4-2016

Superstable Semigroups

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Undergraduate honors thesis under the direction of

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Submitted to the LSU Roger Hadfield Ogden Honors College in partial fulfillment of the Upper Division Honors Program.

April, 2016

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Contents

Abstract

Let A be a linear operator on a Banach space X generating a semigroup $T(t) = e^{tA}$ of bounded linear operators on X. We give examples of semigroups such that $||T(t)|| \leq Me^{-\omega t^{\alpha(t)}}$ with $\omega > 0$ and $\alpha(t) > 1$ (so-called "superstability") and study superstability in terms of general properties of the resolvent $(\lambda I - A)^{-1} f = \int_0^\infty e^{-\lambda t} T(t) f \, dt$ of the generator A. In particular, we answer an open question of A. V. Balakrishnan from 2005 (see [5]) by providing elementary physical examples of superstability which are not of the type of *extinction-in-finite-time* (i.e., not nilpotent) and are less artificial (in the context of semigroup theory) than those given by G. Lumer in [14], [15] or by F. Udwadia in [18].

Chapter 1

Introduction

To be able to present a "natural" class of superstable semigroups, we first discuss semigroups induced by evolutionary flows. Let Ω be a set containing all possible states of an observable deterministic system that is continuous in time; i.e., the evolution of the system depends only on the span of time taken by the system to evolve from a starting point 0, and not on the specific instant of time at which it starts. Let $\sigma : t \to \sigma(t, x)$ be a map (flow) from $[0, \infty) \to \Omega$ that describes the time evolution of a time-autonomous system starting at state $x \in \Omega$ at time $t = 0$. Then

(i) $\sigma(0, x) = x$, and

(ii) $\sigma(t, \sigma(s, x)) = \sigma(t + s, x)$, where $t, s \geq 0$.

By looking at the evolution of the system through the lens of state-observations (measurements) $f : \Omega \to \mathbb{R}$, one is led to the investigation of linear operators $T(t) : \mathcal{F} \to \mathcal{F}$, where $\mathcal{F}$ is a vector space of functions (observations [1]) $f : \Omega \to \mathbb{C}$ defined by

$$T(t)f : x \to f(\sigma(t, x)) \tag{1.1}$$

[1] All possible observations are embedded in the space $\mathcal{F}$.

These $T(t)$ form a semigroup of linear operators; i.e.,

(i) $T(0) = I$, and

(ii) $T(t+s) = T(t)T(s)$ for $t, s \geq 0$.

If the system is non-autonomous, that is, if the time evolution of the system depends not only on the length of time taken by the system to progress from one state to another but also on the specific time instants (dates), then the flow $t \to \gamma(t, s, x)$ describes the state of the system at time $t \geq s \geq 0$ with initial state $x \in \Omega$ at initial time $s \geq 0$. Such a "non-autonomous" flow satisfies

(i) $\gamma(s, s, x) = x$, and

(ii) $\gamma(t, r, \gamma(r, s, x)) = \gamma(t, s, x)$ for all $x \in \Omega$ and $t \geq r \geq s \geq 0$.

Any non-autonomous flow can be viewed as an autonomous flow by adding to the state space (set) Ω the initial times $s \in [0, \infty)$. That is, if one defines a new state $\widetilde{\Omega}$ by $\widetilde{\Omega} = [0, \infty) \times \Omega = \{(s, x) : s \geq 0, x \in \Omega\}$ and a map $t \to \sigma(t, \widetilde{x}) = \sigma(t, (s, x)) := (t+s, \gamma(t+s, s, x))$, then $\sigma(t, \widetilde{x})$ defines an autonomous flow on $\widetilde{\Omega}$ and (1.1) becomes

$$T(t)f : (s, x) \to f(t+s, \gamma(t+s, s, x)), \tag{1.2}$$

where $f \in \mathcal{F} = \{f : [0, \infty) \times \Omega \to \mathbb{R}\}$.

