t_i \) avoids the triangles in Figure 28 and also avoids \( X \). Thus \( t_i \) is a new element unless \( t_i = t_0 \). In the exceptional case, \( \{t_0, a_0, u_0, t_1\} \cap \{t_{i-1}, a_{i-1}, u_{i-1}, t_0\} \), which equals \( \{a_0, u_0, t_1, t_{i-1}, a_{i-1}, u_{i-1}\} \), is a cocircuit. Hence the elements in Figure 28, excluding \( \{t_0, a_i, u_i, v_i\} \), comprise a 3-separating set in \( M \); a contradiction. Thus \( t_i \neq t_0 \).

Next we establish the last part of 8.5.1. Suppose that \( \{1, 2, 3\} \) meets \( (\bigcup_{j=0}^{i} \{a_j, u_j, v_j\}) \cup \{0, t_1, \ldots, t_i\} \). If \( \{1, 2, 3\} \) avoids \( X \), then orthogonality with the cocircuits in Figure 28 implies that \( \{1, 2, 3\} \) contains \( \{t_0, t_1, \ldots, t_i\} \). Hence \( i = 2 \), and \( \lambda(\{a_0, u_0, v_0, a_1, u_1, v_1, t_0, t_1, t_2\}) \leq 2 \); a contradiction. We deduce that \( \{1, 2, 3\} \) meets \( X \). By [5, Lemma 5.4], we see that \( 1 = u_i \), as desired, or \( \{1, 2, 3\} = \{a_k, u_k, v_k\} \) for some \( k \) in \( \{2, 3, \ldots, i\} \). We assume the latter. By orthogonality with the cocircuit \( \{2, 3, a_0, v_0\} \), we see that \( \{2, 3\} \) avoids \( \{u_{k-1}, t_k, a_k\} \), so \( 1 = a_k \) and \( \{2, 3\} = \{u_k, v_k\} \). Now orthogonality with \( \{2, 3, a_0, v_0\} \) implies that \( \{u_k, t_{k+1}, a_{k+1}\} \) is not a triangle. Hence \( k = i \). Thus the last part of 8.5.1 holds.

We can now apply Lemma 6.3 to our structure noting that, by assumption, (ii)(c) of that lemma cannot hold. Since \( \{a_0, v_0, u_0\} \) is the unique triangle of \( M \) containing \( a_0 \), it follows that \( \{a_0, t_0\} \) is not in a triangle of \( M \). Thus \( M \) has a triangle containing \( \{u_{i-1}, t_i\} \). By orthogonality, this triangle meets \( \{v_{i-1}, a_i, v_i\} \). If \( i < n \), then orthogonality with \( \{v_i, u_i, a_{i+1}, v_{i+1}\} \) implies that the third element
of this triangle is $a_i$. If $i = n$, then, up to switching the labels on $a_n$ and $v_n$, we may assume that the third element of this triangle is $a_i$.

To complete the proof of 8.5.1, it remains only to show that $M \setminus u_0, u_1, \ldots, u_i/v_i$ has an $N$-minor. Suppose not. Since $M \setminus u_0, u_1, \ldots, u_{i-1}/v_{i-1}$ has an $N$-minor and has $(a_i, t_i, a_{i-1}, t_{i-1})$ as a 4-fan, we know that $M \setminus u_0, u_1, \ldots, u_{i-1}/v_{i-1} \setminus a_i$ or $M \setminus u_0, u_1, \ldots, u_{i-1}/v_{i-1}/t_{i-1}$ has an $N$-minor. Since the first matroid is isomorphic to $M \setminus u_0, u_1, \ldots, u_{i-1}/v_i$ by Lemma 3.6, we may assume that the second matroid has an $N$-minor. Now Lemma 3.6 implies that $M \setminus u_0, u_1, \ldots, u_{i-1}/v_{i-1}/t_{i-1}$ is isomorphic to $M \setminus a_0, a_1, \ldots, a_{i-1}/v_0/t_{i-1}$. Applying Lemma 3.6 again, this time focusing on the bowtie string at the top of the diagram, we get that the last matroid is isomorphic to $M \setminus a_0, u_0, u_1, \ldots, u_{i-2}/v_0/t_1$. Thus $M \setminus a_0, u_0$ has an $N$-minor. As this matroid has $\{t_0, t_1\}$ as a cocircuit, we deduce that $M/t_0$ has an $N$-minor. Because $\{a_0, u_0, v_0\}$ is the unique triangle containing $a_0$, it follows that $M$ does not contain the structure in Figure 13(A). Thus Lemma 8.3 implies that $M/t_0$ has a 4-fan of the form $(t_1, \beta, \gamma, \delta)$. Hence $\{t_0, t_1, \beta, \gamma\}$ is a circuit of $M$, and $\{\beta, \gamma, \delta\}$ is a triad of $M$. Orthogonality with $\{t_1, a_1, u_1, t_2\}$ implies that $\{\beta, \gamma\}$ meets a triangle of $M$; a contradiction. Thus 8.5.1 holds.

