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Laplace and Segal–Bargmann transforms on Hermitian symmetric spaces and orthogonal polynomials

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Abstract

Let $\mathscr{D} = G/K$ be a complex bounded symmetric domain of tube type in a complex Jordan algebra V and let $\mathscr{D}_{\mathbb{R}} = J \cap \mathscr{D} \subset \mathscr{D}$ be its real form in a formally real Euclidean Jordan algebra $J \subset V$; $\mathscr{D}_{\mathbb{R}} = H/L$ is a bounded realization of the symmetric cone in J. We consider representations of H that are gotten by the generalized Segal–Bargmann transform from a unitary G-space of holomorphic functions on $\mathscr{D}$ to an L^2-space on $\mathscr{D}_{\mathbf{R}}$. We prove that in the unbounded realization the inverse of the unitary part of the restriction map is actually the Laplace transform. We find the extension to $\mathscr{D}$ of the spherical functions on $\mathscr{D}_{\mathbb{R}}$ and find their expansion in terms of the L-spherical polynomials on $\mathscr{D}$, which are Jack symmetric polynomials. We prove that the coefficients are orthogonal polynomials in an L^2-space, the measure being the Harish–Chandra Plancherel measure multiplied by the symbol of the Berezin transform. We prove the difference equation and recurrence relation for those polynomials by considering the action of the Lie algebra and the Cayley transform on the

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polynomials on $\mathscr{D}$. Finally, we use the Laplace transform to study generalized Laguerre functions on symmetric cones.

Keywords: Holomorphic discrete series; Highest weight representations; Branching rule; Bounded symmetric domains; Real bounded symmetric domains; Jordan pairs; Jack symmetric polynomials; Orthogonal polynomials; Laplace transform

1. Introduction

The study of various generalizations of the classical Weyl transform has attracted much interest and has been pursued for some time. As is well-known the Weyl transform maps unitarily from the L^2-space on $\mathbb{C}^d = \mathbb{R}^{2d}$ onto the space of Hilbert–Schmidt operators on Fock-space, $\mathscr{F}(\mathbb{C}^d)$, on $\mathbb{C}^d$. The Weyl transform intertwines the natural actions of the Heisenberg group, $H_n = \mathbb{R} \times \mathbb{C}^d$, and realizes the decomposition of $L^2(H_n)$. However, there is another model for the representation, namely, the so-called Schrödinger model, and the Segal–Bargmann transform is then a unitary map from the Schrödinger model to the Fock model intertwining the action of the Heisenberg group. The Segal–Bargmann and Weyl transforms have been studied for a long time in their connection with geometric quantization. There are several ways to show the unitarity of those transforms and generalize them to other settings. One unifying idea, which also can be used as a guideline for natural generalizations, is the *restriction principle*, i.e. polarization of a suitable restriction map [23–25]. This idea, in turn, also unifies the so-called Wick quantization and Berezin transform in one picture, as we will briefly recall in a moment, and gives interesting connections to physics [16,17]. Other ideas, which also have led to important generalizations, are based on heat-kernel analysis and the Stone von Neumann Theorem [7,11,12,32]. We refer to [23,24] for discussion on the connection between those two ideas.

Consider the tensor product $\mathscr{F}(\mathbb{C}^d) \otimes \overline{\mathscr{F}(\mathbb{C}^d)}$ realized as the space of Hilbert–Schmidt operators with integral kernels $F(z, w)$ holomorphic in z and anti-holomorphic in w. Imbed $\mathbb{C}^d$ into $\mathbb{C}^d \times \overline{\mathbb{C}}^d$, bar denoting opposite complex structure, by $z \mapsto (z, z)$. Consider the restriction mapping from $\mathscr{F}(\mathbb{C}^d) \otimes \overline{\mathscr{F}(\mathbb{C}^d)}$ to real analytic functions on $\mathbb{C}^d$ taking functions $F(z, w)$ to its restriction $F(z, z)$. By taking a multiplier of the Gaussian (which is the restriction of the reproducing kernel of $\mathscr{F}(\mathbb{C}^d)$) into account one gets a bounded injective map $R : \mathscr{F}(\mathbb{C}^d) \otimes \overline{\mathscr{F}(\mathbb{C}^d)} \to L^2(\mathbb{C}^d)$ with dense image. It turns out that the adjoint R^* is the Wick quantization map. Consider the polar decomposition $R^* = U\sqrt{RR^*}$. The map U is the Weyl quantization and RR^* is the Berezin transform. Thus the information about the Wick quantization, the Weyl quantization, and the Berezin transform are encoded in the restriction map.

It is easy to see by a similar calculation, as in [25], that by restriction of the Fock space on $\mathbb{C}^d$ to its real form and taking the unitary part of the restriction we get the

Segal–Bargmann transform. The Weyl transform from $L^2(\mathbb{C}^d) = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ onto $\mathscr{F}(\mathbb{C}^d) \otimes \overline{\mathscr{F}(\mathbb{C}^d)}$ is the tensor product of two Segal–Bargmann transforms, obtained by considering the restriction to two different real forms. The point is then that both transforms may be considered as the unitary part of the adjoint of the restriction of certain holomorphic representations on various real forms of the underlying complex manifolds.

Instead of a flat complex space $\mathbb{C}^d$ we may take a bounded symmetric domain $\mathscr{D} = G/K$ and consider the tensor product of a weighted Bergman space (so-called holomorphic discrete series of G) with its conjugate. One may then perform the polar decomposition $R^* = U\sqrt{RR^*}$ of the restriction map R and get a unitary intertwining operator U from the tensor product onto $L^2(\mathscr{D})$. In terms of representation language this has been studied by Repka [28], and an analytic and explicit approach was started in [37]. In particular, it is realized that the analytic issues are far more subtle when one considers the tensor product of the analytic continuation of weighted Bergman spaces [38]. For that purpose we need to understand the positive part in the polar decomposition, namely the square root of the Berezin transform RR^*. The Berezin transform in the case of weighted Bergman spaces is a positive bounded operator on $L^2(\mathscr{D})$ and its spectral symbol has been calculated by Unterberger–Upmeier [33].

In [24], it is shown how to generalize these ideas to restriction maps from a reproducing kernel Hilbert space of holomorphic function on a complex manifold $M_{\mathbb{C}}$ to a totally real submanifold M. A particular situation is when $M_{\mathbb{C}} = G/K$ is a bounded symmetric domain, the Hilbert space $\mathscr{F}(\mathbb{C}^d)$ is replaced by a weighted Bergman space, and $M = H/H \cap K$ is a totally real homogeneous submanifold. The restriction principle gives a natural way to define a Segal–Bargmann and Berezin transform. The symbol of the Berezin transform is then calculated in [39]. The same result for real tube domains and for classical domains are obtained by van Dijk and Pevzner [6] and, respectively, Neretin [22].

In this paper we will use these ideas to construct and study a natural class of functions and orthogonal polynomials on real symmetric domains. We consider the simplest case when the real symmetric domain is, in the Siegel domain realization, the symmetric cone (Type A in terms of the classification of the root system, see [39]). The starting point is a unitary highest weight representation $(\pi_\nu, \mathscr{H}_\nu)$, with one-dimensional minimal K-type, of the group G. We consider the restriction map to a real form $H/H \cap K$ corresponding to the symmetric cone, both for the bounded realization and the unbounded realization of G/K. The main difference is that in the bounded realization we include the multiplier corresponding to the minimal K-type in our restriction map

$$R : \mathscr{H}_\nu \to L^2(H/H \cap K), \quad f \mapsto D_\nu f|_{H/H \cap K},$$

whereas in the unbounded realization we include this multiplier in the measure, so that the map $R : \mathscr{H}_\nu \to L^2(\Omega, d\mu_\nu)$ is now just the restriction $f \mapsto f|_\Omega$. Then in the unbounded realization the Segal–Bargmann transform turns out to be the Laplace transform (see Theorem 7.4). By considering the unitary image of an orthogonal set of K-finite $H\cap K$-invariant vectors in the holomorphic realization we construct a family of spherical orthogonal functions and polynomials on $H/H\cap K$. One of our main results is the orthogonality relations, recurrence formulas and difference formulas for the polynomials. In the unbounded realization we get functions of the form $\omega \mapsto e^{-\mathrm{Tr}(\omega)}L_\nu(2X)$ where L_ν is a polynomial which agrees with the Laguerre polynomials in the case where $G = \mathrm{SL}(2,\mathbb{R})$ and $H/H\cap K \simeq \mathbb{R}^+$ [5]. These polynomials have also been considered in [9, Chapter XV], but here we relate them to representations of G. In that way we derive differential equations satisfied by these polynomials (see Theorem 7.9 and [4]). Furthermore, applying the spherical Fourier transform we get an orthonormal basis of $L^2(\mathfrak{a}^*, |c(\lambda)|^{-2}\,d\lambda)^W$ and a family of orthogonal polynomials on $\mathfrak{a}^*$ (see Corollary 3.6 and Proposition 4.3). We derive recurrence formulas and difference formulas for these polynomials (see Theorems 5.6 and 6.1).

We mention that a large class of symmetric and non-symmetric polynomials satisfying various difference equations has been recently introduced by using the representation theory of affine Hecke algebras. However, as it is noted by Cherednik [3, p. 484] the meaning of the difference equations needs to be clarified. Our results provide the clarification and meaning of the orthogonality and difference relations for these orthogonal polynomials. It also reveals that the difference relations and the recurrence relations are somewhat dual to each other. We may well expect that this will be true for other types of Macdonald polynomials.

This joint project started in December 2000 as G. Zhang was visiting Louisiana State University. In our discussions we realized that we were working on similar problems using similar ideas, except that G. Zhang was working in the bounded realization for all real tube domains, while the first two authors were working on the restriction to the symmetric cone (Type A case) of unbounded realizations of Siegel domains. The analytic nature involved for the Type A case is somewhat richer, in particular in its connection with the Laplace transform. The treatment of types C and D requires however some different and more combinatorial methods, see [40].

2. Bounded symmetric domains, symmetric cones and Cayley transform

In this section we recall some known necessary background about bounded symmetric domains. We follow the presentation in [19]; see also [9].

2.1. Bounded symmetric domains

Let V be a d-dimensional *complex* Hermitian simple Jordan algebra. Let $\mathscr{D} \subset V$ be an irreducible bounded symmetric domain isomorphic to a tube type domain in V. Let $\mathrm{Aut}(\mathscr{D})$ be the group of all biholomorphic automorphisms of $\mathscr{D}$, let $G = \mathrm{Aut}(\mathscr{D})_0$ be the connected component of the identity in $\mathrm{Aut}(\mathscr{D})$, and let K be the isotropy subgroup of G at the point 0. Then G is semisimple and, as a Hermitian symmetric space, $\mathscr{D} = G/K$. Furthermore, K is a maximal compact subgroup of G. We denote by $\tilde{G}$ the connected simply connected covering group of G and $\tilde{K}$ the pre-image of K in $\tilde{G}$. Then $\tilde{K} \simeq \mathbb{R} \times K_1$ where K_1 is a simply connected compact semisimple Lie group. We start by reviewing the basic structure theory of G in terms of the structure of the Jordan algebra V. The Lie algebra $\mathfrak{g}$ of G is identified with the Lie algebra $\mathrm{aut}(\mathscr{D})$ of all completely integrable holomorphic vector fields on $\mathscr{D}$ equipped with the Lie product

$$[X, Y](z) := X'(z)Y(z) - Y'(z)X(z), \quad X, Y \in \mathrm{aut}(\mathscr{D}), \ z \in \mathscr{D}.$$

Define $\theta : \mathfrak{g} \to \mathfrak{g}$ by $\theta(X)(z) := -X(-z)$. Then θ is a Cartan involution on $\mathfrak{g}$ and

$$\mathfrak{g}^\theta = \{X \in \mathfrak{g} \mid \theta(X) = X\} = \mathfrak{k}$$

is the Lie algebra of K. Let

$$\mathfrak{p} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ corresponding to θ. Every element in $\mathfrak{g}$ is given by a polynomial of degree at most 2 [35]. The Lie algebra $\mathfrak{k}_\mathbb{C}$ corresponds to the elements of degree one and is isomorphic to a subalgebra of $\mathrm{End}(V)$ by $T \mapsto (Tz)\frac{\partial}{\partial z}$. The identity map thus corresponds to the *Euler operator*

$$Z_0 := z\frac{\partial}{\partial z}. \tag{2.1}$$

This element is central in $\mathfrak{k}_\mathbb{C}$ and $\mathrm{ad}(Z_0)$ has eigenvalues ± 1 on $\mathfrak{p}_\mathbb{C}$. The $+1$-eigenspace $\mathfrak{p}^+$ corresponds to the constant polynomials and the -1-eigenspace $\mathfrak{p}^-$ corresponds to the polynomials of degree two. There exists a quadratic form $Q : V \to \mathrm{End}(\bar{V}, V)$ (where $\bar{V}$ is the space V but with the opposite complex structure), such that

$$\mathfrak{p} = \{\xi_v \mid v \in V\}, \quad \text{where } \xi_v(z) := (v - Q(z)\bar{v})\frac{\partial}{\partial z}. \tag{2.2}$$

In the following we will identify elements in $\mathfrak{g}$ with the corresponding polynomials. Let $\{z, \bar{v}, w\}$ be the polarization of $Q(z)\bar{v}$, i.e.,

$$\{z, \bar{v}, w\} = (Q(z+w) - Q(z) - Q(w))\bar{v}.$$

Then

$$\xi_v(z) = v - Q(z)\bar{v} = v - \tfrac{1}{2}\{z, \bar{v}, z\}. \tag{2.3}$$

The space V with this triple product on $V \times \bar{V} \times V$, is a JB*-triple [35]. Define $D : V \times \bar{V} \to \mathrm{End}(V)$ by $D(z, \bar{v})w = \{z, \bar{v}, w\}$. The Lie bracket of two elements

$\xi_v, \xi_w \in \mathfrak{p}$ is then given by

$$[\xi_v, \xi_w] = (D(v, \bar{w}) - D(w, \bar{v})) \in \mathfrak{k}. \tag{2.4}$$

The group K acts on V by linear transformations. Furthermore,

$$z, w \mapsto (z \mid w) := \frac{1}{p} \operatorname{Tr} D(z, \bar{w}) \tag{2.5}$$

is a K-invariant inner product on V. Here Tr is the trace functional on End(V) and $p = p(\mathscr{D})$ is the genus of $\mathscr{D}$ (see (2.8) below). Denote by $||\cdot||$ the corresponding K-invariant operator norm on End(V). Define the *spectral norm* $||\cdot||_{\mathrm{sp}}$ on V by

$$||z||_{\mathrm{sp}} := ||\tfrac{1}{2}D(z, \bar{z})||^{1/2}.$$

Then the domain $\mathscr{D}$ is realized as the open unit ball in V with respect to the spectral norm, i.e. $\mathscr{D} = \{z \in V \mid ||z||_{\mathrm{sp}} < 1\}$.

