9-10-1998

Unitary Representations of Lie Groups with Reflection Symmetry

Palle E.T. Jorgensen
University of Iowa

Gestur Ólafsson
Louisiana State University

JOURNAL OF FUNCTIONAL ANALYSIS **158**, 26–88 (1998)
ARTICLE NO. FU983285

Unitary Representations of Lie Groups with Reflection Symmetry

Palle E. T. Jorgensen

Department of Mathematics, University of Iowa, Iowa City, Iowa 52242
E-mail: jorgen@math.uiowa.edu

and

Gestur Ólafsson

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803
E-mail: olafsson@math.lsu.edu

Received December 30, 1996; accepted March 25, 1998

We consider the following class of unitary representations π of some (real) Lie group G which has a matched pair of symmetries described as follows: (i) Suppose G has a period-2 automorphism τ, and that the Hilbert space $\mathbf{H}(\pi)$ carries a unitary operator J such that $J\pi = (\pi \circ \tau) J$ (i.e., *selfsimilarity*). (ii) An added symmetry is implied if $\mathbf{H}(\pi)$ further contains a closed subspace $\mathbf{K}_0$ having a certain *order-covariance* property, and satisfying the $\mathbf{K}_0$-restricted *positivity*: $\langle v \mid Jv \rangle \geqslant 0$, $\forall v \in \mathbf{K}_0$, where $\langle \cdot \mid \cdot \rangle$ is the inner product in $\mathbf{H}(\pi)$. From (i)–(ii), we get an induced dual representation of an associated dual group G^c. All three properties, selfsimilarity, order-covariance, and positivity, are satisfied in a natural context when G is semisimple and hermitean; but when G is the $(ax+b)$-group, or the Heisenberg group, positivity is incompatible with the other two axioms for the infinite-dimensional irreducible representations. We describe a class of G, containing the latter two, which admits a classification of the possible spaces $\mathbf{K}_0 \subset \mathbf{H}(\pi)$ satisfying the axioms of selfsimilarity and order-covariance.

1. INTRODUCTION

We consider a class of unitary representations of a Lie group G which possess a certain reflection symmetry defined as follows: If π is a representation of G in some Hilbert space $\mathbf{H}$, we introduce the following three structures:

(i) $\tau \in \mathrm{Aut}(G)$ of period 2;

(ii) $J\colon \mathbf{H} \to \mathbf{H}$ is a unitary operator of period 2 such that $J\pi(g)\,J^* = \pi(\tau(g))$, $g \in G$ (this will hold if π is of the form $\pi_+ \oplus \pi_-$ with π_+ and $\pi_- \circ \tau$

0022-1236/98 $25.00

unitarily equivalent); it will further be assumed that there is a closed subspace $\mathbf{K}_0 \subset \mathbf{H}$ which is invariant under $\pi(H)$, $H = G^\tau$, or more generally, under an open subgroup of G^τ;

(iii) positivity is assumed in the sense that $\langle v \mid J(v) \rangle \geqslant 0$, $v \in \mathbf{K}_0$.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{h}$ be the Lie algebra of the fixed-point subgroup $G^\tau = \{g \in G \mid \tau(g) = g\}$. Let $\mathfrak{q} = \{Y \in \mathfrak{g} \mid \tau(Y) = -Y\}$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}.$$

Let H be a closed subgroup of G, $G_o^\tau \subset H \subset G^\tau$. Assume there is an H-invariant, closed, and generating convex cone C in $\mathfrak{q}$ (i.e., $C - C = \mathfrak{q}$) such that C^o consists of hyperbolic elements. We assume that $S(C) = H \exp C$ is a closed semigroup in G which is homeomorphic to $H \times C$, and that

$$H \times C^o \ni (h, Y) \mapsto h \exp Y \in S^o$$

is a diffeomorphism.

We shall consider closed subspaces $\mathbf{K}_0 \subset \mathbf{H}(\pi)$, where $\mathbf{H}(\pi)$ is the Hilbert space of π, such that $\mathbf{K}_0$ is invariant under $\pi(S^o)$. Let J: $\mathbf{H}(\pi) \to \mathbf{H}(\pi)$ be a unitary intertwining operator for π and $\pi \circ \tau$ such that $J^2 = \mathrm{id}$. We assume that $\mathbf{K}_0$ may be chosen such that $\|v\|_J^2 := \langle v \mid Jv \rangle \geqslant 0$ for all $v \in \mathbf{K}_0$. We will always assume our inner product conjugate linear in the first argument. We form, in the usual way, the Hilbert space $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^\sim$ by dividing out with $\mathbf{N} = \{v \in \mathbf{K}_0 \mid \langle v \mid Jv \rangle = 0\}$ and completing in the norm $\|\cdot\|_J$. (This is of course a variation of the Gelfand–Naimark–Segal (GNS) construction.) With the properties of $(G, \pi, \mathbf{H}(\pi), \mathbf{K})$ as stated, we show, using the Lüscher-Mack theorem, that the simply connected Lie group G^c with Lie algebra $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$ carries a *unitary* representation π^c on $\mathbf{K}$ such that $\{\pi^c(h \exp(iY)) \mid h \in H, Y \in C^o\}$ is obtained from π by passing the corresponding operators $\pi(h \exp Y)$ to the quotient $\mathbf{K}_0/\mathbf{N}$. In fact, when $Y \in C$, the selfadjoint operator $d\pi(Y)$ on $\mathbf{K}$ has spectrum contained in $(-\infty, 0]$. As in Corollary 3.4, we show that in the case where C extends to an G^c invariant regular cone in $i\mathfrak{g}^c = i\mathfrak{h} \oplus \mathfrak{q}$ and π^c is injective, then each π^c (as a unitary representation of G^c) must be a direct integral of highest-weight representations of G^c. The examples show that one can relax the condition in different ways, i.e., one can avoid using the Lüscher–Mack theorem by instead constructing local representations and using only cones that are neither generating nor H-invariant.

Let us outline the plan by a simple example. Let $G = \mathrm{SL}(2, \mathbb{R})$, and let P be the parabolic subgroup

$$P = \left\{ p(a, x) = \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{R}^*, x \in \mathbb{R} \right\}.$$

For $s \in \mathbb{C}$, let π_s be the representation of G acting by $[\pi_s(a) f](b) = f(a^{-1}b)$ on the space $\mathbf{H}_s$ of functions $f: G \to \mathbb{C}$,

$$f(gp(a, x)) = |a|^{-s-1} f(g), \qquad \int_{SO(2)} |f(k)|^2 \, dk < \infty,$$

and with inner product

$$\langle f \mid g \rangle = \int_{SO(2)} \overline{f(k)}\, g(k) \, dk,$$

i.e., π_s is the principal series representation of G with parameter s. The representations π_s are unitary in the above Hilbert-space structure as long as $s \in i\mathbb{R}$. For defining a unitary structure for other parameters we need the intertwining operator $A_s: \mathbf{H}_s \to \mathbf{H}_{-s}$ defined by

$$A_s(f)(g) = \int_{-\infty}^{\infty} f(gw\bar{n}_y) \, dy$$

for $\operatorname{Re} s \geqslant 0$ and then generally by analytic continuation. Here w is the Weyl group element $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\bar{n}_y = \left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)$.

By restriction to $\bar{N} = \{\bar{n}_y \mid y \in \mathbb{R}\}$ we can also realize the representations π_s on $\mathbb{R} \simeq \bar{N}$, $y \mapsto \bar{n}_y$. Using that

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha x \\ \alpha y & \alpha y x + \alpha^{-1} \end{pmatrix}$$

we have $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \bar{N}P$ if and only if $a \neq 0$, and in that case

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \bar{n}_{c/a}\, p(a, b/a). \tag{1.1}$$

Thus the intertwinor A_s becomes the singular integral operator

$$A_s f(x) = \int_{-\infty}^{\infty} f(y) \, |x-y|^{s-1} \, dy.$$

In the new inner product

$$\langle f \mid A_s g \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{f(x)}\, g(y) \, |x-y|^{s-1} \, dx \, dy$$

the representation π_s, which is given by

$$\left[\pi_s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f\right](x) = |-bx+d|^{-(s+1)} f\left(\frac{ax-c}{-bx+d}\right),$$

is unitary for $0<s<1$. Notice that we now denote by $\mathbf{H}_s$ the new Hilbert space with the inner product $\langle \cdot \mid A_s(\cdot)\rangle$.

Define an involution τ on G by

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{1.2}$$

The group H is given by

$$H = \pm\left\{h_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \,\middle|\, t \in \mathbb{R}\right\}$$

and the space $\mathfrak{q}$ is

$$\mathfrak{q} = \left\{q(r, s) := \begin{pmatrix} r & s \\ -s & -r \end{pmatrix} \,\middle|\, r, s \in \mathbb{R}\right\}.$$

Take

$$C := \{q(r, s) \mid r \pm s \geqslant 0, r \geqslant 0\} = \overline{\operatorname{conv}\{\mathbb{R}_+ \operatorname{Ad}(H)\, q(1, 0)\}}.$$

as a generating cone. The Cartan involution θ is given by $a \mapsto a^{-t} = waw^{-1}$ and the corresponding maximal compact subgroup is SO(2). Define

$$Jf(a) := f(\tau(a)\, w^{-1}) = f(\tau(aw)).$$

Then $J\colon \mathbf{H}_s \to \mathbf{H}_s$ intertwines π_s and $\pi_s \circ \tau$, and $J^2 = 1$. In our realization of π_s in $\mathbf{L}^2(\mathbb{R})$ we have $J(f)(x) = |x|^{-s-1} f(1/x)$, and

$$A_s(J(g))(x) = \int_{-\infty}^{\infty} g(y)\, |1-xy|^{s-1}\, dy.$$

Hence

$$\langle f \mid g\rangle_J = \langle f \mid A_s Jg\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(x)}\, g(y)\, |1-xy|^{s-1}\, dy\, dx.$$

Let $\mathbf{K}_0$ be the completion of the space of smooth functions with compact support in $I = (-1, 1)$. Notice that the above inner product is defined on $\mathscr{C}_c^\infty(I)$ for every s as we only integrate over compact subsets of $(-1, 1)$.

The Bergman kernel for the domain $\{z \in \mathbb{C} \mid |z| < 1\}$ is $h(z, w) = 1 - z\bar{w}$, and it is well known (cf. [7, p. 268]) that $h(z, w)^{-\lambda}$ is a positive definite kernel function if and only if $\lambda \geqslant 0$. As our kernel is just $h(z, w)^{-(1-s)}$, restricted to the interval I, and $s < 1$, i.e., $1 - s > 0$, it follows that $\langle \cdot \mid \cdot \rangle_J$ is positive definite.

We also know (cf. [16]) that $S = H \exp C$ is a closed semigroup and that $\gamma I \subset I$, and actually S is exactly the semigroup of elements in $\mathrm{SL}(2, \mathbb{R})$ that act by contractions on I. Hence S acts on $\mathbf{K}$. By a theorem of Lüscher and Mack [15, 32], the representation of S on $\mathbf{K}$ extends to a representation of G^c, which in this case is the universal covering of $\mathrm{SU}(1, 1)$ that is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$. We notice that this defines a representation of $\mathrm{SL}(2, \mathbb{R})$ if and only if certain integrality conditions hold; see [25].

We generalize this construction to the non-compactly causal symmetric spaces and in particular to the Cayley-type spaces. Furthermore we identify the resulting representation as an irreducible unitary highest weight representation of the dual group G^c. We restrict ourself to the case of characters induced from a maximal parabolic subgroup, which leads to highest weight modules with one-dimensional lowest K^c-type. This is meant as a simplification and not as a limitation of our method.

Assume now that G is a semidirect product of H and N with N normal and abelian. Define τ: $G \to G$ by $\tau(hn) = hn^{-1}$. Let $\pi \in \hat{H}$ (the unitary dual) and extend π to a unitary representation of G by setting $\pi(hn) = \pi(h)$. In this case, G^c is locally isomorphic to G, and π gives rise to a unitary representation π^c of G^c by the formula $d\pi^c(X) = d\pi(X)$, $X \in \mathfrak{h}$, and $d\pi^c|_{i\mathfrak{q}} = 0$. A special case of this is the 3-dimensional Heisenberg group, and the $(ax + b)$-group. In sections 6 and 7, we show that, if we induce instead a character of the subgroup N to G, then we have $(\mathbf{K}_0/\mathbf{N})^{\sim} = \{0\}$.

Our approach to the general representation correspondence $\pi \mapsto \pi^c$ is related to the integrability problem for representations of Lie groups (see [25]); but the present positivity viewpoint comes from Osterwalder–Schrader positivity; see [50, 51]. In addition the following other papers are relevant in this connection: [9, 22, 23, 27, 55, 59].

2. PRELIMINARIES

The setting for the paper is a general Lie group G with a nontrivial involutive automorphism τ.

Definition 2.1. A unitary representation π acting on a Hilbert space $\mathbf{H}(\pi)$ is said to be *reflection symmetric* if there is a unitary operator J: $\mathbf{H}(\pi) \to \mathbf{H}(\pi)$ such that

(R1) $J^2 = \mathrm{id}$.

(R2) $J\pi(g) = \pi(\tau(g))\, J$, $g \in G$.

If (R1) holds, then π and $\pi \circ \tau$ are equivalent. Furthermore, generally from (R2) we have $J^2\pi(g) = \pi(g)\, J^2$. Thus, if π is irreducible, then we can always renormalize J such that (R1) holds. Let $H = G^\tau = \{g \in G \mid \tau(g) = g\}$ and let $\mathfrak{h}$ be the Lie algebra of H. Then $\mathfrak{h} = \{X \in \mathfrak{g} \mid \tau(X) = X\}$. Define $\mathfrak{q} = \{Y \in \mathfrak{g} \mid \tau(Y) = -Y\}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$.

Definition 2.2. A closed convex cone $C \subset \mathfrak{q}$ is *hyperbolic* if $C^o \neq \emptyset$ and if ad X is semisimple with real eigenvalues for every $X \in C^o$.

We will assume the following for (G, π, τ, J):

(PR1) π is reflection symmetric with reflection J.

(PR2) There is an H-invariant hyperbolic cone $C \subset \mathfrak{q}$ such that $S(C) = H \exp C$ is a closed semigroup and $S(C)^o = H \exp C^o$ is diffeomorphic to $H \times C^o$.

(PR3) There is a subspace $0 \neq \mathbf{K}_0 \subset \mathbf{H}(\pi)$ invariant under $S(C)$ satisfying the positivity condition

$$\langle v \mid v \rangle_J := \langle v \mid J(v) \rangle \geqslant 0, \qquad \forall v \in \mathbf{K}_0.$$

Remark 2.3. In (PR3) we can always assume that $\mathbf{K}_0$ is closed, as the invariance and the positivity pass over to the closure. In (PR2) it is only necessary to assume that $\mathbf{K}_0$ is invariant under exp C, as one can always replace $\mathbf{K}_0$ by $\overline{\langle \pi(H)\, \mathbf{K}_0 \rangle}$, the closed space generated by $\pi(H)\, \mathbf{K}_0$, which is $S(C)$-invariant, as C is H-invariant. For the exact conditions on the cone for (PR2) to hold see the orginal paper by J. Lawson [30], or the monograph [15, pp. 194 ff.].

In some of the examples we will replace (PR2) and (PR3) by the following: weaker conditions

(PR2′) C is (merely) some nontrivial cone in $\mathfrak{q}$.

(PR3′) There is a subspace $0 \neq \mathbf{K}_0 \subset \mathbf{H}(\pi)$ invariant under H and exp C satisfying the positivity condition from (PR3).

(See Section 6 for further details.)

Since the operators $\{\pi(h) \mid h \in H\}$ commute with J, they clearly pass to the quotient by

$$\mathbf{N} := \{v \in \mathbf{K}_0 \mid \langle v \mid Jv \rangle = 0\}$$

and implement unitary operators on $\mathbf{K} := (\mathbf{K}_0/\mathbf{N})^{\sim}$ relative to the inner product induced by

$$\langle u \mid v \rangle_J := \langle u \mid J(v) \rangle. \tag{2.1}$$

which will be denoted by the same symbol. Hence we shall be concerned with passing the operators $\{\pi(\exp Y) \mid Y \in C\}$ to the quotient $\mathbf{K}_0/\mathbf{N}$, and for this we need a basic Lemma.

In general, when $(\mathbf{K}_0, J)$ is given, satisfying the positivity axiom, then the corresponding composite quotient mapping

$$\mathbf{K}_0 \to \mathbf{K}_0/\mathbf{N} \hookrightarrow (\mathbf{K}_0/\mathbf{N})^{\sim} =: \mathbf{K}$$

is *contractive* relative to the respective Hilbert norms. The resulting (contractive) mapping will be denoted β. An operator γ on $\mathbf{H}$ which leaves $\mathbf{K}_0$ invariant is said to *induce* the operator $\tilde{\gamma}$ on $\mathbf{K}$ if $\beta \circ \gamma = \tilde{\gamma} \circ \beta$ holds on $\mathbf{K}_0$. In general, an induced operation $\gamma \mapsto \tilde{\gamma}$ may not exist; and, if it does, $\tilde{\gamma}$ may fail to be bounded, even if γ is bounded.

This above-mentioned operator-theoretic formulation of reflection positivity has applications to the Feynman-Kac formula in mathematical physics, and there is a considerable literature on that subject, with work by E. Nelson, A. Klein and L. J. Landau, B. Simon, and W. B. Arveson. Since we shall not use path space measures here, we will omit those applications, and instead refer the reader to the survey paper [1] (lecture 4) by W. B. Arveson. In addition to mathematical physics, our motivation also derives from recent papers on non-commutative harmonic analysis which explore analytic continuation of the underlying representations; see, e.g., [17, 35, 40, 41, 46].

3. A BASIC LEMMA

LEMMA 3.1. (1) *Let J be a period-2 unitary operator on a Hilbert space $\mathbf{H}$, and let $\mathbf{K}_0 \subset \mathbf{H}$ be a closed subspace such that $\langle v \mid J(v) \rangle \geqslant 0$, $v \in \mathbf{K}_0$. Let γ be an invertible operator on $\mathbf{H}$ such that $J\gamma = \gamma^{-1} J$ and which leaves $\mathbf{K}_0$ invariant and has $(\gamma^{-1})^* \gamma$ bounded on $\mathbf{H}$. Then γ induces a bounded operator $\tilde{\gamma}$ on $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^{\sim}$, where $\mathbf{N} = \{v \in \mathbf{K}_0 \mid \langle v \mid Jv \rangle = 0\}$, and the norm of $\tilde{\gamma}$ relative to the J-inner product in $\mathbf{K}$ satisfies*

$$\|\tilde{\gamma}\| \leqslant \|(\gamma^{-1})^* \gamma\|_{sp}^{1/2}, \tag{3.1}$$

where $\|\cdot\|_{sp}$ is the spectral radius.

(2) *If we have a semigroup S of operators on* $\mathbf{H}$ *satisfying the conditions in* (1), *then*

$$(\gamma_1 \gamma_2)^{\sim} = \tilde{\gamma}_1 \tilde{\gamma}_2, \qquad \gamma_1, \gamma_2 \in S. \tag{3.2}$$

Proof. For $v \in \mathbf{K}_0$, $v \neq 0$, we have

$$\begin{aligned}
\|\gamma(v)\|_J^2 &= \langle \gamma(v) \mid J\gamma(v) \rangle \\
&= \langle \gamma(v) \mid \gamma^{-1} J(v) \rangle \\
&= \langle (\gamma^{-1})^* \gamma(v) \mid J(v) \rangle \\
&= \langle (\gamma^{-1})^* \gamma(v) \mid v \rangle_J \\
&\leqslant \|(\gamma^{-1})^* \gamma(v)\|_J \|v\|_J \\
&\leqslant \|((\gamma^{-1})^* \gamma)^2 (v)\|_J^{1/2} \|v\|_J^{1+1/2} \\
&\;\;\vdots \\
&\leqslant \|((\gamma^{-1})^* \gamma)^{2^n} (v)\|_J^{1/2^n} \|v\|_J^{1+1/2+\cdots+1/2^n} \\
&\leqslant (\|((\gamma^{-1})^* \gamma)^{2^n}\| \|v\|)^{1/2^n} \|v\|_J^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \|((\gamma^{-1})^* \gamma)^{2^n}\|^{1/2^n} = \|(\gamma^{-1})^* \gamma\|_{sp}$, and $\lim_{n\to\infty} \|v\|^{1/2^n} = 1$, the result follows.

By this we get

$$\langle \gamma(v) \mid J\gamma(v) \rangle \leqslant \|(\gamma^{-1})^* \gamma\|_{sp} \langle v \mid J(v) \rangle$$

which shows that $\gamma(\mathbf{N}) \subset \mathbf{N}$, whence γ passes to a bounded operator on the quotient $\mathbf{K}_o/\mathbf{N}$ and then also on $\mathbf{K}$ satisfying the estimate stated in (1). If both the operators in (3.2) leave $\mathbf{N}$ invariant, so does $\gamma_1 \gamma_2$ and the operator induced by $\gamma_1 \gamma_2$ is $\tilde{\gamma}_1 \tilde{\gamma}_2$ as stated. ▮

Corollary 3.2. *Let the notation be as above and assume that γ is unitary on* $\mathbf{H}$. *Then the constant on the right in* (3.1) *is one. Hence $\tilde{\gamma}$ is a contraction on* $\mathbf{K}$.

To understand the assumptions on the space $\mathbf{K}_0$, i.e., positivity and invariance, we include the folowing which is based on an idea of R. S. Phillips [53].

Proposition 3.3. *Let* $\mathbf{H}$ *be a Hilbert space and let J be a period-2 unitary operator on* $\mathbf{H}$. *Let S be a commutative semigroup of unitary operators on* $\mathbf{H}$ *such that $S = S_+ S_-$ with $S_+ = \{\gamma \in S \mid J\gamma = \gamma J\}$ and $S_- = \{\gamma \in S \mid J\gamma = \gamma^{-1} J\}$. Then* $\mathbf{H}$ *possesses a maximal positive and invariant subspace, i.e., a subspace* $\mathbf{K}_0$ *such that $\langle v \mid J(v) \rangle \geqslant 0$, $v \in \mathbf{K}_0$, and $\gamma \mathbf{K}_0 \subset \mathbf{K}_0$, $\gamma \in S$.*

Proof. The basic idea is contained in [53, pp. 386 ff.]. We can represent $\mathbf{H}$ as $\mathbf{L}^2(X, m)$ where X is a Stone space. There is an m-a.e.-defined automorphism $\theta\colon X \to X$ such that

$$J(f) = f \circ \theta, \qquad f \in \mathbf{L}^2(X, m),$$

and S is represented by multiplication operators on $\mathbf{L}^2(X, m)$. From [53, Lemma 5.1], we know that there are clopen subsets A, B in X such that if

$$M_0 := \{x \in X \mid \theta(x) = x\}$$

and

$$M_1 := X \backslash M_0,$$

then A and B are contained in M_1,

$$A \cap B = \varnothing$$

$$A \cup B = M_1$$

and

$$\theta(A) = B.$$

Let $\mathbf{K}_0 := \mathbf{L}^2(M_0 \cup A)$. It is clear that this is a maximal positive and invariant subspace. The positivity follows in the following way: If f is supported in A then $\bar{f} f \circ \theta = 0$ a.e. Hence for $f \in \mathbf{L}^2(M_0 \cup A)$,

$$\begin{aligned}\langle f \mid J(f) \rangle &= \int_{M_0} \bar{f} f \circ \theta \, dm + \int_A \bar{f} f \circ \theta \, dm \\ &= \int_{M_0} |f|^2 \, dm + \int_A \bar{f} f \circ \theta \, dm \\ &= \int_{M_0} |f|^2 \, dm \geqslant 0.\end{aligned}$$

This proves the lemma. ▮

COROLLARY 3.4. *If $M_0 \subset X$ is of measure zero, then the space $\mathbf{K}$ is trivial, i.e., $\langle f \mid J(f) \rangle = 0$ for all $f \in \mathbf{K}_0$.*

Remark 3.5. Assume that we have (PR1) and (PR2). Assume further that we can find an abelian subspace $\mathfrak{a} \subset \mathfrak{q}$ such that $C^o = \mathrm{Ad}(H)(C^o \cap \mathfrak{a})$. Let $S_A = \exp(C^o \cap \mathfrak{a})$. Then S_A is an abelian semigroup, so one can use

Proposition 3.3 to construct a maximal positive and invariant subspace for S_A. But in general we can not expect this space to be invariant under S.

