12-1-2002

The c-function for non-compactly causal symmetric spaces

Bernhard Krötz
The Ohio State University

Gestur Ólafsson
Louisiana State University

The c-function for non-compactly causal symmetric spaces

Bernhard Krötz* **and Gestur Ólafsson****

Introduction

In this paper we prove a product formula for the c-function associated to a non-compactly causal symmetric space $\mathcal{M}$. Let us recall here the basic facts. Let G be a connected semisimple Lie group, $\tau : G \to G$ be a non-trivial involution and $H = G^\tau$. Then $\mathcal{M} := G/H$ is a semsimple symmetric space. The space $\mathcal{M}$ is called non-compactly causal, if $\mathfrak{q} := \{X \in \mathfrak{g} : \tau(X) = -X\}$ contains an open H-invariant hyperbolic cone $C \neq \emptyset$. In this case $S := H\exp(C)$ is a open subsemigroup of G. A *spherical function* on $\mathcal{M}$ is an H-biinvariant continuous function on $S/H \subseteq \mathcal{M}$, which defines an eigendistribution of the algebra of H-invariant differential operators on $\mathcal{M}$, see [FHÓ94], [KNÓ98], [Ól97]. There exists a maximal abelian hyperbolic subspace $\mathfrak{a}$ of $\mathfrak{q}$ such that $C = \mathrm{Ad}(H).(\mathfrak{a} \cap C)$. Let

$$\varphi_\lambda(g.x_0) = \int_H a_H(gh)^{\lambda-\rho}\, d\mu_H(h)$$

be a spherical function given by a convergent integral similar to the expression for the spherical functions on the Riemannian symmetric spaces G/K. Here $x_0 \in \mathcal{M}$ is the coset $\{H\}$ and $a_H(g) \in A := \exp(\mathfrak{a})$ is determined by $g \in Ha_H(g)N$. The asypmtotic behaviour of $\varphi_\lambda(a.x_0)$ along $S \cap A$ is given by $\varphi_\lambda(a.x_0) \sim c(\lambda)a^{\lambda-\rho}$, where ρ is half the sum over the positive roots counted with multiplicities. The function $c(\lambda)$ is the *c-function of the space* $\mathcal{M}$. It turns out that the c-function is a product of two c-function, $c(\lambda) = c_\Omega(\lambda)c_0(\lambda)$ where $c_0(\lambda)$ is the Harish-Chandra c-function of a Riemannian subsymmetric space $G(0)/K(0)$ and $c_\Omega(\lambda)$ is a function associated to the real bounded symmetric domain $H/(H \cap K)$, where K is a τ-stable maximal compact subgroup of G. The c-function was first introduced by Oshima-Sekiguchi in [OS80], whereas $c_\Omega(\lambda)$ was first introduced in [FHÓ94].

The c-function for a Riemannian symmetric space G/K can be written as a product of c-functions of rank one symmetric spaces associated to each restricted root of $\mathfrak{g}$ (Gindikin–Karpelevic formula). For general non-Riemannian symmetric spaces G/H one cannot expect this type of result. However, for non-compactly causal symmetric spaces we show in this paper (cf. Theorem III.5) that such a product formula holds. The case of Cayley type spaces has already been treated by J. Faraut in [Fa95] by the use of Jordan algebra methods and in [Gr97] the case $\mathrm{Sl}(n,\mathbb{R})/\mathrm{SO}(p,q)$ is dealed. The approach presented here is general, different and relies on new insights on the fine convex geometry of the real bounded symmetric domain Ω (cf. Theorem II.5 and Theorem II.7.)

Our result has important applications. The c-function was the last unknown part in the formula for the formal degree of the spherical holomorphic discrete series representations

*Supported by the DFG-project HI 412/5-2

**Supported by LEQSF grant (1996-99)-RD-A-12

representations (cf. [Kr99]). Further it gives us important information on the normalized spherical functions $\widetilde{\varphi}_\lambda := c_\Omega(\lambda)^{-1}\varphi_\lambda$. One knows that the function $\lambda \mapsto \widetilde{\varphi}_\lambda(s.x_0)$ has a meromorphic continuation to $\mathfrak{a}_{\mathbb{C}}^*$ (cf. [Ól97]) and the product formula gives us important information on the poles. In particular, this allows more detailed analysis of the spherical Laplace transform, in particular Paley-Wiener type theorems.

I. Non-compactly causal symmetric spaces and Lie algebras

In this section we introduce notation and recall some facts concerning non-compactly causal symmetric Lie algebras and their associated symmetric spaces. Our source of reference is [HiÓl96].

Algebraic preliminaries

Let $\mathfrak{g}$ be a simple finite dimensional real Lie algebra. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be a non-trivial involution. Then $(\mathfrak{g}, \tau)$ is a *symmetric Lie algebra*. We write $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ for the τ-eigenspace decomposition of $\mathfrak{g}$ corresponding to the eigenvalues $+1$ and -1. Let θ be a Cartan involution of $\mathfrak{g}$ which commutes with τ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the associated Cartan decomposition.

