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Hardy spaces for non-compactly causal symmetric spaces and the most continuous spectrum

Simon Gindikin*, **Bernhard Krötz**** and **Gestur Ólafsson*****

Abstract

Let G/H be a semisimple symmetric space. Then the space $L^2(G/H)$ can be decomposed into a finite sum of series of representations induced from parabolic subgroups of G. The most continuous part of the spectrum of $L^2(G/H)$ is the part induced from the smallest possible parabolic subgroup. In this paper we introduce Hardy spaces canonically related to this part of the spectrum for a class of non-compactly causal symmetric spaces. The Hardy space is a reproducing Hilbert space of holomorphic functions on a bounded symmetric domain of tube type, containing G/H as a boundary component. A boundary value map is constructed and we show that it induces a G-isomorphism onto a multiplicity free subspace of full spectrum in the most continuous part $L^2_{\mathrm{mc}}(G/H)$ of $L^2(G/H)$. We also relate our Hardy space to the classical Hardy space on the bounded symmetric domain .

Introduction

When we transfer from harmonic analysis on Riemannian symmetric spaces to non Riemannian semisimple symmetric spaces G/H one of the most important new phenomena is the fact that different series of unitary representations appear in the decomposition of $L^2(G/H)$ ([BS01ab,D98]). A main objective of harmonic analysis is a *geometric* realization of those *series* of representations; an idea that can be traced back to the article [GG77]. The first step in this program was the realization of the holomorphic discrete series for a group G of Hermitian type in appropriate Hardy spaces in a curved tube in $G_{\mathbb{C}}$. This was accomplished independently in [O82, S86].

There is a natural generalization of the constructions in [GG77, O82, S86] for Hermitian groups G to *compactly causal* symmetric spaces G/H. In [ÓØ88] the holomorphic discrete series for G/H was constructed and the appropriate Hardy space was defined and investigated in [HÓØ91]. The explicit form of the Plancherel density and the corresponding projection operators for the holomorphic discrete series were determined by the two last named authors [K01,KÓ02, Ó00].

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The above mentioned line of work was on the *discrete* part of the spectrum. In this article we give for the first time a geometric realization of the *most continuous* part of the spectrum for a class of non-compactly causal symmetric spaces.

For compactly causal symmetric spaces G/H one has a rich and well understood complex geometry: There exist G-invariant tubes in $G_{\mathbb{C}}/H_{\mathbb{C}}$ which have G/H as Shilov boundary; secondly, most of the compactly causal symmetric spaces can be realized as an open dense orbit in the Shilov-boundary of a bounded Hermitian symmetric space of tube type [Ber96, Bet97, ÓØ99]. This enables us to use complex geometry, holomorphic representations of semigroups, and the well understood structure and harmonic analysis of bounded symmetric domains of tube type. For non-compactly causal symmetric spaces it was believed for a long time that a similar complex geometrical picture does not exist. In [G98] it was conjectured that appropriate tubes (related to complex crowns of Riemannian symmetric spaces) might also exist for non-compactly causal symmetric spaces. It was also conjectured that some Hardy spaces on these tubes are connected with the most continuous spectrum.

In the last year substantial progress was achieved on the geometric part of this program. This gives us now the possibility to start the corresponding analytical investigations – the subject proper of this paper.

The *complex crown* Ξ of a Riemannian symmetric space G/K, first studied in [AG90], is a certain open connected G-invariant Stein domain in $G_{\mathbb{C}}/K_{\mathbb{C}}$ which contains G/K as a totally real submanifold (cf. Section 1 for the definition). The complex crown is universal in many ways. It can be used to parametrize the compact cycles in complex flag manifolds (cf. [GM01]). Further it has the property that the eigenfunctions for the algebra of G-invariant differential operators on G/K extends holomorphically to Ξ [KS01a]. For a more detailed account on the complex crown Ξ we refer to the introduction of [GK02a].

By the recent work of the two first named authors [GK02b] it is known that all non-compactly causal symmetric spaces G/H appear in the *distinguished boundary* $\partial_d\Xi \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}$ of Ξ. Furthermore the non-compactly causal spaces are the only irreducible symmetric spaces which can be realized as G-orbits in $\partial_d\Xi$.

In this paper we consider only those complex crowns that admit a realization as a symmetric space of Hermitian type. By this we mean that there exists a Hermitian group $S \supseteq G$ with maximal compact subgroup $U \supseteq K$ such that Ξ is G-biholomorphic to S/U. For these cases the distinguished boundary $\partial_d\Xi = G(z_1) \simeq G/H$ is exactly one G-orbit and, $\partial_d\Xi$ is equal to the Shilov boundary $\partial_s\Xi$ of Ξ in $G_{\mathbb{C}}/K_{\mathbb{C}}$ [GK02b]. Under the additional assumption that G/H is irreducible and symmetric, hence a non-compactly causal symmetric space, we arrive at the following list:

Table I
$\Xi \simeq S/U$ and $\partial_d\Xi \simeq G/H$ is symmetric

$\mathfrak{g} = \mathrm{Lie}(G)$	$\mathfrak{h} = \mathrm{Lie}(H)$	$\mathfrak{s} = \mathrm{Lie}(S)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R}) \oplus \mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}$	$\mathfrak{su}(n,n) \oplus \mathfrak{su}(n,n)$
$\mathfrak{so}^*(4n)$	$\mathfrak{sl}(n,\mathbb{H}) \oplus \mathbb{R}$	$\mathfrak{so}^*(4n) \oplus \mathfrak{so}^*(4n)$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(1,n-1) \oplus \mathbb{R}$	$\mathfrak{so}(2,n) \oplus \mathfrak{so}(2,n)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$	$\mathfrak{e}_{7(-25)} \oplus \mathfrak{e}_{7(-25)}$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-1)$	$\mathfrak{so}(2,n)$
$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{su}(2n,2n)$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(2n,\mathbb{R})$

We call the triples $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ from the above list *causally symmetric triples.* They can be defined axiomatically starting from a Hermitian Lie algebra $\mathfrak{s}$ of tube type with two commuting involutions τ and σ such that:

(CST1) $\mathfrak{s}^\tau = \mathfrak{g}$ and $(\mathfrak{s}, \tau)$ is compactly causal.

(CST2) $\mathfrak{g}^\sigma = \mathfrak{h}$ and $(\mathfrak{g}, \sigma|_\mathfrak{g})$ is non-compactly causal.

Following É. Cartan the Hermitian symmetric space S/U admits a canonical embedding in the dual compact Hermitian symmetric space and can be realized as a bounded circled domain in $\mathbb{C}^n$. We will use Harish-Chandra's construction of the realization of S/U as a bounded symmetric domain $\mathcal{D}$. One can also realize S/U as an affine homogeneous tube domain $T_\Omega = \mathbb{R}^n + i\Omega$. The Ξ-realization of S/U we use is different from the two mentioned above – it lies inside of $G_\mathbb{C}/K_\mathbb{C}$. The different realizations of S/U give different Shilov boundaries: A compact Shilov boundary $\partial_s\mathcal{D}$ for $\mathcal{D}$ which is a single S-orbit, and a vector space $\mathbb{R}^n$ for the Shilov boundary of the tubes T_Ω. Finally the Shilov boundary for Ξ is a single G-orbit.

Hardy spaces are Hilbert spaces of holomorphic functions which have L^2-boundary values on the Shilov boundary. Since the three realizations mentioned above have different Shilov boundaries, this necessarily leads to different definitions of Hardy spaces. In the first two realizations the constructions of Hardy spaces $\mathcal{H}^2(\mathcal{D})$ and $\mathcal{H}^2(T_\Omega)$ are well known. These two Hardy spaces are isomorphic as irreducible S-modules. We refer to them as *classical Hardy spaces.*

Our definition of the Hardy space on Ξ is new. We use a certain minimal semigroup $\Gamma \subseteq S_\mathbb{C}$ of elements γ with the property $\gamma^{-1}\mathcal{D} \subseteq \mathcal{D}$ (cf. Section 5 for the definition of Γ). Since Ξ is biholomorphic to $\mathcal{D}$, we obtain an action of Γ^{-1} on Ξ by compressions. We define the *Hardy space* $\mathcal{H}^2(\Xi)$ on Ξ by

$$\mathcal{H}^2(\Xi) := \{f \in \mathcal{O}(\Xi) \colon \|f\|^2 := \sup_{\gamma\in\Gamma} \int_{G/H} |f(\gamma^{-1}gz_1)|^2 \, d\mu_{G/H}(gH) < \infty\},$$

where $z_1 \in \partial_d\Xi$ is such that $G/H \simeq Gz_1$. It turns out that $\mathcal{H}^2(\Xi)$ is a Hilbert space of holomorphic functions on Ξ with a natural unitary action L of G by left translations in the arguments. In particular, $\mathcal{H}^2(\Xi)$ admits a reproducing kernel $K_\Xi(z, w)$, the so-called *Cauchy-Szegö kernel*, holomorphic in the first and antiholomorphic in the second variable.

One of our tools is to compare $\mathcal{H}^2(\Xi)$ with the classical Hardy space $\mathcal{H}^2(\mathcal{D})$ which is an irreducible G-spherical unitary highest weight module for the group S. The action of S on $\mathcal{H}^2(\mathcal{D})$ is given through a cocycle representation

$$(\pi_h(s)f)(z) = J_h(s^{-1}, z)^{-1} f(s^{-1}z) \qquad (f \in \mathcal{H}^2(\mathcal{D}), s \in S, z \in \mathcal{D}).$$

Theorem 5.7 now reads:

Theorem A. *There exists an explicit zero free holomorphic function ψ on $\Xi \simeq \mathcal{D}$ such that the mapping*

$$\Psi \colon (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, \mathcal{H}^2(\Xi)), \quad f \mapsto \psi f$$

is a G-equivariant isomorphism of Hilbert spaces.

The fact that G/K is a totally real submanifold of S/U can now be used to construct the generalized Segal-Bargmann transform, which is a unitary G-isomorphism $L^2(G/K) \to \mathcal{H}^2(\Xi)$, [DÓZ01,Ne99,Ó00,ÓØ96,Z01]. This implies that $\mathcal{H}^2(\Xi)$ is a direct integral of principal series representations parametrized by $i\mathfrak{a}^*/\mathcal{W}$. Combining Theorem 5.5 and Theorem 5.8 we come to our main result:

Theorem B. *The boundary value mapping*

$$b \colon \mathcal{H}^2(\Xi) \to L^2(G/H), \quad f \mapsto \big(gH \mapsto \lim_{\substack{\gamma\to\mathbf{1}\\ \gamma\in\Gamma}} f(\gamma^{-1}gz_1)\big)$$

is a G-equivariant isometric embedding. Furthermore the image of b is a multiplicity free full subspace of the most continuous spectrum $L^2(G/H)_{\mathrm{mc}}$ (cf. Definition 4.1).

Our final result, stated in Theorem 5.9, is the identification of the reproducing kernel K_Ξ using the fact that the $\mathcal{H}^2(\Xi)$ is G-isomorphic to $\mathcal{H}^2(\mathcal{D})$. This isomorphism clearly carries K_Ξ into the Szegö kernel K_h for $\mathcal{H}^2(\mathcal{D})$. The same idea was already used for the compactly causal case in [BÓ01,ÓØ99].

Theorem C. *The Cauchy-Szegö kernels K_Ξ and K_h are related through*

$$K_\Xi(z,w) = \psi(z)\overline{\psi(w)}K_h(z,w).$$

To illustrate Theorem C consider the example where $S = G \times G$. Then Ξ is G-biholomorphic to $\mathcal{D}_G \times \mathcal{D}_G^{\mathrm{opp}}$ where $\mathcal{D}_G$ is the Harish-Chandra realization of G/K and $\mathcal{D}_G^{\mathrm{opp}}$ refers to $\mathcal{D}_G$ equipped with the opposite complex structure. Denote by $K_h^G(z,w)$ the Cauchy-Szegö kernel for the classical Hardy space $\mathcal{H}^2(\mathcal{D}_G)$. Then Theorem C says that for all $(z_1,w_1),(z_2,w_2) \in \Xi \simeq \mathcal{D}_G \times \mathcal{D}_G^{\mathrm{opp}}$ we have

$$K_\Xi\big((z_1,w_1),(z_2,w_2)\big) = \frac{K_h^G(z_1,z_2)K_h^G(w_2,w_1)}{K_h^G(z_1,w_1)K_h^G(w_2,z_2)}\,.$$

Note that the formula of K_Ξ is explicit, since the classical Hardy space kernels K_h^G are well known.

In the Plancherel decomposition of $L^2(G/H)$ the most continuous spectrum $L^2_{\mathrm{mc}}(G/H)$ corresponds to one series of representations, the one which is induced off from a minimal parabolic subgroup [BS97]. However, the space $L^2_{\mathrm{mc}}(G/H)$ is not multiplicity free and we encounter the next problem to separate the different multiplicities in $L^2_{\mathrm{mc}}(G/H)$ geometrically. Theorem B implies an embedding

$$\mathcal{H}^2(\Xi) \oplus \mathcal{H}^2(\Xi^{\mathrm{opp}}) \hookrightarrow L^2(G/H)_{\mathrm{mc}}$$

which realizes a multiplicity *two* subspace of the most continuous spectrum of G/H. Presently it is unclear how to obtain the other multiplicities in $L^2(G/H)_{\mathrm{mc}}$ missed by $\mathcal{H}^2(\Xi) \oplus \mathcal{H}^2(\Xi^{\mathrm{opp}})$. In this context notice the striking similarity with the picture for discrete series representations. Only two out of the discrete series, the holomorphic and the antiholomorphic, can be realized as a Hardy space.

We would like to mention, as we know through several talks, that J. Faraut is working on another construction of Hardy spaces for certain non-compactly causal symmetric spaces. Also we would like to mention that our results in Section 2 are related to unpublished work of Y. Neretin on explicit branching formulas for tensor products of highest weight representations.

Our paper is organized as follows:

1. Non-compactly causal symmetric spaces and boundary components of complex domains
2. Continuous branching of spherical highest weight representations
3. Analytical and geometrical constructions on Ξ
4. The classical Hardy space inside $L^2(G/H)_{\mathrm{mc}}$
5. The Hardy space on Ξ

Appendix A: parameter calculations

Appendix B: Structure theory for causally symmetric triples

It is our pleasure to thank the MSRI, Berkeley, for its hospitality during the *Integral geometry program* where this work was accomplished. We also would like to thank the referee for his careful screening of the manuscript and his useful suggestions concerning the readability of the paper.

1. Non-compactly causal symmetric spaces and boundary components of complex domains

The purpose of this section is mostly of preliminary nature. We recall the basic facts concerning causal symmetric spaces. This material is standard and can be found in the monograph [HÓ06]. We then turn to the discussion of the complex crown Ξ of a Riemannian symmetric space G/K. We explain some of the new structural results, in particular those about the distinguished (Shilov) boundary $\partial_d\Xi$ of Ξ (cf. [GK02a]). Subsequently we provide the list of those Ξ which are G-isomorphic to a tube domain $S/U \supseteq G/K$ and which have Shilov boundary $\partial_d\Xi$ isomorphic to a non-compactly causal symmetric space G/H (cf. Table I in the introduction). This will be the class of Ξ's most relevant for this paper. The section is concluded with a discussion of the compactification of $\partial_d\Xi$.

Causal symmetric Lie algebras

Let $\mathfrak{s}$ denote a semisimple real Lie algebra and $\mathfrak{s}_{\mathbb{C}}$ its complexification. We choose a Cartan involution θ on $\mathfrak{s}$ and write $\mathfrak{s} = \mathfrak{u} \oplus \mathfrak{p}_*$ for the associated Cartan decomposition with $\mathfrak{u}$ the maximal compact subalgebra fixed by θ.

In the sequel $\tau: \mathfrak{s} \to \mathfrak{s}$ will denote a non-trivial involution on $\mathfrak{s}$ commuting with θ. Write $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{q}_*$ for the τ-eigenspace decomposition corresponding to the τ-eigenvalues $+1$ and -1. Then $\mathfrak{g}$ is reductive and $\theta|_{\mathfrak{g}}$ is a Cartan involution of $\mathfrak{g}$. With $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$ and $\mathfrak{p} = \mathfrak{g} \cap \mathfrak{p}_*$ we obtain the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

The symmetric pair $(\mathfrak{s}, \mathfrak{g})$ is called *irreducible* if the only τ-invariant ideals in $\mathfrak{s}$ are the trivial ones, $\{0\}$ and $\mathfrak{s}$. In this case either $\mathfrak{s}$ is simple or $\mathfrak{s} \simeq \mathfrak{g} \oplus \mathfrak{g}$, with $\mathfrak{g}$ simple, and $\tau(X, Y) = (Y, X)$ the flip.

