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## The Paley–Wiener Theorem for the Jacobi transform and the local Huygens’ principle for root systems with even multiplicities <sup>☆</sup>

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*Dedicated to Gerrit van Dijk on the occasion of his 65th birthday*

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### ABSTRACT

This note is a continuation of the previous paper by same authors [this issue]. Its purpose is to extend the results to the context of root systems with even multiplicities. Under the even multiplicity assumption, we prove a local Paley–Wiener theorem for the Jacobi transform and the strong Huygens’ principle for the wave equation associated with the modified compact Laplace operator.

### INTRODUCTION

Harish-Chandra’s theory of spherical functions on Riemannian symmetric spaces of the noncompact type, resp. of the compact type, was generalized in the late 1980’s by G. Heckman and E. Opdam into the theory of hypergeometric functions, resp. of Jacobi polynomials, associated with root systems. Correspondingly, the noncompact and compact spherical transforms have as natural generalizations the Opdam’s hypergeometric transform and the Jacobi transform. We refer to [2,4,5] and references therein for more information. It is then a natural question, how

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known results in harmonic analysis on symmetric spaces can be extended to these new integral transforms.

In [1] we proved a Paley–Wiener theorem for the spherical transform on compact symmetric spaces with even multiplicities. The Paley–Wiener theorem was then used to show (by three different methods) that the local strong Huygens’ principle holds for the modified wave equation on these spaces. In this note we show how our results can be extended to prove a local Paley–Wiener theorem for the Jacobi transform and then the strong Huygens principle for the modified Laplacian associated with a root system with even multiplicities.

In the context of special functions associated with root systems, the symmetric space is replaced by a triple  $(\mathfrak{a}, \Delta, m)$ , consisting of an  $n$ -dimensional real Euclidean vector space  $\mathfrak{a}$ , a root system  $\Delta$  in the dual  $\mathfrak{a}^*$  of  $\mathfrak{a}$ , and a multiplicity function  $m$  on  $\Sigma$ . Hence  $\alpha \mapsto m_\alpha$  is a map on  $\Delta$  which is invariant with respect to the Weyl group  $W$  of  $\Delta$ . We will write  $m \geq 0$  to indicate that  $m_\alpha \geq 0$  for all  $\alpha \in \Delta$ . The two results in the title will be proved under the additional assumption that  $\Delta$  is reduced, i.e.  $2\alpha \notin \Delta$  for all  $\alpha \in \Delta$ , and that  $m$  is even, i.e.  $m_\alpha \in 2\mathbb{N}$  for all  $\alpha \in \Delta$ . In the geometric situation of symmetric spaces, even multiplicity functions occur only on reduced root systems. In our more general context, the two conditions have to be imposed. We remark that our notation is based on the theory of symmetric spaces which differs from the Heckman–Opdam notation in the following ways. The root system  $R$  used by Heckman and Opdam is related to our root system  $\Delta$  by  $R = \{2\alpha \mid \alpha \in \Delta\}$  and the multiplicity function  $k$  in Heckman–Opdam’s work is given by  $k_{2\alpha} = m_\alpha/2$ .

Compared to [1], this note contains two new ingredients: an explicit formula for the Jacobi polynomials in the even multiplicity case, and a detailed study of the convergence of certain Jacobi expansions. It is also important to notice that in [1] several results were proved under the more restrictive condition (always satisfied by Riemannian symmetric spaces with even multiplicities) that the multiplicity function is a constant function, i.e. there is  $m \in \mathbb{N}$  such that  $m_\alpha = 2m$  for all  $\alpha \in \Delta$ . This condition will be dropped in the present note. We also remark that we will not work out all details. Our aim is to prove what is necessary to apply the results and methods from [1].

## 1. JACOBI POLYNOMIALS AND THE JACOBI TRANSFORM

Let  $P$  be the lattice of restricted weights. This consists of all elements  $\mu \in \mathfrak{a}_\mathbb{C}^*$  so that  $\mu_\alpha := \langle \mu, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$  for all  $\alpha \in \Delta$ . In particular,  $P \subset \mathfrak{a}^*$ . Then  $A_\mathbb{C} := \text{Hom}_\mathbb{Z}(P, \mathbb{C}^\times)$  is a complex torus with Lie algebra  $\mathfrak{a}_\mathbb{C}$ . It admits the decomposition  $A_\mathbb{C} = AT$  with  $A := \exp(\mathfrak{a})$  and  $T = \exp(\mathfrak{t})$  with  $\mathfrak{t} = i\mathfrak{a}$ . Recall that harmonic analysis of  $K$ -invariant function spaces on  $G/K$ , respectively  $U/K$ , can be reduced to harmonic analysis of Weyl group invariant objects on a maximal flat subspace. Therefore the Lie groups  $A$  and  $T$  are respectively replacements for the symmetric spaces  $G/K$  and  $U/K$ .