An element $v \in V$ is called a *tripotent* if $\{v, \bar{v}, v\} = 2v$. Let us choose and fix a frame $\{e_j\}_{j=1}^r$ of tripotents in V. The number r is called the *rank* of $\mathscr{D}$. Define $H_j = D(e_j, \bar{e}_j) \in \mathfrak{k}_{\mathbb{C}}$. The operator $D(u, \bar{u})$ has real spectrum for each $u \in V$. Hence, by (2.4), $H_j \in i\mathfrak{k}$. Furthermore, the subspace

$$\mathfrak{t}_- = i \bigoplus_{j=1}^{r} \mathbb{R}H_j \subset \mathfrak{k} \tag{2.6}$$

is abelian. Let $\mathfrak{t} = \mathfrak{t}_- \oplus \mathfrak{t}_+$ be a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{g}$ containing $\mathfrak{t}_-$. Let $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the set of roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Recall that a root $\alpha \in \Delta$ is called *compact* if $(\mathfrak{g}_{\mathbb{C}})_\alpha \subset \mathfrak{k}_{\mathbb{C}}$ and *non-compact* if $(\mathfrak{g}_{\mathbb{C}})_\alpha \subset \mathfrak{p}_{\mathbb{C}}$. Denote by Δ_c and Δ_n the set of compact, respectively, non-compact roots. Finally, we recall that two roots $\alpha \neq \pm\beta$ are called *strongly orthogonal* if $\alpha \pm \beta \notin \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. We choose $\gamma_j \in (\mathfrak{t}^*)_{\mathbb{C}}$ so that

$$\gamma_j(H_k) = 2\delta_{jk}$$

and γ_j vanishes on $(\mathfrak{t}_+)_{\mathbb{C}}$. Then $\{\gamma_1, \dots, \gamma_r\}$ is a maximal set of strongly orthogonal non-compact roots. We order them so that

$$\gamma_1 > \cdots > \gamma_r.$$

The element $e := e_1 + \cdots + e_r$ is a *maximal tripotent* and

$$Z_0 = \frac{1}{2}D(e, \bar{e}) = \frac{1}{2}\sum_{r=1}^{r} H_j, \tag{2.7}$$

where Z_0 corresponds to the identity map as before. In this notation the genus of $\mathscr{D}$ is given by

$$p = p(\mathscr{D}) = \frac{2d}{r}. \tag{2.8}$$

2.2. Real bounded symmetric domain

Let $e = e_1 + \cdots + e_r$ be the maximal tripotent as before. The map $\tau : V \to V, \tau(v) = Q(e)\bar{v}$ is a conjugate linear involution of V and induces a decomposition $V = J \oplus iJ$, where J is a formally real Euclidean Jordan algebra with identity e. Notice that τ also defines an involution, also denoted by τ, of $\mathfrak{g}$ and G. The involution on G is simply given by $[\tau(g)](z) = \tau(g(\tau(z)))$, for $z \in \mathscr{D}$. As τ commutes with the Cartan involution θ we have the following decomposition of $\mathfrak{g}$ into eigenspaces:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{q} \\ &= \mathfrak{h}_k \oplus \mathfrak{q}_k \oplus \mathfrak{h}_p \oplus \mathfrak{q}_p \\ &= \mathfrak{k}_h \oplus \mathfrak{k}_q \oplus \mathfrak{p}_h \oplus \mathfrak{p}_q,\end{aligned}$$

where $\mathfrak{h}$ is the $+1$-eigenspace and $\mathfrak{q}$ is the -1-eigenspace of τ. The index k indicates intersection with $\mathfrak{k}$, etc. The restriction of θ to $\mathfrak{h}$ defines a Cartan involution on $\mathfrak{h}$ and the corresponding Cartan decomposition is $\mathfrak{h} = \mathfrak{k}_h \oplus \mathfrak{p}_h$. The space $\mathfrak{p}_q$ is then a real subspace of $\mathfrak{p}$. Recall the definition $\xi_v(z) \coloneqq v - Q(z)\bar{v}$. Then

$$\mathfrak{p}_h = \{\xi_v \mid v \in J\}.$$

Let $\mathscr{D}_{\mathbb{R}} = \mathscr{D} \cap J = \mathscr{D}^{\tau}$. Then $\mathscr{D}_{\mathbb{R}}$ is a real bounded symmetric domain. For the structure of these domains see [15,19]. Let

$$G(\mathscr{D}_{\mathbb{R}}) = \{g \in G \mid g(\mathscr{D}_{\mathbb{R}}) = \mathscr{D}_{\mathbb{R}}\}.$$

Then $G(\mathscr{D}_{\mathbb{R}})$ is a closed subgroup of G with finitely many connected components. Let $H = G(\mathscr{D}_{\mathbb{R}})_o$ be the connected component containing the identity $1 \in G$. Since $e \in \mathscr{D}_{\mathbb{R}}$ the subgroup $\{k \in G(\mathscr{D}_{\mathbb{R}}) \mid k \cdot e = e\}$ is a maximal compact subgroup of $G(\mathscr{D}_{\mathbb{R}})$ and equals $G(\mathscr{D}_{\mathbb{R}}) \cap K$. By replacing $G(\mathscr{D}_{\mathbb{R}})$ by H it follows that $H \cap K$ is a maximal compact subgroup of H and

$$\mathscr{D}_{\mathbb{R}} = H/H \cap K = G(\mathscr{D}_{\mathbb{R}})/G(\mathscr{D}_{\mathbb{R}}) \cap K$$

is a Riemannian symmetric space with the Bergman metric restricted to $\mathscr{D}_{\mathbb{R}}$. Moreover, it is a totally geodesic submanifold. Finally, $H = G_o^{\tau} = \{a \in G \mid \tau(a) = a\}_o$. Thus (G, H) is a symmetric pair. We note that the group H is not semisimple since $\exp(\mathbb{R}\xi_e)$ is in the center of H. The group H is invariant under the Cartan involution θ and $\theta|_H$ is a Cartan involution on H.

2.3. Cayley transform

In this subsection we discuss the realization of $\mathscr{D}$ as a tube-type domain $T(\Omega) = iJ + \Omega$. Let Ω be the cone of positive elements in J:

$$\Omega = \{x^2 \mid x \in J, \det(x) \neq 0\}.$$

Then Ω is a symmetric convex cone. Let

$$T(\Omega) = iJ + \Omega = \{w \in V \mid \mathrm{Re}(w) \in \Omega\}.$$

If $e - z$ is invertible in V let

$$\gamma(z) = \frac{e+z}{e-z} = (e+z)(e-z)^{-1} \tag{2.9}$$

be the Cayley transform. Its inverse is

$$\gamma^{-1}(w) = (w-e)(w+e)^{-1}. \tag{2.10}$$

The following is well known:

Lemma 2.1. *Let the notation be as above. Then the following hold*:

(1) *The Cayley transform γ is a biholomorphic transformation from $\mathscr{D}$ into the Siegel domain $T(\Omega)$.*
(2) *The Cayley transform $\gamma : \mathscr{D} \mapsto T(\Omega)$ maps the real form $\mathscr{D}_{\mathbb{R}}$ of $\mathscr{D}$ onto the real form* Ω of $T(\Omega)$.
(3) *The following diagram commutes where the vertical arrows are the inclusion maps.*

$$\begin{array}{ccc} \mathscr{D}_{\mathbb{R}} & \xrightarrow{\gamma} & \Omega \\ \downarrow & & \downarrow \\ \mathscr{D} & \xrightarrow{\gamma} & T(\Omega) \end{array}$$

Notice that the Cayley transform can be realized by an element, also denoted γ, in $G_{\mathbb{C}}$. In fact, there exists a $X \in \mathfrak{q}_p$ such that $\mathrm{ad}(X)$ has eigenvalues 0, 1, and -1 and $\gamma = \exp(\frac{\pi i}{4}X)$. Define $G^\gamma := \gamma G \gamma^{-1}$ and $H^\gamma = (K_{\mathbb{C}} \cap G^\gamma)_o$. Then

$$G^\gamma(\Omega)_o = \{g \in G^\gamma \mid g(\Omega) = \Omega\} = H^\gamma = \gamma H \gamma^{-1}. \tag{2.11}$$

Let $\mathfrak{g}^\gamma := \mathrm{Ad}(\gamma)(\mathfrak{g})$ be the Lie algebra of G^γ. We collect some important facts in the following lemmas:

Lemma 2.2. *Let the notation be as above. Then the following holds*:

$$\mathfrak{g}^\gamma = \mathfrak{h}_k + i\mathfrak{h}_p + i\mathfrak{q}_k + \mathfrak{q}_p. \tag{2.12}$$

Thus $(\mathfrak{g}^\gamma, \mathfrak{k}^\gamma = \mathfrak{h}_k + i\mathfrak{h}_p, \mathfrak{h}^\gamma = \mathfrak{h}_k + i\mathfrak{q}_k)$ is the *Riemannian dual* of $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h})$.

Lemma 2.3. *With notation as above we have* $\mathfrak{h}^\gamma = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\gamma$ and Z_0 *is central in* $\mathfrak{h}^\gamma$.

We refer to [15] for further information. The group G^γ acts on $T(\Omega)$ by transforming the action of G:

$$g \cdot z = \gamma((\gamma^{-1} g \gamma) \cdot (\gamma^{-1}(z))).$$

For simplicity we will sometimes write γ for the adjoint action of γ on $\mathfrak{g}_{\mathbb{C}}$ and the conjugation with γ in $G_{\mathbb{C}}$.

2.4. Roots

In this subsection we describe the structure of restricted roots for the group H. Let $\xi_j = \xi_{e_j}$ and

$$\xi = \xi_1 + \cdots + \xi_r.$$

By SU(1, 1)-reduction we have the following:

Lemma 2.4. $\xi_j = \mathrm{Ad}(\gamma)^{-1}(H_j)$ *and* $\xi = 2\,\mathrm{Ad}(\gamma)^{-1}(Z_0) = -\mathrm{Ad}(\gamma)(2Z_0)$.

Let

$$\mathfrak{a} = \bigoplus_{j=1}^{r} \mathbb{R}\xi_j,$$

and define $\beta_j \in \mathfrak{a}^*$ by $\beta_j(\xi_i) = 2\delta_{ij}$. Then $\mathfrak{a}$ is maximal abelian in $\mathfrak{h}_p$. In fact, $\mathfrak{a}$ is also maximal abelian in $\mathfrak{p}$. We remark that $\gamma^2 H_j = -H_j$, $\gamma(\mathfrak{t}_-) = \gamma^{-1}(\mathfrak{t}_-) = \mathfrak{a}$ and $\beta_j = \gamma_j \circ \mathrm{Ad}(\gamma)$.

We will often identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}^r$ using the map $\alpha \mapsto \alpha = \alpha_1 \beta_1 + \cdots + \alpha_r \beta_r$. The root system $\Delta(\mathfrak{h}, \mathfrak{a})$ is of type A:

$$\Delta(\mathfrak{h}, \mathfrak{a}) = \left\{ \frac{\beta_j - \beta_k}{2} \,\middle|\, 1 \leqslant j \neq k \leqslant r \right\}.$$

We fix an ordering of the roots so that

$$\beta_1 > \beta_2 > \cdots > \beta_r.$$

Then the corresponding system $\Delta^+ = \Delta^+(\mathfrak{h}, \mathfrak{a})$ of positive roots is given by

$$\Delta^+ = \left\{ \frac{\beta_j - \beta_k}{2} \,\middle|\, 1 \leqslant j < k \leqslant r \right\}.$$

The root spaces $\mathfrak{h}_{(\beta_j - \beta_k)/2}$ all have the same dimension which we denote by a. Then the half sum of the positive roots is

$$\rho = \sum_{j=1}^{r} \rho_j \beta_j = \frac{a}{4} \sum_{j=1}^{r} ((r+1) - 2j)\beta_j. \tag{2.13}$$

Example 2.5. In the case of $\mathscr{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $G = \mathrm{SU}(1,1)$ acting on $\mathscr{D}$ in the usual way

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}},$$

our notation is:

$$\begin{aligned} 2Z_0 = H_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \gamma &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \exp\left(\frac{\pi i}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right), \\ \xi &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \mathscr{D}_{\mathbb{R}} &= (-1, 1), \\ J &= \mathbb{R}, \\ \Omega &= \mathbb{R}^+, \\ T(\Omega) &= \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}. \end{aligned}$$

2.5. Conical functions, spherical functions, and invariant polynomials

Let $\{e_j\}_{j=1}^r$ be the fixed frame as before. Let $u_j := \sum_{k=1}^j e_k$, $j = 1, \ldots, r$. Let $V_j := \{z \in V \mid D(u_j, \bar{u}_j)z = 2z\}$. Then V_j is a Jordan $\bigstar$-subalgebra of V with a determinant polynomial denoted by Δ_j. We extend Δ_j to all of V via $\Delta_j(z) := \Delta_j(\mathrm{pr}_{V_j}(z))$, where pr_{V_j} is the orthogonal projection onto V_j. The polynomials Δ_j are called (principal) *minors*. Notice that $\Delta_r(w) = \Delta(w) = \det(w)$. For any $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$ consider the associated *conical function* [9, p. 122]:

$$\Delta_\alpha(w) := \Delta_1^{\alpha_1-\alpha_2}(w)\Delta_2^{\alpha_2-\alpha_3}(w)\cdots\Delta_r^{\alpha_r}(w), \quad w\in V.$$

Notice that if $w=\sum_{j=1}^r w_je_j$ then $\Delta_\alpha(w)=\prod_{j=1}^r w_j^{\alpha_j}$. Thus the conical functions are generalizations of the power functions.