We read off from the basic Lemma the following Proposition:

PROPOSITION 3.6. *Let π be a unitary representation of G. Assume that $(\tau, J, C, \mathbf{K}_0)$ satisfies the conditions (PR1), (PR2′) and (PR3′). If $Y \in C$, then $\pi(\exp Y)$ induces a contractive selfadjoint operator $\tilde{\pi}(\exp Y)$ on $\mathbf{K}$.*

Proof. If $Y \in C$, then $\pi(\exp Y)\,\mathbf{K}_0 \subset \mathbf{K}_0$, and $\pi(\exp Y)$ is unitary on $\mathbf{H}(\pi)$. Thus

$$\begin{aligned}\langle \pi(\exp Y)\, u \mid J(v)\rangle &= \langle u \mid \pi(\exp(-Y))\, J(v)\rangle \\ &= \langle u \mid J(\pi(\exp Y)\, v)\rangle,\end{aligned}$$

proving that $\pi(\exp Y)$ is selfadjoint in the J-inner product. Since $\pi(\exp Y)$ is unitary on $\mathbf{H}(\pi)$

$$\|\pi(\exp Y)\| = \|\pi(\exp Y)\|_{sp} = 1,$$

and the contractivity property follows. ▮

COROLLARY 3.7. *Let π be a unitary representation of G such that $(\tau, J, C, \mathbf{K}_0)$ satisfies the conditions (PR1), (PR2′) and (PR3′). Then for $Y \in C$ there is a selfadjoint operator $d\tilde{\pi}(Y)$ in $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^{\sim}$, with spectrum contained in $(-\infty, 0]$, such that*

$$\tilde{\pi}(\exp(tY)) = e^{td\tilde{\pi}(Y)}, \qquad t \in \mathbb{R}_+$$

is a contractive semigroup on $\mathbf{K}$. Furthermore the following hold:

(1) *$t \mapsto e^{td\tilde{\pi}(Y)}$ extends to a continuous map $z \mapsto e^{zd\tilde{\pi}(Y)}$ on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geqslant 0\}$ holomorphic on the open right half-plane, and such that $e^{(z+w)\, d\tilde{\pi}(Y)} = e^{zd\tilde{\pi}(Y)} e^{wd\tilde{\pi}(Y)}$.*

(2) *If $Y \in C^o$, then the above map is holomorphic in an open neighborhood of $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geqslant 0\}$.*

(3) *There exists a one-parameter group of unitary operators*

$$\tilde{\pi}(\exp(itY)) := e^{itd\tilde{\pi}(Y)}, \qquad t \in \mathbb{R}$$

on $\mathbf{K}$.

Proof. The last statement follows by the spectral theorem. By construction $\{\tilde{\pi}(\exp(tY)) \mid t \in \mathbb{R}_+\}$ is a semigroup of selfadjoint contractive operators on $\mathbf{K}$. The existence of the operators $d\tilde{\pi}(Y)$ as stated then follows from a general result in operator theory; see, e.g., [8] or [26]. ▮

COROLLARY 3.8. *Let the situation be as in the last corollary. If* $Y \in C \cap -C$ *then* $e^{td\tilde{\pi}(Y)} = \mathrm{id}$ *for all* $t \in \mathbb{R}_+$. *In particular* $d\tilde{\pi}(Y) = 0$ *for every* $Y \in C \cap -C$.

Proof. This follows as the spectrum of $d\tilde{\pi}(Y)$ and $d\tilde{\pi}(-Y)$ is contained in $(-\infty, 0]$. ∎

4. THE LÜSCHER–MACK THEOREM

We use reference [15] for the Lüscher–Mack Theorem, but [9], [10], [22], [23], [25], [27], [32], and [59] should also be mentioned in this connection.

Let π, C, $\mathbf{H}(\pi)$, J and $\mathbf{K}_0$ be as before. We have proved that the operators

$$\{\pi(h \exp(Y)) \mid h \in H,\ Y \in C\}$$

pass to the space $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^{\sim}$ such that $\tilde{\pi}(h)$ is unitary on $\mathbf{K}$, and $\tilde{\pi}(\exp Y)$ is contractive and selfadjoint on $\mathbf{K}$. As a result we arrive at self-adjoint operators $d\tilde{\pi}(Y)$ with spectrum in $(-\infty, 0]$ such that for $Y \in C$, $\tilde{\pi}(\exp Y) = e^{d\tilde{\pi}(Y)}$ on $\mathbf{K}$. As a consequence of that we notice that

$$t \mapsto e^{td\tilde{\pi}(Y)}$$

extends to a continuous map on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geqslant 0\}$ holomorphic on the open right half plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Furthermore,

$$e^{(z+w)\, d\tilde{\pi}(Y)} = e^{zd\tilde{\pi}(Y)} e^{wd\tilde{\pi}(Y)}.$$

As $\mathbf{K}$ is a unitary H-module we know that the H-analytic vectors $\mathbf{K}^{\omega}(H)$ are dense in $\mathbf{K}$. Thus $\mathbf{K}_{oo} := S(C^o)\,\mathbf{K}^{\omega}(H)$ is dense in $\mathbf{K}$. We notice that for $u \in \mathbf{K}_{oo}$ and $X \in C^o$ the function $t \mapsto \tilde{\pi}(\exp tX)\, u$ extends to a holomorphic function on an open neighborhod of the right half-plane. This and the Campbell–Hausdorff formula are among the main tools used in proving the following Theorem of Lüscher and Mack [32]. We refer to [15, p. 292] for the proof. Our present use of Lie theory, cones, and semigroups will follow standard conventions (see, e.g., [6, 11, 30, 63, 66]): the exponential mapping from the Lie algebra $\mathfrak{g}$ to G is denoted exp, the adjoint representation of $\mathfrak{g}$, ad, and that of G is denoted Ad. If π is a representation of G, its differential is denoted $d\pi$, e.g., $d(\mathrm{Ad}) = \mathrm{ad}$. Recall that if π is infinite-dimensional, then $d\pi$ is a representation by unbounded operators on $\mathbf{H}(\pi)$, but the analytic vectors and the C^{∞}-vectors form dense domains for $d\pi$; see [36, 54].

THEOREM 4.1 (Lüscher–Mack). *Let ρ be a strongly continuous contractive representation of $S(C)$ on the Hilbert space $\mathbf{H}$ such that $\rho(s)^* = \rho(\tau(s)^{-1})$. Let G^c be the connected, simply connected Lie group with Lie algebra $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$. Then there exists a continuous unitary representation $\rho^c\colon G^c \to \mathrm{U}(\mathbf{H})$, extending ρ, such that for the differentiated representations $d\rho$ and $d\rho^c$ we have:*

(1) $d\rho^c(X) = d\rho(X) \quad \forall X \in \mathfrak{h}$.

(2) $d\rho^c(iY) = i\, d\rho(Y) \quad \forall Y \in C$.

We apply this to our situation to get the following theorem:

THEOREM 4.2. *Assume that $(\pi, C, \mathbf{H}, J)$ satisfies* (PR1)–(PR3). *Then the following hold:*

(1) *$S(C)$ acts via $s \mapsto \tilde{\pi}(s)$ by contractions on $\mathbf{K}$.*

(2) *Let G^c be the simply connected Lie group with Lie algebra $\mathfrak{g}^c$. Then there exists a unitary representation $\tilde{\pi}^c$ of G^c such that $d\tilde{\pi}^c(X) = d\tilde{\pi}(X)$ for $X \in \mathfrak{h}$ and $i\, d\tilde{\pi}^c(Y) = d\tilde{\pi}(iY)$ for $Y \in C$.*

(3) *The representation $\tilde{\pi}^c$ is irreducible if and only if $\tilde{\pi}$ is irreducible.*

Proof. (1) and (2) follow by the Lüscher-Mack theorem and Proposition 3.6, as the resulting representation of S is obviously continuous.

(3) Let $\mathbf{L}$ be a G^c-invariant subspace in $\mathbf{K}$. Then $\mathbf{L}$ is $\tilde{\pi}(H)$ invariant. Let $Y \in C^o$, $u \in \mathbf{L}^\omega$ and $v \in \mathbf{L}^\perp$. Define $f\colon \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geqslant 0\} \to \mathbb{C}$ by

$$f(z) := \langle v \mid e^{z d\tilde{\pi}(Y)} u \rangle_J.$$

Then f is holomorphic in $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$, and $f(it) = 0$ for every (real) t. Thus f is identically zero. In particular $f(t) = 0$ for every $t > 0$. Thus

$$0 = \langle v \mid e^{t d\tilde{\pi}(Y)} u \rangle_J = \langle v \mid \tilde{\pi}(\exp tY)\, u \rangle_J.$$

As $S^o = H \exp C^o$ it follows that $\tilde{\pi}(S^o)(\mathbf{L}^\omega) \subset (\mathbf{L}^\perp)^\perp = \mathbf{L}$. By continuity we get $\tilde{\pi}(S)\, \mathbf{L} \subset \mathbf{L}$. Thus $\mathbf{K}$ is reducible as an S-module.

The other direction follows in exactly the same way. ∎

Let $(\pi, C, \mathbf{H}, J)$ be as in the last theorem. To identify the resulting representation $\tilde{\pi}^c$ of G^c some facts about holomorhic representations of semigroups and highest weight representations are needed. We refer to [16, Chap. 7] and the references therein, in particular [35], for further references. Define

$$W(\tilde{\pi}^c) := \{X \in \mathfrak{g}^c \mid \forall u \in \mathbf{K}^\infty : i \langle u \mid \tilde{\pi}^c(X)\, u \rangle_J \leqslant 0\}$$

where $\mathbf{K}^\infty$ denotes the $\mathscr{C}^\infty$-vectors for G^c. Then $W(\tilde{\pi}^c)$ is a closed G^c-invariant cone in $\mathfrak{g}^c$. $W(\tilde{\pi}^c)$ is non-trivial as $-iC \subset W(\tilde{\pi}^c)$. Thus $W(\tilde{\pi}^c)$ will always contain the $-\tau$-stable and G-invariant cone generated by $-iC$, i.e., $-i\,\mathrm{Ad}(G)\,C$, but in general $W(\tilde{\pi}^c)$ is neither generating nor pointed. It even does not have to be $-\tau$-invariant. In fact, the Lie algebra of the $(ax+b)$-group, and the Heisenberg group, do not have *any* pointed, generating, invariant cones.

LEMMA 4.3. $W(\tilde{\pi}^c) \cap -W(\tilde{\pi}^c) = \ker(\tilde{\pi}^c)$.

Proof. This is obvious from the spectral theorem. ∎

LEMMA 4.4. *$\mathfrak{g}_1^c := W(\tilde{\pi}^c) - W(\tilde{\pi}^c)$ is an ideal in $\mathfrak{g}^c$. Furthermore,* $[\mathfrak{q}, \mathfrak{q}] \oplus i\mathfrak{q} \subset \mathfrak{g}_1^c$.

Proof. Let $X \in \mathfrak{g}^c$. Then, as $W(\tilde{\pi}^c)$ is invariant by construction, we conclude that

$$e^{t\,\mathrm{ad}(X)}(W(\tilde{\pi}^c) - W(\tilde{\pi}^c)) \subset W(\tilde{\pi}^c) - W(\tilde{\pi}^c), \qquad t \in \mathbb{R}.$$

By differentiation at $t=0$, it follows that $[X, \mathfrak{g}_1^c] \subset \mathfrak{g}_1^c$. This shows that $\mathfrak{g}_1^c$ is an ideal in $\mathfrak{g}^c$. The last part follows as C is generating (in $\mathfrak{q}$). ∎

Remark 4.5. It is not clear if $\mathfrak{g}_1^c$ is τ-stable. To get a τ-stable subalgebra one can replace $W(\tilde{\pi}^c)$ by the cone generated by $-\mathrm{Ad}(G)\,C \subset W(\tilde{\pi}^c)$ or by the maximal G- and $-\tau$-stable cone $W(\tilde{\pi}^c) \cap (-\tau(W(\tilde{\pi}^c)))$ in $W(\pi^c)$.

Let W be a G_1^c invariant cone in $\mathfrak{g}_1^c$. We define $\mathscr{A}(W)$ to be the set of equivalence classes of irreducible unitary representations ρ of G_1^c with $W(\rho) \subset W$.

THEOREM 4.6. *Assume that the analytic subgroup G_1^c of G^c corresponding to $\mathfrak{g}_1^c$ is closed in G and that $W(\tilde{\pi}^c)$ is pointed. Then $\tilde{\pi}^c|_{G_1^c}$ is a direct integral of irreducible representations in $\mathscr{A}(W)$.*

Proof. As G_1^c is closed in G it follows that $\tilde{\pi}^c|_{G_1}$ is a continuous unitary representation of G_1^c. Furthermore $W(\tilde{\pi}^c|_{G_1}) = W(\tilde{\pi}^c)$. The theorem now follows from the theorem of Neeb and Olshanskii [35], to the effect that an injective representation ρ, with $W(\rho)$ pointed and generating, is necessarily a direct integral of representations from $\mathscr{A}(W(\rho))$ (cf. [35]). ∎

5. EXAMPLES OF SEMISIMPLE SYMMETRIC SPACES

We will now generalize the example from the Introduction to a class of semisimple Lie groups. For that we recall some facts about non-compactly causal or ordered semisimple symmetric spaces. We include some ideas of the proofs to make the text more self contained. For more information we refer to [16, 39]. An additional source of inspiration for the present chapter is the following series of papers: [36, 43, 44].

Let G/H be a semisimple symmetric space and let τ be the corresponding involution. *We will assume that G/H is irreducible.* Let θ be a Cartan involution on G commuting with τ. Then

$$\begin{aligned}\mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{q} \\ &= \mathfrak{k} \oplus \mathfrak{p} \\ &= \mathfrak{h}_k \oplus \mathfrak{h}_p \oplus \mathfrak{q}_k \oplus \mathfrak{q}_p\end{aligned}$$

where a subscript denotes the intersection with the corresponding subspace of $\mathfrak{g}$. Let L be a Lie group and $\mathbf{V}$ an L-module. We denote by $\mathbf{V}^L$ the subspace of L-fixed points in $\mathbf{V}$.

Definition 5.1. The irreducible symmetric space G/H is called *non-compactly causal* (NCC) if $\mathfrak{q}_p^{H \cap K} \neq \{0\}$.

Remark 5.2. A NCC-space is also a K_ε-space in the sense of [48, 49].

If G/H is NCC then $\mathfrak{q}_p^{H \cap K}$ is one-dimensional and there exists an element $X^0 \in \mathfrak{q}_p^{H \cap K}$ such that $\mathfrak{h}_k \oplus \mathfrak{q}_p = \mathfrak{z}_\mathfrak{g}(X^0)$. We can normalize X^0 such that ad X^0 has eigenvalues 0, 1, and -1. Let $\mathfrak{a} := \mathbb{R}X^0$, $\mathfrak{n} = \{X \in \mathfrak{g} \mid [X^0, X] = X\}$, and $\bar{\mathfrak{n}} = \{X \in \mathfrak{g} \mid [X^0, X] = -X\} = \theta(\mathfrak{n}) = \tau(\mathfrak{n})$. We also define

$$\mathfrak{m} := \{X \in \mathfrak{z}_\mathfrak{g}(X^0) \mid B(X, X^0) = 0\}$$

where B is the Killing form of $\mathfrak{g}$. Then

$$\mathfrak{p}_{\max} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is a maximal parabolic subalgebra of $\mathfrak{g}$.

Assume from now on that $G \subset G_\mathbb{C}$ where $G_\mathbb{C}$ is the simply connected, connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$. We will also assume that $H = G^\tau$. Then $H \cap K = Z_K(X^0)$. Let $A := \exp \mathfrak{a}$, $N := \exp \mathfrak{n}$ and $\bar{N} := \exp \bar{\mathfrak{n}}$. Let M_o be the analytic subgroup of G corresponding to $\mathfrak{m}$ and let $M = (H \cap K)\, M_o$. Then M is a closed and τ-stable subgroup of G, $M \cap A = \{1\}$ and $MA = Z_G(A)$. Let $P_{\max} := N_G(\mathfrak{p}_{\max})$. Then $P_{\max} = MAN$. We have $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}_{\max}$. The differential of the map $(h, p) \mapsto hp$ is given by $\mathrm{Ad}(p^{-1})(X + \mathrm{Ad}(p)\, Y)_{hp}$, $X \in \mathfrak{h}$ and $Y \in \mathfrak{p}_{\max}$, and this is surjective, as $\mathrm{Ad}(p)\, \mathfrak{p}_{\max} = \mathfrak{p}_{\max}$.

Lemma 5.3. *$HP_{\max}$ is open in G and contained in $\bar{N}P_{\max}$.*

Proof. That $HP_{\max}$ is open in G follows by the above discussion (for the general case see [34]). The proof of the second statement can be found in [16, 39]. The idea is to use a maximal set of strongly orthogonal roots to reduce this to $\mathrm{SL}(2, \mathbb{R})$-calculations as we will explain in a moment. ∎

Let $\mathfrak{a}_q$ be a maximal abelian subalgebra of $\mathfrak{p}$ containing X^0. Then $\mathfrak{a}_q \subset \mathfrak{q}_p$ and $\mathfrak{a}_q$ is maximal abelian in $\mathfrak{q}$. Let Δ be the set of roots of $\mathfrak{a}_q$ in $\mathfrak{g}$. Then $\Delta = \Delta_0 \cup \Delta_+ \cup \Delta_-$, where $\Delta_0 = \{\alpha \in \Delta \mid \alpha(X^0) = 0\}$, $\Delta_\pm = \{\alpha \in \Delta \mid \alpha(X^0) = \pm 1\}$. Choose a positive system Δ_0^+ in Δ_0, and let $\Delta^+ = \Delta_0^+ \cup \Delta_+$. Two roots α, β, $\alpha \neq \pm\beta$ are called *strongly orthogonal* if $\alpha \pm \beta$ is not a root. Choose a maximal set of strongly orthogonal roots $\gamma_1 < \gamma_2 < \cdots < \gamma_r$ in Δ_+ such that γ_r is the maximal root in Δ_+, γ_{r-1} is the maximal root in Δ_+ strongly orthogonal to γ_r, γ_{r-2} is the maximal root in Δ_+ strongly orthogonal to γ_r and γ_{r-1}, etc. Choose $H_j \in \mathfrak{a}_q$ such that $\langle \gamma_i, H_j \rangle = 2\delta_{ij}$ and $H_j \in [\mathfrak{g}_{\gamma_j}, \mathfrak{g}_{-\gamma_j}]$. Choose $X_j \in \mathfrak{g}_{\gamma_j}$ such that, with $X_{-j} := \tau(X_j) = -\theta(X_j)$, we have $H_j = [X_j, X_{-j}]$. In the case of $\mathfrak{sl}(2, \mathbb{R})$ the involution is given by

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

as in the Introduction. In that case

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad X_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Define a homomorphism φ_j: $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to H_j, \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to X_j \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to X_{-j}$$

As the roots γ_j are strongly orthogonal, we get $[\mathrm{Im}(\varphi_j), \mathrm{Im}(\varphi_i)] = \{0\}$ if $i \neq j$. As $\mathrm{SL}(2, \mathbb{C})$ is simply connected the homomorphisms φ_j integrates to homomorphisms $\mathrm{SL}(2, \mathbb{C}) \to G_{\mathbb{C}}$, also denoted by φ_j, such that $\varphi_j(\mathrm{SL}(2, \mathbb{R})) \subset G$, and such that φ_j intertwines the Cartan involution and the above involution τ on $\mathrm{SL}(2, \mathbb{R})$ with the corresponding involutions on G.

The following lemma follows from the maximality of the set of strongly orthogonal roots; see also [42, Lemma 2.3]:

LEMMA 5.4. *Let* $\mathfrak{a}_h = \bigoplus_{j=1}^r \mathbb{R}(X_j + X_{-j})$. *Then* $\mathfrak{a}_h \subset \mathfrak{h}_p$, *and* $\mathfrak{a}_h$ *is maximal abelian in* $\mathfrak{h}_p$.

Let $\log := (\exp|_{\bar{\mathfrak{n}}})^{-1}$: $\bar{N} \to \bar{\mathfrak{n}}$. Define ζ: $\bar{N}P_{\max}/P_{\max} \to \bar{\mathfrak{n}}$ by

$$\zeta(\bar{n}P_{\max}) = \log(\bar{n}).$$

We notice that $\zeta(hx) = \mathrm{Ad}(h)\,\zeta(x)$ for $h \in H \cap K$. We also notice that $H \cap P_{\max} = H \cap K$.

LEMMA 5.5. *Let* $h = \exp \sum_{j=1}^{r} t_j(X_j + X_{-j}) \in A_h := \exp \mathfrak{a}_h$. *Then*

$$hP_{\max} = \left(\exp \sum_{j=1}^{r} \tanh t_j X_{-j}\right) P_{\max}.$$

In particular $\zeta(hP_{\max}) = \sum_{j=1}^{r} \tanh t_j X_{-j}$.

Proof. Assume first that $G = \mathrm{SL}(2, \mathbb{R})$. Then

$$h = h_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

By (1.1), we have $h_t \in \bar{N} P_{\max}$, and $\zeta(h_t P_{\max}) = \tanh t X_{-1}$. Let $t_1, ..., t_r \in \mathbb{R}$: then

$$\begin{aligned} \exp \sum_{j=1}^{r} & t_j(X_j + X_{-j})\, P_{\max} \\ &= \varphi_1(h_{t_1}) \cdots \varphi_r(h_{t_r})\, P_{\max} \\ &= \varphi_1\left(\begin{pmatrix} 1 & 0 \\ \tanh t_1 & 1 \end{pmatrix}\right) \cdots \varphi_r\left(\begin{pmatrix} 1 & 0 \\ \tanh t_r & 1 \end{pmatrix}\right) P_{\max} \\ &= \left(\exp \sum_{j=1}^{r} \tanh(t_j)\, X_{-j}\right) P_{\max}. \end{aligned}$$

From this the lemma now follows. ∎

THEOREM 5.6. *Let* $\Omega = \mathrm{Ad}(H \cap K)\{\sum_{j=1}^{r} t_j X_{-j} \mid \forall j\colon -1 < t_j < 1\}$. *Then* Ω *is convex,*

$$HP_{\max} = (\exp \Omega)\, P_{\max},$$

and ζ *induces an* H*-isomorphism* $H/H \cap K \simeq \Omega$.

