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On the Meromorphic Extension of the Spherical Functions on Noncompactly Causal Symmetric Spaces[1]

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We determine integral formulas for the meromorphic extension in the λ-parameter of the spherical functions φ_λ on a noncompactly causal symmetric space. The main tool is Bernstein's theorem on the meromorphic extension of complex powers of polynomials. The regularity properties of φ_λ are deduced. In particular, the possible λ-poles of φ_λ are located among the translates of the zeros of the Bernstein polynomial. The translation parameter depends only on the structure of the symmetric space. The expression of the Bernstein polynomial is conjectured. The relation between the Bernstein polynomial and the product formula of the c_Ω-function is analyzed. The conjecture is verified in the rank-one case. The explicit formulas obtained in this case yield a detailed description of singularities of φ_λ. In the general higher rank case, the integral formulas are applied to find asymptotic estimates for the spherical functions. In the Appendix, the spherical functions on noncompactly causal symmetric spaces are regarded as a special instance of Harish-Chandra-type expansions associated with roots systems with arbitrary multiplicities. We study expansions obtained by taking averages over arbitrary parabolic subgroups of the Weyl group of the root system. The possible λ-singularities are located in this general context. © 2001 Academic Press

Key Words: spherical functions; Bernstein polynomial; noncompactly causal symmetric spaces; c-functions; Harish-Chandra expansions.

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INTRODUCTION

Noncompactly causal (NCC) symmetric spaces are a special class of noncompact non-Riemannian symmetric spaces. Their particular fine structure permits the extension of the theory of spherical functions on noncompact Riemannian symmetric spaces, which is mainly due to Harish-Chandra and later Gangolli, Godement, Helgason, Varadarajan, and others. The structure of a NCC symmetric space G/H is characterized by the existence of a certain maximal $\mathrm{Ad}(H)$-invariant convex cone $C_{\max}$ in its tangent space $\mathfrak{q}$ at the base point $\{H\}$. The associated semigroup $S = H\exp(C_{\max})$ is a maximal H-bi-invariant domain in G, and one can develop, on its interior S^0, a theory of spherical functions. The common bi-invariant eigendistributions of the G-invariant differential operators on G/H are in fact functions on S^0. They are parametrized (modulo a certain "small" Weyl group W_0) by elements λ in the dual $\mathfrak{a}_{\mathbb{C}}^*$ of the complexification of a Cartan subspace $\mathfrak{a}$ in $\mathfrak{q}$. As in the Riemannian case, these common eigenfunctions φ_λ, called—by analogy—spherical functions, are the building blocks for the bi-invariant harmonic analysis on S^0. However the dependence of the spherical functions on the parameter λ is in general meromorphic, while it is holomorphic in the Riemannian situation. As for the spherical Fourier transform of bi-invariant functions on Riemannian spaces, the spherical Laplace transform $\lambda \mapsto \mathcal{L}(f)(\lambda)$ of a sufficiently regular H-bi-invariant function f on S^0 is defined by integration against φ_λ. So the Laplace transform $\mathcal{L}(f)$ inherits the λ-singularities of the spherical functions. The exact location of these singularities is therefore essential for the understanding of the spherical Laplace transform and for the corresponding development of the harmonic analysis on S^0/H.

The meromorphic dependence of the spherical functions on the parameter λ has been determined by the first author in [26] by means of an expansion formula in terms of Harish-Chandra's generalized hypergeometric functions. This formula has a drawback, it is valid only on a positive Weyl chamber A^+. As H-bi-invariant functions on S^0, the spherical functions are uniquely determined by their W_0-invariant restriction to $S^0 \cap A$, and this set generally contains A^+ strictly. The difficulties in studying the properties of the spherical functions on S^0 using the expansion formula come from the unboundedness of the generalized hypergeometric functions on the walls of A^+. The meromorphic dependence on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and and real analytic dependence on the variable $a \in S^0 \cap A$ can be still deduced from the expansion formula using the monodromy arguments developed by Heckman and Opdam. These arguments are however inexplicit.

In this paper we propose an alternative method for determining the meromorphic extension of the spherical functions in the λ-parameter. The starting point is the integral formula for the spherical functions as determined in [8]. This integral formula is valid only for the λ's in a certain region $\mathscr{E}$ of the parameter space $\mathfrak{a}_{\mathbb{C}}^*$. We then obtain a direct meromorphic continuation of the formula to the whole $\mathfrak{a}_{\mathbb{C}}^*$ by applying Bernstein's theorem on the meromorphic extension of complex powers of polynomials. The resulting integral formulas for the spherical functions hold on the entire $S^0 \cap A$. It follows, in particular, that φ_λ is smooth on $S^0 \cap A$. As an application of the integral formulas, we can determine asymptotic estimates for the spherical functions φ_λ for almost all values of the parameter λ.

For a translation parameter $\delta \in \mathfrak{a}^*$ depending only on the symmetric space, the λ-singularities of the spherical functions are contained in the δ-translates of the zero set of the Bernstein polynomial. This is a locally finite union of hyperplanes. Knowing the explicit expression for the Bernstein polynomial would give the solution to the problem of location of the singularities of the spherical functions and would provide explicit integral formulas for the spherical functions for all values of the parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

At this stage, the Bernstein polynomial is explicitly known only for rank-one NCC spaces. We formulate in this paper the conjecture that the Bernstein polynomial in the general higher-rank case is given by a product formula involving the Bernstein polynomials for the rank-one NCC symmetric subspaces. There are several facts supporting our Conjecture. Some are related to the product formula for the c-function c_Ω associated with the bounded realization of $H/H \cap K$. This is the function governing the asymptotic behavior of the spherical functions. It is defined on $\mathscr{E}$ by an integral formula. The product formula, which provides its meromorphic extension to $\mathfrak{a}_{\mathbb{C}}^*$, has been recently determined by Krötz and the first author (see [24]) by means of a modification of the classical argument of Gindikin and Karpelevic for the determination of Harish-Chandra's c-function. We show in this paper that the same Bernstein polynomial can also be used to directly meromorphically extend the integral formula of c_Ω. As a consequence, we find a divisor of the Bernstein polynomial. The Conjecture states that this divisor coincides (up to a constant multiple) with the Bernstein polynomial itself. As a further indication that the Conjecture is reasonable, we show that, used together with the direct meromorphic continuation of the defining integral formula, it implies the product formula for c_Ω. This analytic proof of the product formula is very easy and of independent interest.

The Conjecture allows us to locate the polar set of the spherical functions as a subset of the polar set of the numerator n_Ω of c_Ω. The function n_Ω is given explicitely as a product over the noncompact positive roots of gamma factors. For $a \in A^+$, the inclusion of the polar set of $\varphi_\lambda(a)$ in the polar set of n_Ω can also be verified by means of the Harish-Chandra-type expansion formula. Working in the Heckman–Opdam's context of hypergeometric functions associated with root systems with arbitrary multiplicities, we can locate the λ-singularities of the expansions obtained from averages over arbitrary subgroups of the Weyl group of the root system. The main tool needed is the method developed by Opdam in [27] for the pole cancellation in Harish-Chandra-type expansions by averaging over Weyl groups. Monodromy arguments then allow us to extend the results to $S^0 \cap A$. The case of the spherical functions on the NCC spaces corresponds to the choice of the small Weyl group W_0 as averaging group.

In the rank-one case we verify the Conjecture as true by direct computation. The explicit formulas obtained in this case allows us to further refine the results on the polar set of the spherical functions. An easy analysis of the non-vanishing of certain functions yields also a complete picture of the removability of the singularities.

Our paper is organized as follows. In Section 1 we recall some structure theory of the NCC spaces $\mathscr{M} := G/H$. Section 2 introduces the spherical functions as H-bi-invariant common eigendistributions of the algebra of G-invariant differential operators on $\mathscr{M}$. The method of Bernstein polynomials for the meromorphic extension of the spherical functions is developed in Section 3. The main results are collected in Theorem 3.2.

The integral formulas for the meromorphic extension is employed in Section 4 to determine asymptotic estimates for the spherical functions. They generalize to arbitrary values of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the estimates proven for $\lambda \in \mathscr{E}$ in [8].

The meromorphic extendibility of the function c_Ω by means of our Bernstein polynomial is the content of Proposition 5.1. This allows us to formulate our Conjecture in Section 5. The product formula for c_Ω is in fact equivalent to a condition related to the Conjecture. This is the content of Theorem 5.1. In Section 6 the rank-one case is studied in details. The Bernstein polynomial is explicitely computed, and the analysis of singularities of the spherical functions is then carried out. The final Section 7 collects the open problems and outlines some possible further developments.

In the Appendix we have studied the λ-singularities of Harish-Chandra-type expansions. They generalize Opdam's results to averages over arbitrary parabolic subgroups of the Weyl group. The inclusion of the polar set of φ_λ in the polar set of n_Ω is a special instance of these general results.

1. PRELIMINARIES

In this section we collect the basic notation for noncompactly causal symmetric spaces. We refer to [19, 26] for a more extended exposition.

Let G be a connected noncompact simple Lie group endowed with a non-trivial involutive automorphism τ. Let H denote a closed subgroup of G satisfying $G_0^\tau \subset H \subset G^\tau := \{g \in G : \tau g = g\}$. Here G_0^τ is the connected component of G^τ containing the identity element $e \in G$. Then $\mathscr{M} := G/H$ is a noncompact symmetric space.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{g}_{\mathbb{C}}$ denote its complexification. The involutions of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ corresponding to τ will be denoted by the same letter τ. Let $G_{\mathbb{C}}$ be an irreducible simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. We will assume $G \subset G_{\mathbb{C}}$. Hence the center $Z(G)$ of G is finite. Let $\mathfrak{h} := \{X \in \mathfrak{g} : \tau X = X\}$ and $\mathfrak{q} := \{X \in \mathfrak{g} : \tau X = -X\}$, so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Furthermore $\mathfrak{h}$ is the Lie algebra of H. Let θ be a Cartan involution of $\mathfrak{g}$ commuting with τ. Set $\mathfrak{k} := \{X \in \mathfrak{g} : \theta X = X\}$ and $\mathfrak{p} := \{X \in \mathfrak{g} : \theta X = -X\}$. The commutativity of θ and τ implies the direct sum decomposition

$$\mathfrak{g} = \mathfrak{h}_k \oplus \mathfrak{h}_p \oplus \mathfrak{q}_k \oplus \mathfrak{q}_p,$$

where $\mathfrak{h}_k := \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{h}_p := \mathfrak{h} \cap \mathfrak{p}$, etc. The involution of G with differential θ will also be denoted by θ. Set $K = G^\theta =: \{g \in G : \theta g = g\}$. Then K is the analytic subgroup of G with Lie algebra $\mathfrak{k}$. It is maximal compact in G and contains $Z(G)$.

For subsets B of G and $\mathfrak{b}$ of $\mathfrak{g}$, the centralizer of $\mathfrak{b}$ in B is $Z_B(\mathfrak{b}) := \{g \in B : \mathrm{Ad}(g)\, X = X \;\forall X \in \mathfrak{b}\}$ and the normalizer of $\mathfrak{b}$ in B is $N_B(\mathfrak{b}) := \{g \in B : \mathrm{Ad}(g)\, X \in \mathfrak{b} \;\forall X \in \mathfrak{b}\}$. We set also $\mathfrak{b}^B := \{X \in \mathfrak{b} : \mathrm{Ad}(g)\, X = X \;\forall g \in B\}$.

We assume that $\mathscr{M}$ is *irreducible* and *noncompactly causal* (NCC), i.e., that $\mathfrak{q}^{H \cap K} \cap \mathfrak{p} \neq \{0\}$. Hence $\mathscr{M}$ is non-Riemannian and $\mathfrak{q}^{H \cap K} \cap \mathfrak{p} = \mathbb{R} Y^0$ is one dimensional. We can normalize Y^0 so that $\mathrm{ad}\, Y^0$ has eigenvalues $0, 1, -1$. The 0-eigenspace of $\mathrm{ad}\, Y^0$ is $\mathfrak{g}(0) = \mathfrak{h}_k \oplus \mathfrak{q}_p$. Let $G(0) = Z_G(Y^0)$. Then $G(0)$ is a θ- and τ-stable subgroup of G with Lie algebra $\mathfrak{g}(0)$ and with Cartan decomposition $(H \cap K) \exp(\mathfrak{q}_p)$ (cf. [19, 3.1.22, (4); 26 Theorems 1.4, (7) and (8)]). In particular, $G(0)/K(0)$ is a Riemannian symmetric space and $K(0) = K \cap H$.

For a subspace $\mathfrak{b}$ of $\mathfrak{g}$, let $\mathfrak{b}_{\mathbb{C}}$ be its complexificaton in $\mathfrak{g}_{\mathbb{C}}$. The dual spaces of $\mathfrak{b}$ and $\mathfrak{b}_{\mathbb{C}}$ and are respectively denoted by $\mathfrak{b}^*$ and $\mathfrak{b}^*_{\mathbb{C}}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{q}_p$ containing Y^0. Then $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and in $\mathfrak{q}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{a}^*$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Then Δ is reduced, that is 2α is not a root for every $\alpha \in \Delta$. For $\alpha \in \Delta$, let m_α denote the dimension of the root space $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)\, X \;\forall H \in \mathfrak{a}\}$. The set of *compact roots* is $\Delta_0 := \{\alpha \in \Delta : \alpha(Y^0) = 0\}$. One has $\Delta_0 = \Delta(\mathfrak{g}(0), \mathfrak{a})$. Set $\Delta_\pm := \{\alpha \in \Delta : \alpha(Y^0) = \pm 1\}$. Then $\Delta_+ \cup \Delta_- = \Delta \backslash \Delta_0$ is the set of *noncompact roots*. We can fix a positive system Δ^+ in Δ by fixing a positive system $\Delta_0^+ := \Delta^+ \cap \Delta_0$ in Δ_0 and setting $\Delta^+ := \Delta_+ \cup \Delta_0^+$.

Set $M := Z_H(\mathfrak{a}) = Z_{H \cap K}(\mathfrak{a})$. The Weyl group for Δ is $W := N_K(\mathfrak{a})/M$, and for Δ_0 is $W_0 := N_H(\mathfrak{a})/Z_H(\mathfrak{a}) = N_{K \cap H}(\mathfrak{a})/Z_{K \cap H}(\mathfrak{a})$. In the literature W and W_0 are often referred to as "the Weyl group" and "the small Weyl group," respectively.

A subset L of G is said to be *essentially connected* if $L=(L\cap M)\,L_0$, L_0 being the connected component of L containing e. Since $\mathscr{M}$ in NCC if follows that H and $G(0)$ are essentially connected (cd. [26, Lemmas 1.6 and 1.7]).

Define subalgebras of $\mathfrak{g}$ by

$$\mathfrak{n}:=\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha,\qquad \mathfrak{n}_+:=\bigoplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha,\qquad \mathfrak{n}_0:=\bigoplus_{\alpha\in\Delta_0^+}\mathfrak{g}_\alpha,$$

$$\bar{\mathfrak{n}}:=\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_{-\alpha},\qquad \mathfrak{n}_-:=\bigoplus_{\alpha\in\Delta_-}\mathfrak{g}_\alpha,\qquad \bar{\mathfrak{n}}_0:=\bigoplus_{\alpha\in\Delta_0^+}\mathfrak{g}_{-\alpha}.$$

Then $\mathfrak{n}$, $\bar{\mathfrak{n}}=\theta(\mathfrak{n})$, $\mathfrak{n}_0$, $\bar{\mathfrak{n}}_0=\theta(\mathfrak{n}_0)$ are nilpotent, and $\mathfrak{n}_+$, $\mathfrak{n}_-=\theta(\mathfrak{n}_+)$ are abelian. Let A, N, $\bar{N}$, N_+, N_-, N_0 and $\bar{N}_0$ be the analytic subgroups of G respectively corresponding to $\mathfrak{a}$, $\mathfrak{n}$, $\bar{\mathfrak{n}}$, $\mathfrak{n}_+$, $\mathfrak{n}_-$, $\mathfrak{n}_0$, and $\mathfrak{n}_0$. Then $N=N_0\rtimes N_+$ and $\bar{N}=\bar{N}_0\rtimes N_-$. Let $\exp\colon\mathfrak{g}\to G$ denote the exponential map. Then exp restricts to a diffeomorphism of $\mathfrak{a}$ onto A, $\mathfrak{n}$ onto N, $\bar{\mathfrak{n}}$ onto $\bar{N}$, etc. The inverse of exp on these spaces will be denoted by log.

Let B denote the Cartan–Killing form of $\mathfrak{g}$, and let $(\cdot,\cdot)$ be the inner product in $\mathfrak{g}$ given by $(X,Y)=-B(X,\theta(Y))$. We write $|X|:=(X,X)^{1/2}$ for the corresponding norm. For $\lambda\in\mathfrak{a}^*$ there is a unique $\bar{H}_\lambda\in\mathfrak{a}$ for which $\lambda(H)=B(H,\bar{H}_\lambda)$ for all $H\in\mathfrak{a}$. Define $(\lambda,\mu):=(\bar{H}_\lambda,\bar{H}_\mu)$ for all $\lambda,\mu\in\mathfrak{a}^*\backslash\{0\}$. Set $H_\lambda=2\bar{H}_\lambda/(\bar{H}_\lambda,\bar{H}_\lambda)$. Hence $\lambda(H_\lambda)=2$. For $\alpha\in\Delta^+$ choose $X_\alpha\in\mathfrak{g}_\alpha$ and $X_{-\alpha}=\tau(X_\alpha)$ so that $H_\alpha=[X_\alpha,X_{-\alpha}]$. Fix a maximal system $\{\gamma_1,...,\gamma_r\}$ of long strongly orthogonal roots in Δ_+. Set $H_j:=H_{\gamma_j}$, $X_j:=X_{\gamma_j}$, $X_{-j}:=X_{-\gamma_j}$ for every $j=1,...,r$.

The *minimal* and *maximal cones* in $\mathfrak{a}$ are respectively

$$c_{\min}:=\sum_{\alpha\in\Delta_+}\mathbb{R}_0^+H_\alpha,$$

$$c_{\max}:=\{X\in\mathfrak{a}:\alpha(X)\geqslant 0\ \forall\alpha\in\Delta_+\},$$

where $\mathbb{R}_0^+$ denotes the set of nonnegative reals. Then $c_{\min}\subset c_{\max}$ and $Y^0\in c_{\min}^0\subset c_{\max}^0$.

Let $C_{\max}=\overline{\mathrm{Ad}(H)\,c_{\max}^0}$. Then $C_{\max}$ is a closed, pointed, generating H-invariant cone in $\mathfrak{q}$ and $S:=H\exp C_{\max}$ is a closed semigroup in G with $S^0=H\exp C_{\max}^0=H(\exp c_{\max}^0)\,H$.

HAN is an open submanifold of G and $S^0\subset HAN$. We have $\bar{N}\cap HAN=N(\Omega)\,\bar{N}_0$ where $N(\Omega):=\exp\Omega$ with Ω a real bounded symmetric domain in $\mathfrak{n}_-$ diffeomorphic to $H/H\cap K$. The domain Ω can be explicitly described as

$$\Omega=\mathrm{Ad}(H\cap K)\left\{\sum_{j=1}^r x_jX_{-j}\colon -1<x_j<1,\ j=1,...,r\right\}. \tag{1}$$

Moreover, S is the compression semigroup of Ω.

The Iwasawa decompositions of the Lie algebra $\mathfrak{g}$ are

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

On the group level, we have diffeomorphisms

$$K \times A \times N \to G, \qquad (k, a, n) \mapsto kan,$$
$$H \times A \times N \to HAN, \qquad (h, a, n) \mapsto han.$$

We write

$$x = k(x)\, a(x)\, n(x) \qquad \text{for} \quad x \in G, \quad k(x) \in K, \quad a(x) \in A, \quad n(x) \in N,$$
$$x = h(x)\, a_H(x)\, n_H(x) \qquad \text{for} \quad x \in HAN, \quad h(x) \in H, \quad a_H(x) \in A, \quad n_H(x) \in N.$$

If $x \in G(0)$, then $h(x) = k(x)$, $a_H(x) = a(x)$ and $n(x) = n_H(x)$.
Define

$$\rho := \tfrac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha, \qquad \rho_+ := \tfrac{1}{2} \sum_{\alpha \in \Delta_+} m_\alpha \alpha, \qquad \rho_0 := \tfrac{1}{2} \sum_{\alpha \in \Delta_0^+} m_\alpha \alpha. \tag{2}$$

We normalize the Haar measures dn, $d\bar{n} := \theta(dn)$, dn_0, and $d\bar{n}_0 := \theta(dn_0)$ respectively, on N, $\bar{N}$, N_0, and $\bar{N}_0$ by the requirements

$$\int_{\bar{N}} a(\bar{n})^{-2\rho}\, d\bar{n} = 1, \qquad \int_{\bar{N}_0} a(\bar{n}_0)^{-2\rho_0}\, d\bar{n}_0 = 1.$$

The Haar measures dn_+ on N_+ and $dn_- := \theta(dn_+)$ on N_- are normalized so that $dn = dn_0\, dn_+$. On $N(\Omega)$ we consider the Haar measure of N_-.

