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Abstract

Let G/H be a semisimple symmetric space. The main tool to embed a principal series representation of G into $L^2(G/H)$ are the H-invariant distribution vectors. If G/H is a non-compactly causal symmetric space, then G/H can be realized as a boundary component of the complex crown Ξ. In this article we construct a minimal G-invariant subdomain Ξ_H of Ξ with G/H as Shilov boundary. Let π be a spherical principal series representation of G. We show that the space of H-invariant distribution vectors of π, which admit a holomorphic extension to Ξ_H, is one dimensional. Furthermore we give a spectral definition of a Hardy space corresponding to those distribution vectors. In particular we achieve a geometric realization of a multiplicity free subspace of $L^2(G/H)_{\rm mc}$ in a space of holomorphic functions.

Introduction

Holomorphic extensions and boundary value maps have been valuable tools to solve problems in representation theory and harmonic analysis on *real* symmetric spaces. Two of the best known constructions are Hardy spaces with their boundary value maps and Cauchy-Szegö-kernels, and Fock space constructions with their corresponding Segal-Barmann transform. It is in this flavour that we establish a correspondence between eigenfunctions on a Riemannian symmetric spaces $X = G/K$ and a non-compactly causal (NCC) symmetric spaces $Y = G/H$ in this paper. In particular we, via analytic continuation, relate a *spherical function* φ_λ on G/K to a *holomorphic* H-invariant distribution on G/H.

Let us explain our results in more detail. On the geometric level we construct a certain minimal G-invariant Stein domain $\Xi_H \subseteq X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ with the following properties: The Riemannian symmetric space X is embedded into Ξ_H as a totally real submanifold and the affine non-compactly causal space Y is isomorphic to the distinguished (Shilov) boundary of Ξ_H. The details of this construction are carried out in Section 1.

The minimal tube Ξ_H is a subdomain of the complex crown $\Xi \subseteq X_{\mathbb{C}}$ of X – an object first introduced in [AG90] which became subject of intense study over the last few years. A

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consequence is that all $\mathbb{D}(X)$-eigenfunctions on X extend holomorphically to Ξ_H [KS01b]. Another key fact is that $\mathbb{D}(X) \simeq \mathbb{D}(Y)$. Thus by taking limits on the boundary Y we obtain a realization of the $\mathbb{D}(X)$-eigenfunctions on X as $\mathbb{D}(Y)$-eigenfunctions on Y. Conversely, eigenfunctions on Y which holomorphically extend to Ξ_H yield by restriction eigenfunctions on X.

It seems to us that the above mentioned transition between eigenfunctions on X and Y is most efficiently described using the techniques from representation theory. To fix the notation let $(\pi, \mathcal{H})$ denote an admissible Hilbert representation of G with finite length. We write $\mathcal{H}^K$ for the space of K-fixed vectors and $(\mathcal{H}^{-\infty})^H$ for the space of H-fixed distribution vectors of π. Using the method of analytic continuation of representations as developed in [KS01a] we establish a bijection

$$\mathcal{H}^K \xrightarrow{\simeq} (\mathcal{H}^{-\infty})^H_{\mathrm{hol}}, \quad v_{\mathrm{K}} \mapsto v_{\mathrm{H}}$$

where $(\mathcal{H}^{-\infty})^H_{\mathrm{hol}} \subseteq (\mathcal{H}^{-\infty})^H$ denotes the subspace characterized through the property that associated matrix coefficients on Y extend holomorphically to Ξ_H (cf. Theorem 2.1.3, Theorem 2.2.4). This bijection and various ramifications are the subject proper of Section 2.

In Section 3 we give an application of our theory towards the geometric realization of the most-continuous spectrum $L^2(Y)_{\mathrm{mc}}$ of $L^2(Y)$. First progress in this direction was achieved in [GKÓ01]. There, for the cases where $\Xi = \Xi_H$, we defined a Hardy space $\mathcal{H}^2(\Xi)$ on Ξ and showed that there is an isometric boundary value mapping realizing $\mathcal{H}^2(\Xi)$ as a multiplicity one subspace of $L^2(Y)_{\mathrm{mc}}$ of full spectrum. It was an open problem how to define Hardy spaces for general NCC symmetric spaces Y and to determine the Plancherel measure explicitely. We solve this problem by giving a spectral definition of the Hardy space, i.e., we take the conjectured Plancherel measure and define a Hilbert space of holomorphic functions $\mathcal{H}^2(\Xi_H)$ on Ξ_H. The identification of $\mathcal{H}^2(\Xi_H)$ as a Hardy space then follows by establishing an isometric boundary value mapping $b\colon \mathcal{H}^2(\Xi_H) \hookrightarrow L^2(G/H)_{\mathrm{mc}}$. In particular we achieve a geometric realization of a multiplicity free subspace of $L^2(Y)_{\mathrm{mc}}$ in holomorphic functions.

It is our pleasure to thank the referee for his very careful work. He pointed out many inaccuracies and made useful remarks on the presentation of the paper.

1. Complex crowns and the domains Ξ_H

The purpose of this section is to give the geometric preliminaries of the analytical constructions to come. Our two main players are a Riemannian symmetric space G/K on the one hand side and on the other hand a non-compactly causal symmetric space G/H. The two symmetric spaces G/K and G/H are "connected" through a complex G-invariant domain $\Xi_H \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}$ in the following way: G/K is a totally real submanifold and G/H constitutes the distinguished (Shilov) boundary of Ξ_H. The domain Ξ_H constructed in this section is an appropriate subdomain of the complex crown Ξ of the Riemannian symmetric space G/K.

This section is organized as follows. We start by briefly recalling the defintion and some standard features of non-compactly causal symmetric spaces . Then we switch to complex crowns Ξ and summarize the main results of [GK02a] on how to realize G/H in the distinguished boundary of Ξ. Finally we give the construction of the domain Ξ_H.

1.1. Non-compactly causal symmetric spaces (NCC)

In this subsection we recall some facts on non-compactly causal symmetric spaces. The material is standard and can be found in the monograph [HÓ96].

Let G be a connected semisimple Lie group and $\mathfrak{g}$ be its Lie algebra. Denote by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ the complexification of $\mathfrak{g}$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then we denote by $\mathfrak{h}_{\mathbb{C}}$ the complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{h}$. We assume that G is contained in a complex group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

If $\sigma: G \to G$ is an involution, then, by abuse of notation, we use the same letter for the derived involution on the Lie algebra $\mathfrak{g}$ and its complex linear extension to $\mathfrak{g}_{\mathbb{C}}$.

Let $\theta: G \to G$ be a Cartan involution and denote by $K < G$ the corresponding maximal compact subgroup. Let $\mathfrak{k} = \{X \in \mathfrak{g}: \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g}: \theta(X) = -X\}$. Then $\mathfrak{k}$ is the Lie algebra of K.

In the sequel we let τ denote an involution on G which we may assume to commute with θ. Let $G^\tau := \{g \in G: \tau(g) = g\}$ and let H be an open subgroup of G^τ. Then G/H is called a *symmetric spaces.* On the Lie algebra level τ induces a splitting $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ with $\mathfrak{h}$ the $+1$ and $\mathfrak{q}$ the -1-eigenspace of τ. Notice that $\mathfrak{h}$ is the Lie algebra of H. The pair $(\mathfrak{g}, \mathfrak{h})$ is called a *symmetric pair.* We have, as θ and τ commute:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{k} + \mathfrak{p}\\ &= \mathfrak{h} + \mathfrak{q}\\ &= \mathfrak{k}\cap\mathfrak{h} + \mathfrak{k}\cap\mathfrak{q} + \mathfrak{p}\cap\mathfrak{h} + \mathfrak{p}\cap\mathfrak{q}\end{aligned}$$

The symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is called *irreducible* if the only τ-invariant ideals in $\mathfrak{g}$ are the trivial ones, $\{0\}$ and $\mathfrak{g}$. In this case either $\mathfrak{g}$ is simple or $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{h}$, with $\mathfrak{h}$ simple, and $\tau(X, Y) = (Y, X)$ the flip. We say that the symmetric space G/H is *irreducible* if the corresponding symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is irreducible.

Let $\emptyset \neq C \subseteq \mathfrak{g}$ be an open subset of $\mathfrak{g}$. Then C is said to be *hyperbolic* if for all $X \in C$ the map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is semisimple with real eigenvalues.

Definition 1.1.1. **(NCC)** Assume that G/H is an irreducible symmetric space. Then the following two conditions are equivalent:

(a) There exists a non-empty H-invariant open hyperbolic convex cone $C \subseteq \mathfrak{q}$ which contains no affine lines;

(b) There exists an element $T_0 \in \mathfrak{q}\cap\mathfrak{p}$, $T_0 \neq 0$, which is fixed by $H \cap K$.

If one of those equivalent conditions are satisfied, then G/H is called *non-compactly causal*, or *NCC* for short. ■

Remark 1.1.2. (a) The element T_0 in Definition 1.1.1 is unique up to multiplication by scalar. We can normalize T_0 such that $\mathrm{ad}(T_0)$ has spectrum $\{0, 1, -1\}$. The eigenspace corresponding to 0 is exactly $\mathfrak{g}^{\theta\tau} = \mathfrak{k}\cap\mathfrak{h} + \mathfrak{p}\cap\mathfrak{q}$.
(b) If G/H is NCC and $\mathfrak{a} \subset \mathfrak{p}\cap\mathfrak{q}$ is maximal abelian, then $T_0 \in \mathfrak{a}$ by (a). Hence, again by (a), it follows that $\mathfrak{a}$ is also maximal abelian in $\mathfrak{p}$ and in $\mathfrak{q}$.
(c) Let T_0 be as above. Then the interior of the convex hull of $\mathbb{R}^+ \mathrm{Ad}(H)T_0$ is a minimal, H-invariant open hyperbolic convex cone in $\mathfrak{q}$.
(d) All the NCC pairs $(\mathfrak{g}, \mathfrak{h})$ are classified and we refer to [HÓ96, Th. 3.2.8] for the complete list. ■

1.2. The complex crown of a Riemannian symmetric space

The NCC spaces are exactly the affine symmetric spaces that can be realized as a symmetric subspace in the distinguished boundary of the complex crown Ξ of the Riemannian symmetric space G/K. We will therefore recall some basic facts about Ξ. We refer to [GK02a] and [GK02b] as a standard source.

Let the notation be as in Subsection 1.1. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra. For $\alpha \in \mathfrak{a}^*$ let $\mathfrak{g}^\alpha = \{X \in \mathfrak{g}\colon (\forall H \in \mathfrak{a})\ [H, X] = \alpha(H)X\}$ and let $\Sigma := \{\alpha \in \mathfrak{a}^*\colon \alpha \neq 0, \mathfrak{g}^\alpha \neq \{0\}\}$ be the corresponding set of restricted roots.

Following [AG90] we define a bounded convex subset of $\mathfrak{a}$ by

$$\Omega = \{X \in \mathfrak{a}\colon (\forall \alpha \in \Sigma)\ |\alpha(X)| < \frac{\pi}{2}\} .$$

Denote by $K_\mathbb{C}$ the analytic subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{k}_\mathbb{C}$. Then we define a $G - K_\mathbb{C}$ double coset domain in $G_\mathbb{C}$ by

$$\widetilde{\Xi} = G\exp(i\Omega)K_\mathbb{C}$$

and recall that $\widetilde{\Xi}$ is open in $G_\mathbb{C}$ [KS01a]. In particular the domain

$$\Xi = \widetilde{\Xi}/K_\mathbb{C}$$

is an open G-invariant subset of $G_\mathbb{C}/K_\mathbb{C}$ containing G/K as a totally real submanifold. We refer to Ξ as the *complex crown* of the Riemannian symmetric space G/K (cf. [AG90]). Observe that the definition of Ξ and $\widetilde{\Xi}$ is independent of the choice of $\mathfrak{a} \subseteq \mathfrak{p}$. For a subset $\omega \subseteq \mathfrak{a}$ we define a tube domain in $A_\mathbb{C} = \exp(\mathfrak{a}_\mathbb{C})$ by

$$T(\omega) = A\exp(i\omega)$$

and notice that $T(2\Omega)$ is biholomorphic to $\mathfrak{a} + i2\Omega$ via the exponential map.

Fix a positive system Σ^+ of Σ and define a subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ by

$$\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha .$$

Write $N_\mathbb{C}$ for the analytic subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{n}_\mathbb{C}$. Then it follows from [GK02b] that

$$\widetilde{\Xi} \subseteq N_\mathbb{C} T(\Omega) K_\mathbb{C} \tag{1.2.1}$$

or even more precisely

$$\widetilde{\Xi} = \left[\bigcap_{g\in G} gN_\mathbb{C} T(\Omega) K_\mathbb{C}\right]_0 \tag{1.2.2}$$

where the subscript $_0$ denotes the connected component of $[\cdot]$ containing G.

1.3. The distinguished boundary of Ξ

Write $\overline{\Xi}$ and $\partial\Xi$ for the closure respectively the boundary of Ξ in $G_{\mathbb{C}}/K_{\mathbb{C}}$. The *distinguished boundary* of Ξ is a certain finite union of G-orbits in $\partial\Xi$ which features many properties of a Shilov boundary. It was introduced and investigated in [GK02a] and the objective of this subsection is to recall its definition and basic properties.

Let $\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the Weyl group of $\mathfrak{a}$ in G. Write $\overline{\Omega}$ for the closure of Ω and notice that $\overline{\Omega}$ is a $\mathcal{W}$-invariant compact convex set. Denote by $\partial_e\Omega$ the set of extreme points of $\overline{\Omega}$. Then there exists $X_1, \ldots, X_n \in \partial_e\Omega$ such that

$$\partial_e\Omega = \mathcal{W}(X_1) \amalg \ldots \amalg \mathcal{W}(X_n) .$$

Define the *distinguished boundary* of Ξ in $G_{\mathbb{C}}/K_{\mathbb{C}}$ by

$$\partial_d\Xi := G\exp(i\partial_e\Omega)K_{\mathbb{C}}/K_{\mathbb{C}} .$$

We refer to [GK02a] for detailed information about $\partial_d\Xi$ and recall here only the facts that we need.

For $1 \leq j \leq n$ let $z_j := \exp(iX_j)K_{\mathbb{C}} \in \partial_d\Xi$. If $G_{\mathbb{C}}$ is not simply connected it can happen that $G(z_j) = G(z_k)$ for some $j \neq k$. But after relabelling the z_j we can assume that there is an $m \leq n$ such that $G(z_j) \neq G(z_k)$ for $1 \leq j, k \leq m$, $j \neq k$, and

$$\partial_d\Xi = G(z_1) \amalg \ldots \amalg G(z_m) .$$

Denote by H_j the isotropy subgroup of G in z_j. Then as G-spaces we have

$$\partial_d\Xi = G/H_1 \amalg \ldots \amalg G/H_m .$$

As a consequence of the complete classification of $\partial_d\Xi$ in [GK02a] we obtain the following fact.

Proposition 1.3.1. *For the distinguished boundary $\partial_d\Xi$ of Ξ the following assertions hold:*

(i) *If one of the boundary components G/H_j of $\partial_d\Xi$ is a symmetric space, then it is a non-compactly causal symmetric space.*

(ii) *Every non-compactly causal symmetric space of the form G/H is locally isomorphic to a G-orbit in the distinguished boundary of $\partial_d\Xi$ of Ξ.* ■

1.4. The domains Ξ_H

We keep the notation from Subsections 1.2 - 1.3. From now on we fix an element $X_{\mathrm{H}} \in \partial_e\Omega$ and set $x_{\mathrm{H}} = \exp(iX_{\mathrm{H}})$, $z_{\mathrm{H}} = x_{\mathrm{H}}K_{\mathbb{C}}$. As the notation suggests we denote by $H < G$ the stabilizer of $z_{\mathrm{H}} \in \partial_d\Xi$ in G.

For the rest of this paper we will employ the following assumptions: $G_{\mathbb{C}}$ is simply connected and $G/H \simeq G(z_{\mathrm{H}})$ is an NCC symmetric space (cf. Proposition 1.3.1). Notice that this implies in particular $H = G^\tau$. Recall the element T_0 from Definition 1.1.1 and notice that we have (up to sign) $X_{\mathrm{H}} = \frac{\pi}{2}T_0$.

We define a domain $\Omega_{\mathrm{H}} \subseteq \Omega$ by

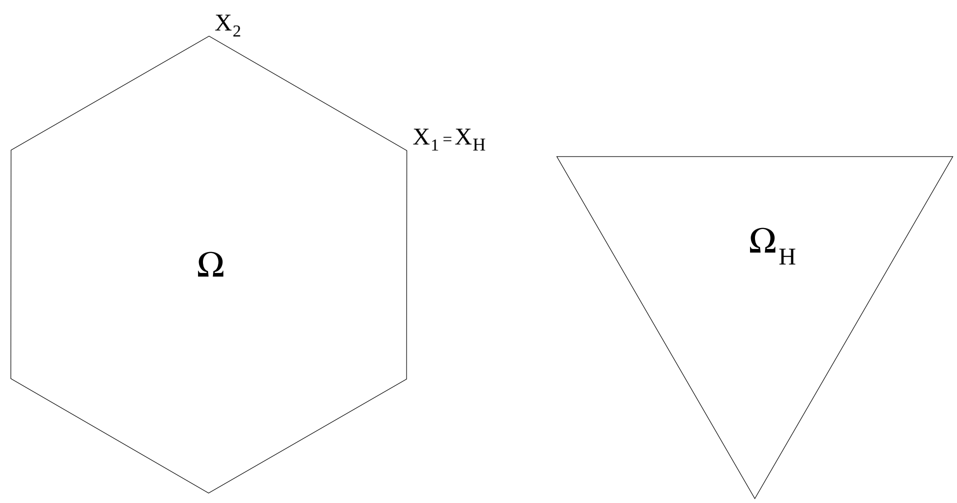
$$\Omega_{\mathrm{H}} = \mathrm{int}\left(\mathrm{conv}\{\mathcal{W}(X_{\mathrm{H}})\}\right) .$$

Here conv$\{\cdot\}$ denotes the convex hull of $\{\cdot\}$ and int$(\cdot)$ denotes the interior of $(\cdot)$. From the definition we immediately obtain that:

(1.4.1) $\Omega_{\rm H}$ is open in $\mathfrak{a}$.
(1.4.2) $0 \in \Omega_{\rm H}$ (because $X_{\rm H} \neq 0$ and $\mathcal{W}(X_{\rm H})$ meets every Weyl chamber).
(1.4.3) The set of extremal points of $\overline{\Omega_{\rm H}}$ is $\mathcal{W}(X_{\rm H})$.