In this thesis, we investigate the elementary case where the flows $t \to \gamma(t, s, x)$ are defined as the solutions of the separable linear equation

$$x'(t) = a(t)x(t), \quad a(s) = x \in \mathbb{R} = \Omega$$

given by $\gamma(t,s,x) = e^{\int_s^t a(r)dr}x$ and where the observations $f : [0,\infty) \times \mathbb{R} \to \mathbb{C}$ are given by $f(t,x) := g(t)x$ for some $g \in C_0([0,\infty),\mathbb{C})$. Then (1.2) becomes

$$\begin{aligned} T(t)f(s,x) :&= f(t+s, \gamma(t+s,s,x)) \\ &= g(t+s) \cdot \gamma(t+s,s,x) \\ &= g(t+s) \cdot e^{\int_s^{t+s} a(r)dr} \cdot x. \end{aligned}$$

That is, the essential semigroup associated to the flow $t \to \gamma(t,s,x)$ is given by

$$T(t)g(s) := g(t+s) \cdot e^{\int_s^{t+s} a(r)dr} \tag{1.3}$$

on $C_0([0,\infty),\mathbb{C})$. Clearly, if $t = 0$, then $T(0)g(s) = g(s)$ and $T(0) = I$. Furthermore, let $h(s) := T(t_1)g(s) = g(t_1+s)e^{\int_s^{t_1+s} a(r)dr}$. Then

$$\begin{aligned} T(t_2)T(t_1)g(s) &= T(t_2)h(s) \\ &= h(t_2+s) \cdot e^{\int_s^{t_2+s} a(r)dr} \\ &= g(t_1+t_2+s) \cdot e^{\int_{t_2+s}^{t_1+t_2+s} a(r)dr} \cdot e^{\int_s^{t_2+s} a(r)dr} \\ &= g(t_1+t_2+s) \cdot e^{\int_s^{t_1+t_2+s} a(r)dr} \\ &= T(t_1+t_2)g(s). \end{aligned}$$

Thus $T(t_2)T(t_1) = T(t_1+t_2)$ for all $t_1, t_2 \geq 0$, which suggests that the map $t \to T(t)$ is an exponential of the form $T(t) = e^{tA}$, where $A = T'(0)$. In our case we will show below that the generator A is given by

$$(Ag)(s) = T'(0)g(s) = g'(s) + g(s)a(s) \tag{1.4}$$

for sufficiently nice functions g and for T' given in Chapter 2, and that

$$T(t)g(s) = \lim_{n\to\infty} \Big(I - \frac{t}{n}A\Big)^{-n} g(s) = e^{tA}g(s)$$

is a strongly continuous semigroup. Moreover, if $a(t) = -(\alpha+1)t^{\alpha}$ (for $\alpha > 0$) and $g \in C_0([0,\infty),\mathbb{C})$, then the semigroup $T(t)$ is superstable; i.e.,

$$\begin{aligned} ||T(t)g|| &= \sup_{s\geq 0} |T(t)g(s)| = \sup_{s\geq 0} |g(t+s)\cdot e^{\int_s^{t+s} -(\alpha+1)r^{\alpha}dr}| \\ &= \sup_{s\geq 0} |g(t+s)\cdot e^{-(t+s)^{\alpha+1}+s^{\alpha+1}}| \leq ||g||\cdot e^{-t^{\alpha+1}}. \end{aligned}$$

This answers a question of A.V. Balakrishnan from his 2005 paper [5], where he posed the following problem.

> *"Open questions: As we can see, superstability in terms of semigroups simply means that the semigroup is quasi-nilpotent. There are many characterizations (necessary and sufficient conditions) of quasi-nilpotent semigroups. See [3]. However none of them would appear to be constructive ... This raises a currently open question: Are there physical examples of superstability which are not of the type of extinction-in-finite-time? For some recent results, including examples, even if artificial, see Lumer [14],[15]. See also the papers of Udwadia [18]."*

Chapter 2

Strongly Continuous Semigroups

We begin with some basic definitions and facts. A linear map $T : X \to X$ on a Banach space X is called a *bounded linear operator* if there exists $C > 0$ such that $||Tf|| \leq C||f||$ for all $f \in X$. The space $\mathcal{L}(X)$ of all bounded linear operators $T : X \to X$ is also a Banach space with norm $||T|| := \sup_{||f||=1} ||Tf||$.