Let $L := H\cap K\subset G\cap G^\gamma$ and define

$$\psi_\alpha(z) := \int_L \Delta_\alpha(lz)\,dl, \quad z\in V. \tag{2.14}$$

The function $\psi_{\lambda+\rho}$ is the spherical function on Ω corresponding to λ [9, Theorem XIV. 3.1]. We identify $(\mathfrak{t}_-^*)_{\mathbb{C}}$ with $\mathbb{C}^r$ via $m_1\gamma_1+\cdots+m_r\gamma_r\leftrightarrow(m_1,\ldots,m_r)\in\mathbb{C}^r$. If $\lambda=\mathbf{m}$, where $\mathbf{m}=(m_1,\ldots,m_r)$ is a tuple of non-negative integers such that $m_1\geqslant m_2\geqslant\cdots\geqslant m_r\geqslant 0$, then the functions Δ_λ and ψ_λ are holomorphic polynomials on the whole space V and ψ_λ is L-invariant. Let $\mathscr{P}(V)$ be the space of holomorphic polynomials on V, considered as K-space by the regular action, $k\cdot p(z)=p(k^{-1}\cdot z)$. Let

$$\mathbf{\Lambda}=\{\mathbf{m}\in\mathbb{N}_0^r\mid m_1\geqslant m_2\geqslant\cdots\geqslant m_r\geqslant 0\}. \tag{2.15}$$

Then we have the well known Schmid decomposition [9, Theorem XI.2.4] [31].

Lemma 2.6. *The space of polynomials $\mathscr{P}(V)$ decomposes as a K-representation into*

$$\mathscr{P}(V)=\sum_{\mathbf{m}\in\mathbf{\Lambda}}\mathscr{P}_{\mathbf{m}}, \tag{2.16}$$

where each $\mathscr{P}_{\mathbf{m}}$ is of lowest weight $-\mathbf{m}=-(m_1\gamma_1+\cdots+m_r\gamma_r)$, with $m_1\geqslant\cdots\geqslant m_r\geqslant 0$ being integers. In this case the polynomial $\Delta_{\mathbf{m}}$ is a lowest weight vector in $\mathscr{P}_{\mathbf{m}}$ and $\psi_{\mathbf{m}}$ is up to constants the unique L-invariant polynomial in $\mathscr{P}_{\mathbf{m}}$. In particular

$$\mathscr{P}(V)^L=\sum_{\mathbf{m}\in\mathbf{\Lambda}}\mathbb{C}\psi_{\mathbf{m}}. \tag{2.17}$$

As $\psi_{\lambda+\rho}$ is spherical on Ω and because the Cayley transform commutes with the L-action we get the following lemma.

Lemma 2.7. *Let $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$. Then the spherical function $\phi_\lambda(x)$ on the real bounded symmetric domain $\mathscr{D}_{\mathbb{R}}$ is given by*

$$\phi_\lambda(x)=\psi_{i\lambda+\rho}\left(\frac{e+x}{e-x}\right)=\psi_{i\lambda+\rho}\circ\gamma(x), \quad x\in\mathscr{D}_{\mathbb{R}}.$$

2.6. The Γ-function on symmetric cones

The H-invariant measure on Ω is given by

$$d\mu_0 = \Delta(x)^{-\frac{d}{r}}\, dx,$$

where $d = \dim_{\mathbb{R}}(J) = \dim_{\mathbb{C}}(V)$. The *Gindikin–Koecher Gamma function* [10] associated with the convex, symmetric cone Ω is defined by

$$\Gamma_\Omega(\lambda) \coloneqq \int_\Omega e^{-\mathrm{Tr}(x)} \Delta_\lambda(x) \Delta(x)^{-d/r}\, dx.$$

The integral converges if and only if $\mathrm{Re}(\lambda_j) > (j-1)a/2$ for $j = 1, 2, \ldots, r$, [9, Theorem VII.1.1]. Using the identification $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}^r$ via the map $(\lambda_1, \ldots, \lambda_n) \leftrightarrow \lambda_1\beta_1 + \cdots + \lambda_r\beta_r$ we have

$$\Gamma_\Omega(\lambda) = (2\pi)^{\frac{d-r}{2}} \prod_{j=1}^{r} \Gamma\Big(\lambda_j - (j-1)\frac{a}{2}\Big),$$

where Γ is the usual Gamma function. In particular, it follows that Γ_Ω has a meromorphic continuation to all of $\mathfrak{a}_{\mathbb{C}}^*$. We will view Γ_Ω as a function on $\mathbb{C}^r$ and on $\mathfrak{a}_{\mathbb{C}}^*$ using our identification above. We also adopt the notation $\beta_0 = \sum_{j=1}^r \beta_j$ and $\Gamma_\Omega(v) = \Gamma_\Omega(v\beta_0)$, where $v \in \mathbb{C}$. Finally, we define

$$(\lambda)_{\mathbf{m}} = \frac{\Gamma_\Omega(\lambda + \mathbf{m})}{\Gamma_\Omega(\lambda)}.$$

2.7. The Laplace transform

We recall here a few facts about the Laplace transform on Ω. Let μ be a (complex) Radon measure on Ω such that $x \mapsto e^{-(t|x)} \in L^1(\Omega, d|\mu|)$ for all $t \in \Omega$. Define the *Laplace transform of* μ by

$$\mathscr{L}(\mu)(w) \coloneqq \int_\Omega e^{-(w|x)}\, d\mu(x) = \int_\Omega e^{-(t|x)} e^{-i(s|x)}\, d\mu(x),$$

for $w = t + is \in T(\Omega)$. Then $\mathscr{L}(\mu)$ is holomorphic on $T(\Omega)$. In particular, if $f \in L^1(\Omega, d\mu)$ then $f(x)\, d\mu$ is a finite measure and hence $\mathscr{L}(f) \coloneqq \mathscr{L}(f\, d\mu)$ is well defined. Furthermore, if $v \in \mathbb{C}$ is such that $\mathrm{Re}(v) > (r-1)\frac{a}{2}$ let $d\mu_v(x) = \Delta(x)^{v-d/r}\, dx$. Notice that $\Delta_{v-d/r}(x) = \Delta(x)^{v-d/r}$. Then $d\mu_v$ is a quasi-invariant measure on Ω and the H-invariant measure corresponds to $d\mu_o$. If $v > 0$ is real, then we let

$$L_v^2(\Omega) = L^2(\Omega, d\mu_v). \tag{2.18}$$

We define $\mathscr{L}_\nu : L^2_\nu(\Omega) \to \mathcal{O}(T(\Omega))$ by

$$\mathscr{L}_\nu(f)(z) = \mathscr{L}(f\, d\mu_\nu)(z) = \int_\Omega e^{-(z|x)} f(x)\, d\mu_\nu(x), \quad z \in T(\Omega).$$

Here $\mathcal{O}(T(\Omega))$ denotes the space of holomorphic functions on $T(\Omega)$. We notice the following, which follows directly from Proposition VII.1.2 of [9, p. 124] by the holomorphicity of both sides in w:

Lemma 2.8. *Let $\nu \in \mathbb{C}$ be such that $\mathrm{Re}(\nu) > (r-1)\frac{a}{2}$. Then for any $w \in T(\Omega)$*

$$\mathscr{L}(\mu_\nu)(w) = \Gamma_\Omega(\nu)\Delta(w)^{-\nu}.$$

2.8. Unitary highest weight representations

In this subsection we review some simple facts on scalar valued unitary highest weight representations. We restrict the discussion to what we will need later on. From now on $v \mapsto \bar{v}$ denotes conjugation with respect to the real form J. Let $\tilde{G}^\gamma$ be the universal covering group of G^γ. Then $\tilde{G}^\gamma$ acts on $T(\Omega)$ by $(g, z) \mapsto \kappa(g) \cdot z$ where $\kappa : \tilde{G}^\gamma \to G^\gamma$ is the canonical projection. For $\nu > 1 + a(r-1)$ let $\mathscr{H}_\nu(T(\Omega))$ be the space of holomorphic functions $F : T(\Omega) \to \mathbb{C}$ such that

$$||F||^2_\nu := \alpha_\nu \int_{T(\Omega)} |F(x+iy)|^2 \Delta(y)^{\nu - 2d/r}\, dx\, dy < \infty, \tag{2.19}$$

where

$$\alpha_\nu = \frac{2^{r\nu}}{(4\pi)^d \Gamma_\Omega(\nu - d/r)}. \tag{2.20}$$

Then $\mathscr{H}_\nu(T(\Omega))$ is a non-trivial Hilbert space. For $\nu \leqslant 1 + a(r-1)$ this space reduces to $\{0\}$. If $\nu = 2d/r$ this is the *Bergman space*. For $g \in \tilde{G}^\gamma$ and $z \in T(\Omega)$, let $J_g(z) = J(g, z)$ be the *complex* Jacobian determinant of the action of g on $T(\Omega)$ at the point z. We will use the same notation for elements $g \in G$ and $z \in \mathscr{D}$. Then

$$J(ab, z) = J(a, b \cdot z) J(b, z)$$

for all $a, b \in \tilde{G}^\gamma$ and $z \in T(\Omega)$. It is well known that for $\nu > 1 + a(r-1)$

$$\pi_\nu(g) f(z) = J(g^{-1}, z)^{\nu/p} f(g^{-1} \cdot z) \tag{2.21}$$

defines a unitary irreducible representation of $\tilde{G}^\gamma$. In [8,30,36] it was shown that this unitary representation $(\pi_\nu, \mathscr{H}_\nu(T(\Omega)))$ has an analytic continuation to the half-interval $\nu > (r-1)\frac{a}{2}$. Here the representation π_ν is given by the same formula (2.21) but the formula for the *norm* in (2.19) is no longer valid. We collect the necessary information from [9, p. 260], in particular, Theorem XIII.1.1. and Proposition XIII.1.2., in the following theorem. We will give a new proof of (4) later using only part (3).

Theorem 2.9. *Let the notation be as above. Assume that* $\nu > 1 + a(r-1)$. *Then the following hold*:

(1) *The space* $\mathscr{H}_\nu(T(\Omega))$ *is a reproducing kernel Hilbert space.*
(2) *The reproducing kernel of* $\mathscr{H}_\nu(T(\Omega))$ *is given by*

$$K_\nu(z,w) = \Gamma_\Omega(\nu)\Delta(z+\bar{w})^{-\nu}.$$

(3) *If* $\nu > (r-1)\frac{a}{2}$ *then there exists a Hilbert space* $\mathscr{H}_\nu(T(\Omega))$ *of holomorphic functions on* $T(\Omega)$ *such that* $K_\nu(z,w)$ *defined in* (2) *is the reproducing kernel of that Hilbert space. The group* $\tilde{G}^\gamma$ *acts unitarily on* $\mathscr{H}_\nu(T(\Omega))$ *by the action defined in* (2.21).
(4) *The map*

$$L^2_\nu(\Omega) \ni f \mapsto F = \mathscr{L}_\nu(f) \in \mathscr{H}_\nu(T(\Omega))$$

is a unitary isomorphism and
(5) *If* $\nu > (r-1)\frac{a}{2}$ *then the functions*

$$q_{\mathbf{m},\nu}(z) \coloneqq \Delta(z+e)^{-\nu}\psi_{\mathbf{m}}\left(\frac{z-e}{z+e}\right), \quad \mathbf{m} \in \mathbf{\Lambda},$$

form an orthogonal basis of $\mathscr{H}_\nu(T(\Omega))^L$, *the space of* L*-invariant functions in* $\mathscr{H}_\nu(T(\Omega))$.

2.9. Weighted Bergman spaces on the bounded symmetric domain

In the last section we concentrated on the unbounded realization. We will now shift our attention to the bounded realization $\mathscr{D}$. Let

$$h(z,w) \coloneqq \Delta(e - z\bar{w}). \tag{2.22}$$

We note the $h(z,w)^{-p}$ is the *Bergman kernel* on $\mathscr{D}$. We collect a few well-known facts about $h(z,w)$ in the following lemma.

Lemma 2.10. *Let* $g \in G$ *and* $z,w \in \mathscr{D}$. *Then the following hold*:

(1) $h(z,w)$ *is holomorphic in the first variable and anti-holomorphic in the second variable.*
(2) $h(w,z) = \overline{h(z,w)}$ *for all* $z,w \in \mathscr{D}$.
(3) $h(g\cdot z, g\cdot w) = J(g,z)^{1/p}\overline{J(g,w)^{1/p}}h(z,w)$. *In particular,* $h(z,z) = |J(g,0)|^{2/p} > 0$ *where* $g \in G$ *is chosen such that* $g\cdot 0 = z$.

(4) *If* $x, y \in \mathscr{D}_{\mathbb{R}}$ *then* $h(x,y) > 0$. *In particular, all powers* $h(x,y)^{\mu}$, $\mu \in \mathbb{R}$, $x, y \in \mathscr{D}_{\mathbb{R}}$, *are well defined.*

Let $h(z) = h(z,z) = \Delta(e - z\bar{z})$. Then the measure

$$dm_\nu(z) = h(z)^{\nu - p}\, dz \tag{2.23}$$

is a quasi-invariant measure on $\mathscr{D}$ and $dm_0(z)$ is invariant. Furthermore,

$$d\eta(x) = h(x)^{-p/2}\, dx \tag{2.24}$$

is an H-invariant measure on $\mathscr{D}_{\mathbb{R}}$. Let $\tilde{G}$ be the universal covering group for G. As for $T(\Omega)$ there exists a Hilbert space $\mathscr{H}_\nu(\mathscr{D})$ of holomorphic functions $\mathscr{D}$ with a unitary $\tilde{G}$-action, given by

$$\pi_\nu(g)f(z) = J(g^{-1}, z)^{\frac{\nu}{p}} f(g^{-1}z)$$

if $\nu > (r-1)a/2$. Here $z \mapsto J(g,z)$ is again the Jacobian of the action of G on V. If $\nu > 1 + a(r-1)$ then the norm on $\mathscr{H}_\nu(\mathscr{D})$ is given by

$$||F||_\nu^2 = ||F||^2_{\mathscr{H}_\nu(\mathscr{D})} := d_\nu \int_{\mathscr{D}} |F(z)|^2\, dm_\nu(z), \tag{2.25}$$

where

$$d_\nu = \frac{1}{\pi^d} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - d/r)}.$$

The constant d_ν is chosen so that the constant function $z \mapsto 1$ has norm one.

Lemma 2.11. *Let the notation be as above. Then the following hold*:

(1) *If* $F \in \mathscr{H}_\nu(T(\Omega))$ *then the function*

$$\pi_\nu(\gamma^{-1})(F)(w) = 2^{\frac{r\nu}{2}} \Delta(e - w)^{-\nu} F \circ \gamma(w) = \Delta(e - w)^{-\nu} F((e + w)(e - w)^{-1})$$

belongs to $\mathscr{H}_\nu(\mathscr{D})$ *and*

$$\pi_\nu(\gamma^{-1}) : \mathscr{H}_\nu(T_\Omega) \to \mathscr{H}_\nu(\mathscr{D})$$

is a linear isomorphism onto $\mathscr{H}_\nu(\mathscr{D})$.