Let us illustrate the geometry for one example.

Example 1.4.1. Let $G = \mathrm{Sl}(3,\mathbb{R})$. Then $\mathfrak{a}$ is two-dimensional and Σ is a root system of type A_2. We have $\partial_e\Omega = \mathcal{W}(X_1) \amalg \mathcal{W}(X_2)$ and the corresponding isotropy subgroups are given by $H_1 = \mathrm{SO}(1,2)$ and $H_2 = \mathrm{SO}(2,1)$. With $H = H_1$ the geometry of Ω and Ω_H is depicted as follows:



■

Let us define a domain $\Xi_H \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}$ by

$$\Xi_H = G\exp(i\Omega_{\rm H})K_{\mathbb{C}}/K_{\mathbb{C}}\,.$$

The domain Ξ_H has the following properties:

(1.4.4) Ξ_H is G-invariant (clear from the definition).
(1.4.5) Ξ_H is open in $G_{\mathbb{C}}/K_{\mathbb{C}}$ (follows from (1.4.1) and [AG90]).
(1.4.6) $G/K \subseteq \Xi_H$ is a totally real submanifold (follows from (1.4.2) and (1.4.5)).
(1.4.7) Ξ_H is Stein (follows from the convexity of $\Omega_{\rm H}$ and [GK02b]).
(1.4.8) $\Xi_H \subseteq \Xi$ (because $\Omega_{\rm H} \subseteq \Omega$).
(1.4.9) $\Xi_H = \Xi$ iff Σ is of type C_n (cf. [KS01b]).

Write $\overline{\Xi_H}$ for the closure of Ξ_H in $G_{\mathbb{C}}/K_{\mathbb{C}}$ and define the *distinguished boundary* of Ξ_H by

$$\partial_d\Xi_H = G(z_{\rm H})\,.$$

Notice that $\partial_d\Xi_H \simeq G/H$ as G-spaces. Let us remark further that $\partial_d\Xi_H \subseteq \partial_d\Xi$.

The distinguished boundary $\partial_d \Xi_H$ of Ξ_H can be considered as some sort of Shilov boundary of Ξ_H. More precisely, mimicking the argument in [GK02a, Th. 2.3] we obtain that

$$\sup_{z\in\Xi_H} |f(z)| = \sup_{z\in\partial_d \Xi_H|} |f(z)| \tag{1.4.10}$$

for all bounded holomorphic functions f on Ξ_H which continuously extend to $\overline{\Xi_H}$.

2. Holomorphic H-spherical distributions

In the section we assume that G/H is NCC symmetric space realized as $G(z_{\rm H})$ in the distinguished boundary of Ξ.

Recall that a representation (π, V) of G is called *admissible* if the multiplicity of each K-type is finite and of *finite length* if the associated Harish-Chandra module of K-finite vectors V_K is of finite length. Our aim in this section is to associate to a non-zero K-fixed vector $v_{\rm K}$ in an admissible representation (π, V) of finite length a certain canonical H-spherical distribution vector $v_{\rm H} \in (V^{-\infty})^H$. For irreducible representations π the vector $v_{\rm H}$ is unique in the sense that it allows analytic continuation of generalized matrix coefficients on G/H to holomorphic functions on Ξ_H.

We let $X_{\rm H}, x_{\rm H}$, and $z_{\rm H}$ be as in the last subsection and recall that $x_{\rm H}^{-1} H_{\mathbb{C}} x_{\rm H} = K_{\mathbb{C}}$.

General Remark: In all results of this section which involve the domain Ξ_H one can replace Ξ_H by the bigger domain Ξ. This holds in particular for the results in Subsection 2.2.

2.1. The definition of the holomorphic H-spherical distribution vector

Before we discuss the general case let us assume for the moment that V is irreducible and finite dimensional. Then (π, V) extends to a holomorphic representation of $G_{\mathbb{C}}$ which we also denote by (π, V). If $L < G$ is a subgroup of G then we write V^L for the subspace of V fixed by L. Then the mapping

$$V^K \to V^H, \quad v_{\rm K} \mapsto v_{\rm H} := \pi(x_{\rm H}) v_{\rm K}$$

sets up an isomorphism between the K-spherical and H-spherical vectors of V. The obvious problem in the general case is, that $\pi(x_{\rm H})v$ is not necessarily defined as an element in V.

We are now going to develop an appropriate generalization of the mapping $v_{\rm K} \mapsto v_{\rm H}$ for an admissible finite length representation $(\pi, \mathcal{H})$ of G in a Hilbert space $\mathcal{H}$. Denote by $\mathcal{H}^\infty$ respectively $\mathcal{H}^\omega$ the space of smooth respectively analytic vectors in $\mathcal{H}$. Their strong anti-duals, i.e., the space of continuous conjugate linear maps into $\mathbb{C}$, are denoted by $\mathcal{H}^{-\infty}$ respectively $\mathcal{H}^{-\omega}$ and referred to as the G-modules of *distribution vectors*, respectively *hyperfunction vectors* of $(\pi, \mathcal{H})$. Notice the chain of continuous inclusions $\mathcal{H}^\omega \hookrightarrow \mathcal{H}^\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\infty} \hookrightarrow \mathcal{H}^{-\omega}$. Here the inclusion $\mathcal{H} \hookrightarrow \mathcal{H}^{-\infty}$ is the natural one, $v \mapsto (u \mapsto \langle v, u\rangle)$. We will also use the notation $u \mapsto \langle v, u\rangle$ for $v \in \mathcal{H}^{-\infty}$. Define a representation π^0 of G on $\mathcal{H}$ by

$$\begin{aligned}\langle \pi^0(g)v, u\rangle &= \langle \pi(g^{-1})^* v, u\rangle \\ &= \langle v, \pi(g^{-1})u\rangle\end{aligned} \qquad (g \in G;\ u, v \in \mathcal{H}) .$$

Hence the natural representation $\pi^{-\infty}$ of G on $\mathcal{H}^{-\infty}$ is an extension of the representation π^0 on $\mathcal{H}$ to $\mathcal{H}^{-\infty}$. The representation π^0 is called the *conjugate dual representation* of π. Notice that $(\pi^0, \mathcal{H})$ is admissible and of finite length if and only if the same holds for $(\pi, \mathcal{H})$. Note also that $\pi = \pi^0$ if and only if π is unitary. Notice that if $u, v \in \mathcal{H}$ then $\langle v, u\rangle = \overline{\langle u, v\rangle}$. Accordingly, if $u \in \mathcal{H}^\infty$ and $v \in \mathcal{H}^{-\infty}$, then we write

$$\langle u, v\rangle := \overline{\langle v, u\rangle}\,.$$

Denote by $\mathcal{H}_K$ the $(\mathfrak{g}, K)$-module of K-finite vectors in $\mathcal{H}$. Note that by our assumption that $\mathcal{H}$ is admissible and of finite length it follows that $\mathcal{H}_K \subseteq \mathcal{H}^\omega$. But usually we cannot find H-fixed vectors in $\mathcal{H}$ but only in the larger space of distribution vectors. Another complication arises as the space of $(\mathcal{H}^{-\infty})^H$ of H-invariants in $\mathcal{H}^{-\infty}$ is finite dimensional but in general not one-dimensional. For "generic" principal series representations of G one has

$$\dim(\mathcal{H}^{-\infty})^H = |\mathcal{W}/\mathcal{W}_0|$$

with $\mathcal{W}_0 = N_{K\cap H}(\mathfrak{a})/Z_{K\cap H}(\mathfrak{a})$ the *little Weyl group.* Our correspondence will be that we associate to a K-fixed vector $v_{\rm K}$ a unique H-fixed distribution vector $v_{\rm H}$. If π is irreducible, then this distribution $v_{\rm H}$ is the (up to scalar) unique element of $(\mathcal{H}^{-\infty})^H$ for which generalized matrix coefficients extend to holomorphic functions from G/H to Ξ. For the proof we will need the following *Automatic Continuity* Theorem of van den Ban, Brylinski and Delorme (c.f. [vdBD88, Th. 2.1] and [BD92, Th. 1] for the version used here).

If V is a complex vector space, then let us denote by $V^{\overline{*}}$ its algebraic anti-dual. Set $(\mathcal{H}^{-\infty})^{\mathfrak{h}} = \{v \in \mathcal{H}^{-\infty}\colon (\forall X \in \mathfrak{h})\, d\pi^{-\infty}(X)v = 0\}$.

Theorem 2.1.1. **(Automatic Continuity)** *Let $(\pi, \mathcal{H})$ be an admissible representation of G with finite length. Then*

$$(\mathcal{H}^{-\infty})^{H_0} \simeq (\mathcal{H}_K^{\overline{*}})^{\mathfrak{h}}\ , \tag{2.1.1}$$

meaning every $\mathfrak{h}$-fixed anti-linear functional on $\mathcal{H}_K$ admits a unique extension to a continuous and H_0-fixed anti-linear functional on $\mathcal{H}^\infty$. In particular we have that

$$(\mathcal{H}^{-\infty})^{H_0} \simeq (\mathcal{H}^{-\omega})^{H_0}\ . \tag{2.1.2}$$

■

Recall the $G - K_{\mathbb{C}}$ double coset domain $\widetilde{\Xi} = G\exp(i\Omega)K_{\mathbb{C}}$ in $G_{\mathbb{C}}$ and the complex crown $\Xi = \widetilde{\Xi}/K_{\mathbb{C}}$. Our methods use holomorphic extensions of representations. We recall therefore some results from [KS01a] §4 and in particular Theorem 3.1:

Theorem 2.1.2. *Let $(\pi, \mathcal{H})$ be an admissible Hilbert representation of G with finite length. Then the following assertions hold:*

(i) *For every $v \in \mathcal{H}_K$ the orbit mapping $G \to \mathcal{H}$, $g \mapsto \pi(g)v$ extends to a G-equivariant holomorphic mapping $\widetilde{\Xi} \to \mathcal{H}$.*

(ii) *Let $v, w \in \mathcal{H}_K$. Then the restricted matrix coefficient $A \to \mathbb{C}$, $a \mapsto \langle \pi(a)v, w\rangle$ extends to a holomorphic mapping to the abelian tube domain $T(2\Omega) = A\exp(2i\Omega)$.* ■

We will from now on assume that $\pi|_K$ is unitary. This is no restriction in view of Weyl's unitarity trick.

As before we identify G/H with the subset $G(z_{\rm H})$ of $\overline{\Xi_H}$. For $0 \leq t < 1$ we set $a_t = \exp(itX_{\rm H}) \in \exp(i\Omega)$ and notice that $\lim_{t\to 1} a_t = x_{\rm H}$.

Theorem 2.1.3. *Let $(\pi, \mathcal{H})$ be a K-spherical admissible representation of G with finite length. Let $v_K \in \mathcal{H}^K$ be non-zero. Then the anti-linear functional*

$$v_{\mathrm{H}}^{\omega}: \mathcal{H}^{\omega} \to \mathbb{C}, \quad v \mapsto \lim_{t \nearrow 1} \langle \pi^0(a_t) v_{\mathrm{K}}, v \rangle$$

is well defined, non-zero and admits a non-trivial extension to an H-fixed distribution vector of $(\pi, \mathcal{H})$. ■

Note: If $(\pi, \mathcal{H})$ is unitary, then $\pi^0(a_t) = \pi(a_t)$ and so $v_{\mathrm{H}}^{\omega}(v) = \lim_{t \nearrow 1} \langle \pi(a_t) v_{\mathrm{K}}, v \rangle$ for all $v \in \mathcal{H}^{\omega}$.

Proof. We first show that v_{H}^{ω} is well defined. Let $v \in \mathcal{H}^{\omega}$. As v is analytic, we find an $0 < \varepsilon < 1$ such that $\pi(a_\varepsilon) v$ is defined. Notice that $\pi^0(a_{1-\varepsilon}) v_{\mathrm{K}}$ is defined by Theorem 2.1.2 (i). Then we have for all $\varepsilon < t < 1$ that

$$\langle \pi^0(a_t) v_{\mathrm{K}}, v \rangle = \langle \pi^0(a_{t-\varepsilon}) v_{\mathrm{K}}, \pi(a_\varepsilon) v \rangle$$

and hence

$$\lim_{t \nearrow 1} \langle \pi^0(a_t) v_{\mathrm{K}}, v \rangle = \langle \pi^0(a_{1-\varepsilon}) v_{\mathrm{K}}, \pi(a_\varepsilon) v \rangle \ .$$

Thus v_{H}^{ω} is defined.

Next we show that v_{H}^{ω} is fixed by H. As G/H is NCC, it follows that $H = H_0 Z_{H \cap K}(\mathfrak{a})$ [HÓ96, p.79]. Further it is clear from the definition that v_{H}^{ω} it is fixed by $Z_{H \cap K}(\mathfrak{a})$. Hence it is enough to prove that v_{H}^{ω} is H_0-fixed, i.e. annihilated by $\mathfrak{h}$. For that let $Y \in \mathfrak{h}$ and $v \in \mathcal{H}^{\omega}$. Then

$$\begin{aligned} v_{\mathrm{H}}^{\omega}(d\pi(Y) v) &= \lim_{t \nearrow 1} \langle \pi^0(a_t) v_{\mathrm{K}}, d\pi(Y) v \rangle = - \lim_{t \nearrow 1} \langle d\pi^0(Y) \pi^0(a_t) v_{\mathrm{K}}, v \rangle \\ &= - \lim_{t \nearrow 1} \langle \pi^0(a_t) d\pi^0(\mathrm{Ad}(a_t)^{-1} Y) v_{\mathrm{K}}, v \rangle \\ &= - \lim_{t \nearrow 1} \langle \pi^0(a_{t-\varepsilon}) d\pi^0(\mathrm{Ad}(a_t)^{-1} Y) v_{\mathrm{K}}, \pi(a_\varepsilon) v \rangle \ . \end{aligned}$$

From Theorem 2.1.2 (i) we now obtain that

$$\pi^0(a_{t-\varepsilon}) d\pi^0(\mathrm{Ad}(a_t)^{-1} Y) v_{\mathrm{K}} \to \pi^0(a_{1-\varepsilon}) d\pi^0(\mathrm{Ad}(x_{\mathrm{H}})^{-1} Y) v_{\mathrm{K}} \ .$$

But $\mathrm{Ad}(x_{\mathrm{H}})^{-1} Y \in \mathfrak{k}_{\mathbb{C}}$ and hence $d\pi^0(\mathrm{Ad}(x_{\mathrm{H}})^{-1} Y) v_{\mathrm{K}} = 0$. We thus get:

$$\begin{aligned} v_{\mathrm{H}}^{\omega}(d\pi(Y) v) &= - \lim_{t \nearrow 1} \langle \pi^0(a_{t-\varepsilon}) d\pi^0(\mathrm{Ad}(a_t)^{-1} Y) v_{\mathrm{K}}, \pi(a_\varepsilon) v \rangle \\ &= - \langle \pi^0(a_{1-\varepsilon}) d\pi^0(\mathrm{Ad}(x_{\mathrm{H}})^{-1} Y) v_{\mathrm{K}}, \pi(a_\varepsilon) v \rangle \\ &= 0 \, . \end{aligned}$$

Finally, let us show that $v_{\mathrm{H}}^{\omega} \neq 0$. Suppose the contrary, i.e. $v_{\mathrm{H}}^{\omega} = 0$. Then it follows that

$$(\forall u \in \mathcal{H}_K) \qquad \langle v_{\mathrm{H}}^{\omega}, u \rangle = 0 \ . \tag{2.1.3}$$

Now consider the function

$$f: \mathbb{R} + i]-2, 2[\to \mathbb{C}, \quad z \mapsto \langle \pi^0(\exp(z X_{\mathrm{H}})) v_{\mathrm{K}}, v_{\mathrm{K}} \rangle \ .$$

According to Theorem 2.1.2 (ii) the function f is well defined and holomorphic. It is clear that $f \not\equiv 0$ as $f(0) = \langle v_{\mathrm{K}}, v_{\mathrm{K}} \rangle > 0$. But (2.1.3) implies that $f^{(n)}(i) = 0$ for all $n \in \mathbb{N}_0$; a contradiction to $f \not\equiv 0$. ■

In the sequel we write v_{H} for the H-fixed distribution vector obtained from v_{H}^{ω}. We will call v_{H} the *holomorphic H-spherical distribution vector* of $(\pi, \mathcal{H})$ corresponding to v_{K}.