A family of operators $T(t) \in \mathcal{L}(X)$ with $t \geq 0$ is called a *strongly continuous semigroup* if

(i) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$,

(ii) $T(0) = I$, and

(iii) $\lim_{t \to 0^+} T(t)f = f$ for every $f \in X$.

Since $||T(t+t_0)f - T(t_0)f|| \leq ||T(t_0)|| \, ||T(t)f - f||$, it follows that (iii) implies that

(iv) $t \to T(t)f$ is continuous on $[0, \infty)$ for all $f \in X$.

If $A \in \mathbb{C}^{n \times n}$ (or $\mathcal{L}(X)$), then

$$t \mapsto T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n \tag{2.1}$$

is given by a power series that is absolutely convergent for all $t \in \mathbb{C}$. Thus the map $T : \mathbb{C} \to \mathcal{L}(X)$ defined by $T : t \to T(t) = e^{tA}$ is analytic and therefore, in particular, $t \to T(t)f$ is continuous (which implies (iii)). Clearly $T(0) = I$ (condition (ii)), and (i) follows from

$$T(t)T(s) = \left(\sum_{n=0}^{\infty} \frac{A^n}{n!} t^n\right)\left(\sum_{n=0}^{\infty} \frac{A^n}{n!} s^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{\infty} \frac{A^{n-i}}{(n-i)!} t^{n-i} \frac{A^i}{i!} s^i\right)$$
$$= \sum_{n=0}^{\infty}\left(\frac{A^n}{n!} \sum_{i=0}^{\infty} \binom{n}{i} t^{n-i} s^i\right) = \sum_{n=0}^{\infty} \frac{A^n}{n!}(t+s)^n = e^{(t+s)A} = T(t+s).$$

Thus, if $A \in \mathbb{C}^{n\times n}$ (or $\mathcal{L}(X)$), then $t \to T(t) = e^{tA}$ is entire and therefore $T(t)$ $(t \geq 0)$ is a strongly continuous semigroup on $\mathbb{C}^n$ (or $\mathcal{L}(X)$).

If $A \in \mathbb{C}^{n\times n}$ (or $\mathcal{L}(X)$), for operators $T(t) = e^{tA}$ we have that $||T(t)|| = ||e^{tA}|| \leq e^{\mathrm{Re}(t)||A||}$ for all $t \in \mathbb{C}$. In general, a semigroup $T(t)$ is called *of type* (M, ω) if there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$||T(t)|| \leq Me^{\omega t} \text{ for all } t \geq 0. \tag{2.2}$$

We will next show that every strongly continuous semigroup is of type (M, ω) for some $M \geq 1$ and $\omega \geq 0$. To do this, we will use one of the pillars of Functional Analysis, the "Uniform Boundedness Principle" (UBP), due to S. Banach and H. Steinhaus (1927). Whereas the standard textbook proof of the UBP uses the Baire Category Theorem (see [11] or [13]), the following lemma and proof of the UBP due to A. Sokal [17] is more elementary.

Lemma (UBP Lemma). *Let T be a bounded linear operator from a normed linear space X to a normed linear space Y. Then, for any $f \in X$ and $r > 0$, we have*

$$\sup_{g\in B(f,r)} ||T(g)|| \geq r||T|| = r \sup_{h\in B(0,1)} \|T(h)\|, \tag{2.3}$$

where $B(f,r) = \{g \in X : ||g - f|| \le r\}$.

Proof. For $\xi \in X$ we have

$$\max\{||T(f+\xi)||, ||T(f-\xi)||\} \ge \frac{1}{2}\left[||T(f+\xi)|| + ||T(f-\xi)||\right] \ge ||T(\xi)||,$$

where the second $\ge$ uses the triangle inequality in the form $||\alpha - \beta|| \le ||\alpha|| + ||\beta||$. Now take the supremum over $\xi \in B(0,r)$. Then

$$\sup_{\|\xi\| \le r} ||T(\xi)\| = \sup_{\|\xi/r\| \le 1} ||T(\xi)\| = \sup_{\|h\| \le 1} ||T(rh)\| = r\|T\|, \quad \text{and}$$

$$\sup_{\|\xi\| \le r} \max\{||T(f+\xi)||, ||T(f-\xi)||\} = \sup_{\|\xi\| \le r} ||T(f+\xi)|| = \sup_{g \in B(f,r)} ||T(g)||.$$