### 1.1. Jacobi polynomials

Let  $\mathbb{C}[A_{\mathbb{C}}]$  denote the space of finite  $\mathbb{C}$ -linear combinations of elements  $e^{\mu}$  with  $\mu \in P$ , and let  $\mathbb{C}[A_{\mathbb{C}}]^W$  be the subspace of  $W$ -invariant elements. We will always assume that  $m \geq 0$ . Fix a set of positive roots  $\Delta^+$  in  $\Delta$  and set  $\delta(m, t) := \prod_{\alpha \in \Delta^+} |e^{\alpha(\log t)} - e^{-\alpha(\log t)}|^{m_{\alpha}}$ . Define an inner product  $\langle \cdot, \cdot \rangle_m$  on  $\mathbb{C}[A_{\mathbb{C}}]^W$  by

$$\langle f, g \rangle_m := \int_T f(t) \overline{g(t)} \delta(m, t) dt,$$

where  $dt$  is the normalized Haar measure on  $T$ .

Let  $P^+ := \{\mu \in \mathfrak{a}^* \mid \mu_{\alpha} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Delta^+\}$ . For  $\mu \in P^+$  and  $t \in T$  let  $P(m, \mu, t)$  denote the Jacobi polynomial with spectral parameter  $\mu$ . This is defined as follows. The orbit sums

$$M(\mu) := \sum_{\nu \in W\mu} e^{\nu}$$

form a basis of  $\mathbb{C}[A_{\mathbb{C}}]^W$  as  $\mu$  varies in  $P^+$  because each  $W$ -orbit in  $P$  intersects  $P^+$  in exactly one point. The Jacobi polynomial  $P(m, \mu)$  is the exponential polynomial

$$(1.1) \quad P(m, \mu) := \sum_{\nu \geq \mu \in P^+} c_{\mu\nu}(m) M(\nu)$$

where the coefficients  $c_{\mu\nu}(m)$  are defined by the following conditions, cf. [2, §1.3]:

- (i)  $c_{\mu\mu}(m) = 1$ ;
- (ii)  $\langle P(m, \mu), M(\nu) \rangle_m = 0$  for all  $\nu \in P^+$  with  $\nu < \mu$ .

Observe that by definition,  $P(m, \mu, t)$  extends holomorphically to  $A_{\mathbb{C}}$  as a function of  $t$ . Moreover,  $\{P(m, \mu) \mid \mu \in P^+\}$  is a basis for  $\mathbb{C}[A_{\mathbb{C}}]^W$  which is orthogonal with respect to the inner product  $\langle \cdot, \cdot \rangle_m$  (cf. [2, Corollary 1.3.13]).

The  $L^2$ -norm of  $P(m, \mu)$  is expressed in terms of two  $\mathbf{c}$ -functions. The first one, denoted  $\mathbf{c}(m, \lambda)$ , is the  $\mathbf{c}$ -function defined by means of the Gindikin–Karpelevic formula as in [1, (2.10)]. In particular, for reduced root systems and even multiplicity functions,

$$(1.2) \quad \frac{1}{\mathbf{c}(m, \lambda)} = C \prod_{\alpha \in \Delta^+} \prod_{k=0}^{m_{\alpha}/2-1} (\lambda_{\alpha} + k)$$

where the constant  $C$  is given by

$$(1.3) \quad C = \prod_{\alpha \in \Delta^+} \prod_{k=0}^{m_{\alpha}/2-1} \frac{1}{\rho_{\alpha} + k}$$

and  $\rho := \rho(m) := \frac{1}{2} \sum_{\alpha \in \Delta^+} m_{\alpha} \alpha$  as usual.

The second function, denoted  $\mathbf{c}^*(m, \lambda)$ , is a dual  $\mathbf{c}$ -function. In the case of reduced root systems and even multiplicities, these two functions are related by the formula

$$\mathbf{c}^*(m, \lambda) = C (-1)^{\sum_{\alpha \in \Delta^+} m_\alpha / 2} \mathbf{c}(m, \lambda)$$

with  $C$  as in (1.3). See [2, (3.5.2)] for the general definition of  $\mathbf{c}^*$ . Again assuming even multiplicities, one has for  $\mu \in P^+$  (see [2, Corollary 3.5.3]):

$$(1.4) \quad \|P(m, \mu)\|_m^2 = \frac{|W| \mathbf{c}^*(m, -\mu - \rho)}{C \mathbf{c}(m, \mu + \rho)} = |W| \prod_{\alpha \in \Delta^+} \prod_{k=0}^{m_\alpha / 2 - 1} \frac{(\mu_\alpha + \rho_\alpha + k)}{(\mu_\alpha + \rho_\alpha - k)}.$$

The positivity of the last member in (1.4) depends on the following lemma.

**Lemma 1.1.** *Assume that  $\Delta$  is a reduced root system and that for all  $\alpha \in \Delta$  the multiplicities  $m_\alpha$  are even. Then  $\rho_\alpha \geq m_\alpha / 2$  for every  $\alpha \in \Delta^+$ .*

**Proof.** Observe first that

$$(1.5) \quad \rho_\alpha > 0 \quad \text{for every } \alpha \in \Delta^+.$$

Indeed, if  $\alpha$  is a simple root in a reduced root system  $\Delta$ , then  $\rho_\alpha = m_\alpha / 2$ . See [1, Lemma 2.7(1)].