Proof. The convexity will follow from Lemma 5.7. That $(\exp \Omega)\, P_{\max} \subset HP_{\max}$ follows from the fact that $\exp t_j X_{-j} P_{\max} \subset HP_{\max}$ by $\mathrm{SL}(2, \mathbb{R})$-reduction. Let $h \in H$. Then h can be written as $h = k_1 a k_2$, with $k_1, k_2 \in H \cap K$ and $a = \exp \sum t_j(X_j + X_{-j}) \in A_h$. As $H \cap K \subset P_{\max}$, it follows that

$$\begin{aligned} hP_{\max} &= k_1 a P_{\max} \\ &= k_1 \exp \sum_{j=1}^{r} \tanh(t_j)\, X_{-j} P_{\max} \\ &= \exp\left(\operatorname{Ad}(k_1) \sum_{j=1}^{r} \tanh(t_j)\, X_{-j}\right) P_{\max} \\ &\in \exp \Omega P_{\max} \subset \bar{N} P_{\max} \end{aligned}$$

Thus $HP_{\max} \subset \exp(\Omega)\, P_{\max}$. ∎

The maximal compactly embedded subalgebra $\mathfrak{k}^c$ in $\mathfrak{g}^c$ corresponding to the Cartan involution $\theta^c = \theta\tau$ has center $i\mathfrak{a}$ and $\mathfrak{z}_{\mathfrak{g}^c}(i\mathfrak{a}) = \mathfrak{k}^c$. It follows that G^c/K^c is a bounded symmetric domain and that τ induces an anti-holomorphic involution on G^c/K^c, i.e., a conjugation. The real form of G^c/K^c corresponding to this conjugation is exactly $H/H \cap K$, (see [18, 19] for classification). In the classical notation of Harish-Chandra (cf. [13]) we have $\mathfrak{p}^- = \bar{\mathfrak{n}}_{\mathbb{C}}$. Thus G^c/K^c can be realized as a bounded symmetric domain $\Omega_{\mathbb{C}}$ in $\bar{\mathfrak{n}}_{\mathbb{C}}$. Let σ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. Then $\sigma \mid \mathfrak{g}^c = \tau \mid \mathfrak{g}^c$. Thus the conjugation given by τ on $\Omega_{\mathbb{C}}$ is also realized by σ. We now have:

Lemma 5.7. *Let $\Omega_{\mathbb{C}}$ be the bounded convex circular realization of G^c/K^c in $\bar{\mathfrak{n}}_{\mathbb{C}}$. Then*

$$\Omega = \Omega_{\mathbb{C}}^{\sigma} = \{X \in \Omega_{\mathbb{C}} \mid \sigma(X) = X\}.$$

We also notice the following for later use:

Lemma 5.8. *Denote the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}^c$ by σ^c. Then σ^c coincides with the conjugate linear extension $\tau \circ \sigma$ of τ to $\mathfrak{g}_{\mathbb{C}}$.*

Proof. We have $\{X \in \mathfrak{g}_{\mathbb{C}} \mid \tau\sigma(X) = X\} = \mathfrak{h} \oplus i\mathfrak{q} = \mathfrak{g}^c$. Hence the lemma. ∎

Let

$$S(H, P_{\max}) := \{g \in G \mid gH \subset HP_{\max}\}.$$

Then $S(H, P_{\max})$ is a closed semigroup invariant under $s \mapsto s^\# := \tau(s)^{-1}$. For $g \in G$ and $X \in \bar{\mathfrak{n}}$ such that $g \exp X \in \bar{N} P_{\max}$ define $g \cdot X \in \bar{\mathfrak{n}}$ and $a(g, X) \in A$ by

$$g \exp X \in \exp(g \cdot X)\, Ma(g, X)\, N.$$

Then $g \cdot X = \zeta(g \exp X)$, where $\zeta\colon H/H \cap K \simeq \Omega$ is introduced in the Theorem 5.6 (see also Lemmas 5.4–5.5). It follows that $a(g, X)$ is defined for $g \in S(H, P_{\max})$ and $X \in \Omega$. The map $(g, X) \mapsto g \cdot X$ transfers the canonical action on $G/P_{\max}$, restricted to the open set $HP_{\max}/P_{\max}$, to Ω. We have the lemma:

LEMMA 5.9. (1) *Let* $s, r \in S(H, P_{\max})$ *and* $X \in \Omega$. *Then* $(sr) \cdot X = (s \cdot (r \cdot X))$ *and* $a(sr, X) = a(s, r \cdot X)\, a(r, X)$.

(2) *Let* $g = ma \in MA$ *and* $X \in \bar{\mathfrak{n}}$. *Then* $g \exp X \in \bar{N} P_{\max}$, $g \cdot X = \mathrm{Ad}(g)\, X$, *and* $a(g, X) = a$.

(3) *Let* C *be an* H*-invariant pointed and generating cone in* $\mathfrak{q}$ *containing* X^0. *Then* $S = H \exp C$ *is a closed semigroup acting on* Ω *by contractions. Furthermore* $H \times C^o \ni (h, X) \mapsto h \exp X \in S^o$ *is a diffeomorphism.*

(4) $S(H, P_{\max}) \subset HP_{\max}$.

Proof. Let s, r and X be as in the lemma. Then on the one hand

$$(sr) \exp X = \exp((sr) \cdot X)\, m(sr, X)\, a(sr, X)\, n(sr, X)$$

for some $m(sr, X) \in M$ and $n(sr, X) \in N$. On the other hand, using the notation

$$\begin{aligned} n &= n(r, X), \\ n_1 &= n(s, r \cdot X), \\ n_2 &= [(m(r, X)\, a(r, X))^{-1}\, n(s, r \cdot X)\, m(r, X)\, a(r, X)]\, n(r, X) \end{aligned}$$

$(n, n_1, n_2 \in N)$, we have

$$\begin{aligned} (sr) \exp(X) &= s(r \exp(X)) \\ &= s \exp(r \cdot X)\, m(r, X)\, a(r, X)\, n \\ &= \exp(s \cdot (r \cdot X))\, m(s, r \cdot X)\, a(s, r \cdot X)\, n_1 m(r, X)\, a(r, X)\, n_1 \\ &= \exp(s \cdot (r \cdot X))\, m(s, r \cdot X)\, m(r, X)\, a(s, r \cdot X)\, a(r, X)\, n_2. \end{aligned}$$

This proves (1).

(2) This follows from $g \exp X = \exp(\mathrm{Ad}(g)\, X)\, g$, and the fact that MA normalizes $\bar{N}$.

(3) Let $\mathfrak{p}$ and $\mathfrak{q}$ be as described before Definition 5.1 above, and let C be a pointed and generating H-invariant cone in $\mathfrak{q}$ such that $C^o \cap \mathfrak{p} \neq \emptyset$. Then by [16, 39]

$$C^o = \mathrm{Ad}(H)(C^o \cap \mathfrak{a}_q). \tag{5.1}$$

Let $X \in C^o \cap \mathfrak{a}$ and $Y \in \Omega$. Then $Y = \sum Y_{-\alpha}$, with $\alpha \in \Delta_+$ and $Y_{-\alpha} \in \mathfrak{g}_{-\alpha}$. Therefore

$$\exp(X) \cdot Y = \mathrm{Ad}(\exp X)\, Y = \sum_{\alpha \in \Delta_+} e^{-\alpha(X)} Y_{-\alpha}$$

As $\alpha(X) > 0$ we see that $\exp(X) \cdot Y \in \Omega$. This also shows that $\exp(C^o \cap \mathfrak{a})$ acts by contractions on Ω. Let now $s \in S^o$. Then $s = h \exp X = h \exp(\mathrm{Ad}(h_1)\, X_1)$ with $h, h_1 \in H$, $X \in C^o$ and $X_1 \in C^o \cap \mathfrak{a}$. Let $Y \in \Omega$. Then

$$s \cdot Y = hh_1 \cdot (\exp X_1 \cdot ((h_1^{-1}) \cdot Y)) \in \Omega.$$

It follows that S acts by contractions on Ω.

(4) Apply $S(H, P_{\max})$ to $eP_{\max}$, e the identity in G. ∎

LEMMA 5.10. *Let $t > 0$ and $Y \in \Omega$. Then $\exp tX^0 \in S$ and $\exp tX^0 \cdot Y = e^{-t} Y$.*

We also notice the following sharpening of (3) in Lemma 5.10 (cf. [15] and [16]):

LEMMA 5.11. *Let $C = C_{\max}$ be the maximal pointed generating cone in $\mathfrak{q}$ containing X^0. Then the following hold:*

(1) $C^o \cap \mathfrak{a} = \{X \in \mathfrak{a}_q \mid \forall \alpha \in \Delta_+ : \alpha(X) > 0\}$;

(2) $S(H, P_{\max}) = H \exp C_{\max}$.

We need to fix the normalization of measures before we discuss the representations that we will use. Let the measure da on A be given by

$$\int_A f(a)\, da = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(a_t)\, dt, \qquad a_t = \exp 2tX_0.$$

We fix the Lebesgue measure dX on $\bar{\mathfrak{n}}$ such that, for $d\bar{n} = \exp(dX)$, we then have

$$\int_{\bar{\mathfrak{n}}} a(\bar{n})^{-2\rho}\, d\bar{n} = 1.$$

Here $\rho(X) = \frac{1}{2} \mathrm{Tr}(\mathrm{ad}(X))|_{\mathfrak{n}}$ as usual, and $a(g)$, $g \in G$, is determined by $g \in KMa(g)\, N$. The Haar measure on compact groups will always be normalized to have total measure one. The measure on N is $\theta(d\bar{n})$. Let us fix a Haar measure dh on H. Then we can normalize the invariant measure on G and M such that for $f \in \mathscr{C}_c(G)$, $\mathrm{Supp}(f) \subset HP_{\max}$, we have

$$\int_G f(g)\, dg = \int_H \int_M \int_A \int_N f(hman)\, a^{2\rho}\, dn\, dm\, da\, dh.$$

The invariant measure $d\dot{x}$ on G/H is then given by

$$\int_G f(x)\, dx = \int_{G/H} \int_H f(xh)\, dh\, d\dot{x}, \qquad f \in \mathscr{C}_c(G)$$

and similarly for $K/H \cap K$. We fix the Haar measure on M such that $dg = a^{2\rho}\, dk\, dm\, da\, dn$.

LEMMA 5.12. *Let the measures be normalized as above. Then the following hold*:

(1) *Let $f \in \mathscr{C}_c(\bar{N}MAN)$. Then*

$$\int_G f(g)\, dg = \int_{\bar{N}} \int_M \int_A \int_N f(\bar{n}man)\, a^{2\rho}\, d\bar{n}\, dm\, da\, dn.$$

(2) *Let $f \in \mathscr{C}_c(\bar{N})$. For $y \in \bar{N}MAN$ write $y = \bar{n}(y)\, m_{\bar{N}}(y)\, a_{\bar{N}}(y)\, n_{\bar{N}}(y)$. Let $x \in G$. Then*

$$\int_{\bar{N}} f(\bar{n}(x\bar{n}))\, a_{\bar{N}}(x\bar{n})^{-2\rho}\, d\bar{n} = \int_{\bar{N}} f(\bar{n})\, d\bar{n}.$$

(3) *Write, for $g \in G$, $g = k(g)\, m(g)\, a(g)\, n(g)$ according to $G = KMAN$. Let $h \in \mathscr{C}(K/H \cap K)$. Then*

$$\int_{K/H \cap K} h(\dot{k})\, d\dot{k} = \int_{\bar{N}} h(k(\bar{n})\, H \cap K)\, a(\bar{n})^{-2\rho}\, d\bar{n}.$$

(4) *Let $h \in \mathscr{C}(K/H \cap K)$ and let $x \in G$. Then*

$$\int_{K/H \cap K} f(k(xk)\, H \cap K)\, a(xk)^{-2\rho}\, d\dot{k} = \int_{K/H \cap K} f(\dot{k})\, d\dot{k}$$

(5) *Assume that* $\operatorname{Supp}(f) \subset H/H \cap K \subset K/H \cap K$. *Then*

$$\int_{K/H \cap K} f(\dot{k})\, d\dot{k} = \int_{H/H \cap K} f(k(h)\, H \cap K)\, a(h)^{-2\rho}\, d\dot{h}.$$

(6) *Let $f \in \mathscr{C}_c(\bar{N})$,* $\operatorname{Supp}(f) \subset \exp \Omega$. *Then*

$$\int_{\bar{N}} f(\bar{n})\, d\bar{n} = \int_{H/H \cap K} f(\bar{n}(h))\, a_{\bar{N}}(h)^{-2\rho}\, d\dot{h}.$$

(7) *For* $x \in HP_{\max}$ *write* $x = h(x)\, m_H(x)\, a_H(x)\, n_H(x)$ *with* $h(x) \in H$, $m_H(x) \in M$, $a_H(x) \in A$, *and* $n_H(x) \in N$. *Let* $f \in \mathscr{C}_c^\infty(H/H \cap K)$ *and let* $x \in G$ *be such that* $xHP_{\max} \subset HP_{\max}$. *Then*

$$\int_{H/H \cap K} f(h(xh)\, H \cap K)\, a_H(xh)^{-2\rho}\, d\dot{h} = \int_{H/H \cap K} f(\dot{h})\, d\dot{h}$$

Proof. Up to normalizing constants this can be found in [38]. Let us show that the constant in (1) is equal to 1. Choose $c > 0$ such that

$$c \int_G f(g)\, dg = \int_{\bar{N}} \int_M \int_A \int_N f(\bar{n}man)\, a^{2\rho}\, d\bar{n}\, dm\, da\, dn.$$

Let $\varphi \in \mathscr{C}_c(P_{\max})$ such that $\int_{MAN} \varphi(man)\, a^{2\rho}\, dm\, da\, dn = 1$ and $\varphi(mp) = \varphi(p)$ for every $m \in M \cap K = H \cap K$ and every $p \in P_{\max}$. Define $f \in \mathscr{C}(G)$ by $f(kman) = \varphi(man)$. Then

$$\begin{aligned}
c &= c \int_K \int_M \int_A \int_N f(kman)\, a^{2\rho}\, dk\, dm\, da\, dn \\
&= c \int_G f(g)\, dg \\
&= \int_{\bar{N}} \int_M \int_A \int_N f(\bar{n}man)\, a^{2\rho}\, d\bar{n}\, dm\, da\, dn \\
&= \int_{\bar{N}} \int_M \int_A \int_N f(k(\bar{n})\, m(\bar{n})\, a(\bar{n})\, n(\bar{n})\, man)\, a^{2\rho}\, d\bar{n}\, dm\, da\, dn \\
&= \int_{\bar{N}} \int_M \int_A \int_N f(k(\bar{n})\, man)\, a(\bar{n})^{-2\rho}\, a^{2\rho}\, d\bar{n}\, dm\, da\, dn \\
&= \int_{\bar{N}} a(\bar{n})^{-2\rho}\, d\bar{n} = 1
\end{aligned}$$

This proves (1). The other claims are proved in a similar way. ∎

Let us now go over to the representations that we are going to use. We identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ by

$$\mathfrak{a}_{\mathbb{C}}^* \ni \nu \mapsto 2\nu(X^0) \in \mathbb{C}.$$

Then ρ corresponds to $\dim \mathfrak{n}$. For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let $\mathscr{C}^\infty(\nu)$ be the space of $\mathscr{C}^\infty$-functions $f: G \to \mathbb{C}$ such that, for $a_t = \exp t(2X^0)$,

$$f(gma_t n) = e^{-(\nu+\rho)\,t} f(g) = a_t^{-(\nu+\rho)} f(g).$$

Define an inner product on $\mathscr{C}^\infty(\nu)$ by

$$\langle f \mid g \rangle_\nu := \int_K \overline{f(k)}\, g(k)\, dk = \int_{K/H \cap K} \overline{f(k)}\, g(k)\, d\dot{k}.$$

Then $\mathscr{C}^\infty(\nu)$ becomes a pre-Hilbert space. We denote by $\mathbf{H}(\nu)$ the completion of $\mathscr{C}^\infty(\nu)$. Define $\pi(\nu)$ by

$$[\pi(\nu)(x)\, f](g) := f(x^{-1}g), \qquad x, g \in G, \qquad f \in \mathscr{C}^\infty(\nu).$$

Then $\pi(\nu)(x)$ is bounded, so it extends to a bounded operator on $\mathbf{H}(\nu)$, which we denote by the same symbol. Furthermore $\pi(\nu)$ is a continuous representation of G which is unitary if and only if $\nu \in i\mathbb{R}$. By [54] we have $\mathbf{H}(\nu)^\infty = \mathscr{C}^\infty(\nu)$. We can realize $\mathbf{H}(\nu)$ as $\mathbf{L}^2(K/H \cap K)$ and as $\mathbf{L}^2(\bar{N}, a(\bar{n})^{2\,\mathrm{Re}(\nu)}\, d\bar{n})$ by restriction (see Lemma 5.15). In the first realization the representation $\pi(\nu)$ becomes

$$[\pi(\nu)(x)\, f](k) = a(x^{-1}k)^{-\nu-\rho}\, f(k(x^{-1}k))$$

and in the second

$$[\pi(\nu)(x)\, f](\bar{n}) = a_{\bar{N}}(x^{-1}\bar{n})^{-\nu-\rho}\, f(\bar{n}(x^{-1}\bar{n})).$$

The following is well known, but for completeness we include the proof.

Lemma 5.13. *The pairing*

$$\mathbf{H}(\nu) \times \mathbf{H}(-\bar{\nu}) \ni (f, g) \mapsto \langle f \mid g \rangle_\nu := \int_K \overline{f(k)}\, g(k)\, dk$$

$$= \int_{K/H \cap K} \overline{f(k)}\, g(k)\, d\dot{k}$$

is G-invariant, i.e.,

$$\langle \pi(\nu)(x)\, f \mid g \rangle_\nu = \langle f \mid \pi(-\bar{\nu})(x^{-1})\, g \rangle_\nu.$$

Proof. Let $x \in G$ and $k \in K$. Then $x(x^{-1}k) = k$, which implies that

$$\begin{aligned}
k &= xk(x^{-1}k)\, a(x^{-1}k)\, n(x^{-1}k)\\
&= k(xk(x^{-1}k))\, a(xk(x^{-1}k))\, n(xk(x^{-1}k))\, a(x^{-1}k)\, n(x^{-1}k)\\
&= k(xk(x^{-1}k))\, a(xk(x^{-1}k))\, a(x^{-1}k)\, n.
\end{aligned}$$

for some $n=N$. Thus $k(xk(x^{-1}k))=k$ and $a(xk(x^{-1}k))=a(x^{-1}k)^{-1}$. Using those relations, and Lemma 5.12, we get

$$\begin{aligned}\langle \pi(v)(x)\, f \mid g\rangle_v &= \int_{K/H\cap K} \overline{f(x^{-1}k)}\, g(k)\, d\dot{k}\\ &= \int_{K/H\cap K} \overline{f(k(x^{-1}k))}\, a(x^{-1}k)^{-(\bar{v}+\rho)}\, g(k(xk(x^{-1}k)))\, d\dot{k}\\ &= \int_{K/H\cap K} \overline{f(k)}[a(xk)^{-(-\bar{v}+\rho)}\, g(k(xk))]\, d\dot{k}\\ &= \langle f \mid \pi(-v)(x^{-1})\, g\rangle_v.\end{aligned}$$

This proves the lemma. ∎

Remark 5.14. We notice that if v is purely imaginary, i.e., $-\bar{v}=v$, the above shows that $(\pi(v), \mathbf{H}(v))$ is then unitary.

LEMMA 5.15. (1) *The restriction map induces an isometry of* $\mathbf{H}(v)$ *onto* $\mathbf{L}^2(\bar{N}, a(\bar{n})^{2\,\mathrm{Re}\,v}\, d\bar{n})$.

(2) *On* $\bar{N}$ *the invariant pairing* $\langle\cdot\mid\cdot\rangle_v$ *is given by*

$$\langle f \mid g\rangle_v = \int_{\bar{N}} \overline{f(\bar{n})}\, g(\bar{n})\, d\bar{n}, \qquad f\in\mathbf{H}(v), \quad g\in\mathbf{H}(-\bar{v}).$$

(3) *Let* $\mathbf{H}_H(v)$ *be the closure of* $\{f\in\mathscr{C}^\infty(v)\mid \mathrm{Supp}(f)\subset HP_{\max}\}$. *Then* $\mathbf{H}_H(v)\ni f\mapsto f|_H\in\mathbf{L}^2(H/H\cap K, a(h)^{2\rho}\, d\dot{h})$ *is an isometry.*

(4) *Let* $f\in\mathbf{H}(v)$, $g\in\mathbf{H}(-\bar{v})$ *and assume that* $\mathrm{Supp}(fg)\subset HP_{\max}$. *Then*

$$\langle f\mid g\rangle_v = \int_{H/H\cap K} \overline{f(h)}\, g(h)\, d\dot{h}.$$

Proof. (1) We have $k(\bar{n})=\bar{n}a(\bar{n})^{-1}n$, $\bar{n}\in\bar{N}$, $n\in MN$. By Lemma 5.12 we get

$$\int_{K/H\cap K} \overline{f(k)}\, g(k)\, d\dot{k} = \int_{\bar{N}} \overline{f(\bar{n})}\, g(\bar{n})\, a(\bar{n})^{\bar{v}+\mu}\, d\bar{n}, \qquad f\in\mathbf{H}(v), \quad g\in\mathbf{H}(\mu).$$

(2)–(4) follow immediately from Lemma 5.12. ∎

Let us assume, from now on, that there exists an element $w\in N_K(\mathfrak{a})$ such that $\mathrm{Ad}(w)(X^0)=-X^0$ on $\mathfrak{a}$. Let us remark the following for later use:

LEMMA 5.16. *Let* $w\in K$ *be such that* $\mathrm{Ad}(w)|_{\mathfrak{a}}=-\mathrm{id}$. *Then* $w^2\in H\cap K$, *and there is a* $m\in H\cap K$ *such that* $\tau(w)=w^{-1}m$.

Proof. As $\mathrm{Ad}(w^2)|_{\mathfrak{a}} = \mathrm{id}$ we get $w^2 \in M \cap K = H \cap K$. Let $X \in \mathfrak{a}$. Then

$$X = \tau(\mathrm{Ad}(w)\, X) = \mathrm{Ad}(\tau(w))(\tau(X)) = -\mathrm{Ad}(\tau(w))\, X.$$

Hence $\mathrm{Ad}(\tau(w)\, w)\, X = X$. Thus $\tau(w)\, w =: m \in M \cap K = H \cap K$. It follows that $\tau(w) = w^{-1}(wmw^{-1})$. The claim follows as $wMw^{-1} = M$. ∎

We recall that G/H is of *Cayley type* if $\mathfrak{h}$ has a one-dimensional center contained in $\mathfrak{h}_p$. This is the case if and only if G/K is a tube-type domain $G/K \simeq \mathbb{R}^n + i\Omega$, where Ω is an open self-dual cone isomorphic to $H/H \cap K$. Thus G/H is locally isomorphic to one of the following spaces (where we denote by the subscript $+$ the group of elements having positive determinant): $\mathrm{Sp}(n, \mathbb{R})/\mathrm{GL}(n, \mathbb{R})_+$, $\mathrm{SU}(n, n)/\mathrm{GL}(n, \mathbb{C})_+$, $\mathrm{SO}^*(4n)/\mathrm{SU}^*(2n)\, \mathbb{R}_+$, $\mathrm{SO}(2, k)/\mathrm{SO}(1, k-1)\, \mathbb{R}_+$ and $E_{7(-25)}/E_{6(-26)}\, \mathbb{R}_+$.

LEMMA 5.17. *Assume that G/H is of Cayley type. Let*

$$w = \varphi_1\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdots \varphi_r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \exp\left(\frac{\pi}{2} \sum_{j=1}^{r} X_j + \theta(X_j)\right).$$

Then $\mathrm{Ad}(w)|_{\mathfrak{a}} = -\mathrm{id}$.

Proof. As G/H is of Cayley type, $X^0 = \frac{1}{2} \sum_{j=1}^{r} H_j$. The claim follows now by simple $\mathfrak{sl}(2, \mathbb{R})$-calculation. ∎

We also recall the following lemma from [16, 39]:

LEMMA 5.18. *Assume that G/H is of Cayley type. Let $Y^0 \in \mathfrak{h}_p$ be such that $\mathfrak{z}_{\mathfrak{g}}(Y^0) = \mathfrak{h}$ and such that* $\mathrm{spec}(\mathrm{ad}\, Y^0) = \{0, 1, -1\}$. *Then* $\mathbf{c} := \mathrm{Ad}(\exp(\pi i/2)\, Y^0)$ *defines a Lie algebra isomorphism $\mathfrak{g} \to \mathfrak{g}^c$ such that*

(1) $\mathbf{c}|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$.