On the group level we denote by S and $S_{\mathbb{C}}$ connected Lie groups with Lie algebra $\mathfrak{s}$ and $\mathfrak{s}_{\mathbb{C}}$ respectively. We will assume – if not otherwise stated – that $S \subseteq S_{\mathbb{C}}$. In addition we will require that τ exponentiates to an involution on $S_{\mathbb{C}}$, again denoted by τ. By G we denote the connected subgroup of S with Lie algebra $\mathfrak{g}$. Finally we call the symmetric space S/G *irreducible* if $(\mathfrak{s}, \mathfrak{g})$ is irreducible.

Definition 1.1. Let C be an open convex subset of $\mathfrak{q}_*$. Then C is called *hyperbolic*, if for all $X \in C$ the operator $\mathrm{ad}(X)$ is semisimple with real eigenvalues. We call C *elliptic*, if all operators $\mathrm{ad}(X)$, $X \in C$ are semisimple and with imaginary spectrum. ∎

We recall some facts on causal symmetric spaces (c.f. [HÓ96], Chapter 3):

Definition 1.2. **(Causal symmetric spaces)** The symmetric space S/G is called *causal* if there exists a non-empty open G-invariant convex cone C, containing no affine lines, in $\mathfrak{q}_*$. ∎

There are two different types of causal symmetric spaces, the non-compactly causal symmetric spaces (NCC) and the compactly causal symmetric spaces (CC). In addition, there is the intersection of those two classes, the Cayley type symmetric spaces (CT).

Definition 1.3. **(NCC)** Assume that S/G is an irreducible causal symmetric space. Then the following two conditions are equivalent:

(a) There exists a non-empty G-invariant open hyperbolic cone $C \subseteq \mathfrak{q}_*$ which contains no affine lines;

(b) There exists an element $T^0 \in \mathfrak{q}_* \cap \mathfrak{p}_*$, $T^0 \neq 0$, which is fixed by K.

If one of those equivalent conditions are satisfied, then S/G is called *non-compactly causal*.■

Definition 1.4. **(CC)** Assume that S/G is an irreducible causal symmetric space. Then the following two conditions are equivalent:

(a) There exists a non-empty G-invariant open elliptic cone $C \subseteq \mathfrak{q}_*$ which contains no affine lines;

(b) There exists an element $X^0 \in \mathfrak{q}_* \cap \mathfrak{u}$, $X^0 \neq 0$, which is fixed by K.

If one of those equivalent conditions are satisfied, then S/G is called *compactly causal*. ■

Definition 1.5. **(CT)** Assume that S/G is an irreducible causal symmetric space. Then S/G is called a *symmetric space of Cayley type*, if it is both non-compactly causal and compactly causal. ■

Remark 1.6. (a) The elements T^0 and X^0 in Definition 1.3 and Definition 1.4 are unique up to multiplication by scalar. If S/G is NCC then we normalize T^0 such that $\mathrm{ad}(T^0)$ has spectrum $\{0, 1, -1\}$. The eigenspace corresponding to 0 is exactly $\mathfrak{s}^{\theta\tau} = \mathfrak{k} \oplus (\mathfrak{p}_* \cap \mathfrak{q}_*)$. If S/G is CC, then we normalize X^0 such that the spectrum of $\mathrm{ad}(iX^0)$ is $\{0, 1, -1\}$. In this case the zero eigenspace is exactly $\mathfrak{u}$.

(b) If S/G is compactly causal, then $\mathfrak{u}$ has a non trivial center $\mathfrak{z}(\mathfrak{u})$ and $\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_* = \mathbb{R}X^0$. If S is simple then $\mathfrak{z}(\mathfrak{u}) = \mathbb{R}X^0$. If $\mathfrak{s} \simeq \mathfrak{g} \oplus \mathfrak{g}$. Then $X^0 = (X^{00}, -X^{00})$ with X^{00} central in $\mathfrak{k}$ and $\mathfrak{z}(\mathfrak{u}) = \mathbb{R}X^0 \oplus \mathbb{R}(X^{00}, X^{00})$. ■

Denote the complex linear extension of τ to $\mathfrak{s}_{\mathbb{C}}$ by τ. The c-dual $(\mathfrak{s}^c, \tau^c)$ of $(\mathfrak{s}, \tau)$ is defined by $\mathfrak{s}^c = \mathfrak{g} \oplus i\mathfrak{q}_* \subseteq \mathfrak{s}_{\mathbb{C}}$ with involution $\tau^c = \tau|_{\mathfrak{s}^c}$. Notice that the c-dual of $(\mathfrak{s}^c, \tau^c)$ is $(\mathfrak{s}, \tau)$. Let S^c denote the analytic subgroup of $S_{\mathbb{C}}$ with Lie algebra $\mathfrak{s}^c$. Then $G = (S \cap S^c)_0$, where the subscript $_0$ denotes the connected component containing $\mathbf{1}$. We recall the following fact:

Proposition 1.7. *Assume that S/G is an irreducible symmetric space. Then the following holds:*

(i) *S/G is non-compactly causal if and only if S^c/G is compactly causal.*

(ii) *S/G is of Cayley type if and only if $S/G \simeq S^c/G$.*

(iii) *Suppose that S/G is irreducible and causal. Then S/G is of Cayley type if and only if G is not simple. In that case $\mathfrak{z}(\mathfrak{g})$ is one dimensional and contained in $\mathfrak{p}$.* ■

Example 1.8. **(The group case)** Let G be a connected semisimple Lie group, $S = G \times G$, and $\tau(a, b) = (b, a)$. Then $S/G \simeq G$ is compactly causal if and only if G is a group of Hermitian type. In this case $S_{\mathbb{C}} = G_{\mathbb{C}} \times G_{\mathbb{C}}$ and $S^c/G \simeq G_{\mathbb{C}}/G$ (cf. [HÓ96], Example 1.2.2). ■

Let us remark that the causal symmetric pairs $(\mathfrak{s}, \mathfrak{g})$ are classified and refer to the list [HÓ96, Th. 3.2.8].

The domain Ξ – the complex crown of G/K

According to [GK02a] every non-compactly causal symmetric space G/H can be realized as a boundary component of the corresponding complex crown Ξ of the Riemannian symmetric space G/K. In the following two subsections we briefly recall the definition of Ξ and explain some of the results of [GK02ab] needed for our purpose.

Let $\mathfrak{g}$ be a reductive Lie algebra with complexification $\mathfrak{g}_{\mathbb{C}}$. We let $G_{\mathbb{C}}$ be a connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and let G be the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}$. Let $K \subseteq G$ be a maximal compact subgroup and let $\mathfrak{k}$ be its Lie algebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition corresponding to $\mathfrak{k}$. Finally we choose a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. Let $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and for $\alpha \in \mathfrak{a}^*$ define

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g} \colon (\forall H \in \mathfrak{a})[H, X] = \alpha(H)X\} .$$

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the corresponding set of (restricted) roots, i.e., $\Sigma = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \colon \mathfrak{g}^{\alpha} \neq \{0\}\}$. Then

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^{\alpha} .$$

We write A respectively N for the analytic subgroup of G corresponding to $\mathfrak{a}$ respectively $\mathfrak{n}$. We also need the *Weyl group* of $\Sigma(\mathfrak{a}, \mathfrak{g})$, $\mathcal{W} := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.

Let

$$\Omega = \{X \in \mathfrak{a} \colon (\forall \alpha \in \Sigma) \ |\alpha(X)| < \frac{\pi}{2}\} .$$

Then the *complex crown* of the Riemannian symmetric space G/K is defined by

$$\Xi := G \exp(i\Omega) K_{\mathbb{C}}/K_{\mathbb{C}} .$$

Note that Ξ is an open domain in $G_{\mathbb{C}}/K_{\mathbb{C}}$ and that G acts properly on it (cf. [AG90]). Write $\partial\Xi$ for the topological boundary of Ξ and notice that $G \exp(i\partial\Omega) K_{\mathbb{C}}/K_{\mathbb{C}} \subseteq \partial\Xi$ (cf. [AG90]).

The distinguished boundary of Ξ

Write $\overline{\Xi}$ for the closure of Ξ in $G_{\mathbb{C}}/K_{\mathbb{C}}$, and note that $\overline{\Xi} = \Xi \amalg \partial\Xi$, where $\amalg$ denotes disjoint union. Write $\partial_e\Omega$ for the extreme points of the compact convex set $\overline{\Omega}$. Recall from [GK02a] that $\partial_e\Omega$ is a finite union of $\mathcal{W}$-orbits:

$$\partial_e\Omega = \mathcal{W}(X_1) \amalg \ldots \amalg \mathcal{W}(X_n) .$$

We define the *distinguished boundary of* Ξ *in* $G_{\mathbb{C}}/K_{\mathbb{C}}$ by

$$\partial_d\Xi := G \exp(i\partial_e\Omega) K_{\mathbb{C}}/K_{\mathbb{C}} .$$

It was shown in [GK02a, Th. 2.3] that $\partial_d\Xi$ has the property that

$$\sup_{z \in \Xi} |f(z)| = \sup_{z \in \partial_d\Xi} |f(z)| \tag{1.1}$$

for all bounded holomorphic functions f on Ξ which continuously extend to $\overline{\Xi}$. Moreover, if $\partial_d\Xi$ is connected, i.e., if $\partial_e\Omega = \mathcal{W}(X_1)$ is a single orbit, then $\partial_d\Xi$ is minimal with respect to the property (1.1). In this case we call $\partial_d\Xi$ the *Shilov boundary* of Ξ and denote it by $\partial_s\Xi$.

Write $z_j := \exp(iX_j)K_{\mathbb{C}} \in \partial_d\Xi$ and write H_j for the isotropy subgroup of G in z_j. Then we have the following (cf. [GK02a, Th. 3.6, Th. 3.26]):

Theorem 1.9. *Assume that $G_{\mathbb{C}}$ is simply connected. Then, with the notation introduced above, the following assertions hold:*

(i) $\partial_d\Xi \simeq G/H_1 \amalg \ldots \amalg G/H_n$.

(ii) *If one of the boundary components G/H_j is an irreducible symmetric space, then it is a non-compactly causal symmetric space. Moreover every non-compactly causal symmetric space G/H attached to G appears G-locally in the distinguished boundary of $\partial_d\Xi$ of Ξ.* ∎

Realization of Ξ as tube domains

We consider now the case where Ξ has a realization as a tube domain. The material in this subsection is taken from [KS01b]. We go back to our notations from the first subsection, as we will be considering causally symmetric triples $\mathfrak{h} \subseteq \mathfrak{g} \subseteq \mathfrak{s}$. Thus throughout this section $(\mathfrak{s}, \tau)$ denotes a compactly causal symmetric Lie algebra. We require $S_{\mathbb{C}}$ to be simply connected and write S, G, U and K for the analytic subgroups of $S_{\mathbb{C}}$ with Lie algebras $\mathfrak{s}, \mathfrak{g}, \mathfrak{u}$ and $\mathfrak{k}$. Note that the embedding

$$G/K \hookrightarrow S/U, \quad gK \mapsto gU$$

realizes G/K as a totally real submanifold of the Hermitian symmetric space S/U.

Recall that $\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_* = \mathbb{R}X^0$ is one dimensional and that we can normalize X^0 such that $\operatorname{Spec}(\operatorname{ad} iX^0) = \{0, 1, -1\}$. We let $\mathfrak{t} \subseteq \mathfrak{u}$ be a compact τ-stable Cartan algebra of $\mathfrak{s}$ and note that $\mathfrak{z}(\mathfrak{u}) \subseteq \mathfrak{t}$. Write $\Delta = \Delta(\mathfrak{s}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ for the root system with respect to $\mathfrak{t}_{\mathbb{C}}$. Call a root α *compact* if $\alpha\,|_{\mathfrak{z}(\mathfrak{u})} = 0$ and *non-compact* otherwise. Thus α is non-compact if and only if $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{p}_{\mathbb{C}}$. Denote by Δ_c the set of compact roots and by Δ_n the set of non-compact roots. Let $\Delta_n^+ := \{\alpha \in \Delta : \alpha(iX^0) = 1\}$ and fix a positive system Δ^+ such that $\Delta_n^+ \subseteq \Delta^+$. Write $\mathfrak{p}^{\pm} = \oplus_{\alpha \in \Delta_n^+} \mathfrak{g}_{\mathbb{C}}^{\pm\alpha} \subseteq \mathfrak{p}_{\mathbb{C}}$. Then we have the triangular decomposition of $\mathfrak{s}_{\mathbb{C}}$:

$$\mathfrak{s}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{u}_{\mathbb{C}} \oplus \mathfrak{p}^- \ .$$

Let $P^{\pm} = \exp(\mathfrak{p}^{\pm})$. Let $U_{\mathbb{C}} \subseteq S_{\mathbb{C}}$ be the complexification of U. Then it is well known that $S \subseteq P^+ U_{\mathbb{C}} P^-$ and we have the Borel embedding

$$S/U \hookrightarrow S_{\mathbb{C}}/U_{\mathbb{C}} P^- \ .$$

The map $P^+ \times U_{\mathbb{C}} \times P^- \ni (p^+, u, p^-) \mapsto p^+ u p^- \in P^- U_{\mathbb{C}} P^+$ is a diffeomorphism onto the open dense subset $P^+ U_{\mathbb{C}} P^- \subseteq S_{\mathbb{C}}$. We write $s = p^+(s)\kappa(s)p^-(s)$ for the unique decomposition of an element $s \in P^+ U_{\mathbb{C}} P^-$. Then we have the Harish Chandra realization of S/U as a bounded symmetric domain $\mathcal{D} \subseteq \mathfrak{p}^+$ given by

$$S/U \ni sU \mapsto \log(p^+(s)) \in \mathfrak{p}^+ \ ,$$

where $\log = (\exp|_{\mathfrak{p}^+})^{-1} : P^+ \to \mathfrak{p}^+$.

Write $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ for the restricted root system with respect to $\mathfrak{a}$ and $\widehat{\Sigma} = \Sigma(\mathfrak{s}, \mathfrak{a})$ for the double restricted root system. If Σ is an abstract root system on $\mathfrak{a}$, then we also write $\Omega = \Omega(\Sigma)$ to indicate the dependency of Ω on Σ. Define

$$\Xi_0 := G \exp(i\Omega(\widehat{\Sigma})) K_{\mathbb{C}}/K_{\mathbb{C}} \ ,$$

and notice that $\Xi_0 \subseteq \Xi$ since $\Sigma \subseteq \widehat{\Sigma}$.

Theorem 1.10. *(cf.* [KS01b, Sect. 2]*) The embedding $G/K \hookrightarrow S/U$ extends to a G-equivariant biholomorphism*

$$\Phi : \Xi_0 \to S/U \subseteq S_{\mathbb{C}}/U_{\mathbb{C}} P^- \ .$$

Furthermore the following statements are equivalent:

1. (1) $\Xi = \Xi_0$.
2. (2) $\Omega(\widehat{\Sigma}) = \Omega(\Sigma)$.
3. (3) $\operatorname{rank}_{\mathbb{R}}(\mathfrak{g}) = \frac{1}{2} \operatorname{rank}_{\mathbb{R}}(\mathfrak{s})$.
4. (4) Σ *is of type* C_n *or* BC_n *for* $n \geq 2$ *or* $(\mathfrak{g}, \mathfrak{s})$ *is one of* $(\mathfrak{su}(1,1), \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1))$ *or* $(\mathfrak{so}(1,n), \mathfrak{so}(2,n))$. ■

The non-compactly causal symmetric spaces G/H which are singled out by the additional requirement that $\Xi = \Xi_0 \simeq S/U$ are the ones that are most important for the rest of this article. We will therefore discuss them in more details now. The non-compactly causal symmetric spaces that satisfy the additional requirement $\Xi = \Xi_0$ are exactly those where Σ is of type C_r, say

$$\Sigma = \{\frac{1}{2}(\pm\gamma_i \pm \gamma_j) \colon 1 \leq i, j \leq r\}\backslash\{0\} .$$

Define $Y^j \in \mathfrak{a}$ by $\gamma_i(Y^j) = 2\delta_{ij}$ and $Y^0 = \frac{1}{2}(Y^1 + \ldots + Y^r)$. In this case $n = 1$, $X_1 = \frac{\pi}{2}Y^0$ and

$$\partial_e\Omega = \mathcal{W}(\frac{\pi}{2}Y^0) .$$

Hence with $z_1 \colon= \exp(i\frac{\pi}{2}Y^0)K_{\mathbb{C}}$ we have that $\partial_s\Xi = G(z_1)$ is a connected G-space. Write H for the isotropy subgroup of G in z_1 and $\mathfrak{h}$ for the Lie algebra of H. In cases where $\mathrm{Ad}(z_1)$ plays the role of a partial Cayley transform we write $\mathbf{c}$ for z_1. It follows from [GK02b] that

$$\sigma\colon G \to G, \quad g \mapsto (\mathrm{Ad}(\mathbf{c}^2) \circ \theta)(g)$$

is an involution on G such that $H = G^\sigma$. Comparing Theorem 1.10 with the list of all causal pairs in [HÓ96, Th.3.2.8] we finally arrive at Table I from the introduction.