![Figure 29](image)

**Figure 29.** $n \geq 2$ and all of the elements shown are distinct except that 1 may be the same as $u_n$.

By repeatedly applying 8.5.1 on our bowtie string, we deduce that $M$ contains the structure in Figure 29 and $M \setminus u_0, u_1, \ldots, u_n/v_n$ has an $N$-minor. By Lemma 3.6 we see that

8.5.2. $M \setminus u_0, u_1, \ldots, u_n/v_n \cong M \setminus a_0, a_1, \ldots, a_n/v_0$.

We also get from 8.5.1 that $\{1, 2, 3\}$ avoids the other elements in Figure 29 except that 1 may be $u_n$, or $\{1, 2, 3\}$ may be $\{a_n, u_n, v_n\}$. If $\{1, 2, 3\} = \{a_n, u_n, v_n\}$, then orthogonality implies that $\{2, 3\} = \{u_n, v_n\}$, so 1 = $a_n$. By 8.5.2 we deduce, since $a_0 = 4$, that $M \setminus 1, 4$ has an $N$-minor; a contradiction. We conclude that all the elements in Figure 29 are distinct, except that 1 may be $u_n$.

Next we show that

8.5.3. $M \setminus u_n$ has no 4-fan with $v_n$ as its coguts element.

Suppose that $M \setminus u_n$ has $(7, 8, 9, v_n)$ as a 4-fan. Then $(\{a_n, u_n, v_n\}, \{7, 8, 9\}, \{8, 9, v_n, u_n\})$ is a bowtie, and 5 Lemma 5.4] implies that $\{7, 8, 9\} = \{a_j, u_j, v_j\}$ for some $j$ in $\{0, 1, \ldots, n - 2\}$. Then $\chi(X \cup \{t_1, t_2, \ldots, t_n\}) \leq 2$. Since $\{1, 2, 3, t_0\}$ avoids $X \cup \{t_1, t_2, \ldots, t_n\}$, to avoid a contradiction, we must have that 1 = $u_n$. Hence $\{1, 2, 3\}$ meets $\{8, 9, v_n, u_n\}$ in a single element; a contradiction. Thus 8.5.3 holds.
Next we show that

8.5.4. \( \{a_n, u_n\} \) is contained in a 4-cocircuit.

Since \( M \setminus u_0, u_1, \ldots, u_n/v_n \cong M \setminus a_0, a_1, \ldots, a_n/v_0 \) by 8.5.2, we deduce that the second matroid has an \( N \)-minor. Thus both \( M \setminus u_n \) and \( M \setminus a_{n-1} \) have \( N \)-minors. Moreover, by Hypothesis VII, \( M \setminus u_n \) is \( (4,4,S) \)-connected. As \( M \) has no quick win, Lemma 8.6 and 8.5.3 imply that 8.5.4 holds unless \( v_n \) is in a triangle \( T \) with \( u_{n-1} \) or \( v_{n-1} \). In the exceptional case, orthogonality with the vertex cocircuits in Figure 29 implies that \( T = \{v_n, u_{n-1}, t_n\} \); a contradiction. Thus 8.5.4 holds.