We prove the inequalities in the statement by induction on the length  $\ell(\alpha)$  of a root  $\alpha \in \Delta^+$ . Recall that, if  $\Pi := \{\alpha_1, \dots, \alpha_n\}$  is the basis of simple roots in  $\Delta^+$  and  $\alpha = \sum_{j=1}^n n_j \alpha_j$ , then  $\ell(\alpha) := \sum_{j=1}^n n_j$ . The inequalities being true when  $\ell(\alpha) = 1$ , we then suppose that  $\ell(\alpha) > 1$ . Because of (1.5), there must be  $\gamma \in \Pi$  so that  $\langle \gamma, \alpha \rangle > 0$ . Since  $\ell(\alpha) > 1$ , the root  $\alpha$  is not simple. Hence  $s_\gamma \alpha := \alpha - 2 \frac{\langle \gamma, \alpha \rangle}{\langle \gamma, \gamma \rangle} \gamma$  satisfies  $s_\gamma \alpha \in \Delta^+$  and  $\ell(s_\gamma \alpha) < \ell(\alpha)$ . We can therefore apply the inductive hypothesis to  $s_\gamma \alpha$ . Noticing that  $s_\gamma \rho = \rho - m_\gamma \gamma$ , we get:

$$\begin{aligned} \frac{\langle \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle} &= \frac{\langle s_\gamma \rho, s_\gamma \alpha \rangle}{\langle s_\gamma \alpha, s_\gamma \alpha \rangle} = \frac{\langle \rho, s_\gamma \alpha \rangle}{\langle s_\gamma \alpha, s_\gamma \alpha \rangle} - m_\gamma \frac{\langle \gamma, s_\gamma \alpha \rangle}{\langle s_\gamma \alpha, s_\gamma \alpha \rangle} \\ &\geq m_{s_\gamma \alpha} / 2 + m_\gamma \frac{\langle \gamma, \alpha \rangle}{\langle s_\gamma \alpha, s_\gamma \alpha \rangle} \geq m_{s_\gamma \alpha} / 2 = m_\alpha / 2. \quad \square \end{aligned}$$

## 1.2. Hypergeometric functions

Let  $F(m, \lambda, a)$  denote the hypergeometric function of spectral parameter  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ . By definition, this is the unique solution of the Heckman–Opdam hypergeometric system of differential equations corresponding to the spectral parameter  $\lambda$  which is analytic in the space parameter  $a \in A$  and normalized by the condition  $F(m, \lambda, e) = 1$ . Here  $e$  denotes the identity element of  $A$ . See, e.g., [2, §4]. If the triple  $(\mathfrak{a}, \Delta, m)$  is geometric (i.e. arises from a Riemannian symmetric space of the noncompact type  $G/K$ ), then  $F(m, \lambda, a)$  agrees with the restriction to  $A = \exp(\mathfrak{a})$  of the spherical function  $\varphi_\lambda$  from [1, (2.7)].

### 1.3. Explicit formulas

The generalization of Lemma 2.5 in [1] and the Weyl group invariance stated in Theorem 2.11 in [1] is given as follows.

**Lemma 1.2.** *Suppose that  $m \geq 0$  is a fixed multiplicity function. Then*

$$(1.6) \quad F(m, \mu + \rho, a) = \mathbf{c}(m, \mu + \rho)P(m, \mu, a).$$

Furthermore,

$$(1.7) \quad c(m, w(\mu - \rho) - \rho)P(m, w(\mu - \rho) - \rho) = c(m, \mu)P(m, \mu)$$

for all  $\mu \in P^+$  and  $w \in W$ .

**Proof.** See [2, (4.4.10)].  $\square$

Formula (1.6) provides a holomorphic extension of the hypergeometric function  $F(m, \mu + \rho)$  to  $A_{\mathbb{C}}$ . Moreover, the second relation shows that we can extend the definition of  $P(m, \mu)$  to  $\mu \in P$ . Observe, however, that in the context of special functions associated with root systems, the functions  $F(m, \lambda)$  and  $P(m, \mu)$  do not generally admit integral representations. Likewise, these functions cannot be considered as matrix coefficients of group representations. Nevertheless, the representation dimensions, or Plancherel density,  $d(\mu) = d(m, \mu)$  can still be introduced by means of Vretare's formula:

$$(1.8) \quad d(m, \mu) := \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{c}(m, -\rho + \varepsilon)}{\mathbf{c}(m, \mu + \rho)\mathbf{c}(m, -(\mu + \rho) + \varepsilon)}.$$

As in [1], this formula simplifies because of Lemma 1.1, which allows us to compute the limit in (1.8) as the quotient of the limits of the  $\mathbf{c}$ -functions appearing in the numerator and in the denominator.