(2) *Let* $\mathfrak{q}^+ := \{X \in \mathfrak{q} \mid [Y^0, X] = X\}$. *Then* $\mathbf{c}|_{\mathfrak{q}^+} = i\, \mathrm{id}$.

(3) *Let* $\mathfrak{q}^- := \{X \in \mathfrak{q} \mid [Y^0, X] = -X\}$. *Then* $\mathbf{c}|_{\mathfrak{q}^-} = -i\, \mathrm{id}$.

(4) *We have* $\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^-$.

Proof. That $\mathbf{c}\colon \mathfrak{g} \to \mathfrak{g}^c$ is an isomorphism follows from (1)–(4). (1)–(3) follow directly. For (4) notice that $\mathrm{ad}\, Y^0$ maps $\mathfrak{q}$ into $\mathfrak{q}$. As the centralizer of Y^0 is exactly $\mathfrak{h}$ it follows that $\mathrm{ad}\, Y^0\colon \mathfrak{q} \to \mathfrak{q}$ is an isomorphism and that $\mathfrak{q}$ is the direct sum of the eigenspaces of $\mathrm{ad}\, Y^0$ for the eigenvalues 1 and -1. From that the claim follows. ∎

Assume now that $\mathfrak{h}$ is one of the Lie algebras $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{su}(n, n)$, $\mathfrak{so}^*(4n)$, $\mathfrak{so}(2, k)$ and $\mathfrak{e}_{7(-25)}$. Let $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ and let $G = H_{\mathbb{C}}$ be the simply connected group with Lie algebra $\mathfrak{g}$. Let $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be the conjugation with respect to $\mathfrak{h}$. Denote the corresponding real analytic involution $G \to G$ by the same letter. Then it is well known that $G^\tau = H$ is connected. We refer to [16, Example 1.2.2].

LEMMA 5.19. *Assume that* $\tau: \mathfrak{g} \to \mathfrak{g}$ *is the conjugation with respect to the real form* $\mathfrak{h}$. *Then*

(1) $\mathfrak{g}^c \simeq \mathfrak{h} \times \mathfrak{h}$ *and* G^c *is locally isomorphic to* $H \times H$.

(2) *Under this isomorphism the involution* τ *corresponds to* $\tau(X, Y) = (Y, X)$, *i.e.,* $\mathfrak{h}$ *corresponds to the diagonal in* $\mathfrak{g}^c$.

(3) *Let* $\tilde{H}$ *be the simply connected Lie group with Lie algebra* $\mathfrak{h}$. *Then* G^c *is* $\tilde{H} \times \tilde{H}$ *and* τ *is given by* $\tau(a, b) = (b, a)$. *In particular* $(G^c)^\tau = \{(a, a) \mid a \in \tilde{H}\} \simeq \tilde{H}$ *and* $G^c/\tilde{H} \ni (a, b)\tilde{H} \mapsto ab^{-1} \in \tilde{H}$ *is an isomorphism.*

Notice that in this case we can construct, using the strongly orthogonal roots, commuting homomorphisms $\varphi_j^{\mathbb{C}}: \mathrm{SL}(2, \mathbb{C}) \to G$ such that actually $\varphi_j^{\mathbb{C}}(\mathrm{SU}(1, 1)) \subset H$ and $X^0 = \frac{1}{2} \sum_j \varphi_j^{\mathbb{C}}(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$. Using this homomorphism instead of φ_j we get

LEMMA 5.20. *Let*

$$w = \prod_j \varphi_j^{\mathbb{C}}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

Then $\mathrm{Ad}(w)(X^0) = -X^0$.

Proof. This follows again by simple $\mathfrak{sl}(2, \mathbb{R})$-calculation as $X^0 = \frac{1}{2} \sum_{j=1}^r H_j$. ∎

For $\mathrm{Re}(v)$ "big," we can construct an intertwining operator $A(v): \mathbf{H}(v) \to \mathbf{H}(-v)$ (cf. [28, 65]) by

$$[A(v) f](x) := \int_{\bar{N}} f(xw\bar{n})\, d\bar{n} \tag{5.2}$$

Let us show that $A(v) f \in \mathbf{H}(-v)$. For that let $x \in G$, $man \in MAN$. Then

$$\begin{aligned}
[A(v) f](xman) &= \int_{\bar{N}} f(xmanw\bar{n})\, d\bar{n} \\
&= \int_{\bar{N}} f(xw(w^{-1}mw)\, a^{-1}(w^{-1}nw)\, \bar{n})\, d\bar{n} \\
&= a^{v+\rho} \int_{\bar{N}} f(xw(a^{-1}\bar{n}a))\, d\bar{n} \\
&= a^{-(-v+\rho)} \int_{\bar{N}} f(xw\bar{n})\, d\bar{n}
\end{aligned}$$

Here the third equation follows by the facts that $w^{-1}Nw = \bar{N}$, $w^{-1}Mw = M$, and M acts unimodularly on $\bar{N}$. The last equation follows by

$$\int_{\bar{N}} f(a^{-1}\bar{n}a)\, d\bar{n} = a^{-2\rho} \int_{\bar{N}} f(\bar{n})\, d\bar{n}.$$

The intertwining property is obvious.

The map $\nu \mapsto A(\nu)$ has an analytic continuation to a meromorphic function on $\mathfrak{a}^*_{\mathbb{C}}$. Because of Lemma 5.13 we can define a new invariant bilinear form on $\mathscr{C}^\infty_c(\nu)$ by

$$\langle f \mid g \rangle := \langle f \mid A(\nu)\, g \rangle_\nu.$$

If there exists a (maximal) constant $R > 0$ such that the invariant bilinear form $\langle \cdot \mid \cdot \rangle$ is positive definite for $|\nu| < R$, we call the resulting unitary representations *the complementary series*. Otherwise we set $R = 0$. We have the following results from [47, 57, 58].

LEMMA 5.21. *For the Cayley-type symmetric spaces the constant* R *is given by*

$$\mathrm{SU}(n,n) : R = \begin{cases} n, & n \text{ odd} \\ 0, & n \text{ even}, \end{cases}$$

$$\mathrm{SO}^*(4n) : R = n$$

$$\mathrm{Sp}(n,\mathbb{R}) : R = \begin{cases} n/2, & n \text{ even} \\ 0, & n \text{ odd}, \end{cases}$$

$$\mathrm{SO}_o(n,2) : R = \begin{cases} 0, & n \equiv 0 \bmod 4 \\ 1, & n \equiv 1, 3 \bmod 4 \\ 2, & n \equiv 2 \bmod 4, \end{cases}$$

$$E_{7(-25)} : R = 3.$$

In the cases where $\langle \cdot \mid A(\nu) \cdot \rangle_\nu$ is positive definite we complete $\mathscr{C}^\infty_c(\nu)$ with respect to this new inner product, but denote the resulting space by the same symbol $\mathbf{H}(\nu)$ as before.

LEMMA 5.22. $w^{-1}\tau(\bar{N})\, w = \bar{N}$, *and* φ: $\bar{N} \ni \bar{n} \mapsto w^{-1}\tau(\bar{n})\, w \in \bar{N}$ *is unimodular.*

Proof. The first claim follows as $\mathrm{Ad}(w)$ and τ act by -1 on $\mathfrak{a}$, and thus map N onto $\bar{N}$, and $\bar{N}$ onto N. The second follows as we can realize φ^2 by conjugation by an element in $M \cap K$. ∎

LEMMA 5.23. *For* $f \in \mathbf{H}(\nu)$ *let* $J(f)(x) := f(\tau(xw))$. *Then the following properties hold*:

(1) $J(f)(x) = f(\tau(x)\, w^{-1})$.

(2) $J(f) \in \mathbf{H}(\nu)$ *and* $A(\nu)\, J = JA(\nu)$.

(3) $J\colon \mathbf{H}(\nu) \to \mathbf{H}(\nu)$ *is an unitary isomorphism.*

(4) $J^2 = \mathrm{id}$.

(5) *For* $x \in G$, *we have* $J \circ \pi(\nu)(x) = \pi(\nu)(\tau(x)) \circ J$.

Proof. (1) This follows from Lemma 5.16, as f is M-right invariant.

(2) Let $x \in G$ and $man \in P_{\mathbf{max}}$. By (1) we get

$$\begin{aligned} J(f)(xman) &= f((x)(m)\, a^{-1}\tau(n)\, w^{-1}) \\ &= f(\tau(x)\, w^{-1}(w\tau(m)\, w^{-1})\, a(w\tau(n)\, w^{-1})) \\ &= a^{-(\nu+\rho)} f(\tau(x)\, w^{-1}), \end{aligned}$$

as $\tau(M) = M$, $w^{-1}Mw = M$, and $w^{-1}Nw = \tau(N) = \bar{N}$. For $\mathrm{Re}(\nu)$ "big" we have

$$\begin{aligned} A(\nu)[Jf](x) &= \int Jf(xw\bar{n})\, d\bar{n} \\ &= \int f(\tau(xw\bar{n})\, w^{-1})\, d\bar{n} \\ &= \int f(\tau(x)\, \tau(w)\, \tau(\bar{n})\, w^{-1})\, d\bar{n} \end{aligned}$$

From Lemma 5.16 it follows easily that $\tau(w) = m_1 w$, for some $m_1 \in M$. Thus by Lemma 5.22,

$$\begin{aligned} A(\nu)[Jf](x) &= \int f(\tau(x)\, m_1 w\tau(\bar{n})\, w^{-1})\, d\bar{n} \\ &= \int f(\tau(x)\, m_1\bar{n})\, d\bar{n} \\ &= \int f(\tau(x)\, w^{-1}w\bar{n})\, d\bar{n} \\ &= J[A(\nu)\, f](x). \end{aligned}$$

The claim now follows by analytic continuation.

(3) Using that $\tau(dk) = dk$ and that K is unimodular it follows by direct calculation and (2) that $J^* = J$. That J is a unitary isomorphism follows now by (4).

(4) This follows as $\tau^2 = \mathrm{id}$ and $w^2 \in H \cap K$.

(5) Let $x, y \in G$. Then

$$\begin{aligned} J[\pi(\nu)(x) f](y) &= [\pi(\nu)(x) f](\tau(yw)) \\ &= f(x^{-1}\tau(yw)) \\ &= f(\tau(\tau(x)^{-1} yw)) \\ &= [\pi(\nu)(\tau(x))(Jf)](y), \end{aligned}$$

which is exactly what we wanted to prove. ∎

Notice that, even if the individual operators $A(\nu)$ and J do not exists, it is always possible to define the composite operator $A(\nu)\,J$ by

$$[A(\nu)\,J](f)(x) := \int_{\bar{N}} f(\tau(x)\,\bar{n})\,d\bar{n}$$

for Re λ "big" and then by analytic continuation for other parameters. By simple calculation we get

Lemma 5.24. *Assume that G/H is non-compactly causal. Then $A(\nu)\,J$ intertwines $\pi(\nu)$, and $\pi(-\nu) \circ \tau$ if $A(\nu)\,J$ has no pole at ν.*

The next theorem shows that the intertwining operator $A(\nu)\,J$ is a convolution operator whith kernel $y, x \mapsto a_{\bar{N}}(\tau(y)^{-1}\,x)^{\nu-\rho}$. The reflection positivity then reduces to the problem to determine those ν for which this kernel is positive semidefinite.

Theorem 5.25. *Let $f \in \mathscr{C}^\infty(\nu)$. Then*

$$[A(\nu)\,J](f)(\bar{n}) = \int_{\bar{N}} f(x)\,a_{\bar{N}}(\tau(\bar{n})^{-1}\,x)^{\nu-\rho}\,dx.$$

If $\mathrm{Supp}(f) \subset HP_{\max}$, *then for $h \in H$*

$$[A(\nu)\,J](f)(h) = \int_{H/H \cap K} f(x)\,a_{\bar{N}}(h^{-1}\,x)^{\nu-\rho}\,d\dot{x}.$$

Proof. We may assume that ν is big enough such that the integral defining $A(\nu)$ converges. The general statement follows then by analytic continuation. We have

$$[A(\nu)\, J]\, f(\bar{n}) = \int_{\bar{N}} Jf(\bar{n}wx)\, dx$$

$$= \int f(\tau(\bar{n})\, w^{-1}\tau(x)\, w)\, dx$$

$$= \int f(\tau(\bar{n})\, x)\, dx$$

$$= \int f(\bar{n}(\tau(\bar{n})\, x))\, a_{\bar{N}}(\tau(\bar{n})\, x)^{-(\nu+\rho)}\, dx.$$

Now $a_{\bar{N}}(\tau(\bar{n})\, x) = a_{\bar{N}}(\tau(\bar{n})^{-1}\, \bar{n}(\tau(\bar{n})\, x))^{-1}$. By Lemma 5.12 we get

$$[A(\nu)\, J]\, f(\bar{n}) = \int f(\bar{n}(\tau(\bar{n})\, x))\, a_{\bar{N}}(\tau(\bar{n})^{-1}\, \bar{n}(\tau(\bar{n})\, x))^{\nu-\rho}\, a_{\bar{N}}(\tau(\bar{n})\, x)^{-2\rho}\, dx$$

$$= \int f(x)\, a_{\bar{N}}(\tau(\bar{n})^{-1}\, x)^{\nu-\rho}\, dx$$

The second statement follows in the same way. ∎

Corollary 5.26. *Let $f, g \in \mathscr{C}^\infty(\nu)$. Then*

$$\langle f \mid g \rangle_J = \int_{\bar{N}} \int_{\bar{N}} \overline{f(x)}\, g(y)\, a_{\bar{N}}(\tau(x)^{-1}\, y)^{\nu-\rho}\, dx\, dy.$$

If f and g both have support in $HP_{\max}$, then

$$\langle f \mid g \rangle_J = \int_{H/H \cap K} \int_{H/H \cap K} \overline{f(h)}\, g(k)\, a_{\bar{N}}(h^{-1}k)^{\nu-\rho}\, dh\, dk.$$

In Theorem 5.33 below, we use this for describing the representations for which the corresponding J sesquilinear form $\langle \cdot \mid \cdot \rangle_J$ is positive semidefinite on the space of functions supported on $HP_{\max}$.

Assume that G/H is non-compactly causal. Let $\mathscr{C}_c^\infty(\Omega)$ be the space of $\mathscr{C}^\infty$-functions on $\bar{N}$ with compact support in Ω. We view this as the subspace in $\mathscr{C}^\infty(\nu)$ consisting of functions f, such that $\mathrm{Supp}(f) \subset HP_{\max}$ and $\mathrm{Supp}(f \mid \bar{N})$ is compact. Then $\langle f \mid g \rangle_J$ is defined for every $f, g \in \mathscr{C}_c^\infty(\Omega)$. In particular we can form the form $\langle \cdot \mid \cdot \rangle_J$ in all cases.

Lemma 5.27. *Suppose that G/H is non-compactly causal. Let $s \in S$ and $f \in \mathscr{C}_c^\infty(\Omega)$. Then $\pi(\nu)(s)\, f \in \mathscr{C}_c^\infty(\Omega)$, i.e., $\mathscr{C}_c^\infty(\Omega)$ is S-invariant.*

Proof. Let $f \in \mathscr{C}_c^\infty(\Omega)$ and $s \in S$. Then $\pi(\nu)(s)\, f(x) = f(s^{-1}x) \neq 0$ only if $s^{-1}x \in \mathrm{Supp}(f) \subset HP_{\max}$. Thus $\mathrm{Supp}(\pi(\nu)(s)\, f) \subset s\,\mathrm{Supp}(f) \subset sHP_{\max} \subset HP_{\max}$. ∎

Let $(\pi, \mathbf{H})$ be an admissible representation of G^c and let $\mathbf{H}_{K^c}$ be the space of K^c-finite elements in $\mathbf{H}$. For $\delta \in \hat{K}^c$ let $\mathbf{H}(\delta)$ be the subspace of K^c-finite vectors of type δ, i.e.,

$$\mathbf{H}(\delta) = \bigcup_{T \in \mathrm{Hom}_{K^c}(\mathbf{H}_\delta, \mathbf{H})} T(\mathbf{H}_\delta),$$

where $\mathbf{H}_\delta$ is the representation space of δ.

DEFINITION 5.28. $(\pi, \mathbf{H})$ is called a *highest-weight representation* of G^c (with respect to Δ^+) if there exists a $\delta \in \hat{K}^c$ such that

(1) $d\pi(\mathfrak{n}_{\mathbb{C}})\, \mathbf{H}(\delta) = 0$,

(2) $d\pi(U(\bar{\mathfrak{n}}))\, \mathbf{H}(\delta) = \mathbf{H}_{K^c}$. Notice that the multiplicity of δ *in π is one if π is irreducible. We call* δ for the minimal K^c-type of π.

Assume that G/H non-compactly causal. By the theorem of Moore (cf. [12]) we know that the roots in Δ_+ restricted to the span of $H_1, ..., H_r$, are given by $\pm\frac{1}{2}(\gamma_i + \gamma_j)$, $1 \leqslant i \leqslant j \leqslant r$ and possibly $\frac{1}{2}\gamma_j$. The root spaces for γ_j are all one-dimensional and the root spaces $\mathfrak{g}_{\pm 1/2(\gamma_i + \gamma_j)}$, $1 \leqslant i < j \leqslant r$, have all the common dimension d.

THEOREM 5.29 (Vergne–Rossi, Wallach). *Assume that G/H is non-compactly causal and that G^c is simple. Let $\lambda_0 \in \mathfrak{a}^*$ be such that $\langle \lambda_0, H_r \rangle = 1$. Let $\gamma = \langle \lambda_0, 2X^0 \rangle$ and let*

$$L_{\mathrm{pos}} := -\frac{\gamma(r-1)\, d}{2}.$$

Then the following hold:

(1) *For $\nu - \rho < L_{\mathrm{pos}}$ there exists a irreducible unitary highest weight representation $(\rho_\nu, \mathbf{K}_\nu)$ of G^c with one-dimensional minimal K^c-type $\nu - \rho$.*

(2) *If G/H is of Cayley-type, then $\gamma = r$. Furthermore $\nu \leqslant L_{\mathrm{pos}}$ if and only if $\nu \leqslant r$.*

Proof. (1) By [62, pp. 41–42] (see also [64]) $(\rho_\nu, \mathbf{K}_\lambda)$ exists if $\langle \nu - \rho, H_r \rangle \leqslant -(r-1)\, d/2$. But $\nu - \rho = \langle \nu - \rho, H_r \rangle\, \lambda_0$. Hence $\langle \nu - \rho, 2X^0 \rangle = \langle \nu - \rho, H_r \rangle \langle \lambda_0, 2X^0 \rangle = \gamma \langle \nu - \rho, H_r \rangle$,

(2) If G/H is of Cayley type then $2X^0 = \sum_{j=1}^r H_j$ and $\gamma_j = \gamma_r - \sum n_\alpha \alpha$, $\alpha \in \Delta_0^+$, $n_\alpha \geqslant 0$.

Thus $\langle \nu-\rho, X^0\rangle = r\langle \nu-\rho, H_r\rangle$. We also have (cf. [45])

$$\rho = \frac{1}{2}\left(1+\frac{(r-1)\,d}{2}\right)(\gamma_1 + \cdots + \gamma_r).$$

From this the theorem follows. ∎

Let us state this more explicitely for the Cayley-type spaces to compare the existence of $(\rho_\nu, \mathbf{K}_\nu)$ to the existence of the complementary series, cf. Lemma 5.21:

LEMMA 5.30. *For the Cayley-type symmetric spaces the highest weight representation* $(\rho_\nu, \mathbf{K}_\nu)$ *exists for* ν *in the following half-line*:

$$\mathrm{SU}(n,n) : \nu \leqslant n.$$

$$\mathrm{SO}^*(4n) : \nu \leqslant 2n$$

$$\mathrm{Sp}(n,\mathbb{R}) : \nu \leqslant n$$

$$\mathrm{SO}_o(n,2) : \nu \leqslant 2$$

$$E_{7(-25)} : \nu \leqslant 3.$$

In particular we have that $(\rho_\nu, \mathbf{K}_\nu)$ *is defined for* $\nu \in [-R, R]$.

Remark 5.31. Let us remind the reader that we have only described here the continuous part of the unitary spectrum. There are also finitely many discrete points, the socalled *Wallach set*, giving rise to unitary highest weight representations.

Let us still assume that G^c is simple. Let $(\rho_\nu, \mathbf{K}_\nu)$ be as above. Let $u \in \mathbf{K}_\lambda(\lambda-\rho)$, $\|u\| = 1$. Let $H^c = (G^c)^\tau$. Then H^c is connected [38]. Let $\tilde{H}$ be the universal covering of H^c and H_o. We notice that

$$H^c/H^c \cap K^c = H/H \cap K.$$

Denote the restriction of ρ_ν to H^c by $\rho_{\nu,H}$. We can then lift $\rho_{\nu,H}$ to a representation of $\tilde{H}$ also denoted by $\rho_{\nu,H}$. We let $C = C_{\min}$ be the minimal H-invariant cone in $\mathfrak{q}$ generated by X^0. We denote by $\tilde{C} = \tilde{C}_{\min}$ the minimal G^c-invariant cone in $i\mathfrak{g}^c$. Then $\tilde{C} \cap \mathfrak{q} = \mathrm{pr}_\mathfrak{q}(\tilde{C}) = C$, where $\mathrm{pr}_\mathfrak{q}\colon \mathfrak{g} \to \mathfrak{q}$ denotes the orthogonal projection (cf. [16, 39]). As $L_{\mathrm{pos}} \leqslant 0$ it follows that ρ_λ extends to a holomorphic representation of the universal semigroup $\Gamma(G^c, \tilde{C})$ corresponding to G^c and $\tilde{C}$, (cf. [15]). Let G_1^c be the analytic subgroup of $G_\mathbb{C}$ corresponding to the Lie algebra $\mathfrak{g}^c$. Let H_1 be the analytic subgroup of G_1^c corresponding to $\mathfrak{h}$. Then—as we are assuming that $G \subset G_\mathbb{C}$—we have $H_1 = H_o$. Let $\kappa: G^c \to G_1^c$ be the canonical projection and let $Z_H = \kappa^{-1}(Z_{G_1^c} \cap H_o)$. Then ρ_ν is trivial on Z_H as $\nu - \rho$ is trivial on $\exp([\mathfrak{k}^c, \mathfrak{k}^c]) \supset H^c \cap K^c$. Thus ρ_ν factors to G^c/Z_H, and to $\Gamma(G^c, \tilde{C})/Z_H$. Notice that $(G^c/Z_H)_o^\tau$ is isomorphic to H_o. Therefore we can view H_o as subgroup of G^c/Z_H, and $S_o(C) = H_o \exp C$ as a subsemigroup of $\Gamma(G^c, \tilde{C})/Z_H$. In particular $\tau_\nu(s)$ is defined for $s \in S_o(C)$. This allows us to write $\rho_\nu(h)$ or $\rho_{\nu, H}(h)$ for $h \in H_o$. As $\mathfrak{n}_{\mathbb{C}} = \mathfrak{p}^+$ and $\mathfrak{p}^- = \bar{\mathfrak{n}}_{\mathbb{C}} = \tau(\mathfrak{n}_{\mathbb{C}})$ it follows, using Lemma 5.8, that

$$a_{\bar{N}}(h)^{\nu - \rho} = \langle u \mid \rho_{\nu, H}(h)\, u \rangle.$$

In particular we get that $(h, k) \mapsto a_{\bar{N}}(h^{-1}k)^{\nu - \rho}$ is positive semidefinite if $\nu - \rho \leqslant L_{\text{pos}}$.