The Lebesgue measure $d\lambda$ on $\mathfrak{a}^*$ and the Haar measure da on A are normalized so that the following formulas hold for the Euclidean Fourier transform,

$$\hat{f}(\lambda) = \int_A f(a)\, e^{-i\lambda(\log a)}\, da, \qquad f(a) = \int_{\mathfrak{a}^*} \hat{f}(\lambda)\, e^{i\lambda(\log a)}\, d\lambda.$$

Haar measures dq on compact groups Q are normalized by $\int_Q dq = 1$. The Haar measures dg on G and dh on H can be normalized so that the following formula holds for $f \in \mathscr{C}_c(HAN)$,

$$\int_G f(g)\, dg = \int_H \int_A \int_N f(han)\, a^{2\rho}\, dh\, da\, dn$$
$$= \int_K \int_A \int_N f(kan)\, a^{2\rho}\, dk\, da\, dn.$$

2. SPHERICAL FUNCTIONS

A function $\varphi\colon S^0 \to \mathbb{C}$ is said to be *H-bi-invariant* if $\varphi(h_1 x h_2) = \varphi(x)$ for all $x \in S^0$ and all $h_1, h_2 \in H$.

Let $\mathbb{D}(\mathcal{M})$ denote the (commutative) algebra of G-invariant differential operators on $\mathcal{M}$. A *spherical function* on $\mathcal{M}$ is a continuous H-bi-invariant function $\varphi\colon S^0 \to \mathbb{C}$ which is common eigendistribution of the all elements of $\mathbb{D}(\mathcal{M})$; that is, there is a character $\chi\colon \mathbb{D}(\mathcal{M}) \to \mathbb{C}$ so that in the sense of distributions $D\varphi = \chi(D)\,\varphi$ for all $D \in \mathbb{D}(\mathcal{M})$.

By H-bi-invariance, spherical functions are uniquely determined by their W_0-invariant restriction to $S^0 \cap A = \exp c^0_{\max}$. Observe that we have fixed no normalization for the spherical functions, so constant multiples of spherical functions are again spherical functions.

For each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we define

$$a_H(x)^\lambda = \begin{cases} 0 & \text{if } \ x \notin HAN \\ e^{\lambda(\log a_H(x))} & \text{if } \ x \in HAN. \end{cases}$$

Let $\mathscr{E}$ denote the set of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ for which the function $h \mapsto a_H(xh)^{\lambda-\rho}$ is integrable on H with respect to dh. For $\lambda \in \mathscr{E}$ the function φ_λ defined on S^0 by

$$\varphi_\lambda(x) = \int_H a_H(xh)^{\lambda-\rho}\, dh = \int_H a_H(\tau(x)^{-1}\, h)^{\lambda-\rho}\, dh \tag{3}$$

is a spherical function. It can be shown [8, Proposition 5.3] that the convergence of the integral defining $\varphi_\lambda(x)$ depends only on λ and not on x. Moreover $\mathscr{E}$ is the set of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ for which the integral

$$c_\Omega(\lambda) := \int_{N(\Omega)} a_H(\omega)^{-(\lambda+\rho)}\, d\omega \tag{4}$$

converges. The set $\mathscr{E}$ has been computed in [24] as

$$\mathscr{E} = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \operatorname{Re} \lambda(H_\alpha) < 2 - m_\alpha \ \forall \alpha \in \Delta_+\}.$$

c_Ω extends as a meromorphic function on $\mathfrak{a}^*_{\mathbb{C}}$ by means of the product formula [24, Theorem III.5]

$$c_\Omega(\lambda) = \kappa \prod_{\alpha \in \Delta_+} B\left(\frac{m_\alpha}{2},\ -\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1\right), \tag{5}$$

where $B(x, y) = \Gamma(x)\,\Gamma(y)/\Gamma(x+y)$ is the Beta function and κ is a positive constant depending only on the symmetric space.

In a few cases an explicit formula for the spherical functions is available.

I. *Rank-one Case* (cf. [8, Sect. 10]; see also Section 6). In this case $G = SO_0(1, n)$ and $H = SO_0(1, n-1)$ with $n \geqslant 2$. We identify $\mathfrak{a}_{\mathbb{C}}^*$ and $\mathbb{C}$ by setting $\alpha \equiv 1$, where α denotes the unique positive root. For $a_t = \exp(tY^0) \in S^0 \cap A \equiv \mathbb{R}_+ := (0, \infty)$,

$$\varphi_\lambda(a_t) = c_\Omega(\lambda)(2 \cosh t)^{\lambda-\rho}\, {}_2F_1\left(\frac{-\lambda+\rho}{2} \frac{-\lambda+\rho+1}{2}; 1-\lambda; \frac{1}{\cosh^2 t}\right), \tag{6}$$

where $\rho = (n-1)/2$ and where ${}_2F_1$ denotes Gauss hypergeometric function. Moreover

$$c_\Omega(\lambda) = 2^{2\rho-1}\Gamma(\rho)\, \frac{\Gamma(-\lambda-\rho+1)}{\Gamma(-\lambda+1)}.$$

II. *Complex Case* (cf. [8, Sect. 9; 19, p. 89] for the classification of G/H). In this case for $a \in S^0 \cap A$

$$\varphi_\lambda(a) = \kappa_1 c_\Omega(\lambda)\, \frac{\sum_{w \in W_0} \varepsilon(w)\, a^{w\lambda}}{\prod_{\alpha \in \Delta_0^+} (\lambda, \alpha) \prod_{\alpha \in \Delta^+} \sinh \alpha(\log a)}$$

and

$$c_\Omega(\lambda) = \frac{\kappa_2}{\prod_{\alpha \in \Delta_+} (\lambda, \alpha)}$$

for constants κ_1 and κ_2 depending only on the symmetric space.

III. $\mathbf{G} = SU(n, n)$, $\mathbf{H} = SL(n, \mathbb{C}) \times \mathbb{R}_+$ (cf. [30, Sect. 5.1]; see also [2, Theorem 3]). Identify the Lie algebra $\mathfrak{a}$ with the space of n-ples $t = (t_1/2, ..., t_n/2) \in \mathbb{R}^n$ and set a_t for the element of A corresponding to t. Define $\gamma_j(t) = t_j$. Then $\Delta_+ = \{\gamma_j\colon j = 1, ..., l\} \cup \{(\gamma_i + \gamma_j)/2 : 1 \leqslant i < j \leqslant n\}$ and $\Delta_0^+ = \{(\gamma_i - \gamma_j)/2\colon 1 \leqslant i < j \leqslant n\}$. In this situation

$$\varphi_\lambda(a) = \kappa_1\, \frac{1}{\prod_{i<j}(\lambda_i^2 - \lambda_j^2)}\, \frac{\det(Q_{-\lambda_j - 1/2}(\cosh t_j))}{\prod_{i<j}(\cosh t_j - \cosh t_i)},$$

Q_v being the Legendre function of second type, and

$$c_\Omega(\lambda) = \kappa_2 \prod_{j=1}^{n} \frac{\Gamma(-\lambda_j + 1/2)}{\Gamma(-\lambda_j + 1)} \prod_{1 \leqslant i < j \leqslant n} \frac{\Gamma((\lambda_i + \lambda_j)/2)}{\Gamma((\lambda_i + \lambda_j)/2 + 1)}$$
$$= \kappa_2 \prod_{j=1}^{n} \frac{\Gamma(-\lambda_j + 1/2)}{\Gamma(-\lambda_j + 1)} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{(\lambda_i + \lambda_j)}.$$

The constants κ_1 and κ_2 depend only on the symmetric space.

In the complex case $\varphi_\lambda c_\Omega^{-1}(\lambda)$ is an entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. The singularities of φ_λ in Example III are located along the hyperplanes $\lambda_j = 1/2 + m$ ($j = 1, ..., n$; $m \in \mathbb{N}$), and $\lambda_i + \lambda_j = 0$ ($1 \leqslant i < j \leqslant n$), cf. [30, Theorem 5.1]. In all these cases, the polar set of $\varphi_\lambda(a)$ is contained for all $a \in S^0 \cap A$ in the polar set of the numerator

$$n_\Omega(\lambda) := \prod_{\alpha \in \Delta_+} \Gamma\left(-\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1\right)$$

the function c_Ω. Examples II and III show that the inclusion is in general strict. On the other hand, the equality holds for example in the rank-one case with $\rho = 1/2$ (see also Section 6).

3. MEROMORPHIC EXTENSION OF THE SPHERICAL FUNCTIONS

The spherical functions φ_λ, initially defined on the set $\mathscr{E}$, extend meromorphically to $\mathfrak{a}_{\mathbb{C}}^*$. The meromorphic extendibility has been proved in [26, Sect. 5] by means of an expansion formula in terms of the Harish-Chandra's generalized hypergeometric functions. In this section we use an alternative method by means of Bernstein polynomials.

For every $j = 1, ..., l$, let $p_j(t) \in \mathbb{R}[t_1, ..., t_s]$ be a nonnegative polynomial with real coefficients. Set $p = \prod_{j=1}^{l} p_j$, and for $z = (z_1, ..., z_l) \in \mathbb{C}^l$ write $p^z = \prod_{j=1}^{l} p_j^{z_j}$. Define $\nabla = (\partial/\partial t_1, ..., \partial/\partial t_s)$. The following theorem states the existence of Bernstein polynomials (cf. [3, Theorem A.3.2; 25, p. 169]).

BERNSTEIN'S THEOREM. *Let* $\varepsilon = (1, ..., 1)$. *Then there is a polynomial* $b(z) \in \mathbb{C}[z_1, ..., z_l]$ *and a polynomial* $Q(z, t, \nabla)$ *so that*

$$b(z+\varepsilon)\, p(t)^z = Q(z+\varepsilon, t, \nabla)\, p(t)^{z+\varepsilon} \tag{7}$$

whenever the right-hand side is defined.

Observe that $p(t)^z$ is a C^∞ function on $\{t \in \mathbb{R}^s : p(t) > 0\}$. It is everywhere a C^k function provided $\operatorname{Re} z_j > k$ for all $j = 1, ..., l$.

A more precise description of the form of the polynomial b has been obtained by Sabbah [28, Proposition 1.2] and by Gyoja [11, Theorem on p. 399]:

THEOREM 3.1. *There is an integer $N > 0$ and, for every $i = 1, ..., N$, there is an l-ple $f_i = (a_{i1}, ..., a_{il}) \in \mathbb{N}_0^l$ and a positive number $a_i \in \mathbb{Q}$, so that the polynomial $b(z) \in \mathbb{C}[z_1, ..., z_l]$ in Bernstein's theorem can be written as a product of N affine forms*:

$$b(z+\varepsilon) = \prod_{i=1}^{N} (\langle f_i z \rangle + a_i).$$

When $l = 1$ we can select, up to constant multiples, a unique polynomial b satisfying (7) by the requirement of having minimal degree. In the higher-rank case $l > 1$ this is not necessarily the case. We call *a Berstein polynomial* every polynomial which satisfies Bernstein's theorem and is a product of affine forms as in Theorem 3.1. We refer to [12, 21] for cases where this polynomial can be in fact uniquely selected. In our situation we will prove that every possible Bernstein polynomial is divided by a certain polynomial B. Therefore proving the conjecture that B is a Bernstein polymonial will also imply the existence of a minimal Bernstein polymonial. This will be *the* Bernstein polynomial for the spherical functions.

The meromorphic continuation of the spherical function φ_λ will be obtained by writing (3) as an integral over $N(\Omega) \subset N_-$, and then applying Bernstein's theorem to the polynomial function $a_H(\omega)^\delta$ on $N(\Omega)$. Here $\delta \in \mathfrak{a}_{\mathbb{C}}^*$ is the dominant integral weight which plays the role of ε.

We will need the following lemmas (similar ideas can be found in [17, Lemma 5.2, p. 251]).

LEMMA 3.1. *Let G be a Lie group and let A, B, C be analytic subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$, respectively. Suppose that the sum $\mathfrak{a} + \mathfrak{b} + \mathfrak{c}$ is direct and that $\mathfrak{b}$ normalizes $\mathfrak{c}$. Then the mapping Φ: $A \times B \times C \to G$ defined by $\Phi(a, b, c) = abc$ is an immersion.*

Proof. Write L_x and R_x for the left and right translations by the element $x \in G$. Identify the tangent space $T_{(a,b,c)}(A \times B \times C)$ of $A \times B \times C$ at (a, b, c) with $(dL_a)_e(\mathfrak{a}) \times (dL_b)_e(\mathfrak{b}) \times (dL_c)_e(\mathfrak{c})$, where $(dL_x)_e$ is the differential of L_x at the identity element $e \in G$. Since for all $a \in A$ and $c \in C$

$$\Phi \circ (L_a \times \mathrm{id} \times \mathrm{id}) = L_a \circ \Phi, \qquad \Phi \circ (\mathrm{id} \times \mathrm{id} \times R_c) = R_c \circ \Phi,$$

it is enough to prove the injectivity of $d\Phi_{(e,b,e)}$. Let $X \in \mathfrak{a}$, $Y \in \mathfrak{b}$ and $Z \in \mathfrak{c}$. Then

$$d\Phi_{(e,b,e)}(X, (dL_b)_e\, Y, Z) = (dR_b)_e\,(X + \mathrm{Ad}(b)\; Y + \mathrm{Ad}(b)\; Z) = 0$$

implies $X = Y = Z = 0$ because $\mathrm{Ad}(b)\; Y \in \mathfrak{b}$, $\mathrm{Ad}(b)\; Z \in \mathfrak{c}$ and the sum $\mathfrak{a} + \mathfrak{b} + \mathfrak{c}$ is direct. ∎

LEMMA 3.2. (1) *The map* $(a, k, \omega) \mapsto ak\omega$ *is an injective immersion of* $A \times K(0) \times N_-$ *into* G.

(2) $(S^0 \cap A)\, K(0)\, N(\Omega)$ *is an immersed submanifold of* G *contained in* HAN.

(3) *Let* $\overline{N(\Omega)}$ *denote the closure of* $N(\Omega)$. *Then, for every* $a \in S^0 \cap A$, *the set* $aK(0)\, \overline{N(\Omega)}$ *is compact in* HAN.

Proof. As $\mathfrak{k}(0) = \mathfrak{h}_k \subset \mathfrak{g}(0)$ normalizes $\mathfrak{n}_-$, part (1) follows immediately from the previous lemma and the Iwasawa decomposition of G corresponding to the choice of $-\Delta^+$ as system of positive roots. Since $S^0 \cap A$ and $N(\Omega)$ are open submanifolds of A and N_-, respectively, $(S^0 \cap A)\, K(0)\, N(\Omega)$ is an immersed submanifold of G by part (1). The description of S as compression semigroup (cf., e.g., [8, Theorem 4.2]) gives $S^0 = \{x \in G : x\overline{HAN} \subset HAN\}$. Since

$$(S^0 \cap A)\, K(0) \subset S^0 = H(S^0 \cap A)\, H,$$

we obtain

$$(S^0 \cap A)\, K(0)\, \overline{N(\Omega)} \subset S^0\, \overline{HAN} \subset HAN.$$

This proves parts (2) and (3) because of the compactness of $K(0)$ and $\overline{N(\Omega)}$. ∎

LEMMA 3.3. (1) *The map* $(a, k, \omega) \mapsto \log a_H(ak\omega)$ *is well defined and continuous from* $(S^0 \cap A) \times K(0) \times \overline{N(\Omega)}$ *into* $\mathfrak{a}$, *and real analytic on* $(S^0 \cap A) \times K(0) \times N(\Omega)$.

(2) *The map* $(\lambda, a, k, \omega) \mapsto a_H(ak\omega)^\lambda$ *is well defined and continuous from* $\mathfrak{a}_{\mathbb{C}}^* \times (S^0 \cap A) \times K(0) \times \overline{N(\Omega)}$ *into* $\mathbb{C}$. *It is entire as a function of* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *and real analytic as a function of* $(a, k, \omega) \in (S^0 \cap A) \times K(0) \times N(\Omega)$.

(3) *For every compact subsets* Q *of* $S^0 \cap A$ *and* Λ *of* $\mathfrak{a}_{\mathbb{C}}^*$ *there exist constants* C_Q, $C_{Q,\Lambda}$ *so that*

$$|\log a_H(ak\omega)| \leqslant C_Q, \qquad a \in Q, \quad k \in K(0), \quad \omega \in \overline{N(\Omega)}$$

$$|a_H(ak\omega)^\lambda| \leqslant C_{Q,\Lambda} \qquad \lambda \in \Lambda, \quad a \in Q, \quad k \in K(0), \quad \omega \in \overline{N(\Omega)}.$$

(4) *Let D be a differential operator on $A \times K(0) \times N_-$ with continuous coefficients. For every compact subsets Q of $S^0 \cap A$ and Λ of $\mathfrak{a}_{\mathbb{C}}^*$ there exist constants $C_{D,Q}$, $C_{D,Q,\Lambda}$ so that*

$$|D \log a_H(ak\omega)| \leqslant C_{D,Q}, \qquad a \in Q, \quad k \in K(0), \quad \omega \in \overline{N(\Omega)}$$

$$|Da_H(ak\omega)^{\lambda}| \leqslant C_{D,Q,\Lambda} \qquad \lambda \in \Lambda, \quad a \in Q, \quad k \in K(0), \quad \omega \in \overline{N(\Omega)}$$

Proof. For part (3), apply the chain rule and observe that $\log a_H$ is real analytic on HAN (which contains the compact set $QK(0)\,\overline{N(\Omega)}$) and that the multiplication map is real analytic on $G \times G \times G$ (which contains the compact set $Q \times K(0) \times \overline{N(\Omega)}$).

Because of Lemma 4.5(3) in [26], for $a \in S^0 \cap A$ and $\lambda \in \mathscr{E}$

$$\varphi_\lambda(a) = \int_{\bar{N} \cap HAN} a_H(a\bar{n})^{\lambda-\rho}\, a_H(\bar{n})^{-(\lambda+\rho)}\, d\bar{n}$$

$$= \int_{N(\Omega)} \int_{\bar{N}_0} a_H(a\omega\bar{n}_k)^{\lambda-\rho}\, a_H(\omega\bar{n}_k)^{-(\lambda+\rho)}\, d\bar{n}_k\, d\omega.$$

Considering the Iwasawa decomposition

$$\bar{n}_0 = k(\bar{n}_0)\, a(\bar{n}_0)\, n(\bar{n}_0) \in G(0) = K(0)AN_0$$

and observing that $a_H(a\omega\bar{n}_0) = a_H(a\omega k(\bar{n}_0))a(\bar{n}_0)$, we have

$$\varphi_\lambda(a) = \int_{N(\Omega)} \int_{\bar{N}_0} a_H(a\omega k(\bar{n}_0))^{\lambda-\rho}\, a_H(\omega k(\bar{n}_0))^{-(\lambda+\rho)}\, a(\bar{n}_0)^{-2\rho}\, d\bar{n}_0\, d\omega$$

$$= \int_{N(\Omega)} \int_{\bar{n}_0} a_H(ak(\bar{n}_0)\omega)^{\lambda-\rho}\, a_H(\omega)^{-(\lambda+\rho)}\, a(\bar{n}_0)^{-2\rho}\, d\bar{n}_0\, d\omega.$$

The last equality has been obtained by substituting $k(\bar{n}_0)\,\omega k(\bar{n}_0)^{-1}$ for ω and by noticing that $a_H(k(\bar{n}_0)\,\omega) = a_H(\omega)$ since $k(\bar{n}_0) \in H$. The group M normalizes $\bar{N}_0$ and $d(m^{-1}\bar{n}_0 m) = d\bar{n}_0$ for every $m \in M$. By the same argument we get $k(m^{-1}\bar{n}_0 m) = m^{-1}k(\bar{n}_0)\, m$, $a(m^{-1}\bar{n}_0 m) = a(\bar{n}_0)$, and $a_H(am^{-1}k(\bar{n}_0)\, m\omega) = a_H(ak(\bar{n}_0)\, m\omega)$. The substitution of $m^{-1}\bar{n}_0 m$ for $\bar{n}_0$ therefore yields

$$\varphi_\lambda(a) = \int_{N(\Omega)} \left[\int_{\bar{n}_0} a_H(ak(\bar{n}_0)m\omega)^{\lambda-\rho}\, a(\bar{n}_0)^{-2\rho}\, d\bar{n}_0 \right] a_H(\omega)^{-(\lambda+\rho)}\, d\omega. \tag{8}$$

By Lemma 3.3, there is a constant C (depending on a and λ) so that

$$a_H(ak(\bar{n}_0)m\omega)^{\mathrm{Re}\,\lambda-\rho} < C$$

for all $\bar{n}_0 \in \bar{N}_0$, $m \in M \subset K(0)$, $\omega \in N(\Omega)$. Since $\lambda \in \mathscr{E}$, Fubini's theorem applies when both sides of (8) are integrated over M. The integral formula

$$\int_{K(0)} f(k)\,dk = \int_{\bar{N}_0 \times M} f(k(\bar{n}_0)m)\, a(\bar{n}_0)^{-2\rho}\, d\bar{n}_0\, dm$$

(cf. [9, Formula (2.5.42), p. 83]) then gives

$$\begin{aligned}\varphi_\lambda(a) &= \int_{N(\Omega)} \left[\int_{\bar{N}_0 \times M} a_H(ak(\bar{n}_0)m\omega)^{\lambda-\rho}\, a(\bar{n}_0)^{-2\rho}\, d\bar{n}_0 \right] a_H(\omega)^{-(\lambda+\rho)}\, d\omega \\ &= \int_{N(\Omega)} \left[\int_{K(0)} a_H(ak\omega)^{\lambda-\rho}\, dk \right] a_H(\omega)^{-(\lambda+\rho)}\, d\omega. \qquad (9)\end{aligned}$$

Let $\mathbb{N}_0$ denote the set of nonnegative integers. According to Helgason's theorem [17, Theorem 4.1, p. 535, and Corollary 4.3, p. 538], the highest weights of the irreducible finite dimensional K-spherical representations of G consist exactly of the $\lambda \in \mathfrak{a}^*$ satisfying $\frac{(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{N}_0$ for all $\alpha \in \Delta^+$. Since $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ are conjugate for NCC spaces (and, more generally, for K_ε-spaces in the sense of Oshima–Sekiguchi) a representation of G is H-spherical if and only if it is K-spherical. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots in Δ^+. Define $\mu_1, \ldots, \mu_l \in \mathfrak{a}^*$ by the conditions

$$(\mu_i, \alpha_j) = \delta_{ij}(\alpha_i, \alpha_i), \qquad i, j = 1, \ldots, l,$$

where δ_{ij} is Kronecker's symbol. Then the set of equivalence classes of finite dimensional irreducible H-spherical representations of G is parametrized by

$$\Lambda_K := \bigoplus_{i=1}^{l} \mathbb{N}_0 \mu_i.$$

For every $j = 1, \ldots, l$, let π_j be the H-spherical finite dimensional irreducible representation of G with highest weight μ_j. We choose an inner product $\langle \cdot, \cdot \rangle$ in the space of π_j such that $\pi_j(x)^* = \pi_j(\theta(x)^{-1})$ for all $x \in G$. Let v_j and u_j be respectively a highest weight vector and a H-fixed vector for π_j. The normalization of u_j and v_j can be fixed so that $\langle v_j, u_j \rangle = 1$.