Remark 2.1.4. By Theorem 2.1.3 we have

$$v_{\mathrm{H}}^{\omega} = \mathrm{w} - \lim_{t \nearrow 1} \pi^0(a_t) v_{\mathrm{K}},$$

i.e. v_{H}^{ω} is the weak limit of $\pi^0(a_t)v_{\mathrm{K}}$ for $t \to 1$ in the locally convex space $\mathcal{H}^{-\omega}$. It is possible to strengthen this convergence: Let $v \in \mathcal{H}^{\omega}$ and $C \subseteq G$ a compact subset. Then for all $\varepsilon > 0$ there exists $0 < s < 1$ such that for all $0 < s < t < 1$:

$$\sup_{g \in C} |\langle v_{\mathrm{H}}^{\omega}, \pi(g)v\rangle - \langle \pi^0(a_t)v_{\mathrm{K}}, \pi(g)v\rangle| < \varepsilon \,. \tag{2.1.4}$$

In fact this follows from a simple modification of the first part of the proof of Theorem 2.1.3: we only have to observe that for $v \in \mathcal{H}^{\omega}$ there exists a $0 < \delta < 1$ such that $\pi(a_\delta)\pi(g)v$ exists for all $g \in C$.

Supported by calculations in the rank one case we conjecture that one actually has $\pi(a_t)v_{\mathrm{K}} \to v_{\mathrm{H}}$ in $\mathcal{H}^{-\infty}$ weakly (and hence strongly by the Banach-Steinhaus Theorem which applies as $\mathcal{H}^{\infty}$ is a Fréchet space). ∎

Let us illustrate the situation by the discussion of one example.

Example 2.1.5. Here we will determine an explicit analytic description of v_{H} for unitary principal series of $G = \mathrm{Sl}(2,\mathbb{R})$. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ denote a unitary spherical principal series of G with parameter $\lambda \in i\mathfrak{a}^*$. Then $\pi_\lambda^0 = \pi_\lambda$ as π_λ is unitary. In the sequel we will identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ in such a way that $\rho \in \mathfrak{a}^*$ corresponds to 1. With our choice of $\mathfrak{a}$ to be

$$\mathfrak{a} = \left\{ \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} : s \in \mathbb{R} \right\}$$

this identification is given by

$$\lambda \mapsto \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

We will use the noncompact realization of $\mathcal{H}_\lambda = L^2(\mathbb{R})$ of π_λ. Then for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ the operator $\pi_\lambda(g)$ is given by

$$(\pi_\lambda(g)f)(x) = |bx+d|^{-1-\overline{\lambda}} f\left(\frac{ax+c}{bx+d}\right) \qquad (f \in L^2(\mathbb{R}), x \in \mathbb{R}) \,.$$

A normalized K-spherical vector is then given by

$$v_{\mathrm{K}}(x) = \frac{1}{\sqrt{\pi}}(1+x^2)^{-\frac{1}{2}(1+\overline{\lambda})} \,.$$

Notice that $H = \mathrm{SO}(1,1)$ and (up to sign) we have $X_{\mathrm{H}} = \begin{pmatrix} \frac{\pi}{4} & 0 \\ 0 & -\frac{\pi}{4} \end{pmatrix}$. Thus for $0 \leq t < 1$ the element $a_t \in \exp(i\Omega)$ is given by

$$a_t = \begin{pmatrix} e^{i\frac{\pi}{4}t} & 0 \\ 0 & e^{-i\frac{\pi}{4}t} \end{pmatrix} .$$

Then we have

$$v_{\mathrm{H}}^{\omega} = \mathrm{w} - \lim_{t \nearrow 1} \pi_\lambda(a_t) v_{\mathrm{K}}$$

or

$$v_{\mathrm{H}}^{\omega}(x) = \mathrm{w} - \lim_{t \nearrow 1} \frac{e^{i\frac{\pi}{4}t(1+\overline{\lambda})}}{\sqrt{\pi}}(1+e^{i\pi t}x^2)^{-\frac{1}{2}(1+\overline{\lambda})} \; .$$

A simple calculation then shows that v_{H}^{ω} and v_{H} are given by the locally integrable function

$$v_{\mathrm{H}}(x) = \begin{cases} \frac{e^{i\frac{\pi}{4}(1+\overline{\lambda})}}{\sqrt{\pi}}(1-x^2)^{-\frac{1}{2}(1+\overline{\lambda})} & \text{for } |x|<1, \\ 0 & \text{for } |x|=1, \\ \frac{e^{-i\frac{\pi}{4}(1+\overline{\lambda})}}{\sqrt{\pi}}(x^2-1)^{-\frac{1}{2}(1+\overline{\lambda})} & \text{for } |x|>1 \; . \end{cases}$$

A basis of $(\mathcal{H}_{\lambda}^{-\infty})^H$ is given by $v_{\mathrm{H},1}, v_{\mathrm{H},\,2}$ where

$$v_{\mathrm{H},1}(x) = \begin{cases} \frac{1}{\sqrt{\pi}}(1-x^2)^{-\frac{1}{2}(1+\overline{\lambda})} & \text{for } |x|<1, \\ 0 & \text{for } |x|\geq 1. \end{cases}$$

and

$$v_{\mathrm{H},2}(x) = \begin{cases} \frac{1}{\sqrt{\pi}}(x^2-1)^{-\frac{1}{2}(1+\overline{\lambda})} & \text{for } |x|>1, \\ 0 & \text{for } |x|\leq 1. \end{cases}$$

These two basis vectors are chosen such that they have support in the open H-orbits on $\mathbb{P}^1(\mathbb{R})$, namely $]-1,1[$ and $\mathbb{P}^1(\mathbb{R})\backslash[-1,1]$. Notice that v_{H} is a non-trivial linear combination of $v_{\mathrm{H},\,1}$ and $v_{\mathrm{H},2}$ and that v_{H} has full support on $\mathbb{R}$. Another interesting feature of this example is that we have here

$$v_{\mathrm{H}} = \mathrm{w} - \lim_{t \nearrow 1} \pi_{\lambda}(a_t)v_{\mathrm{K}} \qquad \text{in } \mathcal{H}_{\lambda}^{-\infty}$$

and hence also strongly by the Banach-Steinhaus Theorem (compare with the conjecture stated at the end of Remark 2.1.4) . ∎

2.2. Holomorphic extension of matrix coefficienst on G/H to Ξ_H

In this subsection we clarify the role of the holomorphic distribution vector v_{H} in view of holomorphic extensions of matrix coefficients from G/H to Ξ. If U is a complex manifold, then we denote by $\mathcal{O}(U)$ the space of holomorphic functions $f : U \to \mathbb{C}$.

Definition 2.2.1. Let f be a continuous function on G/H. Then we say that f has a *holomorphic extension* to Ξ_H if there exists an $\widetilde{f} \in \mathcal{O}(\Xi_H)$ such that for all compact subsets $C \subseteq G$ one has

$$\lim_{t \nearrow 1} \sup\nolimits_{g\in C} |f(gH) - \widetilde{f}(ga_tK_{\mathbb{C}})| = 0 \; . \tag{2.2.1}$$

∎

Notice that (2.2.1) implies that

$$f(gH) = \lim_{t \nearrow 1} \widetilde{f}(ga_tK_{\mathbb{C}})$$

for all $g \in G$. Furthermore the holomorphic extensions are unique by the following lemma:

Lemma 2.2.2. (Identity Theorem for holomorphic extensions) *Let $f \in C(G/H)$ and assume that f has a holomorphic extension $\widetilde{f} \in \mathcal{O}(\Xi_H)$. Then $f \equiv 0$ implies $\widetilde{f} \equiv 0$. In particular, the holomorphic extension $f \in C(G/H)$ is unique if it exists.*

Proof. This is easily reduced to the one-dimensional case as follows. Define an abelian tube domain $T = \exp(\mathbb{R}X_{\rm H} +]-1,1[iX_{\rm H})$ and set $\partial_s T = \exp(\mathbb{R}X_{\rm H} + iX_{\rm H})$. We realize $T \subseteq \Xi_H$ and $\partial_s T \subseteq G/H$ through the T-orbit , respectively $\partial_s T$-orbit, through $K_{\mathbb{C}} \in \Xi_H$, respectively $z_{\rm H} \in G/H$. Let $f \in C(G/H)$ and assume that f has a holomorphic extension $\widetilde{f}$. Let $\varphi = f|_{\partial_s T}$. Then φ has a holomorphic extension to T given by $\widetilde{\varphi} = \widetilde{f}|_T$. By the well known one-dimensional situation we have $\widetilde{\varphi} \equiv 0$ if $\varphi \equiv 0$. Thus if $f \equiv 0$, we obtain $\widetilde{f}|_T \equiv 0$. Replacing f by f_g where $f_g(xH) = f(gxH)$ for $g \in G$, then the above discussion implies that $\widetilde{f}|_{GT} \equiv 0$ and hence $\widetilde{f} \equiv 0$ as GT contains the totally real submanifold G/K of Ξ_H. ∎

We assume that the complex conjugation $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}},\ X \mapsto \overline{X}$ with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ lifts to a conjugation $g \mapsto \overline{g}$ of $G_{\mathbb{C}}$. Notice that this is always satisfied if $G_{\mathbb{C}}$ is the universal complexification of G. Notice also that $\overline{x} \in \widetilde{\Xi}$ for all $x \in \widetilde{\Xi}$. Let $u, v \in \mathcal{H}_K$. Then the functions

$$\widetilde{\Xi} \ni x \mapsto \langle \pi^0(x)u, v\rangle, \langle u, \pi(\overline{x}^{-1})v\rangle \in \mathbb{C}$$

are well defined and holomorphic by Theorem 2.1.2. Both of them agree if $x \in G$ and hence they agree on all of $\widetilde{\Xi}$.

Proposition 2.2.3. *Let $(\pi, \mathcal{H})$ be a admissible K-spherical representation of G with finite length. Let $v \in \mathcal{H}^\omega$. Then the following assertions hold:*

(i) *The matrix coefficient*

$$f_{v,v_{\rm H}}: G/H \to \mathbb{C}, \quad gH \mapsto \langle \pi(g^{-1})v, v_{\rm H}\rangle$$

admits a holomorphic extension $\widetilde{f}_{v,v_{\rm H}}$. Moreover, we have

$$\widetilde{f}_{v,v_{\rm H}}(xK_{\mathbb{C}}) = \langle v, \pi^0(\overline{x})v_{\rm K}\rangle$$

for all $xK_{\mathbb{C}} \in \Xi_H$.

(ii) *The matrix coefficient*

$$g_{v_{\rm H},v}: G/H \to \mathbb{C}, \quad gH \mapsto \langle \pi^0(g)v_{\rm H}, v\rangle$$

admits a holomorphic extension $\widetilde{g}_{v_{\rm H},v}$. Moreover, we have

$$\widetilde{g}_{v_{\rm H},v}(xK_{\mathbb{C}}) = \langle \pi^0(x)v_{\rm K}, v\rangle \quad \text{and} \quad \widetilde{g}_{v_{\rm H},v}(xK_{\mathbb{C}}) = \overline{\widetilde{f}_{v,v_{\rm H}}(\overline{x}K_{\mathbb{C}})}$$

for all $xK_{\mathbb{C}} \in \Xi_H$.

Proof. We will only show (i) as the proof for (ii) is the same. By Theorem 2.1.2 (i) it follows that

$$\widetilde{f}(xK_{\mathbb{C}}) = \langle v, \pi^0(\bar{x})v_{\rm K}\rangle$$

exists and is holomorphic on Ξ_H. Let $g \in G$ and $0 < t < 1$. Then

$$\langle \pi(g^{-1})v, \pi^0(a_t)v_{\rm K}\rangle = \langle v, \pi^0(ga_t)v_{\rm K}\rangle = \widetilde{f}(ga_tK_{\mathbb{C}}).$$

Taking the limit at $t \to 1$ and using the remarks just before (2.1.4) it follows that

$$\lim_{t \nearrow 1} \widetilde{f}(ga_tK_{\mathbb{C}}) = \langle \pi(g^{-1})v, v_{\rm H}\rangle$$

and the convergence is uniform on compact subsets in G. ∎

Denote by $(\mathcal{H}^{-\infty})^H_{\mathrm{hol}} \subset (\mathcal{H}^{-\infty})^H$ the space of H-invariant distribution vectors η such that the function

$$G/H \ni x \mapsto g_{\eta,v}(x) := \langle \pi^0(x)\eta, v\rangle$$

has a holomorphic extension to Ξ_H for all $v \in \mathcal{H}^\omega$. Notice that for $g \in G$ and $x \in \Xi_H$ we have

$$\widetilde{g}_{\eta,\pi(g)v}(x) = \widetilde{g}_{\eta,v}(g^{-1}x)\,. \tag{2.2.2}$$

Our next task is to prove a converse of Proposition 2.2.3, namely that the map

$$\mathcal{H}^K \ni v_{\mathrm{K}} \mapsto v_{\mathrm{H}} \in (\mathcal{H}^{-\infty})^H_{\mathrm{hol}}$$

is an isomorphism. In particular only the H-invariant distribution vectors constructed in Theorem 2.1.3 have holomorphic extension. Thus if $\dim \mathcal{H}^K = 1$, as in the case of the principal series representations of G, the space $(\mathcal{H}^{-\infty})^H_{\mathrm{hol}}$ is also one-dimensional, i.e., there is (up to scalar) a unique H-spherical distribution vector which allows holomorphic extension of the smooth matrix coefficients.

As before we consider the pairing

$$(\mathcal{H}^{-\infty})^H \times \mathcal{H}^\infty \to C^\infty(G/H), \quad (\eta, v) \mapsto g_{\eta,v};\ g_{\eta,v}(gH) = \langle \pi^0(g)\eta, v\rangle\ .$$

Theorem 2.2.4. *Let $(\pi, \mathcal{H})$ be an admissible representation of G with finite length. Then the map*

$$\mathcal{H}^K \ni v_{\mathrm{K}} \mapsto v_{\mathrm{H}} \in (\mathcal{H}^{-\infty})^H_{\mathrm{hol}}$$

is a linear isomorphism.

Proof. It follows from Theorem 2.1.3 that $\mathcal{H}^K \ni v_{\mathrm{K}} \mapsto v_{\mathrm{H}} \in (\mathcal{H}^{-\infty})^H_{\mathrm{hol}}$ is well defined and injective. It is also clear that the map is linear. It remains to show that the map is onto. For that let $\eta \in (\mathcal{H}^{-\infty})^H_{\mathrm{hol}}$. Define a conjugate linear map $\widetilde{\eta} : \mathcal{H}_K \to \mathbb{C}$ by

$$\widetilde{\eta}(u) = \widetilde{g}_{\eta,u}(K_{\mathbb{C}})\,.$$

Then it follows from (2.2.2) that $\widetilde{\eta}$ is K-invariant. Thus we find a unique $v_{\mathrm{K}} \in \mathcal{H}_K$ such that $\widetilde{\eta}(u) = \langle v_{\mathrm{K}}, u\rangle$ for all $u \in \mathcal{H}_K$. In particular, it follows that

$$(\forall u \in \mathcal{H}_K) \qquad \widetilde{g}_{\eta,u}(K_{\mathbb{C}}) = \widetilde{g}_{v_{\mathrm{H}},u}(K_{\mathbb{C}})\ . \tag{2.2.3}$$

Fix $w \in \mathcal{H}_K$. We claim that $\widetilde{g}_{\eta,w} = \widetilde{g}_{v_{\mathrm{H}},w}$. In fact, it follows from (2.2.2) and (2.2.3) that all derivatives of $\widetilde{g}_{\eta,w}$ and $\widetilde{g}_{v_{\mathrm{H}},w}$ coincide at $K_{\mathbb{C}} \in \Xi_H$. Thus $\widetilde{g}_{\eta,w} = \widetilde{g}_{v_{\mathrm{H}},w}$ by Taylor's Theorem.

It follows from our claim that $g_{\eta,w} = g_{v_{\mathrm{H}},w}$ for all $w \in \mathcal{H}_K$. In particular, we obtain that

$$(\forall w \in \mathcal{H}_K) \qquad \langle \eta, w\rangle = g_{\eta,w}(H) = g_{v_{\mathrm{H}},w}(H) = \langle v_{\mathrm{H}}, w\rangle\ ,$$

and so $\eta = v_{\mathrm{H}}$, concluding the proof of the theorem. ∎

Corollary 2.2.5. **(Multiplicity one)** *Let $(\pi, \mathcal{H})$ be an irreducible K-spherical Hilbert representation of G. Then $\dim(\mathcal{H}^{-\infty})^H_{\mathrm{hol}} = 1$.*

Proof. According to [H84, Ch. IV, Th. 4.5(iii)] we have that $\dim \mathcal{H}^K = 1$. Thus the assertion follows from Theorem 2.2.4. ∎

2.3. Distributional characters and boundary values

Denote by dg and dh Haar measures on G and H. Notice that both G and H are unimodular and hence a left Haar measure is also a right Haar measure. Denote by dgH a invariant measure on G/H, which we will normalize in a moment. Recall that the mapping

$$C_c^\infty(G) \to C_c^\infty(G/H), \quad f \mapsto f^H;\ f^H(xH) = \int_H f(xh)\ dh$$

is continuous and onto. We will normalize the measure dgH in such a way that

$$\int_G f(g)\ dg = \int_{G/H} f^H(gH)\ dgH$$

holds for all $f \in C_c(G)$.