Let us now concider the case $G = H_{\mathbb{C}}$ and $G^c = \tilde{H} \times \tilde{H}$. Denote the constant L_{pos} for $\tilde{H}$ by S_{pos} and denote, for $\mu \leqslant S_{\text{pos}}$, the representation with lowest $\tilde{H} \cap \tilde{K}$-type μ by (τ_μ, L_μ). Let $\bar{\tau}_\mu$ be the conjugate representation. Recall that we view $\tilde{H}$ as a subset of G^c by the diagonal embedding

$$\tilde{H} \ni h \mapsto (h, h) \in \Delta(G^c) := \{(x, x) \in G^c \mid x \in \tilde{H}\}.$$

The center of $\mathfrak{k}^c$ is two dimensional (over $\mathbb{R}$) and generated by $i(X^0, X^0)$ and $i(X^0, -X^0)$. We choose $Z^0 = i(X^0, -X^0)$. Then $\mathfrak{p}^+ = \mathfrak{n} \times \bar{\mathfrak{n}}$. Let u again be a lowest weight vector of norm one. Denote the corresponding vector in the conjugate Hilbert space by $\bar{u}$. Then for $h \in \tilde{H}$:

$$\begin{aligned} \langle u \otimes \bar{u} \mid \tau_\mu \otimes \bar{\tau}_\mu(h, h)\, u \otimes \bar{u} \rangle &= \langle u \mid \tau_\lambda(h)\, u \rangle \overline{\langle u \mid \tau_\lambda(h)\, u \rangle} \\ &= |\langle u \mid \tau_\lambda(h)\, u \rangle|^2 \\ &= a_{\bar{N}}(h)^{2\mu} \end{aligned}$$

Thus we define in this case $L_{\text{pos}} := 2S_{\text{pos}}$. As before we notice that $\tau_\nu \otimes \bar{\tau}_\nu(h, h)\, u \otimes \bar{u}$ is well defined on H. We now have:

Lemma 5.32. *Assume that G/H is non-compactly causal. For $\nu - \rho \leqslant L_{\text{pos}}$ there exists an unitary irreducible highest weight representation $(\rho_\nu, \mathbf{K}_\nu)$ of G^c and a lowest K^c-type vector u of norm one such that for every $h \in H$*

$$a_{\bar{N}}(h)^{\nu - \rho} = \langle u \mid \tau_\nu(h)\, u \rangle.$$

Hence the kernel

$$(H \times H) \ni (h, k) \mapsto a_{\bar{N}}(h)^{\nu - \rho} \in \mathbb{R}$$

is positive semidefinite. In particular $\langle \cdot \mid \cdot \rangle_J$ is positive semidefinite on $\mathscr{C}_c^\infty(\Omega)$ for $\lambda - \rho \leqslant L_{\text{pos}}$.

The Basic Lemma and the Lüscher–Mack Theorem, together with the above, now imply the following theorem.

THEOREM 5.33 (Reflection Symmetry for Complementary Series). *Assume that G/H is non-compactly causal and such that there exists a $w \in K$ such that $\mathrm{Ad}(w)|_{\mathfrak{a}} = -1$. Let π_ν be a complementary series such that $\nu \leqslant L_{\mathrm{pos}}$. Let C be the minimal H-invariant cone in $\mathfrak{q}$ such that $S(C)$ is contained in the contraction semigroup of $HP_{\max}$ in $G/P_{\max}$. Let Ω be the bounded realization of $H/H \cap K$ in $\bar{\mathfrak{n}}$. Let $J(f)(x) := f(\tau(x)\, w^{-1})$. Let $\mathbf{K}_0$ be the closure of $\mathscr{C}_c^\infty(\Omega)$ in $\mathbf{H}_\nu$. Then the following hold:*

(1) *$(G, \tau, \pi_\nu, C, J, \mathbf{K}_0)$ satisfies the positivity conditions* (PR1)–(PR2).

(2) *π_ν defines a contractive representation $\tilde{\pi}_\nu$ of $S(C)$ on $\mathbf{K}$ such that $\tilde{\pi}_\nu(\gamma)^* = \tilde{\pi}_\nu(\tau(\gamma)^{-1})$.*

(3) *There exists a unitary representation $\tilde{\pi}_\nu^c$ of G^c such that*

(i) $d\tilde{\pi}_\nu^c(X) = d\tilde{\pi}_\nu(X) \quad \forall X \in \mathfrak{h}$.

(ii) $d\tilde{\pi}_\lambda^c(iY) = i\, d\tilde{\pi}_\lambda(Y) \quad \forall Y \in C$.

We remark that this Theorem includes the results of R. Schrader for $\mathrm{SL}(2n, \mathbb{C})/\mathrm{SU}(n, n)$, [59].

We will now generalize this to all non-compactly causal symmetric spaces and all ν such that $\nu - \rho \leqslant L_{\mathrm{pos}}$. We will also show that actually $\tilde{\pi}_\nu^c \simeq \rho_\nu$, where ρ_ν is the irreducible unitary highest weight representation of G^c such that

$$a(h)^{\nu-\rho} = \langle u \mid \rho_\nu(h)\, u \rangle$$

as before. From now on we assume that $\nu - \rho \leqslant L_{\mathrm{pos}}$. Let $\mathbf{K}_0$ be the completion of $\mathscr{C}_c^\infty(\Omega)$ in the norm $\langle \cdot \mid A(\nu)\, J(\cdot) \rangle$. Let $\mathbf{N}$ be the space of vectors of zero lenght and let $\mathbf{K}$ be the completion of $\mathbf{K}_0/\mathbf{N}$ in the induced norm. First of all we have to show that $\pi_\nu(\gamma)$ passes to a continuous operator $\tilde{\pi}_\nu(\gamma)$ on $\mathbf{K}$ such that $\tilde{\pi}_\nu(\gamma)^* = \tilde{\pi}_\nu(\tau(\gamma)^{-1})$. For that we recall that

$$H/H \cap K = H_o/H_o \cap K = \Omega \tag{5.3}$$

so we my replace the integration over H in $\langle f \mid A(\nu)\, Jf \rangle$ with integration over H_o. For $f \in \mathscr{C}_c^\infty(\Omega)$ define

$$\rho_\nu(f)\, u := \int_{H_o} f(h \cdot 0)\, \rho_\nu(h)\, u\, dh. \tag{5.4}$$

LEMMA 5.34. *Assume that $\nu - \rho \leqslant L_{\mathrm{pos}}$. Let ρ_ν, $\mathbf{K}_\nu$ and u be as specified in Lemma 5.32, and let $f, g \in \mathscr{C}_c^\infty(\Omega)$ and $s \in S(C)$. Then the following hold:*

(1) $\langle f \mid [A(\nu)\, J](g)\rangle_\nu = \langle \rho_\nu(f)\, u \mid \rho_\nu(g)\, u\rangle$;

(2) $\rho_\nu(\pi_\nu(s)\, f)\, u = \rho_\nu(s)\, \rho_\nu(f)\, u$;

(3) $\pi_\nu(s)$ *passes to a contractive operator* $\tilde{\pi}_\nu(s)$ *on* **K** *such that* $\tilde{\pi}_\nu(s)^* = \tilde{\pi}_\nu(\tau(s)^{-1})$.

Proof. (1) Let f and g be as above. Then

$$\begin{aligned}\langle f \mid [A(\nu)\, J](g)\rangle &= \int_{H_o/H_o \cap K} \int_{H_o/H_o \cap K} \overline{f(h)}\, g(k)\, a_{\bar{N}}(h^{-1}k)^{\nu-\rho}\, dh\, dk \\ &= \int_{H_o/H_o \cap K} \int_{H_o/H_o \cap K} \overline{f(h)}\, g(k) \langle u \mid \rho_\nu(h^{-1}k)\, u\rangle\, dh\, dk \\ &= \int_{H_o/H_o \cap K} \int_{H_o/H_o \cap K} \overline{f(h)}\, g(k) \langle \rho_\nu(h)\, u \mid \rho_\nu(k)\, u\rangle\, dh\, dk \\ &= \langle \rho_\nu(f)\, u \mid \rho_\nu(g)\, u\rangle.\end{aligned}$$

This proves (1).

(2) This follows from Lemma 5.12, (7) and the following calculation:

$$\begin{aligned}\rho_\nu(\pi_\nu(s)\, f)\, u &= \int f(s^{-1}h)\, \rho_\nu(h)\, u\, dh \\ &= \int f(h(s^{-1}h))\, a_H(s^{-1}h)^{-(\nu+\rho)}\, \rho_\nu(h)\, u\, dh \\ &= \int f(h(s^{-1}h))\, a_H(sh(s^{-1}))^{\nu-\rho}\, \rho_\nu(h)\, a_H(s^{-1}h)^{-2\rho}\, u\, dh \\ &= \int f(h)\, \rho_\nu(sh)\, u\, dh \\ &= \rho_\nu(s)\, \rho_\nu(f)\, u,\end{aligned}$$

where we have used that

$$\rho_\nu(sh)\, u = a_H(sh)^{\nu-\rho}\, \rho_\nu(h(sh))\, u.$$

(3) By (1) and (2) we get

$$\begin{aligned}\|\pi_\nu(s)\, f\|_J^2 = \|\rho_\nu(s)\, \rho_\nu(f)\, u\|^2 &\leqslant \|\rho_n(f)\, u\|^2 \\ &= \langle f \mid [A(\nu)\, J]\, f\rangle_\nu\ (= \|f\|_J^2).\end{aligned}$$

Thus $\pi_\nu(s)$ passes to a contractive operator on **K**. That $\tilde{\pi}_\nu(s)^* = \tilde{\pi}_\nu(\tau(s)^{-1})$ follows from Lemma 5.24. ∎

THEOREM 5.35 (Identification Theorem). *Assume that G/H is non-compactly causal and that $\nu - \rho \leqslant L_{\mathrm{pos}}$. Let ρ_ν, $\mathbf{K}_\nu$ and $u \in \mathbf{K}_\nu$ be as in Lemma 5.32. Then the following hold*:

(1) *There exists a continuous contractive representation $\tilde{\pi}_\nu$ of $S_o(C)$ on $\mathbf{K}$ such that*

$$\tilde{\pi}_\nu(s)^* = \tilde{\pi}_\nu(\tau(s)^{-1}), \qquad \forall s \in S_o(C).$$

(2) *There exists a unitary representation $\tilde{\pi}^c_\nu$ of G^c such that*

(i) $d\tilde{\pi}^c_\nu(X) = d\tilde{\pi}_\nu(X) \quad \forall X \in \mathfrak{h}$.

(ii) $d\tilde{\pi}^c_\nu(iY) = i\, d\tilde{\pi}_\nu(Y) \quad \forall Y \in C$.

(3) *The map*

$$\mathscr{C}^\infty_c(\Omega) \ni f \mapsto \rho_\nu(f)\, u \in \mathbf{K}_\nu$$

extends to an isometry $\mathbf{K} \simeq \mathbf{K}_\nu$ intertwining $\tilde{\pi}^c_\nu$ and ρ_ν. In particular $\tilde{\pi}^c_\nu$ is irreducible and isomorphic to ρ_ν.

Proof. (1) follows from Lemma 5.34 as obviously $\tilde{\pi}_\nu(sr) = \tilde{\pi}_\nu(s)\, \tilde{\pi}_\nu(r)$.

(2) This follows now from the Theorem of Lüscher–Mack.

(3) By Lemma 5.34 we know that $f \mapsto \rho_\nu(f)\, u$ defines an isometric $S_o(C)$-intertwining operator. Let $f \in \mathscr{C}^\infty_c(\Omega)$. Differentiation and the fact that τ_ν is holomorphic gives

(i) $\rho_\nu(d\tilde{\pi}^c_\nu(X)\, f)\, u = d\rho_\nu(X)\, \rho_\nu(f)\, u, \quad \forall X \in \mathfrak{h}$.

(ii) $\rho_\nu(i\, d\tilde{\pi}^c_\nu(Y)\, f)\, u = i\, d\rho_\nu(Y)\, \rho_\nu(f)\, u, \quad \forall Y \in C$.

But those are exactly the relations that define $\tilde{\pi}^c_\nu$. The fact that $\mathfrak{h} \oplus iC$ generates $\mathfrak{g}^c$ implies that $f \mapsto \rho_\nu(f)\, u$ induces an $\mathfrak{g}^c$-intertwining operator intertwining $\tilde{\pi}^c_\nu$ and ρ_ν. As both are also representations of G^c, it follows that this is an isometric G^c-map. In particular if this is not the zero-map it has to be an isomorphism as ρ_ν is irreducible. Choose a sequence $\{f_j\}$ in $\mathscr{C}^\infty_c(\Omega)$ approximating the Delta function. The usual calculation shows that

$$\rho_\nu(f_j)\, u \to u.$$

Hence there is a j such that $\rho_\nu(f_j)\, u \neq 0$. This proves the theorem. ∎

Remark 5.36. The above theorem realizes the highest weight representation ρ_ν on a function space on $H/H \cap K$. The construction is in some sense inverse to the construction in [44]. The highest weight representation ρ_ν can be realized in a Hilbert space $\mathcal{O}$ of holomorphic functions on $\Omega_{\mathbb{C}}$. The restriction of a holomorphic function to Ω is injective by Lemma 5.7.

Multiplying by a suitable character induces then a injective H-intertwining operator into $\mathbf{L}^2(H/H\cap K)$, at least for ν big enough. We refer to [44] for further details.

We will now explain another view of the above results using local representations instead on the theorem of Lüscher–Mack. This will use the realization as functions on $\Omega\subset\bar{\mathfrak{n}}$ and in particular explain the kernel

$$(X, Y)\mapsto K_\nu(X, Y) := a_{\bar{N}}(\tau(\exp X)^{-1}\exp Y)^{\nu-\rho} \qquad X, Y\in\Omega.$$

For this we recall some results from [4], in particular Theorem 5.1 and Theorem 7.1: We assume that $\nu-\rho\leqslant L_{\mathrm{pos}}$. Let ρ_ν, $\mathbf{K}_\nu$ and u be as before. Then

$$\begin{aligned} K_\nu(X, Y) &= \langle u \mid \rho_\nu(\exp(-\tau(X))\exp(Y))\, u\rangle \\ &= \langle \rho_\nu(\exp(X))\, u \mid \rho_\nu(\exp(Y))\, u\rangle \end{aligned}$$

because of Lemma 5.8.

LEMMA 5.37. *Let the notation be as above. Then the following hold.*

(1) *The map*

$$\Omega\ni X\mapsto q_X u := \rho_\nu(\exp X)\, u\in\mathbf{K}_\nu$$

extends to a holomorphic map on $\Omega_{\mathbb{C}}$ *given by*

$$q_X u = \sum_{n=0}^{\infty} \frac{d\rho_\nu(X)^n\, u}{n!}.$$

(2) *The function* $\langle q_X u \mid q_Y u\rangle$ *is an extension of* $K_\nu(X, Y)$ *to* $\Omega_{\mathbb{C}}\times\Omega_{\mathbb{C}}$, *holomorphic in the second variable, and antiholomorphic in the first variable. We will denote this extension also by* $K_\nu(X, Y)$.

(3) *The function* $K_\nu(X, Y)$ *is positive definite.*

Proof. See [4]. ∎

Let $U\subset\Omega$ be open. We identify $\mathscr{C}_c^\infty(U)$ with the space of elements in $f\in\mathscr{C}^\infty(\Omega)$ such that $f|_{\bar{N}}\circ\zeta^{-1}\in\mathscr{C}_c^\infty(U)$. For $R>0$, let

$$B_R := \mathrm{Ad}(H\cap K)\left\{\sum_{j=1}^{r} t_j X_{-j}\;\middle|\; -R<t_j<R\right\}.$$

Then B_R is open in $\bar{\mathfrak{n}}$. Let $\beta\colon \mathbf{K}_0\to(\mathbf{K}_0/\mathbf{N})^\sim=\mathbf{K}$ be the canonical map. Then β is a contraction ($\|\beta(f)\|_J^2=\langle f\mid Jf\rangle\leqslant\|f\|^2$). For $U\subset\Omega$ open, let $\mathbf{K}(U) := \beta(\mathscr{C}_c^\infty(U))$.

THEOREM 5.38. *Let $U \subset \Omega$ be open. Then $\mathbf{K}(U)$ is dense in $\mathbf{K}$.*

Proof. Let $x \in U$. Then we can choose $h \in H$ such that $hx = 0$. As $\mathscr{C}_c^\infty(U) = h \cdot \mathscr{C}_c^\infty(h \cdot U)$ and H acts unitarily, it follows that we can assume that $0 \in U$. Let $R > 0$ be such that $B_R \subset U$. Then $\mathscr{C}_c^\infty(B_R) \subset \mathscr{C}_c^\infty(U)$. Hence we can assume that $U = B_R$. Let $g \in \mathscr{C}_c^\infty(U)^\perp$ and let $f \in \mathscr{C}_c^\infty(\Omega)$. We want to show that $\langle g \mid f \rangle_J = 0$. Choose $0 < L < 1$ such that $\operatorname{Supp}(f) \subset B_L$. For $t \in \mathbb{R}$ and $a_t = \exp(2tX^0)$ we have $a_t \cdot B_L = B_{e^{-2t}L}$. Thus $\operatorname{Supp}(\pi(v)(a_t) f) \subset B_{e^{-2t}L}$. Choose $0 < s_0$ such that $e^{-2t}L < R$ for every $t > s_0$. Then $\pi(v)(a_t)(f) \in \mathscr{C}_c^\infty(U)$ for every $t > s_0$. It follows that for $t > s_0$:

$$\begin{aligned} 0 &= \langle g \mid \pi(v)(a_t) f \rangle_J \\ &= \iint \overline{g(x)} [\pi(v)(a_t) f](y)\, K_\lambda(x, y)\, dx\, dy \\ &= e^{(\lambda+1)t} \iint \overline{g(x)}\, f(e^{2t}y)\, K_\lambda(x, y)\, dx\, dy \\ &= e^{(\lambda-1)t} \iint \overline{g(x)}\, f(y)\, K_\lambda(x, e^{-2t}y)\, dx\, dy. \end{aligned}$$

By Lemma 5.37 we know that $z \mapsto K_\lambda(x, zy)$ is holomorphic on $D = \{z \mid |z| < 1\}$. As g and f both have compact support it follows that

$$F(z) := \iint \overline{g(x)}\, f(y)\, K_\lambda(x, zy)\, dx\, dy$$

is holomorphic on D. But $F(z) = 0$ for $0 < z < e^{-2s_0}$. Thus $F(z) = 0$ for every z. In particular

$$\langle g \mid \pi(v)(a_t) f \rangle_J = 0$$

for every $t > 0$. By continuity $\langle g \mid f \rangle_J = 0$. Thus $g = 0$. ∎

Let us recall some basic facts from [22]. Let ρ be a local homomorphism of a neighborhood U of e in G into the space of linear operators on the Hilbert space $\mathbf{H}$ such that $\rho(g)$ is densely defined for $g \in U$. Furthermore $\rho|_{U \cap H}$ extends to a strongly continuous representation of H in $\mathbf{H}$. ρ is called a *local representation* if there exists a dense subspace $\mathbf{D} \subset \mathbf{H}$ such that the following hold:

(LR1) $\forall g \in U$, $\mathbf{D} \subset \mathbf{D}(\rho(g))$, where $\mathbf{D}(\rho(g))$ is the domain of definition for $\rho(g)$.

(LR2) If $g_1, g_2, g_1g_2 \in U$ and $u \in \mathbf{D}$ then $\rho(g_2)\,u \in \mathbf{D}(\rho(g_1))$ and

$$\rho(g_1)[\rho(g_2)\,u] = \rho(g_1 g_2)\,u.$$

(LR3) Let $Y \in \mathfrak{h}$ such that $\exp tY \in U$ for $0 \leqslant t \leqslant 1$. Then for every $u \in \mathbf{D}$

$$\lim_{t \to 0} \rho(\exp tY)\,u = u.$$

(LR4) $\rho(Y)\,\mathbf{D} \subset \mathbf{D}$ for every $Y \in \mathfrak{h}$.

(LR5) $\forall u \in \mathbf{D}\ \exists V_u$ an open 1-neighborhood in H such that $UV_u \subset U^2$ and $\rho(h)\,u \in \mathbf{D}$ for every $h \in V_u$.

(LR6) For every $Y \in \mathfrak{q}$ and every $u \in \mathbf{D}$ the function

$$h \mapsto \rho(\exp(\mathrm{Ad}(h)\ Y)\,u$$

is locally integrable on $\{h \in H \mid \exp(\mathrm{Ad}(h)\ Y) \in U\}$.

Ref. [22] now states that every local representation extends to a unitary representation of G^c. We now want to use Theorem 5.38 to construct a local representation of G. For that let $0 < R < 1$ and let $\mathbf{D} = \mathbf{K}(B_R(0))$. Let V be a symmetric open neighborhood of $1 \in G$ such that $V \cdot B_R(0) \subset \Omega$. Let U_1 be a convex symmetric neighborhood of 0 in $\mathfrak{g}$ such that with $U :=$ $\exp U_1$ we have $U^2 \subset V$. If $g \in U$ then obviously (LR1)–(LR3) are satisfied. (LR4) is satisfied as differentiation does not increase support. (LR6) is also clear as $u = \beta(f)$ with $f \in \mathscr{C}_c^\infty(U)$ and hence $\|\rho_c(\exp \mathrm{Ad}(h)\ Y)\,u\|$ is continuous as a function of h.

(LR5) Let $u = \beta(f) \in \mathbf{K}(B_R(0))$. Let $L = \mathrm{Supp}(f) \subset B_R(0)$. Let V_u be such that $V_u^{-1} = V_u$, $V_u L \subset B_R(0)$, and $V_u \subset U$. Then $UV_u \subset U^2$ and $\tilde{\pi}(v)(h)\,u = \beta(\pi(v)(h)\,f)$ is defined and in D. This now implies that $\tilde{\pi}$ restricted to U is a local representation. Hence the existence of $\tilde{\pi}^c$ follows from [22]. We notice that this construction of $\tilde{\pi}^c$ does not use the full semigroup S but only H and $\exp \mathbb{R}_+ X^o$. ∎

6. THE DIAGONAL CASE $\pi \oplus (\pi \circ \tau)$

A special case of the setup in Definition 2.1 above arises as follows: Let the group G, and $\tau \in \mathrm{Aut}_2(G)$ be as described there. Let $\mathbf{H}_\pm$ be two given complex Hilbert spaces, and $\pi_\pm \in \mathrm{Rep}(G, \mathbf{H}_\pm)$ a pair of unitary representations. Suppose $T: \mathbf{H}_- \to \mathbf{H}_+$ is a unitary operator such that $T\pi_- =$ $(\pi_+ \circ \tau)\,T$, or equivalently,

$$T\pi_-(g)\,f_- = \pi_+(\tau(g))\,Tf_- \tag{6.1}$$

for all $g \in G$, and all $f_- \in \mathbf{H}_-$. Form the direct sum $\mathbf{H} := \mathbf{H}_+ \oplus \mathbf{H}_-$ with inner product

$$\langle f_+ \oplus f_- \mid f'_+ \oplus f'_- \rangle := \langle f_+ \mid f'_+ \rangle_+ + \langle f_- \mid f'_- \rangle_- \tag{6.2}$$

where the $\pm$ subscripts are put in to refer to the respective Hilbert spaces $\mathbf{H}_\pm$, and we may form $\pi := \pi_+ \oplus \pi_-$ as a unitary representation on $\mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-$ by

$$\pi(g)(f_+ \oplus f_-) = \pi_+(g)\, f_+ \oplus \pi_-(g)\, f_-, \qquad g \in G, \quad f_\pm \in \mathbf{H}_\pm.$$

Setting

$$J := \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}, \tag{6.3}$$

i.e., $J(f_+ \oplus f_-) = (Tf_-) \oplus (T^* f_+)$, it is then clear that properties (1)–(2) from Definition 2.1 will be satisfied for the pair (J, π). Formula (6.1) may be recovered by writing out the relation

$$J\pi = (\pi \circ \tau)\, J \tag{6.4}$$

in matrix form, specifically

$$\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \begin{pmatrix} \pi_+(g) & 0 \\ 0 & \pi_-(g) \end{pmatrix} = \begin{pmatrix} \pi_+(\tau(g)) & 0 \\ 0 & \pi_-(\tau(g)) \end{pmatrix} \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}.$$

If, conversely, (6.4) is assumed for some unitary period-2 operator J on $\mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-$, and, if the two representations π_+ and π_- are *disjoint*, in the sense that no irreducible in one occurs in the other (or, equivalently, there is no nonzero intertwiner between them), then, in fact, (6.1) will follow from (6.4). The diagonal terms in (6.3) will be zero if (6.4) hold. This last implication is an application of Schur's lemma.