Note that the first five pairs $(\mathfrak{g}, \mathfrak{h})$ in Table I are of Cayley type. We call a triple $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ from Table I a *causally symmetric triple*. Note that we always have

$$\mathrm{rank}_{\mathbb{R}}(\mathfrak{h}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}) = \frac{1}{2}\,\mathrm{rank}_{\mathbb{R}}(\mathfrak{s}) .$$

We write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the σ-eigenspace decomposition of $\mathfrak{g}$. Note that σ extends to an involution on $\mathfrak{s}$ since $\mathrm{Spec}\,\mathrm{ad}_{\mathfrak{s}}(Y^0) = \{-1, 0, 1\}$. Further observe that σ commutes with τ. Our choice of the maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ is then such that $\mathfrak{a} \subseteq \mathfrak{q} \cap \mathfrak{p}$.

Causally symmetric triples can also be defined axiomatically:

Definition 1.11. **(Causally symmetric triples)** A triple of Lie algebras $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ with $\mathfrak{h} \subseteq \mathfrak{g} \subseteq \mathfrak{s}$ is called a *causally symmetric triple* if the following axioms are satisfied:

(CST1) $\mathfrak{s}$ is Hermitian and of tube type.

(CST2) There exists two commuting involutions τ and σ on $\mathfrak{s}$ such that:

(1) $(\mathfrak{s}, \tau)$ is compactly causal and $\mathfrak{g} = \mathfrak{s}^\tau$.

(2) $(\mathfrak{g}, \sigma|_{\mathfrak{g}})$ is non-compactly causal and $\mathfrak{h} = \mathfrak{g}^\sigma$. ■

If we only require that G/K embeds into S/U and that $\partial_d\Xi$ has at least one boundary G/H_j component (cf. Theorem 1.9) which is a non-compactly causal symmetric space, then we arrive at the following more general situation (cf. [KS01b] and [GK02b]).

Theorem 1.12. *Suppose that G/K is a totally real submanifold of S/U and that there exists a symmetric subgroup H of G such that G/H is a non-compactly causal symmetric space and is a G-orbit in $\partial_d\Xi$. With $\partial_d\Xi_0 = G\exp(i\partial_e\Omega(\widehat{\Sigma}))K_{\mathbb{C}}/K_{\mathbb{C}}$ the distinguished boundary of Ξ_0 we arrive at the following four possibilities:*

(i) $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ *is a causally symmetric triple.*

(ii) G *is the structure group of an Euclidian Jordan algebra, i.e.,* $\mathfrak{g}$ *is isomorphic to one of the following*

$$\mathfrak{gl}(n, \mathbb{R}) \qquad \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R} \qquad \mathfrak{sl}(n, \mathbb{H}) \oplus \mathbb{R} \qquad \mathfrak{so}(1, n) \oplus \mathbb{R} \qquad \mathfrak{f}_{4(-20)} \oplus \mathbb{R} ,$$

and $\partial_d\Xi_0 = \partial_d\Xi \amalg G/K \amalg G/K$.

(iii) $G = \mathrm{SO}(n, n)$, $\partial_d\Xi_0 = G/\mathrm{SO}(n, \mathbb{C})$, *and*

$$\partial_d\Xi = \partial_d\Xi_0 \amalg G/\big(\mathrm{SO}(1, n-1) \times \mathrm{SO}(1, n-1)\big) .$$

(iv) $G = \mathrm{SO}(2n, \mathbb{C})$, $\partial_d\Xi_0 = G/\mathrm{SO}^*(2n)$, *and* $\partial_d\Xi = \partial_d\Xi_0 \amalg G/\mathrm{SO}(2, 2n-2)$. ■

The compactification of the Shilov boundary $\partial_s\Xi$

For the remainder of this section we assume that $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ is a causally symmetric triple.

Proposition 1.13. *Assume that $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ is a causally symmetric triple. Then the mapping*

$$\iota\colon G_{\mathbb{C}}/K_{\mathbb{C}} \to S_{\mathbb{C}}/U_{\mathbb{C}}P^-, \quad gK_{\mathbb{C}} \mapsto gU_{\mathbb{C}}P^-$$

is an open $G_{\mathbb{C}}$-equivariant embedding.

Proof. Since $\mathfrak{s}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} + \mathfrak{u}_{\mathbb{C}} + \mathfrak{p}^-$ it is clear that the image of ι is open. It remains to show that ι is injective, which means that $G_{\mathbb{C}} \cap U_{\mathbb{C}}P^- = K_{\mathbb{C}}$. Also denote by τ and θ the holomorphic extension of τ and θ to an involution on $S_{\mathbb{C}}$. Note that $S_{\mathbb{C}}^{\tau} = G_{\mathbb{C}}$ and $S_{\mathbb{C}}^{\theta} = U_{\mathbb{C}}$ since $S_{\mathbb{C}}$ was assumed to be simply connected.

We have $\tau(P^-) = P^+$. Thus we obtain

$$G_{\mathbb{C}} \cap U_{\mathbb{C}}P^- = G_{\mathbb{C}} \cap U_{\mathbb{C}}P^- \cap \tau(U_{\mathbb{C}}P^-) = G_{\mathbb{C}} \cap \left(U_{\mathbb{C}}P^- \cap U_{\mathbb{C}}P^+\right) .$$

Now the well known fact $U_{\mathbb{C}}P^- \cap U_{\mathbb{C}}P^+ = U_{\mathbb{C}}$ implies that

$$G_{\mathbb{C}} \cap U_{\mathbb{C}}P^- = G_{\mathbb{C}} \cap U_{\mathbb{C}} .$$

It remains to show that $G_{\mathbb{C}} \cap U_{\mathbb{C}} = K_{\mathbb{C}}$. Now we are going to use the fact that the pair $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ is from Table I. Quick inspection shows that $G_{\mathbb{C}}$ is simply connected. Hence $G_{\mathbb{C}}^{\theta} = K_{\mathbb{C}}$ is connected and so $G_{\mathbb{C}} \cap U_{\mathbb{C}} = G_{\mathbb{C}} \cap S_{\mathbb{C}}^{\theta} = K_{\mathbb{C}}$, concluding the proof of the proposition. ∎

Write $\overline{\mathcal{D}}$ for the closure in $\mathfrak{p}^+$ of the bounded symmetric domain $\mathcal{D}$. Recall that this is also the closure of S/U in $S_{\mathbb{C}}/U_{\mathbb{C}}P^-$. Hence Proposition 1.13 implies that the isomorphism $\Phi\colon \Xi \to \mathcal{D}$ from Theorem 1.10 extends to a G-equivariant embedding

$$\overline{\Phi}\colon \overline{\Xi} \to \overline{\mathcal{D}} .$$

Thus $\overline{\mathcal{D}}$ is the natural G-invariant compactification of Ξ. Let $z_1 := \exp(i\frac{\pi}{2}Y^0)K_{\mathbb{C}} \in \partial_d\Xi$ as before. It is known that the stabilizer of z_1 in S is a maximal parabolic subgroup P of S. Note that P is given by

$$P = \mathbf{c}U_{\mathbb{C}}P^-\mathbf{c}^{-1} \cap S .$$

Furthermore the Shilov boundary $\partial_s\mathcal{D}$ of $\mathcal{D}$ is the S-orbit through z_1 and

$$\partial_s\mathcal{D} = S(x_1) = U(x_1) \simeq U/K .$$

From the G-equivariance of $\overline{\Phi}$ it hence follows that $\overline{\Phi}(\partial_s\Xi) \subseteq \partial_s\mathcal{D}$.

Theorem 1.14. *The image of the Shilov boundary $\partial_s\Xi$ under the embedding $\overline{\Phi}$ is open and dense in $\partial_s\mathcal{D}$. In particular, $\partial_s\mathcal{D}$ is a compactification of $\partial_s\Xi$.*

Proof. First it is clear that $\overline{\Phi}(\partial_s\Xi)$ is contained in $\partial_s\mathcal{D}$. Further note that $G/H \hookrightarrow S/P$ is an open G-invariant embedding. Hence $\overline{\Phi}(\partial_s\Xi)$ is open.

Let us now show that $\overline{\Phi}(\partial_s\Xi)$ is dense in $\partial_s\mathcal{D}$. For that we realize $\overline{\Xi}$ in $\overline{\mathcal{D}}$ via the embedding $\overline{\Phi}$. If $Y := \overline{\Phi}(\partial_s\Xi)$ is not dense, then we find a continuous function f on $\overline{\mathcal{D}}$, holomorphic on $\mathcal{D} = \Xi$ which does not attain its maximum on $\overline{Y}$. But this contradicts the fact that $\Xi = \mathcal{D}$ and that $\partial_s\Xi$ is the Shilov boundary of Ξ. ∎

2. Continuous branching of spherical highest weight representations

Throughout this section $(\mathfrak{s},\tau)$ denotes a compactly causal symmetric Lie algebra with τ-eigenspace decomposition $\mathfrak{s}=\mathfrak{g}\oplus\mathfrak{q}_*$. Denote by S a simply connected Lie group with Lie algebra $\mathfrak{s}$ and write $G{:}=S^\tau$ for the fixed point group. Observe that G is connected since S is assumed to be simply connected.

In this section we discuss the *generalized Segal-Bargmann transform* introduced in [ÓØ85] (see also [DÓZ01,Ne99,Ó00,Z01]). The generalized Segal-Bargmann transform is a unitary G-isomorphism $U_\lambda:(L,L^2(G/K))\to(\pi_\lambda,\mathcal{H}_\lambda)$ where $\mathcal{H}_\lambda$ is a unitary G-spherical highest weight representation of S. In this section we will give a sharp criterion in terms of λ for the existence of U_λ. In particular this result will imply the continuous branching of $\pi_\lambda|_G$.

Write U for the analytic subgroup of S associated to $\mathfrak{u}$ and $U_\mathbb{C}$ for its complexification. We denote by $P^\pm$ the analytic subgroup of $S_\mathbb{C}$ with Lie algebras $\mathfrak{p}^\pm$. Assume for a moment that S is not simply connected and that $S\subseteq S_\mathbb{C}$. By the Harish Chandra decomposition of S we have $S\subseteq P^+U_\mathbb{C}P^-$. For $s\in S$ write $s=p^+(s)\kappa(s)p^-(s)$ with the obvious notation. Recall that the map $S\ni s\mapsto\log p^+(s)\in\mathfrak{p}^+$ gives the realization of S/U as a bounded symmetric domain in $\mathfrak{p}^+$. Further we have the middle projection $\kappa:S\to U_\mathbb{C}$. By the usual lifting argument we obtain a map $\kappa{:}\,S\to U_\mathbb{C}$ for a simply connected group S. This lifting is unique if we require – as we will – that $\kappa(\mathbf{1})=\mathbf{1}$. We denote by $X\mapsto\overline{X}$ the conjugation in $\mathfrak{s}_\mathbb{C}$ with respect to the real form $\mathfrak{s}$. Then $\overline{\mathfrak{p}^+}=\mathfrak{p}^-$.

Realization of unitary highest weight representations.

Our source of reference for the facts collected below is [N99, Ch. XII]. Let $(\pi_\lambda,\mathcal{H}_\lambda)$ be a unitary highest weight representation of S with highest weight $\lambda\in i\mathfrak{t}^*$ and with respect to the positive system Δ^+. Write $F(\lambda)$ for the finite dimensional lowest U-type in $\mathcal{H}_\lambda$, i.e., the space which is generated by applying U to a highest weight vector v_λ of $\mathcal{H}_\lambda$. We now briefly recall the realization of $\mathcal{H}_\lambda$ in $\mathcal{O}(\mathcal{D},F(\lambda))$. For that we write σ_λ for the representation of $U_\mathbb{C}$ on $F(\lambda)$. Define

$$\mathcal{K}{:}\,\mathcal{D}\times\mathcal{D}\to U_\mathbb{C},\quad (z,w)\mapsto\kappa(\exp(-\overline{w})\exp(z))^{-1}$$
$$J{:}\,S\times\mathcal{D}\to U_\mathbb{C},\quad (s,z)\mapsto\kappa(g\exp(z))\ .$$

Set

$$K_\lambda=\sigma_\lambda\circ\mathcal{K}\quad\text{and}\quad J_\lambda=\sigma_\lambda\circ J\,. \tag{2.1}$$

Note that K_λ is holomorphic in the first variable, antiholomorphic in the second and satisfies the cocyle identity

$$K_\lambda(s(z),s(w))=J_\lambda(s,z)K_\lambda(z,w)J_\lambda(s,w)^* \tag{2.2}$$

for all $s\in S$, $z,w\in\mathcal{D}$. Here A^* denotes the adjoint of an operator $A\in B(F(\lambda))=\mathrm{End}(F(\lambda))$.

The relation to $(\pi_\lambda,\mathcal{H}_\lambda)$ is as follows: π_λ is unitary if and only if K_λ is positive definite. If this is the case, then $(\pi_\lambda,\mathcal{H}_\lambda)$ can be realized in $\mathcal{O}(\mathcal{D},F(\lambda))$ as the reproducing kernel Hilbert space corresponding to K_λ. So henceforth we will assume that $\mathcal{H}_\lambda\subseteq\mathcal{O}(\mathcal{D},F(\lambda))$. The representation π_λ is then given by

$$(\pi_\lambda(s)f)(z)=J_\lambda(s^{-1},z)^{-1}f(s^{-1}z) \tag{2.3}$$

for $f\in\mathcal{H}$, $s\in S$ and $z\in\mathcal{D}$.

G-spherical unitary highest weight representation

A unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ is called *G-spherical* if there exists a non-trivial G-fixed element $\nu \in \mathcal{H}_\lambda^{-\infty}$, where $\mathcal{H}_\lambda^{-\infty}$ denotes the module of distribution vectors of $(\pi_\lambda, \mathcal{H}_\lambda)$. If $(\pi_\lambda, \mathcal{H}_\lambda)$ is a unitary highest weight representation of S, then we call λ *regular* if $\mathcal{H}_\lambda$ is dense in $\mathcal{O}(\mathcal{D}, F(\lambda))$, or, equivalently if $\lambda|_{\mathfrak{z}(\mathfrak{u})}$ lies on the continuous halfline of the Wallach set (this is made precise in Appendix A). We say that $(\pi_\lambda, \mathcal{H}_\lambda)$ is *scalar* if $F(\lambda) = \mathbb{C}$. This is equivalent to $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$.

We now recall some facts concerning G-spherical unitary highest weight representation. For more information and an almost complete classification see [KN02].

- If $(\pi_\lambda, \mathcal{H}_\lambda)$ is G-spherical, then $\nu_0 := \nu|_{F(\lambda)}$ is non-zero and K-fixed. In particular, $F(\lambda)$ is a K-spherical module for U.
- Suppose that λ is regular, then $(\pi_\lambda, \mathcal{H}_\lambda)$ is G-spherical if and only if $F(\lambda)$ is K-spherical.
- Suppose that $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$ is regular, then $(\pi_\lambda, \mathcal{H}_\lambda)$ is G-spherical if and only if $\lambda \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$. In particular, if S is simple, then all regular $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$ correspond to G-spherical representations.
- If we are in the group case $S = G \times G$, then the G-spherical unitary highest weight representations of S are exactly the representations $\pi_\lambda \otimes \pi_\lambda^*$ with π_λ a unitary highest weight representation of G and π_λ^* its dual representation.