For $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}$ let $\mathfrak{z}_\mathfrak{a}(\mathfrak{b}) := \{X \in \mathfrak{a} : [X, Y] = 0, Y \in \mathfrak{b}\}$ be the *centralizer of* $\mathfrak{b}$ *in* $\mathfrak{a}$. We call $(\mathfrak{g}, \tau)$ *non-compactly causal*, or simply NCC, if $\mathfrak{z}_{\mathfrak{q}\cap\mathfrak{p}}(\mathfrak{h} \cap \mathfrak{k}) \neq \{0\}$. We call $(\mathfrak{g}, \tau)$ *non-compactly Riemannian (NCR)* if τ is a Cartan involution. If not otherwise stated from now on $(\mathfrak{g}, \tau)$ denotes a NCC symmetric Lie algebra. Then $\mathfrak{z}_{\mathfrak{q}\cap\mathfrak{p}}(\mathfrak{h} \cap \mathfrak{k}) = \mathfrak{z}(\mathfrak{q} \cap \mathfrak{p}) = \mathbb{R}X_0$ is one dimensional. Let $\mathfrak{a} \subseteq \mathfrak{q} \cap \mathfrak{p}$ be a maximal abelian subspace and note that $\mathbb{R}X_0 \subseteq \mathfrak{a}$ and that $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. We write $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and

$$\mathfrak{g} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha$$

for the corresponding root space decomposition. We write $\mathfrak{g}(0) := \mathfrak{h} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p}$ and note that $(\mathfrak{g}(0), \tau(0))$, with $\tau(0) := \tau|_{\mathfrak{g}(0)}$, is NCR. If $\alpha \in \Delta$ then either $\mathfrak{g}^\alpha \subseteq \mathfrak{g}(0)$ or $\mathfrak{g}^\alpha \subseteq \mathfrak{q}\cap\mathfrak{k} + \mathfrak{h}\cap\mathfrak{p}$. A root $\alpha \in \Delta$ is called *compact* if $\mathfrak{g}^\alpha \subseteq \mathfrak{g}(0)$ and *non-compact* if $\mathfrak{g}^\alpha \subseteq \mathfrak{q}\cap\mathfrak{k} + \mathfrak{h}\cap\mathfrak{p}$. We write Δ_k and Δ_n for the collection of compact and non-compact roots, respectively. Note that $\Delta = \Delta_k \dot\cup \Delta_n$.

We can and will normalize X_0 such that $\mathrm{Spec}(\mathrm{ad}\, X_0) = \{-1, 0, 1\}$. Then $\Delta_k = \{\alpha \in \Delta : \alpha(X_0) = 0\}$ and we can choose a positive system Δ^+ of Δ such that

$$\Delta_n^+ := \Delta_n \cap \Delta^+ = \{\alpha \in \Delta_n : \alpha(X_0) = 1\}$$

and such that $\Delta_k^+ := \Delta_k \cap \Delta^+$ is a positive system in Δ_k. Let $\Delta^- := -\Delta^+$, $\Delta_n^- := -\Delta_n^+$ and $\Delta_k^- := -\Delta_k^+$.

We recall now few facts about the structure of the root system Δ. Two roots $\alpha, \beta \in \Delta$ are said to be *strongly orthogonal* if $\alpha \pm \beta$ is not a root. Let $\Gamma := \{\gamma_1, \ldots, \gamma_r\}$ be a system of strongly orthogonal roots in Δ_n^+ of maximal length, i.e., Γ consists of pairwise strongly orthogonal roots and has maximal number of elements with respect to this property. We set

$$\mathcal{W} := N_{\mathrm{Inn}(\mathfrak{h}\cap\mathfrak{k})}(\mathfrak{a}) / Z_{\mathrm{Inn}(\mathfrak{h}\cap\mathfrak{k})}(\mathfrak{a})$$

and call $\mathcal{W}$ the *small Weyl group* of Δ.

Proposition I.1. *For the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ of a non-compactly causal symmetric Lie algebra $(\mathfrak{g}, \tau)$ the following assertions hold:*

(i) *The root system Δ is reduced, i.e., if $\alpha \in \Delta$ then $2\alpha \notin \Delta$. In particular, there exists at most two root lengths.*

(ii) *All long roots in Δ_n^+ are conjugate under the small Weyl group $\mathcal{W}$. Moreover, all roots γ_i, $1 \leq i \leq r$, are long.*

(iii) *Write $\Delta_{n,s}^+$ for the short roots in Δ_n^+. Then, if $\Delta_{n,s}^+ \neq \emptyset$, one has*

$$\Delta_{n,s}^+ = \{\frac{1}{2}(\gamma_i + \gamma_j): 1 \leq i < j \leq r\}$$

and all elements of $\Delta_{n,s}^+$ are conjugate under $\mathcal{W}$.

Proof. (i) [HiÓl96, Th. 3.2.4] or [NÓ99, Lemma 2.12].
(ii) [NÓ99, Lemma 2.26].
(iii) [NÓ99, Lemma 2.22, Lemma 2.24]. ■

For $\alpha \in \Delta$ let $H_\alpha \in \{[X, \tau(X)]: X \in \mathfrak{g}^\alpha\} \subseteq \mathfrak{a}$ be such that $\alpha(H_\alpha) = 2$. For each $1 \leq i \leq r$ let $H_i = H_{\gamma_i}$. We set $\mathfrak{c} := \mathrm{span}_{\mathbb{R}}\{H_1, \ldots, H_r\} \subseteq \mathfrak{a}$ and write $\mathfrak{b}$ for the orthogonal complement of $\mathfrak{c}$ in $\mathfrak{a}$ with respect to the Cartan-Killing form; in particular $\mathfrak{a} = \mathfrak{c} \oplus \mathfrak{b}$.

Proposition I.2. *The positive system Δ_k^+ can be chsosen such that for the restriction of $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ to $\mathfrak{c}$ the following assertions hold:*

$$\Delta_n^+|_{\mathfrak{c}} = \{\frac{1}{2}(\gamma_i + \gamma_j): 1 \leq i, j \leq r\} \cup \{\frac{1}{2}\gamma_i: 1 \leq i \leq r\},$$

$$\Delta_k^+|_{\mathfrak{c}} \backslash \{0\} = \{\frac{1}{2}(\gamma_i - \gamma_j): 1 \leq j < i \leq r\} \cup \{-\frac{1}{2}\gamma_i: 1 \leq i \leq r\}.$$

Moreover, the second sets in the two unions from above may or may not occur simultaneously.

Proof. [NÓ99, Th. 2.21] or [Kr99, Th. IV.4]. ■

Since we have free choice for Δ_k^+ we assume in the sequel that $\Delta_k^+|_{\mathfrak{c}} \backslash \{0\} \subseteq \{\frac{1}{2}(\gamma_i - \gamma_j): 1 \leq j < i \leq r\} \cup \{-\frac{1}{2}\gamma_i: 1 \leq i \leq r\}$.