LEMMA 6.1. *Let* $0 \neq \mathbf{K}_0$ *be a closed linear subspace of* $\mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-$ *satisfying the positivity condition* (PR3) *in Definition* 2.2, *i.e.,*

$$\langle v \mid Jv \rangle \geqslant 0, \qquad \forall v \in \mathbf{K}_0 \tag{6.5}$$

where

$$J = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \tag{6.6}$$

is given from a fixed unitary isomorphism $T\colon \mathbf{H}_- \to \mathbf{H}_+$ *as in* (6.1). *For* $v = f_+ \oplus f_- \in \mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-$, *set* $P_+ v := f_+$. *The closure of the subspace* $P_+ \mathbf{K}_0$ *in* $\mathbf{H}_+$ *will be denoted* $\overline{P_+ \mathbf{K}_0}$. *Then the subspace*

$$\mathbf{G} = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \mathbf{K}_0 \,\middle|\, f_- \in T^*(\overline{P_+ \mathbf{K}_0}) \right\}$$

is the graph of a closed linear operator M *with domain*

$$\mathbf{D} = \left\{ f_+ \in \mathbf{H}_+ \,\middle|\, \exists f_- \in T^*(\overline{P_+ \mathbf{K}_0}) \text{ s.t. } \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \mathbf{K}_0 \right\}; \tag{6.7}$$

and, moreover, the product operator $L := TM$ *is dissipative on this domain, i.e.,*

$$\langle Lf_+ \mid f_+ \rangle_+ + \langle f_+ \mid Lf_+ \rangle_+ \geqslant 0 \tag{6.8}$$

holds for all $f_+ \in \mathbf{D}$.

Proof. The details will only be sketched here, but the reader is referred to [61] and [21] for definitions and background literature. An important argument in the proof is the verification that, if a column vector of the form $\binom{0}{f_-}$ is in $\mathbf{G}$, then f_- must necessarily be zero in $\mathbf{H}_-$. But using positivity, we have

$$|\langle u \mid Jv \rangle|^2 \leqslant \langle u \mid Ju \rangle \langle v \mid Jv \rangle, \qquad \forall u, v \in \mathbf{K}_0. \tag{6.9}$$

Using this on the vectors $u = \binom{0}{f_-}$ and $v = \binom{k_+}{k_-} \in \mathbf{K}_0$, we get

$$\left\langle \begin{pmatrix} 0 \\ f_- \end{pmatrix} \middle| \begin{pmatrix} Tk_- \\ T^*k_+ \end{pmatrix} \right\rangle = \langle f_- \mid T^*k_+ \rangle = 0, \qquad \forall k_+ = P_+ v.$$

But, since f_- is also in $T^*(\overline{P_+ \mathbf{K}_0})$, we conclude that $f_- = 0$, proving that $\mathbf{G}$ is the graph of an operator M as specified. The dissipativity of the operator $L = TM$ is just a restatement of (PR3). ∎

The above result involves only the operator-theoretic information implied by the data in Definition 2.2, and, in the next lemma, we introduce the representations:

LEMMA 6.2. *Let the representations* $\pi_\pm$ *and the intertwiner* T *be given as specified before the statement of Lemma* 6.1. *Let* $H = G^\tau$; *and suppose we have a cone* $C \subset \mathfrak{q}$ *as specified in* (PR2′). *Assume* (PR1), (PR2′) *and* (PR3′) *and assume further that*

(PR4) $\mathbf{D}$ *is dense in* $\mathbf{H}_+$, *and*

(PR5) *the commutant of* $\{\pi_+(h) \mid h \in H\}$ *is abelian.*

Then it follows that the operator $L = TM$ *is normal.*

Proof. Since T is a unitary isomorphism $\mathbf{H}_- \to \mathbf{H}_+$ we may make an identification and reduce the proof to the case where $\mathbf{H}_+ = \mathbf{H}_-$ and T is the identity operator. We then have

$$\pi_- = T^{-1}(\pi_+ \circ \tau)\, T = \pi_+ \circ \tau;$$

and if $h \in H$, then

$$\pi_-(h) = \pi_+(\tau(h)) = \pi_+(h);$$

while, if $\tau(g) = g^{-1}$, then

$$\pi_-(g) = \pi_+(\tau(g)) = \pi_+(g^{-1}).$$

Using only the H part from (PR2′), we conclude that $\mathbf{K}_0$ is invariant under $\pi_+ \oplus \pi_+(H)$. If the projection $P_{\mathbf{K}_0}$ of $\mathbf{H}_+ \oplus \mathbf{H}_+$ onto $\mathbf{K}_0$ is written as an operator matrix

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

with entries representing operators in $\mathbf{H}_+$, and satisfying

$$P_{11}^* = P_{11},$$

$$P_{22}^* = P_{22},$$

$$P_{12}^* = P_{21},$$

$$P_{ij} = P_{i1} P_{1j} + P_{i2} P_{2j},$$

then it follows that

$$P_{ij}\pi_+(h) = \pi_+(h)\, P_{ij} \qquad \forall i, j = 1, 2, \quad \forall h \in H, \tag{6.10}$$

which puts each of the four operators P_{ij} in the commutant $\pi_+(H)'$ from (PR5). Using (PR4), we then conclude that L is a dissipative operator with $\mathbf{D}$ as dense domain, and that $\mathbf{K}_0$ is the graph of this operator. Using (PR5), and a theorem of Stone [61], we finally conclude that L is a normal operator, i.e., it can be represented as a multiplication operator with dense domain $\mathbf{D}$ in $\mathbf{H}_+$. ∎

We shall consider two cases below (the Heisenberg group, and the $(ax+b)$-group) when conditions (PR4)–(PR5) can be verified from the context of the representations. Suppose G has two abelian subgroups H, N, and the second N also a normal subgroup, such that $G = HN$ is a product representation in the sense of Mackey [33]. Define $\tau \in \mathrm{Aut}_2(G)$ by setting

$$\tau(h) = h, \quad \forall h \in H, \quad \text{and} \quad \tau(n) = n^{-1}, \quad \forall n \in N. \tag{6.11}$$

The Heisenberg group is a copy of $\mathbb{R}^3$ represented as matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

or equivalently vectors $(a, b, c) \in \mathbb{R}^3$. Setting $H = \{(a, 0, 0) \mid a \in \mathbb{R}\}$ and

$$N = \{(0, b, c) \mid b, c \in \mathbb{R}\}, \tag{6.12}$$

we arrive at one example.

The $(ax+b)$-group is a copy of $\mathbb{R}^2$ represented as matrices $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, $a = e^s$, $b \in \mathbb{R}$, $s \in \mathbb{R}$. Here we may take $H = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{R}_+\}$ and

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{R} \right\}, \tag{6.13}$$

and we have a second example of the Mackey factorization. Generally, if $G = HN$ is specified as described, we use the representations of G which are induced from one-dimensional representations of N. If G is the Heisenberg group, or the $(ax+b)$-group, we get all the infinite-dimensional irreducible representations of G by this induction (up to unitary equivalence, of course). For the Heisenberg group, the representations are indexed by $\hbar \in \mathbb{R} \setminus \{0\}$, $\hbar$ denoting Planck's constant. The representation $\pi_\hbar$ may be given in $\mathbf{H} = \mathbf{L}^2(\mathbb{R})$ by

$$\pi_\hbar(a, b, c)\, f(x) = e^{i\hbar(c+bx)} f(x+a), \qquad \forall f \in \mathbf{L}^2(\mathbb{R}), \quad (a, b, c) \in G. \tag{6.14}$$

The Stone-von Neumann uniqueness theorem asserts that every unitary representation π of G satisfying

$$\pi(0, 0, c) = e^{i\hbar c} I_{\mathbf{H}(\pi)} \qquad (\hbar \neq 0)$$

is unitarily equivalent to a direct sum of copies of the representation $\pi_\hbar$ in (6.14).

The $(ax+b)$-group (in the form $\{(\begin{smallmatrix} e^s & b \\ 0 & 1 \end{smallmatrix}) \mid s, b \in \mathbb{R}\}$) has only two inequivalent infinite-dimensional unitary irreducible representations, and they may also be given in the same Hilbert space $\mathbf{L}^2(\mathbb{R})$ by

$$\pi_{\pm}\begin{pmatrix} e^s & b \\ 0 & 1 \end{pmatrix} f(x) = e^{\pm i e^x b} f(x+s), \qquad \forall f \in \mathbf{L}^2(\mathbb{R}). \tag{6.15}$$

There are many references for these standard facts from representation theory; see, e.g., [24].

Lemma 6.3. *Let the group G have the form $G = HN$ for locally compact abelian subgroups H, N, with N normal, and $H \cap N = \{e\}$. Let χ be a one-dimensional unitary representation of N, and let $\pi = \mathrm{ind}_N^G(\chi)$ be the corresponding induced representation. Then the commutant of $\{\pi(H) \mid h \in H\}$ is an abelian von Neumann algebra: in other words, condition* (PR5) *in Lemma* 6.2 *is satisfied.*

Proof. See, e.g., [24]. ∎

In the rest of the present section, we will treat the case of the *Heisenberg group*, and the $(ax+b)$-*group* will be the subject of the next section.

For both groups we get pairs of unitary representations $\pi_{\pm}$ arising from some $\tau \in \mathrm{Aut}_2(G)$ and described as in (6.4) above. But when the two representations π_+ and $\pi_- = \pi_+ \circ \tau$ are irreducible and disjoint, we will show that there are no spaces $\mathbf{K}_0$ satisfying (PR1), (PR2′), and (PR3) such that $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^{\sim}$ is nontrivial. Here (PR2) is replaced by

(PR2′) C is a nontrivial cone in $\mathfrak{q}$.

Since for both groups, and common to all the representations, we noted that the Hilbert space $\mathbf{H}_+$ may be taken as $\mathbf{L}^2(\mathbb{R})$, we can have J from (6.6) represented in the form $J = (\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix})$. Then the J-inner product on $\mathbf{H}_+ \oplus \mathbf{H}_- = \mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R}) \simeq \mathbf{L}^2(\mathbb{R}, \mathbb{C}^2)$ may be brought into the form

$$\left\langle \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \middle| \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \right\rangle_J = 2\,\mathrm{Re}\langle f_+ \mid f_- \rangle = 2 \int_{-\infty}^{\infty} \mathrm{Re}(\overline{f_+(x)}\, f_-(x))\, dx. \tag{6.16}$$

For the two examples, we introduce

$$N_+ = \{(0, b, c) \mid b, c \in \mathbb{R}_+\}$$

where N is defined in (6.12), but N_+ is not H-invariant. Alternatively, set

$$N_+ = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{R}_+ \right\}$$

for the alternative case where N is defined from (6.13), and note that this N_+ is H-invariant. In fact there are the following four invariant cones in $\mathfrak{q}$:

$$C_1^+ = \{(0, 0, t) \mid t \geqslant 0\}$$

$$C_1^- = \{(0, 0, t) \mid t \leqslant 0\}$$

$$C_2^+ = \{(0, x, y) \mid x \in \mathbb{R}, y \geqslant 0\}$$

$$C_2^- = \{(0, x, y) \mid x \in \mathbb{R}, y \leqslant 0\}$$

Let π denote one of the representations of $G = HN$ from the discussion above (see formulas (6.14) and (6.15)) and let $\mathscr{D}$ be a closed subspace of $\mathbf{H} = \mathbf{L}^2(\mathbb{R})$ which is assumed invariant under $\pi(HN_+)$. Then it follows that the two spaces

$$\mathscr{D}_\infty := \bigvee \{\pi(n)\, \mathscr{D} \mid n \in N\} \tag{6.17}$$

$$\mathscr{D}_{-\infty} := \bigwedge \{\pi(n)\, \mathscr{D} \mid n \in N\} \tag{6.18}$$

are invariant under $\pi(G)$, where the symbols $\bigvee$ and $\bigwedge$ are used for the usual lattice operations on closed subspaces in $\mathbf{H}$. We leave the easy verification to the reader, but the issue is resumed in the next section. If P_∞, resp., $P_{-\infty}$, denotes the projection of $\mathbf{H}$ onto $\mathscr{D}_\infty$, resp., $\mathscr{D}_{-\infty}$, then we assert that both projections $P_{\pm\infty}$ are in the commutant of $\pi(G)$. So, if π is irreducible, then each P_∞, or $P_{-\infty}$, must be 0 or I. Since $\mathscr{D}_{-\infty} \subset \mathscr{D} \subset \mathscr{D}_\infty$ from the assumption, it follows that $P_\infty = I$ if $\mathscr{D} \neq \{0\}$.

Lemma 6.4. *Let G be the Heisenberg group, and let the notation be as described above. Let π_+ be one of the representations $\pi_\hbar$ and let π_- be the corresponding $\pi_{-\hbar}$ representation. Let $0 \neq \mathbf{K}_0 \subset \mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R})$ be a closed subspace which is invariant under $(\pi_+ \oplus \pi_-)(HN_+)$. Then it follows that there are only the following possibilities for $\overline{P_+\mathbf{K}_0}$: $\{0\}$, $\mathbf{L}^2(\mathbb{R})$, or $A\mathscr{H}_+$ where $\mathscr{H}_+$ denotes the Hardy space in $\mathbf{L}^2(\mathbb{R})$ consisting of functions f with Fourier transform $\hat{f}$ supported in the half-line $[0, \infty)$, and where $A \in \mathbf{L}^\infty(\mathbb{R})$ is such that $|A(x)| = 1$* a.e. $x \in \mathbb{R}$. *For the space $\overline{P_-\mathbf{K}_0}$, there are the possibilities: $\{0\}$, $\mathbf{L}^2(\mathbb{R})$, and $A\mathscr{H}_-$, where A is a (possibly different) unitary $\mathbf{L}^\infty$-function, and $\mathscr{H}_-$ denotes the negative Hardy space.*

Proof. Immediate from the discussion, and the Beurling–Lax theorem classifying the closed subspaces in $\mathbf{L}^2(\mathbb{R})$ which are invariant under the multiplication operators, $f(x) \mapsto e^{iax} f(x)$, $a \in \mathbb{R}_+$. We refer to [31], or [14], for a review of the Beurling–Lax theorem. ∎

COROLLARY 6.5. *Let $\pi_\pm$ be the representations of the Heisenberg group, and suppose that the subspace* $\mathbf{K}_0$ *from Lemma* 6.4 *is chosen such that* (PR1)–(PR3) *in Definition* 2.2 *hold. Then* $(\mathbf{K}_0/\mathbf{N})^{\sim}=\{0\}$.

Proof. Suppose there are unitary functions $A_\pm\in\mathbf{L}^\infty(\mathbb{R})$ such that $\overline{P_\pm\mathbf{K}_0}=A_\pm\mathscr{H}_\pm$. Then this would violate the Schwarz-estimate (6.9), and therefore condition (PR3). Using irreducibility of $\pi_+=\pi_\hbar$ and of $\pi_-=\pi_+\circ\tau=\pi_{-\hbar}$, we may reduce to considering the cases when one of the spaces $\overline{P_\pm\mathbf{K}_0}$ is $\mathbf{L}^2(\mathbb{R})$. By Lemma 6.2, we are then back to the case when $\mathbf{K}_0$ or $\mathbf{K}_0^{-1}$ is the graph of a densely defined normal and dissipative operator L, or L^{-1}, respectively. We will consider L only. The other case goes the same way. Since

$$(\pi_+\oplus\pi_-)(0,b,0)(f_+\oplus f_-)(x)=e^{i\hbar bx}f_+(x)\oplus e^{-i\hbar bx}f_-(x) \tag{6.19}$$

it follows that L must anti-commute with the multiplication operator ix on $\mathbf{L}^2(\mathbb{R})$. For deriving this, we used assumption (PR3) at this point. We also showed in Lemma 6.2 that L must act as a multiplication operator on the Fourier-transform side. But the anti-commutativity is inconsistent with a known structure theorem in [52], specifically Corollary 3.3 in that paper. Hence there are unitary functions $A_\pm$ in $\mathbf{L}^\infty(\mathbb{R})$ such that $\overline{P_\pm\mathbf{K}_0}=A_\pm\mathscr{H}_\pm$. But this possibility is inconsistent with positivity in the form $\mathrm{Re}\langle f_+\mid f_-\rangle\geqslant 0$, $\forall(f_+,f_-)\in\mathbf{K}_0$ (see (6.16)) if $(\mathbf{K}_0/\mathbf{N})^{\sim}\neq\{0\}$. To see this, note that $\mathbf{K}_0$ is invariant under the unitary operators (6.19) for $b\in\mathbb{R}_+$. The argument from Lemma 6.4, now applied to $\pi_+\oplus\pi_-$, shows that the two subspaces

$$\mathbf{K}_0^\infty:=\bigvee_{b\in\mathbb{R}}(\pi_+\oplus\pi_-)(0,b,0)\,\mathbf{K}_0$$

and

$$\mathbf{K}_0^{-\infty}:=\bigwedge_{b\in\mathbb{R}}(\pi_+\oplus\pi_-)(0,b,0)\,\mathbf{K}_0$$

are both invariant under the whole group $(\pi_+\oplus\pi_-)(G)$. But the commutant of this is 2-dimensional: the only projections in the commutant are represented as one of the following,

$$\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad\begin{pmatrix}I&0\\0&0\end{pmatrix},\quad\begin{pmatrix}0&0\\0&I\end{pmatrix},\quad\text{or}\quad\begin{pmatrix}I&0\\0&I\end{pmatrix},$$

relative to the decomposition $\mathbf{L}^2(\mathbb{R})\oplus\mathbf{L}^2(\mathbb{R})$ of $\pi_+\oplus\pi_-$. The above analysis of the anti-commutator rules out the cases $\left(\begin{smallmatrix}I&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right)$, and if $(\mathbf{K}_0/\mathbf{N})^\sim \neq \{0\}$, we are left with the cases $\mathbf{K}_0^\infty = \{0\}$ and $\mathbf{K}_0^\infty = \mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R})$. Recall, generally $\mathbf{K}_0^{-\infty} \subset \mathbf{K}_0 \subset \mathbf{K}_0^\infty$, as a starting point for the analysis. A final application of the Beurling–Lax theorem (as in [31]; see also [5]) to (6.19) then shows that there must be a pair of unitary functions $A_\pm$ in $\mathbf{L}^\infty(\mathbb{R})$ such that

$$\mathbf{K}_0 = A_+ \mathscr{H}_+ \oplus A_- \mathscr{H}_- \tag{6.20}$$

where $\mathscr{H}_\pm$ are the two Hardy spaces given by having $\hat{f}$ supported in $[0, \infty)$, respectively, $(-\infty, 0]$. The argument is now completed by noting that (6.20) is inconsistent with the positivity of $\mathbf{K}_0$ in (6.5); that is, we clearly do not have $\langle \binom{A_+h_+}{A_-h_-} \mid J\binom{A_+h_+}{A_-h_-} \rangle = 2\,\mathrm{Re}\langle A_+h_+ \mid A_-h_- \rangle$ semi-definite, for all $h_+ \in \mathscr{H}_+$ and all $h_- \in \mathscr{H}_-$. This concludes the proof of the corollary. ∎

Remark 6.6. At the end of the above proof of Corollary 6.5, we arrived at the conclusion (6.20) for the subspace $\mathbf{K}_0$ under consideration. Motivated by this, we define a closed subspace $\mathbf{K}_0$ in a direct sum Hilbert space $\mathbf{H}_+ \oplus \mathbf{H}_-$ to be *uncorrelated* if there are closed subspaces $\mathbf{D}_\pm \subset \mathbf{H}_\pm$ in the respective summands such that

$$\mathbf{K}_0 = \mathbf{D}_+ \oplus \mathbf{D}_- \tag{6.21}$$

Contained in the corollary is then the assertion that every semigroup-invariant $\mathbf{K}_0$ in $\mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R})$ is uncorrelated, where the semigroup here is the subsemigroup S in the Heisenberg group G given by

$$S = \{(a, b, c) \mid b \in \mathbb{R}_+,\, a, c \in \mathbb{R}\}, \tag{6.22}$$

and the parameterization is the one from (6.12). We also had the representation π in the form $\pi_+ \oplus \pi_-$ where the respective summand representations $\pi_\pm$ of G are given by (6.14) relative to a pair $(h, -h)$, $h \in \mathbb{R}\backslash\{0\}$ some fixed value of Planck's constant. In particular, it is assumed in Corollary 6.5 that each representation $\pi_\pm$ is *irreducible*. But for proving that some given semigroup-invariant $\mathbf{K}_0$ must be uncorrelated, this last condition can be relaxed considerably; and this turns out to be relevant for applications to Lax-Phillips scattering theory for the wave equation with obstacle scattering [31]. In that context, the spaces $\mathbf{D}_\pm$ will be outgoing, respectively, incoming subspaces; and the wave equation translates backwards, respectively forwards, according to the unitary one-parameter groups $\pi_-(0, b, 0)$, respectively, $\pi_+(0, b, 0)$, with $b \in \mathbb{R}$ representing the time-variable t for the unitary time-evolution one-parameter group which solves the wave equation under consideration. The unitary-equivalence identity (6.4) stated before Lemma 6.1 then implies equivalence of the wave-dynamics before, and after, the obstacle scattering.

Before stating our next result, we call attention to the $(2n+1)$-dimensional Heisenberg group G_n in the form $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, in parameter form: $a, b \in \mathbb{R}^n$, $c \in \mathbb{R}$, and product rule

$$(a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c' + a \cdot b')$$

where $a + a' = (a_1 + a'_1, ..., a_n + a'_n)$ and $a \cdot b' = \sum_{j=1}^{n} a_j b'_j$. For every (fixed) $b \in \mathbb{R}^n \backslash \{0\}$, we then have a subsemigroup

$$S(b) = \{(a, \beta b, c) \mid \beta \in \mathbb{R}_+, a \in \mathbb{R}^n, c \in \mathbb{R}\}; \tag{6.23}$$

and we show in the next result that it is enough to have invariance under such a semigroup in G_n, just for a single direction, defined from some fixed $b \in \mathbb{R}^n \backslash \{0\}$.

THEOREM 6.7. *Let $\pi_\pm$ be unitary representations of the Heisenberg group G on respective Hilbert spaces $\mathbf{H}_\pm$, and let $T: \mathbf{H}_- \to \mathbf{H}_+$ be a unitary isomorphism which intertwines π_- and $\pi_+ \circ \tau$ as in (6.1) where*

$$\tau(a, b, c) = (a, -b, -c), \qquad \forall (a, b, c) \in G \simeq \mathbb{R}^{2n+1}. \tag{6.24}$$

Suppose there is $\hbar \in \mathbb{R} \backslash \{0\}$ such that

$$\pi_+(0, 0, c) = e^{i\hbar c} I_{\mathbf{H}_+}. \tag{6.25}$$

If $\mathbf{K}_0 \subset \mathbf{H}_+ \oplus \mathbf{H}_-$ is a closed subspace which is invariant under

$$\{(\pi_+ \oplus \pi_-)(a, \beta b, c) \mid a \in \mathbb{R}^n, \beta \in \mathbb{R}_+, c \in \mathbb{R}\}$$

from (6.23), $b \in \mathbb{R}^n \backslash \{0\}$, then we conclude that $\mathbf{K}_0$ must automatically be uncorrelated.