□

Theorem (Uniform Boundedness Principle (UBP))**.** *Let* T_α *be a family of bounded linear operators in* $\mathcal{L}(X)$ *that is pointwise bounded (i.e.,* $||T_\alpha f|| \le M_f$ *for all* α *in some value set* I *and some* $M_f > 0$*). Then it is also uniformly bounded (i.e.,* $||T_\alpha|| \le M$ *for all* $\alpha \in I$ *and some* $M > 0$*).*

Proof. Suppose that $\sup_{T \in \mathcal{F}} ||T|| = \infty$, and choose $(T_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $||T_n|| \ge 4^n$. Then set $f_0 = 0$. By the UBP lemma,

$$\sup_{g \in B(f_0, \frac{1}{3})} ||T_1(g)|| \ge \frac{1}{3}||T_1||.$$

Thus there exists $f_1 \in B(f_0, \frac{1}{3})$ such that $||T_1(f_1)|| \ge \frac{2}{3}\frac{1}{3}||T_1||$. Applying the UBP lemma again,

$$\sup_{g \in B(f_1, \frac{1}{3^2})} ||T_2(g)|| \ge \frac{1}{3^2}||T_2||.$$

Thus there exists $f_2 \in B(f_1, \frac{1}{3^2})$ such that $||T_2(f_2)|| \ge \frac{2}{3}\frac{1}{3^2}||T_2||$. By induction, there exists a sequence f_n $(n \geq 0)$ such that

$$||f_n - f_{n-1}|| \leq \frac{1}{3^n} \text{ and } ||T_n f_n|| \geq \frac{2}{3}\frac{1}{3^n}||T_n||.$$

Since

$$\|f_{n+k} - f_n\| \leq \sum_{i=1}^{k} \|f_{n+k+1-i} - f_{n+k-i}\| \leq \sum_{i=1}^{k} \frac{1}{3^{n+k+1-i}} \leq \frac{1}{2}\frac{1}{3^n},$$

the sequence f_n is Cauchy, hence convergent to some $f \in X$, and

$$||f - f_n|| = \lim_{k \to \infty} \|f_{n+k} - f_n\| \leq \frac{1}{2}\frac{1}{3^n}.$$

Since $\|T_n(f - f_n)\| \leq \|T_n\|\|f - f_n\| \leq \frac{1}{2}\frac{1}{3^n}\|T_n\|$ and $||T_n(f_n)|| \geq \frac{2}{3}\frac{1}{3^n}||T_n||$, it follows from $\|a - b\| \geq \|a\| - \|b\|$ that

$$\begin{aligned}\|T_n(f)\| &= \|T_n(f_n) - T_n(f - f_n)\| \geq \|T_n(f_n)\| - \|T_n(f - f_n)\| \\ &\geq \frac{2}{3}\frac{1}{3^n}||T_n|| - \frac{1}{2}\frac{1}{3^n}\|T_n\| = \frac{1}{6}\frac{1}{3^n}\|T_n\| \geq \frac{1}{6}\frac{4^n}{3^n} \to \infty,\end{aligned}$$

which is a contradiction. □

Now, if $t \to T(t)f$ is continuous, then $t \to ||T(t)f||$ is also continuous since $|| \cdot || : X \to \mathbb{R}$ is a continuous map. Since continuous, real-valued functions are bounded on compact intervals, it follows that $||T(t)f|| \leq M_f$ for all $t \in [0, 1]$. By the UBP, there exists a constant $M \geq 1$ such that $||T(t)|| \leq M$ for all $t \in [0, 1]$. Now let $t = n + s$ for some $n \in \mathbb{N}$ and $s \in [0, 1]$. Then

(v) $$\begin{aligned}||T(t)|| &= ||T(s)T(n)|| = ||T(s)T(1)^n|| \leq ||T(s)|| * ||T(1)||^n \\ &\leq M * M^n \leq M * M^t = Me^{t \ln M} = Me^{\omega t}\end{aligned}$$

for $\omega := \ln M$.