**Corollary 1.3.** *Assume that  $\Delta$  is reduced and that  $m_{\alpha}$  is even for each  $\alpha \in \Delta$ . Then the following properties hold:*

(1) *For all  $\mu \in P^+$  we have*

$$d(m, \mu) = \frac{\mathbf{c}(m, -\rho)}{\mathbf{c}(m, \mu + \rho)\mathbf{c}(m, -(\mu + \rho))}.$$

(2) *The function  $d(m, \mu)$  extends as a polynomial function on  $\mathfrak{a}_{\mathbb{C}}^*$  given by*

$$d(m, \lambda) = \prod_{\alpha \in \Delta^+} \prod_{k=0}^{m_{\alpha}/2-1} \frac{k^2 - (\lambda + \rho)_{\alpha}^2}{k^2 - \rho_{\alpha}^2}.$$

Assuming that  $\Delta$  is reduced and all multiplicities are even, the function  $\delta(m, t)$  extends to a  $W$ -invariant holomorphic function on  $A_{\mathbb{C}}$ . Moreover, because of Theorem 5.1(c) in [3], there exists a  $W$ -invariant differential operator  $D$  on  $A$  with coefficients which are holomorphic on  $A_{\mathbb{C}}$  so that for all  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  and all  $a \in A$  we have

$$(1.9) \quad \delta(m, a)F(m, \lambda, a) = \frac{1}{d(m, \lambda - \rho)} D\left(\sum_{w \in W} a^{w\lambda}\right).$$

The right-hand side of (1.9) is holomorphic in  $a \in A_{\mathbb{C}}$  and therefore provides a holomorphic extension of  $\delta(m, a)F(m, \lambda, a)$ . It also follows from (1.9) that

$$(1.10) \quad D\left(\sum_{w \in W} a^{w\lambda}\right) = 0$$

for all  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  satisfying  $\lambda_{\alpha} \in \pm\{0, 1, \dots, m_{\alpha}/2 - 1\}$  for some  $\alpha \in \Delta^+$ .

As an easy corollary of the above, we obtain the following formulas for the Jacobi polynomials.

**Corollary 1.4.** *Suppose that the root system is reduced and that all multiplicities are even. Then for all  $\mu \in P$  and all  $a \in A_{\mathbb{C}}$  we have*

$$\delta(m, a)P(m, \mu, a) = \frac{1}{\mathbf{c}(m, \mu + \rho)d(m, \mu)} D\left(\sum_{w \in W} a^{w(\mu + \rho)}\right)$$

and

$$(1.11) \quad \delta(m, a) \frac{P(m, \mu, a)}{\|P(m, \mu)\|_m^2} = \frac{C}{|W| \mathbf{c}^*(m, -\mu - \rho)d(m, \mu)} D\left(\sum_{w \in W} a^{w(\mu + \rho)}\right) \\ = \tilde{C} \mathbf{c}(m, \mu + \rho) D\left(\sum_{w \in W} a^{w(\mu + \rho)}\right),$$

where

$$(1.12) \quad \tilde{C} = \frac{C}{|W|} \prod_{\alpha \in \Delta^+} \prod_{k=0}^{m_{\alpha}/2-1} (\rho_{\alpha} - k).$$

#### 1.4. Estimates for the Jacobi polynomials

Estimates for the Jacobi polynomials and their derivatives can be deduced from (1.6), from the estimate

$$(1.13) \quad \frac{1}{|\mathbf{c}(m, \lambda)|} \leq \text{const.} (1 + \|\lambda\|)^{\sum_{\alpha \in \Delta^+} m_{\alpha}/2}$$

for the  $\mathbf{c}$ -function, and from Opdam's estimates for the hypergeometric function and its derivatives. We note that the latter estimates are only valid inside  $\{X + iH \in \mathfrak{a}_{\mathbb{C}} \mid |\alpha(H)| \leq \pi/2 \text{ for all } \alpha \in \Delta\}$ , which is smaller than the domain used in [1].

**Lemma 1.5.** *Suppose that  $m \geq 0$  is a fixed multiplicity function.*

(1) *Let  $t = \exp(iH) \in T$  satisfy  $|\alpha(H)| \leq \pi/2$  for all  $\alpha \in \Delta$ . Then we have for all  $\mu \in P^+$*

$$(1.14) \quad \begin{aligned} |P(m, \mu, t)| &\leq C_1 |W|^{1/2} (1 + \|\mu\|)^{\sum_{\alpha \in \Delta^+} m_{\alpha}/2} e^{\max_{w \in W} w\rho(H)} \\ &\leq C_2 (1 + \|\mu\|)^{\sum_{\alpha \in \Delta^+} m_{\alpha}/2}, \end{aligned}$$

where  $C_1$  and  $C_2$  are some positive constants.

(2) *Let  $I = (i_1, \dots, i_n)$  be a multi-index and set  $|I| = \sum_{j=1}^n i_j$ . Let  $\partial^I = \partial(H_1)^{i_1} \dots \partial(H_n)^{i_n}$  be the corresponding partial differential operator associated with an orthonormal basis  $\{H_1, \dots, H_n\}$  of  $\mathfrak{t}$ . Let  $t = \exp(iH) \in T$  satisfy  $|\alpha(H)| \leq \pi/2$  for all  $\alpha \in \Delta$ . Then there is a constant  $C_3(t)$ , depending on  $t$ , such that for all  $\mu \in P^+$  we have*

$$|\partial^I P(m, \mu, t)| \leq C_3(t) (1 + \|\mu\|)^{|I| + \sum_{\alpha \in \Delta^+} m_{\alpha}/2}.$$

**Proof.** The first estimate is an immediate consequence of [4, Theorem 3.15 and Proposition 6.1], together with (1.13).