(2) *The inverse* $\pi_\nu(\gamma) : \mathscr{H}_\nu(\mathscr{D}) \to \mathscr{H}_\nu(T(\Omega))$ *is given by*

$$\pi_\nu(\gamma)(F)(z) = 2^{\frac{r\nu}{2}} \Delta(z + e)^{-\nu} F \circ \gamma^{-1}(z) = 2^{\frac{r\nu}{2}} \Delta(z + e)^{-\nu} F((z - e)(z + e)^{-1}).$$

(3) *Let* $F \in \mathscr{H}_\nu(T(\Omega))$ *then*

$$||\pi_\nu(\gamma^{-1})(F)||^2_{\mathscr{H}_\nu(\mathscr{D})} = \Gamma_\Omega(\nu) ||F||^2_{\mathscr{H}_\nu(T(\Omega))}.$$

(4) *If* $g \in G$ *and* $F \in \mathscr{H}_\nu(\mathscr{D})$ *then*

$$\pi_\nu(\gamma)(\pi_\nu(g)F) = \pi_\nu(\gamma g \gamma^{-1})\pi_\nu(\gamma)(F).$$

(5) *If* $\nu > (r-1)\frac{a}{2}$ *then* $\mathscr{P}(V) \subset \mathscr{H}_\nu(\mathscr{D})$ *and* $\mathscr{P}(V)$ *is dense in* $\mathscr{H}_\nu(\mathscr{D})$.

(6) *If* $\nu > (r-1)\frac{a}{2}$ *then the functions* $\psi_{\mathbf{m}}$, $\mathbf{m} \in \mathbf{\Lambda}$, *form an orthogonal basis for* $\mathscr{H}_\nu(\mathscr{D})^L$.

Proof. That the map is an isomorphism follows from [9, Proposition XIII.1.3]. The intertwining relation in property (4) is a simple calculation and is left to the reader. □

In particular, it follows that [9, Proposition XIII.1.4]:

Lemma 2.12. *The reproducing kernel of* $\mathscr{H}_\nu(\mathscr{D})$ *is given by* $K_\nu(z,w) = h(z,w)^{-\nu} = \Delta(e - z\bar{w})^{-\nu}$.

Define the *Fock-space* on V to be the space of holomorphic functions on V such that

$$||F||^2_{\mathscr{F}(V)} = \pi^{-d} \int_V |F(z)|^2 e^{-||z||^2}\, dz < \infty. \tag{2.26}$$

The following result is proved by Faraut and Koranyi [8], see also [9, Propositions XI.4.1 and XIII.2.2]. This will play an essential role in our work.

Theorem 2.13. *Assume that* $\nu > 1 + a(r-1)$ *and* $\mathbf{m} \geq 0$. *Then the norms of* $\psi_{\mathbf{m}}$ *in the Fock-space and weighted Bergman spaces are given by*

$$||\psi_{\mathbf{m}}||^2_{\mathscr{F}(V)} = \frac{1}{d_{\mathbf{m}}}\left(\frac{d}{r}\right)_{\mathbf{m}}$$

and

$$||\psi_{\mathbf{m}}||^2_{\mathscr{H}_\nu(\mathscr{D})} = \frac{1}{d_{\mathbf{m}}} \frac{(\frac{d}{r})_{\mathbf{m}}}{(\nu)_{\mathbf{m}}}$$

respectively, where $d_{\mathbf{m}}$ *is the dimension of the space* $\mathscr{P}_{\mathbf{m}}$.

Corollary 2.14. *Assume that* $\nu > 1 + a(r-1)$. *Then the functions*

$$\sqrt{\frac{d_{\mathbf{m}}(\nu)_{\mathbf{m}}}{(\frac{d}{r})_{\mathbf{m}}}}\,\psi_{\mathbf{m}}, \quad \mathbf{m} \in \mathbf{\Lambda}$$

form an orthonormal basis for $\mathscr{H}_\nu(\mathscr{D})^L$.

Proof. This follows from Lemma 2.11, part 6, and the above theorem. □

3. Berezin transform and generalized Segal–Bargmann transform for $\mathcal{D}_{\mathbb{R}}$

3.1. Restriction to $\mathcal{D}_{\mathbb{R}}$ of holomorphic functions on $\mathcal{D}$

In this subsection we discuss the restriction principle for the bounded symmetric space $\mathcal{D}_{\mathbb{R}}$ [23,24,39]. In particular, we give an exact bound for the parameter v such that the restriction map R, defined below, maps $\mathcal{P}(V)$ into $L^2(H/L)$.

Recall that the H-invariant measure on $\mathcal{D}_{\mathbb{R}}$ is given by $d\eta(x) = h(x)^{-\frac{p}{2}}\,dx$. The group H acts unitarily on $L^2(\mathcal{D}_{\mathbb{R}}, d\eta)$ by $g \cdot f(x) = f(g^{-1} \cdot x)$. Furthermore,

$$L^2(\mathcal{D}_{\mathbb{R}}, d\eta) \simeq_H L^2(H/L, d\dot{h}),$$

where $d\dot{h}$ denotes an H-invariant measure on H/L.

Lemma 3.1. *Let $v \in \mathbb{R}$. Then $p(x)h(x)^{v/2} \in L^2(\mathcal{D}_{\mathbb{R}}, d\eta)$, for all $p \in \mathcal{P}(V)$, if and only if $v > \frac{a(r-1)}{2}$.*

Proof. Let $p \in \mathcal{P}(V)$. As the closure of $\mathcal{D}_{\mathbb{R}}$ is compact it follows that p is bounded on $\mathcal{D}_{\mathbb{R}}$. Hence it is enough to show that $h^{v/2} \in L^2(\mathcal{D}_{\mathbb{R}}, d\eta)$ if and only if $v > (r-1)\frac{a}{2}$. Note that $h^{v/2}$ is L-invariant. Writing the invariant measure on H/L using polar coordinates gives for every L-invariant function:

$$\int_{H/L} f(h \cdot 0)\, d\dot{h} = \int_{t_1 > t_2 > \cdots > t_r} f\left(\exp\left(\sum_{j=1}^{r} t_j \xi_j\right) \cdot 0\right) \left(\prod_{i<j} \sinh(t_i - t_j)\right)^a dt_1 \ldots dt_r.$$

If $G = \mathrm{SU}(1,1)$ then $\xi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$g_t \coloneqq \exp(t\xi_1) = \begin{pmatrix} \cosh(t) & \sinh(t) \\ -\sinh(t) & \cosh(t) \end{pmatrix}$$

and hence

$$g_t \cdot 0 = \tanh(t).$$

A similar calculation in the general case gives

$$\exp\left(\sum_{j=1}^{r} t_j \xi_j\right) \cdot 0 = \sum_{j=1}^{r} \tanh(t_j) e_j.$$

Thus

$$h\left(\exp\left(\sum_{j=1}^{r} t_j\xi_j\right)\cdot 0\right) = \Delta\left(e - \sum_{j=1}^{r} \tanh^2(t_j)e_j\right)$$
$$= \prod_{j=1}^{r} (1 - \tanh^2(t_j))$$
$$= \prod_{j=1}^{r} \frac{1}{\cosh^2(t_j)}.$$

It follows that

$$\int_{\mathscr{D}_{\mathbb{R}}} |h(x)^{v/2}|^2 h(x)^{-p/2}\, dx = \int_{t_1>\cdots>t_r} \prod_{j=1}^{r} \cosh(t_j)^{-2v} \times \left(\prod_{i<j} \sinh(t_i - t_j)\right)^a dt_1 \dots dt_r.$$

Let $\phi(t) = \frac{1-e^{-2t}}{2}$ and observe that ϕ is non-negative and increasing, $\phi(t) = 0$ if and only if $t = 0$, ϕ is bounded (by $\frac{1}{2}$) on $[0, \infty)$, and $\sinh(t_i - t_j) = e^{t_i - t_j}\phi(t_i - t_j)$. In a similar way let $\psi(t) = \frac{1+e^{-2t}}{2}$. Observe that $\psi(t) \in (\frac{1}{2}, 1]$, for all $t \in [0, \infty)$, and $\cosh(t_j) = e^{t_j}\psi(t_j)$. Let $t = (t_1, \dots, t_r)$ and define $\Phi(t) = \prod_{i<j} \phi(t_i - t_j)$ and $\Psi(t) = \prod_{j=1}^{r} \psi(t_j)$. In this notation we have

$$\int_{\mathscr{D}_{\mathbb{R}}} |h(x)^{v/2}|^2 h(x)^{-p/2}\, dx = \int_{t_1>\cdots>t_r} \Phi(t)\Psi(t) e^{\sum_{j-1}^{r} (-2v+(r+1-2j)a)t_j}\, dt_1 \dots dt_r.$$

Now suppose $v > (r-1)\frac{a}{2}$. Then $-2v + (r+1-2j)a < 0$ for each $j = 1, \dots, r$. Since Ψ and Φ are bounded the integral converges. On the other hand, suppose the integral converges. Let $\varepsilon > 0$. Then the integral converges when integrated over the complement of the set where $t_i - t_j < \varepsilon$, for all i, j such that $1 < i < j < r$. Since Φ and Ψ are bounded away from zero on such a set it must be that $-2v + (r+1-2j)a < 0$ for each $j = 1, \dots, r$. This implies $v > (r-1)\frac{a}{2}$. $\square$

We remark that the constant in the last lemma is exactly the same as the end-point of the continuous set of parameters, $(\frac{a}{2}(r-1), \infty)$, for the unitary highest weight modules. Assume that $v > (r-1)\frac{a}{2}$. Let R_v be the restriction map $R_v : \mathscr{H}_v(\mathscr{D}) \to C^\infty(\mathscr{D}_{\mathbb{R}})$ given by

$$R_v f(x) = f(x)h(x)^{\frac{v}{2}}. \tag{3.1}$$

As v will be fixed most of the time, we will often write R for R_v. Consider the restriction of the group action π_v of $\tilde{G}$ to $\tilde{H}$, where $\tilde{H}$ is the subgroup corresponding to the Lie algebra $\mathfrak{h}$. Using the method from [23, Lemma 3.4], we can now prove the following lemma.

Lemma 3.2. *Assume that $v > (r-1)\frac{a}{2}$. Then the map R_v is a closed densely defined $\tilde{H}$-intertwining operator from $\mathscr{H}_v(\mathscr{D})$ into $L^2(\mathscr{D}_{\mathbb{R}}, d\eta)$.*

Proof. That R intertwines the action of $\tilde{H}$ on $\mathscr{H}_v(\mathscr{D})$ and the regular action of $\tilde{H}$ on $L^2(\mathscr{D}, d\eta)$ follows by the transformation properties of $h(x)$. If F is a polynomial, $RF \in L^2(\mathscr{D}_{\mathbb{R}}, d\eta)$ by Lemma 3.1. Hence R is densely defined. Let $g \in C_c^\infty(\mathscr{D}_{\mathbb{R}})$ and $\varepsilon > 0$. Then there exists a polynomial F such that $||F - h^{-v/2}g||_\infty < \varepsilon/||h^{v/2}||$. Hence

$$||RF - g||^2 = \int_{\mathscr{D}_{\mathbb{R}}} |h(x)^v|F(x) - h(x)^{-v/2}g(x)|^2 \, d\eta(x)$$

$$\leqslant \frac{\varepsilon^2}{||h^{v/2}||^2} \int h(x)^v \, d\eta(x) = \varepsilon^2.$$

Hence $\mathrm{Im}(R)$ is dense in $L^2(\mathscr{D}_{\mathbb{R}}, d\eta)$. Finally, R is closed because point evaluations in $\mathscr{H}_v(\mathscr{D})$ are continuous. □

We can now consider the adjoint $R^* : L^2(\mathscr{D}_{\mathbb{R}}, d\eta) \to \mathscr{H}_v(\mathscr{D})$ as a densely defined operator.

Theorem 3.3. *Assume that $v > (r-1)\frac{a}{2}$. Let f be in the domain of definition of R_v^*. Then the following holds:*

(1) $R_v R_v^* f(y) = \int_{\mathscr{D}_{\mathbb{R}}} \frac{h(y)^{v/2} h(x)^{v/2}}{h(y,x)^v} f(x) \, d\eta(x)$.
(2) *If $g \in H$, then this is the same as*

$$R_v R_v^* f(g \cdot 0) = \int_H J(h^{-1}g, 0)^{v/p} f(h) \, d\dot{h} = D_v * f(g),$$

where $D_v(h) = J(h, 0)^{v/p}$ is in $L^2(H/L, d\dot{h})$ and $D_v * f$ stands for the group convolution.
(3) *Assume that $v > (r-1)a$. Then $D_v \in L^1(\mathscr{D}_{\mathbb{R}}, d\eta)$.*
(4) *If $v > (r-1)a$.* Then *$R_v R_v^* : L^2(\mathscr{D}_{\mathbb{R}}, d\eta) \to L^2(\mathscr{D}_{\mathbb{R}}, d\eta)$ is continuous with norm $||R_v R_v^*|| \leqslant ||D_v||_{L^1}$.*
(5) *If $v > (r-1)a$. Then $R_v R_v^* : L^\infty(\mathscr{D}_{\mathbb{R}}, d\eta) \to L^\infty(\mathscr{D}_{\mathbb{R}}, d\eta)$ is continuous with norm $||R_v R_v^*|| \leqslant ||D_v||_{L^1}$.*

Proof. (1) Let f be in the domain of definition of R^*. Then $R^*f \in \mathscr{H}_\nu(\mathscr{D})$ and for $w \in \mathscr{D}$ we get:

$$\begin{aligned} R^*f(w) &= (R^*f, h(\cdot, w)^{-\nu})_{\mathscr{H}_\nu(\mathscr{D})} \\ &= (f, R(h(\cdot, w)^{-\nu}))_{L^2(\mathscr{D}_\mathbb{R})} \\ &= \int_{\mathscr{D}_\mathbb{R}} f(x) h(x)^{\nu/2} h(w, x)^{-\nu}\, d\eta(x), \end{aligned}$$

where we have used that $\overline{h(z, w)} = h(w, z)$ (cf. Lemma 2.10). Thus for $y \in \mathscr{D}_\mathbb{R}$ we get:

$$RR^*f(y) = \int_{\mathscr{D}_\mathbb{R}} f(x) h(y)^{\nu/2} h(x)^{\nu/2} h(y, x)^{-\nu}\, d\eta(x)$$

which proves the first statement.