Lemma I.3. *Assume that $\Delta_{n,s} \neq \emptyset$ and let Π_k be the set of simple roots corresponding to Δ_k^+. Then there exits $\beta_1, \ldots, \beta_m \in \mathfrak{b}^*$, $\beta_j(X_0) = 0$, and $\delta_1, \ldots \delta_l \in \mathfrak{b}^*$, $\delta_i(X_0) = \frac{1}{2}$, such that*

$$\Pi_k = \{\frac{1}{2}(\gamma_{i+1} - \gamma_i): 1 \leq i \leq r-1\} \cup \{\beta_1, \ldots, \beta_m\} \cup \{-\frac{1}{2}\gamma_r + \delta_i: 1 \leq i \leq l\}.$$

Here the last set occurs if and only if there exits half roots in $\Delta|_{\mathfrak{c}}$.

Proof. For each $\alpha \in \Delta$ let s_α denote the corresponding reflection. Then $s_{\gamma_j}(\frac{1}{2}(\gamma_i + \gamma_j)) = \frac{1}{2}(\gamma_i - \gamma_j)$, $i \neq j$ together with Proposition I.1(iii) shows that $\Delta_k \supseteq \{\frac{1}{2}(\gamma_i - \gamma_j): 1 \leq i \neq j \leq r\}$. Thus Proposition I.2 yields that

$$\begin{aligned}\Delta_k^+ \subseteq \{\frac{1}{2}(\gamma_i - \gamma_j): 1 \leq j < i \leq r\} + (\mathfrak{b}^* \cap X_0^\perp) \\ \cup \{-\frac{1}{2}\gamma_i: 1 \leq i \leq r\} + \{\delta \in \mathfrak{b}^*: \delta(X_0) = \frac{1}{2}\} \cup \mathfrak{b}^*.\end{aligned}$$

Now the assertion follows easily from Proposition I.2 and the fact that Δ is a root system. ■

We define the *maximal cone* in $\mathfrak{a}$ is defined by

$$C_{\max}:=\{X\in\mathfrak{a}:(\forall\alpha\in\Delta_n^+)\,\alpha(X)\geq 0\}.$$

Lemma I.4. *Let $X_0=X_0^b+X_0^c$ with $X_0^b\in\mathfrak{b}$ and $X_0^c\in\mathfrak{c}$. Then we have $X_0^b,X_0^c\in C_{\max}$.*

Proof. First note that $X_0^c=\frac{1}{2}(H_1+\ldots+H_r)$ and so $X_0^c\in C_{\max}$ by Proposition I.2. To show $X_0^b\in C_{\max}$ let $\alpha\in\Delta_n^+$. Then Proposition I.2 shows that $\alpha=\frac{1}{2}(\gamma_i+\gamma_j)+\beta$ with $\beta\in\mathfrak{b}^*$, $\beta(X_0)=\beta(X_0^b)=0$, or $\alpha=-\frac{1}{2}\gamma_i+\delta$ with $\delta\in\mathfrak{b}^*$ and $\delta(X_0)=\delta(X_0^b)=\frac{1}{2}$. In any case we have $\alpha(X_0^b)\geq 0$ concluding the proof of the lemma. ■

Finally we define subalgebras of $\mathfrak{g}$ by

$$\mathfrak{n}:=\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}^\alpha,\quad \overline{\mathfrak{n}}:=\bigoplus_{\alpha\in\Delta^-}\mathfrak{g}^\alpha,\quad \mathfrak{n}_k^\pm:=\bigoplus_{\alpha\in\Delta_k^\pm}\mathfrak{g}^\alpha,\quad \mathfrak{n}_n^\pm:=\bigoplus_{\alpha\in\Delta_n^\pm}\mathfrak{g}^\alpha$$

and note that $\mathfrak{n}=\mathfrak{n}_n^+\rtimes\mathfrak{n}_k^+$ and $\overline{\mathfrak{n}}=\mathfrak{n}_n^-\rtimes\mathfrak{n}_k^-$ are semidirect products.

Analytic preliminaries

Let $G_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and let G be the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}$. Let $H=G^\tau=\{X\in G:\tau(g)=g\}$. We write A, K, N, $\overline{N}$, $N_k^\pm$, $N_n^\pm$ for the analytic subgroups of G which correspond to $\mathfrak{a}$, $\mathfrak{g}(0)$, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{n}$, $\overline{\mathfrak{n}}$, $\mathfrak{n}_k^\pm$, $\mathfrak{n}_n^\pm$. Note that the groups A, N, $\overline{N}$, $N_k^\pm$, $N_n^\pm$ are all simply connected and that the corresponding exponential mappings $\exp_A:\mathfrak{a}\to A$, $\exp_N:\mathfrak{n}\to N$ etc. are all diffeomorphisms. Let $G(0)=Z_G(X_0)=\{g\in G:\mathrm{Ad}(g).X_0=X_0\}$. Then H and $G(0)$ are τ and θ invariant, $H=(H\cap K)\exp(\mathfrak{h}\cap\mathfrak{p})$ and $G(0)=(H\cap K)\exp(\mathfrak{q}\cap\mathfrak{p})$.

The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g}=\mathfrak{h}+\mathfrak{a}+\mathfrak{n}$ and the multiplication mapping

$$H\times A\times N\to G,\quad (h,a,n)\mapsto han$$

is an analytic diffeomorphism onto its open image HAN.

Note that $\overline{N}=N_n^-\rtimes N_k^-$. We have

$$\overline{N}\cap HAN=\exp(\Omega)N_k^-=N_k^-\exp(\Omega) \tag{1.1}$$

with $\Omega\cong H/H\cap K$ a real bounded symmetric domain in $\mathfrak{n}_n^-$.