Fix an orthogonal basis $\{Y_1, ..., Y_s\}$ for $\mathfrak{n}_-$ so that $Y_h = X_{-h}$ for $h = 1, ..., r$. Recalling that $|X_{-h}|^2 = 2/(\gamma_1, \gamma_1)$ for all $h = 1, ..., r$ (cf. [19, p. 96]), we can also request $|Y_h|$ to be constant for all $h = 1, ..., s$. We identify N_- with $\mathbb{R}^s$ via the diffeomorphism

$$T: t = (t_1, ..., t_s) \mapsto \bar{n}_t := \exp\left(\sum_{h=1}^{s} t_h Y_h\right).$$

A function $f: N_- \to \mathbb{C}$ is called a *polynomial on* N_- if $X \mapsto f(\exp X)$ is a polynomial function on $\mathfrak{n}_-$. Since N_- is a nilpotent group, the matrix coefficients of the representations π_j are polynomials on N_- (cf., e.g., [4, p. 13]). We can therefore define nonnegative polynomials $p_j, p \in \mathbb{R}[t_1, ..., t_s]$ by

$$p_j(t) := |\langle \pi_j(\bar{n}_t) v_j, u_j \rangle|^2, \qquad j = 1, ..., l, \tag{10}$$

$$p(t) := \prod_{j=1}^{l} p_j(t). \tag{11}$$

We will consider the restriction of these polynomials to $T^{-1}(N(\Omega))$. We write ω_t for $\bar{n}_t \in N(\Omega)$ and set $B(\sqrt{r}) := \{t = (t_1, ..., t_s) \in \mathbb{R}^s : \|t\|^2 := \sum_{h=1}^{s} t_h^2 \leqslant r\}$.

LEMMA 3.4. *$T^{-1}(N(\Omega)) \subset B(\sqrt{r})$ is a bounded open subset of $\mathbb{R}^s$.*

Proof. $T^{-1}(N(\Omega))$ is open in $\mathbb{R}^s$ since $N(\Omega)$ is open in N_-. Recall the description (1) of $N(\Omega)$. Because of the choice of the basis in $\mathfrak{n}_-$, the map T identifies $\{\sum_{j=1}^{r} x_j X_{-j}: -1 < x_j < 1, j = 1, ..., r\}$ with $(-1, 1)^r \times \{0\}^{s-r} \subset \mathbb{R}^s$, and the $\mathrm{Ad}(K \cap H)$-action corresponds to the action of a subgroup of the orthogonal group $O(s)$. ∎

Identify $\mathbb{C}^l$ and $\mathfrak{a}_{\mathbb{C}}^*$ via

$$z = (z_1, ..., z_l) \mapsto 2 \sum_{j=1}^{l} z_j \mu_j.$$

In particular $\varepsilon := (1, ..., 1)$ corresponds to

$$\delta := 2 \sum_{j=1}^{l} \mu_j. \tag{12}$$

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and set

$$z_j(\lambda) := -\frac{(\lambda + \rho \alpha_j)}{2(\alpha_j, \alpha_j)} = -\frac{(\lambda + \rho)(H_{\alpha_j})}{4}, \qquad z(\lambda) = (z_1(\lambda), ..., z_l(\lambda)). \tag{13}$$

Then $z(\lambda)$ corresponds to $2\sum_{j=1}^{l} z_j(\lambda)\,\mu_j = -(\lambda+\rho)$ and $z(\lambda-\delta)$ corresponds to $-(\lambda-\delta+\rho)$, so $z(\lambda-\delta)=z(\lambda)+\varepsilon$. Observe that H is θ-stable, $\pi_j(N)\,v_j=v_j$ and $\pi_j(H)\,u_j=u_j$. Hence we have for every $\omega_t\in N(\Omega)$

$$p(t)^{z(\lambda)}=\prod_{j=1}^{l} p_j(t)^{z_j(\lambda)}=\prod_{j=1}^{l} a_H(\omega_t)^{2z_j(\lambda)\,\mu_j}\,|\langle v_j, u_j\rangle|^{2z_j(\lambda)}$$
$$=a_H(\omega_t)^{-(\lambda+\rho)}. \tag{14}$$

Notice that

$$p_j(t)=a_H(\omega_t)^{2\mu_j} \qquad \text{and} \qquad p(t)=a_H(\omega_t)^{\delta}. \tag{14}$$

We have $\delta\in c^0_{\max}$ because $\delta(H_\alpha)>0$ for all $\alpha\in\Delta^+$. It follows that $\mathfrak{a}^*_{\mathbb{C}}=\bigcup_{m\in\mathbb{N}}(\mathscr{E}+m\delta)$, with $\mathscr{E}+m\delta\subset\mathscr{E}+m'\delta$ if $m\leqslant m'$.

Applying Bernstein's theorem to $p(t)$ on $T^{-1}(N(\Omega))$, we obtain the following Lemma.

LEMMA 3.5. *There exists a polynomial $b(\lambda):=b(z(\lambda))$ in the variables $z_j(\lambda)$ and a polynomial $Q(\lambda, t, \nabla):=Q(z(\lambda), t, \nabla)$ in the variables $z_j(\lambda)$, t_j and $\partial/\partial t_j$, so that on $N(\Omega)$*

$$b(\lambda-\delta)\,a_H(\omega_t)^{-(\lambda+\rho)}=Q(\lambda-\delta, t, \nabla)\,a_H(\omega_t)^{-(\lambda-\delta+\rho)}. \tag{15}$$

Moreover, there is an integer $N>0$ so that $b(\lambda-\delta)$ can be written as

$$b(\lambda-\delta)=\prod_{i=1}^{N}\left(-\sum_{j=1}^{l} a_{ij}\,\frac{(\lambda+\rho)(H_{\alpha_j})}{4}+a_i\right), \tag{16}$$

where $a_{ij}\in\mathbb{N}_0$ and a_i are positive rationals.

The idea for the meromorphic continuation of φ_λ is the following. Let D^* denote the formal adjoint of a differential operator D. By inserting (15) in (9), we get

$$b(\lambda-\delta)\,\varphi_\lambda(a)$$
$$=\int_{N(\Omega)}\left[\int_{K(0)} a_H(ak\omega_t)^{\lambda-\rho}\,dk\right] Q(\lambda-\delta, t, \nabla)\,a_H(\omega_t)^{-(\lambda-\delta+\rho)}\,d\omega_t$$
$$=\int_{N(\Omega)} Q(\lambda-\delta, t, \nabla)^*\left[\int_{K(0)} a_H(ak\omega_t)^{\lambda-\rho}\,dk\right] a_H(\omega_t)^{-(\lambda-\delta+\rho)}\,d\omega_t, \tag{17}$$

provided λ is chosen in a suitable open subdomain of $\mathscr{E}$ in which the function $a_H(\omega_t)^{-(\lambda+\rho)}$ vanishes on the boundary of $N(\Omega)$. The right hand side of (17) is holomorphic in $\mathscr{E}+\delta$. It provides a holomorphic extension of the left hand side of (17), and hence a meromorphic extension of φ_λ on this domain. The procedure can then be iterated to get a meromorphic extension of φ_λ on the whole $\mathfrak{a}^*_{\mathbb{C}}$. The precise statement is given in Theorem 3.2 below. We first need some lemmas.

Let $\mathbb{N}$ denote the set of positive integers. For $m \in \mathbb{N}$, the composition $D_1 \circ \cdots \circ D_m$ of differential operators $D_1, ..., D_m$ will be shortly written as $\prod_{h=1}^m D_h$.

LEMMA 3.6. (1) *$Q(\lambda - \delta, t, \nabla)^* = P(\lambda - \delta, t, \nabla)$, where P is a polynomial.*

(2) *For every $m \in \mathbb{N}$, define*

$$P_m(\lambda, t, \nabla) := P(\lambda - m\delta, t, \nabla) \circ \cdots \circ P(\lambda - 2\delta, t, \nabla) \circ P(\lambda - \delta, t, \nabla). \tag{18}$$

Then

$$\left[\prod_{h=1}^m Q(\lambda - h\delta, t, \nabla) \right]^* = P_m(\lambda, t, \nabla).$$

Define

$$I_m(\lambda, a) := \int_{N(\Omega)} F_m(\lambda, t, a)\, a_H(\omega_t)^{-(\lambda + \rho - m\delta)}\, d\omega_t \tag{19}$$

with

$$F_m(\lambda, a, t) := \int_{K(0)} P_m(\lambda, t, \nabla)\, a_H(ak\omega_t)^{\lambda - \rho}\, dk.$$

PROPOSITION 3.1. *$I_m(\lambda, a)$ is well defined on $(\mathscr{E} + m\delta) \times (S^0 \cap A)$. It is holomorphic in $\lambda \in \mathscr{E} + m\delta$ and real analytic in $a \in S^0 \cap A$.*

Proof. Let d denote the total degree of P_m. If $\mathbf{q} = (q_1, ..., q_s)$ is a multi-index, write $|\mathbf{q}| := \sum_{h=1}^s q_h$ and set

$$\left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} := \frac{\partial^{q_1}}{\partial t_1^{q_1}} \cdots \frac{\partial^{q_s}}{\partial t_s^{q_s}}.$$

There exists a polynomial $\tilde{P}_m(\lambda, t, \{s(t, \mathbf{q})\})$—polynomial in the variables $z_j(\lambda)$, t_h and $s_j(t, \mathbf{q}) := (\frac{\partial}{\partial t})^{\mathbf{q}} \mu_j(\log a_H(ak\omega_t))$ for $j = 1, ..., l$, $h = 1, ..., s$ and for the multi-indices $\mathbf{q}$ with $|\mathbf{q}| \leqslant d$—so that

$$P_m(\lambda, t, \nabla)\, a_H(ak\omega_t)^{\lambda - \rho} = \tilde{P}_m(\lambda, t, \{s(t, \mathbf{q})\})\, a_H(ak\omega_t)^{\lambda - \rho}. \tag{20}$$

Let $\lambda_o \in \mathscr{E} + m\delta$. There exists $\varepsilon_0 > 0$ so that $\lambda_0 + \varepsilon_0 \delta \in \mathscr{E} + m\delta$. Write

$$\Upsilon := \{ \nu \in \mathfrak{a}_{\mathbb{C}}^* : \operatorname{Re} \nu(H_\alpha) < \operatorname{Re} \lambda_0(H_\alpha) + \varepsilon_0 \delta(H_\alpha) \ \forall \alpha \in \Delta_+ \}.$$

Υ is an open neighborhood of λ_0 entirely contained in $\mathscr{E} + m\delta$. Let $a_0 \in S^0 \cap A$. Choose open neighborhoods Λ of λ_0 and U of a_0 with compact closures $\bar{\Lambda}$ and $\bar{U}$ so that $\bar{\Lambda} \subset \Upsilon$ and $\bar{U} \subset S^0 \cap A$. The choice of Λ and Lemmas 3.3, part (3), and 3.4 imply the existence of a constant C_m (depending on Λ and U) for which

$$\begin{aligned} &|[P_m(\lambda, t, \nabla)\, a_H(ak\omega_t)^{\lambda - \rho}]\, a_H(\omega_t)^{-(\lambda + \rho - m\delta)}| \\ &\quad \leqslant C_m a_H(\omega_t)^{-(\operatorname{Re} \lambda_0 + \varepsilon_0 \delta + \rho - m\delta)} \end{aligned} \tag{21}$$

for all $\lambda \in \Lambda$, $\omega_t \in N(\Omega)$, $a \in U$, $k \in K(0)$ (recall from [19, p. 152] that $\log a_H(\omega_t) \in -c_{\min}$ for $\omega_t \in N(\Omega)$). We have $\lambda_0 + \varepsilon_o \delta \in \mathscr{E} + m\delta$, so $a_H(\omega)^{-(\operatorname{Re} \lambda_0 + \varepsilon_0 \delta + \rho - m\delta)}$ is integrable on $N(\Omega)$. The function

$$(\lambda, a) \mapsto [P_m(\lambda, t, \nabla)\, a_H(ak\omega_t)^{\lambda - \rho}]\, a_H(\omega_t)^{-(\lambda + \rho - m\delta)}$$

is entire in $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \equiv \mathbb{C}^l$ and real analytic in $a \in S^0 \cap A$. The estimate (21) and the dominated convergence theorem prove that the integral defining $I_m(\lambda, a)$ converges to a continuous function on $\Lambda \times U$. Morera's theorem applied to each variable separately shows moreover that $I_m(\lambda, a_0)$ is holomorphic in $\lambda \in \Lambda$.

Let D be a differential operator on A. Formula (20) and Lemmas 3.3 and 3.4 give for some constant $C_{m, D}$ (depending also on U)

$$\begin{aligned} &|D[P_m(\lambda_0, t, \nabla)\, a_H(ak\omega_t)^{\lambda_0 - \rho}]\, a_H(\omega_t)^{-(\lambda_0 + \rho - m\delta)}| \\ &\quad \leqslant C_{m, D} a_H(\omega_t)^{-(\operatorname{Re} \lambda_0 + \varepsilon_0 \delta + \rho - m\delta)} \end{aligned}$$

for all $a \in U$, $\omega_t \in N(\Omega)$, $k \in K(0)$. This allows us to differentiate $I_m(\lambda_0, a)$ under integral sign. Thus $I_m(\lambda_0, a)$ is a C^∞ function of $a \in U$.

Let $x_1, ..., x_l$ be the coordinates in A. To prove that $I_m(\lambda_0, a)$ is real analytic near $a_0 \equiv (x_1^0, ..., x_l^0)$ it suffices to determine a constant C_0 so that for every multi-index $\mathbf{q} = (q_1, ..., q_l)$

$$\left| \left(\frac{\partial}{\partial x} \right)^{\mathbf{q}} I_m(\lambda_0, x_1^0, ..., x_l^0) \right| \leqslant C_0^{|\mathbf{q}|}\, |\mathbf{q}|\,! \tag{22}$$

(cf. [23, Lemma 3.3.5 and Theorem 3.3.3]). Since

$$P_m(\lambda_0, t, \nabla)\, a_H(ak\omega_t)^{\lambda_0 - \rho}$$

is real analytic in a, it has a power series expansion in some polydisc $\Delta(x^0, r^0) := \{x \in A : |x_j - x_j^0| < r^0 \ \forall j\} \subset U$. Applying Cauchy's Estimates to the complexification of the power series, one obtains for every multi-index $\mathbf{q}$ and every $a \equiv x \in \Delta(x^0, r^0)$

$$\left|\left(\frac{\partial}{\partial x}\right)^{\mathbf{q}} P_m(\lambda_0, t, \nabla)\, a_H(ak\omega_t)^{\lambda_0-\rho}\right| \leqslant M\, \frac{\mathbf{q}!}{(r^0)^{|\mathbf{q}|}},$$

where

$$M := \max_{\substack{a \in \bar{\Delta}(x^0, r^0) \\ t \in T^{-1}(N(\Omega))}} |P_m(\lambda_0, t, \nabla)\, a_H(ak\omega_t)^{\lambda_0-\rho}|.$$

Differentiation under integral sign yields (22) with

$$C_0 = \frac{1}{r^0} \max\left\{M \int_{N(\Omega)} a_H(\omega)^{-(\mathrm{Re}\,\lambda_0+\rho-m\delta)}\, d\omega,\ 1\right\}.$$

The result follows since (λ_0, a_0) has been chosen arbitrarily in $(\mathscr{E}+m\delta)\times(S^0 \cap A)$. ∎

Lemma 3.7. *Suppose* $\mathrm{Re}\,\lambda(H_\alpha)+\rho(H_\alpha)<0$ *for all* $\alpha \in \Delta_+$. *Let* $\omega_0 \in \partial N(\Omega)$. *Then*

$$\lim_{\substack{\omega \in N(\Omega) \\ \omega \to \omega_0}} a_H(\omega)^{-(\lambda+\rho)} = 0.$$

Proof. Suppose $\omega_n \in N(\Omega)$ converges to ω_0 as $n \to \infty$. Then $Z_n := \log \omega_n$ converges to $Z_0 := \log \omega_0$. Write $Z_n := \mathrm{Ad}(k_n) \sum x_{nj} X_{-j}$ with $k_n \in K \cap H$ and $x_{nl} \in (-1, 1)$. Since $K \cap H$ and $[-1, 1]$ are compact, by possibly passing to subsequences, we can assume that k_n converges to $k \in K \cap H$ and x_{nj} converges to $x_j \in [-1, 1]$ for all $j = 1, ..., l$. Then $Z_0 = \mathrm{Ad}(k) \sum x_j X_{-j}$. Since $Z_0 \notin \Omega$, it follows that for at least one j we have $|x_j| = 1$. Arguing as in the proof of Theorem 3.10 in [26], one can estimate

$$0 \leqslant a_H(\omega_n)^{-(\mathrm{Re}\,\lambda+\rho)} \leqslant \prod_{j=1}^{r} (1-x_{nj}^2)^{-C/2},$$

with $\lim_{n\to\infty} \prod_{j=1}^{r} (1-x_{nj}^2)^{-C/2} = 0$ because $C<0$. ∎

Definition 3.1. We say that $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is *b-regular* if it does not belong to the set of hyperplanes $\bigcup_{h=1}^{\infty} \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : b(\lambda - h\delta) = 0\}$.

Notice that the set of b-regular elements is open and dense in $\mathfrak{a}_{\mathbb{C}}^*$. Recall the differential operator $P_m(\lambda, t, \nabla)$ in Lemma 3.6 and

$$I_m(\lambda, a) = \int_{N(\Omega)} F_m(\lambda, t, a)\, a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t$$

as in (19).

THEOREM 3.2. (1) *Let $m \in \mathbb{N}$. Define the polynomial b_m by*

$$b_m(\lambda) := \prod_{h=1}^{m} b(\lambda - h\delta). \tag{23}$$

Then for all $\lambda \in \mathscr{E}$, we have

$$b_m(\lambda)\, \varphi_\lambda(a) = I_m(\lambda, a), \qquad a \in S^0 \cap A. \tag{24}$$

Hence $I_m(\lambda, a)/b_m(\lambda)$ is a meromorphic extension of $\varphi_\lambda(a)$ to $\mathscr{E} + m\delta$.

(2) *Set $I_0(\lambda, a) := \varphi_\lambda(a)$. Then for all $m \in \mathbb{N}$ and $a \in S^0 \cap A$,*

$$b(\lambda - m\delta)\, I_{m-1}(\lambda, a) = I_m(\lambda, a), \qquad \lambda \in \mathscr{E} + (m-1)\,\delta. \tag{25}$$

(3) *For every fixed $a \in S^0 \cap A$, the function $\lambda \mapsto \varphi_\lambda(a)$ extends to a meromorphic function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, which we still denote $\varphi_\lambda(a)$. It satisfies the functional equations* (24) *for all $m \in \mathbb{N}$. Its poles are contained in the locally finite union of hyperplanes which are the zero set of the polynomials $b(\lambda - h\delta)$, $h \in \mathbb{N}$.*

For every pole λ_0 of $\varphi_\lambda(a)$, let m_0 be the smallest positive integer such that $\lambda_0 \in \mathscr{E} + m_0\delta$. The order of the pole λ_0 is $\leqslant$ the order of λ_0 as a root of $b_{m_0}(\lambda)$.

For every b-regular $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\varphi_\lambda(a)$ is a real analytic function of $a \in S^0 \cap A$.