To prove the estimates for the derivatives, we proceed as in [4, Corollary 6.2]. Let  $t = \exp(iH)$  be fixed as in the statement. Choose  $\delta = \delta(t) > 0$  so that

$$\begin{aligned} T_{\mu} &:= \{z \in \mathfrak{t}_{\mathbb{C}} \mid |z_j - iH_j| = \delta/\|\mu\|\} \\ &\subset \{z = Y + iX \in \mathfrak{t} \mid |\alpha(X)| < \pi/2 \text{ for all } \alpha \in \Delta^+\} \end{aligned}$$

for all  $\mu \in P^+$  with  $\mu \neq 0$ . Cauchy's integral formula then gives

$$\partial^I P(m, \mu, t) = \int_{iH + T_{\mu}} \frac{P(m, \lambda, z)}{(z - iH)^{I+(1, \dots, 1)}} dz.$$

The result follows then easily from (1.14).  $\square$

### 1.5. The Jacobi transform

The Jacobi transform of  $f \in L^2(T)^W$  is the function  $\hat{f}(m, \cdot) : P^+ \rightarrow \mathbb{C}$  defined by

$$\hat{f}(m, \mu) := \langle f, P(m, \mu) \rangle_m = \int_T f(t) P(m, \mu, t^{-1}) \delta(m, t) dt.$$



Here we have used the property that  $\overline{P(m, \mu, t)} = P(m, \mu, t^{-1})$ . The inversion formula is given by

$$(1.15) \quad f = \sum_{\mu \in P^+} \hat{f}(m, \mu) \frac{P(m, \mu)}{\|P(m, \mu)\|_m^2}$$

with convergence in the sense of  $L^2$ . We remark that  $\delta(m, t) = \delta(m, t^{-1})$  in the even multiplicity case. If  $f \in C^\infty(T)^W$  we therefore obtain from Corollary 1.4 that

$$(1.16) \quad \mathbf{c}(m, \mu + \rho)d(m, \mu)\hat{f}(m, \mu) = |W| \int_T (D^* f)(t)t^{-(\mu+\rho)} dt,$$

where  $D^*$  denotes the formal adjoint of the differential operator  $D$  with respect to the measure  $dt$ .

## 2. THE PALEY-WIENER THEOREM FOR THE JACOBI TRANSFORM

In the following we shall assume that a constant  $R > 0$  has been chosen so that the ball  $B_R := \{X \in \mathfrak{t} \mid \|X\| \leq R\}$  is contained in the set

$$(2.1) \quad S := \{X = iH \in \mathfrak{t} \mid |\alpha(H)| \leq \pi/2 \text{ for all } \alpha \in \Delta\}.$$

Then the map  $\exp : \mathfrak{t} \rightarrow T$  is a diffeomorphism of  $B_R$  onto its image, say  $D_R$ . We shall refer to this condition by saying that  $R$  is *small*. We set  $C_R^\infty(T)^W := \{f \in C^\infty(T)^W \mid \text{Supp}(f) \subseteq D_R\}$ . Moreover, we denote by  $PW_R(\mathfrak{t}^*)$  the space of functions on  $P^+$  admitting an extension  $F : \mathfrak{a}_\mathbb{C}^* = \mathfrak{t}_\mathbb{C}^* \rightarrow \mathbb{C}$  so that  $\lambda \mapsto \mathbf{c}(m, \lambda)F(\lambda - \rho)$  is a  $W$ -invariant holomorphic function of exponential type  $R$  (see [1, Definition 3.2]).

The following theorem is the main result of this note.

**Theorem 2.1** (Local Paley–Wiener theorem for the Jacobi transform). *Suppose that the root system is reduced, all multiplicities  $m_\alpha$  are even and that  $R > 0$  is small. Then the Jacobi transform is a bijection of  $C_R^\infty(T)^W$  onto  $PW_R(\mathfrak{t}^*)$ .*

The remainder of this section is devoted to the proof of Theorem 2.1.

To show that the Jacobi transform maps into  $PW_R(\mathfrak{t}^*)$  one can follow the same lines as in Theorem 3.8 of [1]. We only remark here that the exponential type  $R$  for  $\mathbf{c}(m, \lambda)\hat{f}(m, \lambda - \rho)$  depends on (1.16) together with the fact that  $d(m, \lambda)$  is a polynomial function (cf. also [1, Lemma 3.3]).

To prove the surjectivity, let  $F \in PW_R(\mathfrak{t}^*)$  and set, according to the inversion formula (1.15),

$$(2.2) \quad f(t) := \sum_{\mu \in P^+} F(\mu) \frac{P(m, \mu, t)}{\|P(m, \mu)\|_m^2}.$$

We need to show:

- (1)  $f$  is smooth and  $W$ -invariant;

- (2)  $\hat{f} = F$ ;
- (3)  $\text{Supp}(f) \subseteq B_R$ .

For each  $\alpha \in \Delta$ , let  $A_\alpha \in \mathfrak{a}$  be defined by the condition that  $\alpha(H) = \langle A_\alpha, H \rangle$  for all  $H \in \mathfrak{a}$ . Then  $\Gamma := \text{span}_{\mathbb{Z}}\{2\pi i A_\alpha / \langle \alpha, \alpha \rangle \mid \alpha \in \Delta^+\}$  is a lattice in  $\mathfrak{t}$  so that  $A_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} / \Gamma$ .