In this section $(\pi, \mathcal{H})$ will denote a *unitary* admissible representation representation of G with finite length. Further we will assume that $\mathcal{H}^K \neq \{0\}$. Notice that π unitary implies $\pi = \pi^0$.

For $f \in C_c^\infty(G)$ let us recall the mollifying property: $\pi(f)\mathcal{H}^{-\infty} \subseteq \mathcal{H}^\infty$. Thus the mapping

$$\Theta_{\pi, v_{\mathrm{H}}}: C_c^\infty(G) \to \mathbb{C}, \quad f \mapsto \langle \pi(f) v_{\mathrm{H}}, v_{\mathrm{H}} \rangle .$$

is well defined. It is known that $\Theta_\pi = \Theta_{\pi, v_{\mathrm{H}}}$ is a H-bi-invariant positive definite distribution on G. The H-bi-invariance implies that $\Theta_\pi(f)$ does only depend on f^H. We can therefore define a H-invariant distribution on G/H, also denoted by Θ_π, by

$$\Theta_\pi(f^H) = \Theta_\pi(f) .$$

On the other hand we notice that $\pi(x)v_{\mathrm{K}} \in \mathcal{H}^\omega$ for all $x \in G\exp(i\Omega_H)K_{\mathbb{C}}$ and hence $x \mapsto \langle v_{\mathrm{H}}, \pi(x)v_{\mathrm{K}} \rangle$ descends to a well defined and anti-holomorphic function on Ξ_H. Here v_{K} and v_{H} correspond to each other according to Theorem 2.2.4. We can therefore define the holomorphic function $\theta_\pi = \theta_{\pi, v_H} : \Xi_H \to \mathbb{C}$ by

$$\theta_\pi(xK_{\mathbb{C}}) = \overline{\langle v_{\mathrm{H}}, \pi(x)v_{\mathrm{K}} \rangle} = \langle \pi(x)v_{\mathrm{K}}, v_{\mathrm{H}} \rangle .$$

Then θ_π is left H-invariant.

Our next aim is to show that Θ_π is given by the limit operation and convolution:

$$\Theta_\pi(f) = \lim_{t \nearrow 1} \int_{G/H} f(gH)\ \overline{\theta_\pi(g^{-1}a_t)}\ dgH$$

for suitable regular functions f.

As we have only established the convergence $\pi(a_t)v_{\mathrm{K}} \to v_{\mathrm{H}}$ in $\mathcal{H}^{-\omega}$ and not in $\mathcal{H}^{-\infty}$, we cannot work with test-functions but must use an appropriate space of analytic vectors. For that let us write $L^1(G)^{\omega,\omega}$ for the space of analytic vectors for the left-right regular representation of $G \times G$ on $L^1(G)$. Notice that $L^1(G)^{\omega,\omega}$ is an algebra under convolution which is invariant under the natural involution $f \mapsto f^*$ with $f^*(x) = \overline{f(x^{-1})}$. Its importance lies in the fact that $\pi(f)\mathcal{H}^{-\omega} \subseteq \mathcal{H}^\omega$ holds for all $f \in L^1(G)^{\omega,\omega}$ (cf. Proposition A.4.1 in the appendix).

Write $L^1(G/H)^\omega$ for the space of analytic vectors for the left regular representation of G on $L^1(G/H)$. According to Proposition A.3.2 below, the averaging map $f \mapsto f^H$ maps $L^1(G)^{\omega,\omega}$ into $L^1(G/H)^\omega$. Finally let us define the space:

$$\mathcal{A}^1(G/H) = \{f^H \in L^1(G/H)^\omega \mid f \in L^1(G)^{\omega,\omega}\} .$$

From our discussion above we conclude that the mapping

$$\Theta_\pi^\omega : L^1(G)^{\omega,\omega} \to \mathbb{C}, \quad f \mapsto \langle \pi(f)v_{\mathrm{H}}, v_{\mathrm{H}} \rangle$$

is well defined and H bi-invariant. In particular $\Theta_\pi^\omega(f)$ depends only on f^H and therefore factors to $\mathcal{A}^1(G/H)$. We denote the corresponding map again by Θ_π^ω.

Theorem 2.3.1. *Let $(\pi, \mathcal{H})$ be an unitary admissible representation of G of finite length with $\mathcal{H}^K \neq \{0\}$. Let $v_{\rm K} \in \mathcal{H}^K$. Then*

$$(\forall f \in \mathcal{A}^1(G/H)) \qquad \Theta_\pi^\omega(f) = \lim_{t \nearrow 1} \int_{G/H} f(gH)\, \overline{\theta_\pi(g^{-1}a_t)}\, dgH\ .$$

Proof. Let $F \in L^1(G)^{\omega,\omega}$ be such that $f = F^H$. As $\pi(F)v_{\rm H} \in \mathcal{H}^\omega$ (cf. Proposition A.4.1), it follows from Theorem 2.1.3 that

$$\Theta_\pi^\omega(F) = \lim_{t \nearrow 1} \langle \pi(F)v_{\rm H}, \pi(a_t)v_{\rm K}\rangle\ .$$

Thus:

$$\begin{aligned}
\Theta_\pi^\omega(F) &= \lim_{t \nearrow 1} \langle v_{\rm H}, \int_G \overline{F(g^{-1})}\pi(g)\pi(a_t)v_{\rm K}\ dg\rangle \\
&= \lim_{t \nearrow 1} \int_G F(g^{-1})\ \langle v_{\rm H}, \pi(g)\pi(a_t)v_{\rm K}\rangle\ dg \\
&= \lim_{t \nearrow 1} \int_G F(g)\ \langle v_{\rm H}, \pi(g^{-1})\pi(a_t)v_{\rm K}\rangle\ dg \\
&= \lim_{t \nearrow 1} \int_G F(g)\ \overline{\theta_\pi(g^{-1}a_t)}\ dg\ .
\end{aligned} \tag{2.3.1}$$

Fix $0 \le t < 1$. We claim that $g \mapsto \theta_\pi(g^{-1}a_t)$ is a bounded function on G. Indeed, for $g \in G$ we have

$$\begin{aligned}
|\theta_\pi(g^{-1}a_t)| &= |\langle \pi(g^{-1}a_t)v_{\rm K}, v_{\rm K}\rangle| = |\langle \pi(a_t)v_{\rm K}, \pi(g)v_{\rm K}\rangle| \\
&\le \|\pi(a_t)v_{\rm K}\| \cdot \|\pi(g)v_{\rm K}\| \le \|\pi(a_t)v_{\rm K}\| \cdot \|v_{\rm K}\|
\end{aligned}$$

and our claim follows from $\|\pi(a_t)v_{\rm K}\| < \infty$. Combining (2.3.1) with our claim then yields

$$\begin{aligned}
\Theta_\pi^\omega(F) &= \lim_{t \nearrow 1} \int_G F(g)\ \overline{\theta_\pi(g^{-1}a_t)}\ dg \\
&= \lim_{t \nearrow 1} \int_{G/H} f(gH)\ \overline{\theta_\pi(g^{-1}a_t)}\ dgH\ ,
\end{aligned}$$

completing the proof of the theorem. ■

We finish this subsection by the following simple remark.

Lemma 2.3.2. *Suppose that π is irreducible. Then Θ_π is an eigendistribution of the algebra $\mathbb{D}(G/H)$ of invariant differential operators on G/H.*

Proof. Recall the surjective homomorphisms $\mathcal{U}(\mathfrak{g}_\mathbb{C})^\mathfrak{h} \to \mathbb{D}(G/H)$ and $\mathcal{U}(\mathfrak{g}_\mathbb{C})^\mathfrak{k} \to \mathbb{D}(G/K)$. We have $x_{\rm H}^{-1}H_\mathbb{C}x_{\rm H} = K_\mathbb{C}$. Hence, $\mathrm{Ad}(x_{\rm H}^{-1})$ defines an isomorphism

$$\mathrm{Ad}(x_{\rm H}^{-1})\colon \mathcal{U}(\mathfrak{g}_\mathbb{C})^\mathfrak{h} \to \mathcal{U}(\mathfrak{g}_\mathbb{C})^\mathfrak{k}\ .$$

In order to prove the lemma it is sufficient to show that $v_{\rm H}$ is an eigenvector for each $d\pi^0(u)$, $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})^\mathfrak{h}$.

Notice that for each $\widetilde{u} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})^\mathfrak{k}$ there exists a constant $c(\widetilde{u})$ such that $d\pi^0(\widetilde{u})v_{\rm K} = c(\widetilde{u})v_{\rm H}$. For $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})^\mathfrak{h}$ we now obtain that

$$\begin{aligned} d\pi^0(u)v_{\rm H} &= \lim_{t\nearrow 1} d\pi^0(u)\pi^0(a_t)v_{\rm K} \\ &= \lim_{t\nearrow 1} \pi^0(a_t)(d\pi^0(\mathrm{Ad}(a_t^{-1})u)v_{\rm K} \\ &= \lim_{t\nearrow 1} \pi^0(a_t)(d\pi^0(\mathrm{Ad}(x_{\rm H})^{-1}u)v_{\rm K} \\ &= c(\mathrm{Ad}(x_{\rm H}^{-1})u)v_{\rm H}\ . \end{aligned}$$

This completes the proof of the lemma. ■

2.4. Principal series representations

In this section we consider the case where $\pi = \pi_\lambda$ is a *spherical principal series representation.* In particular we will be discuss the dependence of $v_{\rm H}$ on the spectral parameter λ.

Let us first recall some well known facts about the principal series representations. For $\alpha \in \Sigma$ let $m_\alpha = \dim \mathfrak{g}^\alpha$ and $\rho := \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$. Write $M = Z_K(\mathfrak{a})$ and denote by $\kappa: G \to K$ and $a: G \to A$ the projections onto K, resp. A, associated to the Iwasawa decomposition $G = NAK$. Note that a and κ have unique holomorphic extension to $\widetilde{\Xi}$ also denoted by a and κ (cf. (1.2.1) and [KS01a]). As we are assuming that $G \subseteq G_{\mathbb{C}}$ with $G_{\mathbb{C}}$ simply connected and $H = G^\tau$, we have $M = Z_H(A)$. In particular $M \subseteq H \cap K$.

Define a minimal parabolic subgroup of G by $P_{\rm min} = MAN$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let

$$\mathcal{D}_\lambda = \{f \in C^\infty(G) : (\forall g \in G)(\forall man \in P)\ f(mang) = a^{\rho-\lambda} f(g)\}\ .$$

The group G acts on $\mathcal{D}_\lambda$ by right translation. Denote the corresponding representation by π_λ^∞, i.e., $\pi_\lambda^\infty(g)f(x) = f(xg)$. Denote by $\widetilde{\mathcal{H}}_\lambda$ the completion of $\mathcal{D}_\lambda$ in the norm corresponding to the inner product

$$\langle f, g\rangle = \int_K f(k)\overline{g(k)}\, dk\,.$$

Then π_λ^∞ extends to a representation π_λ of G in $\mathcal{H}_\lambda$. We refer to $(\widetilde{\mathcal{H}}_\lambda, \pi_\lambda)$ as the *spherical principal series representation of G with parameter λ*. The principal series representations $(\pi_\lambda, \widetilde{\mathcal{H}}_\lambda)$ are admissible and of finite length; they are irreducible for generic parameters λ and unitary if $\lambda \in i\mathfrak{a}^*$. It is also well known that $\widetilde{\mathcal{H}}_\lambda^\infty = \mathcal{D}_\lambda$ excusing our above notation π_λ^∞.

The restriction map $\widetilde{\mathcal{H}}_\lambda \ni f \mapsto f\mid_K \in L^2(K)$ is injective by the left NA-covariance of f. Furthermore $f\mid_K$ is left M-invariant. Hence the restriction map defines an isometry $\widetilde{\mathcal{H}}_\lambda \hookrightarrow L^2(M\backslash K)$. On the other hand if $F \in L^2(M\backslash K)$ then we can define $f \in \widetilde{\mathcal{H}}_\lambda$ by $f(nak) = a^{\rho-\lambda}F(Mk)$. Hence $\widetilde{\mathcal{H}}_\lambda \simeq L^2(M\backslash K)$. In this realization we have

$$[\pi_\lambda(g)f](Mk) = a(kg)^{\rho-\lambda} f(M\kappa(kg))\,.$$

Hence the Hilbert space is the same for all λ but the formula for the representation depends on λ. Notice that for $k \in K$ this simplifies to $(\pi_\lambda(k)f)(Mx) = f(Mxk)$. We write $\mathcal{H}_\lambda = L^2(M\backslash K)$ to indicate the role of $L^2(M\backslash K)$ as the representation space for π_λ and call it the *compact realization of* π_λ. As a consequence of this discussion, we see that $v_{{\rm K},\lambda} = \mathbf{1}_{M\backslash K}$ is a normalized K-fixed vector in $\mathcal{H}_\lambda$ and in fact $\mathcal{H}_\lambda^K = \mathbb{C}v_{{\rm K},\lambda}$. We write $v_{{\rm H},\lambda}$ and $v_{{\rm H},\lambda}^\omega$ instead of $v_{\rm H}$ and $v_{\rm H}^\omega$ to indicate the dependence of λ.

We have $\pi_\lambda^0 = \pi_{-\overline{\lambda}}$. Thus in the compact realization we have

$$[\pi_\lambda^0(a_t)v_{{\rm K},\lambda}](Mk) = a(ka_t)^{\rho+\overline{\lambda}} \tag{2.4.1}$$

for all $0 \leq t < 1$ and so

$$\langle v_{\mathrm{H},\lambda}^{\omega}, v\rangle = \lim_{t \nearrow 1} \int_{M\backslash K} a(ka_t)^{\rho+\overline{\lambda}}\, \overline{v(Mk)}\, dMk \qquad (v \in \mathcal{H}_{\lambda}^{\omega}) \ . \tag{2.4.2}$$

In the sequel it will be importnat that $\mathcal{H}_{\lambda}^{\omega} = C^{\omega}(M\backslash K)$ is independent of λ. The following theorem specifies the dependence of $v_{\mathrm{H},\lambda}$ on λ:

Theorem 2.4.1. *The mapping*

$$\mathfrak{a}_{\mathbb{C}}^* \to \coprod_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} (\mathcal{H}_{\lambda}^{-\infty})^H, \quad \lambda \mapsto v_{\mathrm{H},\lambda}$$

is weakly anti-holomorphic in the sense that for all $v \in C^{\omega}(M\backslash K)$ the mapping

$$\mathfrak{a}_{\mathbb{C}}^* \to \mathbb{C}, \quad \lambda \mapsto \langle v_{\mathrm{H},\lambda}, v\rangle$$

is anti-holomorphic.

Proof. Let $v \in C^{\omega}(M\backslash K)$. It is convenient to consider v as an M-invariant function on K. As v is analytic, there exists an open $K \times K$-invariant neighborhood $\mathcal{U}$ of K in $K_{\mathbb{C}}$ such that v extends to a holomorphic M-invariant function $\widetilde{v}$ on $\mathcal{U}$.

Let $\varepsilon > 0$. Then it follows from the compactness of Ka_{ε} and (1.2) that we can choose $\varepsilon > 0$ small enough such that $\kappa(Ka_{\varepsilon}) \subseteq \mathcal{U}$.

Now consider v as an element of $\mathcal{H}_{\lambda}^{\omega}$. We claim that $\pi_{\lambda}(a_{\varepsilon})v$ exists. In fact, using our introductory remarks, we have

$$[\pi_{\lambda}(a_{\varepsilon})v](Mk) = a(ka_{\varepsilon})^{\rho-\lambda}\widetilde{v}(\kappa(ka_{\varepsilon})) \ .$$

With (2.4.1) we now compute

$$\begin{aligned} \langle v_{\mathrm{H},\lambda}, v\rangle &= \langle \pi_{\lambda}(a_{1-\varepsilon})^0 v_{\mathrm{K},\lambda}, \pi_{\lambda}(a_{\varepsilon})v\rangle \\ &= \int_{M\backslash K} a(ka_{1-\varepsilon})^{\rho+\overline{\lambda}} \cdot \overline{a(ka_{\varepsilon})^{\rho-\lambda}} \cdot \overline{\widetilde{v}(\kappa(ka_{\varepsilon}))}\, dMk \ . \end{aligned} \tag{2.4.3}$$

By our remarks at the beginning of the proof, we have

$$\sup_{k\in K} |\widetilde{v}(\kappa(ka_{\varepsilon}))| < \infty \ . \tag{2.4.4}$$

Notice that (1.2) implies that both $a(Ka_{1-\varepsilon})$ and $a(Ka_{\varepsilon})$ are compact subsets of $T(\Omega)$. Thus, if $C \subseteq \mathfrak{a}_{\mathbb{C}}^*$ is a compact subset, then

$$\sup_{\lambda\in C}\sup_{k\in K} |a(ka_{1-\varepsilon})^{\rho+\overline{\lambda}}| < \infty, \qquad \text{and} \qquad \sup_{\lambda\in C}\sup_{k\in K} |a(ka_{\varepsilon})^{\rho-\lambda}| < \infty \ . \tag{2.4.5}$$

Therefore, if we use the estimates (2.4.4) and (2.4.5), it follows from (2.4.3) that $\lambda \mapsto \langle v_{\mathrm{H},\lambda}, v\rangle$ is anti-holomorphic. ∎

2.5. Integral representation and asymptotic behaviour of θ_π

Previously we have defined a H-invariant holomorphic function θ_π for unitary representations π. For non-unitary π we define θ_π by

$$\theta_\pi(xK_{\mathbb{C}}) = \langle \pi(x)v_{\mathrm{K}}, v_{\mathrm{H}}\rangle \qquad (xK_{\mathbb{C}} \in \Xi_H)\ .$$

Clearly, θ_π is a holomorphic function on Ξ_H. Moreover, if π is unitary, then θ_π is H-invariant.