Proof. Let $g(t) \in \mathbb{R}[t_1, ..., t_s]$ be a nonnegative polynomial and let $w \in \mathbb{C}$. Then $t \mapsto g(t)^w$ is smooth on $\{t\colon g(t) > 0\}$. Suppose that $g(t_0) = 0$. Let $\mathbf{q} = (q_1, ..., q_s)$ be a nonzero multi-index. If $\operatorname{Re} w > |\mathbf{q}|$, then

$$\lim_{t \to t_0} \left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} g(t)^w = 0.$$

Hence q is C^k near t_0 if $\operatorname{Re} w > k$. To prove part 1, apply the above results to $g(t) := p_j(t)$ as in (10). It is equal to $a_H(\omega_t)^{2\mu_j}$, and hence positive, on $T^{-1}(N(\Omega))$. Let p be as in (11), and let Q be the polynomial from Bernstein's theorem applied to p. Let $m \in \mathbb{N}$ be fixed. Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfies

$$(\operatorname{Re} \lambda + \rho)(H_{\alpha_j}) < -4m \deg Q \qquad \text{for all} \quad j = 1, ..., l. \tag{26}$$

This implies in particular $(\mathrm{Re}\,\lambda - h\delta + \rho)(H_\alpha) \leqslant (\mathrm{Re}\,\lambda + \rho)(H_\alpha) < 0$ for all $\alpha \in \Delta_+$ and all $h = 0, 1, ..., m$. Lemma 3.7 then ensures that

$$\lim_{\omega_t \to \omega_0} a_H(\omega_t)^{-(\lambda - h\delta + \rho)} = 0$$

for every $\omega_0 \in \partial N(\Omega)$ and all $h = 0, 1, ..., m$. Moreover, because of the above argument,

$$\lim_{\omega_t \to \omega_0} \left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} a_H(\omega_t)^{2z_j(\lambda - h\delta)\,\mu_j} = 0$$

for all $|\mathbf{q}| \leqslant m \deg Q$ and all $j = 1, ..., l$, $h = 0, 1, ..., m$. Thus

$$\lim_{\omega_t \to \omega_0} \left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} a_H(\omega_t)^{-(\lambda - h\delta + \rho)} = 0$$

for all $|\mathbf{q}| \leqslant m \deg Q$ and all $h = 0, 1, ..., m$. It follows that, when λ satisfies (26),

$$\begin{aligned}\int_{N(\Omega)} f(\omega_t) &\left[\left(\prod_{h=1}^{m} Q(\lambda - h\delta, t, \nabla)\right) a_H(\omega_t)^{-(\lambda - m\delta + \rho)}\right] d\omega_t \\ &= \int_{N(\Omega)} \left[\left(\prod_{h=1}^{m} Q(\lambda - h\delta, t, \nabla)\right)^{*} f(\omega_t)\right] a_H(\omega_t)^{-(\lambda - m\delta + \rho)}\, d\omega_t \end{aligned} \tag{27}$$

for every smooth function f on $N(\Omega)$ for which the integrals on both sides of the equality converge. Iterating Eq. (15) we obtain

$$\left[\prod_{h=1}^{m} b(\lambda - h\delta)\right] a_H(\omega_t)^{-(\lambda + \rho)} = \left[\prod_{h=1}^{m} Q(\lambda - h\delta, t, \nabla)\right] a_H(\omega_t)^{-(\lambda - m\delta + \rho)}. \tag{28}$$

Observe that Lemma 3.3 ensures that the function

$$k \mapsto \int_{K(0)} a_H(ak\omega_t)^{\lambda - \rho}\, dk$$

can be differentiated under integral sign. Inserting (28) in (27), we therefore obtain on the set of all $\lambda \in \mathscr{E}$ which also satisfy the relations (26)

$$\left[\prod_{h=1}^{m} b(\lambda - h\delta)\right] \varphi_\lambda(a) = I_m(\lambda, a). \tag{29}$$

Morera's theorem ensures that for every $a \in S^0 \cap A$, $\varphi_\lambda(a)$ given by the integral (3) is a holomorphic function of $\lambda \in \mathscr{E}$. Thus the equality (29) extends to all of $\mathscr{E}$.

Part (2) is an immediate consequence of the fact that both sides are equal to $b_m(\lambda)\,\varphi_\lambda(a)$ on $\mathscr{E}$ and holomorphic on $\mathscr{E}+(m-1)\,\delta$.

Part (3) follows immediately from (1) and Proposition 3.1. ▮

4. ASYMPTOTICS

The functional equations (24) can be employed to determine asymptotic estimates for the spherical functions φ_λ on A^+. These estimates generalize to arbitrary $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ those given for $\lambda \in \mathscr{E}$ in [8, Theorem 6.8] and formula (6.1).

According to (19), for all $a \in S^0 \cap A$ and $\lambda \in \mathscr{E}+m\delta$,

$$I_m(\lambda, a) = \int_{N(\Omega)} \left[\int_{K(0)} P_m(\lambda, t, \nabla)\, a_H(ak\omega_t)^{\lambda-\rho}\, dk \right] a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t .$$

For $x, y \in G$, set $y^x := xyx^{-1}$. When $\omega_t = \exp(\sum_{h=1}^s t_h Y_h) \in N(\Omega)$ and $a \in S^0 \cap A$, $k \in K \cap H$, we have

$$\begin{aligned} a_H(ak\omega_t) &= a_H(ak\omega_t k^{-1}a^{-1}ak) \\ &= a_H(ak\omega_t k^{-1}a^{-1}k(ak))\, a(ak) \\ &= a_H(\omega_t^{k(ak)^{-1}ak})\, a(ak). \end{aligned}$$

Let $\{\alpha_1, ..., \alpha_s\}$ be an enumeration with multiplicities of Δ_+ such that $Y_h \in \mathfrak{g}^{-\alpha_h}$ for all $h = 1, ..., s$. We can then write

$$\begin{aligned} \omega_t^{k(ak)^{-1}ak} &= \exp\left(\mathrm{Ad}(k(ak)^{-1})\, \mathrm{Ad}(a)\, \mathrm{Ad}(k) \sum_{h=1}^s t_h Y_h \right) \\ &= \exp\left(\sum_{v,\,u,\,h=1}^s t_h c_{hu}(k)\, c_{uv}(a, k)\, e^{-\alpha_u(\log a)} Y_v \right) \\ &= \omega_{f(t,\,a,\,k)}, \end{aligned}$$

where

$$f(t, a, k) = (f_1(t, a, k), ..., f_s(t, a, k)),$$

and

$$f_v(t, a, k) = \sum_{h,\,u=1}^s t_h c_{hu}(k)\, c_{uv}(a, k)\, e^{-\alpha_u(\log a)} \qquad \text{for all} \quad v = 1, ..., s. \tag{30}$$

Note that

$$|c_{hu}(k)| \leqslant 1 \qquad \text{and} \qquad |c_{uv}(a, k)| \leqslant 1 \qquad \text{for all } h, u, v \tag{31}$$

because $\mathrm{Ad}(k)$ is an orthogonal transformation for $k \in K(0) = K \cap H$. It follows that

$$I_m(\lambda, a) = \int_{N(\Omega)} \left[\int_{K(0)} P_m(\lambda, t, \nabla)\, a_H(\omega_{f(t,a,k)})^{\lambda-\rho}\, a(ak)^{\lambda-\rho}\, dk \right] \times a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t. \tag{32}$$

We introduce the notation "$a \in S^0 \cap A$, $a \to \infty$" to mean that $a \in S^0 \cap A$ and $\lim e^{-\alpha(\log a)} = 0$ for all $\alpha \in \Delta_+$.

Lemma 4.1. *For every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and every multi-index $\mathbf{q}$, the function*

$$a \mapsto \left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} a_H(\omega_{f(t,a,k)})^{\lambda-\rho}$$

converges to 1 *if* $\mathbf{q} = \mathbf{0}$ *and to* 0 *if* $\mathbf{q} \neq \mathbf{0}$ *as* $a \in S^0 \cap A$, $a \to \infty$. *The convergence is uniform on* $K(0)$, *on* $T^{-1}(N(\Omega)) \subset B(\sqrt{r})$, *and on compact subsets of* $\mathfrak{a}_{\mathbb{C}}^*$. *In fact, define*

$$\min(a) := \min_{\alpha \in \Delta_+} \alpha(\log a),$$

which is positive for $a \in S^0 \cap A$. *Then for every compact set* $\Lambda \subset \mathfrak{a}_{\mathbb{C}}^*$ *there are constants* C *and* $M_{\Lambda, \mathbf{q}}$ *so that for all* $\lambda \in \Lambda$ *and* $a \in S^0 \cap A$ *with* $\min(a) \geqslant C$

$$|a_H(\omega_{f(t,a,k)})^{\lambda-\rho} - 1| \leqslant M_{\Lambda, \mathbf{0}}\, e^{-\min(a)} \tag{33}$$

$$\left| \left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} a_H(\omega_{f(t,a,k)})^{\lambda-\rho} \right| \leqslant M_{\Lambda, \mathbf{q}}\, e^{-\min(a)}. \tag{34}$$

Proof. Recall from (14) that for every $j = 1, ..., l$ and every $\omega_t \in N(\Omega)$, we have $a_H(\omega_t)^{2\mu_j} = p_j(t)$. The polynomial $p_j(t)$ satisfies $p_j(0) = 1$. So we can write it as

$$p_j(t) = 1 + \sum_{\mathbf{q} \neq \mathbf{0}} a_{j, \mathbf{q}}\, t^{\mathbf{q}}, \tag{35}$$

with at most finitely many nonzero coefficients $a_{j,\mathbf{q}} \in \mathbb{C}$. In (35) we have set $t^{\mathbf{q}} := t_1^{q_1} \cdots t_s^{q_s}$ for every multi-index $\mathbf{q} = (q_1, ..., q_s)$. Since $|t_h| \leqslant \sqrt{r}$ for all $h = 1, ..., s$ (cf. Lemma 3.4) and because of (31),

$$|f_h(t, a, k)| \leqslant \sqrt{r}\, s^2 e^{-\min(a)},$$

and hence for every $\mathbf{q} \neq \mathbf{0}$

$$\begin{aligned} |f(t, a, k)|^{\mathbf{q}} := \prod_{h=1}^{s} |f_h(t, k, a)|^{q_h} &\leqslant \prod_{h=1}^{s} (\sqrt{r}\, s^2)^{q_k}\, e^{-\min(a)\, q_k} \\ &\leqslant r^{|\mathbf{q}|/2} s^{2\,|\mathbf{q}|}\, e^{-\min(a)}. \end{aligned} \tag{36}$$

Thus

$$|a_H(\omega_{f(t,k,a)})^{2\mu_j} - 1| = |p_j(f(t, k, a)) - 1| \leqslant C_{j,\mathbf{0}}\, e^{-\min(a)}, \tag{37}$$

where we have set $C_{j,\mathbf{0}} := \sum_{\mathbf{q} \neq \mathbf{0}} |a_{j,\mathbf{q}}|\, r^{|\mathbf{q}|/2} s^{2\,|\mathbf{q}|}$.

Assume now that $\mathbf{q}$ is a nonzero multi-index. Using (30) and (31), one can prove that for every nonzero multi-index $\mathbf{q}'$ there is a constant $C_{\mathbf{q},\mathbf{q}'}$ for which

$$\left|\left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} f(t, k, a)\right|^{\mathbf{q}'} \leqslant C_{\mathbf{q},\mathbf{q}'}\, e^{-\min(a)}.$$

It follows that

$$\left|\left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} a_H(\omega_{f(t,k,a)})^{2\mu_j}\right| = \left|\left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} p_j(f(t, k, a))\right| \leqslant C_{j,\mathbf{q}}\, e^{-\min(a)}. \tag{38}$$

The constant C in the statement is chosen in order to have $p_j(f(t, k, a))$ uniformly bounded below by a positive constant. We can choose for instance $C := \log(2C_{\mathbf{0}})$ with $C_{\mathbf{0}} := \max_{j=1,...,l} C_{j,\mathbf{0}}$. With this choice (37) gives in fact $|p_j(f(t, k, a)) - 1| \leqslant 1/2$ for all $t \in T^{-1}(N(\Omega))$, $k \in K(0)$ and $a \in S^0 \cap A$ with $\min(a) \geqslant C$.

The estimates (33) and (34) then follow with a straightforward computation from (37) and (38), respectively, and from the equality

$$a_H(\omega_{f(t,k,a)})^{\lambda - \rho} = \prod_{j=1}^{l} a_H(\omega_{f(t,k,a)})^{2z_j(-\lambda)\,\mu_j}. \quad \blacksquare$$

Separate the terms in $P_m(\lambda, t, \nabla)$ which are independent of ∇ by writing

$$P_m(\lambda, t, \nabla) = P_m^0(\lambda, t) + P_m^{>}(\lambda, t, \nabla). \tag{39}$$

Lemma 4.1 shows that

$$\lim_{\substack{a \in S^0 \cap A \\ a \to \infty}} P_m(\lambda, t, \nabla)\, a_H(\omega_{f(t,a,k)})^{\lambda-\rho} = P_m^0(\lambda, t) \tag{40}$$

uniformly in $\omega_t \in N(\Omega)$, in $k \in K \cap H$, and in λ on compact subsets of $\mathscr{E} + m\delta$.

We introduce the notation "$a \in A^+$, $a \to \infty$" to mean that $a \in A^+$ and $\lim e^{-\alpha(\log a)} = 0$ for all $\alpha \in \Delta^+$. When G has real rank bigger than 1, we write "$a \in A^+$, $a \xrightarrow[\Delta_0]{} \infty$" to mean that $a \in A^+$ and $\lim e^{-\alpha(\log a)} = 0$ for all $\alpha \in \Delta_0^+$.

Assume G has real rank bigger than one, and let ψ_λ^0 denote the spherical function for the Riemannian symmetric space $G(0)/K(0)$. Recall the definition of ρ_0 and ρ_+ in (2), and set

$$\mathfrak{a}_+^* := \{\lambda \in \mathfrak{a} : \langle \alpha, \lambda \rangle > 0 \ \forall \alpha \in \Delta_0^+\}.$$

For $a \in A$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ (cf. [17, Proposition 6.3, Chap. IV])

$$\begin{aligned} \psi_\lambda^0(a) &:= \int_{K(0)} a(ak)^{\lambda-\rho_0}\, dk \\ &= a^{\lambda-\rho_0} \int_{\bar{N}_0} a(a\bar{n}_0 a^{-1})^{\lambda-\rho_0}\, a(\bar{n}_0)^{-(\lambda+\rho_0)}\, d\bar{n}_0 . \end{aligned} \tag{41}$$

Moreover if $\lambda \in \mathfrak{a}_+^*$ and $a \in A^+$, then (cf. [17, Theorem 6.14, Chap. IV])

$$\lim_{\substack{a \in A^+ \\ a \xrightarrow[\Delta^0]{} \alpha}} a^{\rho_0-\lambda} \psi_\lambda^0(a) = c_0(\lambda), \tag{42}$$

the convergence being uniform on compacta of $\mathfrak{a}_+^*$. In (42), c_0 denotes Harish-Chandra's c-function of $G(0)/K(0)$.

Lemma 4.2. $a^{\rho_0-\lambda}\psi_\lambda^0(a) = a^{\rho-\lambda} \int_{K(0)} a(ak)^{\lambda-\rho}\, dk.$

Proof. Observe that $a(\bar{n}_0)^{\rho_+} = 1$ for all $\bar{n}_0 \in \bar{N}_0$. Therefore (41) gives

$$\begin{aligned} a^{\rho_0-\lambda}\psi_\lambda^0(a) &= \int_{\bar{N}_0} a(a\bar{n}_0 a^{-1})^{\lambda-\rho_0}\, a(\bar{n}_0)^{-(\lambda+\rho_0)}\, d\bar{n}_0 \\ &= a^{\rho-\lambda} \int_{\bar{N}_0} a(a\bar{n}_0 a^{-1})^{\lambda-\rho}\, a^{\lambda-\rho} a(\bar{n}_0)^{-\lambda+\rho} a(\bar{n}_0)^{-2\rho_0}\, d\bar{n}_0 \\ &= a^{\rho-\lambda} \int_{K(0)} a(ak)^{\lambda-\rho}\, dk. \quad \blacksquare \end{aligned}$$

THEOREM 4.1. *For all* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *with* $\operatorname{Re}\lambda \in \mathfrak{a}_+^*$

$$\lim_{\substack{a \in A^+ \\ a \to \infty}} a^{\rho-\lambda}\varphi_\lambda(a) = c_0(\lambda)\, c_\Omega(\lambda) \tag{43}$$

as meromorphic functions in λ.

Proof. Choose $m \in \mathbb{N}$ so that $\lambda \in \mathscr{E} + m\delta$. From formula (32) we have for all $a \in A^+ \subset S^0 \cap A$

$$\begin{aligned} b_m(\lambda)\, a^{\rho-\lambda}\varphi_\lambda(a) &= a^{\rho-\lambda} I_m(\lambda, a) \\ &= \int_{N(\Omega)} \left[a^{\rho-\lambda} \int_{K(0)} P_m(\lambda, t, \nabla)\, a_H(\omega_{f(t,a,k)})^{\lambda-\rho}\, a(ak)^{\lambda-\rho}\, dk \right] \\ &\quad \times a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t. \end{aligned} \tag{44}$$

The right-hand side of (44) converges as $a \in A^+$, $a \to \infty$, to

$$c_0(\lambda) \int_{N(\Omega)} P_m^0(\lambda, t)\, a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t = b_m(\lambda)\, c_0(\lambda)\, c_\Omega(\lambda),$$

the convergence being uniform on compacta of $\mathfrak{a}_+^* + i\mathfrak{a}^*$. In fact, because of formulas (33), (42), and Lemma 4.2,

$$a^{\rho-\lambda} \int_{K(0)} a_H(\omega_{f(t,a,k)})^{\lambda-\rho}\, a(ak)^{\lambda-\rho}\, dk$$

converges to $c_0(\lambda)$, uniformly in $\omega_t \in N(\Omega)$ and uniformly on compacta of $\mathfrak{a}_+^* + i\mathfrak{a}^*$.

Lemma 4.2 and (42) guarantee that $|a^{\rho-\lambda}| \int_{K(0)} |a(ak)^{\lambda-\rho}|\, dk$ converges to $c_0(\operatorname{Re}\lambda)$ and therefore remains bounded on compacta of $\mathfrak{a}_+^* + i\mathfrak{a}^*$. Thus, by (34),

$$a^{\rho-\lambda} \int_{K(0)} P_m^{>}(\lambda, t, \nabla)\, a_H(\omega_{f(t,a,k)})^{\lambda-\rho}\, a(ak)^{\lambda-\rho}\, dk$$

converges to 0 as $a \in A^+$, $a \to \infty$, uniformly in $\omega_t \in N(\Omega)$ and uniformly on compacta of $\mathfrak{a}_+^* + i\mathfrak{a}^*$. Finally, the uniformity in $\omega_t \in N(\Omega)$ allows us to pass the limit under the integral sign.

The uniformity of the limit on compacta of $\mathfrak{a}_+^* + i\mathfrak{a}^*$ proves also that the left hand-side of (43) is a meromorphic function of $\lambda \in \mathfrak{a}_+^* + i\mathfrak{a}^*$. Thus (43) holds as equality of meromorphic functions on $\mathfrak{a}_+^* + i\mathfrak{a}^*$. ∎

Let Y^0 be the element of $\mathfrak{q}^{K\cap H}\cap\mathfrak{p}$ selected in Section 1. Set $a_\sigma := \exp(\sigma Y^0)$ for $\sigma > 0$. Observe that for $a\in A$ we have $aa_\sigma \in S^0\cap A$ for sufficiently large σ. Because of the $\mathrm{Ad}(K\cap H)$-invariance of Y^0, we get

$$a_H(\omega_t^{k(aa_\sigma k)^{-1}aa_\sigma k})\, a(aa_\sigma k) = a_H(\omega_{e^{-\sigma}t}^{k(ak)^{-1}ak})\, a(ak)\, a_\sigma.$$

for all $k\in K\cap H$, $a\in S^0\cap A$ and all $\omega_t\in N(\Omega)$. We have set $e^{-\sigma}t := (e^{-\sigma}t_1, ..., e^{-\sigma}t_s)$. Hence formula (32) becomes

$$I_m(\lambda, aa_\sigma) = a_\sigma^{\lambda-\rho}\int_{N(\Omega)}\left[\int_{K(0)} P_m(\lambda, t, \nabla)\, a_H(\omega_{f(e^{-\sigma}t, a, k)})^{\lambda-\rho}\, a(ak)^{\lambda-\rho}\, dk\right] \times a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t. \tag{45}$$

Lemma 4.3. *For every $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ and every multi-index $\mathbf{q}$, the function $(\frac{\partial}{\partial t})^{\mathbf{q}}\, a_H(\omega_{f(e^{-\sigma}t, a, k)})^{\lambda-\rho}$ converges to 1 if $\mathbf{q}=\mathbf{0}$ and to 0 if $\mathbf{q}\neq\mathbf{0}$ as $\sigma\to+\infty$ and $a\in A$. The convergence is uniform on $K(0)$, on $T^{-1}(N(\Omega))\subset B(\sqrt{r})$, and on compact subsets of $S^0\cap A$ and $\mathfrak{a}_{\mathbb{C}}^*$. In fact, define $\min(a) := \min_{\alpha\in\Delta_+}\alpha(\log a)$ (which is positive only for $a\in S^0\cap A$). Then for every compact set $\Lambda\subset\mathfrak{a}_{\mathbb{C}}^*$ and $a\in A$ there are constants $M_{\Lambda,\mathbf{q}}>0$ so that for all $\lambda\in\Lambda$*

$$|a_H(\omega_{f(e^{-\sigma}t, a, k)})^{\lambda-\rho} - 1 \leqslant M_{\Lambda,\mathbf{0}}\, e^{-\min(a)}e^{-\sigma}$$

$$\left|\left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} a_H(\omega_{f(e^{-\sigma}t, a, k)})^{\lambda-\rho}\right| \leqslant M_{\Lambda,\mathbf{q}}\, e^{-\min(a)}e^{-\sigma}.$$

Proof. The proof works essentially as in Lemma 4.1. Notice that (36) now becomes

$$|f(e^{-\sigma}t, a, k)|^{\mathbf{q}} \leqslant r^{|\mathbf{q}|/2} s^{2|\mathbf{q}|}\, e^{-\min(a)}\, e^{-\sigma}. \quad \blacksquare$$

Theorem 4.2. *For every b-regular $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ and every $a\in A$*

$$\lim_{\sigma\to\infty} a_\sigma^{\rho-\lambda}\varphi_\lambda(aa_\sigma) = \psi_{\lambda-\rho_+}^{\mathbf{0}}(a)\, c_\Omega(\lambda).$$

Proof. The theorem follows from (45) by the same argument used in the proof of Theorem 4.1. We only remark that

$$\int_{K(0)} a(ak)^{\lambda-\rho}\, dk = \psi_{\lambda-\rho_+}^{\mathbf{0}}(a)$$

for every $a\in A$. $\blacksquare$

5. BERNSTEIN POLYNOMIALS AND THE FUNCTION C_Ω

In Section 3 we have used the method of Bernstein polynomials to extend $\varphi_\lambda(a)$, initially given on $\mathscr{E}$ by the integral (9), to a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$. Recall from (4) that the function c_Ω is defined on $\mathscr{E}$ by integration over $N(\Omega)$ of the same polynomial power $a_H(\omega)^{-(\lambda+\rho)}$ to which we applied Bernstein's theorem for the meromorphic extension of φ_λ. The same procedure for the defining integral (4) then provides a meromorphic continuation of c_Ω to $\mathfrak{a}_{\mathbb{C}}^*$. The comparison of the results so obtained with the known product formula gives some piece of information on the Bernstein polynomials.