(2) Let $g_1, g_2 \in H$. According to Lemma 2.10 we have

$$h(g \cdot 0) = h(g \cdot 0, g \cdot 0) = J(g \cdot 0, 0)^{2/p}.$$

Thus

$$h(g_1 \cdot 0, g_2 \cdot 0) = h(g_2 g_2^{-1} g_1 \cdot 0, g_2 \cdot 0) = J(g_2, g_2^{-1} g_1 \cdot 0)^{1/p} J(g_2, 0)^{1/p},$$

where we have used that $J(g_2, 0)$ and $J(g_1, 0)$ are real, and that $h(z, 0) = 1$ for all z. The cocycle relation $J(ab, z) = J(a, bz) J(b, z)$ gives

$$J(g_1, 0) = J(g_2(g_2^{-1} g_1), 0) = J(g_2, g_2^{-1} g_1 \cdot 0) J(g_2^{-1} g_1, 0).$$

Hence

$$J(g_2, g_2^{-1} g_1 \cdot 0) = J(g_1, 0) J(g_2^{-1} g_1, 0)^{-1}.$$

Thus the integral kernel for RR^* becomes

$$\begin{aligned} \frac{h(g_1 \cdot 0)^{\nu/2} h(g_2 \cdot 0)^{\nu/2}}{h(g_1 \cdot 0, g_2 \cdot 0)^\nu} &= \frac{J(g_1, 0)^{\nu/p} J(g_2, 0)^{\nu/p}}{J(g_1, 0)^{\nu/p} J(g_2^{-1} g_1, 0)^{-\nu/p} J(g_2, 0)^{\nu/p}} \\ &= J(g_2^{-1} g_1, 0)^{\nu/p} = D_\nu(g_2^{-1} g_1). \end{aligned}$$

As $D_\nu(g) = J(g, 0)^{\nu/p} = h(g \cdot 0)^{\nu/2}$ it follows by Lemma 3.1 that $D_\nu \in L^2(H/L, d\dot{h})$.

(3) Notice that $D_\nu^2 = D_{2\nu}$. Hence the claim follows from Lemma 3.1.

(4) and (5) are now obvious. $\square$

If $\nu > a(r-1)$ then

$$RR^*1(g \cdot 0) = \int_H D_\nu(g^{-1} h)\, dh = ||D_\nu||_{L^1} < \infty.$$

Since $||D_\nu||_{L^1} > 0$ we define c_ν by the relation $\frac{1}{c_\nu} = ||D_\nu||_{L^1}$. The operator RR^* is called *the Berezin transform* on $\mathscr{D}_{\mathbb{R}}$ and $B_\nu = c_\nu RR^*$ is called *the normalized Berezin transform* because $B_\nu(1) = 1$. By Theorem 3.3

$$\begin{aligned} B_\nu f(x) &= c_\nu \int_{\mathscr{D}_{\mathbb{R}}} \frac{h(x)^{\nu/2} h(y)^{\nu/2}}{h(x,y)^\nu} f(y)\, d\eta(y) \\ &= c_\nu \int_{\mathscr{D}_{\mathbb{R}}} \frac{h(x)^{\nu/2} h(y)^{\nu/2}}{|h(x,y)^{\nu/2}|^2} f(y)\, d\eta(y). \end{aligned} \tag{3.2}$$

The space $L^2(\mathscr{D}_{\mathbb{R}}, d\eta)^L$ is decomposed into a direct integral of principle series representations of H via the spherical Fourier transform

$$\tilde{f}(\lambda) = \mathscr{F}(f)(\lambda) = \int_{\mathscr{D}_{\mathbb{R}}} f(x)\phi_\lambda(x)\, d\eta(x) \tag{3.3}$$

or, in terms of the spectral decomposition of commuting self-adjoint operators, as a direct integral of eigenspaces of the invariant differential operators. The L-invariant eigenfunctions of B_ν are precisely the spherical functions ϕ_λ, $\lambda \in a^*_{\mathbb{C}}$. Thus

$$B_\nu \phi_\lambda = b_\nu(\lambda)\phi_\lambda.$$

The symbol b_ν is explicitly calculated in [39].

3.2. *The generalized Segal–Bargmann transform*

We assume in this section that $\nu > (r-1)\frac{a}{2}$. The operator RR^* is well defined and by definition positive. We can therefore define $\sqrt{RR^*}$. Then there exists a partial isometry U_ν such that $R^* = U_\nu \sqrt{RR^*}$. To simplify notation we will often write U for U_ν. As $R = \sqrt{RR^*}\, U^*$ and $\mathrm{Im}(R)$ is dense it follows that U is actually a unitary isomorphism.

Definition 3.4. Let $\nu > (r-1)\frac{a}{2}$. The unitary isomorphism $U_\nu : L^2(\mathscr{D}_{\mathbb{R}}, d\eta) \to \mathscr{H}_\nu(\mathscr{D})$ is called the generalized Segal–Bargmann transform [24].

Let $W = N_L(\mathfrak{a})/Z_L(\mathfrak{a})$ be the Weyl group of $\mathfrak{a}$ corresponding to the root system $\Delta(\mathfrak{h}, \mathfrak{a})$ and let $f \to \tilde{f}$ denote the spherical Fourier transform. Then $f \mapsto \tilde{f} = \mathscr{F}(f)$ extends to a unitary isomorphism

$$\mathscr{F} : L^2(\mathscr{D}_{\mathbb{R}}, d\eta)^L \to L^2\left(\mathfrak{a}^*/W, \frac{d\lambda}{|c(\lambda)|^2}\right) \simeq L^2\left(\mathfrak{a}^*, \frac{d\lambda}{w|c(\lambda)|^2}\right)^W, \tag{3.4}$$

where

$$c(\lambda) = c_0 \prod_{j<k} \frac{\Gamma(i(\lambda_j - \lambda_k))}{\Gamma((\rho_j - \rho_k))} \frac{\Gamma(\frac{a}{2} + i(\lambda_j - \lambda_k))}{\Gamma(\frac{a}{2} + \rho_j - \rho_k)}$$

is the Harish–Chandra c-function, c_0 is a constant whose value can be evaluated by using known integral formulas (see [40] for the case of type C and D domains), and w is the order of the Weyl group W. Combining all of this together we now get the following proposition.

Proposition 3.5. *Suppose* $v > (r-1)\frac{a}{2}$. *Then*

$$\mathscr{F} \circ U_v^* : \mathscr{H}_v(\mathscr{D})^L \to L^2\left(\mathfrak{a}^*, \frac{1}{w}|c(\lambda)|^{-2}\, d\lambda\right)^W$$

is a unitary isomorphism.

Corollary 3.6. *Let the notation be as above. Then the following hold*:

(1) *If* $v > (r-1)\frac{a}{2}$ *then the functions*

$$\mathscr{F} U_v^*(\psi_{\mathbf{m}}), \quad \mathbf{m} \in \Lambda$$

form an orthogonal basis for the Hilbert space $L^2(\mathfrak{a}^*/W, |c(\lambda)|^{-2}\, d\lambda)$.

(2) *If* $v > 1 + a(r-1)$ *then the functions*

$$\sqrt{\frac{d_{\mathbf{m}}(v)_{\mathbf{m}}}{(\frac{d}{r})_{\mathbf{m}}}}\, \mathscr{F} U_v^*(\psi_{\mathbf{m}}), \quad \mathbf{m} \in \Lambda$$

form an orthonormal basis for the Hilbert space $L^2(\mathfrak{a}^*/W, |c(\lambda)|^{-2}\, d\lambda)$.

Proof. This follows Lemma 2.11, part 6, and Corollary 2.14. □

Our first main goal of this paper is to identify the functions $\mathscr{F} U^* \psi_{\mathbf{m}}$ and study their analytical properties.

4. Generating functions and orthogonality relations for the branching coefficients

In this section we derive the orthogonality relations for the branching coefficients. These results follow somewhat easily from a general consideration [40]. In the case where one considers the branching rules for the tensor product $\mathscr{H}_v(\mathscr{D}) \otimes \overline{\mathscr{H}_v(\mathscr{D})}$ of G, considered as the restriction of the representation of $\tilde{G} \times \tilde{G}$ on the diagonal, the branching coefficients, also called the Clebsch–Gordan coefficients, are studied in [25,37,38]. The results here parallel those obtained there. Thus, we will be brief.

Define $p_{\mathbf{m}} \in \mathscr{P}(\mathfrak{a}^*)$ by the Rodrigue's type formula

$$p_{v,\mathbf{m}}(\lambda) = p_{\mathbf{m}}(\lambda) := ||\psi_{\mathbf{m}}||^{-2}_{\mathscr{F}(V)}\psi_{\mathbf{m}}(\partial_x)(\Delta(e-x^2)^{-v/2}\phi_\lambda(x))|_{x=0}. \tag{4.1}$$

We can consider the polynomials $p_{\mathbf{m}}(\lambda)$ as a generalization of the Hermite polynomials [40].

Lemma 4.1. *Consider the expansion of* $h(x)^{-v/2}\phi_\lambda(x) = \Delta(e-x^2)^{-v/2}\phi_\lambda(x)$ *in terms of the L-invariant polynomials* $\psi_{\mathbf{m}}$. *Then*

$$\Delta(e-x^2)^{-v/2}\phi_\lambda(x) = \sum_{\mathbf{m}\in\Lambda} p_{v,\mathbf{m}}(\lambda)\psi_{\mathbf{m}}(x). \tag{4.2}$$

Proof. Define $P_{\mathbf{m}}(\lambda)$ by

$$\Delta(e-x^2)^{-v/2}\phi_\lambda(x) = \sum_{\mathbf{m}\in\Lambda} P_{\mathbf{m}}(\lambda)\psi_{\mathbf{m}}(x).$$

Differentiating both side with respect to $\psi_{\mathbf{m}}(\partial_x)$ and setting $x=0$ gives

$$||\psi_{\mathbf{m}}||^2_{\mathscr{F}(V)}p_{\mathbf{m}}(\lambda) = P_{\mathbf{m}}(\lambda)(\psi_{\mathbf{m}}(\partial_x)\psi_{\mathbf{m}}(x))|_{x=0} = P_{\mathbf{m}}(\lambda)||\psi_{\mathbf{m}}||^2_{\mathscr{F}(V)}.$$

Hence the claim. □

Lemma 4.2. *Assume that* $v > a(r-1)$. *Let* $F \in \mathscr{H}_v(\mathscr{D})^L$. *Then*

$$\mathscr{F}(\sqrt{R_v R_v^*}U_v^*(F))(\lambda) = c_v^{-1/2}b_v(\lambda)^{1/2}\mathscr{F}(U_v^*(F))(\lambda).$$

Proof. Recall that RR^* is a convolution by D_{v^*}. Hence, for all $f \in L^2(\mathscr{D}_{\mathbb{R}}, d\eta)^L$,

$$\mathscr{F}(RR^*f)(\lambda) = \mathscr{F}(D_v)(\lambda)\mathscr{F}(f)(\lambda).$$

Furthermore,

$$\mathscr{F}(D_v)(\lambda) = \int D_v(h)\phi_\lambda(h)\,d\dot{h} = c_v^{-1}B_v(\phi_\lambda)(1) = c_v^{-1}b_v(\lambda),$$

since $\phi_\lambda(1) = 1$. Consequently,

$$\mathscr{F}(\sqrt{RR^*}f)(\lambda) = c_v^{-\frac{1}{2}}\sqrt{b_v(\lambda)}\tilde{f}(\lambda).$$

Applying this to a function $f = U^*F$, $F \in \mathscr{H}_v(\mathscr{D})$, gives the lemma. □

The next proposition states that the polynomials $p_{\mathbf{m}}(\lambda)$ can be also obtained via the Segal–Bargman transform of $\psi_{\mathbf{m}}$.

Proposition 4.3. *Assume that* $\nu > a(r-1)$. *Then*

$$\mathscr{F}(U_\nu^* \psi_{\mathbf{m}})(\lambda) = c_\nu^{-\frac{1}{2}} \sqrt{b_\nu(\lambda)} ||\psi_{\mathbf{m}}||_\nu^2 \, p_{\nu,\mathbf{m}}(\lambda).$$

Proof. Since $\nu > a(r-1)$, we have

$$\int_D h(x)^{\frac{\nu}{2}} \, d\eta(x) < \infty.$$

Let $C(\rho)$ be the convex hull of $W \cdot \rho$. By the Helgason–Johnson theorem ([13, Theorem 8.1, p. 458], [14], see also [1] for some L^p-results), if $\lambda \in \mathfrak{a}^* + iC(\rho)$ then ϕ_λ is a bounded function on $\mathscr{D}_{\mathbb{R}}$. Thus the integral defining $B_\nu \phi_\lambda$ is absolutely convergent and

$$B_\nu \phi_\lambda(x) = c_\nu \int_{\mathscr{D}_{\mathbb{R}}} \frac{h(x)^{\nu/2} h(y)^{\nu/2}}{h(x,y)^\nu} \phi_\lambda(y) \, d\eta(y) = b_\nu(\lambda) \phi_\lambda(x).$$

We divide by $c_\nu h(x)^{\nu/2}$. Furthermore, extend $||\psi_{\mathbf{m}}||_\nu^{-1} \psi_{\mathbf{m}}$, $\mathbf{m} \in \mathbf{\Lambda}$, to an orthonormal basis $F_{\mathbf{n}}$ of $\mathscr{H}_\nu(\mathscr{D})$. Then

$$h(z,w)^{-\nu} = \sum_{\mathbf{n}} F_{\mathbf{n}}(z) \overline{F_{\mathbf{n}}(w)}.$$

This gives:

$$c_\nu^{-1} b_\nu(\lambda) h(x)^{-\nu/2} \phi_\lambda(x) = \int_{\mathscr{D}_{\mathbb{R}}} \sum_{\mathbf{n}} h(y)^{\nu/2} F_{\mathbf{n}}(y) \phi_\lambda(y) \overline{F_{\mathbf{n}}(x)} \, d\eta(y).$$