In this subsection we will give an integral representation of the functions θ_π for principal series represntations π. This will also allow us to read off the asymptotic behaviour of θ_π.

Recall the definition of the spherical function φ_λ of parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by

$$\varphi_\lambda(g) = \langle \pi_\lambda(g)v_{\mathrm{K},\lambda}, v_{\mathrm{K},\lambda}\rangle = \int_K a(kg)^{\rho-\lambda}\ dk \qquad (g \in G)\ .$$

It follows from Theorem 2.1.2 that φ_λ admits a holomorphic extension to $\widetilde{\Xi}_H$ (or Ξ_H if we wish to consider φ_λ as a function on G/K). Also $\varphi_\lambda|_A$ extends holomorphically to the tube $T(2\Omega) = A\exp(2i\Omega)$. Notice that $z_{\mathrm{H}} \in T(2\Omega)$. All mentioned holomorphic extensions of φ_λ are also denoted by φ_λ.

In the sequel we abbreviate and write θ_λ instead of θ_{π_λ}. The next result is immediate from the definitions, Theorem 2.1.2 and the formula (2.4.1).

Theorem 2.5.1. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a principal series representation with parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then for all $xK_{\mathbb{C}} \in \Xi_H$ we have*

$$\begin{aligned}\theta_\lambda(xK_{\mathbb{C}}) &= \lim_{t\nearrow 1}\int_K a(kx)^{\rho-\lambda}\overline{a(ka_t)^{\rho+\overline{\lambda}}}\ dk \\ &= \lim_{t\nearrow 1}\varphi_\lambda(a_t x)\,.\end{aligned}$$

Furthermore,

$$\theta_\lambda(aK_{\mathbb{C}}) = \varphi_\lambda(x_{\mathrm{H}}a)$$

for all $a \in T(\Omega) = A\exp(i\Omega)$. Here φ_λ denotes the holomorphically extended spherical function to $T(2\Omega)$.

Proof. Let $x \in \Xi_H$. Then we have

$$\begin{aligned}\theta_\lambda(x) &= \langle \pi_\lambda(x)v_{\mathrm{K}}, v_{\mathrm{H}}\rangle \\ &= \lim_{t\nearrow 1}\langle \pi_\lambda(x)v_{\mathrm{K}}, \pi_\lambda^0(a_t)v_{\mathrm{K}}\rangle \\ &= \lim_{t\nearrow 1}\int_K a(kx)^{\rho-\lambda}\ \overline{a(ka_t)^{\rho+\overline{\lambda}}}\, dk\,.\end{aligned}$$

But we can also write the third line as

$$\begin{aligned}\theta_\lambda(x) &= \lim_{t\nearrow 1}\langle \pi_\lambda(x)v_{\mathrm{K}}, \pi_\lambda^0(a_t)v_{\mathrm{K}}\rangle \\ &= \lim_{t\nearrow 1}\langle \pi_\lambda(a_t x)v_{\mathrm{K}}, v_{\mathrm{K}}\rangle \\ &= \lim_{t\nearrow 1}\varphi_\lambda(a_t x)\,.\end{aligned}$$

The last statement follows now from Theorem 2.1.2, part (ii). ∎

To discuss the asymptotic expansions of θ_λ along a positive Weyl chamber we first have to recall some facts on the Harish-Chandra expansion of the spherical functions. For that let $\mathfrak{a}_+ = \{X \in \mathfrak{a} \colon (\forall \alpha \in \Sigma^+)\ \alpha(X) > 0\}$ and set $A^+ = \exp(\mathfrak{a}_+)$. Further we define $\Lambda = \mathbb{N}_0[\Sigma^+]$. If $\mu \in \Lambda$, then we define a meromorphic function $\Gamma_\mu(\lambda)$ in the parameter $\lambda \in \mathfrak{a}_\mathbb{C}^*$ by $\Gamma_0(\lambda) = 1$ and then recursively by

$$\Gamma_\mu(\lambda) = \frac{2}{\langle \mu, \mu - \lambda \rangle} \sum_{\alpha \in \Sigma^+} m_\alpha \sum_{k \in \mathbb{N}} \Gamma_{\mu - 2k\alpha} \langle \mu + \rho - 2k\alpha - \lambda, \alpha \rangle \ .$$

We call $\lambda \in \mathfrak{a}_\mathbb{C}^*$ generic if $\Gamma_\mu(\cdot)$ is holomorphic at λ for all $\mu \in \Lambda$. For generic $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we define the Harish-Chandra Φ-function on A^+ by

$$\Phi_\lambda(a) = a^{\lambda - \rho} \sum_{\mu \in \Lambda} \Gamma_\mu(\lambda) a^{-\mu} \qquad (a \in A^+) \ .$$

This series is locally absolutely convergent. In particular, we see that Φ_λ extends to a holomorphic function on $A^+ \exp(2i\Omega) \subseteq A_\mathbb{C}$ which we also denote by Φ_λ. Finally, with $\mathbf{c}(\lambda)$ the familiar Harish-Chandra $\mathbf{c}$-function on G/K, we have for all generic parameters $\lambda \in \mathfrak{a}_\mathbb{C}^*$ that

$$\varphi_\lambda(a) = \sum_{w \in \mathcal{W}} \mathbf{c}(w\lambda) \Phi_{w\lambda}(a) \qquad (a \in A^+) \ .$$

Combining these facts with Theorem 2.5.1 we now obtain that:

Theorem 2.5.2. *Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ be a generic parameter. Then the following assertions hold.*

(i) *For $a \in A^+ \exp(i\Omega)$ we have*

$$\theta_\lambda(aK_\mathbb{C}) = \varphi_\lambda(z_{\mathrm{H}} a) = \sum_{w \in W} \mathbf{c}(w\lambda)(z_{\mathrm{H}} a)^{w\lambda - \rho} \Phi_{w\lambda}(z_{\mathrm{H}} a) \ .$$

(ii) *Suppose that $\langle \mathrm{Re}\, \lambda, \alpha \rangle > 0$ for all $\alpha \in \Sigma^+$. Fix $Y \in \mathfrak{a}_+$. Then*

$$\lim_{t \to \infty} e^{t(\rho - \lambda)(Y)} \theta_\lambda(\exp(tY) K_\mathbb{C}) = \mathbf{c}(\lambda) \cdot z_{\mathrm{H}}^{\lambda - \rho} \ .$$

■

2.6. H-orbit coefficients of the holomorphic distribution vector

As we have remarked already earlier the space $(\mathcal{H}_\lambda^{-\infty})^H$ has dimension $|\mathcal{W}/\mathcal{W}_0|$ for generic λ. One can parametrize $(\mathcal{H}_\lambda^{-\infty})^H$ through the open H-orbits in the flag manifold $P_{\min}\backslash G$. These orbits haven been parametrized by Rossmann and Matsuki (cf. [M79]); they are given by

$$P_{\min} w H \qquad (w \in \mathcal{W}/\mathcal{W}_0) \ .$$

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $w \in \mathcal{W}/\mathcal{W}_0$ define a right H-invariant function on G by

$$\eta_{\lambda,w}(x) = \begin{cases} a^{\rho + \overline{\lambda}} & \text{for } x = manwh \in MANwH \\ 0 & \text{otherwise} \ . \end{cases}$$

For $\lambda \in \mathfrak{a}^*$ we will use the notation $\lambda << 0$ if $\langle \lambda, \alpha \rangle << 0$ for all $\alpha \in \Sigma^+$. Then it is known that for $\lambda << 0$ the functions $\eta_{\lambda,w}$ are continuous and define H-fixed distribution vectors of π_λ

[Ó87]. Moreover, the distributions $\eta_{\lambda,w}$ admit continuation in λ to a weakly anti-meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$. For generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have

$$(\mathcal{H}_\lambda^{-\infty})^H = \bigoplus_{w\in\mathcal{W}/\mathcal{W}_0} \mathbb{C}\eta_{\lambda,w}$$

and the mapping

$$j_\lambda\colon \mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|} \to (\mathcal{H}_\lambda^{-\infty})^H, \quad (c_w)_w \mapsto \sum_{w\in\mathcal{W}/\mathcal{W}_0} c_w \eta_{\lambda,w}$$

is a bijection for generic λ, weakly anti-meromorphic in λ [vdB88]. For $\lambda << 0$ the inverse of j_λ is given by the evaluation mapping

$$\mathrm{ev}\colon (\mathcal{H}_\lambda^{-\infty})^H \to \mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, \quad \eta \mapsto (\eta(w))_w \ .$$

On the other hand we know that the weakly holomorphic distribution vector $v_{\mathrm{H},\lambda}$ depends weakly anti-holomorphically on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ (cf. Theorem 2.4.1). The next theorem gives us the coefficients of $v_{\mathrm{H},\lambda}$ in terms of the basis $(\eta_{\lambda,w})$ of $(\mathcal{H}_\lambda^{-\infty})^H$. We note that for $\lambda << 0$ the distribution $v_{\mathrm{H},\lambda}$ is given through the bounded measurable function

$$v_{\mathrm{H},\lambda}(k) = \lim_{t\nearrow 1} a(ka_t)^{\rho+\overline{\lambda}} \ . \tag{2.6.1}$$

Theorem 2.6.1. *For generic parameters $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have*

$$v_{\mathrm{H},\lambda} = \sum_{w\in\mathcal{W}/\mathcal{W}_0} z_{\mathrm{H}}^{w^{-1}(\rho+\overline{\lambda})} \cdot \eta_{\lambda,w} \ .$$

Proof. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ generic write

$$v_{\mathrm{H},\lambda} = \sum_{w\in\mathcal{W}/\mathcal{W}_0} c_{\lambda,w} \cdot \eta_{\lambda,w}$$

for the basis expansion. As the coefficients $c_{\lambda,w}$ depend weakly anti-meromorphically on λ, it is sufficient to show that $c_{\lambda,w} = z_{\mathrm{H}}^{w^{-1}(\rho+\overline{\lambda})}$ for $\lambda << 0$. Then (2.6.1) implies that

$$c_{\lambda,w} = v_{\mathrm{H}}(w) = \lim_{t\nearrow 1} v_{\mathrm{H},\lambda}(wa_t) = a(wz_{\mathrm{H}})^{\rho+\overline{\lambda}} = z_{\mathrm{H}}^{w^{-1}(\rho+\overline{\lambda})} \ ,$$

as was to be shown. ∎

Remark 2.6.2. Let us go back to Example 2.1.5 for $G = \mathrm{Sl}(2,\mathbb{R})$. Here $\mathcal{W}_0 = \{\mathbf{1}\}$ and so $\mathcal{W}/\mathcal{W}_0 = \{\mathbf{1}, w\}$ where w is the non-trivial element in the Weyl group which acts by multiplication by -1. The distributions $v_{\mathrm{H},1}$ and $v_{\mathrm{H},2}$ from Example 2.1.5 are given in the above notation by

$$v_{\mathrm{H},1} = \eta_{\lambda,\mathbf{1}} \qquad \text{and} \qquad v_{\mathrm{H},2} = \eta_{\lambda,w} \ .$$

In Example 2.1.5 we did show that

$$v_{\mathrm{H}} = v_{\mathrm{H},\lambda} = e^{i\frac{\pi}{4}(1+\overline{\lambda})} v_{\mathrm{H},1} + e^{-i\frac{\pi}{4}(1+\overline{\lambda})} v_{\mathrm{H},2}.$$

As $z_{\mathrm{H}}^{\rho+\overline{\lambda}} = e^{i\frac{\pi}{4}(1+\overline{\lambda})}$, we hence see that the above formula is a special case of Theorem 2.6.1. ∎

2.7. Transformation under the intertwining matrix

For $w \in \mathcal{W}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ generic we have an intertwining operator

$$A(\lambda, w\lambda): (\pi_\lambda, \mathcal{H}_\lambda^\infty) \to (\pi_{w\lambda}, \mathcal{H}_{w\lambda}^\infty) .$$

Notice that the Hilbert space $\mathcal{H}_\lambda$ is independent of λ because we use the compact realzation. We can therefore speak about meromorphic maps from $\mathfrak{a}_{\mathbb{C}}^*$ into the space of bounded operators from $\mathcal{H}_\lambda$ into $\mathcal{H}_\mu$. In this sense it is well known that the map $\mathfrak{a}_{\mathbb{C}}^* \ni \lambda \mapsto A(\lambda, w\lambda)$ is meromorphic. Dualizing, we obtain an anti-meromorphic family of intertwining operators

$$A(\lambda, w\lambda)^*: \mathcal{H}_{w\lambda}^{-\infty} \to \mathcal{H}_\lambda^{-\infty} .$$

Restricting $A(\lambda, w\lambda)^*$ to the space of H-invariant distribution vectors we obtain a linear bijection, say

$$A_H(\lambda, w\lambda)^*: (\mathcal{H}_{w\lambda}^{-\infty})^H \to (\mathcal{H}_\lambda^{-\infty})^H .$$

Often one refers to $A_H^*(\lambda, w\lambda)$ as the *intertwining matrix*. In terms of the basic distribution vectors $(\eta_{w\lambda,w'})_{w'}$ respectively $(\eta_{\lambda,w'})_{w'}$ the operator $A_H(\lambda, w\lambda)^*$ has an unknown, seemingly complicated expression. In this section we will show that the the intertwining matrix maps the holomorphic distribution vector $v_{\mathrm{H},w\lambda}$ to a multiple of $v_{\mathrm{H},\lambda}$. In order to describe this multiple more precisely we need more notations.

For $w \in \mathcal{W}$ define a subgroup of $\overline{N} = \theta(N)$ by

$$\overline{N}_w = \overline{N} \cap wNw^{-1} .$$

For $\mathrm{Re}\,\lambda << 0$ we define functions

$$\mathbf{c}_w(\lambda) = \int_{\overline{N}_w} a(\overline{n})^{\rho-\lambda}\ d\overline{n} .$$

If $w = w_0$ is the longest element in $\mathcal{W}$, then we write $\mathbf{c}(\lambda)$ instead of $\mathbf{c}_{w_0}(\lambda)$ and remark that $\mathbf{c}(\lambda)$ is the familiar Harish-Chandra c-function on G/K. The functions $\mathbf{c}_w(\lambda)$ admit meromorphic continuation to $\mathfrak{a}_{\mathbb{C}}^*$ and can be explicitely computed (Gindikin-Karpelevic formula).

With this notation the intertwinig operators $A(\lambda, w\lambda)$ for $\lambda << 0$ are defined by

$$[A(\lambda, w\lambda)f](x) = \int_{\overline{N}_w} f(\overline{n}wx)\ d\overline{n} \qquad (f \in \mathcal{D}_\lambda, x \in G) .$$

Theorem 2.7.1. *Assume that $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is generic. Then*

$$A_H(\lambda, w\lambda)^* v_{\mathrm{H},w\lambda} = \mathbf{c}_w(\overline{\lambda}) \cdot v_{\mathrm{H},\lambda} .$$

Proof. It is well known - and follows immediately from the definition - that $A(\lambda, w\lambda)v_{\mathrm{K},\lambda} = \mathbf{c}_w(\lambda)v_{\mathrm{K},w\lambda}$. But then, as $A(\lambda, w\lambda)^* v_{\mathrm{K},w\lambda}$ is K-invariant, we also get

$$\begin{aligned}\langle A(\lambda, w\lambda)^* v_{\mathrm{K},w\lambda}, v_{\mathrm{K},\lambda}\rangle &= \langle v_{\mathrm{K},w\lambda}, A(\lambda, w\lambda)v_{\mathrm{K},\lambda}\rangle \\ &= \langle v_{\mathrm{K},w\lambda}, \mathbf{c}_w(\lambda)v_{\mathrm{K},w\lambda}\rangle \\ &= \overline{\mathbf{c}_w(\lambda)} .\end{aligned}$$

Noticing that $\overline{\mathbf{c}_w(\lambda)} = \mathbf{c}_w(\overline{\lambda})$ it follows that

$$A(\lambda, w\lambda)^* v_{\mathrm{K},w\lambda} = \mathbf{c}_w(\overline{\lambda}) v_{\mathrm{K},\lambda}\,.$$

Finally, using that $A(\lambda, w\lambda)^*$ is an intertwining operator, we get for a K-finite u :

$$\begin{aligned}
\langle A_H^*(\lambda, w\lambda) v_{\mathrm{H},w\lambda}, u\rangle &= \langle v_{\mathrm{H},w\lambda}, A(\lambda, w\lambda)u\rangle \\
&= \lim_{t\nearrow 1} \langle \pi^0_{w\lambda}(a_t) v_{\mathrm{K},w\lambda}, A(\lambda, w\lambda)u\rangle \\
&= \lim_{t\nearrow 1} \langle v_{\mathrm{K},w\lambda}, \pi_{w\lambda}(a_t)^{-1} A(\lambda, w\lambda)u\rangle \\
&= \lim_{t\nearrow 1} \langle v_{\mathrm{K},w\lambda}, A(\lambda, w\lambda)\pi_{w\lambda}(a_t)^{-1}u\rangle \\
&= \lim_{t\nearrow 1} \langle A(\lambda, w\lambda)^* v_{\mathrm{K},w\lambda}, \pi_{w\lambda}(a_t)^{-1}u\rangle \\
&= \mathbf{c}_w(\overline{\lambda}) \lim_{t\nearrow 1} \langle \pi_\lambda(a_t) v_{\mathrm{K},\lambda}, u\rangle \\
&= \mathbf{c}_w(\overline{\lambda}) \langle v_{\mathrm{H},\lambda}, u\rangle\,.
\end{aligned}$$

Hence $A_H(\lambda, w\lambda)^* v_{\mathrm{H},w\lambda} = \mathbf{c}_w(\overline{\lambda}) \cdot v_{\mathrm{H},\lambda}$, as was to be shown. ∎

2.8. Relation to the horospherical picture

In this subsection we explain the construction of the holomorphic distribution vector v_{H} from the horospherical point of view.