The generalized Segal-Bargmann transform

In this subsection $(\pi_\lambda, \mathcal{H}_\lambda)$ denotes a G-spherical unitary highest weight representation of S. Note that this means in particular that there exists a non-trivial K-fixed vector $\nu_0 \in F(\lambda)^*$. Write $\nu_0 = \langle \cdot, v_0 \rangle$ where $0 \neq v_0 \in F(\lambda)$ is a K-fixed vector. Define

$$D_\lambda : G \to B(F(\lambda)), \quad g \mapsto J_\lambda(g, 0)^{-1} = \sigma_\lambda(\kappa(g))^{-1}$$

and

$$\|D_\lambda\| : G/K \to \mathbb{R}^+, \quad gK \to \|D_\lambda(g)\|$$

where $\|\cdot\|$ denotes the operator norm on $B(F(\lambda))$. Write $\mathcal{A}(G/K)$ for the space of analytic functions on G/K. Define a restriction map

$$R_\lambda : \mathcal{H}_\lambda \to \mathcal{A}(G/K), \quad R_\lambda(f)(gK) := \langle D_\lambda(g) f(gK), v_0 \rangle .$$

Note that this map is well defined, since v_0 is K-fixed. We write L for the left regular representation of G on functions on G/K, i.e., $(L(g)f)(xK) = f(g^{-1}xK)$ for $g, x \in G$ and f a function on G/K. We then have the following lemma, see [Ó00], Lemma 3.4:

Lemma 2.1. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a G-spherical unitary highest weight representation of S.*

(i) *The map $R_\lambda : \mathcal{H}_\lambda \to \mathcal{A}(G/K)$ intertwines $\pi_\lambda|_G$ with L. Moreover, if $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$, then R_λ is injective.*

(ii) *If λ is regular and $\|D_\lambda\| : G/K \to \mathbb{R}^+$ is square integrable, then* $\operatorname{im} R_\lambda \subseteq L^2(G/K)$ *is dense and $R_\lambda : \mathcal{H}_\lambda \to L^2(G/K)$ is continuous.* ■

Assume now that $\|D_\lambda\| \in L^2(G/K)$ and that λ is regular. Then $R_\lambda\colon \mathcal{H}_\lambda \to L^2(G/K)$ is continuous with dense image. Thus we can consider the polar decomposition of the continuous operator R_λ given by $R_\lambda = U_\lambda P_\lambda$ with $P_\lambda = (R_\lambda^* R_\lambda)^{\frac{1}{2}}$ and U_λ a partial isometry. If $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$, then U_λ is an isometry and the unitary G-equivariant isomorphism

$$U_\lambda^* : L^2(G/K) \to \mathcal{H}_\lambda$$

is called the *generalized Segal-Bargmann transform* (cf. [Ó00]).

We will now specify the set of parameters for which $\|D_\lambda\|$ is square integrable. We will distinguish the two cases of S simple and $S = G \times G$. The group case is much simpler, since in that case the square integrability of D_λ can be easily reduced to known integrals. We therefore treat the group case first since the general proof requires some knowledge of advanced structure theory the general reader might not be so familiar with.

The square integrability of $\|D_\lambda\|$: group case

In this section we consider the case where $S = G \times G$. We let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary representation of G. Then the S-representation we consider is $\widehat{\pi}_\lambda = \pi_\lambda \otimes \pi_\lambda^*$. Recall that these are precisely the G-spherical unitary highest weight representations of S. As explained before we can realize $\mathcal{H}_\lambda$ in $\mathcal{O}(G/K, F(\lambda))$. As before we write K_λ for its reproducing kernel and J_λ for the corresponding cocycle. We recall now some well known facts on representations of the holomorphic discrete series on G. Write Z for the center of G. We say $(\pi_\lambda, \mathcal{H}_\lambda)$ belongs to the *relative discrete series* if all the matrix coefficients $g \mapsto (\pi_\lambda(g)u, v)$, $u, v \in \mathcal{H}_\lambda$, are square integrable modulo Z. By abuse of notation we will for the moment denote by $\mathfrak{t}$ a compact Cartan algebra of $\mathfrak{g}$ and Δ for the corresponding root system. We write β for the highest root in Δ^+ and set $\rho := \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

Lemma 2.2. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of the simply connected Hermitian Lie group G. Then the following assertions are equivalent:*

(1) *$(\pi_\lambda, \mathcal{H}_\lambda)$ belongs to the relative discrete series.*

(2) *The highest weight $\lambda \in i\mathfrak{t}^*$ satisfies the Harish Chandra condition*

$$\langle \lambda + \rho, \beta \rangle < 0 \ .$$

(3) *The integral*

$$\int_{G/K} \|K_\lambda(z,z)\|^{-1} \, d\mu_{G/K}(z) < \infty$$

is finite. Here $\|\cdot\|$ denotes the operator norm on $B(F(\lambda))$ and $\mu_{G/K}$ the invariant measure on G/K.

(4) *The integral*

$$\int_{G/Z} \|J_\lambda(g,0)^{-1}(J_\lambda(g,0)^{-1})^*\| \, d\mu_{G/Z}(gZ)$$

is finite.

Proof. (1) $\iff$ (2) is the criterion of Harish Chandra (cf. [N99, Th. XII.5.12]). For the equivalence of (2) and (3) see [N99, Th. XII.5.6]. Finally, the equivalence of (3) and (4) is immediate from (2.1) and (2.2). ■

As before we realize $(\widehat{\pi}_\lambda, \mathcal{H}_\lambda \widehat{\otimes} \mathcal{H}_\lambda^*)$ in $\mathcal{O}(G/K \times \overline{G/K}, B(F(\lambda)))$. Here we used the identification $B(F(\lambda)) = F(\lambda) \otimes F(\lambda)^*$. The $U = K \times K$-spherical vector in $B(F(\lambda))$ is the trace in this case. The function $\|D_\lambda\|$ is given by

$$\|D_\lambda\|\colon G/K \to \mathbb{R}^+, \quad gK \mapsto \mathrm{tr}(J_\lambda(g,0)^{-1}(J_\lambda(g,0)^{-1})^*) \ .$$

If G is a locally compact group, then we write μ_G for a left invariant Haar measure on G.

Lemma 2.3. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of G. Then D_λ is square integrable if and only if*

$$\langle 2\lambda + \rho, \beta\rangle < 0$$

for β the highest root in Δ^+.

Proof. That D_λ is square integrable means that

$$\int_{G/Z} \|J_\lambda(g,0)^{-1}(J_\lambda(g,0)^{-1})^*)\|^2 \, d\mu_{G/Z}(gZ) < \infty. \tag{2.4}$$

Assume for a moment that $\lambda \in i\mathfrak{z}(\mathfrak{k})^*$, i.e., $F(\lambda) \simeq \mathbb{C}$. Then $J_\lambda(g,0)^2 = J_{2\lambda}(g,0)$ and (2.4) becomes

$$\int_{G/Z} \|J_{2\lambda}(g^{-1},0)J_{2\lambda}(g^{-1},0)^*\| \, d\mu_{G/Z}(gZ) < \infty. \tag{2.5}$$

From the equivalence (2) $\Longleftrightarrow$ (4) in Lemma 2.2 we obtain that (2.5) is equivalent to

$$\langle 2\lambda + \rho, \beta\rangle < 0$$

for β the highest root in Δ^+. This proves the lemma for the case of $\lambda \in i\mathfrak{z}(\mathfrak{k})^*$. The general case can be easily reduced to the special case discussed above (see (2.6) below). ■

Lemma 2.4. *If the highest weight λ of $(\pi_\lambda, \mathcal{H}_\lambda)$ satisfies the condition of $\langle 2\lambda+\rho, \beta\rangle < 0$, then λ is regular.*

Proof. This follows from the classification of unitary highest weight modules (cf. [EHW83]). In the scalar case this is proved in Lemma A.2 (iv) in Appendix A. ■

Assume now that $\langle 2\lambda+\rho, \beta\rangle < 0$ holds. In view of Lemma 2.3 and Lemma 2.4, this implies in particular that D_λ is square integrable. In view of Lemma 2.1 we have thus proved:

Theorem 2.5. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation with highest weight λ satisfying the condition $\langle 2\lambda + \rho, \beta\rangle < 0$ with β the highest root. Then we have an onto G-equivariant partial isometry*

$$U_\lambda \colon \mathcal{H}_\lambda \widehat{\otimes} \mathcal{H}_\lambda^* \to L^2(G/K)$$

which is an isomorphism when $\lambda \in i\mathfrak{z}(\mathfrak{k})^*$. In particular, for $\lambda \in i\mathfrak{z}(\mathfrak{k})^*$ the branching of the irreducible unitary representation $\pi_\lambda \otimes \pi_\lambda^*$ of $G \times G$ to the diagonal subgroup G is completely continuous and multiplicity free. ■

The square integrability of $\|D_\lambda\|$: general case

In this subsection we will prove a theorem that will give us the exact range of parameters for the square integrability of D_λ. For that recall some results from [ÓØ88] applied to the symmetric Lie algebra $(\mathfrak{s}, \theta\tau)$, which is also compactly causal. The proof of the statements uses simple $\mathfrak{su}(1,1)$-reduction (cf. Appendix B). Choose the Cartan subspace $\mathfrak{t}$ such that $\mathfrak{e} := \mathfrak{t} \cap \mathfrak{q}_*$ is maximal abelian in $\mathfrak{q}_*$. Write $\Delta_\mathfrak{e} = \Delta(\mathfrak{s}_\mathbb{C}, \mathfrak{e}_\mathbb{C})$ for the restricted root system with respect to $\mathfrak{e}_\mathbb{C}$. Note that $\Delta_\mathfrak{e}$ is always of type C_n or BC_n. Define a positive system $\Delta_\mathfrak{e}^+$ by $\Delta_\mathfrak{e}^+ := \Delta^+|_\mathfrak{e} \backslash \{0\}$. Further set $\Delta_{\mathfrak{e},n}^+ := \Delta_n^+|_\mathfrak{e}$. Note that $\gamma|_\mathfrak{e} \neq 0$ for all $\gamma \in \Delta_n$.

Let $\gamma_1, \ldots, \gamma_r$ be a maximal set of long strongly orthogonal roots in $\Delta^+_{\mathfrak{c},n}$. Then we can choose $E^{\pm j} \in \mathfrak{s}_{\mathbb{C}}^{\pm\gamma_j}$ such that with $Y^j = -i(E^j - E^{-j})$ the space $\mathfrak{a} = \bigoplus_{j=1}^r \mathbb{R}Y^j$ is maximal abelian in $\mathfrak{p}$. Let $H^j \in i\mathfrak{c}$ be such that $\gamma_i(H^j) = 2\delta_{ij}$. Then

$$\kappa(\exp(\sum_{j=1}^r t_j Y^j)) = \exp\left(-\frac{1}{2}\sum_{j=1}^r \log(\cosh(2t_j))H^j\right) .$$

Hence

$$\|D_\lambda(\exp(\sum_{j=1}^r t_j Y^j))\| = \exp\left(\frac{1}{2}\langle\lambda, \sum_{j=1}^r \log(\cosh(2t_j))H^j\rangle\right) .$$

Thus with $a = \exp(\sum_{j=1}^r t_j Y^j)$ and $k_1, k_2 \in K$:

$$\|D_\lambda(k_1 a k_2)\|^2 = \prod_{j=1}^r \cosh(2t_j)^{\langle\lambda, H^j\rangle} . \tag{2.6}$$

The root vectors E^j determine a Cayley transform c such that $c(\sum_{j=1}^r \mathbb{R}H^j) = \mathfrak{a}$. Recall that $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ is the set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$. Choose Σ^+ such that $\Sigma^+ \subseteq \Delta^+ \circ c^{-1}$. Let ρ_G be the half sum of positive roots counted with multiplicities and let $\rho_G^c = \rho_G \circ c^{-1} \in i\mathfrak{c}^*$.

Theorem 2.6. *We have $D_\lambda \in L^2(G/K)$ if and only if*

$$\langle\lambda + \rho_G^c, \beta\rangle < 0$$

for β the highest root in Δ^+.

Proof. First note that $\langle\lambda + \rho_G^c, \beta\rangle < 0$ is equivalent to

$$\langle\lambda + \rho_G^c, \alpha\rangle < 0 \quad (\forall\alpha \in \Delta_n^+) .$$

Further the fact that $\lambda, \rho_G^c \in i\mathfrak{c}^*$ implies that this is equivalent to

$$\langle\lambda + \rho_G^c, \alpha\rangle < 0 \quad (\forall\alpha \in \Delta_{\mathfrak{c},n}^+) .$$

Let

$$\Delta_G(k_1 a k_2) := \prod_{\alpha\in\Sigma^+} \sinh(\alpha(\log a))^{m_\alpha} \qquad k_1, k_2 \in K, a \in A$$

where $m_\alpha = \dim \mathfrak{g}^\alpha$. Then for $f \in L^1(G)$:

$$\int_G f(g)\ dg = \int_K \int_{A^+} \int_K f(k_1 a k_2)\ \Delta_G(a)\ d\mu_K(k_1)\ d\mu_A(a)\ d\mu_K(k_2)$$

Let $\varphi(t) = \frac{1}{2}(1 - e^{-2t})$, $t \geq 0$. Then φ is increasing, $0 \leq \varphi(t) \leq 1/2$, and $\varphi(t) = 0$ if and only if $t = 0$. Furthermore $\sinh(t) = \varphi(t)e^t$. Define

$$\Phi(a) := \prod_{\alpha\in\Sigma^+} \varphi(\alpha(\log a))^{m_\alpha}\ a \in A^+ .$$

Then there exists a positive constant C such that

$$\Phi(a)a^{2\rho_G} = \Delta(a) \leq Ca^{2\rho_G}$$

for all $a \in A^+$. Notice that by (2.6) there are constants $C_1, C_2 > 0$ such that

$$C_1 e^{\lambda(\sum_{j=1}^r t_j H^j)} \leq D_\lambda(k_1 \exp(\sum_{j=1}^r t_j Y^j) k_2) \leq C_2 e^{\lambda(\sum_{j=1}^r t_j H^j)} .$$

Hence for $a = \exp(\sum_{j=1}^r t_j Y^j) \in A^+$

$$C_1^2 \Phi(a) \prod_{j=1}^r \exp(2t_j\langle\lambda + \rho_G^c, H^j\rangle) \leq \|D_\lambda(k_1 a k_2)\|^2 \Delta(a) \leq CC_2^2 \prod_{j=1}^r \exp(2t_j\langle\lambda + \rho_G^c, H^j\rangle) .$$

The claim now follows as $t_j > 0$ for $a \in A^+$. ∎

From Theorem 2.6 and Lemma 2.1 we now obtain the following result:

Theorem 2.7. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a G-spherical unitary highest weight representation of S. Suppose the highest weight λ is regular and satisfies the condition*

$$\langle \lambda + \rho_G^c, \beta \rangle < 0$$

for β the highest root in Δ^+. Then we have an onto G-equivariant partial isometry

$$U_\lambda \colon \mathcal{H}_\lambda \to L^2(G/K)$$

which is an isomorphism when $\lambda \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$. In particular, for $\lambda \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$ the representation $\pi_\lambda|_G$ of G is multiplicity free with completely continuous spectrum. ■

Remark 2.8. (a) The condition in Theorem 2.7 is the same as the one in Lemma 2.3 in the group case. The condition on the regularity of λ in Theorem 2.7 is probably superfluous, but we did not check it.
(b) The results in [GrKo01] for $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ show that the condition $\langle \lambda + \rho_G^c, \beta \rangle < 0$ on λ in Theorem 2.7 is also necessary for continuous branching. It would be interesting to know if this is generally true. ■

Example 2.9. Let us consider $S = \widetilde{\mathrm{Sl}}(2, \mathbb{R})$ with $G = \mathrm{SO}(1,1)$. In this case $\rho_G^c = 0$ and this means that all unitary highest weight representations π_λ have continuous branching for $\pi_\lambda|_G$.

On the other hand consider the group case $S = \widetilde{\mathrm{Sl}}(2, \mathbb{R}) \times \widetilde{\mathrm{Sl}}(2, \mathbb{R})$. Here the condition $\langle \lambda + \rho_G^c, \beta \rangle < 0$ gives a real restriction. Further by the results in [GrK01] we do not have continuous branching for $\pi_\lambda \otimes \pi_\lambda^*$ to G on the full half line. ■

Applications to the classical Hardy space

Write $\partial_s \mathcal{D}$ for the Shilov boundary of $\mathcal{D} \simeq S/U$ and recall that $\partial_s \mathcal{D}$ is the U-orbit through a certain $E \in \partial_s \mathcal{D}$. Note that $\mathcal{D}$ is a circular domain. In particular we have $r\partial_s \mathcal{D} \subseteq \mathcal{D}$ for all $0 \leq r < 1$.

The *Hardy space parameter* $\lambda_h \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$ is defined by

$$\lambda_h = -\rho_n := -\frac{1}{2} \sum_{\alpha \in \Delta_n^+} \alpha \, .$$

The corresponding Hilbert space $\mathcal{H}_{\lambda_h}$ is the *classical Hardy space* on $\mathcal{D}$:

$$\mathcal{H}^2(\mathcal{D}) := \{ f \in \mathcal{O}(\mathcal{D}) \colon \sup_{0 \leq r < 1} \int_U |f(ru(E))|^2 \, d\mu_U(u) < \infty \} \, . \tag{2.7}$$

Here we identified the Shilov boundary of $\partial_s \mathcal{D}$ with $U(E)$ as explained before. We write $\pi_h := \pi_{\lambda_h}$ for the unitary irreducible representation of S on $\mathcal{H}^2(\mathcal{D})$.