Proof. The group-law in the Heisenberg group yields the following commutator rule:

$$(a, 0, 0)(0, b, 0)(-a, 0, 0) = (0, b, a \cdot b)$$

for all $a, b \in \mathbb{R}^n$. We now apply $\pi = \pi_+ \oplus \pi_-$ to this, and evaluate on a general vector $f_+ \oplus f_- \in \mathbf{K}_0 \subset \mathbf{H}_+ \oplus \mathbf{H}_-$: abbreviating $\pi(a)$ for $\pi(a, 0, 0)$, and $\pi(b)$ for $\pi(0, b, 0)$, we get

$$\pi(a)\, \pi(\beta b)\, \pi(-a)(f_+ \oplus f_-) = e^{i\hbar\beta a \cdot b} \pi_+(\beta b)\, f_+ \oplus e^{-i\hbar\beta a \cdot b} \pi_-(\beta b)\, f_- \in \mathbf{K}_0$$

valid for all $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}_+$. Note, in (6.25), we are assuming that π_+ takes on some specific value $e^{i\hbar c}$ on the one-dimensional center. Since π_-

is unitarily equivalent to $\pi_+ \circ \tau$ by assumption (see (6.25)), we conclude that

$$\pi_-(0, 0, c) = e^{-i\hbar c} I_{\mathbf{H}_-}, \qquad \forall c \in \mathbb{R}.$$

The argument really only needs that the two representations $\pi_\pm$ define *different* characters on the center. (Clearly $\hbar \neq -\hbar$ since $\hbar \neq 0$.) Multiplying through first with $e^{-i\hbar\beta a \cdot b}$, and integrating the resulting term

$$\pi_+(\beta b) f_+ \oplus e^{-i2\hbar\beta a \cdot b} \pi_-(\beta b) f_- \in \mathbf{K}_0$$

in the a-variable, we get $\pi_+(\beta b) f_+ \oplus 0 \in \mathbf{K}_0$. The last conclusion is just using that $\mathbf{K}_0$ is a closed subspace. But we can do the same with the term

$$e^{i2\hbar\beta a \cdot b} \pi_+(\beta b) f_+ \oplus \pi_-(\beta b) f_- \in \mathbf{K}_0,$$

and we arrive at $0 \oplus \pi_-(\beta b) f_- \in \mathbf{K}_0$. Finally letting $\beta \to 0_+$, and using strong continuity, we get $f_+ \oplus 0$ and $0 \oplus f_-$ both in $\mathbf{K}_0$. Recalling that $f_\pm$ are general vectors in $P_\pm \mathbf{K}_0$, we conclude that $P_+ \mathbf{K}_0 \oplus P_- \mathbf{K}_0 \subset \mathbf{K}_0$, and therefore $\overline{P_+ \mathbf{K}_0} \oplus \overline{P_- \mathbf{K}_0} \subset \mathbf{K}_0$. Since the converse inclusion is obvious, we arrive at (6.21) with $\mathbf{D}_\pm = \overline{P_\pm \mathbf{K}_0}$. ∎

The next result shows among other things that there are representations π of the Heisenberg group G_n (for each n) such that the reflected representation π^c of $G_n^c \simeq G_n$ (see Theorem 4.2) acts on a nonzero Hilbert space $\mathbf{H}^c = (\mathbf{K}_0/\mathbf{N})^\sim$. However, because of Lemma 4.3, $\pi^c(G_n^c)$ will automatically be an *abelian* group of operators on $\mathbf{H}^c$. To see this, note that the proof of Theorem 6.7 shows that π^c must act as the identity operator on $\mathbf{H}^c$ when restricted to the one-dimensional center in $G_n^c \simeq G_n$.

It will be convenient for us to read off this result from a more general context: We shall consider a general Lie group G, and we fix a right-invariant Haar measure on G.

Definition 6.8. A distribution F on the Lie group G will be said to be *positive definite* (PD) if

$$\int_G \int_G F(uv^{-1}) \overline{f(u)} f(v) \, du \, dv \geqslant 0 \qquad \text{(PD)}$$

for all $f \in \mathscr{C}_c^\infty(G)$; and we say that f is PD on some open subset $\Omega \subset G$ if this holds for all $f \in \mathscr{C}_c^\infty(\Omega)$. The interpretation of the expression in (PD) is in the sense of distributions. But presently measurable functions F will serve as the prime examples.

We say that the distribution is *reflection-positive* (RP) on Ω ((RP$_\Omega$) for emphasis) if, for some period-2 automorphism τ of G, we have

$$F \circ \tau = F \tag{6.26}$$

and

$$\int_G \int_G F(\tau(u)\, v^{-1})\, \overline{f(u)}\, f(v)\, du\, dv \geqslant 0 \tag{RP$_\Omega$}$$

for all $f \in \mathscr{C}_c^\infty(\Omega)$.

We say that some element x in G is (RP$_\Omega$)-*contractive* if (RP$_\Omega$) holds, and

$$0 \leqslant \int_G \int_G F(\tau(u)\, v^{-1})\, \overline{f(ux)}\, f(vx)\, du\, dv \leqslant \int_G \int_G F(\tau(u)\, v^{-1})\, \overline{f(u)}\, f(v)\, du\, dv,$$

$\forall f \in \mathscr{C}_c^\infty(\Omega)$. Note that, since

$$\int_G \int_G F(\tau(u)\, v^{-1})\, \overline{f(ux)}\, f(vx)\, du\, dv$$
$$= \int_G \int_G F(\tau(u)\, \tau(x)^{-1}\, xv^{-1})\, \overline{f(u)}\, f(v)\, du\, dv,$$

it follows that every x in $H = G^\tau$ is contractive: in fact, isometric. If instead $\tau(x) = x^{-1}$, then contractivity amounts to the estimate

$$0 \leqslant \int_G \int_G F(\tau(u)\, x^2 v^{-1})\, \overline{f(u)}\, f(v)\, du\, dv \leqslant \int_G \int_G F(\tau(u)\, v^{-1})\, \overline{f(u)}\, f(v)\, du\, dv,$$

$\forall f \in \mathscr{C}_c^\infty(\Omega)$. Using the basic Lemma one can also show that x acts by contractions.

The following result is useful, but an easy consequence of the definitions and standard techniques for positive definite distributions; see for example [24, 55].

THEOREM 6.9. *Let F be a distribution on a Lie group G with a period-2 automorphism τ, and suppose F is τ-invariant*, (PD) *holds on G, and* (RP$_\Omega$) *holds on some open, and semigroup-invariant, subset Ω in G. Then define*

$$(\pi(u)\, f)(v) := f(vu), \qquad \forall u, v \in G, \qquad \forall f \in \mathscr{C}_c^\infty(G);$$

and

$$Jf := f \circ \tau.$$

Let $\mathbf{H}(F)$ *be the Hilbert space obtained from the GNS construction, applied to* (PD), *with inner product on* $\mathscr{C}_c^\infty(G)$ *given by*

$$\langle f \mid g \rangle := \int_G \int_G F(uv^{-1})\, \overline{f(u)}\, g(v)\, du\, dv.$$

Then π *extends to a unitary representation of* G *on* $\mathbf{H}(F)$, *and* J *to a unitary operator, such that*

$$J\pi = (\pi \circ \tau)\, J.$$

If (RP_Ω) *further holds, as described, then* π *induces* (*via Theorem* 4.2) *a unitary representation* π^c *of* G^c *acting on the new Hilbert space* $\mathbf{H}^c$ *obtained from completing in the new inner product from* (RP_Ω), *and dividing out with the corresponding kernel.*

The simplest example of a function F on the Heisenberg group G_n satisfying (PD), but not (RP_Ω), for nontrivial Ω's, may be obtained from the Green's function for the sub-Laplacian on G_n; see [60, p. 599] for details.

If complex coordinates are introduced in G_n, the formula for F takes the following simple form: let $z \in \mathbb{C}^n$, $c \in \mathbb{R}$, and define

$$F(z, c) = \frac{1}{(|z|^4 + c^2)^n}.$$

Then we adapt the product in G_n to the modified definition as follows:

$$(z, c) \cdot (z', c') = (z + z', c + c' + \langle z, z' \rangle) \qquad \forall z, z' \in \mathbb{C}^n, \quad \forall c, c' \in \mathbb{R},$$

where $\langle z, z' \rangle$ is the symplectic form

$$\langle z, z' \rangle := 2\, \mathrm{Im}(z \cdot \bar{z}').$$

The period-2 automorphism τ on G_n we take as

$$\tau(z, c) = (\bar{z}, -c)$$

with $\bar{z}$ denoting complex conjugation, i.e., if $z = (z_1, ..., z_n)$, then $\bar{z} := (\bar{z}_1, ..., \bar{z}_n)$.

The simplest example where both (PD) and (RP_Ω) hold on the Heisenberg group G_n is the following:

EXAMPLE 6.10. Let $\zeta=(\zeta_1,...\zeta_n)\in\mathbb{C}^n$, $\xi_j=\operatorname{Re}\zeta_j$, $\eta_j=\operatorname{Im}\zeta_j$, $j=1,..., n$. Define

$$F(z, c)=\int_{\mathbb{R}^{2n}} \frac{e^{i\,\mathrm{Re}(z\cdot\bar{\zeta})}}{\prod_{j=1}^{n}(|\zeta_j|^2+1)}\, d\xi_1\cdots d\xi_n\, d\eta_1\cdots d\eta_n.$$

Let $\Omega:=\{(z, c)\in G_n \mid z=(z_j)_{j=1}^n,\ \operatorname{Im} z_j>0\}$. Then (PD) holds on G_n, and (RP_Ω) holds, referring to this Ω. Since the expression for $F(z, c)$ factors, the problem reduces to the $(n=1)$ special case. There we have

$$F(z, c)=\int_{\mathbb{R}^2} \frac{e^{i(x\xi+y\eta)}}{\xi^2+\eta^2+1}\, d\xi\, d\eta;$$

and if $f\in\mathscr{C}_c^\infty(\Omega)$ with $\Omega=\{(z, c)\mid y>0\}$, then

$$\int_{G_1}\int_{G_1} F(\tau(u)\,v^{-1})\,\overline{f(u)}\,f(v)\,du\,dv$$

$$=\int_{\mathbb{R}^8} \frac{e^{i(x-x')\,\xi}\,e^{-i(y+y')\,\eta}}{\xi^2+\eta^2+1}\,\overline{f(x+iy, c)}$$

$$\times f(x'+iy', c')\,d\xi\,d\eta\,dx\,dy\,dc\,dx'\,dy'\,dc'.$$

Let $\tilde{f}$ denote the Fourier transform in the x-variable, keeping the last two variables (y, c) separate. Then the integral transforms as follows:

$$\int_{\mathbb{R}^5} \frac{e^{-(y+y')\sqrt{1+\xi^2}}}{\sqrt{1+\xi^2}}\,\overline{\tilde{f}(\xi, y, c)}\,\tilde{f}(\xi, y', c')\,d\xi\,dy\,dy'\,dc\,dc'.$$

Introducing the Laplace transform in the middle variable y, we then get (since f is supported in $y>0$)

$$\int_0^\infty e^{-y\sqrt{1+\xi^2}}\,\tilde{f}(\xi, y, c)\,dy=\tilde{f}_\lambda(\xi, \sqrt{1+\xi^2}, c);$$

the combined integral reduces further:

$$\int_{\mathbb{R}}\left|\int_{\mathbb{R}} \tilde{f}_\lambda(\xi, \sqrt{1+\xi^2}, c)\,dc\right|^2 \frac{d\xi}{1+\xi^2}$$

which is clearly positive; and we have demonstrated that (RP_Ω) holds. It is immediate that F is τ-invariant (see (6.26)), and also that it satisfies (PD) on G_n.

7. THE $(ax+b)$-GROUP

We showed that in general we get a unitary representation π^c of the group G^c from an old one π of G, provided π satisfies the assumptions of reflection positivity. The construction as we saw uses a certain cone C and a semigroup $H \exp C$, which are part of the axiom system. What results is a new class of unitary representations π^c satisfying a certain spectrum condition (semi-bounded spectrum).

But, for the simplest non-trivial group G, this semi-boundness turns out *not* to be satisfied in the general case. Nonetheless, we still have a reflection construction getting us from unitary representations π of the $(ax+b)$-group, such that $\pi \circ \tau \simeq \pi$ (unitary equivalence), to associated unitary representations π^c of the same group. The (up to conjugation) unique non-trivial period-2 automorphism τ of G, where G is the $(ax+b)$-group, is given by

$$\tau(a, b) = (a, -b).$$

Recall that the G may be identified with the matrix-group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a > 0,\ b \in \mathbb{R} \right\}$$

and (a, b) corresponds to the matrix $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$. In this realization the Lie algebra of G has the basis

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have $\exp(tX) = (e^t, 0)$ and $\exp(sY) = (1, s)$. Hence $\tau(X) = X$ and $\tau(Y) = -Y$. Thus $\mathfrak{h} = \mathbb{R}X$ and $\mathfrak{q} = \mathbb{R}Y$. We notice the commutator relation $[X, Y] = Y$. The possible H-invariant cones in $\mathfrak{q}$ are $\pm\{tY \mid t \geqslant 0\}$. It is known from Mackey's theory that G has two inequivalent, unitary, irreducible, infinite-dimensional representations $\pi_\pm$, and it is immediate that we have the unitary equivalence (see details below):

$$\pi_+ \circ \tau \simeq \pi_-. \tag{7.1}$$

Hence, if we set $\pi := \pi_+ \oplus \pi_-$, then $\pi \circ \tau \simeq \pi$, so we have the setup for the general theory. We show that π may be realized on $\mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R}) \simeq \mathbf{L}^2(\mathbb{R}, \mathbb{C}^2)$, and we find and classify the invariant positive subspaces $\mathbf{K}_0 \subset \mathbf{L}^2(\mathbb{R}, \mathbb{C}^2)$. To understand the interesting cases for the $(ax+b)$-group G, we need to relax the invariance condition: We shall *not* assume invariance of $\mathbf{K}_0$ under the semigroup $\{\pi(1, b) \mid b \geqslant 0\}$, but only under the infinitesimal unbounded generator $\pi(Y)$. With this, we still get the correspondence $\pi \mapsto \pi^c_{\mathbf{K}_0}$ as described above.

We use the above notation. We know from Mackey's theory [33] that there are two inequivalent irreducible infinite-dimensional representations of G, and we shall need them in the following alternative formulations: Let $\mathscr{L}_\pm$ denote the respective Hilbert space $\mathbf{L}^2(\mathbb{R}_\pm)$ with the multiplicative invariant measure $d\mu_\pm = dp/|p|$, $p \in \mathbb{R}_\pm$. Then the formula

$$f \mapsto e^{ipb} f(pa) \tag{7.2}$$

for functions f on $\mathbb{R}$ restricts to two unitary irreducible representations, denoted by $\pi_\pm$ of G on the respective spaces $\mathscr{L}_\pm$. Let $Q(f)(p) := f(-p)$ denote the canonical mapping from $\mathscr{L}_+$ to $\mathscr{L}_-$, or equivalently from $\mathscr{L}_-$ to $\mathscr{L}_+$. Then we have for $g \in G$ (cf. (7.1)):

$$Q\pi_+(g) = \pi_-(\tau(g))\, Q \tag{7.3}$$

For the representation $\pi := \pi_+ \oplus \pi_-$ on $\mathbf{H} := \mathscr{L}_+ \oplus \mathscr{L}_-$ we therefore have

$$J\pi(g) = \pi(\tau(g))\, J, \qquad g \in G, \tag{7.4}$$

where J is the unitary involutive operator on $\mathbf{H}$ given by

$$J = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}. \tag{7.5}$$

Instead of the above p-realization of π we will mainly use the following x-formalism. The map $t \mapsto \pm e^t$ defines an isomorphism $L_\pm : \mathscr{L}_\pm \to \mathbf{L}^2(\mathbb{R})$, where we use the (additive) Lebesgue measure dx on $\mathbb{R}$. For $g = (e^s, b) \in G$ and $f \in \mathbf{L}^2(\mathbb{R})$, set

$$(\pi_\pm(g)\, f)(x) := e^{\pm i e^x b} f(x+s), \qquad x \in \mathbb{R}. \tag{7.6}$$

A simple calculation shows that $L_\pm$ intertwines the old and new construction of $\pi_\pm$, excusing our abuse of notation. In this realization Q becomes simply the identity operator $Q(f)(x) = f(x)$. The involution J: $\mathbf{L}^2(\mathbb{R}, \mathbb{C}^2)$ is now simply given by

$$J(f_0, f_1) = (f_1, f_0)$$

or $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

In this formulation the operator

$$L := \pi_\pm(\Delta_H - \Delta_q) = \pi_\pm(X^2 - Y^2) \tag{7.7}$$

takes the form

$$L = \left(\frac{d}{dx}\right)^2 + e^{2x}, \tag{7.8}$$

but it is on $\mathbf{L}^2(\mathbb{R})$ and $-\infty < x < \infty$. This operator is known to have defect indices $(1, 0)$ [20, 37], which means that it cannot be extended to a self-adjoint operator on $\mathbf{L}^2(\mathbb{R})$. Using a theorem from [20, 56] we can see this by comparing the quantum mechanical problem for a particle governed by $-L$ as a Schrödinger operator (i.e., a strongly repulsive force) with the corresponding classical one governed (on each energy surface) by

$$E_{\text{kin}} + E_{\text{pot}} = \left(\frac{dx}{dt}\right)^2 - e^{2x} = E.$$

The escape time for this particle to $x = \pm\infty$ *is*

$$t_{\pm} = \int_{\text{finite}}^{\pm\infty} \frac{dx}{\sqrt{E + e^{2x}}}, \tag{7.9}$$

i.e., t_∞ is finite, and $t_{-\infty} = \infty$. We elaborate on this point below. The non-zero defect vector for the quantum mechanical problem corresponds to a boundary condition at $x = \infty$ since this is the singularity which is reached in finite time.

The fact from [20] we use for the defect index assertion is this: The Schrödinger operator $H = -(d/dx)^2 + V(x)$ for a single particle has non-zero defect solutions $f_{\pm} \in \mathbf{L}^2(\mathbb{R})$ to $H^* f_{\pm} = \pm i f_{\pm}$ iff there are solutions $t \mapsto x(t)$ to the corresponding classical problem

$$E = \left(\frac{dx(t)}{dt}\right)^2 + V(x(t))$$

with finite travel-time to $x = +\infty$, respectively, $x = -\infty$. The respective (possibly infinite) travel-times are

$$t_{\pm\infty} = \int_{\text{finite}}^{\pm\infty} \frac{dx}{\sqrt{E - V(x)}}.$$

The correspondence principle states that one finite travel-time to $+\infty$ (say) yields a dimension in the associated defect space, and similarly for the other travel-time to $-\infty$.

In the x-formalism, (7.3) from above then simplifies to the following identity for operators on the *same* Hilbert space $\mathbf{L}^2(\mathbb{R})$ (carrying the two inequivalent representations π_+ and π_-):

$$\pi_+(g) = \pi_-(\tau(g)), \qquad g \in G. \tag{7.10}$$

We realize the representation $\pi = \pi_+ \oplus \pi_-$ in the Hilbert space $\mathbf{H} = \mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R}) = \mathbf{L}^2(X_2)$ where $X_2 = 0 \times \mathbb{R} \cup 1 \times \mathbb{R}$. We may represent J by an automorphism θ: $X_2 \to X_2$ (as illustrated in Proposition 3.3):

$$\theta(0, x) := (1, x) \qquad \text{and} \qquad \theta(1, y) = (0, y), \qquad x, y \in \mathbb{R},$$

and

$$J(f)(\omega) = f(\theta(\omega)), \qquad \omega \in X_2.$$

Notice that the subset

$$X_2^\theta = \{\omega \in X_2 \mid \theta(\omega) = \omega\}$$

is empty. Define for $f \in \mathbf{L}^2(X_2)$, $f_k(x) = f(k, x)$, $k = 0, 1$, $x \in \mathbb{R}$. We have for $g = (e^s, b) \in G$:

$$(\pi(g)\, f)_0\,(x) = e^{ibe^x} f_0(x+s) = (\bar{\pi}_+(g)\, f_0)(x)$$

and

$$(\pi(g)\, f)_1\,(x) = e^{-ibe^x} f_1(x+s) = (\bar{\pi}_-(g)\, f_1)(x).$$

Proposition 7.1. *Let* $\pi = \pi_+ \oplus \pi_-$ *be the representation from* (7.1)–(7.4) *above of the* $(ax+b)$*-group* G. *Then the only choices of reflections* $\mathbf{K}_0$ *as in Remark* 2.3 *for the sub-semigroup* $S = \{(a, b) \in G \mid b > 0\}$ *will have* $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^\sim$ *equal to* 0.

Proof. Let $\mathbf{K}_0$ be as specified in Remark 2.3 relative to the semigroup S, and let $P_{\mathbf{K}_0}$ be the representation of the corresponding orthogonal projection operator as given in (7.12)–(7.13) in terms of the measurable field $\mathbb{R} \ni \xi \mapsto Q(\xi)$. Specifically, the space $\mathbf{K}_0 \subset \mathbf{H}$ (with the positivity and invariance properties from Section 2) will then be translation invariant, i.e., invariant under the translation group

$$\begin{pmatrix} f_0(x) \\ f_1(x) \end{pmatrix} \mapsto \begin{pmatrix} f_0(x+s) \\ f_1(x+s) \end{pmatrix}, \qquad x, y, s \in \mathbb{R}. \tag{7.11}$$

Hence the projection in $\mathbf{L}^2(\mathbb{R}, \mathbb{C}^2)$ onto $\mathbf{K}_0$, denoted by $P_{\mathbf{K}_0}$, may be represented as a multiplication operator in the Fourier transform space

$$f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \qquad \hat{f}(\xi) = \begin{pmatrix} f_0(\xi) \\ f_1(\xi) \end{pmatrix}, \qquad \xi \in \mathbb{R},$$

where as usual

$$\hat{f}_k(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f_k(x)\, dx, \qquad k = 0, 1.$$

LEMMA 7.2. *Let Q be the projection in $\mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R})$ onto a translation-invariant J-positive subspace. Then Q is represented by a measurable field of 2×2 complex matrices $\mathbb{R} \ni \xi \mapsto (Q_{ij}(\xi))_{ij=1}^2$ such that $|Q_{12}(\xi)|^2 = Q_{11}(\xi)\, Q_{22}(\xi)$ a.e. on $\mathbb{R}$, and $Q_{12}(\xi) + Q_{21}(\xi) \geqslant 0$ a.e.; and conversely.*

Proof. Since all the operators commuting with the translation group (7.12) are known (see, e.g., [31]), there is a measurable field of projections $Q(\xi)\colon \mathbb{C}^2 \to \mathbb{C}^2$, i.e., $Q(\xi)^2 = Q(\xi) = Q(\xi)^*$, such that (7.12)–(7.13) hold:

$$(P_{\mathbf{K}_0} f)^\wedge\,(\xi) = Q(\xi)\, \hat{f}(\xi). \tag{7.12}$$

With $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\colon \mathbb{C}^2 \to \mathbb{C}^2$ as before, we have the basic positivity:

$$Q(\xi)\, JQ(\xi) \geqslant 0, \qquad \xi \in \mathbb{R}. \tag{7.13}$$

Hence

$$\det(Q(\xi)\, JQ(\xi)) \geqslant 0 \tag{a}$$

and

$$\operatorname{Tr}(Q(\xi)\, JQ(\xi)) \geqslant 0 \tag{b}$$

Since $\det(QJQ) = -\det(Q) = -\det(Q^2) = -(\det Q^2) \leqslant 0$, it follows from (a) that $\det Q(\xi) = 0$, and, from (a)–(b), that $Q(\xi)$ is for each ξ a projection into a subspace in $\mathbb{C}^2$ of dimension 0 or 1. Write $Q = (Q_{ij})$, with $Q_{ij}\colon \mathbb{R} \to \mathbb{C}$ measurable. Then $Q = Q^*$ gives, for $\xi \in \mathbb{R}$,

$$Q_{11}(\xi),\ Q_{22}(\xi) \in \mathbb{R} \qquad \text{and} \qquad Q_{21}(\xi) = \overline{Q_{12}(\xi)}.$$

The relation $Q^2(\xi) = Q(\xi)$ implies

$$Q_{11}(\xi)^2 + |Q_{12}(\xi)|^2 = Q_{11}(\xi),$$
$$Q_{22}(\xi)^2 + |Q_{12}(\xi)|^2 = Q_{22}(\xi),$$

and

$$(Q_{11}(\xi)+Q_{22}(\xi))\,Q_{12}(\xi)=Q_{12}(\xi).$$

In particular

$$0\leqslant Q_{11}(\xi),\,Q_{22}(\xi)\leqslant 1$$

and

$$|Q_{12}(\xi)|^2=Q_{11}(\xi)(1-Q_{11}(\xi))=Q_{22}(\xi)(1-Q_{22}(\xi)).$$

From det $Q(\xi)=0$, we finally get

$$|Q_{12}(\xi)|^2=Q_{11}(\xi)\,Q_{22}(\xi).$$

COROLLARY 7.3. *These relations imply the following for the matrix Q*:

(1) *If $Q_{12}(\xi)=0$ then we have the three possibilities*:

$$Q(\xi)=0,$$

$$Q(\xi)=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad \textit{and}$$

$$Q(\xi)=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

In all those cases, we have $Q(\xi)\,JQ(\xi)=0$.