**Lemma 2.2.** *Let  $R > 0$  be small according to the definition given at the beginning of this section. For  $F \in PW_R(\mathfrak{t}^*)$  define  $h : \mathfrak{t} \rightarrow \mathbb{C}$  by*

$$h(H) := \sum_{\mu \in P} \mathbf{c}(m, \mu) F(\mu - \rho) e^{\mu(H)}.$$

Then  $h$  is a  $\Gamma$ -periodic function on  $\mathfrak{t}$  and  $\text{Supp}(h) \subseteq B_R + \Gamma$ .

**Proof.** See [1, Lemma 3.14].  $\square$

**Lemma 2.3.** *Let  $R > 0$  be small and let  $S$  be as in (2.1). Let  $F \in PW_R(\mathfrak{t}^*)$  and define  $f$  by (2.2). Then the following properties hold:*

- (1)  $f$  is smooth and  $W$ -invariant on  $S$ .
- (2) For all  $t \in T$  we have, with  $\tilde{C}$  as in (1.12),

$$(2.3) \quad \delta(m, t) f(t) = \tilde{C} D \left( \sum_{\mu \in P} \mathbf{c}(m, \mu) F(\mu - \rho) t^\mu \right).$$

- (3)  $f$  extends as a  $W$ -invariant smooth function on  $T$  so that  $\hat{f} = F$  and  $\text{Supp}(f) \subseteq D_R$ .

**Proof.** Observe first that  $\|P(m, \mu)\|_m^{-2}$  is uniformly bounded in  $\mu \in P^+$ . According to Lemma 1.5, for each multi-index  $I$ , each  $t \in S$  and each  $N \in \mathbb{N}$ , there is a constant  $C_{t, I, N} > 0$  such that

$$\left| F(\mu) \frac{\partial_I P(m, \mu, t)}{\|P(m, \mu)\|_m^2} \right| \leq C_{t, I, N} (1 + \|\mu\|)^{-N + |I| + \sum_{\alpha \in \Delta^+} m_\alpha / 2}.$$

By choosing  $N$  large enough, it follows that the series  $\sum_{\mu} F(\mu) \partial_I P(m, \mu) / \|P(m, \mu)\|_m^2$  converges uniformly. Thus  $f$  is smooth. This proves (1).

The set  $P^+$  is a fundamental domain for the action of  $W$  on  $P$ , and  $\rho \in P$  since  $m$  is even. Because of (1.10), we obtain as in [1, Lemma 3.13]:

$$D \left( \sum_{\mu \in P^+, w \in W} \mathbf{c}(m, \mu + \rho) F(\mu) t^{w(\mu + \rho)} \right) = D \left( \sum_{\mu \in P} \mathbf{c}(m, \mu) F(\mu - \rho) t^\mu \right).$$

It follows by Lemma 2.2 that

$$\begin{aligned}
\delta(m, t)f(t) &= \sum_{\mu \in P^+} F(\mu) \frac{\delta(m, t)P(m, \mu, t)}{\|P(m, \mu)\|_m^2} \\
&= \tilde{C} \mathbf{c}(m, \mu + \rho) \sum_{\mu \in P^+} F(\mu) D \left( \sum_{w \in W} t^{w(\mu + \rho)} \right) \\
&= \tilde{C} D \left( \sum_{\mu \in P} \mathbf{c}(m, \mu) F(\mu - \rho) t^\mu \right).
\end{aligned}$$

By Lemma 2.2, the right-hand side of (2.3) is supported in  $D_R$ . Thus  $f$  extends as a smooth  $W$ -invariant function on  $T$  with  $\text{Supp}(f) \subseteq D_R$ . Finally, as  $f \in L^2(T)^W$ , we have

$$\sum_{\mu \in P^+} F(\mu) \frac{P(m, \mu, t)}{\|P(m, \mu)\|_m^2} = f(t) = \sum_{\mu \in P^+} \hat{f}(m, \mu) \frac{P(m, \mu, t)}{\|P(m, \mu)\|_m^2}$$

and hence  $F(\mu) = \hat{f}(m, \mu)$  for all  $\mu \in P^+$ .  $\square$

As a corollary of the proof of the Paley–Wiener theorem, we obtain two integral formulas for functions in  $C_R^\infty(T)^W$  for  $R > 0$  small and even multiplicities.

**Lemma 2.4.** *Suppose that  $R > 0$  is small and that  $f \in C_R^\infty(T)^W$ . Then*

$$(2.4) \quad \delta(m, t)f(t) = \tilde{C} D \left( \int_{it^*} \mathbf{c}(m, \lambda) \hat{f}(m, \lambda - \rho) t^\lambda d\lambda \right),$$

where  $D$  is the differential operator of (1.9) and  $\tilde{C}$  is as in (1.12). Moreover,

$$f(t) = \int_{it^*} \hat{f}(m, \lambda - \rho) F(m, \lambda, t) e(m, \lambda) d\lambda,$$

where

$$e(m, \lambda) := \frac{C}{|W|^2} \prod_{\alpha \in \Delta^+} \prod_{k=0}^{m_\alpha/2-1} (\lambda_\alpha - k).$$