For $A \in \mathbb{C}^{n\times n}$ (or $\mathcal{L}(X)$), the operators $T(t) = e^{tA}$ are determined by the matrix (or operator) A, and A can be obtained from $T(t) = e^{tA}$ by taking the derivative of $t \to e^{tA}$ at $t = 0$; i.e., $A = T'(0)$. Based on this, we define the *infinitesimal generator* A of a strongly continuous semigroup T as the operator

$$Af := \lim_{h\searrow 0} \frac{T(h)f - f}{h}$$

defined for all f in the domain

$$D(A) := \{f \in X | \lim_{h\searrow 0} \frac{T(h)f - f}{h} \text{ exists}\}.$$

The following example introduces the fundamental semigroup investigated in this thesis.

Example 2.1 (The Fundamental Example)**.** Let $X = C_0([0,\infty),\mathbb{C})$ be the space of all continuous paths $f : [0,\infty) \to \mathbb{C}$ that vanish at infinity; i.e., $\lim_{r\to\infty} f(r) = 0$. This is a Banach space with norm $||f||_\infty := \sup_{r\geq 0} |f(r)|$. Let $a : [0,\infty) \to \mathbb{C}$ be locally integrable with

$$\text{Re } a(r) \leq \omega \tag{2.4}$$

for all $r \geq 0$ and for some $\omega \geq 0$. Define a family of linear operators $T(t) : X \to X$ $(t \geq 0)$ by

$$T(t)f(s) := f(t+s)e^{\int_s^{t+s} a(r)\,dr}. \tag{2.5}$$

As seen in the discussion following (1.3), the operators $T(t)$ constitute a semigroup on X. Moreover,

$$||T(t)f||_\infty = \sup_{s\geq 0} |T(t)f(s)| = \sup_{s\geq 0} |f(t+s)e^{\int_s^{t+s} a(r)\,dr}|$$
$$\leq ||f||_\infty \, e^{\mathrm{Re}\left(\int_s^{t+s} a(r)\,dr\right)} \leq e^{\omega t}||f||_\infty$$

for all $f \in X$. Thus, $||T(t)|| \leq e^{\omega t}$.

It remains to be shown that the map $t \to T(t)f$ is continuous for all $f \in X$ and $t \geq 0$; that is, we must show that $||T(t)f - f|| \to 0$ as $t \to 0$ for all $f \in X$. To do so, we first assume that f has compact support; i.e., $f(s) = 0$ for all $s > s_0$ for some $s_0 > 0$. This implies that the function $s \to f(s)$ is not only continuous, but uniformly continuous. Now,

$$||T(t)f - f|| = \sup_{s\geq 0} |T(t)f(s) - f(s)| = \sup_{0\leq s\leq s_0} |f(t+s)e^{\int_s^{t+s} a(r)\,dr} - f(s)|$$
$$\leq \sup_{0\leq s\leq s_0} |f(t+s) - f(s)|\, |e^{\int_s^{t+s} a(r)\,dr}| + \sup_{0\leq s\leq s_0} |e^{\int_s^{t+s} a(r)\,dr} - 1|\, |f(s)|$$
$$\leq e^{\omega t} \sup_{0\leq s\leq s_0} |f(t+s) - f(s)| + ||f||_\infty \sup_{0\leq s\leq s_0} |e^v - 1|$$

where $v = \int_s^{t+s} a(r)\,dr = A(t+s) - A(s)$ with $A(t) := \int_0^t a(r)\,dr$ and $0 \leq s \leq s_0$. Since $A(\cdot)$ is uniformly continuous on compact intervals, we can choose $\mu > 0$ such that $|v| = |A(t+s) - A(s)|| \leq 1$ for all $t \in [0, \mu]$ and all $0 \leq s \leq s_0$. Moreover, since the map $v \to \frac{e^v - 1}{v} = \sum_{n=1}^{\infty} \frac{1}{n!} v^{n-1}$ is entire, there exists a constant $C > 0$ such that $|\frac{e^v-1}{v}| \leq C$ for all v with $|v| \leq 1$. Thus

$$||T(t)f - f|| \leq e^{\omega\mu} \sup_{0\leq s\leq s_0} |f(t+s) - f(s)| + C\,||f||_\infty \sup_{0\leq s\leq s_0} |A(t+s) - A(s)|$$