II. The geometry of the real bounded symmetric domain Ω

We denote by κ the Cartan-Killing form on $\mathfrak{g}$ and define an inner product on $\mathfrak{g}$ by $\langle X,Y\rangle:=-\kappa(X,\theta(Y))$ for $X,Y\in\mathfrak{g}$. Let $X_i\in\mathfrak{g}^{\gamma_i}$ be such that $H_i=[X_i,X_{-i}]$, with $X_{-i}=\tau(X_i)$. By [HiÓl96] and Herman's Convexity Theorem we have

$$\Omega=\{X\in\mathfrak{n}_n^-:\|\,\mathrm{ad}(X+\tau(X))\|<1\} \tag{2.1}$$

$$=\mathrm{Ad}(H\cap K).\{\sum_{j=1}^r t_jX_{-j}:-1<t_j<1,j=1,\ldots,r\}, \tag{2.2}$$

where $\|\cdot\|$ denotes the operator norm corresponding to the scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Note that (2.1) implies that Ω is a convex balanced subset of $\mathfrak{n}_n^-$.

Remark II.1. Recall the definition of the maximal cone C_{rmmax} in $\mathfrak{a}$. Then it is clear from the characterization (2.1) of Ω that $e^{\mathrm{ad}\, X}.\Omega \subseteq \Omega$ for all $X \in C_{\max}$. We also have a *minimal cone* in $\mathfrak{a}$ defined by

$$C_{\min} := \mathrm{cone}(\{[X, \tau(X)]: X \in \mathfrak{g}^\alpha, \alpha \in \Delta^+\}) = \overline{\sum_{\alpha \in \Delta_n^+} \mathbb{R}^+ H_\alpha} \,.$$

We note that $C_{\min} \subseteq C_{\max}$ and in particular $H_i \in C_{\max}$ for each $1 \leq i \leq r$. ∎

The following concept turns out to be very useful for the investigation of the fine convex geometry of Ω.

Definition II.2. (Oshima-Sekiguchi) By a *signature of* Δ we understand a map $\varepsilon: \Delta \to \{-1, 1\}$ with the following properties:

(S1) $\varepsilon(\alpha) = \varepsilon(-\alpha)$ for all $\alpha \in \Delta$.

(S2) $\varepsilon(\alpha + \beta) = \varepsilon(\alpha)\varepsilon(\beta)$ for all $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$. ∎

If $\varepsilon : \Delta \to \{-1, 1\}$ is a signature then $\theta_\varepsilon : \mathfrak{g} \to \mathfrak{g}$ defined by $\theta_\varepsilon(X) = \varepsilon(\alpha)\theta(X)$, $X \in \mathfrak{g}^\alpha$ and $\theta_\varepsilon|_{\mathfrak{z}_\mathfrak{g}(\mathfrak{a})} = \theta|_{\mathfrak{z}_\mathfrak{g}(\mathfrak{a})}$ is an involution on $\mathfrak{g}$ that commutes with θ (see [OS80, Def. 1.2]). As $\tau|_{\mathfrak{z}_\mathfrak{g}(\mathfrak{a})} = \theta|_{\mathfrak{z}_\mathfrak{g}(\mathfrak{a})}$ and $\tau|_{\mathfrak{g}^\alpha} = \pm\theta|_{\mathfrak{g}^\alpha}$, with $+$ if α is compact and -1 if α non-compact, it follows that θ_ε also commutes with τ.

Lemma II.3. *Keep the notation of Definition II.2.*

(i) *If ε is a signature of Δ, then the prescription*

$$\sigma_\varepsilon(X) := \begin{cases} X & \text{for } X \in \mathfrak{z}_\mathfrak{g}(\mathfrak{a}), \\ \varepsilon(\alpha)X & \text{for } X \in \mathfrak{g}^\alpha,\ \alpha \in \Delta \end{cases}$$

defines an involutive automorphism of $\mathfrak{g}$. The involution σ_ε commutes with both τ and θ.

(ii) *Let $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ be a basis of Δ. Then for any collection $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ one can define a signature ε of Δ by setting*

$$\varepsilon(\pm \sum_{i=1}^n n_i\alpha_i) := \prod_{i=1}^n \varepsilon_i^{n_i} \quad \textit{for} \quad \sum_{i=1}^n n_i\alpha_i \in \Delta.$$

(iii) *Let the notation be as in (ii). Then $\varepsilon \mapsto (\varepsilon(\alpha_i))_{i=1}^n$ defines a bijection between the set of signatures of Δ and $\{-1, 1\}^n$.*

Proof. (i) This follows by the Oshima-Sekiguchi construction because $\sigma_\varepsilon = \tau_\varepsilon\theta$. (ii) is clear and (iii) follows from (ii). ∎

In the sequel we identify signatures with elements in $\{-1, 1\}^n$.

Lemma II.4. *Let ε be a signature of Δ. Then $\sigma_\varepsilon(\Omega) = \Omega$.*

Proof. Let $X \in \Omega$. By (2.2) there is a $k \in H \cap K$ and $Y = \sum_{j=1}^r t_j X_{-j}$, $-1 < t_j < 1$ such that $\mathrm{Ad}(k).Y = X$. As σ_ε commutes with τ and θ it follows that $\sigma_\varepsilon(k) \in K \cap H$. Hence $\sigma_\varepsilon(X) = \mathrm{Ad}(\sigma_\varepsilon(k)). \sum_{j=1}^r \varepsilon(\gamma_j)t_j X_{-j} \in \Omega$. ∎

Recall that there is basis $\Pi \subseteq \Delta^+$ having the form

$$\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$$

with α_0 long and non-compact and α_i, $1 \leq i \leq n$ compact. Thus every non-compact negative root $\gamma \in \Delta_n^-$ can be written as $\gamma = -\alpha_0 - \sum_{i=1}^n m_i\alpha_i$, $m_i \in \mathbb{N}_0$. By our choice of Δ_k^+ we have $\alpha_0 = \gamma_1$.