(2) *If $Q_{12}(\xi)\neq 0$, then $0<Q_{22}(\xi)=1-Q_{11}(\xi)<1$. Let $\mu(\xi)=Q_{12}(\xi)/Q_{11}(\xi)$. Then by $\mathrm{Tr}(Q(\xi)\,JQ(\xi))\geqslant 0$ we have $\mathrm{Re}\,\mu(\xi)\geqslant 0$ and*

$$Q(\xi)=\frac{1}{1+|\mu(\xi)|^2}\begin{pmatrix}1&\mu(\xi)\\\overline{\mu(\xi)}&|\mu(\xi)|^2\end{pmatrix}. \tag{7.14}$$

With $\lambda=\bar{\mu}$ we get that the image of $Q(\xi)$ is given by

$$\left\{u(\xi)\begin{pmatrix}1\\\lambda(\xi)\end{pmatrix}\;\middle|\;u(\xi)\in\mathbb{C}\right\}.$$

Specifying to our situation, $f=\binom{f_0}{f_1}\in\mathbf{K}_0$ if and only if

$$\hat{f}_1(\xi)=\lambda(\xi)\,\hat{f}_0(\xi). \tag{7.15}$$

Since $Q(\xi)$ is a measurable field of projections, the function $\mathbb{R} \ni \xi \mapsto \lambda(\xi)$ must be measurable, but it may be unbounded. This also means that $P_{\mathbf{K}_0}$ is the projection onto the graph of the operator $T_0\colon f_0 \mapsto f_1$ where f_0 and f_1 are related as in (7.15), and the Fourier transform $\hat{\cdot}$ is in the $\mathbf{L}^2$-sense.

Proof of Proposition 7.1 (*continued*). We first assume that $\mathbf{K}_0$ arises this way as the graph of an operator T_0 as described. This assumption will then be "removed" later.

The assumed invariance of $\mathbf{K}_0$ under $\pi = \pi_+ \oplus \pi_-$ takes the form

$$\begin{pmatrix} \pi_+(b) & 0 \\ 0 & \pi_-(b) \end{pmatrix} \mathbf{K}_0 \subset \mathbf{K}_0, \qquad \forall b > 0. \tag{7.16}$$

Let $\mathscr{D} \subset \mathbf{L}^2(\mathbb{R})$ consist of the $\mathbf{L}^2(\mathbb{R})$-closure of the functions f_0 such that

$$\left\langle \begin{pmatrix} \hat{f}_0 \\ \lambda \hat{f}_0 \end{pmatrix} \middle| J \begin{pmatrix} \hat{f}_0 \\ \lambda \hat{f}_0 \end{pmatrix} \right\rangle = 0.$$

This may also be expressed in the form

$$\int_{-\infty}^{\infty} \operatorname{Re} \lambda(\xi) \, |\hat{f}_0(\xi)|^2 \, d\xi = 0. \tag{7.17}$$

It follows from (7.16) and Lemma 3.1 (the Basic Lemma) that

$$\gamma_b := \begin{pmatrix} \pi_+(b) & 0 \\ 0 & \pi_-(b) \end{pmatrix} = \begin{pmatrix} \pi_+(b) & 0 \\ 0 & \pi_+(-b) \end{pmatrix}$$

for $b > 0$ satisfies

$$\langle \gamma_b(v) \mid J \gamma_b(v) \rangle \leqslant \langle v \mid Jv \rangle \tag{7.18}$$

for all $v \in \mathbf{K}_0$ and $b \in \mathbb{R}_+$. When the explicit operators are substituted into the latter estimate, we get

$$\int_{-\infty}^{\infty} \operatorname{Re} \lambda(\xi) \, |(\pi_+(b) \, f_0)^\wedge \, (\xi)|^2 \, d\xi \leqslant \int_{-\infty}^{\infty} \operatorname{Re} \lambda(\xi) \, |\hat{f}_0(\xi)|^2 \, d\xi,$$

valid for $b \in \mathbb{R}_+$, and

$$\begin{pmatrix} f_0 \\ T_0 f_0 \end{pmatrix} \in \mathbf{K}_0 \subset \begin{pmatrix} \mathbf{L}^2(\mathbb{R}) \\ \mathbf{L}^2(\mathbb{R}) \end{pmatrix}.$$

It follows that $\pi_+(b)$ maps the subspace $\mathscr{D}$ into itself when $b \in \mathbb{R}_+$; and, as a consequence, the Lax–Phillips setup applies to $\mathscr{D}$ as a closed subspace in $\mathbf{L}^2(\mathbb{R})$, relative to the unitary one-parameter group $\{\pi_+(b) \mid b \in \mathbb{R}\}$ of operators in $\mathbf{L}^2(\mathbb{R})$. Let

$$\mathscr{D}_\infty := \bigvee_{b \in \mathbb{R}} \pi_+(b)\, \mathscr{D}, \tag{7.19}$$

$$\mathscr{D}_{-\infty} := \bigwedge_{b \in \mathbb{R}} \pi_+(b)\, \mathscr{D}, \tag{7.20}$$

where $\bigvee$ and $\bigwedge$ denote the lattice operations on closed subspaces in $\mathbf{L}^2(\mathbb{R})$, and

$$(\pi_+(b)\, f)(x) = e^{ibe^x} f(x), \qquad f \in \mathbf{L}^2(\mathbb{R}), \quad b, x \in \mathbb{R}.$$

It follows from the ansatz (7.19)–(7.20) that both of the spaces $\mathscr{D}_\infty$ and $\mathscr{D}_{-\infty}$ are invariant under $\{\pi_+(b) \mid b \in \mathbb{R}\}$, and moreover that

$$\mathscr{D}_{-\infty} \subset \mathscr{D} \subset \mathscr{D}_\infty. \tag{7.21}$$

It is enough to show that the assumption $\mathscr{D} \neq \{0\}$ leads to a trivial quotient space $(\mathbf{K}_0/\mathbf{N})^{\sim}$. Let

$$\tau(s)\, f(x) = f(x+s), \qquad f \in \mathbf{L}^2(\mathbb{R}), \quad s, x \in \mathbb{R}$$

be the translation part. We have

$$\tau(s)\, \pi_+(b) = \pi_+(e^s b)\, \tau(s) \tag{7.22}$$

and we conclude that $\mathscr{D}_{\pm\infty}$ are also both invariant under $\{\tau(s) \mid s \in \mathbb{R}\}$. Since, as we noted, the system (7.50) is irreducible in $\mathbf{L}^2(\mathbb{R})$, we conclude by Schur's lemma that $\mathscr{D}_\infty = \mathbf{L}^2(\mathbb{R})$. Recall $\mathscr{D} \neq 0$ was assumed at the outset. For the space $\mathscr{D}_{-\infty}$, we then have only two possibilities, $\mathscr{D}_{-\infty} = \{0\}$ and $\mathscr{D}_{-\infty} = \mathbf{L}^2(\mathbb{R})$, again by Schur's lemma, and the first possibility must be ruled out by virtue of the Lax–Phillips theorem [31]. Notice that the spectrum of $\{\pi_+(b) \mid b \in \mathbb{R}\}$ is evidently a half-line, and the two properties, $\mathscr{D}_{-\infty} = \{0\}$ and $\mathscr{D}_\infty = \mathbf{L}^2(\mathbb{R})$, would contradict the conclusion in the Lax–Phillips theorem, to the effect that the spectrum would then necessarily have to be two-sided, i.e., all $\mathbb{R} = (-\infty, \infty)$, and of homogeneous Lebesgue type, i.e., unitarily equivalent, up to multiplicity, with translation on the line.

Only the possibility $\mathscr{D}_{-\infty} = \mathbf{L}^2(\mathbb{R})$ remains to be considered. But we have

$$\mathscr{D}_{-\infty} \subset \mathscr{D} \subset P_0 \mathbf{K}_0,$$

so it would follow that $\mathscr{D} = \mathbf{L}^2(\mathbb{R})$, and we are then reduced back again to the case $Q = P_{\mathbf{K}_0} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ from part I of the present proof; i.e., to a trivial induced Hilbert space $(\mathbf{K}_0/\mathbf{N})^{\sim}$ as already noted. ∎

The following argument deals with the general case, avoiding the separation of the proof into the two cases (I) and (II): If vectors $v \in \mathbf{K}_0$ are expanded as $v = \binom{h}{k}$, $h = Q_{11}h + Q_{12}k$, $k = Q_{21}h + Q_{22}k$, we can introduce $\mathscr{D} = \{h \in \mathbf{L}^2(\mathbb{R}) \mid \exists k \in \mathbf{L}^2(\mathbb{R}) \text{ s.t. } \binom{h}{k} \in \mathbf{N}\}$. If $b > 0$, we then have from (7.16):

$$\pi_+(b)\, h = Q_{11}\pi_+(b)\, h + Q_{12}\pi_+(-b)\, k,$$
$$\pi_+(-b)\, k = Q_{21}\pi_+(b)\, h + Q_{22}\pi_+(-b)\, k,$$

valid for any $\binom{h}{k} \in \mathbf{K}_0$, and $b \in \mathbb{R}_+$. So it follows from Lemma 3.1 again that $\mathscr{D}$ is invariant under $\{\pi_+(b) \mid b > 0\}$, and also under the whole semigroup $\{\pi_+(g) \mid g \in S\}$ where π_+ is now denoting the corresponding (see (7.22)) irreducible representation of G on $\mathbf{L}^2(\mathbb{R})$. Hence, we may apply the Lax–Phillips argument to the induced spaces $\mathscr{D}_{\pm\infty}$ from (7.19)–(7.20). If $(\mathbf{K}_0/\mathbf{N})^\sim$ should be $\neq \{0\}$, then $\mathscr{D} = \{0\}$ by the argument. Since we are assuming $(\mathbf{K}_0/\mathbf{N})^\sim \neq \{0\}$, we get $\mathscr{D} = \{0\}$, and as a consequence the following operator graph representation for $\mathbf{K}_0$: $(\mathbf{K}_0/\mathbf{N})^\sim = \beta(G(L))$ where $G(L)$ is the graph of a closed operator L in $\mathbf{L}^2(\mathbb{R})$. Specifically, this means that the linear mapping $\mathbf{K}_0/\mathbf{N} \ni \binom{h}{k} + \mathbf{N} \mapsto h$ is well defined as a linear closed operator. This in turn means that $\mathbf{K}_0$ may be represented as the graph of a closable operator in $\mathbf{L}^2(\mathbb{R})$ as discussed in the first part of the proof. Hence such a representation could have been assumed at the outset.

Remark 7.4. In a recent paper on local quantum field theory [2], Borchers considers in his Theorem II.9 a representation π of the $(ax+b)$-group G on a Hilbert space $\mathbf{H}$ such that there is a *conjugate linear* J (i.e., a period-2 antiunitary) such that $J\pi J = \pi \circ \tau$ where τ is the period-2 automorphism of G given by $\tau(a, b) := (a, -b)$. In Borchers's example, the one-parameter subgroup $b \mapsto \pi(1, b)$ has semibounded spectrum, and there is a unit-vector $v_0 \in \mathbf{H}$ such that $\pi(1, b)\, v_0 = v_0$, $\forall b \in \mathbb{R}$. The vector v_0 is cyclic and separating for a von Neumann algebra M such that $\pi(1, b)\, M\pi(1, -b) \subset M$, $\forall b \in \mathbb{R}_+$. Let $a = e^t$, $t \in \mathbb{R}$. Then, in Borchers's construction, the other one-parameter subgroup $t \mapsto \pi(e^t, 0)$ is the modular group Δ^{it} associated with the cyclic and separating vector v_0 (from Tomita–Takesaki theory [3, Vol. I]). Finally, J is the corresponding modular conjugation satisfying $JMJ = M'$ when M' is the commutant of M.

ACKNOWLEDGMENTS

The authors would like to thank the University of Iowa (May 1994), the Louisiana State University (Nov. 1995), the University of Odense (June 1995), and the Mittag-Leffler Institute (Spring 1996) for their support. We would also want to thank Bent Ørsted and Steen Pedersen for helpful discussions. We would like to thank Brian Treadway for his excellent typesetting of the final version of the paper in LATEX. Both authors were also supported in part by the National Science Foundation. G.O. was supported by NSF grant DMS-9626541, LEQSF grant (1996-99)-RD-A-12 and the Danish Research Council.

REFERENCES

1. W. B. Arveson, Feynman–Kac formula, *in* “Ten Lectures on Operator Algebras,” CBMS, Vol. 55, Chap. 4, NSF and the Amer. Math. Soc., Providence, RI, 1984 (reprinted 1988).
2. H. J. Borchers, The CPT-theorem in two-dimensional theories of local observables, *Comm. Math. Phys.* **143** (1992), 315–332.
3. O. Bratteli and D. W. Robinson, “Operator Algebras and Quantum Statistical Mechanics,” Vol. I, 2nd ed., Springer-Verlag, Berlin/New York, 1987.
4. M. D. Davidson and R. C. Fabec, Geometric realization for highest weight representations, *Contemp. Math.* **191** (1995), 13–32.
5. H. Dym and H. P. McKean, “Fourier Series and Integrals,” Academic Press, New York, 1970.
6. J. Faraut, J. Hilgert, and G. Ólafsson, Spherical functions on ordered symmetric spaces, *Ann. Inst. Fourier* **44** (1995), 927–966.
7. J. Faraut and A. Korányi, “Analysis on Symmetric Cones,” Clarendon, Oxford, 1994.
8. J. Fröhlich, Unbounded symmetric semigroups on a separable Hilbert space are essentially selfadjoint, *Adv. Appl. Math.* **1** (1980), 237–256.
9. J. Fröhlich, K. Osterwalder, and E. Seiler, On virtual representations of symmetric spaces and their analytic continuation, *Ann. Math.* **118** (1983), 461–489.
10. F. M. Goodman and P. E. T. Jorgensen, Lie algebras of unbounded derivations, *J. Funct. Anal.* **52** (1983), 369–384.
11. S. Helgason, “Differential Geometry and Symmetric Spaces,” Academic Press, New York, 1962.
12. S. Helgason, “Differential Geometry, Lie Groups and Symmetric Spaces,” Academic Press, San Diego, 1978.
13. S. Helgason, “Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators and Spherical Functions,” Academic Press, San Diego, 1984.
14. H. Helson, “Lectures on Invariant Subspaces,” Academic Press, San Diego, 1964.
15. J. Hilgert and K.-H. Neeb, “Lie Semigroups and Their Applications,” Lecture Notes in Math., Vol. 1552, Springer-Verlag, Berlin/New York, 1993.
16. J. Hilgert and G. Ólafsson, “Causal Symmetric Spaces, The Geometry and Harmonic Analysis,” Perspect. Math., Vol. 18, Academic Press, San Diego, 1996.
17. J. Hilgert, G. Ólafsson, and B. Ørsted, Hardy spaces on affine symmetric spaces, *J. Reine Angew. Math.* **415** (1991), 189–218.
18. H. Jaffee, Real forms of Hermitian symmetric spaces, *Bull. Amer. Math. Soc.* **81** (1975), 456–458.
19. H. Jaffee, Anti-holomorphic automorphisms of the exceptional symmetric domains, *J. Differential Geom.* **13** (1978), 79–86.
20. P. E. T. Jorgensen, Representations of differential operators on a Lie group, *J. Funct. Anal.* **20** (1975), 105–135.
21. P. E. T. Jorgensen, Unbounded operators: perturbations and commutativity problems, *J. Funct. Anal.* **39** (1980), 281–307.
22. P. E. T. Jorgensen, Analytic continuation of local representations of Lie groups, *Pacific J. Math.* **125** (1986), 397–408.
23. P. E. T. Jorgensen, Analytic continuation of local representations of symmetric spaces, *J. Funct. Anal.* **70** (1987), 304–322.
24. P. E. T. Jorgensen, “Operators and Representation Theory,” North-Holland, Amsterdam, 1988.
25. P. E. T. Jorgensen and R. T. Moore, “Operator Commutation Relations,” Kluwer Academic, Dordrecht/Norwell, MA, 1984.

26. A. Klein and L. J. Landau, Construction of a unique selfadjoint generator for a symmetric local semigroup, *J. Funct. Anal.* **44** (1981), 121–137.
27. A. Klein and L. J. Landau, From the Euclidean group to the Poincaré group via Osterwalder–Schrader positivity, *Comm. Math. Phys.* **87** (1983), 469–484.
28. A. W. Knapp and E. M. Stein, Intertwining operators for semisimple groups, II, *Invent. Math.* **60** (1980), 9–84.
29. A. Korányi and J. A. Wolf, Realization of Hermitian symmetric spaces as generalized half-planes, *Ann. of Math.* **81** (1965), 265–288.
30. J. D. Lawson, Polar and Olshanskii decompositions, *J. Reine Angew. Math.* **448** (1994), 191–219.
31. P. D. Lax and R. S. Phillips, "Scattering Theory," 2nd ed., Academic Press, New York, 1989.
32. M. Lüscher and G. Mack, Global conformal invariance and quantum field theory, *Comm. Math. Phys.* **41** (1975), 203–23.
33. G. W. Mackey, "Induced Representations of Groups and Quantum Mechanics," Benjamin, Elmsford, NY, 1968.
34. T. Matsuki, Orbits on affine symmetric spaces under the action of parabolic subgroups, *Hiroshima Math. J.* **12** (1982), 307–320.
35. K.-H. Neeb, Holomorphic representation theory, II, *Acta Math.* **173** (1994), 103–133.
36. E. Nelson, Analytic vectors, *Ann. Math.* **70** (1959), 572–615.
37. E. Nelson and W. F. Stinespring, Representations of elliptic operators, *Amer. J. Math.* **81** (1959), 547–560.
38. G. Ólafsson, Fourier and Poisson transformation associated to a semisimple symmetric space, *Invent. Math.* **90** (1987), 605–629.
39. G. Ólafsson, Causal symmetic spaces, *Math. Gotting.* **15** (1990).
40. G. Ólafsson, Symmetric spaces of Hermitian type, *Differential Geom. Appl.* **1** (1991), 195–233.
41. G. Ólafsson, Spherical functions on ordered symmetric space; A Harish-Chandra type formula, in preparation.
42. G. Ólafsson and B. Ørsted, The holomorphic discrete series for affine symmetric spaces, I, *J. Funct. Anal.* **81** (1988), 126–159.
43. G. Ólafsson and B. Ørsted, Is there an orbit method for affine symmetric spaces? *in* "The Orbit Method in Representation Theory" (M. Duflo, N. V. Pedersen, and M. Vergne, Eds.), Birkhäuser, Boston, Cambridge, MA, 1990.
44. G. Ólafsson and B. Ørsted, Generalizations of the Bargmann transform, *in* "Proceedings of 'Workshop on Lie Theory and Its Applications in Physics,' Clausthal, August 1996" (Dobrev, Doebner, and J. Hilgert, Eds.), World Science Publishing Co, 1996.
45. G. Ólafsson and B. Ørsted, "Causal Compactification and Hardy Spaces," Report No. 19, Institut Mittag-Leffler, submitted, 1995/1996, to appear in Transaction of the AMS.
46. Ol'shanskii, Invariant cones in Lie algebras, Lie semigroups, and the holomorphic discrete series, *Funct. Anal. Appl.* **15** (1982), 275–285.
47. B. Ørsted and G. Zhang, Generalized principal series representations and tube domains, *Duke Math. J.* **78** (1995), 335–357.
48. T. Oshima and J. Sekiguchi, Eigenspaces of invariant differential operators on an affine symmetric space, *Invent. Math.* **57** (1980), 1–81.
49. T. Oshima and J. Sekiguchi, The restricted root system of a semisimple symmetric pair, *Adv. Stud. Pure Math.* **4** (1984), 313–359.
50. K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions, *Comm. Math. Phys.* **31** (1973), 83–112.
51. K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions, *Comm. Math. Phys.* **42** (1975), 281–305.

52. S. Pedersen, Anticommuting selfadjoint operators, *J. Funct. Anal.* **89** (1990), 428–443.
53. R. S. Phillips, The extension of dual subspaces invariant under an algebra, *in* “Proc. Int. Symp. Linear Spaces,” Pergamon, Elmsford, NY, 1961.
54. N. S. Poulsen, On C^{∞}-vectors and intertwining bilinear forms for representations of Lie groups, *J. Funct. Anal.* **9** (1992), 87–120.
55. H. Prado, University of Iowa Thesis, 1989.
56. M. Reed and B. Simon, “Methods of Modern Math. Physics, II: Fourier Analysis, Self-adjointness,” Academic Press, New York, 1975.
57. S. Sahi, Explicit Hilbert spaces for certain unipotent representations, *Invent. Math.* **110** (1992), 409–418.
58. S. Sahi, Unitary representations on the Shilov boundary of a symmetric tube domain, *Contemp. Math.* **145** (1003), 275–286.
59. R. Schrader, Reflection positivity for the complementary series of $\mathrm{SL}(2n, \mathbb{C})$, *Publ. Res. Inst. Math. Sci.* **22** (1986), 119–141.
60. E. M. Stein, “Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals,” Princeton Univ. Press, Princeton, NJ, 1993.
61. M. H. Stone, On unbounded operators in Hilbert space, *J. Indian Math. Soc. (N.S.)* **15** (1951), 155–192.
62. M. Vergne and H. Rossi, Analytic continuation of the holomorphicdiscrete series of a semi-simple Lie group, *Acta Math.* **136** (1976), 1–59.
63. G. Warner, “Harmonic Analysis on Semisimple Lie Groups, I,” Springer-Verlag, Berlin/New York, 1972.
64. N. Wallach, The analytic continuation of the discrete series I, II, *Trans. Amer. Math. Soc.* **251** (1979), 1–17, 19–37.
65. N. R. Wallach, “Real Reductive Groups, II,” Academic Press, San Diego, 1992; *Pure Appl. Math.* **132**-II.
66. K. Yosida, “Lectures on Differential and Integral Equations,” Dover, New York, 1991.