PROPOSITION 5.1. (1) *Let $m \in \mathbb{N}$ and let $P_m^0(\lambda, t)$ be as in formula* (39). *Then the function*

$$\int_{N(\Omega)} P_m^0(\lambda, t)\, a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t$$

is holomorphic in $\lambda \in \mathscr{E} + m\delta$.

(2) *For $\lambda \in \mathscr{E} + m\delta$, the function $\lambda \mapsto c_\Omega(\lambda)$ extends to a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ given by*

$$b_m(\lambda)\, c_\Omega(\lambda) = \int_{N(\Omega)} P_m^0(\lambda, t)\, a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t. \tag{46}$$

Proof. Part (1) can be either obtained by a direct argument as in the proof of Proposition 3.1, or by using formula (45) with $a = e$. For, since

$$\omega_{f(e^{-\sigma}t,\, e,\, k)} = \omega_{e^{-\sigma}t},$$

one obtains from (45)

$$I_m(\lambda, a_\sigma) = a_\sigma^{\lambda-\rho} \int_{N(\Omega)} [P_m(\lambda, t, \nabla)\, a_H(\omega_{e^{-\sigma}t})^{\lambda-\rho}]\, a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t. \tag{47}$$

Lemma 4.1 proves that

$$\lim_{\sigma\to\infty} P_m(\lambda, t, \nabla)\, a_H(\omega_{e^{-\sigma}t})^{\lambda-\rho} = P_m^0(\lambda, t)$$

uniformly in $\omega_t \in N(\Omega)$ and in λ on compact subsets of $\mathscr{E} + m\delta$. Since $N(\Omega)$ has compact closure, we can pass the limit under the integral sign and get

$$\lim_{\sigma \to \infty} a_\sigma^{\rho-\lambda} I_m(\lambda, a_\sigma) = \int_{N(\Omega)} P_m^0(\lambda, t)\, a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t. \tag{48}$$

As a uniform limit of holomorphic functions, the right-hand side of (48) is holomorphic on $\mathscr{E} + m\delta$.

For all $\lambda \in E$ satisfying the inequalities (26), formula (46) follows as in the proof of Theorem 3.2 by observing that

$$P_m(\lambda, t, \nabla)\, 1 = P_m^0(\lambda, t).$$

For $\lambda \in \mathscr{E} + m\delta$ with $b_m(\lambda) \neq 0$ define

$$c_\Omega(\lambda) := \frac{1}{b_m(\lambda)} \int_{N(\Omega)} P_m^0(\lambda, t)\, a_H(\omega_t)^{-(\lambda+\rho-m\delta)}\, d\omega_t. \tag{49}$$

If $\lambda \in \mathscr{E}$, then the definition of b_m in (15) shows that this agrees with the old definition of $c_\Omega(\lambda)$. Hence (49) gives a meromorphic extension of $c_\Omega(\lambda)$ to $\mathscr{E} + m\delta$. ∎

Formula (46) shows that the zero set of $b_m(\lambda)$ contains the polar set of $c_\Omega(\lambda)$ in $\mathscr{E} + m\delta$, which can be determined by means of the product formula (5).

Consider in particular the case $m = 1$. When all multiplicities m_α ($\alpha \in \Delta_+$) are odd, we conclude that the polynomial

$$B(\lambda) := \prod_{\alpha \in \Delta_+} \left(-\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1 \right) \cdots \left(-\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + \frac{\delta(H_\alpha)}{2} \right) \tag{50}$$

divides $b(\lambda - \delta)$. (Observe that $\delta(H_\alpha)/2$ is always a positive integer).

Suppose now that $\alpha \in \Delta_+$ has even multiplicity m_α. Then some poles of $\Gamma(-\lambda(H_\alpha)/2 - m_\alpha/2 + 1)$ in $\mathscr{E} + \delta$ are cancelled by zeros of $\Gamma(-\lambda(H_\alpha)/2 + 1)$ when $m_\alpha < \delta(H_\alpha)$. The factor corresponding to the root α in (50) should then be replaced by

$$\left(-\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1 \right) \cdots \left(-\frac{\lambda(H_\alpha)}{2} + \min\left\{ -\frac{m_\alpha}{2} + \frac{\delta(H_\alpha)}{2}, 0 \right\} \right).$$

When G has rank one, the polynomial $b(\lambda - \delta)$ can be directly computed (see Section 6). In this case there is a unique root $\alpha \in \Delta_+$ and $\delta(H_\alpha) = 4$. Therefore the case m_α even $< \delta(H_\alpha)$, occurs for $m_\alpha = 2$. We will prove that also in this case $B(\lambda)$ divides $b(\lambda - \delta)$, and this is due to vanishing of the right-hand side of (46) at the points $-\lambda(H_\alpha)/2 = 1, ..., m_\alpha/2 - \delta(H_\alpha)/2 + 1$. We will even show that the equality $B(\lambda) = b(\lambda - \delta)$ holds in this case.

Also motivated by the results in the Appendix, one could be tempted to conjecture that in fact this equality holds for every NCC symmetric space.

Conjecture. For every NCC symmetric space we have (up to a constant multiple)

$$B(\lambda) = b(\lambda - \delta).$$

An immediate consequence of this conjecture would be the following corollary, which describes the singularities of the spherical functions.

Corollary 5.1. *The polar set of the spherical functions $\varphi_\lambda(a)$ is contained for all $a \in S^0 \cap A$ in the polar set of the numerator*

$$n_\Omega(\lambda) = \prod_{\alpha \in \Delta_+} \Gamma\left(-\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1\right) \tag{51}$$

of the function $c_\Omega(\lambda)$.

In the Appendix, Corollary 5.1 will be proven as true by means of different methods.

Separate the terms in $P^0_m(\lambda, t)$ which do not contain the variable t by writing

$$P^0_m(\lambda, t) = P^{00}_m(\lambda) + P^{01}_m(\lambda, t).$$

Because of (46)

$$b_m(\lambda)\, c_\Omega(\lambda) = P^{00}_m(\lambda)\, c_\Omega(\lambda - m\delta) + f_m(\lambda), \tag{52}$$

where

$$f_m(\lambda) := \int_{N(\Omega)} P^{01}_m(\lambda, t)\, a_H(\omega_t)^{-(\lambda + \rho - m\delta)}\, d\omega_t$$

is a holomorphic function in $\mathscr{E} + m\delta$.

Suppose that $P^0_1(\lambda, t) = P^{00}_1(\lambda)$ is in fact independent of the variable t, a fact that is true for instance in the rank-one case. Let $P(\lambda)$ be the polynomial defined by

$$P(\lambda) := \prod_{\alpha \in \Delta_+} \left(-\frac{\lambda(H_\alpha)}{2} + \frac{\delta(H_\alpha)}{2}\right) \cdots \left(-\frac{\lambda(H_\alpha)}{2} + 1\right). \tag{53}$$

Then $f_1(t)=0$, so the product formula for c_Ω implies the equality

$$\frac{b(\lambda-\delta)}{P_1^{00}(\lambda)}=\frac{B(\lambda)}{P(\lambda)}, \qquad \lambda\in\mathfrak{a}_{\mathbb{C}}^{*}. \tag{54}$$

The following Theorem 5.1 states that also the converse is true: equality (54) and the product formula for c_Ω are equivalent. As a consequence, we can obtain a very elementary analytic proof of the product formula, every time the condition (54) can be directly verified. This should be (at least in theory) possible when an explicit expression for the function a_H is available. See also Section 7 on this matter.

THEOREM 5.1. *Suppose $P_1^0(\lambda,t)=P_1^{00}(\lambda)$ is independent of the variable t. Let c_Ω denote the meromorphically continued function on $\mathfrak{a}_{\mathbb{C}}^{*}$ considered in Proposition* 5.1. *Then the following are equivalent:*

(1) *The equality of meromorphic functions*

$$\frac{b(\lambda-\delta)}{P_1^{00}(\lambda)}=\frac{B(\lambda)}{P(\lambda)}$$

holds for all $\lambda\in\mathfrak{a}_{\mathbb{C}}^{*}$.

(2) *There is a constant κ, depending only on the symmetric space and on the normalization of the measures, so that for all $\lambda\in\mathfrak{a}_{\mathbb{C}}^{*}$*

$$c_\Omega(\lambda)=\kappa\prod_{\alpha\in\Delta_+}B\left(-\frac{\lambda(H_\alpha)}{2}-\frac{m_\alpha}{2}+1,\frac{m_\alpha}{2}\right),$$

where B denotes the Beta function.

Proof. We have already shown that the product formula implies the equality in part (1).

The method employed to prove the converse is a modification of argument used by Cohn [4] for the analytic determination of the Harish-Chandra's c-function.

The validity of Eq. (54) together with the assumption $P_1^0(\lambda,t)=P_1^{00}(\lambda)$ implies (without knowledge of the product formula) that c_Ω, initially defined on $\mathscr{E}$ by means of the integral (4), extends as a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^{*}$ satisfying the functional equation

$$B(\lambda)\,c_\Omega(\lambda)=P(\lambda)\,c_\Omega(\lambda-\delta), \qquad \lambda\in\mathfrak{a}_{\mathbb{C}}^{*}.$$

We now formulate as separate lemmas the various steps that bring us to the product formula as the unique solution of this functional equation.

LEMMA 5.4. *The general solution of the functional equation*

$$B(\lambda)\, f(\lambda) = P(\lambda)\, f(\lambda-\delta), \qquad \lambda \in \mathfrak{a}^*_{\mathbb{C}} \tag{55}$$

is

$$f(\lambda) := \prod_{\alpha \in \Delta_+} \frac{\Gamma(-\lambda(H_\alpha)/2 - m_\alpha/2 + 1)}{\Gamma(-\lambda(H_\alpha)/2 + 1)}\, F(\lambda),$$

where $F(\lambda)$ *is a periodic function of period* δ: $F(\lambda+\delta) = F(\lambda)$ *for all* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

Proof. The functional equation $x\Gamma(x) = \Gamma(x+1)$ for the Γ-function implies that every function f as above is a solution to (55). Conversely, let f be a solution, and set

$$F(\lambda) := \prod_{\alpha \in \Delta_+} \frac{\Gamma(-\lambda(H_\alpha)/2 + 1)}{\Gamma(-\lambda(H_\alpha)/2 - m_\alpha/2 + 1)}\, f(\lambda).$$

Then

$$\begin{aligned} F(\lambda-\delta) &:= \frac{P(\lambda)}{B(\lambda)}\, f(\lambda-\delta) \prod_{\alpha \in \Delta_+} \frac{\Gamma(-\lambda(H_\alpha)/2 + 1)}{\Gamma(-\lambda(H_\alpha)/2 - m_\alpha/2 + 1)} \\ &= f(\lambda) \prod_{\alpha \in \Delta_+} \frac{\Gamma(-\lambda(H_\alpha)/2 + 1)}{\Gamma(-\lambda(H_\alpha)/2 - m_\alpha/2 + 1)} \\ &= F(\lambda). \quad \blacksquare \end{aligned}$$

COROLLARY 5.2. *There is a holomorphic function* F *of period* δ *so that*

$$c_\Omega(\lambda) = \prod_{\alpha \in \Delta_+} \frac{\Gamma(-\lambda(H_\alpha)/2 - m_\alpha/2 + 1)}{\Gamma(-\lambda(H_\alpha)/2 + 1)}\, F(\lambda), \qquad \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{56}$$

Proof. Lemma 5.1 ensures the existence of a δ-periodic function F so that (56) holds. c_Ω is meromorphic, with poles at most included in the singular set of $\prod_{\alpha \in \Delta_+} \Gamma(-\lambda(H_\alpha)/2 - m_\alpha/2 + 1)$. Hence F is meromorphic with poles at most on the singular set of $\prod_{\alpha \in \Delta_+} \Gamma(-\lambda(H_\alpha)/2 + 1)$. In particular, F is holomorphic on $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \operatorname{Re} \lambda(H_\alpha) < 0\ \forall \alpha \in \Delta_+\}$, so holomorphic in the whole $\mathfrak{a}^*_{\mathbb{C}}$ by periodicity. $\blacksquare$

The product formula for c_Ω will follow, if we can show that F is constant. The crucial step for this purpose is the following lemma, which is an adaptation of a lemma by Harish-Chandra (cf. [13, Lemma 19.4, proven in Sect. 21]).

LEMMA 5.2. *Let σ denote a real variable. Then for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying* $\operatorname{Re} \lambda(H_\alpha) < 0$ *for all* $\alpha \in \Delta_+$, *we have*

$$\lim_{\sigma \to +\infty} \frac{c_\Omega(\lambda - \sigma\delta)}{c_\Omega(-\sigma\delta)} = 1.$$

The proof of Lemma 5.2 requires some preparation. We therefore postpone it to the end of the section and deduce instead from it the product formula of [24] for c_Ω, so concluding the proof of Theorem 5.1.

Recall the following asymptotic estimate for the ratio of gamma functions (see [5, 1.18(4)]): For every complex numbers a, b and for $\eta > 0$

$$\frac{\Gamma(a+s)}{\Gamma(b+s)} = s^{(a-b)} \left[1 + \frac{(a-b)(a+b-1)}{2s} + O(s^{-2}) \right]$$

$$\text{as} \quad |s| \to \infty, \quad \arg(s) \leqslant \pi - \eta.$$

Conclusion of the proof of Theorem 5.18. We have to prove that the function F of Corollary 5.2 is constant. By holomorphy, it is enough to prove $F(\lambda_1) = F(\lambda_2)$ for all $\lambda_1, \lambda_2 \in \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \operatorname{Re} \lambda(H_\alpha) < 0 \ \forall \alpha \in \Delta_+\}$. Observe that

$$\prod_{\alpha \in \Delta_+} \frac{\Gamma(-\lambda_2(H_\alpha)/2 - m_\alpha/2 + 1 + \sigma(\delta(H_\alpha)/2))}{\Gamma(-\lambda_2(H_\alpha)/2 + 1 + \sigma(\delta(H_\alpha)/2))}$$
$$\times \frac{\Gamma(-\lambda_1(H_\alpha)/2 + 1 + \sigma(\delta(H_\alpha)/2))}{\Gamma(-\lambda_1(H_\alpha)/2 - m_\alpha/2 + 1 + \sigma(\delta(H_\alpha)/2))}$$
$$\approx \prod_{\alpha \in \Delta_+} \left[\sigma \frac{\delta(H_\alpha)}{2} \right]^{-m_\alpha/2} \left[\sigma \frac{\delta(H_\alpha)}{2} \right]^{m_\alpha/2} = 1$$

for $\sigma \in \mathbb{R}$, $\sigma \to +\infty$. By periodicity, for every $h \in \mathbb{N}$, $F(\lambda - h\delta) = F(\lambda)$. Lemma 5.2 then gives

$$\frac{F(\lambda_1)}{F(\lambda_2)} = \lim_{h \to \infty} \frac{F(\lambda_1 - h\delta)}{F(\lambda_2 - h\delta)}$$
$$= \lim_{h \to \infty} \prod_{\alpha \in \Delta_+} \frac{\Gamma(-\lambda_2(H_\alpha)/2 - m_\alpha 2 + 1 + h(\delta(H_\alpha)/2))}{\Gamma(-\lambda_2(H_\alpha)/2 + 1 + h(\delta(H_\alpha)/2))}$$
$$\times \frac{\Gamma(-\lambda_1(H_\alpha)/2 + 1 + h(\delta(H_\alpha)/2))}{\Gamma(-\lambda_1(H_\alpha)/2 - m_\alpha/2 + 1 + h(\delta(H_\alpha)/2))} \frac{c_\Omega(\lambda_1 - h\delta)}{c_\Omega(\lambda_2 - h\delta)}$$
$$= \lim_{h \to \infty} \frac{c_\Omega(\lambda_1 - h\delta)}{c_\Omega(\lambda_2 - h\delta)} = 1. \quad \blacksquare$$

The remainder of this section is devoted to the proof of Lemma 5.2. Neeb's convexity theorem ensures that for all $\omega \in N(\Omega)$

$$\log a_H(\omega) \in -\sum_{\alpha \in \Delta_+} \mathbb{R}_0^+ H_\alpha. \tag{57}$$

(cf. [19, Corollary 5.5.14]). It follows that, with δ as in (12),

$$\delta(\log a_H(\omega)) \leqslant 0$$

for all $\omega \in N(\Omega)$.

Let $T \geqslant 0$ and define

$$\Omega_T := \{X \in \Omega : -\delta(\log a_H(\exp X)) \leqslant T\},$$
$$N(\Omega)_T := \{\omega \in N(\Omega) : -\delta(\log a_H(\omega)) \leqslant T\}.$$

In particular, $\Omega_0 = \{0\}$. The following lemma proves that the set of Ω_T's with $T > 0$ is a basis for the neighborhoods of 0 in $\mathfrak{n}_-$.

Lemma 5.3. *For every $\varepsilon > 0$ there is $T_\varepsilon > 0$ so that $\Omega_{T_\varepsilon} \subset \{X \in \Omega : |X| < \varepsilon\}$.*

Proof. Set $d := \min_{\alpha \in \Delta_+} \delta(H_\alpha)$. Then $d > 0$. Proceeding as in the proof of Theorem 3.10 in [26], we have for $X = \mathrm{Ad}(k)(\sum_{j=1}^r x_j X_{-j}) \in \Omega$,

$$\log a_H(\exp X) = \sum_{w \in W_0} \lambda_w w \left(\tfrac{1}{2} \sum_{j=1}^r \log(1 - x_j^2)\, H_j \right)$$

with $\lambda_w \geqslant 0$ and $\sum_{w \in W_0} \lambda_w = 1$. Since $wH_j \in \Delta_+$ and $\log(1 - x_j^2) \leqslant 0$,

$$\delta(\log a_H(\exp X)) \leqslant \frac{d}{2} \sum_{j=1}^r \log(1 - x_j^2). \tag{58}$$

Observe that $|X|^2 = \sum_{j=1}^r x_j^2\, |X_{-j}|^2$. Let $M := \max_{j=1, ..., r} |X_j|^2$. We can assume without loss of generality that $0 < \varepsilon < \sqrt{rM}$. Then it is easy to check from (58) that any $0 < T_\varepsilon < -\frac{d}{2} \log(1 - \varepsilon^2/Mr)$ works. ∎

Lemma 5.4. *Let $s := \dim \mathfrak{n}_-$. There exists a constant $C > 0$ such that for every $\varepsilon \in (0, 1]$*

$$\int_{N(\Omega)_\varepsilon} d\omega \geqslant C\varepsilon^s.$$

Proof. Let $Y^0 \in \mathfrak{p} \cap \mathfrak{q}^{H \cap K}$ be cone generating element fixed in Section 1, and set $a_\sigma := \exp(\sigma Y^0)$ for $\sigma > 0$. Then $\log a_H(a_\sigma \omega\, a_{-\sigma}) = e^{-\sigma} \log a_H(\omega)$.