Let $\kappa \in \mathfrak{a}_{\mathbb{C}}^*$ be a complex parameter. We call the holomorphic function

$$\widetilde{\psi}_\kappa(nak) = a^\kappa = \exp(<\kappa, \log a>), \qquad n \in N_{\mathbb{C}}, a \in T(\Omega), k \in K_{\mathbb{C}}$$

on $\widetilde{\Xi}$ (see (1.2.1)) the *holomorphic horospherical function with parameter* κ.

This function can be pushed down to Ξ as a holomorphic locally $N_{\mathbb{C}}$-invariant function ψ_κ. The function ψ_κ is a holomorphic extension of the usual horospherical (N-invariant, A-homogeneous) function on G/K, corresponding to spherical principal representations related to κ. For certain values of the parameter κ the function ψ_κ has boundary distribution values $\psi_{\kappa,\mathrm{H}}$ on G/H. To understand the structure of those distributions, let us remark that a Zariski open part of G/H is the disjoint union of domains $Y_j = NAy_j$ where $y_1, \cdots, y_k$ correspond to the vertices of Ω_H ($\mathcal{W}$-equivalent). On each Y_j we have an N-invariant, A-homogeneous distribution with parameter κ (such distributions on Y_j are unique up to a multiplicative constant).

The function $\widetilde{\psi}_\kappa$ can be also pushed down on a domain D_H in $N_{\mathbb{C}}M_{\mathbb{C}}\backslash G_{\mathbb{C}}$ as a holomorphic function ψ_κ. This function with parameters holomorphically extends K-invariant vectors in the principal spherical representations on $MN\backslash G$. The boundary values of this holomorphic function give an H-invariant distribution, the domains Y_j correspond to the H-orbits on $MN\backslash G$ and we have the corresponding decomposition of ψ_κ.

3. An application: Hardy spaces for NCC symmetric spaces

In this section we apply our theory developed in Section 2 to associate to every NCC symmetric space G/H a Hardy space $\mathcal{H}^2(\Xi_H)$. The Hardy space is a G-invariant Hilbert space of holomorphic functions on Ξ_H featuring a boundary value mapping which gives an isometric embedding of $\mathcal{H}^2(\Xi_H)$ into the most-continuous spectrum $L^2(G/H)_{\rm mc}$ of $L^2(G/H)$. Hence we give a realization of a part of $L^2(G/H)_{\rm mc}$ in a space of holomorphic function on Ξ_H, generalizing and extending our previous results from [GKÓ01] to all NCC spaces.

This section is organized as follows: After a brief digression on Hilbert spaces of holomorphic functions on Ξ_H, we give an adhoc definition of the Hardy space through the spectral measure. Then, after recalling the theory of the most-continuous spectrum, we will show that there is a boundary value mapping embedding $\mathcal{H}^2(\Xi_H)$ isometrically into $L^2(G/H)_{\rm mc}$.

3.1. G-invariant Hilbert spaces of holomorphic functions on Ξ_H

In this section we briefly recall the abstract theory of G-invariant Hilbert spaces of holomorphic functions, specialized to the complex manifold Ξ_H (see also [FT99] and [K99] for the general theory).

In the sequel we will consider $\mathcal{O}(\Xi_H)$ as a Fréchet space with the topology of compact convergence. We let G act on $\mathcal{O}(\Xi_H)$ by the left regular representation L:

$$(L(g)f)(z) = f(g^{-1}z) \qquad (g\in G, f\in\mathcal{O}(\Xi_H), z\in\Xi_H)\ . \tag{3.1.1}$$

By a *G-invariant Hilbert space of holomorphic functions on* Ξ_H we understand a Hilbert space $\mathcal{H}\subseteq\mathcal{O}(\Xi_H)$ such that:

(IH1) The inclusion $\mathcal{H}\hookrightarrow\mathcal{O}(\Xi_H)$ is continuous.

(IH2) The Hilbert space $\mathcal{H}$ is invariant under L and the the corresponding representation of G is unitary.

It follows from (IH1) that for every $z\in\Xi_H$ the point evaluation $\mathcal{H}\to\mathbb{C},\ f\mapsto f(z)$, is continuous. Thus, there exists a $\mathcal{K}_z\in\mathcal{H}$ such that $\langle f,\mathcal{K}_z\rangle = f(z)$ holds for every $f\in\mathcal{H}$. In this way we obtain a function

$$\mathcal{K}\colon \Xi_H\times\Xi_H\to\mathbb{C},\quad (z,w)\mapsto\mathcal{K}(z,w)=\langle\mathcal{K}_w,\mathcal{K}_z\rangle\ .$$

The function $\mathcal{K}$ is holomorphic in the first variable and anti-holomorphic in the second variable. It follows from (IH2) that $\mathcal{K}$ is G-invariant, i.e., $\mathcal{K}(gz,gw)=\mathcal{K}(z,w)$ holds for all $g\in G$ and all $z,w\in\Xi_H$. We call $\mathcal{K}$ the *Cauchy-Szegö kernel* of $\mathcal{H}$ and note that $\mathcal{H}$ is determined by $\mathcal{K}$.

To describe the spectral resolution of $\mathcal{K}$ denote by $\widehat{G}_s$ the K-spherical unitary dual of G. We view $\widehat{G}_s$ as a subset of $\mathfrak{a}^*_{\mathbb{C}}/\mathcal{W}$ using the parametrization of the spherical principal series. Notice that the topology on $\widehat{G}_s$ coincides with the topology induced from $\mathfrak{a}^*_{\mathbb{C}}/\mathcal{W}$. Slightly abusing our notation from Subsection 2.4, we denote by $(\pi_\lambda,\mathcal{H}_\lambda)$ a representative of $\lambda\in\widehat{G}_s$. For $\lambda\in\widehat{G}_s$ define the G-invariant kernel $\mathcal{K}_\lambda$ by

$$\mathcal{K}_\lambda(xK_{\mathbb{C}},yK_{\mathbb{C}}) = \langle\pi_\lambda(\overline{y})v_{\rm K,\lambda},\pi_\lambda(\overline{x})v_{\rm K,\lambda}\rangle \qquad (xK_{\mathbb{C}},yK_{\mathbb{C}}\in\Xi_H)\ .$$

Write $\Xi_H^{\rm opp}$ for Ξ_H but endowed with the opposite complex structure. We recall that the map

$$\widehat{G}_s\to\mathcal{O}(\Xi_H\times\Xi_H^{\rm opp}),\quad \lambda\mapsto\mathcal{K}_\lambda$$

is continuous [KS01b, Sect. 5] (it follows from the fact that the spherical functions φ_λ and their holomorphic continuations are continuous in λ). Then by [KS01b, Th. 5.1] there exists a unique Borel measure μ on $\widehat{G}_s$ such that

$$\mathcal{K}(z,w) = \int_{\widehat{G}_s} \mathcal{K}_\lambda(z,w)\, d\mu(\lambda) \qquad (z,w \in \Xi_H) \tag{3.1.2}$$

with the right hand side converging absolutely on compact subsets of $\Xi_H \times \Xi_H$. Equivalently phrased, the mapping

$$\Phi\colon \int_{\widehat{G}_s}^{\oplus} \mathcal{H}_\lambda\, d\mu(\lambda) \to \mathcal{H}, \quad s = (s_\lambda)_\lambda \mapsto \left(xK_{\mathbb{C}} \mapsto \int_{\widehat{G}_s} \langle \pi_\lambda(x^{-1})s_\lambda, v_{\mathrm{K},\lambda}\rangle\, d\mu(\lambda)\right) \tag{3.1.3}$$

is a G-equivariant unitary isomorphism. In the sequel we refer to the measure μ as the *Plancherel measure* of $\mathcal{H}$.

In [KS01b] a criterion was given on a Borel measure μ on $\widehat{G}_s$ to be a Plancherel measure for an invariant Hilbert space $\mathcal{H} = \mathcal{H}(\mu)$ on Ξ. This criterion can be easily adapted to invariant Hilbert spaces on Ξ_H. Let us provide the necessary modifications.

Define a norm $\|\cdot\|_{\mathrm{H}}$ on $\mathfrak{a}_{\mathbb{C}}^*$ by

$$\|\lambda\|_{\mathrm{H}} := \sup_{w\in\mathcal{W}/\mathcal{W}_0} |\lambda(wX_{\mathrm{H}})| \qquad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*)\ .$$

Then [KS01b, Prop. 5.4] and its proof readily gives the following generalization:

Proposition 3.1.1. *Let μ be a Borel measure on $\widehat{G}_s$ with the property*

$$(\forall 0 \le c < 2) \qquad \int_{\widehat{G}_s} e^{c\|\operatorname{Im}\lambda\|_{\mathrm{H}}}\, d\mu(\lambda) < \infty\ . \tag{3.1.4}$$

Then μ is the Plancherel measure of an invariant Hilbert space $\mathcal{H}(\mu)$ on Ξ_H. ■

3.2. The definition of the Hardy space

We are now ready to give the definition of the Hardy space on Ξ_H. Let us denote by $i\mathfrak{a}_+^*$ an open Weyl chamber in $i\mathfrak{a}^*$. In the sequel we will consider $i\mathfrak{a}_+^*$ mainly as a subset of $\widehat{G}_s$. Let $\mathcal{O}$ be a neighborhood of $i\mathfrak{a}^*$ such that $\sum_{w\in\mathcal{W}/\mathcal{W}_0} z_{\mathrm{H}}^{-2w^{-1}\lambda}$ has a holomorphic square root $\mathbf{z}_{\mathrm{H}}(\lambda)$ on $\mathcal{O}$. Define a holomorphic function $\mathbf{c}_{\mathrm{G/H}}$ on $\mathcal{O}$ by

$$\mathbf{c}_{\mathrm{G/H}}(\lambda) = \mathbf{c}(\lambda)\cdot \mathbf{z}_{\mathrm{H}}(\lambda)$$

and define a Borel measure μ on $i\mathfrak{a}_+^*$ by

$$d\mu(\lambda) = \frac{d\lambda}{|\,\mathbf{c}_{\mathrm{G/H}}(\lambda)|^2} \tag{3.2.1}$$

where $d\lambda$ denotes the Lebesgue measure. Then we have:

Lemma 3.2.1. *The measure μ satisfies the condition (3.1.4); in particular μ is the Plancherel measure of an invariant Hilbert space $\mathcal{H}(\mu)$ on Ξ_H.*

Proof. Recall the growth behaviour of the $\mathbf{c}$-function on the imaginary axis: There exists constants $C, N > 0$ such that

$$(\forall \lambda \in i\mathfrak{a}^*) \qquad \frac{1}{|\,\mathbf{c}(\lambda)|^2} \leq C(1+|\lambda|^N)\ .$$

Moreover for $\lambda \in i\mathfrak{a}^*$ one has $z_{\mathrm{H}}^{\lambda} = e^{\lambda(iX_{\mathrm{H}})} > 0$. Hence

$$\sum_{w\in\mathcal{W}/\mathcal{W}_0} |z_{\mathrm{H}}^{-w^{-1}\lambda}|^2 \geq e^{2\|\mathrm{Im}\lambda\|_{\mathrm{H}}}.$$

Combining these two facts now yields that μ satisfies (3.1.4). ∎

Using Proposition 3.1.1 and Lemma 3.2.1 we now can give an adhoc-definition of the Hardy space on Ξ_H.

Definition 3.2.2. **(Hardy space)** Let G/H be a NCC symmetric space and Ξ_H its associated domain in $G_{\mathbb{C}}/K_{\mathbb{C}}$. Then we define the *Hardy space* $\mathcal{H}^2(\Xi_H)$ on Ξ_H by

$$\mathcal{H}^2(\Xi_H) = \mathcal{H}(\mu)$$

with μ as in (3.2.1). ∎

Recall the Cauchy-Szegö kernel $\mathcal{K}(z,w)$ of the invariant Hilbert space $\mathcal{H}^2(\Xi_H)$ from Subsection 3.1.

Lemma 3.2.3. *Let $\mathcal{K}$ be the Cauchy-Szegö kernel of $\mathcal{H}^2(\Xi_H)$. Then the limits*

$$\Psi(z) = \lim_{t\nearrow 1} \mathcal{K}(z, a_t K_{\mathbb{C}}) \qquad (z\in\Xi_H)$$

exist locally uniformly. In particular, $\Psi\colon \Xi_H \to \mathbb{C}$ is an H-invariant holomorphic function.

Proof. Fix $z \in \Xi_H$ and let $U \subseteq \Xi_H$ be a compact neighborhood of z. Choose $\varepsilon > 0$ small enough such that $a_\varepsilon U \subseteq \Xi_H$. Then, by G-invariance, we have

$$\mathcal{K}(z, a_t K_{\mathbb{C}}) = \mathcal{K}(a_\varepsilon z, a_{-\varepsilon} a_t K_{\mathbb{C}}) = \mathcal{K}(a_\varepsilon z, a_{t-\varepsilon} K_{\mathbb{C}})$$

for all $\varepsilon < t < 1$. The claim follows now, because $]\varepsilon, 1+\varepsilon[\ni t \mapsto \mathcal{K}(a_\varepsilon z, a_{t-\varepsilon}K_{\mathbb{C}}) \in \mathbb{C}$ is continuous, and hence $\lim_{t\nearrow 1} \mathcal{K}(z, a_t K_{\mathbb{C}}) = \mathcal{K}(a_\varepsilon z, a_{1-\varepsilon} K_{\mathbb{C}})$ exists and the convergence is uniform on compact subsets. ∎

We refer to Ψ as the *Cauchy-Szegö function* of $\mathcal{H}^2(\Xi_H)$. As $\mathcal{K}$ is G-invariant, it follows that $\mathcal{K}$ can be reconstructed from Ψ. Moreover, as $H_{\mathbb{C}}T(\Omega_{\mathrm{H}})K_{\mathbb{C}}/K_{\mathbb{C}}$ meets Ξ_H in an open set, we conclude that Ψ is uniquely determined by its restriction to $T(\Omega_{\mathrm{H}})K_{\mathbb{C}}/K_{\mathbb{C}} \subseteq \Xi_H$.

Using Theorem 2.5.2 (i) we finally obtain the spectral resolution of Ψ.

Theorem 3.2.4. *For $a \in T(\Omega_{\mathrm{H}})$ we have*

$$\Psi(aK_{\mathbb{C}}) = \int_{i\mathfrak{a}_+^*} \varphi_\lambda(z_{\mathrm{H}} a)\ \frac{d\lambda}{|\,\mathbf{c}_{\mathrm{G/H}}(\lambda)|^2},$$

where the integrals on the right hand side converge uniformly and absolutely on compact subsets of $T(\Omega_{\mathrm{H}})$. ∎

We now discuss the boundary value map $b\colon \mathcal{H}^2(\Xi_H) \to L^2(G/H)_{\rm mc}$. As usual, this boundary value map can be nicely defined pointwise only on an appropriate dense subspace of $\mathcal{H}^2(\Xi_H)$. Write $\mathcal{H}^2(\Xi_H)^\omega$ for the analytic vectors of the left regular representation $(L, \mathcal{H}^2(\Xi_H))$. Fix $f \in \mathcal{H}^2(\Xi_H)^\omega$. Then for every compact subset $C \subseteq G$ there exists an $0 < \varepsilon < 1$ such that $L(a_{-\varepsilon}g^{-1})f$ exists for all $g \in G$. In particular, if $0 < \varepsilon \leq t < 1$, then

$$f(ga_tK_{\mathbb{C}}) = f(ga_\varepsilon a_{t-\varepsilon}K_{\mathbb{C}}) = [L(a_{-\varepsilon}g^{-1})f)](a_{t-\varepsilon}K_{\mathbb{C}}) \ .$$

and so

$$\lim_{t\nearrow 1} f(ga_tK_{\mathbb{C}}) = [L(a_{-\varepsilon}g^{-1})f)](a_{1-\varepsilon}K_{\mathbb{C}}) \ .$$

It follows that we have a well defined G-equivariant boundary value map:

$$b^\omega\colon \mathcal{H}^2(\Xi_H)^\omega \to C(G/H), \quad b^\omega(f)(gH) = \lim_{t\nearrow 1} f(ga_tK_{\mathbb{C}}) \ . \tag{3.2.2}$$

Recall from (3.1.3) the isomorphism $\Phi\colon \int_{i\mathfrak{a}_+^*}^{\oplus} \mathcal{H}_\lambda \ d\mu(\lambda) \to \mathcal{H}^2(\Xi_H) = \mathcal{H}^2(\mu)$:

$$s = (s_\lambda)_\lambda \mapsto \left(xK_{\mathbb{C}} \mapsto \int_{\widehat{G}_s} \langle \pi_\lambda(x^{-1})s_\lambda, v_{\mathrm{K},\lambda}\rangle \ d\mu(\lambda)\right) \ .$$

It is useful to have the corresponding formula for b^ω on the space of sections with values in $\left(\int_{i\mathfrak{a}_+^*}^{\oplus} \mathcal{H}_\lambda \ d\mu(\lambda)\right)^\omega$, i.e., on the space of analytic sections. In this regard, it is better to replace $\mathcal{H}^2(\Xi_H)^\omega$ by some smaller but dense subspace $\mathcal{H}^2(\Xi_H)_0$. In order to define $\mathcal{H}^2(\Xi_H)_0$ we have to introduce some terminology. For a section $s = (s_\lambda)_\lambda \in \int_{i\mathfrak{a}_+^*}^{\oplus} \mathcal{H}_\lambda \ d\mu(\lambda)$ we define its support by $\mathrm{supp}(s) = \overline{\{\lambda \in i\mathfrak{a}_+^*\colon s_\lambda \neq 0\}}$. Futhermore we shall use the identifications $\mathcal{H}_\lambda^\omega = C^\omega(M\backslash K)$ for $\lambda \in i\mathfrak{a}_+^*$. Recall that if $f \in \mathcal{H}^2(\Xi_H)^\omega$ and $s = (s_\lambda)_\lambda = \Phi^{-1}(f)$, then almost each stalk s_λ is an analytic vector, i.e. $s_\lambda \in C^\omega(M\backslash K)$. The subspace $\mathcal{H}^2(\Xi_H)_0$ is then defined by

$$\mathcal{H}^2(\Xi_H)_0 = \left\{ f \in \mathcal{H}^2(\Xi_H)^\omega \quad : \quad \begin{array}{ll} f & \text{is } K\text{-finite,} \\ s = (s_\lambda)_\lambda = \Phi^{-1}(f) & \text{has compact support,} \\ s\colon i\mathfrak{a}_+^* \to C^\omega(M\backslash K) & \text{is weakly smooth.} \end{array} \right\}$$

It is an easy verification that $\mathcal{H}^2(\Xi_H)_0$ is a dense subspace of $\mathcal{H}^2(\Xi_H)$. Write $b_0\colon \mathcal{H}^2(\Xi_H)_0 \to C(G/H)$ for the restriction of b^ω to $\mathcal{H}^2(\Xi_H)_0$.