Lemma 2.10. *Assume that $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ is a causally symmetric triple. Then λ_h is regular and satisfies the condition $\langle \lambda_h + \rho_G^c, \beta \rangle < 0$.*

Proof. See Proposition A.3 in the appendix. ■

From this we obtain the following important corollary to Theorem 2.8:

Corollary 2.11. *Let $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ be a causally symmetric triple and $\mathcal{H}^2(\mathcal{D})$ be the classical Hardy space on $\mathcal{D} \simeq S/U$. Then the inverse of the Segal-Bargmann transform*

$$U\colon (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, L^2(G/K))$$

is a G-equivariant unitary isomorphism. In particular, the branching $\pi_h \mid_G$ is completely continuous and multiplicity free. ■

3. Analytical and geometrical constructions on Ξ

From now on we will assume that $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ is a causally symmetric triple. Further we will require $S_{\mathbb{C}}$ to be simply connected and henceforth S, U, G, K and H will denote the analytic subgroups of $S_{\mathbb{C}}$ with Lie algebras $\mathfrak{s}$, $\mathfrak{u}$, $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$.

We have seen in Section 1 that $\partial_s\Xi \simeq G/H$ is open and dense in $\partial_s\mathcal{D}$ (cf. Theorem 1.14). Now on $\partial_s\Xi$ we have a natural G-invariant measure $\mu_{G/H}$ while on $\partial_s\mathcal{D}$ we have a natural U-invariant measure $\mu_{\partial_s\mathcal{D}}$. These two measures are related through a density function, i.e. $d\mu_{G/H}(z) = \frac{1}{|\psi(z)|^2} d\mu_{\partial_s\mathcal{D}}(z)$. In this section we will show that one can choose the measurable function ψ on $\partial_s\mathcal{D}$ in such a way that it admits a continuous extension to $\mathcal{D} \amalg \partial_s\mathcal{D}$ which in addition is holomorphic on $\mathcal{D}$. The function ψ will satisy a cocycle property which allows us to identify the L^2-spaces $L^2(G/H)$ and $L^2(\partial_s\mathcal{D}, \mu_{\partial_s\mathcal{D}})$ in a natural G-equivariant way (cf. Lemma 3.11 below). Moreover the fact that ψ has a holomorphic extension to $\Xi \simeq \mathcal{D}$ will be used for comparing the classical Hardy space from (2.7) with our Hardy space on Ξ in Section 5.

The function ψ

In this subsection we give the construction of the function ψ mentioned above. For that we first have to collect some facts on certain finite dimensional $K_{\mathbb{C}}$-spherical representations of $U_{\mathbb{C}}$.

Denote by (π_m, V_m) the irreducible representation of $U_{\mathbb{C}}$ with lowest weight $-m\rho_n$. If such a representation exists it is $K_{\mathbb{C}}$-spherical. Note that $V_m \simeq \mathbb{C}$ is one dimensional since $\rho_n \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$. Also notice that the representation $\det \mathrm{Ad}|_{\mathfrak{p}^-}$ of U has lowest weight $-2\rho_n$. Let

$$\zeta = \frac{1}{2}(\gamma_1 + \ldots + \gamma_r) \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$$

where $\gamma_1, \ldots, \gamma_r$ is a maximal set of long strongly orthogonal roots in Δ_n^+. Let

$$d = \dim \mathfrak{s}_{\mathbb{C}}^{\frac{1}{2}(\gamma_i+\gamma_j)} = \dim \mathfrak{s}_{\mathbb{C}}^{\frac{1}{2}(\gamma_i-\gamma_j)}$$

(cf. Table II in Appendix A). Then

$$\rho_n = (1 + \frac{d(r-1)}{2})\zeta \tag{3.1}$$

(cf. Appendix A).

Proposition 3.1. *Let the notation be as above. Let $m \in \mathbb{N}$ be minimal such that (π_m, V_m) exists. Then the following holds:*

(i) *If $(\mathfrak{g}, \mathfrak{h})$ is of Cayley type and $\mathfrak{g} \neq \mathfrak{so}(2, 2k+1), \mathfrak{sp}(2n, \mathbb{R})$ then $m = 1$.*

(ii) *If* $(\mathfrak{g},\mathfrak{h})$ *is of Cayley type and* $\mathfrak{g}=\mathfrak{so}(2,2k+1),\mathfrak{sp}(2n,\mathbb{R})$ *then* $m=2$.

(iii) *If* $(\mathfrak{s},\mathfrak{g})=(\mathfrak{su}(2n,2n),\mathfrak{sp}(n,n))$ *then* $m=1$.

(iv) *If* $(\mathfrak{s},\mathfrak{g})=(\mathfrak{sp}(2n,\mathbb{R}),\mathfrak{sp}(n,\mathbb{C}))$ *then* $m=2$.

(v) *If* $(\mathfrak{s},\mathfrak{g})=(\mathfrak{so}(2,n),\mathfrak{so}(1,n))$ *for* $n>2$ *then* $m=1$ *if* n *is even and* $m=2$ *for* n *odd.*

Proof. Using Table II for d in Appendix A, this follows from (3.1) by straightforward computation. ∎

Lemma 3.2. *The set* $G_{\mathbb{C}}U_{\mathbb{C}}P^-$ *is open and dense in* $S_{\mathbb{C}}$. *Furthermore* $S\subseteq G_{\mathbb{C}}U_{\mathbb{C}}P^-$.

Proof. This is Theorem 2.4 in [ÓØ88]. ∎

Assume now that (π_m,V_m) exists. Then $V_m=\mathbb{C}v_m$ and v_m is both a lowest weight vector and a $K_{\mathbb{C}}$-spherical vector. Normalize v_m such that $(v_m,v_m)=1$. Let

$$\mathcal{U}:=\{z\in\mathfrak{p}^+ : \exp(z)\in G_{\mathbb{C}}U_{\mathbb{C}}P^-\}.$$

Since $G_{\mathbb{C}}U_{\mathbb{C}}P^-$ is open and dense in $S_{\mathbb{C}}$, it follows from Lemma 3.2 that $\mathcal{U}$ is an open dense and locally $G_{\mathbb{C}}$-invariant subset of $\mathfrak{p}^+$. Moreover $\mathcal{D}\subseteq\mathcal{U}$.

For $s\in\mathcal{U}$ define $K_{\mathbb{C}}u_G(s)\in K_{\mathbb{C}}\backslash U_{\mathbb{C}}$ by

$$s\in G_{\mathbb{C}}u_G(s)P^- . \tag{3.2}$$

Define a function $\psi_m:\mathcal{U}\to\mathbb{C}$ by

$$\psi_m(z):=(\pi_m(u_G(\exp(z))v_m,v_m). \tag{3.3}$$

Lemma 3.3. *The function* ψ_4 *extends to an holomorphic function on* $\mathfrak{p}^+$.

Proof. This follows from [BÓ01], Proposition 4.1 and the table on p. 294, but we will give a short proof here. First notice that there exists a finite dimensional representation (σ,W) of $S_{\mathbb{C}}$ with lowest weight $-2\rho_n$. As $-2\rho_n\in i(\mathfrak{z}(\mathfrak{u})\cap\mathfrak{q}_*)^*$ it follows by Helgason's Theorem that the irreducible representation with lowest weight $-4\rho_n$ is $G_{\mathbb{C}}$-spherical. Denote this representation by (σ_4,W_4). Let $v\in W_4$ be a lowest weight vector and $u\in W_4$ be a $G_{\mathbb{C}}$-invariant vector. We can normalize v and u such that $(u,v)=1$. It then follows that

$$(\forall s\in S_{\mathbb{C}})\qquad (\sigma_4(s)v,u)=(\sigma_4(u_G(s))v,u).$$

Now note that the representation of $U_{\mathbb{C}}$ generated by $\sigma_4(U_{\mathbb{C}})v$ is in fact irreducible and hence equivalent to π_4. Thus

$$(\forall z\in\mathcal{U})\qquad (\sigma_4(u_G(\exp(z)))v,u)=\psi_4(z)$$

and the claim follows as $z\mapsto(\sigma_4(\exp z)v,u)$ is holomorphic on $\mathfrak{p}^+$. ∎

Proposition 3.4. *Let* ψ_m *be as above. Then the following assertions hold:*

(i) *Let* $g\in G_{\mathbb{C}}$ *and* $z\in\mathcal{U}$ *such that* $gz\in\mathcal{U}$. *Then*

$$\psi_m(gz)=J_{m\rho_n}(g,z)\psi_m(z).$$

(ii) ψ_m *is holomorphic and has no zeros on* $\mathcal{D}$.

Proof. (i) We have $g\exp(z)=\exp(gz)J(g,z)p$ with $p\in P^-$. Hence

$$\exp(gz)=g\exp(z)J(g,z)^{-1}p'$$

for some $p'\in P^-$. It follows that

$$\psi_m(gz)=(\pi_m(\exp(z))\pi_m(J(g,z)^{-1})v_m,v_m))=J_{m\rho_n}(g,z)\psi_m(z)$$

because v_m is a weight vector with weight $-m\rho_n$.

(ii) The holomorphicity of ψ_m is clear by construction. To see that $\psi_m(z)\neq 0$ for $z\in\mathcal{D}$ write $z=g(0)$ for some $g\in G\exp(i\Omega)$. Since $\mathcal{D}\subseteq\mathcal{U}$, we obtain from (i) that

$$\psi_m(z)=\psi_m(g(0))=J_{m\rho_n}(g,0)\psi_m(0).$$

Now $\psi_m(0)=(v_m,v_m)=1\neq 0$ and $J_{m\rho_n}(g,0)\neq 0$ by construction. ∎

For the rest of this subsection we consider ψ_m as a function on $\mathcal{D}$ only. Note that ψ_2 always exists by Proposition 3.1. Since $\mathcal{D}$ is simply connected and ψ_m is zero-free it follows from Proposition 3.4(ii) that we can define a holomorphic square root $\psi(z) := \sqrt{\psi_2(z)}$ which becomes unique under the requirement $\psi(0) = 1$. Note that $\psi = \psi_1$ in case (π_1, V_1) exists.

Write $\widetilde{S}$ for the universal covering of S and G_1 for the analytic subgroup of $\widetilde{S}$ with Lie algebra $\mathfrak{g}$. Note that $J_{\rho_n}(s,z)$ exists for all $s \in \widetilde{S}$ and $z \in \mathcal{D}$ (cf. Section 2). From Proposition 3.4(i) we thus obtain that $\psi(gz) = J_{\rho_n}(g,z)\psi(z)$ for all $g \in G_1$ and $z \in \mathcal{D}$. In particular, $J_{\rho_n}(g,z) = \frac{\psi(z)}{\psi(gz)}$ and so $J_{\rho_n}(g,z)$, initially defined only on $G_1 \times \mathcal{D}$, factors to a function on $G \times \mathcal{D}$, which we denote by J_h^{-1}. Summarizing our discussions we have proved:

Proposition 3.5. *There exist an unique holomorphic function $\psi: \mathcal{D} \to \mathbb{C}^*$ and a unique analytic function $J_h: G \times \mathcal{D} \to \mathbb{C}^*$ with the following properties:*

(i) *$\psi(gz) = J_h^{-1}(g,z)\psi(z)$ for all $g \in G$, $z \in \mathcal{D}$.*

(ii) *$\psi^2 = \psi_2$ and $\psi(0) = 1$.*

(iii) *J_h satisfies the cocycle property $J_h(g_1g_2, z) = J_h(g_1, g_2z)J_h(g_2, z)$ for $g_1, g_2 \in G$, $z \in \mathcal{D}$ as well as $J_h^{-2} = J_{2\rho_n}$ and $J_h(\mathbf{1}, z) = 1$ for all $z \in \mathcal{D}$.* ∎

Example 3.6. Let $S = G \times G$, and G the diagonal group. Write $G \subset P_G^+ K_{\mathbb{C}} P_G^-$ for the triangular decomposition for $G_{\mathbb{C}}$ and let $k_1(g) \in K_{\mathbb{C}}$ be the corresponding Harish-Chandra projection. If $\mathcal{D}_G \subseteq \mathfrak{p}_G^+$ is the bounded realization of G/K, then $\mathcal{D} = \mathcal{D}_G \times \mathcal{D}_G^{\rm opp}$ where $\mathcal{D}_G^{\rm opp}$ denotes $\mathcal{D}_G$ equipped with the opposite complex structure. Note that $P^+ = P_G^+ \times P_G^-$ and $P^- = P_G^- \times P_G^+$. In particular we are realizing S/U inside $(G_{\mathbb{C}}/K_{\mathbb{C}}P^-) \times (G_{\mathbb{C}}/K_{\mathbb{C}}P^+)$. Recall that the conjugation $g \mapsto \overline{g}$ with respect to G satisfies $\overline{P^+} = P^-$.

Let $(s,t) \in G \times G$. Then

$$(s,t) = (t,t)(t^{-1}s, \mathbf{1}) = (t,t)(p_1^+ k_1(t^{-1}s)p_1^-, \mathbf{1}) = (tp_1^+, tp_1^+)(k_1(t^{-1}s)p_1^-, (p_1^+)^{-1}) .$$

Hence

$$G_{\mathbb{C}} u_G(s,t) = G_{\mathbb{C}}(k_1(t^{-1}s), \mathbf{1}) .$$

As $U_{\mathbb{C}} = K_{\mathbb{C}} \times K_{\mathbb{C}}$ it follows that any $K_{\mathbb{C}}$-spherical representation π_U of $U_{\mathbb{C}}$ is of the form $\pi_U = \pi \otimes \pi^*$ where π is an irreducible representation of $K_{\mathbb{C}}$. Hence

$$\psi(z,w) = \pi_1(k_1(\exp(-\bar{w})\exp(z))) = K_{-\rho_n^G}(z,w)^{-1} \qquad (z,w \in \mathcal{D}_G) \tag{3.4}$$

where ρ_n^G is the ρ_n for the Hermitian group G. Notice that $K_{-\rho_n^G}$ is nothing else but the reproducing kernel of the classical Hardy space on $\mathcal{D}_G$. ∎

SU(1, 1)-reduction

In this subsection we will discuss the case $S = \mathrm{SU}(1,1)$ even if it is not in our list of causally symmetric triples. The reason is, that many calculations can be reduced to this situation using τ-equivariant embedding of $\mathrm{SU}(1,1)$ into S. Let

$$S = \mathrm{SU}(1,1) = \{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \}$$

Define $\tau : S \to S$ by conjugation with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. On $S_{\mathbb{C}} = SL(2,\mathbb{C})$ the involution τ is given by

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

gko6. In particular

$$G = \{\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t)\end{pmatrix} : t \in \mathbb{R}\} .$$

In this situation we have

$$Y^0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad X^0 = -i\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} \quad \text{and} \quad Z^0 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix} .$$

For the general situation of a causally symmetric triple recall the $\mathfrak{su}(1,1)$-triple $\{X^j, Y^j, Z^j\}$ from Appendix B. From the relations (B.1) and (B.2) in Appendix B it now follows:

Lemma 3.7. *Let the notation be as above. Then the map* $X^j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$, $Y^j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$, *and* $Z^j \mapsto \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}$ *defines a Lie algebra homomorphism into* $\mathfrak{su}(1,1)$ *intertwining the involution* τ *on* $\mathfrak{s}$ *and the above involution on* $\mathfrak{su}(1,1)$. *Furthermore it also intertwines the Cartan involution* θ *on* $\mathfrak{s}$ *and the Cartan involution* $X \mapsto -X^*$ *on* $\mathfrak{su}(1,1)$. ■

On the group level we have:

$$U_{\mathbb{C}} = \{\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1}\end{pmatrix} : \gamma \in \mathbb{C}^*\} ,$$

$$P^+ = \{\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix} : z \in \mathbb{C}\} ,$$

and

$$P^- = \{\begin{pmatrix} 1 & 0 \\ w & 1\end{pmatrix} : w \in \mathbb{C}\} .$$

Furthermore

$$\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ w & 1\end{pmatrix} = \begin{pmatrix} \gamma + zw\gamma^{-1} & z\gamma^{-1} \\ \gamma^{-1}w & \gamma^{-1}\end{pmatrix} .$$

Consequently, if $d \neq 0$ we have

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 1 & b/d \\ 0 & 1\end{pmatrix}\begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d\end{pmatrix}\begin{pmatrix} 1 & 0 \\ c/d & 1\end{pmatrix} .$$

For the $G_{\mathbb{C}}U_{\mathbb{C}}P^-$-decomposition we notice that

$$\begin{pmatrix} a & b \\ b & a\end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ w & 1\end{pmatrix} = \begin{pmatrix} a\gamma + bw\gamma^{-1} & b\gamma^{-1} \\ b\gamma + a\gamma^{-1}w & a\gamma^{-1}\end{pmatrix} .$$

Hence, using that $a^2 - b^2 = 1$, we obtain that

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} \frac{d}{\sqrt{d^2-b^2}} & \frac{b}{\sqrt{d^2-b^2}} \\ \frac{b}{\sqrt{d^2-b^2}} & \frac{d}{\sqrt{d^2-b^2}}\end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{d^2-b^2}} & 0 \\ 0 & \sqrt{d^2-b^2}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{cd-ab}{d^2-b^2} & 1\end{pmatrix} .$$

Thus the open sets $G_{\mathbb{C}}K_{\mathbb{C}}P^-$ and $P^+K_{\mathbb{C}}P^-$ in $S_{\mathbb{C}}$ are given by:

$$G_{\mathbb{C}}U_{\mathbb{C}}P^- = \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : b^2 - d^2 \neq 0\}$$

$$P^+U_{\mathbb{C}}P^- = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid d \neq 0\}$$

Furthermore we notice that the $P^+, U_{\mathbb{C}}$, and P^- components, whenever defined, are given by

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto u_G(s) = \begin{pmatrix} \frac{1}{\sqrt{d^2-b^2}} & 0 \\ 0 & \sqrt{d^2-b^2} \end{pmatrix}$$

and

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix} .$$

Notice that a double covering is needed in general for u_G, but as an element of the coset $K_{\mathbb{C}} \backslash U_{\mathbb{C}}$, with $K_{\mathbb{C}} = \{\pm 1\}$ it is well defined. In the following we will identify P^+ with $\mathbb{C}$ by $z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $\mathbb{C}^*$ with $U_{\mathbb{C}}$ by $\gamma \mapsto \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$. In particular we get the following lemma:

Lemma 3.8. *Identify $\mathfrak{p}^+$ and $\mathfrak{p}^-$ with $\mathbb{C}$, and $U_{\mathbb{C}}$ with $\mathbb{C}^*$ in the way explained above. Let $z, w \in \mathbb{C}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the following assertions hold:*

(i) *If $z\overline{w} \neq 1$ then $\mathcal{K}(z,w) \in U_{\mathbb{C}}$ is defined and given by*

$$\mathcal{K}(z,w) = \frac{1}{1 - z\overline{w}} .$$

(ii) *If $az + d \neq 0$ then $J(g,z)$ is defined and*

$$J(g,z) = cz + d .$$

(iii) *Assume that $|z| < 1$. Then $u_G(z)$ is defined and*

$$u_G(z) = \frac{1}{\sqrt{1 - z^2}} .$$

∎

Description of $\partial_s\Xi$ in $\partial_s\mathcal{D}$

In this subsection we will use the $\mathfrak{su}(1,1)$-reduction to identify $\partial_s\Xi$ in $\partial_s\mathcal{D}$ as the non-vanishing locus of the function ψ. Recall that ψ_4 extends to a holomorphic function on $\mathfrak{p}^+$.