If $\omega \in N(\Omega)_1$, then

$$-\delta(\log a_H(a_{-\log \varepsilon} \omega a_{\log \varepsilon})) = -\varepsilon\delta(\log a_H(\omega)) \leqslant \varepsilon.$$

So $a_{-\log \varepsilon} \omega a_{\log \varepsilon} \in N(\Omega)_\varepsilon$. Observe that

$$2\rho_+(\log a_\sigma) = \sigma \sum_{\alpha \in \Delta_+} m_\alpha \alpha(Y^0) = \sigma \sum_{\alpha \in \Delta_+} m_\alpha = \sigma s.$$

Therefore

$$\int_{N(\Omega)_\varepsilon} d\omega \geqslant \int_{a_{-\log \varepsilon} N(\Omega)_1 \, a_{\log \varepsilon}} d\omega = e^{-2\rho_+(\log a_{-\log \varepsilon})} \int_{N(\Omega)_1} d\omega = C\varepsilon^s,$$

with

$$C := \int_{N(\Omega)_1} d\omega. \quad \blacksquare$$

LEMMA 5.5. *Let* $s := \dim \mathfrak{n}_-$. *There is a constant* $C_0 > 0$ *such that for all* $\sigma \geqslant 1$

$$\int_{N(\Omega)} a_H(\omega)^{\sigma\delta} \, d\omega \geqslant C_0 \sigma^{-s}. \tag{59}$$

Consequently, for all $\sigma \geqslant 1$

$$c_\Omega(-\sigma\delta) \geqslant C_0 \sigma^{-s}.$$

Proof. Let $\mu(T) := \int_{N(\Omega)_T} d\omega$. Fix $\varepsilon \in (0, 1]$ and define for $r \geqslant 1$

$$\begin{aligned} N(\Omega)_{\varepsilon, r} &:= N(\Omega)_{r\varepsilon} - N(\Omega)_{(r-1)\varepsilon} \\ &= \{\omega \in N(\Omega) : (r-1)\,\varepsilon < -\delta(\log a_H(\omega)) \leqslant r\varepsilon\}. \end{aligned}$$

Then

$$\int_{N(\Omega)_{\varepsilon, r}} e^{\sigma\delta(\log a_H(\omega_t))} \, d\omega \geqslant e^{-\sigma r\varepsilon} [\mu(r\varepsilon) - \mu((r-1)\,\varepsilon)].$$

Since $N(\Omega) = \bigcup_{r=1}^\infty N(\Omega)_{\varepsilon, r}$, it follows that

$$\int_{N(\Omega)} a_H(\omega)^{\sigma\delta} \, d\omega \geqslant \sum_{r=1}^\infty e^{-\sigma r\varepsilon} [\mu(r\varepsilon) - \mu((r-1)\,\varepsilon)].$$

The series $\sum_{r=1}^{\infty} e^{-\sigma r\varepsilon}\mu(r\varepsilon)$ is convergent because $\mu(r\varepsilon) \leqslant \int_{N(\Omega)} d\omega < \infty$. Since $\mu(0)=0$, we obtain from Lemma 5.4

$$\begin{aligned}
&\sum_{r=1}^{\infty} e^{-\sigma r\varepsilon}[\mu(r\varepsilon)-\mu((r-1)\,\varepsilon)]\\
&\quad=\sum_{r=1}^{\infty} e^{-\sigma r\varepsilon}\mu(r\varepsilon)-e^{-\sigma\varepsilon}\sum_{r=1}^{\infty} e^{-\sigma(r-1)\,\varepsilon}\mu((r-1)\,\varepsilon)\\
&\quad=(1-e^{-\sigma\varepsilon})\sum_{r=1}^{\infty} e^{-\sigma r\varepsilon}\mu(r\varepsilon)\\
&\quad\geqslant e^{-\sigma\varepsilon}\mu(\varepsilon)(1-e^{-\sigma\varepsilon})\\
&\quad\geqslant Ce^{-\sigma\varepsilon}(1-e^{-\sigma\varepsilon})\,\varepsilon^{s}.
\end{aligned}$$

Inequality (59) follows by setting $\varepsilon=1/\sigma$.

The lower bound for $c_\Omega(-\sigma\delta)$ is an immediate consequence of (59). In fact, $\rho(H_\alpha)\geqslant 0$ for all $\alpha\in\Delta^+$. Hence (57) implies $a_H(\omega)^{-\rho}\geqslant 1$ for all $\omega\in N(\Omega)$. ∎

Proof of Lemma 5.2. Choose $\sigma_0>0$ so that $-\sigma\delta\in\varepsilon$ for $\sigma\geqslant\sigma_0$. Let $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ satisfy $\operatorname{Re}\lambda(H_\alpha)<0$ for all $\alpha\in\Delta_+$. Given $\varepsilon>0$, choose $T_\varepsilon>0$ so that

$$\sup_{\omega\in N(\Omega)_{T_\varepsilon}} |a_H(\omega)^{-\lambda}-1|<\frac{\varepsilon}{2}.$$

Set $M_\lambda:=\sup_{\omega\in N(\Omega)}|a_H(\omega)^{-\lambda}-1|$, which is finite by Lemma 3.7. Then for $\sigma\geqslant\sigma_0$

$$\begin{aligned}
\left|\frac{c_\Omega(\lambda-\sigma\delta)}{c_\Omega(-\sigma\delta)}-1\right| &\leqslant \int_{N(\Omega)} \frac{a_H(\omega)^{\sigma\delta-\rho}}{c_\Omega(-\sigma\delta)}\,|a_H(\omega)^{-\lambda}-1|\,d\omega\\
&\leqslant\frac{\varepsilon}{2}+M_\lambda\int_{N(\Omega)\backslash N(\Omega)_{T_\varepsilon}}\frac{a_H(\omega)^{\sigma\delta-\rho}}{c_\Omega(-\sigma\delta)}\,d\omega.
\end{aligned}$$

If $\sigma\geqslant\frac{1}{2}\max_{\alpha\in\Delta_+}(\rho(H_\alpha)/\delta(H_\alpha))$, then $a_H(\omega)^{\sigma\delta-\rho}\leqslant a_H(\omega)^{\sigma\delta/2}$ for all $\omega\in N(\Omega)$. For all $\omega\in N(\Omega)\backslash N(\Omega)_{T_\varepsilon}$ is $a_H(\omega)^{\sigma\delta/2}\leqslant e^{-\sigma T_\varepsilon/2}$. If $\sigma\geqslant 1$, then $c_\Omega(-\sigma\delta)^{-1}\leqslant C_0^{-1}\sigma^s$ by Lemma 5.5. We can therefore select $\sigma_\varepsilon>0$ such that

$$M_\lambda\int_{N(\Omega)\backslash N(\Omega)_{T_\varepsilon}}\frac{a_H(\omega)^{\sigma\delta-\rho}}{c_\Omega(-\sigma\delta)}\,d\omega\leqslant M_\lambda C_0^{-1}e^{-\sigma T_\varepsilon/2}\,|N(\Omega)|\,\sigma^s<\frac{\varepsilon}{2}$$

for all $\sigma>\sigma_\varepsilon$.

6. THE RANK-ONE CASE

In the rank-one case, $G = \mathrm{SO}_0(1, n)$ and $H = \mathrm{SO}_0(1, n-1)$ with $n \geqslant 2$. The involution τ and the Cartan involution θ are

$$\tau(g) = I_{1,n} g I_{1,n}, \qquad g \in G$$
$$\theta(g) = (g^\top)^{-1}, \qquad g \in G$$

where

$$I_{1,n} := \begin{bmatrix} -1 & 0 \\ 0 & \mathbf{I} \end{bmatrix},$$

$\mathbf{I}$ is the $n \times n$-identity matrix, and $^\top$ denotes transposition. Hence $K = \mathrm{SO}(n)$, with

$$K \cap H = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{k} & 0 \\ 0 & 0 & 1 \end{bmatrix} : \mathbf{k} \in \mathrm{SO}(n-1) \right\}.$$

Let

$$Y^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathbf{0} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $\mathfrak{a} = \mathbb{R} Y^0$. There is exactly one positive root α, which is noncompact. It satisfies $\alpha(Y^0) = 1$, so $H_\alpha = 2Y^0$. The Weyl group W is of order 2 and the nontrivial element of W maps Y^0 into $-Y^0$. The Killing form B of $\mathfrak{g} = \mathfrak{so}(n, 1)$ is given by $B(X, Y) := (n-1)\,\mathrm{tr}(XY)$. The multiplicity of α is $m_\alpha = n-1$ and

$$\bar{\mathfrak{n}} = \mathfrak{n}_- = \mathfrak{g}_{-\alpha} = \left\{ X_t := \begin{bmatrix} 0 & -t & 0 \\ t^\top & \mathbf{O} & t^\top \\ 0 & t & 0 \end{bmatrix} : t \in \mathbb{R}^{n-1} \right\}.$$

Let $\mathrm{e} := (1, 0, \dots, 0)$. Then

$$\Omega = \mathrm{Ad}(K \cap H)\{(-1, 1)\, X_{\mathrm{e}}\} = \left\{ X_t : t \in \mathbb{R}^{n-1},\ \|t^2\| := \sum_{j=1}^{n-1} t_j^2 < 1 \right\},$$

$$N(\Omega) = \left\{ \omega_t := \begin{bmatrix} 1 - \|t\|^2/2 & -t & -\|t\|^2/2 \\ t^\top & \mathbf{I} & t^\top \\ \|t\|^2/2 & t & 1 + \|t\|^2/2 \end{bmatrix} : t \in \mathbb{R}^{n-1},\ \|t\| < 1 \right\}.$$

We identify $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\lambda \in \mathbb{C}$ if $\lambda(H_\alpha)/2 = \lambda$. Hence $\mu = \alpha \equiv 1$ and $\delta = 2\alpha \equiv 2$.

We will compute the Bernstein polynomial for the rank-one case by means of a modified Mellin transformation. Set $\mathbb{R}_+^m := (0, \infty)^m$. For $f\colon \mathbb{R}_+^m \to \mathbb{C}$ define $\mathscr{M}f$ on $\{z \in \mathbb{C} : \operatorname{Re} s > 0\}$ by

$$\mathscr{M}f(s) := \int_{\mathbb{R}_+^m} f(t)\, \|t\|^{s-1}\, dt, \tag{60}$$

provided the integral converges. When $m=1$, then $\mathscr{M}$ coincides with the usual Mellin transformation on $(0, \infty)$.

Let

$$E := \sum_{j=1}^{m} t_j \frac{\partial}{\partial t_j}$$

denote the Euler operator. Observe that E maps radial functions into radial functions. Indeed, if $f(t) = F(\|t\|)$ is radial, then $Ef(t) = \|t\|\, F'(\|t\|)$.

Lemma 6.1. *Suppose that f is separately absolutely continuous in each variable on $\mathbb{R}_+^m$. In particular, each partial derivative $(\partial/\partial t_j) f$ is a.e. defined on $(0, \infty)$. Assume that the integrals defining $\mathscr{M}f$ and $\mathscr{M}((\partial/\partial t_j) f)$ converge absolutely for* $\operatorname{Re} s > 0$ *and for all $j = 1, ..., m$. Then the integral defining $\mathscr{M}(Ef)$ converges absolutely on* $\operatorname{Re} s > 0$ *and*

$$\mathscr{M}(Ef)(s) = -(s+m-1)\, \mathscr{M}f(s). \tag{61}$$

Proof. Observe that

$$\begin{aligned}\sum_{j=1}^{m} \frac{\partial}{\partial t_j} (t_j\, \|t\|^{s-1}) &= \sum_{j=1}^{m} (\|t\|^{s-1} + (s-1)\, t_j^2\, \|t\|^{s-3})\\ &= (s+m-1)\, \|t\|^{s-1}.\end{aligned}$$

Integration by parts then gives

$$\begin{aligned}\mathscr{M}(Ef)(s) &= \sum_{j=1}^{m} \int_{\mathbb{R}_+^m} t_j \left(\frac{\partial}{\partial t_j} f\right) \|t\|^{s-1}\, dt\\ &= \sum_{j=1}^{m} \int_{\mathbb{R}_+^m} \left\{\frac{\partial}{\partial t_j} (t_j\, \|t\|^{s-1} f) - f \frac{\partial}{\partial t_j} (t_j\, \|t\|^{s-1})\right\} dt\\ &= -(s+m-1) \int_{\mathbb{R}_+^m} f(t)\, \|t\|^{s-1}\, dt.\end{aligned}$$

In the previous computations we have used the fact that $t_j\, \|t\|^{s-1} f$ has limit zero either for $t_j \to 0$ (since f is absolutely continuous with integrable partial derivatives) or for $t_j \to \infty$ (since $t_j \|t\|^{s-1} f$ is absolutely continuous and integrable). ∎

Let $\sigma(S_+^{m-1})$ denote the surface area of the intersection S_+^{m-1} of the unit sphere in $\mathbb{R}^m$ with $\mathbb{R}_+^m$. If $f(t) = F(\|t\|)$ is a radial function, then by passing to polar coordinates

$$\begin{aligned}\mathscr{M}f(s) &= \int_{\mathbb{R}_+^m} f(t)\, \|t\|^{s-1}\, dt \\ &= \sigma(S_+^{m-1}) \int_0^\infty F(x)\, x^{s+m-2}\, dx \\ &= \sigma(S_+^{m-1})(\mathscr{M}_1 F)(s+m-1),\end{aligned}$$

where $\mathscr{M}_1$ denotes the classical Mellin transformation on $(0, \infty)$:

$$\mathscr{M}_1 F(s) := \int_0^\infty F(x)\, x^{s-1}\, dx, \qquad \operatorname{Re} s > 0.$$

$\mathscr{M}_1$ is injective on the set of functions on $(0, \infty)$ with support in some interval $(0, r]$ and having a continuous extension on $[0, r)$. Indeed, if F is any such function, then

$$\mathscr{M}_1 F(a+ib) = \mathscr{F}(x^a F \circ \mu)(b),$$

where $\mathscr{F}$ is the Fourier transformation and $\mu(y) := e^y$. The assumptions on F guarantee that $x^a F \circ \mu$ belongs to $L^1(\mathbb{R})$, on which $\mathscr{F}$ is injective. It follows that $\mathscr{M}$ is injective on the set R_r of radial functions with support contained in the ball $\{t \in \mathbb{R}_+^m : 0 < \|t\| \leqslant r\}$ in $\mathbb{R}_+^m$ and having a continuous extension on $\{t \in [0, \infty)^m : \|t\| < r\}$.

Computations of a_H for the rank-one group G can be reduced to computations in $\mathrm{SL}(2, \mathbb{R})$ by means of the following standard proposition, which holds also in a higher rank setting (cf. [22, pp. 321 and 331]).

PROPOSITION 6.1. *Let α be a root and $0 \neq X_\alpha \in \mathfrak{g}^\alpha$.*

(1) *$[X_\alpha, \theta X_\alpha] = B(X_\alpha, \theta X_\alpha)\, \bar{H}_\alpha$, where $\bar{H}_\alpha \in \mathfrak{a}$ is the unique element satisfying $B(H, \bar{H}_\alpha) = \alpha(H)$ for all $H \in \mathfrak{a}$. Moreover $B(X_\alpha, \theta X_\alpha) < 0$.*

(2) *Suppose $\alpha \in \Delta_+$. Set $H_\alpha := 2\bar{H}_\alpha/(\alpha, \alpha)$ (hence $\alpha(H_\alpha) = 2$) and $X_{-\alpha} = \tau(X_\alpha) = -\theta(X_\alpha)$. Then $\mathbb{R}H_\alpha \oplus \mathbb{R}X_\alpha \oplus \mathbb{R}X_{-\alpha} =: \mathfrak{sl}_X$ is a θ- and τ-stable Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.*

(3) *Suppose we have normalized* X_α *so that* $B(X_\alpha, \theta X_\alpha) = -2/(\alpha, \alpha)$. *Then an isomorphism between* $\mathfrak{sl}(2, \mathbb{R})$ *and* $\mathfrak{sl}_X$ *is given by* $\phi\colon \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}_X$ *defined via*

$$X := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mapsto X_\alpha$$

$$Y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mapsto X_{-\alpha}$$

$$H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mapsto H_\alpha.$$

The following proposition makes Eq. (15) explicit for groups of rank one. It also proves the Conjecture in Section 5 as true in this case.

PROPOSITION 6.2. (1) $a_H(\omega_t)^{-(\lambda+\rho)} = p(t)^{z(\lambda)}$ *with*

$$p(t) = (1 - \|t\|^2)^2, \tag{62}$$

$$z(\lambda) = -\tfrac{1}{2}(\lambda + \rho). \tag{63}$$

(2) *The Bernstein polynomial for* $p(t)$ *is*

$$b(\lambda - \delta) = (-\lambda - \rho + 1)(-\lambda - \rho + 2). \tag{64}$$

Equation (15) *holds with*

$$Q(\lambda - \delta, t, \nabla) = \left(-\lambda - \rho + 2 - \frac{1}{2} \sum_{j=1}^{m_\alpha} t_j \frac{\partial}{\partial t_j}\right)\left(-\lambda - \rho + 1 - \frac{1}{2} \sum_{j=1}^{m_\alpha} t_j \frac{\partial}{\partial t_j}\right). \tag{65}$$

(3) *For all* $m \in \mathbb{N}$, *one has* $\mathscr{E} + m\delta = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < -\rho + 2m - 1\}$. *The polynomial*

$$b(\lambda - m\delta) = (-\lambda - \rho + 2m - 1)(-\lambda - \rho + 2m)$$

has roots $-\rho + 2m - 1$ *and* $-\rho + 2m$ *belonging to* $(\mathscr{E} + m\delta) \backslash (\mathscr{E} + (m-1)\delta)$.

(4) *For every* $m \in \mathbb{N}$ *the polynomial* $b_m(\lambda)$ *in* (23) *is*

$$b_m(\lambda) = (-\lambda - \rho + 1)(-\lambda - \rho + 2) \cdots (-\lambda - \rho + 2m - 1)(-\lambda - \rho + 2m)$$

and $P_m(\lambda, t, \nabla) = P(\lambda - m\delta, t, \nabla) \circ \cdots \circ P(\lambda - \delta, t, \nabla)$ *with*

$$P(\lambda - h\delta, t, \nabla) = \left(-\lambda + 2h + \frac{1}{2} \sum_{j=1}^{m_\alpha} t_j \frac{\partial}{\partial t_j} \right) \left(-\lambda + 2h - 1 + \frac{1}{2} \sum_{j=1}^{m_\alpha} t_j \frac{\partial}{\partial t_j} \right). \tag{66}$$

Hence

$$P_m^0(\lambda, t) = \prod_{h=1}^{m} (-\lambda + 2h - 1)(-\lambda + 2h) = b_m(\lambda - \rho). \tag{67}$$

In particular,

$$P_1^0(\lambda, t) = P_1^{00}(\lambda) = (-\lambda + 1)(-\lambda + 2). \tag{68}$$

Proof. Part (1) follows from $\mathfrak{sl}(2, \mathbb{R})$-reduction and the direct computation of a_H for $\mathrm{SL}(2, \mathbb{R})$ in [26, Lemma 3.9].

To prove part (2), set $m := m_\alpha = n - 1$, and define the function f on $\mathbb{R}_+^m$ by

$$f(t) := \begin{cases} 1 - \|t\|^2 & \text{if} \quad \|t\| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $\mathrm{Re}\, z > -1$ and $\mathrm{Re}\, s > 0$,

$$\begin{aligned} \mathcal{M}f^{z+2}(s) &= \int_{\|t\| < 1} (1 - \|t\|^2)^{z+2} \, \|t\|^{s-1} \, dt \\ &= \sigma(S_+^{m-1}) \int_0^1 (1 - x^2)^{z+2} \, x^{s+m-2} \, dx \\ &= \frac{\sigma(S_+^{m-1})}{2} B\left(z + 3, \frac{s}{2} + \rho - \frac{1}{2} \right) \\ &= \frac{\sigma(S_+^{m-1})}{2} \frac{(z+2)(z+1)}{(z + \rho + 3/2 + s/2)(z + \rho + 1/2 + s/2)} B\left(z + 1, \frac{s}{2} + \rho - \frac{1}{2} \right) \\ &= \frac{\sigma(S_+^{m-1})}{2} \frac{(z+2)(z+1)}{(z + 2 + (2\rho + s - 1)/2)(z + 1 + (2\rho + s - 1)/2)} \mathcal{M}f^z(s). \end{aligned}$$

We have used the integral formula (cf. [6, 6.1(31)])

$$\int_0^1 (1 - x^h)^{\nu - 1} \, x^{s-1} \, dx = \frac{1}{h} B(\nu, s/h), \qquad h > 0, \quad \mathrm{Re}\, \nu > 0, \quad \mathrm{Re}\, s > 0.$$

Applying Lemma 6.1 to f^{z+2}, we obtain

$$\mathcal{M}\left[\left(z+2-\frac{1}{2}\sum_{j=1}^{m_\alpha} t_j\frac{\partial}{\partial t_j}\right)\left(z+1-\frac{1}{2}\sum_{j=1}^{m_\alpha} t_j\frac{\partial}{\partial t_j}\right)f^{z+2}\right]$$
$$=\mathcal{M}[(z+2)(z+1)f^z]. \tag{69}$$

In (69), the transformation $\mathcal{M}$ is applied to radial functions belonging to the space R_1. By injectivity, it follows that

$$\left(z+2-\frac{1}{2}\sum_{j=1}^{m_\alpha} t_j\frac{\partial}{\partial t_j}\right)\left(z+1-\frac{1}{2}\sum_{j=1}^{m_\alpha} t_j\frac{\partial}{\partial t_j}\right)f^{z+2}=(z+2)(z+1)f^z.$$

Equations (64) and (65) thus follow by setting $z:=2z(\lambda)$.