Orthogonality implies that the 4-cocircuit containing \( \{a_n, u_n\} \) meets \( \{u_{n-1}, t_n\} \), and Lemma 8.3 implies that the cocircuit avoids \( \{a_{n-1}, u_{n-1}, v_{n-1}\} \). Hence it contains \( t_n \). Let \( t_{n+1} \) be the fourth element in this cocircuit. Orthogonality with the triangles in Figure 29 implies that \( t_{n+1} \) avoids all of the elements in the figure with the possible exception of \( t_0 \). Now we apply [5, Lemma 6.5] and conclude that one of the following holds: \( M \setminus u_0, u_1, \ldots, u_n \) is internally 4-connected; or \( M \setminus u_0, u_1, \ldots, u_n \) is \( (4,4,S) \)-connected and every 4-fan in this matroid is also a 4-fan of \( M \setminus u_n \) with \( v_n \) as its coguts element or is a 4-fan of \( M \setminus u_0 \) with \( a_0 \) in its triangle; or \( M \) is a quartic Möbius ladder with \( a_0 \) in two triangles. Since \( \{a_0, u_0, v_0\} \) is the only triangle of \( M \) containing \( a_0 \), we deduce that either \( M \) has a ladder win, a contradiction; or \( M \setminus u_n \) has a 4-fan with \( v_n \) as its coguts element, which contradicts 8.5.3. We conclude that the lemma holds.

We continue to consider the case when \( M \) contains the structure in Figure 27. \( M \setminus 4 \) is \( (4,4,S) \)-connected with an \( N \)-minor, \( M \setminus 1, 4 \) does not have an \( N \)-minor, and \( M \setminus 6, 5 \) is \( (4,4,S) \)-connected with an \( N \)-minor. Now \( M \setminus 6, 5 \) has \( \{a, 4, c, d\} \) as a 4-fan, and the preceding lemma dealt with the case when \( M \setminus 6, 5 \setminus a \) has an \( N \)-minor. Our final lemma deals with the case when the last matroid does not have an \( N \)-minor.

Lemma 8.6. Let \( M \) and \( N \) be internally 4-connected binary matroids with \(|E(M)| \geq 16\) and \(|E(N)| \geq 7\). Suppose that \( M \) contains the structure in Figure 27, where the elements are all distinct except that \( u_2 \) may be 1, or \( \{a_2, u_2, v_2\} \) may be \( \{1, 2, 3\} \). Suppose that \( M \setminus 4 \) and \( M \setminus 6 \) are \( (4,4,S) \)-connected having \( N \)-minors, and that \( M \setminus 1, 4 \) does not have an \( N \)-minor. Suppose that Hypothesis VII holds, that \( \{4, 5, 6\} \) is the only triangle containing 4, and that \( M \setminus 6, 5 \setminus a \) does not have an \( N \)-minor. Then

(i) \( M \) has a quick win; or
(ii) \( M \) or \( M^* \) has an open-rotor-chain win, a bowtie-ring win, or a ladder win; or
(iii) \( M \) or \( M^* \) has an enhanced-ladder win; or
(iv) deleting the central cocircuit of an augmented 4-wheel in \( M^* \) gives an internally 4-connected matroid with an \( N^* \)-minor.

Proof. Assume that the lemma fails. Then, by Lemma 8.2, \( N \preceq M \setminus 6, u \). As before, we relabel 4, 5, 6, a, u, and v as \( a_0, v_0, u_0, a_1, u_1, \) and \( v_1 \), respectively. Let \( \{a_0, v_0, u_0\}, \{v_0, u_0, a_1, v_1\}, \{u_1, v_1, u_1\}, \{v_1, u_1, a_2, v_2\}, \ldots, \{a_n, v_n, u_i\} \) be a right-maximal bowtie string. We show next that

8.6.1. \( M \setminus u_0, u_1, \ldots, u_i/v_i \) has no \( N \)-minor for all \( i \) in \( \{1, 2, \ldots, n\} \) and \( M \setminus u_0, u_1, \ldots, u_j/a_j \) has no \( N \)-minor for all \( j \) in \( \{2, 3, \ldots, n\} \), but \( M \setminus u_0, u_1, \ldots, u_n \) has an \( N \)-minor.
Since $M \backslash u_0, u_1/v_1 \cong M \backslash 6, a/5$, the first matroid does not have an $N$-minor. It follows by Lemma 3.7 that 8.6.1 holds.

By 8.6.1 and Hypothesis VII, it follows that $M \backslash u_i$ is $(4,4,S)$-connected for all $i$ in $\{0,1,\ldots,n\}$. Establishing the next assertion will occupy most of the rest of the proof of Lemma 8.6.