In the sequel we will often identify a function $f \in \mathcal{H}^2(\Xi_H)_0$ with its corresponding section $s = (s_\lambda) = \Phi^{-1}(f)$. We then claim that

$$b_0\colon \mathcal{H}^2(\Xi_H)_0 \to C(G/H), \quad s = (s_\lambda) \mapsto \left(gH \mapsto \int_{i\mathfrak{a}_+^*} \langle \pi_\lambda(g^{-1})s_\lambda, v_{\mathrm{H},\lambda}\rangle \ d\mu(\lambda)\right) \tag{3.2.3}$$

Notice that it is a priori not even clear that the right hand side of (3.2.3) is well defined. To establish (3.2.3) fix $f \in \mathcal{H}^2(\Xi_H)_0$ and $g \in G$. As f is an analytic vector for the left regular representation $(L, \mathcal{H}^2(\Xi_H))$ it follows that there exists an $0 < \varepsilon < 1$ such that $L(a_\varepsilon g^{-1})f$ exists. Using standard procedures one deduces that $\pi_\lambda(a_\varepsilon g^{-1})s_\lambda$ exists for almost all λ. In particular $s_\lambda \in \mathcal{H}_\lambda$ is analytic for almost all λ. Furthermore, $L(a_\varepsilon g^{-1})f$ corresponds to the section $(\pi_\lambda(a_\varepsilon g^{-1})s_\lambda)_\lambda$ and so

$$\|L(a_\varepsilon g^{-1})f\|^2 = \int_{i\mathfrak{a}_+^*} \|\pi_\lambda(a_\varepsilon g^{-1})s_\lambda\|^2 \ d\mu(\lambda) < \infty \ . \tag{3.2.4}$$

With the convention $\pi_\lambda(a_1)v_{K,\lambda} = v_{H,\lambda}$ we then have for all $\varepsilon \le t \le 1$ and almost all λ the estimate

$$\begin{aligned} |\langle \pi_\lambda(g^{-1})s_\lambda, \pi_\lambda(a_t)v_{K,\lambda}\rangle| &= |\langle \pi_\lambda(a_\varepsilon g^{-1})s_\lambda, \pi_\lambda(a_{t-\varepsilon})v_{K,\lambda}\rangle| \\ &\le \|\pi_\lambda(a_\varepsilon g^{-1})s_\lambda\| \cdot \|\pi_\lambda(a_{t-\varepsilon})v_{K,\lambda}\| \\ &\le M \cdot \|\pi_\lambda(a_\varepsilon g^{-1})s_\lambda\| \end{aligned} \tag{3.2.5}$$

with $M = \sup_{\substack{\lambda\in\mathrm{supp}(s)\\ \varepsilon\le t\le 1}} \|\pi_\lambda(a_{t-\varepsilon})v_{K,\lambda}\| < \infty$ as $\mathrm{supp}(s)$ is compact.

Recall our notion of holomorphic extension from Definition 2.2.1. As almost each stalk s_λ is an analytic vector in $\mathcal{H}_\lambda$, it follows from estimates (3.2.4-5) and the compactness of $\mathrm{supp}(s)$ that

$$\begin{aligned} \int_{i\mathfrak{a}_+^*} \langle \pi_\lambda(g^{-1})s_\lambda, v_{H,\lambda}\rangle \; d\mu(\lambda) &= \int_{\mathrm{supp}(s)} \lim_{t\nearrow 1} \langle \pi_\lambda(g^{-1})s_\lambda, \pi(a_t)v_{K,\lambda}\rangle \; d\mu(\lambda) \\ &= \lim_{t\nearrow 1} \int_{\mathrm{supp}(s)} \langle \pi_\lambda(g^{-1})s_\lambda, \pi_\lambda(a_t)v_{K,\lambda}\rangle \; d\mu(\lambda) \\ &= b^\omega(f)(gH) \; . \end{aligned}$$

As f and g were arbitray, this completes the proof of (3.2.3).

3.3. The Plancherel Theorem for $L^2(G/H)_{\mathrm{mc}}$

Before we can show that b_0 has image in $L^2(G/H)_{\mathrm{mc}}$ and extends to an isometric embedding, we need to recall some facts about the most continuous spectrum $L^2(G/H)_{\mathrm{mc}}$ of $L^2(G/H)$ (cf. [vdBS97a] and [D98]). All the results collected below are proved in [vdBS97a] or might be considered as special cases of [D98]. The crucial way where our assumption that G/H is NCC, $H = G^\tau$, and $G \subseteq G_\mathbb{C}$ with $G_\mathbb{C}$ simply connected, enters is the fact that $Z_H(\mathfrak{a}) = Z_K(\mathfrak{a})$ and $H = Z_H(\mathfrak{a})H_0$.

Recall from Subsection 2.7 the mapping $j(\lambda)\colon \mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|} \to (\mathcal{H}_\lambda^{-\infty})^H$ and the intertwining matrix $A_H(\lambda, w\lambda)^*\colon (\mathcal{H}_{w\lambda}^{-\infty})^H \to (\mathcal{H}_\lambda^{-\infty})^H$ both defined for generic $\lambda \in \mathfrak{a}_\mathbb{C}^*$, and all $w \in \mathcal{W}$. For generic λ we define

$$j^0(\lambda)\colon \mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|} \to (\mathcal{H}_\lambda^{-\infty})^H$$

by

$$j^0(\lambda) := [A_H(w_0\lambda, \lambda)^*]^{-1} \circ j(w_0\lambda) \tag{3.3.1}$$

with $w_0 \in \mathcal{W}$ the longest element. Then j^0 has no poles on $i\mathfrak{a}^*$ (cf. [vdBS97b, Th. 1]). Denote by $(\mathbf{e}_w)_{w\in\mathcal{W}/\mathcal{W}_0}$ the canonical basis of the Hilbert space $\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}$. For $w \in \mathcal{W}/\mathcal{W}_0$ define

$$\eta^0_{\lambda,w} := j^0(\lambda)\,\mathbf{e}_w \in (\mathcal{H}_\lambda^{-\infty})^H \; .$$

Define a Hilbert space structure on $\mathrm{Hom}(\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, \mathcal{H}_\lambda)$ using the identification

$$\mathrm{Hom}(\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, \mathcal{H}_\lambda) \simeq \mathcal{H}_\lambda \otimes [\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}]^* \; .$$

Write $\mathcal{S}(G/H)$ for the Schwartz space on G/H and $p_{\mathrm{mc}}\colon L^2(G/H) \to L^2(G/H)_{\mathrm{mc}}$ for the orthogonal projection on the most continuous spectrum. Set $\mathcal{S}_{\mathrm{mc}}(G/H) = p_{\mathrm{mc}}(\mathcal{S}(G/H))$. Then for functions $f \in \mathcal{S}_{\mathrm{mc}}(G/H)$ the Fourier transform is defined by

$$\mathcal{F}(f) = \big(\pi_\lambda(f)j^0(\lambda)\big)_\lambda \; . \tag{3.3.2}$$

By [D98, Th. 3] or [vdBS97a,Cor. 18.2 and Prop. 18.3], $\mathcal{F}$ extends to a G-equivariant unitary isomorphism

$$\mathcal{F}\colon L^2(G/H)_{\rm mc} \to \int_{i\mathfrak{a}^*_+}^{\oplus} \mathrm{Hom}(\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, \mathcal{H}_\lambda)\ d\lambda\ . \tag{3.3.3}$$

In particular, we have (using suitable normalization of measures) that

$$\|f\|^2 = \int_{i\mathfrak{a}^*_+} \|\mathcal{F}(f)(\lambda)\|^2\ d\lambda \tag{3.3.4}$$

for all $f \in \mathcal{S}_{\rm mc}(G/H)$.

Next we wish to describe $\mathcal{F}^{-1}$. Let $(\mathbf{e}^*_w)_{w\in\mathcal{W}/\mathcal{W}_0}$ be the dual basis of $(\mathbf{e}_w)_{w\in\mathcal{W}/\mathcal{W}_0}$. Then a section s of $\int_{i\mathfrak{a}^*_+}^{\oplus} \mathrm{Hom}(\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, \mathcal{H}_\lambda)\ d\lambda$ can be written as $s = (\sum_{w\in\mathcal{W}/\mathcal{W}_0} s_{\lambda,w} \otimes \mathbf{e}^*_w)_\lambda$ with $s_{\lambda,w} \in \mathcal{H}_\lambda$ for all $\lambda \in i\mathfrak{a}^*_+$ and $w \in \mathcal{W}/\mathcal{W}_0$. Recall that if $s = (\sum_{w\in\mathcal{W}/\mathcal{W}_0} s_{\lambda,w} \otimes \mathbf{e}^*_w)_\lambda$ is a smooth vector, then $s_{\lambda,w}$ is a smooth vector in $\mathcal{H}_\lambda$ for almost all λ. In the sequel we will use the identification $\mathcal{H}^\infty_\lambda = C^\infty(M\backslash K)$. Define a subspace of $\left(\int_{i\mathfrak{a}^*_+}^{\oplus} \mathrm{Hom}(\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, \mathcal{H}_\lambda)\ d\lambda\right)^\infty$ by

$$\mathcal{H}_0 = \left\{ s \in \left(\int_{i\mathfrak{a}^*_+}^{\oplus} \mathrm{Hom}(\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, \mathcal{H}_\lambda)\ d\lambda\right)^\infty \quad : \quad \begin{array}{l} s \text{ is } K\text{-finite, } \mathrm{supp}(s) \text{ is compact,} \\ s\colon i\mathfrak{a}^*_+ \to \mathrm{Hom}(\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, C^\infty(M\backslash K)) \\ \text{is weakly smooth.} \end{array} \right\}$$

It is not hard to see that $\mathcal{H}_0$ is a dense subspace in $\int_{i\mathfrak{a}^*_+}^{\oplus} \mathrm{Hom}(\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, \mathcal{H}_\lambda)\ d\lambda$. Then for an element $s \in \mathcal{H}_0$ the inverse Fourier-transform is given by [D98, Th. 3]

$$\mathcal{F}^{-1}(s)(gH) = \int_{i\mathfrak{a}^*_+} \sum_{w\in\mathcal{W}/\mathcal{W}_0} \langle \pi_\lambda(g^{-1}) s_{\lambda,w}, \eta^0_{\lambda,w}\rangle\ d\lambda\ . \tag{3.3.5}$$

Moreover [D98, Th. 3] implies that

$$\mathcal{F}^{-1}(\mathcal{H}_0) \subseteq \mathcal{S}_{\rm mc}(G/H)\ ; \tag{3.3.6}$$

in particular $\mathcal{F}(\mathcal{F}^{-1}(s))$ is given by the formula (3.3.2) for $s \in \mathcal{H}_0$.

Remark 3.3.1. We have normalized the invariant measure on G/H and the measure $d\lambda$ on $\mathfrak{a}^*$ so that (3.3.3) and (3.3.4) holds without any additional constants. This is possible, because we are only working with the principial series of representations and the most continuous part of the spectrum. In general, one has to take into account the order of several Weyl groups. We refer to Theorem 31 and Remark 32 in [vdB00] for general discussion on the normalization of measures. ■

3.4. Isometry of the boundary value mapping

In this subsection we complete our discussion of the boundary value mapping begun in Subsection 3.2.

Theorem 3.4.1. **(Isometry of the boundary value mapping)** *The boundary value mapping, initially defined by*

$$b_0: \mathcal{H}^2(\Xi_H)_0 \to C(G/H), \quad b_0(f)(gH) = \lim_{t \nearrow 1} f(g a_t K_{\mathbb{C}})$$

(cf. (3.2.2-3)*) extends to a G-equivariant isometric embedding*

$$b: \mathcal{H}^2(\Xi_H) \to L^2(G/H)_{\rm mc} \ .$$

Proof. For each $\lambda \in i\mathfrak{a}_+^*$ define a vector $\mathbf{b}(\lambda) \in (\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|})^*$ by

$$\mathbf{b}(\lambda) = \mathbf{c}(w_0\lambda) \sum_{w \in \mathcal{W}/\mathcal{W}_0} z_{\rm H}^{-w^{-1}(w_0\lambda+\rho)} \, \mathbf{e}_w^* \ .$$

Notice, that for $\lambda \in i\mathfrak{a}^*$ we have

$$|z_{\rm H}^{-2w^{-1}\lambda}| = z_{\rm H}^{-w^{-1}(\lambda+\rho)} \overline{z_{\rm H}^{-w^{-1}(\lambda+\rho)}} = z_{\rm H}^{-w^{-1}(\lambda+\rho)} z_{\rm H}^{w^{-1}(\bar\lambda+\rho)} \ .$$

Therefore, employing the Maass-Selberg relation for $\mathbf{c}(\lambda)$ we obtain

$$\| \, \mathbf{b}(\lambda)\|^2 = | \, \mathbf{c}(\lambda)|^2 \cdot \sum_{w \in \mathcal{W}/\mathcal{W}_0} |z_{\rm H}^{-w^{-1}\lambda}|^2 = | \, \mathbf{c}_{G/H}(\lambda)|^2 \ .$$

In particular we see that we have an G-equivariant isometric embedding of direct integrals

$$\iota: \int_{i\mathfrak{a}_+^*}^{\oplus} \mathcal{H}_\lambda \ d\mu(\lambda) \to \int_{i\mathfrak{a}_+^*}^{\oplus} \mathrm{Hom}(\mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|}, \mathcal{H}_\lambda) \ d\lambda, \quad s = (s_\lambda)_\lambda \mapsto (s_\lambda \otimes \mathbf{b}(\lambda))_\lambda \ .$$

From the definition of the spaces $\mathcal{H}^2(\Xi_H)_0$ and $\mathcal{H}_0$ it is then clear that

$$\iota\left(\Phi^{-1}(\mathcal{H}^2(\Xi_H)_0)\right) \subseteq \mathcal{H}_0 \ . \qquad (3.4.1)$$

As

$$\iota(s)_\lambda = \mathbf{c}(w_0\lambda) \sum_{w \in \mathcal{W}/\mathcal{W}_0} z_{\rm H}^{-w^{-1}(w_0\lambda+\rho)} s_\lambda \otimes \mathbf{e}_w^* \ ,$$

we get by (3.2.3), (3.3.5), Theorem 2.6.1 and Theorem 2.7.1 that

$$\begin{aligned}
[\mathcal{F}^{-1}(\iota(s))](gH) &= \int_{i\mathfrak{a}_+^*} \sum_{w \in \mathcal{W}/\mathcal{W}_0} \mathbf{c}(w_0\lambda) z_{\rm H}^{-w^{-1}(w_0\lambda+\rho)} \langle \pi_\lambda(g^{-1}) s_\lambda, \eta^0_{\lambda,w} \rangle \ d\lambda \\
&= \int_{i\mathfrak{a}_+^*} \sum_{w \in \mathcal{W}/\mathcal{W}_0} \langle \pi_\lambda(g^{-1}) s_\lambda, \mathbf{c}(w_0\bar\lambda) z_{\rm H}^{w^{-1}(w_0\bar\lambda+\rho)} \eta^0_{\lambda,w} \rangle \ d\lambda \\
&= \int_{i\mathfrak{a}_+^*} \langle \pi_\lambda(g^{-1}) s_\lambda, v_{{\rm H},\lambda} \rangle \ d\lambda \\
&= b_0(s)(gH)
\end{aligned}$$

From this and (3.3.6) it follows that $b_0(f) \in S_{\rm mc}(G/H)$; in particular we have $b_0(f) \in L^2(G/H)_{\rm mc}$. Finally,

$$\|b_0(f)\|_{L^2(G/H)_{\rm mc}} = \|\mathcal{F}^{-1}(\iota(s))\| = \|\iota(s)\| = \|s\| = \|\Phi(s)\| = \|f\|$$

as $\mathcal{F}$, ι and Φ are isometric. This completes the proof of the theorem. ■

Remark 3.4.2. The domain Ξ_H is maximal in the sense that generic functions in $\mathcal{H}^2(\Xi_H)$ do not extend holomorphically over Ξ_H. ■

3.5. Concluding remarks and the example of $G = \mathrm{Sl}(2,\mathbb{R})$

In [GKÓ01] we defined a Hardy space $\mathcal{H}^2(\Xi)$ on Ξ for the cases where $\Xi = \Xi_H$. Let us briefly summarize its construction in order to put it into perspective with the results in this section.