Theorem II.5. *For each $\gamma \in \Delta_n^-$ let $p_\gamma\colon \mathfrak{n}_n^- \to \mathfrak{g}^\gamma$ be the orthogonal projection. Then*

$$X \in \Omega \Rightarrow p_\gamma(X) \in \Omega.$$

Proof. Let $X = \sum_{\gamma\in\Delta_n^-} X_\gamma \in \Omega$ with $X_\gamma \in \mathfrak{g}^\gamma$, $\gamma \in \Delta_n^-$. We have to show that $X_\gamma \in \Omega$. Recall that there are at most two root length in Δ (cf. Proposition I.1(i)).

Case 1: γ is a long root.

By Proposition I.1(ii) there exists an element $h \in N_{\mathrm{Inn}(\mathfrak{h}\cap\mathfrak{k})}(\mathfrak{a})$ such that $h.\gamma = -\alpha_0$. Thus we may assume that $\gamma = -\alpha_0 = -\gamma_1$. Let $H := \sum_{j=2}^r H_j$. By Remark II.1 we have

$$X_1 := \lim_{t\to+\infty} e^{t\,\mathrm{ad}\,H}.X \in \Omega.$$

If we express $X_1 = \sum_{\beta\in\Delta_n^-} X_\beta$ as a sum of root vectors, then Proposition I.2 implies that $\beta|_\mathfrak{c} = -\gamma_1$ or $\beta = \frac{1}{2}\gamma_1 - \delta$ with $\delta(X_0^b) = \frac{1}{2}$. Since $X_0^b \in C_{\max}$ (cf. Lemma I.4), we now get

$$X_\gamma = \lim_{t\to+\infty} e^{t\,\mathrm{ad}\,X_0^b}.X_1 \in \Omega.$$

Case 2: γ is a short root.

By Proposition I.1(iii) we may assume that $\gamma = \frac{1}{2}(\gamma_1+\gamma_2)$ and by Lemma I.3 we may suppose $\alpha_0 = \gamma_1$, $\alpha_j = \frac{1}{2}(\gamma_{j+1}-\gamma_j)$ for $1 \le j \le r-1$. Write

$$X = \sum_{m_i\ge 0} X_{m_1,\ldots,m_n},$$

where $X_{m_1,\ldots,m_n} \in \mathfrak{g}^{-(\alpha_0+\sum_{i=1}^n m_i\alpha_i)}$. Then we have to show that $X_{1,0,\ldots,0} \in \Omega$. Set

$$X_{\mathrm{ev}} := \sum_{m_n\equiv 0(2)} X_{m_1,\ldots,m_n} \quad \text{and} \quad X_{\mathrm{odd}} := \sum_{m_n\equiv 1(2)} X_{m_1,\ldots,m_n}.$$

Then $X = X_{\mathrm{ev}} + X_{\mathrm{odd}}$ and we claim that X_{ev}, $X_{\mathrm{odd}} \in \Omega$. Let $\varepsilon = (1,1,1,\ldots,-1)$. Then by Lemma II.4 we get:

$$\sigma_\varepsilon(X) = \sigma_\varepsilon(X_{\mathrm{ev}} + X_{\mathrm{odd}}) = X_{\mathrm{ev}} - X_{\mathrm{odd}} \in \Omega\,.$$

Since Ω is balanced and convex we moreover have

$$X_{\mathrm{ev}} = \frac{1}{2}(X + \sigma_\varepsilon(X)) \in \Omega \quad \text{and} \quad X_{\mathrm{odd}} = \frac{1}{2}(X - \sigma_\varepsilon(X)) \in \Omega.$$

By repeating this argument we thus my assume that

$$X = \sum_{\substack{m_1\equiv 1(2)\\ m_j\equiv 0(2),\, j>1}} X_{m_1,\ldots,m_n}.$$

Now we apply the contraction semigroup generated by $H = \sum_{j=3}^r H_j \in C_{\max}$ and obtain

$$X_1 := \lim_{t\to+\infty} e^{t\,\mathrm{ad}\,H}.X \in \Omega.$$

Thus we may assume $X = X_1$ and $X = \sum_{\beta\in\Delta_n^-} X_\beta$ with $-\beta = \gamma, \gamma_1, \gamma_2, \frac{1}{2}(\gamma_1+\gamma_2)+\beta, -\frac{1}{2}\gamma_1+\sigma_1, -\frac{1}{2}\gamma_2+\sigma_2$ and $\beta, \sigma_1, \sigma_2 \in \mathfrak{b}^*$, $\sigma_1(X_0) = \sigma_2(X_0) = \frac{1}{2}$ (cf. Proposition I.2). Write $\beta = -\gamma_1 - \sum_{j=1} m_j\alpha_j$. The cases $\beta = \gamma_1$ and $\beta = \gamma_2$ are excluded, since we have $m_1 = 0$, resp. $m_1 = 2$, contradicting $m_1 \equiv 1(2)$. Applying to X the contraction semigroup generated by $X_0^b \in C_{\max}$ excludes the case $\beta = -\frac{1}{2}\gamma_1 + \sigma_1$ and $\beta = -\frac{1}{2}\gamma_2 + \sigma_2$. Let now $Y \in \mathfrak{b}$ such that $\delta_j(Y) > 0$, $1 \le j \le l$, and $\beta_j(Y) > 0$, $1 \le j \le m$ (cf. Lemma I.3). Then $\Delta_n^+ \subseteq \mathbb{N}_0[\Pi]$ shows that $Y \in C_{\max}$. But then

$$X_\gamma = \lim_{t\to+\infty} e^{t\,\mathrm{ad}\,Y}.X \in \Omega,$$

completing the proof Case 2 and hence of the theorem. ∎

Subdomains of rank one

For $\alpha \in \Delta^+$ we set
$$\mathfrak{g}(\alpha) := \big(\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]\big)'$$
and $\tau(\alpha) := \tau\,|_{\mathfrak{g}(\alpha)}$. Then $(\mathfrak{g}(\alpha), \tau(\alpha))$ is a symmetric subalgebra of $(\mathfrak{g}, \tau)$ of real rank one, that is $\mathfrak{a}(\alpha) := \mathfrak{a} \cap \mathfrak{g}(\alpha)$ is one dimensional. Further we set $\mathfrak{h}(\alpha) := \mathfrak{h} \cap \mathfrak{g}(\alpha)$ etc. We denote by $G(\alpha)$, $A(\alpha)$ etc. the analytic subgroups of G corresponding to $\mathfrak{g}(\alpha)$, $\mathfrak{a}(\alpha)$ etc. Let $H(\alpha) = G(\tau)^{\tau(\alpha)} = G(\alpha) \cap H$.