for all $0 \leq t \leq \mu$. Let $\varepsilon > 0$. Because of the uniform continuity of $s \to f(s)$ and $s \to A(s)$, it follows that there exists $0 < \delta \leq \mu$ such that $|f(t+s) - f(s)| < \frac{\varepsilon}{2e^{\omega\mu}}$

and $|A(t+s) - A(s)| \leq \frac{\varepsilon}{2C||f||_\infty}$ for all $0 \leq s \leq s_0$ and for all $0 \leq t \leq \delta$. Thus

$$||T(t)f - f|| \leq \varepsilon$$

for all $t \in [0, \delta]$ or $\lim_{t \searrow 0} T(t)f = f$ for all $f \in X$ with compact support. Since the continuous functions with compact support are dense in X, there exists for each $f \in X$ and $\varepsilon > 0$ a function $f_0 \in X$ with compact support such that $||f_0 - f|| < \varepsilon$. Then $||T(t)|| \leq e^{\omega t}$ implies

$$||T(t)f - f|| \leq ||T(t)|| \, ||f - f_0|| + ||T(t)f_0 - f_0|| + ||f - f_0|| \to 0$$

as $t \to 0$ all $f \in X$. Thus the semigroup (2.5) is strongly continuous on $X = C_0([0,\infty), \mathbb{C})$. The domain of the generator

$$A = T'(0) : f \to f' + af \tag{2.6}$$

contains all functions $f \in X$ such that f' exists and that the function $s \to f'(s) + a(s)f(s)$ is again a function in $C_0([0,\infty), \mathbb{C})$. Clearly, $D(A) \neq X$.

There are two important observations to be made. First, the operator A is the sum of the two operators $A_1 : f \to f'$ and $A_2 : f \to af$, where A_1 is the generator of the strongly continuous shift semigroup $T_1(t)f : s \to f(t+s)$ with $\|T_1(t)\| = 1$ and (if a is assumed to be continuous) A_2 is the generator of the strongly continuous multiplication semigroup $T_2(t)f : s \to e^{ta(s)}f(s)$ with $\|T_2(t)\| \leq e^{\omega t}$ (see Example 2 below). If one takes

$$a(t) := -\alpha t^{\alpha - 1}$$

for some $\alpha > 1$, then $\mathrm{Re}(a(t)) \leq 0 = \omega$ for all $t \geq 0$ and $\|T_2(t)\| = 1$ for all $t \geq 0$.

However, the semigroup $T(t)$ $(t \geq 0)$ generated by $A = A_1 + A_2$ is given by

$$T(t)f : s \to f(t+s)e^{-(t+s)^{\alpha}+s^{\alpha}}$$

and satisfies

$$||T(t)|| = e^{-t^{\alpha}}(t \geq 0) \tag{2.7}$$

since $\max_{s\geq 0} e^{-(t+s)^{\alpha}+s^{\alpha}} = e^{-t^{\alpha}}$. Thus the semigroup is "superstable", where the following definition is due to A.V. Balakrishnan (see [2]).

Definition 2.1 (Superstability)**.** A strongly continuous semigroup $T(t)$ $(t \geq 0)$ is called *superstable* if

$$\lim_{t\to\infty} \frac{\ln ||T(t)||}{t} = -\infty.$$

Before we discuss "superstability" in more detail in Chapter 3, we continue collecting basic information about strongly continuous semigroups.

Proposition 2.1 (Das Allerweltslemma)**.** *Let A be the generator of a strongly continuous semigroup $T(t)$ $(t \geq 0)$ on a Banach space X. Then*

(i) The semigroup $T(t)$ commutes with A on $D(A)$ and for all $t \geq 0$,

$$T(t)f - f = \begin{cases} A\int_0^t T(s)f\,ds & \text{if } f \in X, \\ \int_0^t T(s)Af\,ds & \text{if } f \in D(A), \text{ and} \end{cases} \tag{2.8}$$

(ii) $D(A)$ is dense in X (i.e., every element in X is the limit of elements in $D(A)$) and A is closed (i.e., the graph (f, Af) for $f \in D(A)$ is closed in $X \times X$ or, equivalently, if $f_n \in D(A)$ and $(f_n, Af_n) \to (f, g)$ implies that $f \in D(A)$ and $g = Af$).