We claim that $B(\lambda):=(-\lambda-\rho+1)(-\lambda-\rho+2)$ divides the Bernstein polynomial. This follows from the results of Section 5 for all $\rho\neq 1$. Suppose now $\rho=1$. From Section 5 we can only deduce that λ is a divisor of the Bernstein polynomial, and we need therefore to prove that also $-\lambda+1$ is a divisor. Equivalently, setting $z:=2z(\lambda)$, we have to show the following property:

If $h(z)$ is a polynomial for which exists a polynomial differential operator $\tilde{Q}(\lambda-\delta, t, \nabla)$ satisfying

$$\tilde{Q}(\lambda-\delta, t, \nabla)(1-t_1^2-t_2^2)^{z+2}=h(z)(1-t_1^2-t_2^2)^z,$$

then $z+1$ divides $h(z)$. (Here $t=(t_1, t_2)$.)

Write

$$\tilde{Q}(\lambda-\delta, t, \nabla)=\sum_{\mathbf{q}} f_{\mathbf{q}}(z, t)\left(\frac{\partial}{\partial t}\right)^{\mathbf{q}} \tag{70}$$

with $\mathbf{q}=(q_1, q_2)\in\mathbb{N}_0\times\mathbb{N}_0$. Observe that $z+1$ divides $(\frac{\partial}{\partial t})^{\mathbf{q}}(1-t_1^2-t_2^2)^{z+2}$ for $|\mathbf{q}|\geqslant 3$. Evalution of both sides of (70) at $z=-1$ then gives

$$f_{(0,0)}(-1, t)(1-t_1^2-t_2^2)-2f_{(1,0)}(-1, t)\,t_1-2f_{(0,1)}(-1, t)\,t_2-4$$
$$=h(-1)\frac{1}{1-t_1^2-t_2^2}.$$

The left-hand hand side is a polynomial in t_1 and t_2, whereas the $(1-t_1^2-t_2^2)^{-1}$ on the right-hand side has singularities. For the equality to hold, we must have $h(-1)=0$. Thus $z+1$ divides $h(z)$, which proves the claim.

Having determined a differential operator $Q(\lambda, t, \nabla)$ and a polynomial $b(\lambda-\delta)=B(\lambda)$ for which Equation (15) holds, we can conclude that $B(\lambda)$

is indeed minimal among the monic polynomials satisfying (15), and hence the Bernstein polynomial of $p(t)$.

Parts (3) and (4) are an immediate consequence of (2). ∎

Proposition 6.2 implies that the polar set of $\varphi_\lambda(a)$ is contained in $\{k-\rho : k \in \mathbb{N}\}$ for all $a \in S^0 \cap A$. In fact, this can be also immediately seen from the explicit formula (6), which shows that $\Gamma(-\lambda-\rho+1)^{-1}\,\varphi_\lambda(a)$ is an entire function of $\lambda \in \mathbb{C}$ for all $a \in S^0 \cap A$. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Using [5, 2.1.1 (4)], one can moreover deduce from (6) that $\lambda = \rho + k$, $k \in \mathbb{N}_0$, is a removable singularity of $\varphi_\lambda(a)$ for all $a \in S^0 \cap A$ when $\rho \in \mathbb{N}$. We now see how removable singularities can be determined by means of the method of Bernstein polynomials. This should provide a hint for dealing with removable singularities in higher rank cases where explicit formulas are not available.

The main idea is the following. Let $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ be a possible pole for $\varphi_\lambda(a)$, that is a zero of some $b(\lambda - m\delta)$. The polynomial $b_m(\lambda)$ has simple zeros in $\mathscr{E} + m\delta$. The functional relation (24) implies that λ_0 is a removable singularity for the meromorphic extension of $\varphi_\lambda(a)$ if and only if $I_m(\lambda_0, a) = 0$. Otherwise λ_0 is a simple pole.

Set $a_\sigma := \exp(\sigma Y^0)$ for $\sigma > 0$. Then $I_m(\lambda, a_\sigma)$ is given by (47) for $\lambda \in \mathscr{E} + m\delta$. Hence

$$I_m(k-\rho, a_\sigma) = a_\sigma^k \int_{\|t\|<1} [P_m(k-\rho, t, \nabla)(1 - e^{-2\sigma}\|t\|^2)^{k-2\rho}](1 - \|t\|^2)^{2m-k}\, dt$$

for $2m > k-1$.

The polar set of $\varphi_\lambda(a_\sigma)$ is given by the following corollary.

Corollary 6.1. (1) *Let $\rho \in \mathbb{N}$. Then $\varphi_\lambda(a_\sigma)$ has:*

(a) *Simple poles at $\lambda = 1-\rho, \ldots, -1, 0, 1, \ldots, \rho-1$ for all $\sigma > 0$;*

(b) *Removable singularities at $k-\rho$ with $k \geqslant 2\rho$ for all $\sigma > 0$.*

(2) *Let $\rho \notin \mathbb{N}$. Then $\varphi_\lambda(a_\sigma)$ has:*

(a) *Simple poles at $\lambda = k-\rho$ with $1 \leqslant k < \rho$ for all $\sigma > 0$;*

(b) *Simple poles at $\lambda = k-\rho$ with $k > \rho$ for all $\sigma > 0$ except at most countably many.*

Proof. Fix $2m > k-1$. Observe that

$$P_m(k-\rho, t, \nabla)(1 - e^{-2\sigma}\|t\|^2)^{k-2\rho} = \sum_{h=0}^{\infty} \frac{(2\rho-k)_h}{h!}\, e^{-2\sigma h} \left[\prod_{q=1}^{2m} (-k+\rho+q+h)\right] \|t\|^{2h}. \tag{71}$$

We have set

$$(2\rho - k)_0 := 1,$$
$$(2\rho - k)_h := (2\rho - k)(2\rho - k + 1) \cdots (2\rho - k + h - 1), \qquad h \in \mathbb{N}.$$

When $k < 2\rho$,

$$(2\rho - k)_h > 0 \qquad \text{for all} \quad h \in \mathbb{N}_0.$$

When $k \geqslant 2\rho$,

$$(-1)^h (2\rho - k)_h = \begin{cases} (k - 2\rho - h + 1)_h > 0 & \text{for all} \quad h \in \{0, ..., k - 2\rho\}, \\ 0 & \text{for all} \quad h > k - 2\rho. \end{cases} \tag{72}$$

Consider $\prod_{q=1}^{2m} (-k + \rho + q + h)$. It is positive for all $h \in \mathbb{N}_0$ when $k \leqslant \rho$. When $\rho \in \mathbb{N}$ and $k < 2\rho$, it is non-negative for all $h \in \mathbb{N}_0$ and positive for h large. In these cases, $P_m(k - \rho, t, \nabla)(1 - e^{-2\sigma} \|t\|^2)^{k - 2\rho}$ is then a positive function and therefore $I_m(k - \rho, a_\sigma) > 0$. This proves (2)(a) for $1 \leqslant k < \rho$ and (1)(a).

When $k \geqslant 2\rho$, the sum in (71) is finite because of (72). Suppose also $\rho \in \mathbb{N}$. Then $k - \rho - h$ is an integer $\geqslant \rho$ and $\leqslant k - 1$ for all $h \in \{0, ..., k - 2\rho\}$. Since $2m > k - 1$, it follows that $\prod_{q=1}^{2m} (-k + \rho + q + h) = 0$ for all h. Thus $P_m(k - \rho, t, \nabla)(1 - e^{-2\sigma} \|t\|^2)^{k - 2 - \rho} \equiv 0$ and $I_m(k - \rho, a_\sigma) = 0$ for all $\sigma > 0$, which proves (1)(b).

Suppose $\rho \notin \mathbb{N}$. Because of (40),

$$\lim_{\sigma \to +\infty} P_m(k - \rho, t, \nabla)(1 - e^{-2\sigma} \|t\|^2)^{k - 2\rho}$$
$$= P_m^0(k - \rho) = \prod_{q=1}^{2m} (-k + \rho + q) \neq 0$$

uniformly in $\{\|t\| < 1\}$. Hence, for every $k \in \mathbb{N}$, $I_m(k - \rho, a_\sigma)$ is never identically 0 as a function of a_σ when $\rho \notin \mathbb{N}$. It follows that $k - \rho$ is in this case always a pole of $\varphi_\lambda(a_s)$ for all $\sigma > 0$ except for at most countably many. In particular, (2)(b) holds. ▮

Remark 6.1. The result in Corollary 6.1 for the case $\rho \notin \mathbb{N}$ can be improved by using the explicit formula for the spherical functions and the classical results in [31] on the zeros $x \in (0, 1)$ of the hypergeometric function ${}_2F_1(\frac{-\lambda + \rho}{2}, \frac{-\lambda + \rho + 1}{2}; 1 - \lambda; x)$ for $\lambda \in \mathbb{R}$. It is in fact possible to deduce from [31] that, when $\rho \notin \mathbb{N}$, the singularity of $\varphi_\lambda(a_\sigma)$ at $\lambda = k - \rho$ is removable for exactly one value of σ in the following cases:

(a) k odd, $k > \rho$ and $2\rho \equiv 1 (\mathrm{mod}\ 4)$;

(b) k even, $\rho < k < 2\rho$ and $2\rho \equiv 3 (\mathrm{mod}\ 4)$.

In all other cases the singularity is not removable.

On the other hand, compared to the classical arguments for the hypergeometric functions, it is surely remarkable how easily the removability or not removability of singularities is established in Corollary 6.1 when $\rho \in \mathbb{N}$.

7. REMARKS AND OPEN PROBLEMS

The most important problem that this paper leaves open is the verification of the conjecture on the Bernstein polynomial. Many Gröbner-basis-related algorithms for computing Bernstein polynomials have been recently appeared in the literature (cf. [29] and references therein). However, to our knowledge, they deal only with the case $l = 1$, which for us corresponds to the rank-one case. These algorithms are therefore of no help for approaching the higher-rank situation. Explicit computations of Bernstein polynomials when $l > 1$ have been done with D-module techniques by Kashiwara in [21] and Gyoja in [12]. The polynomial powers they use arise from semi-invariant corresponding to dominant integral weights for algebraic and semisimple simply connected complex Lie groups, a setting with certain similarities with ours. The Bernstein polynomial they obtain are products over the set of the positive roots of Bernstein polynomial corresponding to the rank-one case, the same structure we conjecture as true for our Bernstein polynomial (the polynomials are however not the same!).

For rank-one NCC spaces, the starting point for the direct computation the Bernstein polynomial in Section 6 was the explicit formula for the function a_H on $N(\Omega)$. The next class of NCC symmetric spaces on which conjectures are usually tested are the spaces with a complex structure. On them the spherical functions are essentially weighted averages of exponentials, and their harmonic analysis reduces to the classical Fourier analysis on cones. A general explicit formula for a_H is however not known on these spaces, so there is no direct method to check the expression of the Bernstein polynomials on them without going into a case by case approach.

On the other hand, a suitable testing class is that of spaces of Cayley type: for these spaces Jordan algebraic techniques are available and can be used to calculate a_H (cf. for instance [7]). Very close to the Cayley type space are the series $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$ (with a complex structure) and

$SL(2, \mathbb{H})$. The corresponding function a_H on the bounded domains $N(\Omega)$ has been computed in [10].

Besides the asymptotic behavior of the spherical functions determined in Section 4, a possible interesting application of the integral formulas is the determination of estimates for the spherical functions for all values of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. For $\lambda \in \mathscr{E}$ these estimates have been determined in [18], and have been applied for determining the spectra of weighted Bergman spaces. Such estimates could also be useful in Paley–Wiener type theorems.

Regarding the removability of the singularities, the results in the rank-one case depend not only on the form of the Bernstein polynomial, but also on the fact that the method employed yields a "very easy" family of differential operators $P_m(\lambda, t, \nabla)$. These differential operators are (commuting) polynomials in the Euler operator, and their analysis is not harder than explicit one-variable polynomial computation. One should however notice that the operator $Q(\lambda - \delta, t, \nabla)$ in Bernstein's theorem (and hence its formal adjoint P) is not uniquely determined. In fact for every polynomial differential operator $A(\lambda, t, \nabla)$ which annihilates $a_H(\omega_t)^{\lambda+\rho-\delta}$, the differential operator $\tilde{Q}(\lambda - \delta, t, \nabla) := Q(\lambda - \delta, t, \nabla) + A(\lambda, t, \nabla)$ still satisfies the conclusions of Bernstein's theorem. For example, one can easily check by direct computation that in the rank-one case the differential operator

$$\tilde{Q}(\lambda - \delta, t, \nabla) := \frac{1}{4} \sum_{j=1}^{2\rho} \frac{\partial^2}{\partial t_j^2} + \frac{\lambda - 1}{2} \sum_{j=1}^{2\rho} t_j \frac{\partial}{\partial t_j} + (\lambda - 1)(\lambda + \rho - 2)$$

has this property. The formal adjoint of $\tilde{Q}(\lambda - \delta, t, \nabla)$ is

$$\tilde{P}(\lambda - \delta, t, \nabla) = \frac{1}{4} \sum_{j=1}^{2\rho} \frac{\partial^2}{\partial t_j^2} - \frac{\lambda - 1}{2} \sum_{j=1}^{2\rho} t_j \frac{\partial}{\partial t_j} + (-\lambda + 1)(-\lambda + 2).$$

(It is remarkable the fact that $P_1^{00}(\lambda)$ and $\tilde{P}_1^{00}(\lambda)$ are the same.) An analysis of the removable singularities so easy as in the proof of Corollary 6.1 would not be possible with the operator $\tilde{P}(\lambda - \delta, t, \nabla)$. Therefore, if one wishes to improve the results of Corollary 5.1 using the Bernstein polynomial, one needs to have precise pieces of information on the differential operator $Q(\lambda - \delta, t, \nabla)$, and also to have found a method to select a "good" $Q(\lambda - \delta, t, \nabla)$.

8. APPENDIX: HARISH-CHANDRA-TYPE EXPANSIONS

In this appendix we employ the methods of pole cancellation developed by E. Opdam for the study of the polar set of certain Harish-Chandra-type weighted averages of generalized hypergeometric functions. The specific application of this study in our paper is to verify the inclusion of the polar set of the spherical functions in the polar set of the numerator n_Ω of the function c_Ω (as predicted by our Conjecture) by means of an alternative method. Recall in fact from [26] that, for all λ's in the complement of a locally finite union of hyperplanes, the spherical functions φ_λ admit on A^+ the Harish-Chandra-type expansion

$$\varphi_\lambda(a) = c_\Omega(\lambda) \sum_{w \in W_0} c_0(w\lambda)\, \Phi_{w\lambda}(a), \tag{73}$$

where Φ_λ is a generalized hypergeometric function (see formula (74) below for the precise definition of Φ_λ).

We will however work in a much more general setting. There are two reasons for doing this: (a) Heckman–Opdam's methods are natural in a non-symmetric-space situation in which the multiplicities are allowed to assume complex values. Many of the proofs rely on the fact that the generalized hypergeometric functions are also meromorphic in the multiplicity parameter. (b) The restriction to a positive Weyl chamber of the spherical functions on Riemannian noncompact symmetric spaces and on NCC spaces occur as particular cases of the setting we consider. It is reasonable to expect that weighed averages of generalized hypergeometric functions might occur as special functions also in the more general setting of K_ε-symmetric spaces of Oshima and Segikuchi.

One should remark that the expansion formulas hold for almost all $\lambda \in \mathfrak{a}_\mathbb{C}^*$, but only on A^+, because on the walls of this set the generalized hypergeometric functions become unbounded. On the other hand, the integral formulas (25) hold on the whole $S^0 \cap A$.

The authors express their gratitude to Eric Opdam for several discussions on the subject and for having pointed their attention to his Corollary 2.10 in [27].

Let $\mathfrak{a}$ be an l-dimensional real Euclidean vector space with inner product $(\cdot, \cdot)$. For every non-zero $\alpha \in \mathfrak{a}^*$, let $\bar{H}_\alpha \in \mathfrak{a}$ be determined by $\beta(\bar{H}_\alpha) = (\beta, \alpha)$ and set $H_\alpha := 2\bar{H}_\alpha/(\bar{H}_\alpha, \bar{H}_\alpha)$. Let Δ be a (possibly non-reduced) root system in $\mathfrak{a}^*$ and Δ^+ a choice of positive roots. Set $\Delta^- := -\Delta^+$. We denote by $\Pi = \{\alpha_1, ..., \alpha_l\}$ the set of simple positive roots corresponding to Δ^+. The Weyl group W of Δ is the finite group generated by the reflections w_α $(\alpha \in \Delta)$ given by

$$w_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\, \alpha, \qquad \lambda \in \mathfrak{a}^*.$$

A *multiplicity function* is a W-invariant $\mathbb{C}$-valued function m on Δ: if we set $m_\alpha := m(\alpha)$, then $m_{w\alpha} = m_\alpha$ for all $w \in W$ and $\alpha \in \Delta$.

Consider the complexification $\mathfrak{a}_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} \mathfrak{a}$ of $\mathfrak{a}$ viewed as the Lie algebra of the complex torus $H := \mathfrak{a}_\mathbb{C}/2\pi i\mathbb{Z}\{H_\alpha \colon \alpha \in \Delta\}$. The real form $A := \mathfrak{a}$ of H

is an abelian subgroup with Lie algebra $\mathfrak{a}$. We write $\exp\colon \mathfrak{a} \to A$ for the exponential map and set $A^+ := \exp \mathfrak{a}^+$, where $\mathfrak{a}^+$ denotes the open Weyl chamber in $\mathfrak{a}$ on which all elements of Δ^+ are strictly positive.

For an orthonormal basis $\{H_j\}_{j=1}^{l}$ of $\mathfrak{a}$, we consider the differential operator on A^+

$$L(m) := \sum_{j=1}^{l} \partial_j^2 + \sum_{\alpha \in \Delta^+} m_\alpha \coth \alpha \, \partial_\alpha .$$

Here ∂_j ($j = 1, ..., l$) and ∂_α ($\alpha \in \Delta^+$) respectively denote the differential operators on A^+ corresponding to H_j and $\bar{H}_\alpha$.

The prototypical situation in which $L(m)$ physically occurs is the so-called "symmetric case." Suppose G is a connected noncompact real semi-simple Lie group with finite center and let K be a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ ($\subset \mathfrak{g}$) respectively denote the Lie algebras of G and K, and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Fix a maximal subspace $\mathfrak{a}$ of $\mathfrak{p}$, and consider the set Δ of (restricted) roots of the pair $(\mathfrak{g}, \mathfrak{a})$. For every $\alpha \in \Delta$, let m_α be the multiplicity of the root α as a common eigenfunction of all elements ad H ($H \in \mathfrak{a}$). In this context, $L(m)$ coincides with the radial part on A^+ of the Laplace–Beltrami operator of G/K with respect to the left action of K (cf. [17, Proposition 3.9, p. 267])

Set $\Lambda := \mathbb{N}_0 \Delta^+$. Let m be a fixed multiplicity function and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ so that $(\mu, \mu - 2\lambda) \neq 0$ for all $\mu \in 2\Lambda$. The *generalized hypergeometric function* $\Phi_\lambda(m; \cdot)$ on A^+ is the power series

$$\Phi_\lambda(m; a) = a^{\lambda - \rho} \sum_{\mu \in \Lambda} \Gamma_\mu(m; \lambda)\, a^\mu, \qquad a \in A^+, \tag{74}$$

in which the coefficients $\Gamma_\mu(m; \lambda)$ are obtained by means of the recurrence relations

$$\Gamma_0(m; \lambda) = 1$$

$$(\mu, \mu - 2\lambda)\, \Gamma_\mu(m; \lambda) = 2 \sum_{\alpha \in \Delta^+} m_\alpha \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in \Lambda}} \Gamma_{\mu - 2k\alpha}(m; \lambda) \times (\mu + \rho - 2k\alpha - \lambda, \mu - 2\lambda), \qquad \mu \in \Lambda.$$

Observe that the recurrence relations imply $\Gamma_\mu(m; \lambda) = 0$ unless $\mu = \sum_{j=1}^{l} n_j \alpha_j$ with $n_j \geqslant 0$ and n_j even for all $j = 1, ..., l$. Hence the generalized hypergeometric function $\Phi_\lambda(m; a)$ is in fact a sum over $\mu \in 2\Lambda$ and coincides with the one defined by Heckman and Opdam when one replaces 2α with α and $m_{2\alpha}$ with $2k_\alpha$. In the symmetric case $\Phi_\lambda(m; a)$ is the Harish-Chandra's series (cf. [9, Sect. 4.3; 17, Chap. IV]).

THEOREM 8.1 ([27, Corollary 2.3]; see also [16, Corollary 4.2.4]). *For every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying $(\mu, \mu - 2\lambda) \neq 0$ (all $\mu \in 2\Lambda$) and for every multiplicity function m, the series (74) converges to a real analytic function of $a \in A^+$. As a function of λ, it is meromorphic, with at most simple poles located on hyperplanes of the form*

$$\mathscr{H}_\mu := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : (2\lambda - \mu, \mu) = 0\}$$

for some $\mu \in 2\Lambda$.