Assume that $\alpha \in \Delta_n^+$. Then $(\mathfrak{g}(\alpha), \tau(\alpha))$ is NCC and $\mathfrak{n}(\alpha) = \mathfrak{n}_n^+(\alpha) = \mathfrak{g}^\alpha$. Let $\Omega(\alpha) \cong H(\alpha)/\big(K(\alpha) \cap H(\alpha)\big)$ be the real bounded symmetric domain in $\overline{\mathfrak{n}}(\alpha) = \mathfrak{n}_n^-(\alpha)$.

Lemma II.6. *Let $\alpha \in \Delta_n^+$ and $s_\alpha \in G(\alpha)$ be a representaive of the one element big Weyl group $N_{G(\alpha)}(\mathfrak{a}(\alpha))/Z_{G(\alpha)}(\mathfrak{a}(\alpha))$ of $\mathfrak{g}(\alpha)$. Then*
$$\big(\overline{N}(\alpha) \cap H(\alpha)A(\alpha)N(\alpha)\big)\dot{\cup}\big(\overline{N}(\alpha) \cap H(\alpha)s_\alpha A(\alpha)N(\alpha)\big)$$
is open and dense in $\overline{N}(\alpha)$.

Proof. This follows by Matsukis Theorem (cf. [Ma79, Theorem 3]), if we can show that $M(\alpha) := Z_{K(\alpha)}(\mathfrak{a}(\alpha)) \subseteq H(\alpha)$ because $s_\alpha M(\alpha) = M(\alpha)s_\alpha$. Let $F = \exp(i\mathfrak{a}(\alpha)) \cap G(\alpha)$. Then one has $M(\alpha) = FZ_{H(\alpha)_o}(\mathfrak{a}(\alpha))$ by [NÓ99, Lemma 5.7]. But if $f \in F$ then $\tau(\alpha)(f) = f^{-1} = f$, by the same lemma. Hence $F \subseteq H(\alpha)$, which implies that $M(\alpha) \subseteq H(\alpha)$. ∎

Theorem II.7. *Let $\alpha \in \Delta_n^+$. Then $\Omega \cap \overline{\mathfrak{n}}(\alpha) = \Omega(\alpha)$.*

Proof. "$\supseteq$": This is clear.
"$\subseteq$": Note that $\Omega \cap \overline{\mathfrak{n}}(\alpha)$ is open and convex in $\overline{\mathfrak{n}}(\alpha)$. We have

$$\exp(\Omega) \cap \big(H(\alpha)s_\alpha A(\alpha)N(\alpha)\big) = \emptyset, \qquad (2.3)$$

since $\exp(\Omega) \subseteq HAN$ and $HAN \cap Hs_\alpha AN = \emptyset$ by Matsukis Theorem. In view of (2.3), Lemma II.7 implies that there exists an open dense subset Ω_α of $\Omega \cap \overline{\mathfrak{n}}(\alpha)$ such that $\Omega_\alpha \subseteq \Omega(\alpha)$. Now the assertion follows from the fact that both $\Omega(\alpha)$ and $\Omega \cap \overline{\mathfrak{n}}(\alpha)$ are open and convex. ∎

III. The product formula for the c-function

Recall the HAN-decomposition in G from Section I. For each $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and g in G we set
$$a_H(g)^\lambda := \begin{cases} 0 & \text{if } g \notin HAN, \\ e^{\lambda(\log a)} & \text{if } g = han \in HAN \end{cases}.$$

For a locally compact group G we write μ_G for a left Haar measurwe on G.

Definition III.1. (The c-functions) For each $\alpha \in \mathfrak{a}^*$ let $m_\alpha := \dim \mathfrak{g}^\alpha$ and put $\rho := \frac{1}{2}\sum_{\alpha \in \Delta^+} m_\alpha \alpha$. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we now set
$$c(\lambda) := \int_{\overline{N}} a_H(\overline{n})^{-(\lambda+\rho)}\; d\mu_{\overline{N}}(\overline{n}) = \int_{\overline{N} \cap HAN} a_H(\overline{n})^{-(\lambda+\rho)}\; d\mu_{\overline{N}}(\overline{n}),$$

$$c_\Omega(\lambda) := \int_{N_n^-} a_H(\overline{n})^{-(\lambda+\rho)}\; d\mu_{N_n^-}(\overline{n}) = \int_\Omega a_H(\overline{n})^{-(\lambda+\rho)}\; d\mu_{N_n^-}(\overline{n}),$$

and

$$c_0(\lambda) := \int_{N_k^-} a_H(\overline{n})^{-(\lambda+\rho)}\; d\mu_{N_k^-}(\overline{n})$$

whenever the defining integral exist. We write $\mathcal{E}$, $\mathcal{E}_\Omega$ and $\mathcal{E}_0$ for the domain of definition of c, c_Ω and c_0, respectively. We call c the *c-function of the non-compactly causal symmetric space G/H* and c_Ω the *c-function of the real bounded symmetric domain Ω*, while c_0 is the usual *c-function of the non-compact Riemannian symmetric space $G(0)/K(0)$*. ■

Remark III.2. (a) The choice of the particular analytic realization G/H of $(\mathfrak{g}, \tau)$ as a symmetric space is immaterial for the definition of the c-function.
(b) We have $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\Omega$ and for all $\lambda \in \mathcal{E}$ one has the splitting

$$c(\lambda) = c_0(\lambda) c_\Omega(\lambda)$$

(cf. [FHÓ94, Lemma 9.2]).
(c) The c-functions can be written as Laplace transforms (cf. [KNÓ98]). Let us explain this for the c-function c. For c_0 and c_Ω one has analogous statements.