Lemma 3.9. *We have:*

$$\partial_s\Xi = \{z \in \partial_s\mathcal{D}\colon \psi_4(z) \neq 0\} .$$

Proof. "$\subseteq$": Recall the element $z_1 \in \partial_s\Xi$ with isotropy subgroup H, i.e., $\partial_s\Xi = G(z_1) \simeq G/H$. We first show that $\psi_4(z_1) \neq 0$ by using SU(1,1)-reduction. For $S = \mathrm{SU}(1,1)$ we have $\mathcal{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and $z_1 = i$. In particular it follows from Lemma 3.8(iii) that $\psi_4(z_1) \neq 0$. Now $\psi_4(z_1) \neq 0$ in the general case follows from simple SU(1,1)-reduction (cf. Lemma 3.7 and our structural results in Appendix B). From the covariance property of ψ_4 (cf. Proposition 3.5) we hence get that $\psi_4(gz_1) \neq 0$ for all $g \in G$. In particular ψ_4 has no zeros on $G(z_1) = \partial_s\Xi$.

"$\supseteq$": By Theorem 1.14 we know that $\partial_s\Xi$ is open and dense in $\partial_s\mathcal{D}$. Let Y^j and $\mathfrak{a} = \sum_{j=1}^r \mathbb{R}Y^j$ be as in Appendix B. Let $A = \exp(\mathfrak{a})$. Then $G = KAH$. Let g_n be a sequence in G such that $g_n(z_1) \to \partial_s\mathcal{D}\backslash\partial_s\Xi$. Write $g_n = k_n a_n h_n$ with $k_n \in K$, $h_n \in H$ and $a_n = \exp(\sum_{j=1}^r t_{r,n}Y^j) \in A$. Note that $g_n(z_1) \to \partial_s\mathcal{D}\backslash\partial_s\Xi$ means precisely that $a_n \to \infty$.

We have $g_n(z_1) = k_n a_n(z_1)$. We also have that $|J_{4\rho_n}(k,z)| = 1$ for all $k \in K$. Hence it follows from Proposition 3.5 that

$$|\psi_4(k_n a_n(z_1))| = |J_{4\rho_n}(k_n, a_n(z_1))| \cdot |\psi_4(a_n(z_1))| = |\psi_4(a_n(z_1))| = c|J_{4\rho_n}(a_n, z_1)|$$

with $c{:}= |\psi_4(z_1)| > 0$. By SU(1,1)-reduction (cf. Lemma 3.7 and Lemma 3.8(ii)) we get

$$J_{4\rho_n}(a_n, z_1) = \prod_{j=1}^r (i\sinh(t_{n,j}) + \cosh(t_{n,j}))^{-4\rho_n(H^j)}\,.$$

As $a_n \to \infty$, there exists an index j and a subsequence $s_n = t_{n,j}$ such that $|s_n| \to \infty$. Therefore

$$\prod_{j=1}^r (i\sinh(t_{n,j}) + \cosh(t_{n,j}))^{-4\rho_n(H^j)} \to 0$$

because $\rho_n(H^j) > 0$. This concludes the proof of the Lemma. ■

Theorem 3.10. *The function ψ extends to a continuous function on $\mathcal{D} \cup \partial_s\mathcal{D}$ such that*

$$\partial_s\Xi = \{z \in \partial_s\mathcal{D}{:}\, \psi(z) \neq 0\}\,.$$

Proof. Notice that the set $\mathcal{D} \cup \partial_s\Xi$ is simply connected. Thus Lemma 3.9 implies that there exists a unique continuous function $\psi : \mathcal{D} \cup \partial_s\Xi \to \mathbb{C}^*$ such that $\psi(0) = 1$ and $\psi(z)^4 = \psi_4(z)$. In particular this function agrees with our old definition of ψ on $\mathcal{D}$. For $z \in \partial_s\mathcal{D} \setminus \partial_s\Xi$ define $\psi(z) = 0$. It follows from Lemma 3.9 that ψ is continuous. ■

L^2-isomorphism on the Shilov boundaries

Let $\mu_{\partial_s\mathcal{D}}$ be the unique (up to constant) U-invariant measure on $\partial_s\mathcal{D}$. If $\lambda_h = -\rho_n$ is analytically integral, then S acts unitarily on $L^2(\partial_s\mathcal{D}, \mu_{\partial_s\mathcal{D}})$ by

(3.5) $$(\pi_h(s)f)(z) = J_h(s^{-1}, z)^{-1} f(s^{-1}z) \qquad (s \in S,\ z \in \mathcal{D},\ f \in \mathcal{H}^2(\mathcal{D}))$$

with $J_h = J_{-\rho_n}$. In the case where $m = 2$ (see Proposition 3.1) we need to go to a double covering of S. But notice, that $\pi_h(s)$ is always defined for $s \in G$ according to Proposition 3.5. One of the consequences of this observation is the following identification of L^2-spaces:

Lemma 3.11. *The following assertions hold:*

(i) *The G-invariant measure on $G/H \simeq \partial_s\Xi$ as a subset of $\partial_s\mathcal{D}$ is given by*

$$f \mapsto \int_{\partial_s\mathcal{D}} f(z)|\psi(z)|^{-2}\, d\mu_{\partial_s\mathcal{D}}(z)$$

where $d\mu_{\partial_s\mathcal{D}}$ is the unique (up to constant) U-invariant measure on $\partial_s\mathcal{D}$.

(ii) *The mapping $f \mapsto \frac{1}{\psi}f$ is a G-equivariant isomorphism of $L^2(G/H)$ onto $L^2(\partial_s\mathcal{D}, \mu_{\partial_s\mathcal{D}})$.*

Proof. This follows from the covariance of ψ in Proposition 3.5 (cf. [BÓ01], Theorem 5.1). ■

4. The classical Hardy space inside $L^2(G/H)_{\rm mc}$

In this section we will show how to realize the classical Hardy space $\mathcal{H}^2(\mathcal{D})$ (cf. (2.7)) in the most continuous spectrum of $L^2(G/H)$.

The most continuous spectrum $L^2(G/H)_{\rm mc}$

Recall the involution σ on $\mathfrak{g}$ associated to $\mathfrak{h}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the σ-eigenspace decomposition. Note that we can choose $\mathfrak{a} \subseteq \mathfrak{q} \cap \mathfrak{p}$ since $(\mathfrak{g}, \sigma)$ is non-compactly causal (cf. [HÓ96]). In particular, the minimal parabolic subgroup $P_{\min} = MAN$ is also a minimal $\theta\sigma$-*stable* parabolic subgroup of G. Here, as usual, $M = Z_K(\mathfrak{a})$. If $\mathfrak{m}$ is the Lie algebra of M, then note that $\mathfrak{m} \subseteq \mathfrak{h}$ since $\mathfrak{a} \subseteq \mathfrak{q} \cap \mathfrak{p}$. Moreover we have

$$M = Z_H(\mathfrak{a}) \subseteq H$$

as $G_{\mathbb{C}}$ is simply connected (cf. [HÓ96, Lemma 3.1.22]).

Write $\pi_\delta = \operatorname{Ind}_{MAN}^G(\delta)$ for the K-spherical unitary principal series of parameter $\delta \in i\mathfrak{a}^*$. Notice that $\pi_\delta \simeq \pi_{w\delta}$ for $w \in \mathcal{W}$. The *most continuous part* $L^2(G/H)_{\rm mc}$ in $L^2(G/H)$ is by definition the G-invariant subspace in $L^2(G/H)$ which corresponds in the Plancherel formula to all principal series induced from a minimal $\theta\sigma$-stable parabolic subgroup. Denote by $\mathcal{W}_0 = N_{K\cap H}(\mathfrak{a})/Z_{K\cap H}(\mathfrak{a})$ the very little Weyl group. Since $M = M \cap H$ we now obtain from [BS97] or [D98] that

$$(L, L^2(G/H)_{\rm mc}) \simeq \Big(\int_{i\mathfrak{a}^*/\mathcal{W}}^{\oplus} \pi_\delta \otimes \mathrm{id}\ d\mu(\delta), \int_{i\mathfrak{a}^*/\mathcal{W}}^{\oplus} \mathcal{H}_\delta \otimes \mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}\ d\mu(\delta) \Big)$$

where μ is a Borel measure that is completely continuous with respect to the Lebesgue measure on $i\mathfrak{a}^*$. In particular, up to a set of measure zero, the multiplicity of π_δ in $L^2(G/H)_{\rm mc}$ is $|\mathcal{W}/\mathcal{W}_0|$. Let $\omega \subseteq i\mathfrak{a}^*/\mathcal{W}$ be an open subset. Then we write

$$P_\omega \colon L^2(G/H)_{\rm mc} \to \int_\omega \mathcal{H}_\delta \otimes \mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}\ d\mu(\delta)$$

for the orthogonal G-invariant projection associated to ω.

Definition 4.1. We call a closed G-invariant subspace of $W \subseteq L^2(G/H)_{\rm mc}$ a subspace with *full spectrum* if $P_\omega(W) \neq \{0\}$ for all non-empty open subets $\omega \subseteq i\mathfrak{a}^*/\mathcal{W}$. ■

It is interesting to compare the most-continuous spectrum $L^2(G/H)_{\rm mc}$ with the spectrum of $L^2(G/K)$. The Plancherel Theorem for $L^2(G/K)$ has the form

$$(L, L^2(G/K)) \simeq \int_{i\mathfrak{a}^*/\mathcal{W}} \pi_\delta\ d\mu'(\delta)$$

with μ' a Borel measure completely continuous with respect to $d\delta$. Hence we see that the spectrum of $L^2(G/K)$ is the same as the most continuous part of the spectrum of $L^2(G/H)$, but the difference is that the spectrum of $L^2(G/K)$ is multiplicity free.

Embedding of $\mathcal{H}^2(\mathcal{D})$ in the most continuous spectrum

Recall the U-invariant measure $\mu_{\partial_s\mathcal{D}}$ on the Shilov boundary of $\mathcal{D}$ and the boundary value map of the classical Hardy space:

$$b\colon (\pi_h, \mathcal{H}^2(\mathcal{D})) \to (\pi_h, L^2(\partial_s\mathcal{D}, \mu_{\partial_s\mathcal{D}})), \quad f \mapsto (z \mapsto \lim_{\substack{r\to 1\\ r<1}} f(rz))$$

which is an S-equivariant isometric embedding. On the other hand we have the G-equivariant isomorphism from Lemma 3.11:

$$\Psi\colon (\pi_h, L^2(\partial_s\mathcal{D})) \to (L, L^2(G/H)), \quad f \mapsto \psi f \ .$$

Theorem 4.2. *The mapping*

$$\Psi \circ b\colon (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, L^2_{\rm mc}(G/H))$$

is a G-equivariant isometric embedding. Moreover the image $\operatorname{im}\Psi\circ b$ *is a mulitiplicity free subspace of full spectrum in* $L^2_{\rm mc}(G/H)$.

Proof. From Lemma 3.11 it is clear that $\Psi\circ b$ is a G-equivariant map from $\mathcal{H}^2(\mathcal{D})$ into $L^2(G/H)$. It remains to specify the image. From Corollary 2.11 we obtain that $\pi_h|_G \simeq L^2(G/K)$. Thus the remaining assertions of the theorem follow from our discussion of the most continuous spectrum in the previous subsection. ■

5. The Hardy space on Ξ

This final section of the paper is devoted to the Hardy space $\mathcal{H}^2(\Xi)$ on Ξ. After a brief disgression on compression semigroups of $\mathcal{D}\simeq\Xi$ we give the definition of the Hardy space and show that it is in fact a Hilbert space. Also we show the existence of a boundary value map $b\colon\mathcal{H}^2(\Xi)\to L^2(G/H)$. Subsequently, using the results of Section 3, we show that there is a natural identification of $\mathcal{H}^2(\Xi)$ and $\mathcal{H}^2(\mathcal{D})$. This together with Theorem 4.2 will then give us that $\operatorname{im} b\subseteq L^2(G/H)_{\rm mc}$, the main result of this paper.

Definition of $\mathcal{H}^2(\Xi)$ and first properties

In order to define the Hardy space on Ξ we first have to recall some facts on semigroups compressing $\mathcal{D}$, resp. Ξ.

Recall the space $\mathfrak{c}=\mathfrak{t}\cap\mathfrak{q}_*$ and associated restricted root system $\Delta_{\mathfrak{c}}$ from Section 2. For $\alpha\in\Delta_{\mathfrak{c}}$ we denote by $\check\alpha\in i\mathfrak{c}$ the coroot of α. Define

$$C_{\rm min}\colon= \operatorname{cone}(\{\check\alpha\colon \alpha\in\Delta^+_{\mathfrak{c},n}\}) \ ,$$

where $\operatorname{cone}(\cdot)$ refers to the convex cone generated by $(\cdot)$. Then

$$W_{\rm min}\colon= \operatorname{Ad}(G)C_{\rm min}$$

is a minimal open convex cone in $i\mathfrak{q}_*$ and

$$\Gamma := G\exp(W_{\min})$$

is a G-biinvariant subsemigroup of $S_{\mathbb{C}}$ (cf. [HÓ96, Ch. 4]). The closure of Γ in $S_{\mathbb{C}}$ is given by $\overline{\Gamma} = G\exp(\overline{W_{\min}})$. The polar mapping

$$G \times \overline{W_{\min}} \to \overline{\Gamma}, \quad (g, X) \mapsto g\exp(X)$$

is a homeomorphism (cf. [N99, Th. XI.1.7]). Moreover, $\overline{\Gamma}$ is an involutive semigroup with involution

$$\overline{\Gamma} \to \overline{\Gamma}, \quad \gamma = g\exp(X) \mapsto \gamma^* = \exp(X)g^{-1} .$$

Note that Γ^{-1} compresses $\mathcal{D}$:

$$(\forall \gamma \in \Gamma^{-1}) \qquad \gamma(\overline{\mathcal{D}}) \subseteq \mathcal{D}$$

(cf. [N99, Th. XII.3.3]). From the realization of $\overline{\Xi}$ in $\overline{\mathcal{D}}$ we therefore obtain an action of Γ^{-1} on $\overline{\Xi}$ with the property

$$(\forall \gamma \in \Gamma^{-1}) \qquad \gamma(\overline{\Xi}) \subseteq \Xi. \tag{5.1}$$

Definition 5.1. **(Hardy space on Ξ)** The Hardy space on Ξ is defined as

$$\mathcal{H}^2(\Xi) = \{f \in \mathcal{O}(\Xi) : \|f\|^2 := \sup_{\gamma\in\Gamma} \int_{G/H} |f(\gamma^{-1}gz_1)|^2 \, d\mu_{G/H}(gH) < \infty\}. \qquad \blacksquare$$

From the definition of the Hardy space it is not clear yet that $\mathcal{H}^2(\Xi)$ is a Hilbert space. This will follow from the following geometric fact and its proof:

Lemma 5.2. *The semigroup orbit $\Gamma^{-1}(z_1) \subseteq \Xi$ is open and measure dense.*

Proof. Recall the elements $H^j \in i\mathfrak{c}$ and note that $\sum_{j=1}^n t_j H^j \in C_{\min}$ whenever $t_j \geq 0$ and not all $t_j = 0$. By simple $\mathrm{SU}(1,1)$-reduction we obtain for all j that

$$\exp(-t_j H^j)(z_1) = \exp(i\arctan(e^{-t_j})Y^j)(0). \tag{5.2}$$

Write $D := \{z \in \mathbb{C} : |z| < 1\}$ for the unit disc and identify $A\exp(i\Omega)(0)$ with D^n in the obvious way through $\mathrm{SU}(1,1)$-reduction. With $D^- = D\backslash\mathbb{R}$ equation (5.2) then gives

$$A\mathcal{W}\exp(-C_{\min})(z_1) = (D^-)^n. \tag{5.3}$$

In particular $A\mathcal{W}\exp(-C_{\min})(z_1)$ is open and of full measure in $A\exp(i\Omega)(0) = D^n$. Sweeping out Ξ by G-orbits through $\exp(i\Omega)(0)$ proves the lemma. $\blacksquare$

If M is a complex manifold, then we write $\mathcal{O}(M)$ for the Fréchet space of holomorphic functions on M.