8.6.2. $M$ has no bowtie of the form $(\{a_n, u_n, v_n\}, \{a_0, v_0, u_0\}, \{x, u_n, a_0, v_0\})$ with $x$ in $\{a_n, v_n\}$.

Suppose that $M$ does have such a bowtie. Then $a_0 \neq u_n$. By possibly interchanging the labels on $a_n$ and $v_n$, we may assume that $x = v_n$. Next we show the following.

8.6.3. Either $\{2,3\} = \{u_n, v_n\}$; or $\{u_n, v_n, 2,3\}$ is a cocircuit of $M$.

To see this, observe that both $\{u_n, v_n, a_0, v_0\}$ and $\{2,3, a_0, v_0\}$ are cocircuits of $M$. Taking their symmetric difference, we immediately get 8.6.3.

We now eliminate the first possibility in 8.6.3.

8.6.4. $\{2,3\} \neq \{u_n, v_n\}$.

Suppose $\{2,3\} = \{u_n, v_n\}$. Then $a_n = 1$ and $M$ has $(\{a_0, v_0, u_0\}, \{u_0, v_0, a_1, v_1\}, \{a_1, u_1, v_1\}, \ldots, \{a_n, u_n, v_n\}, \{v_n, a_0, u_0, v_0\})$ as a ring of bowties. We now apply [5, Lemma 5.5] noting that, since $M$ does not have a bowtie-ring win, part (i) of that lemma does not hold. Moreover, by 8.6.1, part (iii) of that lemma does not hold. Thus part (ii) of that lemma holds, that is, $M \backslash u_0, u_1, \ldots, u_n$ is sequentially 4-connected but not internally 4-connected, and every 4-fan of it has the form $(\alpha, \beta, \gamma, \delta)$ where $\{\alpha, \beta, \gamma\}$ avoids $\{a_0, u_0, v_0, a_1, u_1, v_1, \ldots, a_n, u_n, v_n\}$, and $M$ has a cocircuit $\{\beta, \gamma, \delta, u_i\}$ for some $i$ in $\{0,1,\ldots,n\}$ and some $\delta$ in $\{a_i, v_i\}$.

With a view to applying Lemma 10.4 of [5], we show next that $i \neq 0$. Assume the contrary. Then $M$ has $\{\beta, \gamma, \delta, u_0\}$ as a cocircuit and $\{u_0, c, a_1\}$ as a triangle where $\delta \in \{a_0, v_0\}$. By orthogonality, $c \in \{\beta, \gamma\}$. Then orthogonality between $\{\alpha, \beta, \gamma\}$ and $\{d, a_0, u_0, c\}$ implies that $d \in \{\alpha, \beta, \gamma\}$. Hence $M \backslash a_0$ has a 5-fan; a contradiction. We conclude that $i \neq 0$.

We now apply [5, Lemma 10.4] noting that we get a contradiction using 8.6.1 unless $i = 1$ and $\delta = a_i$. In the exceptional case, orthogonality between $\{u_0, c, a_1\}$ and $\{a_1, u_1, \beta, \gamma\}$ implies that $\{u_0, c\}$ meets $\{\beta, \gamma\}$. By construction, $u_0 \notin \{\alpha, \beta, \gamma\}$. Hence $c \in \{\beta, \gamma\}$. By orthogonality with $\{a_0, v_0, c, d\}$, the triangle $\{\alpha, \beta, \gamma\}$ contains $\{c, d\}$, so $M \backslash 4$ has a 5-fan; a contradiction. Thus 8.6.4 holds.