Geometrically the situation $\Xi = \Xi_H$ is equivalent to the fact that Ξ is homogeneous for a bigger Hermitian group $S \supseteq G$ (cf. [KS01b]). More precisely, if $U < S$ denotes an appropriate maximal compact subgroup with $K \subseteq U$ then Ξ is G-biholomorphic to the Hermitian symmetric space S/U. For example if G is Hermitian, then $S = G \times G$ and $\Xi \simeq G/K \times \overline{G/K}$.

The assumption $\Xi = \Xi_H$ thus allows us to identify Ξ with a bounded symmetric domain $\mathcal{D} \simeq S/U$. Within this identification one shows that $\partial_d \Xi \simeq G/H$ becomes a Zariski-open subset in the Shilov boundary $\partial_s \mathcal{D}$ of $\mathcal{D}$.

The identification of Ξ with $\mathcal{D}$ was used in [GKÓ01] in a crucial way: One can transfer the action of an appropriate compression-semigroup $\Gamma \supseteq G$ on $\mathcal{D}$ to Ξ and use this to give a definition of a Hardy space as follows:

$$\mathcal{H}^2(\Xi) = \{f \in \mathcal{O}(\Xi) \colon \|f\|^2 = \sup_{\gamma \in \operatorname{int}\Gamma} \int_{G/H} |f(\gamma g z_{\mathrm{H}})|^2 \, dgH < \infty\} . \tag{3.5.1}$$

In [GKÓ01] we have shown – with entirely different methods – that the Hardy space defined as in (3.5.1) has the following properties:

(3.5.2) $\mathcal{H}^2(\Xi)$ is a Hilbert space of holomorphic functions featuring an isometric boundary value mapping $b \colon \mathcal{H}^2(\Xi) \hookrightarrow L^2(G/H)_{\mathrm{mc}}$. Moreover, $\operatorname{im} b$ is a multiplicity one subspace of *full spectrum*.

(3.5.3) The Hardy space $\mathcal{H}^2(\Xi)$ is G-isometric to $L^2(G/K)$ through a transform of Segal-Bargmann type.

(3.5.4) $\mathcal{H}^2(\Xi)$ is G-isometric to the classical Hardy space $\mathcal{H}^2(\mathcal{D})$ through an explicitely given mapping.

In particular for $\Xi = \Xi_H$ it follows from Theorem 3.4.1 and (3.5.2) that the definition of (3.5.1) coincides with our spectral definition of the Hardy space in Definition 3.2.2. For the cases where $\Xi \neq \Xi_H$ there is no apparent semigroup action on Ξ_H and a definition of $\mathcal{H}^2(\Xi_H)$ in the flavour of (3.5.1) seems presently not possible.

Notice that (3.5.3) implies that the Plancherel measure of $\mathcal{H}^2(\Xi)$ has support equal to $i\mathfrak{a}_+^*$. However, in [GKÓ01] we could not determine this measure explicitely. With the new approach given in this section this difficulty is already taken care of with the definition of the Hardy space.

The explicit isomorphism of $\mathcal{H}^2(\Xi)$ with the classical Hardy space $\mathcal{H}^2(\mathcal{D})$ allows us to find also a nice closed expression for the Cauchy-Szegö function Ψ (cf. [GKÓ01, Th. 5.7 and Ex. 5.10]). Combining this closed expression with the spectral resolution of Ψ in Theorem 3.2.4 one obtains interesting identities for (generalized) hypergeometric functions. For example for $G = \mathrm{Sl}(2,\mathbb{R})$ one obtains the following formula:

$$\frac{1-\tanh^2 t}{1+\tanh^2 t} = \frac{\pi}{2}\int_0^\infty F\left(\frac{1}{4}+i\frac{\lambda}{4}, \frac{1}{4}-i\frac{\lambda}{4}, 1; -\sinh^2\left(2t+i\frac{\pi}{2}\right)\right) \cdot \left|\frac{\Gamma\left(\frac{i\lambda+1}{2}\right)}{\Gamma\left(\frac{i\lambda}{2}\right)}\right|^2 \frac{d\lambda}{\cosh\frac{\pi}{2}\lambda}$$

for all $t \in \mathbb{R} + i]-\frac{\pi}{4}, \frac{\pi}{4}[$. Here F denotes the Gauß hypergeometric function.

A. Appendix: Analytic vectors for representations

In this appendix we will summarize some facts on analytic vectors for representations. None of the results collected below is new, however some of them might be hard to find explicitely in the literature. In order to keep the exposition short, we will omit proofs and often do not make the most general assumptions. A more detailed account containing complete proofs can be found in the forthcoming survey [KÓ03].

A.1. Definition and topology of analytic vectors

Throughout this appendix G will denote a connected unimodular Lie group with $G \subseteq G_{\mathbb{C}}$.

Let E be a complex Banach space and $\mathrm{Gl}(E)$ the group of continuous invertible operators on E. By a (Banach) representation of (π, E) of G we will understand a group homomorphism $\pi\colon G \to \mathrm{Gl}(E)$ such that for all $v \in E$ the orbit mapping

$$\gamma_v\colon G \to E, \quad g \mapsto \pi(g)v$$

is continuous.

A vector $v \in E$ is called *analytic* if γ_v is an analytic E-valued map or, equivalently, if there exists an open neighborhood U of $\mathbf{1}$ in $G_{\mathbb{C}}$ and a G-equivariant holomorphic mapping

$$\gamma_{v,U}\colon GU \to E$$

such that $\gamma_{v,U}(\mathbf{1}) = v$. In particular, $\gamma_{v,U}|_G = \gamma_v$.

The vector space of all analytic vectors for (π, E) is denoted by E^ω. We recall a fundamental result of Nelson which states that E^ω is dense in E.

Next we are going to recall the definition of the topology on E^ω.

For a complex manifold M let us denote by $\mathcal{O}(M, E)$ the space of all E-valued holomorphic mappings on E. Topologically we consider $\mathcal{O}(M, E)$ as a Fréchet space with the topology of compact convergence.

For any open neighborhood U of $\mathbf{1}$ in $G_{\mathbb{C}}$ we write E_U for the subspace of E^ω for which $\gamma_{v,U}$ exists. Then we obtain a linear embedding

$$\eta_U\colon E_U \to \mathcal{O}(GU, E), \quad v \mapsto \gamma_{v,U} .$$

The image of η_U is closed and hence $\mathcal{O}(GU, E)$ induces a Fréchet topology on E_U. Notice that for $U_1 \subseteq U_2$ we obtain a continuous embedding $E_{U_2} \to E_{U_1}$ via restriction. Thus

$$E^\omega = \lim_{U \to \{1\}} E_U = \bigcup_U E_U$$

and we can equip E^ω with the inductive limit topology, i.e. the finest topology on E^ω for which all inclusion mappings $E_U \to E^\omega$ become continuous. Notice that this turns E^ω into a locally convex topological vector space.

By $E^{-\omega}$ we will denote the antidual of E^ω, i.e. the space of all antilinear continuous functionals on E^ω. The space $E^{-\omega}$ is referred to as the space of *hyperfunction vectors* of the representation (π, E). We equip $E^{-\omega}$ with the topology of bounded convergence.

A.2. Analytic vectors for $L^1(G/H)$

Let $H < G$ be a closed subgroup such that G/H carries a G-invariant measure. We write $L^1(G/H)$ for the corresponding Banach space of integrable functions and $(L, L^1(G/H))$ for the left regular representation of G on $L^1(G/H)$, i.e.,

$$(L(g)f)(xH) = f(g^{-1}xH) \qquad (g, x \in G, f \in L^1(G/H)) \ .$$

Further it is convenient to assume that $G/H \subseteq G_{\mathbb{C}}/H_{\mathbb{C}}$. Then we have the following characterization of the analytic vectors:

Proposition A.2.1. *Let U be an open neighborhood of $\mathbf{1}$ in $G_{\mathbb{C}}$. Then $f \in L^1(G/H)_U$ if and only if there exists a holomorphic function $\widetilde{f}$ on the open set*

$$U^{-1}GH_{\mathbb{C}}/H_{\mathbb{C}} \subseteq G_{\mathbb{C}}/H_{\mathbb{C}}$$

with the following properties:

(1) $\widetilde{f}|_{G/H} = f$.

(2) *For all $x \in U$ the map*

$$\widetilde{f}_x : G/H \to \mathbb{C}, \quad gH \mapsto \widetilde{f}(x^{-1}gH)$$

belongs to $L^1(G/H)$.

(3) *For all compact subsets $U^c \subseteq U$ we have*

$$\sup_{x \in U^c} \|\widetilde{f}_x\| < \infty \ .$$

■

There are two types of homogeneous spaces G/H which will be of particular interest for us. The first is when $H = \mathbf{1}$. Then $L^1(G)^\omega$ denotes the analytic for the left regular representation of G on $L^1(G)$. The second case is for $G = H \times H$ and $H < G$ the diagonal subgroup. In this case $G/H \simeq H$ and L becomes left-right regular representation of $H \times H$ on H. Here we shall write $L^1(H)^{\omega,\omega}$ for the analytic vectors.

A.3. Averaging properties

Recall that the average map

$$C_c(G) \to C_c(G/H), \quad f \mapsto f^H; f^H(xH) = \int_H f(xh) \ dh$$

is contiunuous and onto. Further, this map extends to a surjective contraction of Banach spaces $L^1(G) \to L^1(G/H)$. We will show that the averaging operator maps analytic vectors into analytic vectors.

A standard application of the Bergman estimate gives:

Lemma A.3.1. *Let $U \subseteq G_{\mathbb{C}}$ be an open neighborhood of $\mathbf{1}$. Then for any pair of compact subsets $U_1, U_2 \subseteq U$ with $U_1 \subseteq \operatorname{int} U_2$ there exists a constant $C > 0$ such that for all $f \in L^1(G)_U$ we have that*

$$(\forall x \in U_1^{-1}G) \qquad \int_H |\widetilde{f}(xh)| \ dh \leq C \sup_{x \in U_2} \|\widetilde{f}_x\|,$$

where $\widetilde{f}$ denotes the extension of f to a holomorphic function on $U^{-1}G$ (cf. Proposition A.2.1). ■

Combining Lemma A.3.1 with Proposition A.2.1 we obtain:

Proposition A.3.2. *Let $U \subseteq G_{\mathbb{C}}$ be an open neighborhood of $\mathbf{1}$. Then for every $f \in L^1(G)_U$ and $g \in G$ the integral $f^H(g) = \int_H f(gh)\, dh$ converges absolutely and $f^H \in L^1(G/H)_U$. In particular, there is a well defined mapping*

$$L^1(G)^\omega \to L^1(G/H)^\omega, \quad f \mapsto f^H .$$

■

A.4. Mollifying properties

In this section $E = \mathcal{H}$ will be a Hilbert space and $(\pi, \mathcal{H})$ a unitary representation of G. For $f \in L^1(G)$ one defines a continuous operator $\pi(f): \mathcal{H} \to \mathcal{H}$ by

$$\pi(f)v = \int_G f(g)\pi(g)v \, dg \qquad (v \in \mathcal{H}) .$$

Notice that this defines a $*$-representation of the Banach algebra $L^1(G)$, i.e. we have $\pi(f * g) = \pi(f)\pi(g)$ and $\pi(f)^* = \pi(f^*)$ with $f^*(x) = \overline{f(x^{-1})}$.

Recall that $L^1(G)^{\omega,\omega}$ denotes the analytic vectors for the left-right regular representation of $G \times G$ on $L^1(G)$. It is easy to see that $L^1(G)^{\omega,\omega}$ is $*$-closed subalgebra of $L^1(G)$.

Let $f \in L^1(G)^{\omega,\omega}$. It follows readily from Proposition A.2.1 and the definition of analytic vectors that $\pi^\omega(f)$ maps $\mathcal{H}$ continuously into $\mathcal{H}^\omega$. In particular the restriction $\pi^\omega(f) := \pi(f)|_{\mathcal{H}^\omega}$ gives rise to a continuos operator $\pi^\omega(f): \mathcal{H}^\omega \to \mathcal{H}^\omega$. Hence we have an algebra representation:

$$\pi^\omega : L^1(G)^{\omega,\omega} \to \mathrm{End}(\mathcal{H}^\omega), \quad f \mapsto \pi^\omega(f) .$$

The corresponding dual representation is given by

$$\pi^{-\omega} : L^1(G)^{\omega,\omega} \to \mathrm{End}(\mathcal{H}^{-\omega}); \quad \pi^{-\omega}(f)\lambda = \lambda \circ \pi^\omega(f^*) .$$

Another application of Proposition A.2.1 then gives us the mollifying property:

Proposition A.4.1. *Let $(\pi, \mathcal{H})$ be a unitary representation of a unimodular Lie group G. Then we have for all $f \in L^1(G)^{\omega,\omega}$ that*

$$\pi^{-\omega}(f)\mathcal{H}^{-\omega} \subseteq \mathcal{H}^\omega .$$

■

Note: For $f \in L^1(G)^{\omega,\omega}$, it is often convenient to write $\pi(f)$ instead of $\pi^{-\omega}(f)$. We will use this convention throughout Section 2 in the main text.

References

[AG90] Akhiezer, D. N., and S. G. Gindikin, *On Stein extensions of real symmetric spaces*, Math. Ann. **286**, 1–12, 1990.

[vdB88] van den Ban, E., *The principal series for a reductive symmetric space I, H-fixed distribution vextors*, Ann. sci. Éc. Norm. Sup. **4**, **21** (1988), 359–412.

[vdB00] —, *The Plancherel theorem for a reductive symmetric space*, Lectures for the European School of Group Theory. August 14–26, 2000, SDU-Odense University. http://www.math.uu.nl/people/ban/publ.html.

[vdBD88] van den Ban, E., and P. Delorme, *Quelques propriétés des représentations sphériques pour les espaces symétriques réductifs*, J. Funct. Anal. **80** (1988), 284–307.

[vdBS97a] van den Ban, E., and H. Schlichtkrull, *The most continuous part of the Plancherel decomposition for a reductive symmetric space*, Ann. of Math. **(2) 145** (1997), no. **2**, 267–364.

[vdBS97b] —, *Fourier transform on a semisimple symmetric space*, Invent. Math. **130** (1997), no. **3**, 517–574.

[BD92] Brylinski, J.-L., and P. Delorme, *Vecteurs distributions H-invariants pour les séries principales généralisées d'espaces symétriques réductifs et prolongement méromorphe d'intégralesd'Eisenstein*, Invent. Math. **109** (1992), no. **3**, 619–664.

[D98] Delorme, P., *Formule de Plancherel pour les espaces symétriques réductifs*, Ann. of Math. **(2) 147** (1998), no. **2**, 417–452.

[FT99] Faraut, J., and E. G. F. Thomas, *Invariant Hilbert spaces of holomorphic functions*, J. Lie Theory **9** (1999), no. **2**, 383–402.

[GK02a] Gindikin, S., and B. Krötz, *Complex crowns of Riemannian symmetric spaces and non-compactly causal symmetric spaces*, Trans. Amer. Math. Soc. **354** (2002), no. **8**, 3299–3327.

[GK02b] —, *Invariant Stein domains in Stein symmetric spaces and a non-linear complex convexity theorem*, IMRN **18** (2002), 959–971.

[GKÓ083] Gindikin, S., B. Krötz and G. Ólafsson, *Hardy spaces for non-compactly causal symmetric spaces and the most continuous spectrum*, Math. Ann. **327** (2003), 25–66.

[H84] Helgason, S., "Groups and Geometric Analysis", Academic Press, 1984.

[HÓ96] Hilgert, J. and G. Ólafsson, "Causal Symmetric Spaces, Geometry and Harmonic Analysis," Acad. Press, 1996.

[K99] Krötz, B., *The Plancherel theorem for biinvariant Hilbert spaces*, Publ. Res. Inst. Math. Sci. **35** (1999), no. **1**, 91–122.

[KÓ03] Krötz, B., and G. Ólafsson, *Analytic vectors for representations – a survey*, in preparation.

[KS01a] Krötz, B., and R.J. Stanton, *Holomorphic extension of representations: (I) automorphic functions*, Annals of Mathematics, to appear.

[KS01b] Krötz, B., and R.J. Stanton, *Holomorphic extensions of representations: (II) geometry and harmonic analysis*, preprint.

[M79] Matsuki, T., *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), 331–357.

[Ó87] Ólafsson, G., *Fourier and Poisson transformation associated to semisimple symmetric space*, Invent Math. **90** (1987), 605–629.

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