Corollary 5.3. *The Hardy space $\mathcal{H}^2(\Xi)$ is a Hilbert space of holomorphic functions. Moreover, the inclusion mapping $\mathcal{H}^2(\Xi) \hookrightarrow \mathcal{O}(\Xi)$ is continuous.*

Proof. If we know that $\mathcal{H}^2(\Xi)$ is complete, then the standard semigroup techniques imply that $\mathcal{H}^2(\Xi)$ is a Hilbert space (cf. [HÓØ91] or [N99, Ch. XIV]). The completeness and the continuity of the embedding $\mathcal{H}^2(\Xi) \hookrightarrow \mathcal{O}(\Xi)$ is easily obtained from (5.3) and Cauchy's Theorem (use $\Gamma = G\exp(C_{\min})G$ and choose local coordinates – recall the definition of the Hardy space). $\blacksquare$

Knowing that $\mathcal{H}^2(\Xi)$ is a Hilbert space, it is straightforward from the definition of $\mathcal{H}^2(\Xi)$ that the left regular representation L of G on $\mathcal{H}^2(\Xi)$ is unitary. In the terminology of [FT99] or [K99] this means that $\mathcal{H}^2(\Xi)$ is a *G-invariant Hilbert space of holomorphic functions on Ξ*. Those Hilbert spaces feature the following properties:

Remark 5.4. (a) The inclusion mapping $\mathcal{H}^2(\Xi) \hookrightarrow \mathcal{O}(\Xi)$ is continuous and so all point evaluations

$$\mathrm{ev}_z\colon \mathcal{H}^2(\Xi) \to \mathbb{C}, \quad f \mapsto f(z) \qquad (z \in \Xi)$$

are continuous. In particular for every $z \in \Xi$ there exists an element $K_z \in \mathcal{H}^2(\Xi)$ such that $f(z) = \langle f, K_z \rangle$.
(b) The Hardy space admits a reproducing kernel

$$K_\Xi\colon \Xi \times \Xi \to \mathbb{C}, \quad (z,w) \mapsto \langle K_w, K_z \rangle \ .$$

Note that K_Ξ is holomorphic in the first and antiholomorphic in the second variable. We call K_Ξ the *Cauchy-Szegö kernel* of Ξ.
(c) By construction the representation $(L, \mathcal{H}^2(\Xi))$ of G is unitary. This can also be phrased by saying that K_Ξ is G-invariant:

$$K_\Xi(gz, gw) = K_\Xi(z,w)$$

for all $g \in G$, $z, w \in \Xi$. ■

The boundary value map

Using the standard semigroup techniques for Hardy spaces (cf. [HÓØ91] or [N99, Ch. XIV]) one easily shows that the prescription

$$L\colon \overline{\Gamma} \to B(\mathcal{H}^2(\Xi)), \quad (L(\gamma)f)(z) := f(\gamma^{-1}z)$$

defines a strongly continuous involutive contractive representation of $\overline{\Gamma}$ whose restriction to Γ is a holomorphic mapping. Here *strongly continuous* means that the mapping $\overline{\Gamma} \to \mathcal{H}^2(\Xi)$, $\gamma \mapsto L(\gamma)f$ is continuous for all $f \in \mathcal{H}^2(\Xi)$; *involutive* means $L(\gamma^*) = L(\gamma)^*$ for all $\gamma \in \overline{\Gamma}$, and *contractive* means that $\|L(\gamma)\| \leq 1$ for all $\gamma \in \overline{\Gamma}$. Note that L is involutive implies in particular that $L|_G$ is unitary.

In order to define the boundary value map we first have to exhibit a good G-invariant subspace of $\mathcal{H}^2(\Xi)$ whose elements extend continuously to the boundary. A good choice herefore is the space of analytic vectors $\mathcal{H}^2(\Xi)^\omega$ of the representation $(L, \mathcal{H}^2(\Xi))$ of G. According to [KNÓ97, App.] we have

$$\mathcal{H}^2(\Xi)^\omega = \bigcup_{\gamma \in \Gamma} L(\gamma)\mathcal{H}^2(\Xi) \ .$$

In particular it follows from the compression property (5.1) that all functions in $\mathcal{H}^2(\Xi)^\omega$ holomorphically extend over the closure $\overline{\Xi}$ of Ξ. As a consequence we obtain a well defined boundary value mapping

$$b^\omega\colon \mathcal{H}^2(\Xi)^\omega \to L^2(G/H), \quad f \mapsto \big(gH \mapsto \lim_{\substack{\gamma \to \mathbf{1} \\ \gamma \in \Gamma}} f(\gamma^{-1}gz_1)\big)$$

which is G-equivariant and continuous.

Theorem 5.5. *The boundary value mapping*

$$b^\omega\colon \mathcal{H}^2(\Xi)^\omega \to L^2(G/H), \quad f \mapsto \big(gH \mapsto \lim_{\substack{\gamma \to \mathbf{1} \\ \gamma \in \Gamma}} f(\gamma^{-1}gz_1)\big)$$

extends to a G-equivariant isometric embedding

$$b\colon \mathcal{H}^2(\Xi) \to L^2(G/H) \ .$$

Proof. This follows from our discussion above and the standard semigroup techniques (cf. [HÓO91] or [N99, Ch. XIV]). ■

The isomorphism between $\mathcal{H}^2(\Xi)$ and $\mathcal{H}^2(\mathcal{D})$

In the sequel we will identify Ξ with $\mathcal{D}$ under the biholomorphism $\Phi\colon \Xi \to \mathcal{D}$ from Section 2. We let G act on $\mathcal{O}(\Xi)$ via the left regular representation L and we let S (or its double covering if necessary) act on $\mathcal{O}(\mathcal{D})$ via the cocycle action π_h

$$(\pi_h(s)f)(z) = J_h(s^{-1}, z)^{-1} f(s^{-1}z)$$

for all $s \in S$, $z \in \mathcal{D}$ and $f \in \mathcal{O}(\mathcal{D})$. Then we have an isomorphism of topological vector spaces

$$\Psi\colon \mathcal{O}(\mathcal{D}) \to \mathcal{O}(\Xi), \quad F \mapsto \psi F$$

with inverse mapping

$$\Psi^{-1}\colon \mathcal{O}(\Xi) \to \mathcal{O}(\mathcal{D}), \quad f \mapsto \frac{1}{\psi} f$$

Note that Ψ and Ψ^{-1} are defined since ψ is a zero free holomorphic function according to Proposition 3.5.

Lemma 5.6. *The maps Ψ and Ψ^{-1} are G-equivariant.*

Proof. Since $J_h = J_{-\rho_n}$ this follows from Proposition 3.5. ∎

From Lemma 5.6 we obtain immediately that the mapping

$$\Psi\colon (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, \mathcal{O}(\Xi)), \quad f \mapsto \psi f$$

is a continuous G-equivariant embedding. More precisely we have:

Theorem 5.7. *The mapping*

$$\Psi\colon (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, \mathcal{H}^2(\Xi)), \quad f \mapsto \psi f$$

is a G-equivariant isomorphism of Hilbert spaces.

Proof. Recall the element $X^0 \in \mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*$. For $s \in]-\infty, 0[$ define

$$\gamma_s := \exp(isX^0)$$

and note that $\gamma_s \in Z(U_{\mathbb{C}}) \cap \Gamma$. Furthermore we have $\gamma_s^{-1}(z_1) = z_{e^s}$ which follows by simple $\mathfrak{su}(1,1)$-reduction using the results in Appendix B. We first show that $\frac{1}{\psi} f \in \mathcal{H}^2(\mathcal{D})$ for all $f \in \mathcal{H}^2(\Xi)$ and that $\|f\| = \|\frac{1}{\psi} f\|$. For every $s < 0$ set $f_s(gH) := f(\gamma_s^{-1} g z_1)$ for $g \in G$. Now it follows from the definition of $\mathcal{H}^2(\Xi)$ and Lemma 3.11 that

$$\begin{aligned}
\|f\|^2 &= \lim_{\substack{\gamma \to \mathbf{1} \\ \gamma \in \Gamma}} \int_{G/H} |f(\gamma^{-1} g z_1)|^2 \, d\mu_{G/H}(gH) \\
&= \sup\nolimits_{s<0} \int_{G/H} |f(\gamma_s^{-1} g z_1)|^2 \, d\mu_{G/H}(gH) \\
&= \sup\nolimits_{s<0} \int_{G/H} |f_s(g z_1)|^2 \, d\mu_{G/H}(gH) \\
&= \sup\nolimits_{s<0} \int_{\partial_s \mathcal{D}} |f_s(z)|^2 |\psi(z)|^{-2} \, d\mu_{\partial_s \mathcal{D}}(z) \\
&= \sup\nolimits_{0 \le r < 1} \int_{\partial_s \mathcal{D}} \left| \frac{f(rz)}{\psi(z)} \right|^2 \, d\mu_{\partial_s \mathcal{D}}(z).
\end{aligned} \tag{5.4}$$

Now consider the function

$$F\colon [0,1]\times \partial_s\mathcal{D}, \quad (r,z)\mapsto \left|\frac{\psi(z)}{\psi(rz)}\right| .$$

Note that F is continuous, hence bounded since defined on a compact set. In particular we find a constant $C>0$ such that $\frac{1}{|\psi(z)|}\geq C\frac{1}{|\psi(rz)|}$ for all r,z. With this information we obtain from (5.4) that

$$\|f\|^2 \geq C\sup\nolimits_{0\leq r<1}\int_{\partial_s\mathcal{D}}\left|\frac{f(rz)}{\psi(rz)}\right|^2\ d\mu_{\partial_s\mathcal{D}}(z), \tag{5.5}$$

i.e., $\frac{1}{\psi}f\in\mathcal{H}^2(\mathcal{D})$. Moreover from $F(1,z)=1$ for all $z\in\partial_s\mathcal{D}$ we now obtain from (5.4) and (5.5) that

$$\begin{aligned}\|f\|^2 &= \lim_{r\to 1}\int_{\partial_s\mathcal{D}}\left|\frac{f(rz)}{\psi(z)}\right|^2\ d\mu_{\partial_s\mathcal{D}}(z)\\ &= \lim_{r\to 1}\int_{\partial_s\mathcal{D}}|F(r,z)|^2\left|\frac{f(rz)}{\psi(rz)}\right|^2\ d\mu_{\partial_s\mathcal{D}}(z).\end{aligned}$$

Thus we have shown that Ψ^{-1} maps $\mathcal{H}^2(\Xi)$ isometrically into $\mathcal{H}^2(\mathcal{D})$. Finally, reversing the arguments from above, also yields that Ψ maps $\mathcal{H}^2(\mathcal{D})$ isometrically into $\mathcal{H}^2(\Xi)$, concluding the proof of the theorem. ∎

An immediate conclusion from Theorem 5.7 in conjunction with Theorem 5.5 and Theorem 4.2 now is:

Theorem 5.8. *The image of the boundary value map $b:\mathcal{H}^2(\Xi)\to L^2(G/H)$ is a mulitiplicity free subspace of $L^2(G/H)_{\rm mc}$ of full spectrum.* ∎

The Cauchy-Szegö kernel

We conclude this section with the derivation of the Cauchy-Szegö kernel for $\mathcal{H}^2(\Xi)$.

Theorem 5.9. *If K_h is the reproducing kernel of $\mathcal{H}^2(\mathcal{D})$ then the Cauchy-Szegö kernel K_Ξ is given by*

$$K_\Xi(z,w)=\psi(z)\overline{\psi(w)}K_h(z,w).$$

Proof. In this proof denote by $(\cdot|\cdot)$ the scalar product on $\mathcal{H}^2(\Xi)$ and by $\langle\cdot,\cdot\rangle$ the scalar product on $\mathcal{H}^2(\mathcal{D})$. Fix $z\in\Xi$ and let $f\in\mathcal{H}^2(\Xi)$. Consider the function $w\mapsto\overline{\psi(z)}\psi(w)K_h(w,z)$ and note that this function lies in $\mathcal{H}^2(\Xi)$. We now have

$$\begin{aligned}(f|\overline{\psi(z)}\psi(\cdot)K_h(\cdot,z)) &= \psi(z)(f|\psi(\cdot)K_h(\cdot,z))\\ &= \psi(z)\langle\frac{1}{\psi(\cdot)}f, K_h(\cdot,z)\rangle\\ &= \psi(z)\frac{1}{\psi(z)}f(z)=f(z).\end{aligned}$$

This concludes the proof of the theorem. ∎

Example 5.10. Let us consider the case of $S = G \times G$. We keep the notation of Example 3.6 and identify $\Xi = \mathcal{D}$ with $\mathcal{D}_G \times \mathcal{D}_G^{\mathrm{opp}}$. Write $K_h^G(z,w)$ for the kernel of the classical Hardy space on $\mathcal{D}_G$. Then the kernel K_h of the classical Hardy space on $\mathcal{D}$ is given by

$$K_h\big((z_1,w_1),(z_2,w_2)\big) = K_h^G(z_1,z_2)\overline{K_h^G(w_1,w_2)}$$

for all (z_1,w_1), (z_2,w_2) in $\mathcal{D}$. From Theorem 5.8 and (3.4) we thus obtain that

$$K_\Xi\big((z_1,w_1),(z_2,w_2)\big) = \frac{K_h^G(z_1,z_2)\overline{K_h^G(w_1,w_2)}}{K_h^G(z_1,w_1)\overline{K_h^G(z_2,w_2)}}\,. \tag{5.6}$$

In particular for $G = \mathrm{SU}(1,1)$ where $\mathcal{D}_G = D$ is the unit disc we have $K_h(z,w) = \frac{1}{1-z\overline{w}}$ and so

$$K_\Xi\big((z_1,w_1),(z_2,w_2)\big) = \frac{(1-z_1\overline{w_1})(1-\overline{z_2}w_2)}{(1-z_1\overline{z_2})(1-\overline{w_1}w_2)}. \qquad \blacksquare$$

Final remarks

It is possible to extend the results in this paper to the cases in Theorem 1.12 (ii), (iii) and (iv). Let us explain this for groups G which are structure groups of Euclidean Jordan algebras (the case in Theorem 1.12(ii)). Here one has

$$\partial_d \Xi_0 = \underbrace{\coprod_{j=1}^{n} G/H_j}_{\partial_s \Xi} \amalg G/K \amalg G/K$$

with each G/H_j non-compactly causal (for more precise information see [GK02b, Sect. 4]). For every boundary component G/H_j of $\partial_d\Xi_0$ one can define a Hardy space

$$\mathcal{H}_j^2(\Xi_0) = \{f \in \mathcal{O}(\Xi_0)\colon \sup_{\gamma\in\Gamma} \int_{G/H_j} |f(\gamma^{-1} g z_j)|^2 \, d\mu_{G/H_j}(gH_j) < \infty\}\ .$$

We expect a boundary value map

$$b_j\colon \mathcal{H}_j^2(\Xi_0) \hookrightarrow L^2(G/H_j)$$

with image a full subspace in $L^2(G/H_j)_{\mathrm{mc}}$. We also expect for each boundary component G/H_j a G-invariant isometric isomorphism

$$\mathcal{H}_j^2(\Xi_0) \hookrightarrow \mathcal{H}^2(\mathcal{D})$$

with the classical Hardy space.