By 8.6.3 we now know that $M$ has $\{2,3, u_n, v_n\}$ as a cocircuit. If $\{1,2,3\}$ avoids $\{a_0, u_0, v_0, a_1, u_1, v_1, \ldots, a_n, u_n, v_n\}$, then we can adjoin $\{2,3, u_n, v_n\}$ to $\{1,2,3\}$ to our right-maximal bowtie string to get a contradiction. Thus $\{1,2,3\}$ meets $\{a_0, u_0, v_0, a_1, u_1, v_1, \ldots, a_n, u_n, v_n\}$. By [5, Lemma 5.4], $\{1,2,3\} = \{a_j, u_j, v_j\}$ for some $j$ with $0 \leq j \leq n - 2$. Certainly $j \neq 0$. Moreover, $j \neq 1$ otherwise the cocircuit $\{2,3, a_0, v_0\}$ contradicts Lemma 3.3. If $u_j \in \{2,3\}$, then $M \backslash u_0, u_1, \ldots, u_n$ has $v_n$ in a 2-cocircuit; a contradiction to 8.6.1. Thus $\{2,3\} = \{a_j, v_j\}$ and $\{2,3, u_n, v_n\} \Delta \{v_j, u_j-1, v_j\}$, which is $\{v_j, u_j-1, u_n, v_n\}$, is a cocircuit in $M$. Again $M \backslash u_0, u_1, \ldots, u_n$ has $v_n$ in a 2-cocircuit; a contradiction. Thus 8.6.2 holds.

We can now apply [5, Lemma 10.1]. Since $n \geq 2$, we conclude that either $M \backslash u_0, u_1/v_1$ has an $N$-minor, or $M$ has $a_0$ in a triangle other than $\{a_0, u_0, v_0\}$. The
former option gives a contradiction to 8.6.1 and the latter gives a contradiction to the assumptions of the lemma.

9. The proof of the main theorem

In this section, as we shall see, it is quite straightforward to assemble the parts from earlier sections to complete the proof of the main result.

Proof of Theorem 1.4. Assume that the theorem fails. Theorem 1.3 implies that Hypothesis VII holds. Now \( M \) has a bowtie \((\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})\) where \( M \setminus 4 \) is \((4, 4, S)\)-connected with an \( N \)-minor and \( M \setminus 1, 4 \) has no \( N \)-minor. Lemma 3.1 implies that \( M \setminus 4/5 \) is \((4, 4, S)\)-connected with an \( N \)-minor. By Lemma 5.1 we know that \( M \) contains (A), (B), or (C) in Figure 13 that \( M \setminus 6 \) is \((4, 4, S)\)-connected, and that \( \{4, 5, 6\} \) is the only triangle in \( M \) containing 5. Moreover, the elements in (A), (B), or (C) are all distinct except that \( a \) may equal 1 in (B) or (C).

Suppose that \( M \) contains the configuration in Figure 13(C). Lemma 6.1 implies that \( M \setminus 4/5, 6 \) does not have an \( N \)-minor. Since \( M \setminus 4 \) is \((4, 4, S)\)-connected with an \( N \)-minor and has \( (1, 2, 3, 5) \) as 4-fans, and \( M \setminus 1, 4 \) does not have an \( N \)-minor, we deduce that \( M \setminus 4/5 \setminus a \) has an \( N \)-minor. Then Lemma 6.3 gives a contradiction. We conclude that \( M \) does not contain the configuration in Figure 13(C).

If \( M \) contains the configuration in Figure 13(A), then Lemma 7.3 gives a contradiction. Hence we may assume that \( M \) does not contain either of the configurations in Figure 13(A) or Figure 13(C). Thus \( M \) contains the configuration in Figure 13(B). Lemma 8.1 implies that \( M \) contains the configuration in Figure 26 where \( M \setminus 6 \) is \((4, 4, S)\)-connected and all of the elements are distinct except that 1 may be \( u \) or \( \{1, 2, 3\} \) may equal \( \{a, u_2, v_2\} \). By Lemma 8.4, \( M \) contains the configuration in Figure 27 where \( \{4, 5, 6\} \) is the only triangle containing 4, and all of the elements are distinct except that \( u_4 \) may be 1. If \( M \setminus 6/5 \setminus a \) has an \( N \)-minor, then Lemma 8.5 gives a contradiction. Thus we may assume that \( N \not\preceq M \setminus 6/5 \setminus a \). Now \( M \setminus 4/5 \cong M \setminus 6/5 \), so \( M \setminus 6/5 \) is \((4, 4, S)\)-connected with an \( N \)-minor. Using Lemma 8.6 we get a contradiction that completes the proof of the theorem.

Acknowledgements

The authors thank Dillon Mayhew for numerous helpful discussions.

References


School of Mathematical Sciences, Brunel University, London, England
*E-mail address: chchchun@gmail.com*

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana, USA
*E-mail address: oxley@math.lsu.edu*