**Proof.** As Corollary 3.16 of [1], using (2.3) and (1.9).  $\square$

### 3. THE LOCAL HUYGENS' PRINCIPLE FOR THE MODIFIED LAPLACE OPERATOR ON $T$ IN THE EVEN MULTIPLICITY CASE

Let  $m \geq 0$  be fixed. The Laplace operator on the compact torus  $T$  is the  $W$ -invariant differential operator

$$L(m) = L_T - \sum_{\alpha \in \Delta^+} m_\alpha \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} \partial_{i\alpha}$$

where  $L_T$  is the Laplace operator on the abelian Lie group  $T$  and  $\partial_{i\alpha}$  denotes the derivative in the direction of the vector  $iA_\alpha$ . Note that the second term vanishes if  $\alpha(H) \in i\frac{\pi}{2} + i\pi\mathbb{Z}$  for all  $\alpha$  and has singularities if there is a root  $\alpha$  such that  $\alpha(H) \in i\pi\mathbb{Z}$ .

The Jacobi polynomials are eigenfunctions of  $L(m)$ , with  $L(m)P(m, \mu) = -\langle \mu + 2\rho, \mu \rangle P(m, \mu)$  for all  $\mu \in P^+$ . The *modified wave equation* on  $T$  is the partial differential equation

$$(3.1) \quad (L(m) - \|\rho\|^2)u = u_{\tau\tau}$$

where  $u = u(t, \tau)$  is a function of  $(t, \tau) \in T \times I$  and  $I \subseteq \mathbb{R}$  is an interval containing 0. As far as we know, there is no statement in the literature about finite propagation speed for the solutions of this wave equation. This will follow from the following lemma; cf. Lemma 4.3 in [1]:

**Lemma 3.1.** *Let  $T$  be the torus associated with a triple  $(\mathfrak{a}, \Delta, m)$  with reduced root system  $\Delta$  and even multiplicity function  $m$ . Let  $R > 0$  be small according to the previous section. Let  $0 < \varepsilon < R$ , and let  $f \in C^\infty(T)^W$ . Assume that  $u(t, \tau)$  is a distributional solution to the Cauchy problem*

$$(3.2) \quad \begin{cases} (L(m) - \|\rho\|^2)u = u_{\tau\tau}, \\ u(t, 0) = 0, \\ u_\tau(t, 0) = f(t). \end{cases}$$

*We suppose that  $u$  is smooth and  $W$ -invariant in the variable  $t \in T$ . Then for  $t = \exp(X) \in T$  and  $\tau \in [0, R - \varepsilon]$  we have*

$$(3.3) \quad \delta(m, t)u(t, \tau) = \tilde{C}D \int_{i\mathfrak{t}^*} \mathbf{c}(m, \lambda) \hat{f}(m, \lambda - \rho) \frac{\sin(\|\lambda\|\tau)}{\|\lambda\|} t^\lambda d\lambda,$$

*where  $D$  is the differential operator of (1.9) and  $\tilde{C}$  is as in (1.12).*

*In particular, the following hold for  $\tau \in [0, R - \varepsilon]$ :*

- (1)  $u$  is a smooth function of  $\tau$ ;
- (2) (Finite propagation speed)  $u(\exp X, \tau) = 0$  if  $\|X\| \geq |\tau| + \varepsilon$ ;
- (3) (Local strong Huygens' principle)  $u(\exp X, \tau) = 0$  if  $\mathfrak{t}$  is odd dimensional and  $|\tau| \geq \|X\| + \varepsilon$ .

**Proof.** Since the domain of integration  $T$  for the Jacobi transform is compact, we can interchange differentiation and integration. Applying the Jacobi transform in

the  $t$ -variable to both sides of (3.2) implies therefore the following initial value problem in the  $\tau$ -variable for  $\hat{u}(m, \mu, \tau)$ :

$$(3.4) \quad \begin{cases} \hat{u}_{\tau\tau}(m, \mu, \tau) = -\|\mu + \rho\|^2 \hat{u}(m, \mu, \tau) \\ \hat{u}(m, \mu, 0) = 0 \\ \hat{u}_\tau(m, \mu, 0) = \hat{f}(m, \mu). \end{cases}$$

Thus

$$\hat{u}(m, \mu, \tau) = \hat{f}(m, \mu) \frac{\sin(\|\mu + \rho\|\tau)}{\|\mu + \rho\|}.$$

The local Paley–Wiener theorem now implies that  $\mu \mapsto \hat{f}(m, \mu)$  belongs to  $PW_\varepsilon(t^*)$ . The function  $\mu \mapsto \sin(\|\mu\|\tau)/\|\mu\|$  has also a  $W$ -invariant holomorphic extension of exponential type  $\tau$ . Thus the function  $\mu \mapsto \mathbf{c}(m, \mu) \hat{f}(m, \mu - \rho) \sin(\|\mu\|\tau)/\|\mu\|$  admits a  $W$ -invariant holomorphic extension of exponential type  $\varepsilon + \tau$ , that is  $\hat{u}(m, \mu, \tau) = \hat{f}(m, \mu) \sin(\|\mu + \rho\|\tau)/\|\mu + \rho\|$  belongs to  $PW_{\varepsilon+\tau}(t^*)$ . Since  $\varepsilon + \tau$  is small for  $\tau \in [0, R - \varepsilon]$ , another application of the local Paley–Wiener theorem yields  $u(\cdot, \tau) \in C_{\varepsilon+\tau}^\infty(T)^W$ . This proves the finite propagation speed and allows us to apply Lemma 2.4 to  $u(\cdot, \tau)$ . Thus (3.3) holds.