Appendix A: parameter calculations

In this appendix we will derive some technical results on parameters of unitary highest weight representations which are needed in Section 2.

Unless otherwise specified $\mathfrak{s}$ denots a simple Hermitian Lie algebra with compact Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{u}$. Write $\gamma_1, \ldots, \gamma_n$ for a maximal system of strongly orthogonal roots. Define

$$\zeta := \frac{1}{2}(\gamma_1 + \ldots + \gamma_r)$$

and note that $\zeta \in i\mathfrak{z}(\mathfrak{u})^*$. Note that we can choose the positive system Δ^+ such that $\beta = \gamma_1$ is the highest root. We write $\check{\gamma}_j \in i\mathfrak{t}$ for the coroot of γ_j. Denote by $\mathfrak{c}$ the subspace of $\mathfrak{t}$ which is spanned by the $i\check{\gamma}_j$. Write $\Sigma := \Delta\,|_{\mathfrak{c}} \backslash\{0\}$ for the restricted root system. Define $\Sigma_n := \Delta_n\,|_{\mathfrak{c}}$, $\Sigma_c := \Delta_c\,|_{\mathfrak{c}} \backslash\{0\}$. Similarly define Σ^+, Σ_n^+, Σ_c^+. By results of Harish-Chandra and Moore we have:

$$\Sigma_n^+ = \{\frac{1}{2}(\gamma_i + \gamma_j) : 1 \leq i, j \leq r\}$$

and

$$\Sigma_c^+ = \{\frac{1}{2}(\gamma_i - \gamma_j) : 1 \leq i < j \leq r\} \cup \{\frac{1}{2}\gamma_i : 1 \leq i \leq r\}\ .$$

The second term in Σ_c^+ only appears if $\mathfrak{s}$ is not of tube type.

Define the number $d := \dim_{\mathbb{C}} \mathfrak{s}_{\mathbb{C}}^{\frac{1}{2}(\gamma_i - \gamma_j)} = \dim_{\mathbb{C}} \mathfrak{s}_{\mathbb{C}}^{\frac{1}{2}(\gamma_i + \gamma_j)}$ for $i \neq j$. This number does not depend on i and j, only on $\mathfrak{s}$. The table is as follows:

Table II

$\mathfrak{s}$	d
$\mathfrak{sp}(n, \mathbb{R})$	1
$\mathfrak{su}(p, q)$	2
$\mathfrak{so}(2, n)$	$n - 2 \quad (n > 2)$
$\mathfrak{so}^*(2n)$	4
$\mathfrak{e}_{6(-14)}$	8
$\mathfrak{e}_{7(-25)}$	8

Theorem A.1. (Wallach) (cf. [W79]) *Let $\lambda_z := z\zeta$ for $z \in \mathbb{R}$. Then the set of scalar unitary highest weights $\lambda_z \in i\mathfrak{z}(\mathfrak{u})^*$ is parametrized by*

$$z \in \mathbb{W} \quad \textit{with} \quad \mathbb{W} =]-\infty, -\frac{d(r-1)}{2}[\cup \{-\frac{d(r-1)}{2}, -\frac{d(r-2)}{2}, \ldots, 0\}. \qquad \blacksquare$$

The set $\mathbb{W}$ in Theorem A.1 is called the *Wallach set*. Note that λ_z is regular means precisely that $z < -\frac{d(r-1)}{2}$.

Lemma A.2. *If $\mathfrak{s}$ is of tube type, i.e, if Σ is of type C_r, then the following assertions hold:*

(i) $\rho_n = (1 + \frac{d(r-1)}{2})\zeta$.

(ii) $\langle \rho, \beta \rangle = \frac{1}{2}(1 + d(r-1))$.

(iii) *If* $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$ *is such that* $\langle 2\lambda + \rho, \beta\rangle < 0$*, then* λ *is regular.*
(iv) *The Hardy space weight* $\lambda_h = -\rho_n$ *satisfies the condition* $\langle 2\lambda_h + \rho, \beta\rangle < 0$*.*

Proof. (i) [ÓØ99, Lemma 2.5].
(ii) Note that $\rho = \rho_n + \rho_c$. Thus (ii) is immediate from (i) and the structure of Σ_c^+.
(iii) Write $\lambda = \lambda_z$. Then by (ii) the condition $\langle 2\lambda + \rho, \beta\rangle < 0$ means that

$$z < -\frac{1}{2}(1 + d(r-1))$$

and Wallach's Theorem A.1 implies the assertion.
(iv) We obtain from (i) that

$$\langle 2\lambda_h + \rho, \beta\rangle = \langle -\rho_n + \rho_c, \gamma_1\rangle = -\frac{1}{2}(1 + \frac{d(r-1)}{2}) + \langle \rho_c, \gamma_1\rangle \ .$$

Now $\langle \rho_c, \gamma_1\rangle = \frac{d(r-1)}{4}$. Thus

$$\langle 2\lambda_h + \rho, \beta\rangle = -\frac{1}{2}(1 + \frac{d(r-1)}{2}) + \frac{d(r-1)}{4} = -\frac{1}{2} < 0 \ ,$$

as was to be shown. ■

Proposition A.3. *Let* $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ *be a causally symmetric triple. Then the following assertions hold:*

(i) *The Hardy space parameter* $\lambda_h = -\rho_n$ *is regular.*
(ii) *The Hardy space parameter* $\lambda_h = -\rho_n$ *satisfies the condition* $\langle \lambda_h + \rho_G^c, \beta\rangle < 0$*.*

Proof. If $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{g}$ is the group case, then $\mathfrak{g}$ is of tube type and $\rho_G^c = \frac{1}{2}\rho$. Thus (i) and (ii) follow in this case from Lemma A.2. Hence we may assume that $\mathfrak{s}$ is simple. Since $\mathfrak{s}$ is of tube type, (i) follows from Lemma A.2(i) and Theorem A.1.

It remains to check that (ii) is satisfied for the pairs

$$(\mathfrak{s}, \mathfrak{g}) = (\mathfrak{sp}(2n, \mathbb{R}),\ \mathfrak{sp}(n, \mathbb{R})),\ (\mathfrak{su}(2n, 2n),\ \mathfrak{sp}(n, \mathbb{C})), \quad \text{and} \quad (\mathfrak{so}(2, n), \mathfrak{so}(1, n)) \,.$$

Let us start with $(\mathfrak{sp}(2n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R}))$. In this case it is easy to see that $\langle \rho_G^c, \beta\rangle = \frac{1}{2}\langle \rho, \beta\rangle$. Thus we obtain

$$\langle \lambda_h + \rho_G^c, \beta\rangle = \frac{1}{2}\langle 2\lambda_h + \rho, \beta\rangle < 0$$

by Lemma A.2 (iv).

Next consider the case $(\mathfrak{su}(2n, 2n), \mathfrak{sp}(n, \mathbb{C}))$. Note that the restricted root system of $\mathfrak{sp}(n, \mathbb{C})$ is of type C_n with with root space dimension 2 for all roots. Thus we obtain from Lemma A.2(ii) that

$$\langle \rho_G^c, \beta\rangle = \frac{1}{2}\big(\frac{1}{2}(1 + 2(n-1))\big) = -\frac{1}{4} + \frac{n}{2} \ .$$

On the other hand we have

$$\langle \lambda_h, \beta\rangle = -\frac{1}{2}(1 + 1 \cdot \frac{2n-1}{2}) = -(\frac{1}{4} + \frac{n}{2}) \ .$$

Thus $\langle \lambda_h + \rho_G^c, \beta\rangle = -\frac{1}{2} < 0$.

Finally consider the pair $(\mathfrak{so}(2, n), \mathfrak{so}(1, n))$. Here one has

$$\langle \lambda_h + \rho_g^c, \beta\rangle = \langle \lambda_h, \beta\rangle + \langle \rho_g^c, \beta\rangle = -\frac{n}{4} + \frac{n-1}{4} = -\frac{1}{4} < 0 \ ,$$

concluding the proof of the proposition. ■

Appendix B: Structure theory for totally symmetric triples

In this appendix we will develop some structure theory for causally symmetric triples $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$. The results will be useful for certain $\mathfrak{su}(1,1)$-reductions used throughout the paper. Also we will give a structural description of the maximal parabolic subgroup P of S. We will state all results without proof since they entirely consist of minor variations of results already in the literature (cf. [HÓ96, Ch. 1, Ch. 4, App. A]).

As in Section 2 let $\mathfrak{t} \subseteq \mathfrak{u}$ denote a compact τ-stable Cartan subalgebra of $\mathfrak{s}$. Set $\mathfrak{c} := \mathfrak{t} \cap \mathfrak{q}_*$ and write $\Delta_\mathfrak{c} = \Delta(\mathfrak{s}_\mathbb{C}, \mathfrak{c}_\mathbb{C})$ for the restricted root system with respect to $\mathfrak{c}_\mathbb{C}$. We can choose Δ^+ (without changing Δ_n^+) such that $\Delta_\mathfrak{c}^+ := \Delta^+|_\mathfrak{c} \backslash \{0\}$ is a positive system for $\Delta_\mathfrak{c}$. Further we set $\Delta_{\mathfrak{c},n}^+ := \Delta_n^+|_\mathfrak{c}$. Note that $\Delta_\mathfrak{c}$ is a root system of type C_r. Write $\gamma_1, \ldots, \gamma_r$ for a maximal set of long positive strongly orthogonal roots in $\Delta_\mathfrak{c}$. Define $H^j \in i\mathfrak{c}$ by $\gamma_k(H^j) = 2\delta_{kj}$. Note that

$$X^0 = -i\frac{1}{2}(H^1 + \ldots + H^r) .$$

We can choose $E^{\pm j} \in \mathfrak{s}_\mathbb{C}^{\pm\gamma_j}$ such that

$$(B.1) \qquad \overline{E^j} = E^{-j}, \quad \tau(E^j) = E^{-j}, \quad \theta(E^j) = -E^{-j} \quad \text{and} \quad [E^j, E^{-j}] = H^j.$$

Note that the elements $\{H^j, E^j, E^{-j}\}$ form an $\mathfrak{su}(1,1)$-triple in $\mathfrak{s}_\mathbb{C}$. Set

$$Z^j := -i(E^j - E^{-j}) \qquad \text{and} \qquad Y^j := E^j + E^{-j}$$

and note that $Z^j \in \mathfrak{q}_* \cap \mathfrak{p}_*$ and $Y^j \in \mathfrak{g} \cap \mathfrak{p}_* = \mathfrak{p}$.

If we set $X^j := -iH^j$ then the elements $\{X^j, Y^j, Z^j\}$ form an $\mathfrak{su}(1,1)$-triple in $\mathfrak{s}$ with relations

$$(B.2) \qquad [X^j, Y^j] = 2Z^j, \quad [X^j, Z^j] = 2Y^j, \quad [Y^j, Z^j] = -2X^j.$$

Our choice of the maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ then is

$$(B.3) \qquad \mathfrak{a} := \bigoplus_{j=1}^r \mathbb{R}Y^j.$$

Further

$$(B.4) \qquad \mathfrak{b} := \bigoplus_{j=1}^r \mathbb{R}Z^j$$

defines a maximal abelian subspace in $\mathfrak{q}_* \cap \mathfrak{p}_*$. Define elements

$$Y^0 := \frac{1}{2}(Y^1 + \ldots + Y^r) \quad \text{and} \quad Z^0 := \frac{1}{2}(Z^1 + \ldots + Z^r) .$$

Note that $\mathrm{Spec}(\mathrm{ad}\, Y^0) = \mathrm{Spec}(\mathrm{ad}\, Z^0) = \{-1, 0, 1\}$. Next we define a Cayley transform

$$\mathbf{C}\colon \mathfrak{s}_{\mathbb{C}} \to \mathfrak{s}_{\mathbb{C}}, \quad X \mapsto e^{i\frac{\pi}{2}\,\mathrm{ad}\,Y^0}(X)\ .$$

Notice the relations

$$\mathbf{C}(H^j) = Z^j, \quad \mathbf{C}(Z^j) = -H^j, \quad \mathbf{C}(Y^j) = Y^j\ .$$

Since $\mathrm{Spec}(\mathrm{ad}\,Y^0) = \{-1, 0, 1\}$, the prescription

$$\sigma(X) = (e^{i\pi\,\mathrm{ad}\,Y^0} \circ \theta)(X) \qquad (X \in \mathfrak{s})$$

defines an involution on $\mathfrak{s}$. Note that σ commutes with τ by construction. Write $\mathfrak{s} = \mathfrak{h}^\star \oplus \mathfrak{q}^\star$ for the σ-eigenspace decomposition of $\mathfrak{s}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the σ-eigenspace decomposition of $\mathfrak{g}$, i.e., $\mathfrak{h} = \mathfrak{h}^\star \cap \mathfrak{g}$ and $\mathfrak{q} = \mathfrak{g} \cap \mathfrak{q}^\star$. Then

$$(B.5) \qquad \mathfrak{a} \subseteq \mathfrak{p} \cap \mathfrak{q} \qquad \text{and} \qquad \mathfrak{b} \subseteq \mathfrak{q}_* \cap \mathfrak{q}^\star \cap \mathfrak{p}_*.$$

As in Section 2 we let $S_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{s}_{\mathbb{C}}$ and S, G, U, K the analytic subgroups of $S_{\mathbb{C}}$ corresponding to $\mathfrak{s}$, $\mathfrak{g}$, $\mathfrak{u}$ and $\mathfrak{k}$. Our final goal is to describe the maximal parabolic subgroup $P = \mathbf{C}(U_{\mathbb{C}}P^-) \cap S$ of S. For that write $\Sigma_{\mathfrak{b}} = \Sigma(\mathfrak{s}, \mathfrak{b})$ for the restricted root system of $\mathfrak{s}$ with respect to $\mathfrak{b}$. Note that $\Sigma_{\mathfrak{b}} = \mathbf{C}^t(\Delta_{\mathfrak{c}})$ and so $\Sigma_{\mathfrak{b}}$ is of type C_r. Define $\Sigma_{\mathfrak{b},n}^+ := C^t(\Delta_{\mathfrak{c},n}^+)$ and write

$$(B.6) \qquad \mathfrak{n}_{\mathfrak{b}}^{\pm} := \bigoplus_{\alpha \in \pm\Sigma_{\mathfrak{b},n}^+} \mathfrak{s}^\alpha.$$

Note that $\mathfrak{n}_{\mathfrak{b}}^{\pm}$ are both abelian and real forms of $\mathbf{C}(\mathfrak{p}^{\pm})$. Write $N_{\mathfrak{b}}^{\pm}$ for the analytic subgroups of S corresponding to $\mathfrak{n}_{\mathfrak{b}}^{\pm}$. Write $L := \mathbf{C}(U_{\mathbb{C}}) \cap S$. Then L is the Levi factor of P and we have

$$(B.7) \qquad P = L \rtimes N_{\mathfrak{b}}^-.$$

We will show that $L = H^\star := S^\sigma$. First we describe $\mathfrak{l} := \mathrm{Lie}(\mathfrak{l})$. For that define subalgebras

$$\mathfrak{s}(0) := \mathfrak{z}_{\mathfrak{s}}(Y^0) \quad \text{and} \quad \mathfrak{g}(0) = \mathfrak{z}_{\mathfrak{g}}(Y^0)\ .$$

Then $\mathfrak{s}(0) = \mathfrak{s}^{\sigma\theta}$ and $\mathfrak{g}(0) = \mathfrak{g}^{\sigma\theta}$ and therefore

$$(B.8) \qquad \mathfrak{g}(0) = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{p} \cap \mathfrak{q}) \quad \text{and} \quad \mathfrak{s}(0) = (\mathfrak{h}^\star \cap \mathfrak{u}) \oplus (\mathfrak{q}^\star \cap \mathfrak{p}_*).$$

Thus we obtain

$$\begin{aligned}\mathfrak{l} = \mathbf{C}(\mathfrak{u}_{\mathbb{C}}) \cap \mathfrak{s} &= (\mathfrak{u}_{\mathbb{C}} \cap \mathfrak{s}(0)) \oplus \big(\mathbf{C}(\mathfrak{u}_{\mathbb{C}}) \cap (\mathfrak{h}^\star \cap \mathfrak{p}_* \oplus \mathfrak{q}^\star \cap \mathfrak{u})\big)\\ &= (\mathfrak{h}^\star \cap \mathfrak{u}) \oplus (\mathfrak{h}^\star \cap \mathfrak{p}_*) = \mathfrak{h}^\star.\end{aligned}$$

Hence $L = H^\star$. Summarizing our discussion we have shown:

Lemma B.1. *Let $P = \mathbf{C}(U_{\mathbb{C}}P^-) \cap S$ be the stabilizer of $x_1 \in \partial_s\mathcal{D}$ in S. Then the Levi-decomposition of the maximal parabolic subgroup P is given by*

$$P = H^\star \rtimes N_{\mathfrak{b}}^-.$$

■

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