There exists a positive Radon measure μ on $\mathfrak{a}$ such that

$$(\forall \lambda \in \mathcal{E}) \quad c(\lambda) = \mathcal{L}_\mu(\lambda) := \int_{\mathfrak{a}} e^{\lambda(X)}\; d\mu(X),$$

i.e., c is the Laplace transform of μ. In particular we see that the domain of definition $\mathcal{E}$ is a tube domain over a convex set, i.e., one has

$$\mathcal{E} = i\mathfrak{a}^* + \mathcal{E}_{\mathbb{R}}$$

with $\mathcal{E}_{\mathbb{R}} \subseteq \mathfrak{a}^*$ a convex subset of $\mathfrak{a}^*$. One knows that $\operatorname{int} \mathcal{E}$ is non-empty. Moreover, the fact that c is a Laplace transform implies that c is holomorphic on $\operatorname{int} \mathcal{E}$ and that c has no holomorphic extension to a connected open tube domain strictly larger than $\operatorname{int} \mathcal{E}$. ■

Now we are going to prove the product formula for the c-function c_Ω. Our srategy is a modified Gindikin-Karpelevic approach as presented in [GaVa88, p. 175–177] or [Hel84, Ch. IV].

For a positive system $R \subseteq \Delta$ we set $\overline{\mathfrak{n}}_R := \bigoplus_{\alpha \in -(\Delta^+ \cap R)} \mathfrak{g}^\alpha$ and write $\overline{N}_R$ for the corresponding analytic subgroup of G. We define an auxiliary c-function by

$$c_R(\lambda) := \int_{\overline{N}_R} a_H(\overline{n})^{-(\lambda+\rho)}\; d\mu_{\overline{N}_R}(\overline{n})$$

whenever the integral exists.

For a single root $\alpha \in \Delta^+$ we set $\rho_\alpha := \frac{1}{2} m_\alpha \alpha$ and write

$$c_\alpha(\lambda) := \int_{\overline{N}(\alpha)} a_{H(\alpha)}(\overline{n})^{-(\lambda+\rho_\alpha)}\; d\mu_{\overline{N}(\alpha)}(\overline{n}).$$

We denote by $\mathcal{E}_\alpha \subseteq \mathfrak{a}_{\mathbb{C}}^*$ the domain of definition of c_α.

Proposition III.3. *For any positive system $R \subseteq \Delta$ we have that*

$$c_R(\lambda) = \prod_{\alpha \in (R \cap \Delta^+)} c_\alpha(\lambda)$$

and $c_R(\lambda)$ is defined if and only if $\lambda \in \bigcap_{\alpha \in (R \cap \Delta^+)} \mathcal{E}_\alpha$.

Proof. We proceed by induction on $|R \cap \Delta^+|$. If $R \cap \Delta^+ = \emptyset$, then the assertion is clear.

Assume that $R \cap \Delta^+ \neq \emptyset$. Then we find an element $\beta \in R \cap \Delta^+$ which is simple in R. Set $Q := s_\beta.R$. Then $Q = R \backslash \{\beta\} \cup \{-\beta\}$ since Δ is reduced (cf. Proposition I.1(i)). Thus we have $(Q \cap \Delta^+) \dot{\cup} \{\beta\} = R \cap \Delta^+$. We now have to distinguish to cases.
Case 1: β is compact.
In this case, the HAN-decomposition of $G(\beta)$ coincides with the Iwasasa decomposition, i.e. $G(\beta) = K(\beta)A(\beta)N(\beta)$. Thus $c_R(\lambda) = c_\beta(\lambda) c_Q(\lambda)$ follow as in [GaVa88, Prop. 4.7.6].
Case 2: β is non-compact.
Set $\overline{N}_Q^k := \overline{N}_Q \cap N_k^-$, $\overline{N}_Q^n := \overline{N}_Q \cap N_n^-$ and note that $\overline{N}_Q = \overline{N}_Q^n \rtimes \overline{N}_Q^k$. Since $\overline{N}_R = \overline{N}(\beta)\overline{N}_Q$ we thus get

$$c_R(\lambda) = \int_{\overline{N}(\beta)} \int_{\overline{N}_Q^n} \int_{\overline{N}_Q^k} a_H(\overline{n}_\beta \overline{n}_n \overline{n}_k)^{-(\lambda+\rho)} \, d\mu_{\overline{N}(\beta)}(\overline{n}_\beta) \, d\mu_{\overline{N}_Q^n}(\overline{n}_n) \, d\mu_{\overline{N}_Q^k}(\overline{n}_k).$$

If $\overline{n}_\beta \overline{n}_n \overline{n}_k \in \overline{N} \cap HAN$, then (1.1) implies that $\overline{n}_\beta \overline{n}_n \in \exp(\Omega)$. Since $\mathfrak{n}_n^-$ is abelian, Theorem II.5 therefore implies that $\overline{n}_\beta \in \exp(\Omega)$ and so $\overline{n}_\beta \in \exp(\Omega(\beta))$ by Theorem II.7. Therefore we can write $\overline{n}_\beta = h_\beta a_\beta n_\beta$ with $h_\beta \in H(\beta)$, $a_\beta \in A(\beta)$ and $n_\beta \in N(\beta)$. Now one can proceed as in [GaVa88, p. 175–177] and one gets $c_R(\lambda) = c_\beta(\lambda) c_Q(\lambda)$. ■