The λ-singular set of $\Phi_\lambda(m; a)$ is in fact much smaller: all singularities are removable unless $\mu = 2n\alpha$, $n \in \mathbb{N}$. $\alpha \in \Delta^{++}$. Here Δ^{++} denotes the set of positive indivisible roots (that is the roots $\alpha \in \Delta^+$ with $\alpha/2 \notin \Delta^+$).

THEOREM 8.2 [27, Corollary 2.10; 16, Proposition 4.2.5]. *For every $a \in A^+$, the function $\Phi_\lambda(m; a)$ is meromorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with at most simple poles located along hyperplanes of the form*

$$\mathscr{H}_{n,\alpha} := \left\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \frac{\lambda(H_\alpha)}{2} = n\right\}$$

with $\alpha \in \Delta^{++}$ and $n \in \mathbb{N}$.

Let Θ be any subset of the set $\Pi = \{\alpha_1, ..., \alpha_l\}$ of simple roots in Δ^+, and let W_Θ be the subgroup of W generated by the reflections $w_i := w_{\alpha_i}$ $(\alpha_i \in \Theta)$. We write $\langle \Theta \rangle$ for the set of elements in Δ which can be written as linear combinations of elements of Θ. We set

$$\langle \Theta \rangle^\pm := \langle \Theta \rangle \cap \Delta^\pm \qquad \text{and} \qquad \langle \Theta \rangle^{++} := \langle \Theta \rangle \cap \Delta^{++}.$$

LEMMA 8.1. *$W_\Theta(\Delta^\pm \setminus \langle \Theta \rangle^\pm) \subset \Delta^\pm \setminus \langle \Theta \rangle^\pm$ and $W_\Theta(\Delta^{++} \setminus \langle \Theta \rangle^{++}) \subset \Delta^{++} \setminus \langle \Theta \rangle^{++}$.*

Proof. For the first inclusion, it is enough to show that $w_i(\Delta^\pm \setminus \langle \Theta \rangle^\pm) \subset \Delta^\pm \setminus \langle \Theta \rangle^\pm$ for every generator w_i of W_Θ. As the claim is obvious if $\langle \Theta \rangle^\pm = \Delta^\pm$, we may assume that $\Delta^\pm \setminus \langle \Theta \rangle^\pm \neq \emptyset$. If $\alpha \in \Delta^+ \setminus \langle \Theta \rangle^+$, then $\alpha = \sum_{j=1}^l n_j(\alpha)\, \alpha_j$ with $n_j(\alpha) \in \mathbb{N}_0$ for all j, and there is at least an index j_0 with $\alpha_{j_0} \in \Theta$ for which $n_{j_0}(\alpha) > 0$. The j_0-th coefficient of $w_i\alpha = \alpha - \alpha(H_{\alpha_i})\, \alpha_i$ is also equal to $n_{j_0}(\alpha) > 0$. Hence $w_i\alpha \in \Delta^+ \setminus \langle \Theta \rangle^+$. The case of $\Delta^- \setminus \langle \Theta \rangle^-$ is similar. The second inclusion is a consequence of the first and of the fact that W maps indivisible roots into indivisible roots. ∎

For a fixed multiplicity function m, we define the following c-functions associated with $\langle \Theta \rangle$,

$$c_{\Theta}(m;\lambda) := \prod_{\alpha \in \langle \Theta \rangle^{++}} \frac{2^{-(\lambda(H_\alpha)/2)-(m_\alpha/2)}\Gamma(\lambda(H_\alpha)/2)}{\Gamma(\lambda(H_\alpha)/4+m_\alpha/4+1/2)\,\Gamma(\lambda(H_\alpha)/4+m_\alpha/4+m_{2\alpha}/2)}$$

$$c_{\Delta\setminus\Theta}(m;\lambda) := \prod_{\alpha \in \Delta^{++}\setminus\langle \Theta \rangle^{++}} \frac{\begin{pmatrix}\Gamma(-\lambda(H_\alpha)/4-m_\alpha/4+1/2)\\ \times \Gamma(-\lambda(H_\alpha)/4-m_\alpha/4-m_{2\alpha}/2+1)\end{pmatrix}}{2^{\lambda(H_\alpha)/2+m_\alpha/2}\Gamma(-\lambda(H_\alpha)/2+1)}.$$

When Δ is reduced, that is if $2\alpha \notin \Delta$ for all $\alpha \in \Delta$, the duplication formula

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\,\Gamma(z+1/2)$$

gives

$$c_{\Theta}(m;\lambda) := \prod_{\alpha \in \langle \Theta \rangle^{++}} \frac{\Gamma(\lambda(H_\alpha)/2)}{2\sqrt{\pi}\,\Gamma(\lambda(H_\alpha)/2+m_\alpha/2)}$$

$$c_{\Delta\setminus\Theta}(m;\lambda) := \prod_{\alpha \in \Delta^{++}\setminus\langle \Theta \rangle^{++}} \frac{\sqrt{\pi}\,\Gamma(-\lambda(H_\alpha)/2-m_\alpha/2+1)}{\Gamma(-\lambda(H_\alpha)/2+1)}.$$

We want to study the λ-poles of the W_Θ-invariant weighted averages

$$\varphi_\lambda(a;m) := c_{\Delta\setminus\Theta}(m;\lambda) \sum_{w \in W_\Theta} c_\Theta(m;w\lambda)\,\Phi_{w\lambda}(a;m) \tag{75}$$

for all $a \in A^+$.

EXAMPLE 8.1. (1) In the symmetric case with $\Theta = \Pi$, the functions $\varphi_\lambda(a;m)$ are (up to multiplicative constants) Harish-Chandra's spherical functions on the Riemannian symmetric space G/K. In the general non-symmetric case, the $\varphi_\lambda(a;m)$'s have been studied by Heckman and Opdam (cf. [16] and references therein).

(2) Let G/K be the Riemannian dual of a NCC symmetric space G/H. Then Δ is reduced, so $\Delta^{++} = \Delta^+$. We can choose the ordering such that the set of simple roots is of the form $\Pi = \{\gamma_1\} \cup \Pi_0$, where $\gamma_1 \in \Delta_+$ and Π_0 is a set of simple positive roots for the Riemannian symmetric space $G(0)/K(0)$. Selecting $\Theta = \Pi_0$, we obtain $\langle \Theta \rangle = \Delta_0$ and $W_\Theta = W_0$. Up to a constant multiple, we have $c_\Theta \equiv c_\Omega$. With the product formula for c_Ω, it follows from [26, Theorem 5.7], that the functions $\varphi_\lambda(a;m)$ agree (up to multiplicative constants) with the spherical functions on the NCC symmetric space G/H. In the case of arbitrary multiplicities, the functions $\varphi_\lambda(a;m)$ have been introduced in [30, Chap. 1]. Both [26, 30] studied the

regularity and the analytic extendibility of $\varphi_\lambda(a; m)$ in $S^0 \cap A$. The analysis of the λ-singularities of $\varphi_\lambda(a; m)$ has not been considered.

(3) If $\Theta = \emptyset$, then $W_\Theta = \{1\}$, $c_\Theta \equiv 1$, and the averaged sum reduces to the single term expression $c_\Delta(m; \lambda)\, \Phi_\lambda(m, a)$.

The numerator

$$n_{\Delta\setminus\langle\Theta\rangle}(m; \lambda) := \prod_{\alpha \in \Delta^{++}\setminus\langle\Theta\rangle^{++}} \Gamma\left(-\frac{\lambda(H_\alpha)}{4} - \frac{m_\alpha}{4} + \frac{1}{2}\right) \times \Gamma\left(-\frac{\lambda(H_\alpha)}{4} - \frac{m_\alpha}{4} - \frac{m_{2\alpha}}{2} + 1\right)$$

of the function $c_{\Delta\setminus\langle\Theta\rangle}$ has at most poles located along the hyperplanes

$$\mathcal{H}_{-m_\alpha/2+(2n-1),\,\alpha}, \qquad \alpha \in \Delta^{++}\setminus\langle\Theta\rangle^{++}, \quad n \in \mathbb{N}$$
$$\mathcal{H}_{-m_\alpha/2-m_{2\alpha}+2n,\,\alpha}, \qquad \alpha \in \Delta^{++}\setminus\langle\Theta\rangle^{++}, \quad n \in \mathbb{N}.$$

The poles associated with $\alpha \in \Delta^{++}\setminus\langle\Theta\rangle^{++}$ are simple when $m_{2\alpha}$ is not an odd integer. In the symmetric case, it is known that if 2α is a root, then $m_{2\alpha}$ is odd. In this case the poles of $n_{\Delta\setminus\langle\Theta\rangle}$ are simple exactly when 2α is not a root for all $\alpha \in \Delta^{++}\setminus\langle\Theta\rangle^{++}$. This occurs for instance when Δ is reduced.

The denominator

$$\prod_{\alpha \in \Delta^{++}\setminus\langle\Theta\rangle^{++}} 2^{\lambda(H_\alpha)/2 + m_\alpha/2}\, \Gamma\left(-\frac{\lambda(H_\alpha)}{2} + 1\right)$$

of $c_{\Delta\setminus\langle\Theta\rangle}$ contributes to $\varphi_\lambda(m; a)$ with simple zeros located along the hyperplanes

$$\mathcal{H}_n, \alpha, \qquad \alpha \in \Delta^{++}\setminus\langle\Theta\rangle^{++}, \quad n \in \mathbb{N}.$$

Notice that for $\alpha \in \Delta$ and $w \in W$, the equality $wH_\alpha = H_{w\alpha}$ implies that $w\lambda \in \mathcal{H}_{n,\alpha}$ if and only if $\lambda \in \mathcal{H}_{n,w\alpha}$.

The function $c_\Theta(m; \lambda)$ has no singularities along hyperplanes associated with roots in $\Delta^{++}\setminus\langle\Theta\rangle^{++}$. Lemma 8.1 ensures that $c_\Theta(m; w\lambda)$ has the same property for all $w \in W_\Theta$. The above observations and Theorem 8.2 imply that for every $w \in W_\Theta$ the function $c_\Theta(m; w\lambda)\, \Phi_{w\lambda}(m; a)$ has at most simple poles along the hyperplanes

$$\mathcal{H}_{n,w\alpha}, \qquad \alpha \in \langle\Theta\rangle^{++}, \qquad n \in \mathbb{Z},$$
$$\mathcal{H}_{n,w\alpha}, \qquad \alpha \in \Delta^{++}\setminus\langle\Theta\rangle^{++}, \quad n \in \mathbb{N}.$$

Because of Lemma 8.1, the latter family of hyperplanes is included in the set of zeros coming from the denominator of $c_{\Delta\setminus\langle\Theta\rangle}$.

The possible singularities of $\varphi_\lambda(m; a)$ are therefore those of $n_{\Delta\setminus\langle\Theta\rangle}(m; \lambda)$ together with possible simple poles along hyperplanes

$$\mathscr{H}_{n,\alpha}, \qquad \alpha \in \langle\Theta\rangle^{++}, \quad n \in \mathbb{Z}.$$

Using the method of pole cancellation developed by Opdam in [27], we now prove that the singularities associated with roots in $\langle\Theta\rangle^{++}$ are in fact removable.

Introduce the function

$$c^c_\Theta(m; \lambda) := \prod_{\alpha \in \Delta^{++}\setminus\langle\Theta\rangle^{++}} \frac{2^{-\lambda(H_\alpha)/2 - m_\alpha/2}\,\Gamma(\lambda(H_\alpha)/2)}{\left(\begin{array}{c}\Gamma(\lambda(H_\alpha)/4 + m_\alpha/4 + 1/2)\,\Gamma(\lambda(H_\alpha)/4 \\ + m_\alpha/4 + m_{2\alpha}/2)\end{array}\right)}.$$

Then

$$\tilde{c}(m; \lambda) := c_\Theta(m; \lambda)\, c^c_\Theta(m; \lambda)$$

agrees (up to a function of m) with the c-function of the root system Δ (cf. Formula 6.4 in [15]).

The zeros and poles of $c^c_\Theta(m; \lambda)$ are along hyperplanes associated with roots in $\Delta^{++}\setminus\langle\Theta\rangle^{++}$. The intersection of an hyperplane associated with a root in $\langle\Theta\rangle^{++}$ and an hyperplane associated with a root in $\Delta^{++}\setminus\langle\Theta\rangle^{++}$ is a variety of codimension >1. It follows from Hartogs's theorem that the singularities of $\varphi_\lambda(m; a)$ along $\mathscr{H}_{n,\alpha}$ ($\alpha \in \langle\Theta\rangle^{++}, n \in \mathbb{Z}$) are removable if and only if the singularities along these hyperplanes of the function

$$\tilde{\varphi}_\lambda(a; m) := \sum_{w \in W_\Theta} \tilde{c}(m; w\lambda)\, \Phi_{w\lambda}(a; m), \qquad a \in A^+,$$

are removable. Since the claim is obviously true when $\Theta = \emptyset$, we will exclude this case from the following discussion. In particular, W_Θ will never be trivial.

Lemma 8.2. *The singularities of* $\tilde{\varphi}_\lambda(a; m)$ *along the hyperplanes*

$$\mathscr{H}_{n,\alpha} \qquad \text{with} \quad \alpha \in \langle\Theta\rangle^{++}, \qquad n \in \mathbb{Z},$$

are all removable.

Proof. We argue as in the proof of Corollary 2.7 in [27] (see also Theorem 4.3.14 in [16]). The function $\tilde{\varphi}_\lambda(a; m)$ cannot have simple poles along hyperplanes of the form $\mathscr{H}_{0,\alpha}$ ($\alpha \in \langle \Theta \rangle^{++}$) because it is W_Θ-invariant. By W_Θ-invariance, we can also restrict ourselves to prove the removability of the poles along $\mathscr{H}_{n,\alpha}$ with $\alpha \in \langle \Theta \rangle^{++}$ and $n \in \mathbb{N}$.

Let then $\alpha \in \langle \Theta \rangle^{++}$ and $n \in \mathbb{N}$ be fixed. We choose λ_0 in $\mathscr{H}_{n,\alpha}$ with the property that λ_0 does not belong to any other hyperplane $\mathscr{H}_{m,\beta}$ with $\beta \in \langle \Theta \rangle^{++}$ and $n \in \mathbb{Z}$. We claim that for $a \in A^+$ the residue

$$\operatorname*{Res}_{\lambda_0} \tilde{\varphi}_\lambda(a; m) := \lim_{\lambda \to \lambda_0} \{(\lambda - n\alpha, n\alpha)\, \tilde{\varphi}_\lambda(a; m)\}$$

of $\tilde{\varphi}_\lambda(a; m)$ along $\mathscr{H}_{0,\alpha}$ vanishes at λ_0. Let w_α denote the reflection associated with the root α. As in [27, Corollary 2.7 (second case)], we compute

$$\operatorname*{Res}_{\lambda_0} \tilde{\varphi}_\lambda(a; m) = \sum_{\substack{w \in W_\Theta \\ w\alpha > 0}} d(m, w; n, \alpha, \lambda_0)\, \Phi_{ww_\alpha \lambda_0}(m; a)$$

with coefficients

$$d(m, w; n, \alpha, \lambda_0) = \lim_{\lambda \to \lambda_0} (\lambda - n\alpha, n\alpha)\{\tilde{c}(m; ww_\alpha \lambda) + \tilde{c}(m; w\lambda_0)\, \Gamma_{2nw\alpha}(m; w\lambda)\}.$$

Up to a constant multiple, these coefficients agree with those in [27], and they are hence equal to zero.

The result then follows from Hartogs's extension theorem. ∎

Corollary 8.1. *For a fixed multiplicity function m and all $a \in A^+$, the function $\varphi_\lambda(a; m)$ is meromorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Its polar set is contained (counting multiplicities) in the polar set of the function $n_{\Delta \setminus \langle \Theta \rangle}(m; \lambda)$. In particular,*

$$\frac{\varphi_\lambda(a; m)}{n_{\Delta \setminus \langle \Theta \rangle}(m; \lambda)}$$

extends as an entire function on $\mathfrak{a}_{\mathbb{C}}^*$.

When Δ is a reduced system of roots, then

$$n_{\Delta \setminus \langle \Theta \rangle}(m; \lambda) = \prod_{\alpha \in \Delta^{++} \setminus \langle \Theta \rangle^{++}} \sqrt{\pi}\, 2^{\lambda(H_\alpha)/2 + m_\alpha/2 + 1}\, \Gamma\left(-\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1\right),$$

and the possible poles of $\tilde{\varphi}_\lambda(a; m)$ are all simple.

Corollary 8.1 can be extended to $W_0 \cdot A^+$ by W_0-invariance. For results on the entire $S^0 \cap A$ one needs the monodromy arguments developed by

Heckman in [14] (see also [16, Sect. 4.3]). In the special case of Harish-Chandra expansions for spherical functions on causal symmetric spaces, the continuation to $S^0 \cap A$ for regular values of the parameter λ has been proved in [26, Theorem 5.8]; the statement for arbitrary values of the multiplicities can also be found in [30, Chap. 2].

THEOREM 8.3. *For a fixed multiplicity function m, there exists a W_Θ-invariant tubular neighborhood U of $[W_\Theta(\overline{A^+})]^0$ in H so that the function*

$$\frac{\varphi_\lambda(a; m)}{n_{\Delta \setminus \langle\Theta\rangle}(m; \lambda)} = \frac{c_{\Delta\setminus\Theta}(m; \lambda)}{n_{\Delta\setminus\langle\Theta\rangle}(m; \lambda)} \sum_{w \in W_\Theta} c_\Theta(m; w\lambda)\, \Phi_{w\lambda}(a; m)$$

extends as a W_Θ-invariant holomorphic function of $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times U$.

Proof. The argument is essentially the same as in Corollary 4.3.9 in [16]. Let w_i denote the reflection associated with the simple root $\alpha_i \in \Theta$. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfies $\lambda(H_\alpha)/2 \notin \mathbb{Z}$ for all $\alpha \in \Delta^+$, then Theorem 4.3.6 in [16] ensures that there is a neighborhood of $\overline{A^+ \cup w_i A^+}^0$ in H on which

$$\tilde{c}(m; w\lambda)\, \Phi_{w\lambda}(m; a) + \tilde{c}(m; w_i w\lambda)\, \Phi_{w_i w\lambda}(m; a)$$

extends as a holomorphic w_i-invariant function. Set

$$W_\Theta(i) := \{w \in W_\Theta : w^{-1}\alpha_i \in \langle\Theta\rangle^+\}.$$

Then

$$W_\Theta = W_\Theta(i) \,\dot\cup\, w_i W_\Theta(i).$$

Hence, for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying $\lambda(H_\alpha)/2 \notin \mathbb{Z}$ for all $\alpha \in \Delta^+$ and which are not zeros of $c_\Theta^c(m; \lambda)$,

$$\sum_{w \in W_\Theta} c_\Theta(m; w\lambda)\, \Phi_{w\lambda}(a; m)$$

$$= \frac{1}{c_\Theta^c(m; \lambda)} \sum_{w \in W_\Theta(i)} [\tilde{c}(m; w\lambda)\, \Phi_{w\lambda}(m; a) + \tilde{c}(m; w_i w\lambda)\, \Phi_{w_i w\lambda}(m; a)]$$

extends to a holomorphic w_i-invariant function on some neighborhood of $\overline{A^+ \cup w_i A^+}^0$ in H. Since α_i is arbitrary in Θ, we conclude that the function $\varphi_\lambda(a; m)/n_{\Delta\setminus\langle\Theta\rangle}(m; \lambda)$ is a holomorphic in a on a tubular neighborhood U of $[W_\Theta(\overline{A^+})]^0$ in H for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ which do not lie on a locally finite union of hyperplanes. The result then follows from Corollary 8.1, W_Θ-invariance in the λ-parameter, and Lemma 2.6 in [27]. ∎

In the case of NCC symmetric spaces, Corollary 8.3 can be reformuled as follows. Recall the numerator n_Ω of the function c_Ω from formula (51).

COROLLARY 8.2. *Let* φ_λ $(\lambda \in \mathfrak{a}_{\mathbb{C}}^*)$ *denote the meromorphically continued spherical functions on a NCC symmetric space* G/H.

(1) (*cf. Corollary* 5.1) *There is a* W_0*-invariant tubular neighborhood* U *of* $S^0 \cap A$ *so that*

$$\frac{\varphi_\lambda(a)}{n_\Omega(\lambda)}$$

extends as a W_0*-invariant holomorphic function of* $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times U$. *In particular, for all* $a \in S^0 \cap A$, *the function* $\varphi_\lambda(a)$ *has at most simple poles located in the polar set of* $n_\Omega(\lambda)$.

(2) *For all* $a \in S^0 \cap A$, *the function*

$$\lambda \mapsto \frac{\varphi_\lambda(a)}{c_\Omega(\lambda)}$$

is holomorphic on

$$\left\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \operatorname{Re} \frac{\lambda(H_\alpha)}{2} < 1 \ \forall \alpha \in \Delta_+\right\} \supset i\mathfrak{a}^* - c_{\max}.$$

Part (2) of Corollary 8.2 is an immediate consequence of the first part, but finds important applications in Paley–Wiener type theorems for the spherical Laplace transform (cf. [1]).

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