Finally, recall from (4.13) in [1] that the function

$$v(X, \tau) := \int_{it^*} \mathbf{c}(m, \lambda) \hat{f}(m, \lambda - \rho) \frac{\sin(\|\lambda\|\tau)}{\|\lambda\|} e^{\lambda(X)} d\lambda$$

is the solution of the Cauchy problem for the wave equation on  $t \cong \mathbb{R}^n$ :

$$\begin{cases} L_t v(X, \tau) = v_{\tau\tau}(X, \tau) \\ v(X, 0) = 0 \\ v_\tau(X, 0) = g(X), \end{cases}$$

where  $g \in C_\varepsilon^\infty(T)^W$  is the inverse Fourier transform of  $\lambda \mapsto \mathbf{c}(m, \lambda) \hat{f}(m, \lambda - \rho)$ . By the standard results for the wave equation on Euclidean spaces it follows that, in case  $\mathfrak{k}$  is odd dimensional,  $v(X, \tau) = 0$  for  $|\tau| \geq \|X\| + \varepsilon$  (the strong Huygens principle). Let  $U$  be the periodization of  $v$  in the  $X$ -variable, and note that the translates of the support of  $g$  are all disjoint. Then (3.3) implies that  $u(\exp X, \tau) = \tilde{C}DU(X, \tau)$  satisfies (3).  $\square$

Different proofs of the local strong Huygens principle can be obtained from the results in the previous section together with same arguments employed in [1]. More precisely, one can prove the following theorem.

**Theorem 3.2.** *Let  $T$  be the torus associated with a triple  $(\mathfrak{a}, \Delta, m)$  with reduced root system  $\Delta$  and even multiplicity function  $m$ . Let  $R > 0$  be small according to*

the previous section. Let  $0 < \varepsilon < R$ , and let  $f \in C^\infty(T)^W$ . Assume that  $T$  is odd dimensional.

Suppose, moreover, that  $u(t, \tau)$  is a smooth solution of the Cauchy problem

$$(3.5) \quad \begin{cases} (L(m) - \|\rho\|^2)u = u_{\tau\tau}, \\ u(t, 0) = 0, \\ u_\tau(t, 0) = f(t). \end{cases}$$

Then the following properties are satisfied:

- (a) (Local exponential Huygens' principle) *There is a constant  $C > 0$  so that for all  $(t, \tau) \in T \times [0, R - \varepsilon]$  and all  $\gamma \in [0, \infty)$  we have*

$$(3.6) \quad |\delta(m, t)u(t, \tau)| \leq C e^{-\gamma(\tau - \|H\| - \varepsilon)}$$

with  $t = \exp H$ .

- (b) (Local strong Huygens' principle)

$$\text{Supp}(u) \cap (T \times [0, R - \varepsilon]) = \text{Supp}(u) \cap (D_R \times [0, R - \varepsilon]) \subseteq S_\varepsilon,$$

where  $S_\varepsilon$  denotes the  $\varepsilon$ -shell

$$(3.7) \quad S_\varepsilon := \{(t = \exp H, \tau) \in T \times [0, \infty) \mid \tau - \varepsilon \leq \|H\| \leq \tau + \varepsilon\}.$$

- (c) *Suppose  $\dim(T) \geq 3$ . Let  $D$  be the differential operator of (1.9). Then for all  $t = \exp H \in T$  and  $\tau \in [0, R - \varepsilon]$  the smooth solution  $u(t, \tau)$  to (3.5) is given by the formula*

$$(3.8) \quad \delta(m, t)u(t, \tau) = \frac{\Omega_n/2}{[(n-3)/2]!\Omega_{n-1}} D\left(\frac{\partial}{\partial(\tau^2)}\right)^{(n-3)/2} (\tau^{n-2}(M^\tau g)(H)).$$

Here  $g \in C^\infty(\mathfrak{t})^W$  is the inverse Euclidean Fourier transform of the function  $\lambda \mapsto \mathbf{c}(m, \lambda)\hat{f}(m, \lambda - \rho)$ , and

$$(3.9) \quad (M^\tau g)(H) := \frac{1}{\Omega_{n-1}(r)} \int_{S_r(H)} g(s) d\sigma(s)$$

is the mean value of a function  $g : \mathfrak{t} \rightarrow \mathbb{C}$  on the Euclidean sphere  $S_r(H) := \{H \in \mathfrak{t} \mid \|H\| = r\}$  in  $\mathfrak{t} \cong \mathbb{R}^n$  with respect to the  $O(n)$ -invariant surface measure  $d\sigma$ . Moreover,  $\Omega_{n-1}(r)$  denotes the surface area of  $S_r(H)$ , and  $\Omega_{n-1} := \Omega_{n-1}(1)$ .

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