Remark III.4. If we choose $R = -\Delta_n^+ \cup \Delta_k^+$ (this is a positive system since Δ_n^+ is $\mathcal{W}$-invariant), then we have $c_0 = c_R$ and Proposition III.3 results in the Gindikin-Karpelevic product formula

$$c_0(\lambda) = \prod_{\alpha \in \Delta_k^+} c_\alpha(\lambda)$$

of the c-function c_0 on $G(0)/K(0)$ (cf. [GaVa88, Th. 4.7.5] or [Hel84, Ch. IV, Th. 6.13, 6.14]).■

Theorem III.5. (The product formula for c_Ω) *For the c-function c_Ω of the real bounded symmetric domain Ω one has*

$$\mathcal{E}_\Omega = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : (\forall \alpha \in \Delta_n^+) \operatorname{Re} \lambda(H_\alpha) < 2 - m_\alpha\}$$

and

$$c_\Omega(\lambda) = \kappa \prod_{\alpha \in \Delta_n^+} B\Big(\frac{m_\alpha}{2}, -\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1\Big)$$

where B denotes the Beta function and κ is a positive constant only depending on $(\mathfrak{g}, \tau)$.

Proof. Set $\mathcal{E}'_\Omega := \bigcap_{\alpha \in \Delta_n^+} \mathcal{E}_\alpha$. We want to apply Proposition III.3 to $R = \Delta^+$. In view of Remark III.2(b) and Remark III.4, we thus get

$$(3.1) \qquad (\forall \lambda \in \mathcal{E} \cap \mathcal{E}'_\Omega) \quad c_\Omega(\lambda) = \prod_{\alpha \in \Delta_n^+} c_\alpha(\lambda) = \prod_{\alpha \in \Delta_n^+} c_{\Omega(\alpha)}(\lambda).$$

By [FHÓ94, (10.3)] one has

$$c_{\Omega(\alpha)}(\lambda) = 2^{m_\alpha - 1} B\Big(\frac{m_\alpha}{2}, -\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1\Big) \tag{3.2}$$

and

$$\mathcal{E}_{\Omega(\alpha)} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \colon \operatorname{Re}\lambda(H_\alpha) < 2 - m_\alpha\}. \tag{3.3}$$

It follows from (3.3) that

$$\mathcal{E}'_\Omega = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \colon (\forall \alpha \in \Delta_n^+) \operatorname{Re}\lambda(H_\alpha) < 2 - m_\alpha\}. \tag{3.4}$$

Besides $\mathcal{E}_\Omega = \mathcal{E}'_\Omega$ all assertions of the theorem now follow from (3.1)-(3.4). Finally, $\mathcal{E}_\Omega = \mathcal{E}'_\Omega$ follows from the fact that all c-functions involved are Laplace transforms (cf. Remark III.2(c)). ■

The following simple fact that shows that we can split off all the non-compact roots to get the c_Ω-function before we come to the compact roots.

Lemma III.6. *Let R be a any positive system of roots in Δ. If $R \cap \Delta_n^+ \neq \emptyset$, then $R \cap \Delta_n^+$ contains a a root that is simple in R.*

Proof. Let $\{\beta_0, \ldots, \beta_n\}$ be the set of simple roots in R. Let $\gamma \in R \cap \Delta_n^+$. Then $\gamma = \sum_{i=0}^n n_i \beta_i$ with $n_i \in \mathbb{N}_0$. Thus $1 = \gamma(X_0) = \sum_{i=0}^n n_i \beta_i(X_0)$ which implies that $\beta_i(X_0) > 0$ for at least one β_i. But then $\beta_i \in \Delta_n^+$. ■

References

[Fa95] Faraut, J., *Fonctions Sphériques sur un Espace Symétrique Ordonné de Type Cayley*, Contemp. Math. **191** (1995), 41–55.

[FHÓ94] Faraut, J., J. Hilgert, and G. Ólafsson, *Spherical functions on ordered symmetric spaces*, Ann. Inst. Fourier **44** (1994), 927–966.

[GaVa88] Gangolli, R., and V.S. Varadarajan, "Harmonic Analysis of Spherical Functions on Real Reductive Groups," Ergebniss der Mathematik **101**, Springer, 1988.

[Gr97] Graczyk, P., *Function c on an ordered symmetric space*, Bull. Sci. math. **121** (1997), 561–572.

[Hel84] Helgason, S., "Groups and Geometric Analysis", Acad. Press, London, 1984.

[HiÓl96] Hilgert, J. and G. Ólafsson, "Causal Symmetric Spaces, Geometry and Harmonic Analysis," Acad. Press, 1996.

[Kr99] Krötz, B., *Formal dimension of semisimple symmetric spaces*, Compositio math., to appear.

[KNÓ98] Krötz, B., K.–H. Neeb, and G. Ólafsson, *Spherical Functions on Mixed Symmetric Spaces*, submitted.

[Ma79] Matsuki, T., *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Jpn. **31**, 331–357 (1979).

[NÓ99] Neumann, A., and G. Ólafsson, *Minimal and Maximal Semigroups Related to Causal Symmetric Spaces*, Semigroup Forum, to appear.

[Ól97] Ólafsson, G., *Spherical Functions and Spherical Laplace Transform on Ordered Symmetric Spaces*, submitted.

[OS80] Oshima, S., Sekiguchi, J, *Eigenspaces of Invariant Differential Operators on an Affine Symmetric Spaces*, Invent. math. **57** (1980), 1–81.

Bernhard Krötz
Mathematical Institute
TU Clausthal
Erzstraße 1
D-38678 Clausthal–Zellerfeld
Germany
e-mail: mabk@math.tu-clausthal.de

Gestur Ólafsson
Department of Mathematics
Louisiana State University
Baton Rouge
LA 70803
e-mail: olafsson@math.lsu.edu