We now let $\psi_{\mathbf{m}}(\partial_x)|_{x=0}$ act on both sides. For the left-hand side we use Lemma 4.1. For the right-hand side we use the fact that $||\psi_{\mathbf{m}}||_\nu^{-1} \psi_{\mathbf{m}}(\partial_x) \overline{F_{\mathbf{n}}(x)}|_{x=0} = \delta_{\mathbf{mn}} \frac{||\psi_{\mathbf{m}}||^2_{\mathscr{F}(V)}}{||\psi_{\mathbf{m}}||_\nu^2}$, when $F_{\mathbf{m}}$ is one of the $||\psi_{\mathbf{m}}||_\nu^{-1} \psi_{\mathbf{m}}$, and 0 otherwise. We thus get

$$||\psi_{\mathbf{m}}||^2_{\mathscr{F}(V)} c_\nu^{-1} b_\nu(\lambda) p_{\mathbf{m}}(\lambda) = \frac{||\psi_{\mathbf{m}}||^2_{\mathscr{F}(V)}}{||\psi_{\mathbf{m}}||_\nu^2} \int_{\mathscr{D}_{\mathbb{R}}} h(y)^{\nu/2} \psi_{\mathbf{m}}(y) \phi_\lambda(y) \, d\eta(y).$$

That is,

$$\mathscr{F}(R\psi_{\mathbf{m}})(\lambda) = ||\psi_{\mathbf{m}}||_\nu^2 c_\nu^{-1} b_\nu(\lambda) p_{\mathbf{m}}(\lambda).$$

The claim follows now from Lemma 4.2 as $R = \sqrt{RR^*} U^*$. $\square$

5. Recurrence formulas for the branching coefficients

Propositions 3.5 and 4.3 together imply that when the measure $b(\lambda)|c(\lambda)|^{-2}\,d\lambda$ is associated with $\mathfrak{a}^*/W$ then the polynomials $p_{\mathbf{n}}(\lambda)$, $\mathbf{n}\in\mathbf{\Lambda}$, are orthogonal. When reference is made to the orthogonality of these polynomials it is understood to be with respect to this measure. In this section and the next we will derive the recurrence and difference formulas for the orthogonal polynomials $p_{\mathbf{n}}(\lambda)$. In most cases, our method will be to prove our formula first for λ in a certain integral cone where we can easily deal with the equations. To pass to the general formula we then need the following elementary result, which can be easily proved by induction. Recall the lattice $\mathbf{\Lambda}$ defined in (2.15).

Lemma 5.1. *Suppose $p_1(\lambda)$ and $p_2(\lambda)$ are two polynomials in $\mathfrak{a}^*_{\mathbb{C}}$. Let $\mu\in\mathfrak{a}^*_{\mathbb{C}}$. Suppose that $p_1(\lambda)=p_2(\lambda)$ for all $\lambda=\mathbf{n}+\mu$, $\mathbf{n}\in\mathbf{\Lambda}$. Then $p_1(\lambda)=p_2(\lambda)$ for all $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$.*

To simplify certain arguments we assume, in this and next section, that v is an even positive integer. To state our result we need the binomial coefficient

$$\binom{\mathbf{n}}{\mathbf{n}-\gamma_k}=\left(n_k+\frac{a}{2}(r-k)\right)\prod_{j\neq k}\frac{n_k-n_j+\frac{a}{2}(j-k-1)}{n_k-n_j+\frac{a}{2}(j-k)},$$

(cf. [18,20,21]). Note that $\binom{\mathbf{n}}{\mathbf{n}-\gamma_k}$ is actually a rational function of $\mathbf{n}$ and can be defined on all $\mathbb{C}^r$ with singularities on some lower dimensional hyperplanes. We let

$$c_{\mathbf{n}}(k)=\prod_{j\neq k}\frac{n_j-n_k-\frac{a}{2}(j+1-k)}{n_j-n_k-\frac{a}{2}(j-k)}.$$

The following theorem is essentially proved in [26], although the holomorphic discrete series π_v is realized there in a degenerate principal series representation. We will not reproduce the same proof here. In the case of $G=\mathrm{SU}(1,1)$ we have $\psi_m(z)=z^m$. This theorem and the lemma that follows are generalizations of the simple facts that

$$\pi_v\left(-\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)z^m=-(v+m)z^{m-1}+mz^{m-1}$$
$$\pi_v\left(-\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)z^m=(v+2m)z^m,$$

which can be verified by a simple calculation.

Theorem 5.2. *Recall that* $\xi = \mathrm{Ad}(\gamma)(-2Z_0)$. *In the bounded realization* $\mathscr{H}_\nu(\mathscr{D})$ *of the representation of* $\mathfrak{g}_{\mathbb{C}}$,

$$\pi_\nu(-\xi)\psi_{\mathbf{n}} = \sum_{j=1}^{r} \binom{\mathbf{n}}{\mathbf{n}-\gamma_j} \psi_{\mathbf{n}-\gamma_j} - \sum_{j=1}^{r} \Big(\nu + n_j - \frac{a}{2}(j-1)\Big) c_{\mathbf{n}}(j)\psi_{\mathbf{n}+\gamma_j}. \tag{5.1}$$

Remark 5.3. As is proved in [26] one may use the above formula to derive the results of Faraut–Koranyi [8] on the norm of $\phi_{\mathbf{n}}$ in the Hilbert space $\mathscr{H}_\nu(\mathscr{D})$. Conversely, knowing that $\pi_\nu(\xi)\phi_\lambda$ is of the above form, it is possible also to find recursively the coefficients by using the results in [8], the dimension formula in [34], and the fact that $\pi_\nu(\xi)$ is a skew-symmetric operator.

The element $2Z_0 = D(e, \bar{e})$ is in the center of $\mathfrak{k}$, and thus acts on each $\psi_{\mathbf{n}}$ by a scalar. A routine calculation gives the following lemma:

Lemma 5.4. *We have*

$$\pi_\nu(-2Z_0)\psi_{\mathbf{m}} = (r\nu + 2|\mathbf{m}|)\psi_{\mathbf{m}}. \tag{5.2}$$

We can draw several important relations from these facts. Recall first from Theorem 2.9, part 5 that

$$q_{\mathbf{m},\nu}(z) = \Delta(z+e)^{-\nu}\psi_{\mathbf{m}}((z-e)(z+e)^{-1}) = 2^{r\nu/2}\pi_\nu(\gamma)\psi_{\mathbf{m}}(z).$$

Recall that $\pi_\nu(\gamma)$ is well defined and that

$$\pi_\nu(\gamma)\pi_\nu(X) = \pi_\nu(\mathrm{Ad}(\gamma)X)\pi_\nu(\gamma)$$

for all $X \in \mathfrak{g}_{\mathbb{C}}$. As $\mathrm{Ad}(\gamma^{-1})(2Z_0) = \xi$ and $\mathrm{Ad}(\gamma^{-1})(\xi) = -2Z_0$, by Lemma 2.4, we get by applying the Cayley transform to (5.1) and (5.2):

Lemma 5.5. *Let* $q_{\mathbf{m},\nu}(z) = \Delta(z+e)^{-\nu}\psi_{\mathbf{m}}((z-e)(z+e)^{-1})$ *then the following holds*:

(1) $\pi_\nu(\xi)q_{\mathbf{m},\nu} = (r\nu + 2|\mathbf{m}|)q_{\mathbf{m},\nu}$.

(2) $\pi_\nu(-2Z_0)q_{\mathbf{m},\nu} = \sum_{j=1}^{r}\binom{\mathbf{m}}{\mathbf{m}-\gamma_j} q_{\mathbf{m}-\gamma_j,\nu} - \sum_{j=1}^{r}(\nu + n_j - \frac{a}{2}(j-1))c_{\mathbf{m}}(j)q_{\mathbf{m}+\gamma_j,\nu}$.

The following theorem gives the recursion relations for the polynomials $p_{\nu,\mathbf{m}}$.

Theorem 5.6. *The following recurrence formula holds*:

$$\Big(2\sum_{j=1}^{r}(i\lambda_j + \rho_j)\Big)p_{\nu,\mathbf{m}}(\lambda) = \sum_{j=1}^{r}\binom{\mathbf{m}+\gamma_j}{\mathbf{m}} p_{\nu,\mathbf{m}+\gamma_j}(\lambda)$$
$$- \Big(\nu + m_j - 1 - \frac{a}{2}(j-1)\Big)c_{\mathbf{m}-\gamma_j}(j)p_{\nu,\mathbf{m}-\gamma_j}(\lambda).$$

Proof. We prove the result for

$$\lambda = -i\left(\mathbf{m} + \frac{\nu}{2} - \rho\right), \tag{5.3}$$

where $\mathbf{m} \in \mathbf{\Lambda}$ and $\frac{\nu}{2}$ is viewed as $\frac{\nu}{2}\sum_{j=1}^{r}\beta_j$. The general result for $\lambda \in \mathbb{C}^r = \mathfrak{a}_{\mathbb{C}}^*$ follows from Lemma 5.1. As $\mathrm{Ad}(\gamma)\xi = -\mathrm{Ad}(\gamma)^{-1}\xi = 2Z_0$ we get from Lemma 5.5:

$$\pi_\nu(\xi)\pi_\nu(\gamma^{-1})\psi_{\mathbf{m}}(x) = -(r\nu + 2|\mathbf{m}|)\pi_\nu(\gamma^{-1})\psi_{\mathbf{m}}(x).$$

Now simplify this expression before actually performing the differentiation:

$$\begin{aligned}\pi_\nu(\gamma^{-1})\psi_{\mathbf{m}}(x) &= 2^{r\nu/2}\psi_{\mathbf{m}}\left(\frac{e+x}{e-x}\right)\Delta(e-x)^{-\nu}\\ &= 2^{r\nu/2}\psi_{\mathbf{m}+\frac{\nu}{2}}\left(\frac{e+x}{e-x}\right)\Delta\left(\frac{e+x}{e-x}\right)^{-\frac{\nu}{2}}\Delta(e-x)^{-\nu}\\ &= 2^{r\nu/2}\psi_{\mathbf{m}+\frac{\nu}{2}}\left(\frac{e+x}{e-x}\right)\Delta(e-x^2)^{-\frac{\nu}{2}}\\ &= 2^{r\nu/2}\phi_\lambda(x)\Delta(e-x^2)^{-\frac{\nu}{2}}.\end{aligned} \tag{5.4}$$

Here we have used that

$$\begin{aligned}\Delta\left(\frac{e+x}{e-x}\right)^{-\nu/2}\Delta(e-x)^{-\nu} &= \Delta(e+x)^{-\nu/2}\Delta(e-x)^{\nu/2}\Delta(e-x)^{-\nu}\\ &= \Delta(e+x)^{-\nu/2}\Delta(e-x)^{-\nu/2}\\ &= \Delta(e-x^2)^{-\frac{\nu}{2}}.\end{aligned}$$

Hence

$$\pi_\nu(\xi)(\Delta(e-x^2)^{-\nu/2}\phi_\lambda(x)) = -(r\nu + 2|\mathbf{m}|)\Delta(e-x^2)^{-\nu/2}\phi_\lambda(x). \tag{5.5}$$

On the other hand, by Lemma 4.1:

$$\phi_\lambda(x)\Delta(e-x^2)^{-\frac{\nu}{2}} = \sum_{\mathbf{n}\in\mathbf{\Lambda}} p_{\mathbf{n}}(\lambda)\psi_{\mathbf{n}}(x). \tag{5.6}$$

Thus (5.5) reads as

$$\pi_\nu(\xi)\sum_{\mathbf{n}\in\mathbf{\Lambda}} p_{\mathbf{n}}(\lambda)\psi_{\mathbf{n}}(x) = -(r\nu + 2|\mathbf{m}|)\sum_{\mathbf{n}\in\mathbf{\Lambda}} p_{\mathbf{n}}(\lambda)\psi_{\mathbf{n}}(x). \tag{5.7}$$

Notice that (5.6) is the power series expansion of an analytic function and the operator $\pi_\nu(\xi)$ is a differential operator; it commutes with the summation. We thus have, in a neighborhood of $x = 0$,

$$\sum_{\mathbf{n}\in\mathbf{\Lambda}} p_{\mathbf{n}}(\lambda)\pi_\nu(\xi)\psi_{\mathbf{n}}(x) = -(r\nu + 2|\mathbf{m}|)\sum_{\mathbf{n}\in\mathbf{\Lambda}} p_{\mathbf{n}}(\lambda)\psi_{\mathbf{n}}(x). \tag{5.8}$$

The left-hand side, in view of Theorem 5.2, is

$$\begin{aligned}
&\sum_{\mathbf{n}\in\mathbf{\Lambda}} p_{\mathbf{n}}(\lambda)\left(\sum_{j=1}^{r}\binom{\mathbf{n}}{\mathbf{n}-\gamma_j}\psi_{\mathbf{n}-\gamma_j}(x) - \sum_{j=1}^{r}\left(\nu + n_j - \frac{a}{2}(j-1)\right)c_{\mathbf{n}}(j)\psi_{\mathbf{n}+\gamma_j}(x)\right)\\
&\quad= \sum_{\mathbf{n}\in\mathbf{\Lambda}}\psi_{\mathbf{n}}(x)\left(\sum_{j=1}^{r}\binom{\mathbf{n}+\gamma_j}{\mathbf{n}}p_{\mathbf{n}+\gamma_j}(\lambda)\right.\\
&\qquad\left.-\left(\nu + n_j - 1 - \frac{a}{2}(j-1)\right)c_{\mathbf{n}-\gamma_j}(j)p_{\mathbf{n}-\gamma_j}(\lambda)\right).
\end{aligned} \tag{5.9}$$

Equating the coefficients of $\psi_{\mathbf{n}}(x)$ in (5.8) and (5.9) we get

$$(r\nu + 2|\mathbf{m}|)p_{\mathbf{n}}(\lambda) = \sum_{j=1}^{r}\binom{\mathbf{n}+\gamma_j}{\mathbf{n}}p_{\mathbf{n}+\gamma_j}(\lambda) - \left(\nu + n_j - 1 - \frac{a}{2}(j-1)\right)c_{\mathbf{n}-\gamma_j}(j)p_{\mathbf{n}-\gamma_j}(\lambda).$$

The relation (5.3) implies that $(r\nu + 2|\mathbf{m}|) = 2|i\lambda + \rho| = 2\sum_{j=1}^{r}(i\lambda_j + \rho_j)$. This finishes the proof. $\square$

Example 5.7. If $\mathscr{D}$ is the unit disk, then we can take $\mathscr{D}_{\mathbb{R}}$ as the unit interval $(-1, 1)$ on the real line. The spherical function on the unit disk is $\phi_\lambda(x) = (\frac{1+x}{1-x})^{i\lambda}$ and expansion (4.2) reads as

$$(1-x^2)^{-\frac{\nu}{2}}\left(\frac{1+x}{1-x}\right)^{i\lambda} = (1-x)^{-\frac{\nu}{2}-i\lambda}(1+x)^{-\frac{\nu}{2}+i\lambda} = \sum_{n=0}^{\infty} p_{n,\nu}(\lambda)x^n \tag{5.10}$$

with

$$p_{n,\nu}(\lambda) = \left(\frac{\nu}{2} + i\lambda\right)_n {}_2F_1\left(\frac{\nu}{2} - i\lambda, -n; -\frac{\nu}{2} - i\lambda - n + 1, 1\right)$$

being the Meixner–Pollacyck polynomials, (cf. [2]). The action of $\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on functions on $(-1, 1)$ is

$$\pi_\nu(\xi)f(x) = \nu x f(x) - (1-x^2)f'(x).$$

Writing the function in equation (5.10) as $G_{\nu,\lambda}(x)$ we have

$$\pi_\nu(\xi)G_{\nu,\lambda} = (-2i\lambda)G_{\nu,\lambda}.$$

This can be proved easily by a direct computation. It exemplifies Eq. (5.5) (here, $\rho = 0$ and so $i\lambda = m + \frac{v}{2}$). It then implies the recurrence relation

$$(2i\lambda)p_{v,n}(\lambda) = (n+1)p_{v,n+1}(\lambda) - (v+n-1)p_{v,n-1}(\lambda),$$

and this coincides with Theorem 5.6.

Remark 5.8. While deriving the recurrence formula we have extended the action of $\mathfrak{g}_{\mathbb{C}}$ and γ on $\mathscr{H}_v(T(\Omega))$ (or $\mathscr{H}_v(\mathscr{D})$) to the space of meromorphic functions on V. Indeed the operator $\pi_v(\gamma^{-1})$ is up to a constant, cf., Lemma 2.11, a unitary operator from $\mathscr{H}_v(T(\Omega))$ onto $\mathscr{H}_v(\mathscr{D})$ so it is initially defined on $\mathscr{H}_v(T(\Omega))$. However, in our formulas the action of $\pi_v(\gamma^{-1})$ on ψ_λ for $\lambda = \mathbf{m}$ is viewed as the extended action since ψ_λ is not an element in $\mathscr{H}_v(T(\Omega))$. It suggests some interesting applications of the idea of extending holomorphic functions on a domain to meromorphic functions to a larger domain.

6. Difference formulas for the branching coefficients

In this section we state and prove the difference equation for the polynomials $p_{v,\mathbf{m}}(\lambda)$

Theorem 6.1. *The polynomials $p_{v,\mathbf{m}}(\lambda)$ satisfy the following difference equation*

$$\begin{aligned}-(rv + 2|\mathbf{n}|)p_{v,\mathbf{n}}(\lambda) = {} & \sum_{j=1}^{r} \binom{i\lambda + \rho - \frac{v}{2}}{i\lambda + \rho - \frac{v}{2} - \gamma_j} p_{v,\mathbf{n}}(\lambda + i\gamma_j) \\ & - \sum_{j=1}^{r} \Big(\frac{v}{2} + i\lambda_j + \rho_j - \frac{a}{2}(j-1)\Big)\Big) c_{i\lambda+\rho-\frac{v}{2}}(j) p_{v,\mathbf{n}}(\lambda - i\gamma_j).\end{aligned}$$

Proof. As in the proof of Theorem 5.6 it suffices to prove the theorem for those λ satisfying $i\lambda + \rho = \mathbf{m} + v/2$. Eqs. (5.5) and (5.6) in the proof above combine to give

$$2^{\frac{-rv}{2}} \pi_v(\gamma^{-1})\psi_{\mathbf{m}} = \sum_{\mathbf{n}\in\Lambda} p_{\mathbf{n}}(\lambda)\psi_{\mathbf{n}}. \tag{6.1}$$

Let the operator $\pi_v(-2Z_{\mathrm{o}})$ act on both sides. For the left-hand side we use Theorem 5.6, Lemma 2.4, and Eq. (6.1) applied to the case $i(\lambda \pm i\gamma_j) + \rho = (\mathbf{m} \mp \gamma_j) + \frac{v}{2}$

to obtain

$$\begin{aligned}
\mathrm{LHS} &= 2^{\frac{-rv}{2}}\pi_v(-2Z_o)\pi_v(\gamma^{-1})\psi_{\mathbf{m}} \\
&= 2^{\frac{-rv}{2}}\pi_v(\gamma^{-1})\pi_v(\xi)\psi_{\mathbf{m}} \\
&= -2^{\frac{-rv}{2}}\pi_v(\gamma^{-1})\left(\sum_{j=1}^{r}\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}\psi_{\mathbf{m}-\gamma_j} - \sum_{j=1}^{r}\Big(v+m_j-\frac{a}{2}(j-1)\Big)c_{\mathbf{m}}(j)\psi_{\mathbf{m}+\gamma_j}\right) \\
&= -\sum_{\mathbf{n}\in\Lambda}\left(\sum_{j=1}^{r}\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}p_{\mathbf{n}}(\lambda+i\gamma_j)\right. \\
&\quad \left. -\sum_{j=1}^{r}\Big(v+m_j-\frac{a}{2}(j-1)\Big)c_{\mathbf{m}}(j)p_{\mathbf{n}}(\lambda-i\gamma_j)\right)\psi_{\mathbf{n}}.
\end{aligned}$$

For the right-hand side we obtain by Lemma 5.4

$$\begin{aligned}
\mathrm{RHS} &= \pi_v(-2Z_o)\sum_{\mathbf{n}\in\Lambda}p_{\mathbf{n}}(\lambda)\psi_{\mathbf{n}} \\
&= \sum_{\mathbf{n}\in\Lambda}(rv+2|\mathbf{n}|)p_{\mathbf{n}}(\lambda)\psi_{\mathbf{n}}.
\end{aligned}$$

The proof is completed by equating the coefficients, rewriting each occurrence of $\mathbf{m}$ in terms of λ, and then applying Lemma 5.1. □

Example 6.2. We continue Example 5.7. The above difference relation can be proved by simple however tricky computations, which in turn reveal the advantage of using the representation theoretic method. Using the notation there, write $G_\lambda(x) = G_{v,\lambda}(x)$ the generating function. Differentiating the expansion we get

$$2x\frac{d}{dx}G_\lambda(x) = \sum_{n=0}^{\infty} 2np_{n,v}(\lambda)x^n. \tag{6.2}$$

On the other hand, differentiating the formula for $G_\lambda(x)$ results in

$$2x\frac{d}{dx}G_\lambda(x) = \Big(\frac{v}{2}+i\lambda\Big)\frac{2x}{1+x}G_{\lambda-i}(x) + \Big(-\frac{v}{2}+i\lambda\Big)\frac{2x}{1-x}G_{\lambda+i}(x). \tag{6.3}$$

We observe that

$$G_\lambda(x) = \frac{1-x}{1+x}G_{\lambda-i}(x) = \frac{1+x}{1-x}G_{\lambda+i}(x),$$

which then imply that

$$\begin{aligned} \nu G_\lambda(x) &= \left(\frac{\nu}{2} + i\lambda\right) G_\lambda(x) + \left(\frac{\nu}{2} - i\lambda\right) G_\lambda(x) \\ &= \left(\frac{\nu}{2} + i\lambda\right) \frac{1-x}{1+x} G_{\lambda-i}(x) + \left(\frac{\nu}{2} - i\lambda\right) \frac{1+x}{1-x} G_{\lambda-i}(x). \end{aligned} \tag{6.4}$$

Summing Eqs. (6.3) and (6.4) gives

$$\left(2x\frac{d}{dx} + \nu\right) G_\lambda(x) = \left(\frac{\nu}{2} + i\lambda\right) G_{\lambda-i}(x) + \left(\frac{\nu}{2} - i\lambda\right) G_{\lambda+i}(x)$$

and consequently

$$(2n+\nu) p_{\nu,n}(\lambda) = \left(\frac{\nu}{2} + i\lambda\right) p_{\nu,n}(\lambda - i) + \left(\frac{\nu}{2} - i\lambda\right) p_{\nu,n}(\lambda + i)$$

or

$$-(2n+\nu) p_{\nu,n}(\lambda) = \left(-\frac{\nu}{2} + i\lambda\right) p_{\nu,n}(\lambda + i) - \left(\frac{\nu}{2} + i\lambda\right) p_{\nu,n}(\lambda - i)$$

which coincides with Theorem 6.1. The proof of Theorem 6.1 is thus conceptually clearer.

7. The restriction principle in the unbounded realization

In this section we discuss the application of the restriction principle to the unbounded realization of G/K. In particular, we use this to introduce the Laguerre functions of the cone Ω and derive some relations they satisfy. For a function F defined on the Siegel domain $T(\Omega) = \Omega + iJ$ let

$$R_\nu F(x) = RF(x) = F(x),$$

where $x \in \Omega$. Since the functions in $\mathscr{H}_\nu(T(\Omega))$ are holomorphic it follows as before that R is injective on $\mathscr{H}_\nu(T(\Omega))$. For $y \in \Omega$, let $k_y(z) = K_\nu(z, y) = K(z, y)$, where $z \in T(\Omega)$. Then $k_y \in \mathscr{H}_\nu(T(\Omega))$ and

$$Rk_y(x) = \Gamma_\Omega(\nu) \Delta(x+y)^{-\nu}.$$

Lemma 7.1. *The linear span of* $\{k_y \mid y \in \Omega\}$ *is dense in* $\mathscr{H}_\nu(T(\Omega))$.

Proof. Assume that $f \in \mathscr{H}_\nu(T(\Omega))$ is perpendicular to all k_y, $y \in \Omega$. Then $f(y) = 0$, for all $y \in \Omega$. As f is holomorphic it follows that $f = 0$. $\square$

Recall the *Beta-function* for Ω:

$$B_\Omega(\nu,\mu)=\int_\Omega \Delta(x+e)^{-\nu-\mu}\Delta(x)^{\nu-d/r}\,dx,$$

which is finite for ν and μ having real parts greater than $(r-1)\frac{a}{2}$ [9, p. 141].

Lemma 7.2. *Let $y\in\Omega$. Then $Rk_y\in L^2_\nu(\Omega)$, for all $\nu>(r-1)\frac{a}{2}$. In fact*

$$||Rk_y||=\frac{\Gamma_\Omega(\nu)}{\Delta(y)^{\nu/2}}\sqrt{B_\Omega(\nu,\nu)}.$$

Proof. Let $y\in\Omega$. Then, performing the change of variables $x\mapsto Q(y^{\frac{1}{2}})x$ where Q is the Jordan quadratic operator in Section 2.1,

$$\begin{aligned}||Rk_y||^2&=\Gamma^2_\Omega(\nu)\int_\Omega\frac{\Delta(x)^{\nu-d/r}}{\Delta(x+y)^{2\nu}}\,dx\\&=\Gamma^2_\Omega(\nu)\frac{1}{\Delta(y)^\nu}\int_\Omega\frac{\Delta(x)^{\nu-d/r}}{\Delta(x+e)^{2\nu}}\,dx\quad(x\mapsto Q(y^{\frac{1}{2}})x)\\&=\frac{\Gamma^2_\Omega(\nu)}{\Delta(y)^\nu}B_\Omega(\nu,\nu)<\infty,\end{aligned}$$

finishing the proof. □

We need to distinguish the Laplace transform $\mathscr{L}_\nu f$ as a mapping on functions on Ω to itself and as a mapping on functions on Ω to holomorphic functions on $T(\Omega)$. We write the former as $\mathscr{L}^\Omega_\nu f$, which is the restriction of the latter.

Lemma 7.3. *Let $\nu>(r-1)\frac{a}{2}$. Then the set $\{Rk_y\mid y\in\Omega\}$ is dense in $L^2_\nu(\Omega)$.*

Proof. Let $f\in L^2_\nu(\Omega)$ and suppose f is orthogonal to all Rk_y, $y\in\Omega$. Then

$$\begin{aligned}0&=(f\mid Rk_y)\\&=\Gamma_\Omega(\nu)\int_\Omega f(x)\Delta(x+y)^{-\nu}\Delta(x)^{\nu-d/r}\,dx\\&=\int_\Omega f(x)\mathscr{L}_\nu(e^{-(y|\cdot)})(x)\Delta(x)^{\nu-d/r}\,dx\\&=\int_\Omega e^{-(y|x)}\mathscr{L}^\Omega_\nu(f)(x)\Delta(x)^{\nu-d/r}\,dx\\&=\mathscr{L}^\Omega_\nu(\mathscr{L}^\Omega_\nu(f))(y).\end{aligned}$$

From this and the injectivity of the Laplace transform it follows that $f=0$. □

By the previous lemmas it follows that the restriction map

$$R : \mathscr{H}_\nu(T(\Omega)) \to L^2_\nu(\Omega)$$

is injective, densely defined and has dense range. Since convergence in $\mathscr{H}_\nu(T(\Omega))$ implies uniform convergence on compact sets it follows easily that R is closed. We can thus polarize R^*, $R^* = U|R^*|$, and obtain a unitary map U from $L^2_\nu(\Omega)$ onto $\mathscr{H}_\nu(T(\Omega))$. We now prove that $U = \mathscr{L}_\nu$.

Theorem 7.4. *Suppose $\nu > \frac{a}{2}(r-1)$. The polar decomposition of R^* is given by $R^* = \mathscr{L}_\nu \mathscr{L}^\Omega_\nu$, where the operator $\mathscr{L}_\nu$ extends to a unitary operator from $L^2_\nu(\Omega)$ onto $\mathscr{H}_\nu(T\Omega)$.*

Proof. Denote temporarily $\mathscr{L}^\Omega = \mathscr{L}^\Omega_\nu$. We define $\mathscr{L}^\Omega$ on the domain

$$\left\{ f \in \mathscr{L}^2_\nu(\Omega) : \int_\Omega e^{-(\cdot|x)} f(x)\, d\mu_\nu(x) \in L^2_\nu(\Omega) \right\}.$$

Note that by Cauchy–Schwarz inequality the condition $f \in L^2_\nu(\Omega)$ implies that $\mathscr{L}^\Omega f$ is a well-defined function on Ω since $e^{-(y|x)}$, for fixed $y \in \Omega$ is a function in $L^2_\nu(\Omega)$. It is then easy to prove that $\mathscr{L}^\Omega$ so defined is a self-adjoint positive operator. Let $f \in L^2_\nu(\Omega)$ be in the domain of RR^*, then

$$\begin{aligned} RR^* f(y) &= R^* f(y) \\ &= (R^* f \mid k_y)_{\mathscr{H}_\nu(T(\Omega))} \\ &= (f \mid Rk_y)_{L^2_\nu(\Omega)} \\ &= \int_\Omega f(x) \overline{K(x,y)}\, d\mu_\nu(x) \\ &= \Gamma_\Omega(\nu) \int_\Omega f(x) \Delta(x+y)^{-\nu}\, d\mu_\nu(x) \\ &= \int_\Omega f(x) \mathscr{L}^\Omega(e^{-(y|\cdot)})(x)\, d\mu_\nu(x) \\ &= \int_\Omega e^{-(y|x)} \Delta(x)^{\nu - d/r} \mathscr{L}^\Omega(f)(x)\, dx \\ &= \mathscr{L}^\Omega(\mathscr{L}^\Omega f)(y), \end{aligned} \tag{7.1}$$

and

$$(\mathscr{L}^\Omega f, \mathscr{L}^\Omega f) = (R^* f, R^* f) < \infty.$$

Thus f is in the domain of $(\mathscr{L}^{\Omega})^2$ and $RR^* = (\mathscr{L}^{\Omega})^2$, since $(\mathscr{L}^{\Omega})^2$ is a self-adjoint extension of RR^*, which is also self-adjoint by the von Neumann theorem (see e.g. [27, VIII, Problem 45]).

Consider the inverse operator R^{-1} acting on the image of R. For a function g in the image of R, $R^{-1}g$ is the unique extension of g to a holomorphic function on $T(\Omega)$. Thus $R^{-1}\mathscr{L}^{\Omega} = \mathscr{L}_\nu$. Let R^{-1} act on the previous equality (7.1)

$$R^* f = \mathscr{L}_\nu \mathscr{L}^{\Omega} f.$$

This proves the polar decomposition formula. Since R^* is densely defined and R is an injective closed operator we have that the unitary part $\mathscr{L}_\nu$ extends to a unitary operator. □

Remark 7.5. The multiplication map $f(x)\to\Delta(x)^{\frac{\nu}{2}}f(x)$ induces a unitary isomorphism between $L^2_\nu(\Omega)$ to $L^2(\Omega, d\mu_o)$. In turn, $L^2(\Omega, d\mu_o)$ is unitarily equivalent to $L^2(\mathscr{D}_{\mathbb{R}}, d\eta)$ via a scalar multiple of the Cayley transform. This is likewise true for $\mathscr{H}_\nu(T(\Omega))$ and $\mathscr{H}_\nu(\mathscr{D})$ (see Lemma 2.11). Furthermore, these isomorphisms intertwine the corresponding restriction maps. It follows then from Theorem 3.3 that R is a continuous operator for $\nu > a(r-1)$.

Corollary 7.6. *Assume that $\nu > 1 + a(r-1)$. Then $\mathscr{L}^*_\nu : \mathscr{H}_\nu(T(\Omega)) \to L^2_\nu(\Omega)$ is given by the integral operator*

$$\mathscr{L}^*_\nu F(x) = \alpha_\nu \int_{T(\Omega)} F(z) e^{-(\bar{z}|x)} \Delta(y)^{\nu - 2d/r}\, dx\, dy,$$

where $\alpha_\nu = \frac{2^{r\nu}}{(4\pi)^d \Gamma_\Omega(\nu - d/r)}$.

Proof. Write $\mathscr{L}$ for $\mathscr{L}_\nu$. Let $F \in \mathscr{H}_\nu(T(\Omega))$ and $f \in L^2_\nu(\Omega)$. Then

$$\begin{aligned}
(\mathscr{L}^* F \,|\, f) &= (F \,|\, \mathscr{L} f)_{\mathscr{H}_\nu(T(\Omega))} \\
&= \alpha_\nu \int_{T(\Omega)} F(z) \overline{\mathscr{L} f(z)} \Delta(y)^{\nu-2d/r}\, dx\, dy \\
&= \alpha_\nu \int_{T(\Omega)} F(z) \int_\Omega \overline{e^{-(z|t)} f(t)} \Delta(t)^{\nu - d/r} \Delta(y)^{\nu-2d/r}\, dt\, dx\, dy \\
&= \alpha_\nu \int_\Omega \int_{T(\Omega)} F(z) e^{-(\bar{z}|t)} \Delta(y)^{\nu-2d/r}\, dx\, dy\, \overline{f(t)}\, d\mu_\nu(t). \qquad \square
\end{aligned}$$

Using the Laplace transform we transfer the G^γ-action, π_ν, on $\mathscr{H}_\nu(T(\Omega))$ to an equivalent action, denoted by λ_ν, on $L^2_\nu(\Omega)$ and note the following simple fact:

Lemma 7.7. *The H^γ-action on $L^2_\nu(\Omega)$ is given by the formula*

$$\lambda_\nu(h)f(x) = \mathrm{Det}(h)^{\frac{\nu}{p}} f(h^t x),$$

where the determinant is taken as a real linear transformation on J and h^t denotes the transpose with respect to the real form $(x \mid y)$.

Proof. For $h \in H^\gamma$ a straightforward calculation gives that $J(h,z) = \mathrm{Det}(h)$ and $d\mu_\nu(hx) = \mathrm{Det}(h)^{\frac{\nu r}{d}} d\mu_\nu(x)$. Thus, for $f \in L^2_\nu(\Omega)$ we have

$$\begin{aligned}\pi_\nu(h)\mathscr{L}_\nu f(z) &= J(h^{-1}, z)^{\frac{\nu r}{2d}} \int_\Omega e^{-(h^{-1}z|x)} f(x)\, d\mu_\nu(x)\\ &= \mathrm{Det}(h)^{-\frac{\nu r}{2d}} \int_\Omega e^{-(z|x)} f(h^t x)\, d\mu_\nu(h^t x)\\ &= \mathrm{Det}(h)^{\frac{\nu r}{2d}} \mathscr{L}_\nu(f \circ h^t)(z).\end{aligned}$$

This calculation now implies the lemma. □

We follow [9, p. 343], and define the generalized Laguerre polynomials by the formula

$$L^\nu_{\mathbf{m}}(x) = (\nu)_{\mathbf{m}} \sum_{|\mathbf{n}| \leqslant |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \frac{1}{(\nu)_{\mathbf{n}}} \psi_{\mathbf{n}}(-x),$$

and the generalized Laguerre functions by

$$\ell^\nu_{\mathbf{m}}(x) = e^{-\mathrm{Tr}(x)} L^\nu_{\mathbf{m}}(2x).$$

By Proposition XV.4.2 in [9, p. 344], we get

Theorem 7.8. *The Laguerre functions form an orthogonal basis of $L^2_\nu(\Omega)^L$. Furthermore,*

$$\mathscr{L}_\nu(\ell^\nu_{\mathbf{m}}) = \Gamma_\Omega(\mathbf{m} + \nu) q^\nu_{\mathbf{m}}.$$

Let E be the Euler operator on Ω (or V). Specifically,

$$Ef(x) = \frac{d}{dt} f(tx)|_{t=1} = \frac{d}{dt} f(\exp(tZ_0) \cdot x)|_{t=0}.$$

We now obtain the following recursion formula.

Theorem 7.9. *The Laguerre functions are related by the following recursion relations*:

$$2E\ell^{\nu}_{\mathbf{m}} = -\nu r\ell^{\nu}_{\mathbf{m}} - \sum_{j=1}^{r} \binom{\mathbf{m}}{\mathbf{m}-\gamma_{\mathbf{j}}} \left(m_j - 1 + \nu - (j-1)\frac{d}{2}\right)\ell^{\nu}_{\mathbf{m}-\gamma_{\mathbf{j}}} + \sum_{j=1}^{r} c_{\mathbf{m}}(j)\ell^{\nu}_{\mathbf{m}+\gamma_{\mathbf{j}}}.$$

Proof. By Lemma 7.7 the H^{γ}-action is given by

$$\lambda_{\nu}(h)f(x) = \det(h)^{\frac{\nu r}{2d}}f(h^t x).$$

The infinitesimal action is then given in the usual way by differentiation. Thus by Lemma 2.2 $Z_0 \in \mathfrak{h}^{\gamma}$ and Z_0 acts on the smooth vectors in $L^2_{\nu}(\Omega)$ by

$$\lambda_{\nu}(Z_0)f(x) = \left(\frac{\nu r}{2} + E\right)f(x).$$

According to Lemma 5.5, part 2, we have

$$-\pi_{\nu}(2Z_0)q_{\mathbf{m},\nu} = \sum_{j=1}^{r} \binom{\mathbf{m}}{\mathbf{m}-\gamma_{\mathbf{j}}} q_{\mathbf{m}-\gamma_{\mathbf{j}},\nu} - \sum_{j=1}^{r} \left(\nu + \mathbf{m}_j - \frac{a}{2}(j-1)c_{\mathbf{m}}(j)\right) q_{\mathbf{m}+\gamma_{\mathbf{j}},\nu}.$$

The inverse Laplace transform of the above equation gives

$$-2\left(\frac{r\nu}{2} + E\right)\frac{\ell^{\nu}_{\mathbf{m}}}{\Gamma(\nu+\mathbf{m})} = \sum_{j=1}^{r} \binom{\mathbf{m}}{\mathbf{m}-\gamma_{\mathbf{j}}} \frac{\ell^{\nu}_{\mathbf{m}-\gamma_{\mathbf{j}}}}{\Gamma(\nu+\mathbf{m}-\gamma_{\mathbf{j}})}$$
$$- \sum_{j=1}^{r} \left(\nu + \mathbf{m}_j - \frac{d}{2}(j-1)c_{\mathbf{m}}(j)\right) \frac{\ell^{\nu}_{\mathbf{m}-\gamma_{\mathbf{j}}}}{\Gamma(\nu+\mathbf{m}+\gamma_{\mathbf{j}})}.$$

This simplifies to

$$-2\left(\frac{\nu r}{2} + E\right)\ell^{\nu}_{\mathbf{m}} = \sum_{j=1}^{r} \binom{\mathbf{m}}{\mathbf{m}-\gamma_{\mathbf{j}}} \left(m_j - 1 + \nu - (j-1)\frac{d}{2}\right)\ell^{\nu}_{\mathbf{m}-\gamma_{\mathbf{j}}} - \sum_{j=1}^{r} c_{\mathbf{m}}(j)\ell^{\nu}_{\mathbf{m}+\gamma_{\mathbf{j}}},$$

and proves the theorem. □

In the above Theorem we used the action of Z_0, but to directly derive a differential equation satisfied by the Laguerre functions one uses the element ξ and part 1 in Lemma 5.5:

Remark 7.10. If the Laguerre polynomials were defined by the formula

$${}^{\circ}L^{\nu}_{\mathbf{m}}(x) = \frac{(\nu)_{\mathbf{m}}}{\Gamma(\nu+\mathbf{m})} \sum_{|\mathbf{n}| \leqslant |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \frac{1}{(\nu)_{\mathbf{n}}} \psi_{\mathbf{n}}(-x) = \frac{L^{\nu}_{\mathbf{m}}(x)}{\Gamma(\nu+\mathbf{m})},$$

it would agree with our definition given in [5]. With this definition the formula in the proceeding theorem would become

$$2E^{\circ}\ell^{\nu}_{\mathbf{m}} = -\nu r^{\circ}\ell^{\nu}_{\mathbf{m}} - \sum_{j=1}^{r}\binom{\mathbf{m}}{\mathbf{m}-\gamma_{\mathbf{j}}}{}^{\circ}\ell^{\nu}_{\mathbf{m}-\gamma_{\mathbf{j}}} + \sum_{j=1}^{r}\left(m_j - 1 + \nu - (j-1)\frac{d}{2}\right)c_{\mathbf{m}}(j)^{\circ}\ell^{\nu}_{\mathbf{m}+\gamma_{\mathbf{j}}}.$$

Remark 7.11. Theorem 7.9 involves both a creation and an annihilation operator, i.e., it involves both a step up $\mathbf{n}\mapsto\mathbf{n}+\gamma_j$ and a step down $\mathbf{n}\mapsto\mathbf{n}-\gamma_j$. This is related to the fact that the element $Z_0\in\mathfrak{z}(\mathfrak{h})$ which is used to derive the relation in Theorem 7.9 has a decomposition $\xi = E_+ + E_-$ into H^{γ} invariant element where E_+ is in the $\mathfrak{p}^{\gamma+}$ and steps down and E_- is in $\mathfrak{p}^{\gamma-}$ and steps up. The elements E_+ and E_- are not in $\mathfrak{h}^{\gamma}_{\mathbb{C}}$ and hence they act as a *second-order* differential operator. In the case of the upper half-plane (in particular, see [5, Proposition 2.7, Theorem 3.4] where we are using $\mathbb{R}+i\mathbb{R}^+$ as a realization of the tube domain) this corresponds to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$\frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix} \leftrightarrow \frac{-i}{2}(tD^2 + (2t+\nu)D + (t+\nu))$$
$$\frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -1 \end{pmatrix} \leftrightarrow \frac{-i}{2}(tD^2 - (2t-\nu)D + (t-\nu))$$

on $i\mathbb{R}^+$. Furthermore, one can in a similar way get a differential equation satisfied by the Laguerre functions by applying the Cayley transform to part 1 in Lemma 5.5

$$\pi_\nu(\xi)p_{\mathbf{m},\nu} = (r\nu + 2|\mathbf{m}|)p_{\mathbf{m},\nu}.$$

This equation is also a second-order differential equation, which in the case of the upper half-plane is

$$(tD^2 + \nu D - t)\ell^{\nu}_n = -(\nu + 2n)\ell^{\nu}_n.$$

All of these equations can be carried over to the general case to find the radial part of the corresponding operators. We think that it is an interesting problem to find an explicit formula for the creation operator, annihilation operator and the operator $\pi_\nu(\xi)$, in the general case.

To this end, the paper of Ricci and Vignati [29] considers the cases of $SU(n,n)$ and $Sp(n,\mathbb{R})$ and derive a system of differential operators which are diagonal on the

Laguerre functions. The paper of Davidson and Ólafsson [4] consider the case of $SU(n,n)$. There the Lie algebraic action is derived from which the creation and annihilation operators are deduced. Further recursion relations that the Laguerre functions satisfy are thus produced.

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