Proof. (i) Since $T(t)$ commutes with $\frac{1}{h}(T(h) - I)$ for $t \geq 0$ and $h > 0$, it follows that $f \in D(A)$ implies $T(t)f \in D(A)$ and $AT(t)f = T(t)Af$. Now, for $f \in X$,

$$\frac{1}{h}\left(T(h)\int_0^t T(s)f\,ds - \int_0^t T(s)f\,ds\right) = \frac{1}{h}\int_0^t T(s+h)f\,ds - \frac{1}{h}\int_0^t T(s)f\,ds$$
$$= \frac{1}{h}\int_t^{t+h} T(s)f\,ds - \frac{1}{h}\int_0^h T(s)f\,ds \to T(t)f - f \text{ as } h \searrow 0.$$

Therefore, $\int_0^t T(s)f\,ds \in D(A)$ and

$$T(t)f - f = A\int_0^t T(s)f\,ds$$

for all $f \in X$. If $f \in D(A)$, then $\frac{1}{h}T(s)(T(h)-I)f \to T(s)Af$ uniformly for $s \in [0,t]$. So,

$$A\int_0^t T(s)f\,ds = \lim_{h\searrow 0}\frac{1}{h}(T(h)-I)\int_0^t T(s)f\,ds$$
$$= \int_0^t \lim_{h\searrow 0}\frac{1}{h}(T(h)-I)\,T(s)f\,ds = \int_0^t AT(s)f\,ds.$$

(ii) Clearly, A is a linear operator. Now, suppose $f_n \to f$ and $Af_n \to g$. Since $T(\cdot)Af_n \to T(\cdot)g$ uniformly on $[0,t]$, statement (i) implies that

$$T(t)f - f = \lim_{n\to\infty}(T(t)f_n - f_n) = \lim_{n\to\infty}\int_0^t T(s)Af_n\,ds = \int_0^t T(s)g\,ds.$$

Dividing both sides by t and taking the limit as $t \searrow 0$, we conclude that $f \in D(A)$ and $Af = g$. So, A is closed. Finally, since $\frac{1}{t}\int_0^t T(s)f\,ds \in D(A)$ for all $f \in X$ and $\lim_{t\searrow 0}\frac{1}{t}\int_0^t T(s)f\,ds = f$, the domain $D(A)$ is dense in X.

□

By (2.5), the function $t \mapsto T(t)f$ is continuously differentiable for $f \in D(A)$ and

$$\frac{d}{dt}T(t)f = T(t)Af = AT(t)f \text{ (for } t \geq 0). \tag{2.9}$$

Thus, if $f \in D(A)$, then $u(t) := T(t)f$ is a classical solution of the Abstract Cauchy Problem

$$\text{(ACP)} \quad \begin{cases} u'(t) &= Au(t) \;\; (t \geq 0), \\ u(0) &= f. \end{cases}$$

A continuous function $u : [0, \infty) \to X$ with $\int_0^t u(s)\,ds \in D(A)$ for all $t \geq 0$ and

$$u(t) - f = A \int_0^t u(s)\,ds$$

is called a *mild solution* of (ACP). Note that if u is a mild solution, we have

$$\frac{1}{h}(u(t+h) - u(t)) = A\frac{1}{h}\int_t^{t+h} u(s)\,ds$$

for all arbitrarily small $h > 0$. Since $\lim\limits_{h\to 0} \dfrac{1}{h}\displaystyle\int_t^{t+h} u(s)\,ds = u(t)$ and A is closed, it follows that $u'(t) = Au(t)$ for all mild solutions for which $u'(t)$ exists. On the other hand, if u is a classical solution of (ACP) and A is closed, then u is a mild solution of (ACP). Indeed, since $u(t) \in D(A)$ for all $t \geq 0$ and u is continuous, the Riemann sums $\sum_\pi u(\xi_i)(t_i - t_{i-1})$ are in $D(A)$ and u is Riemann integrable with $\int_0^t u(s)\,ds = \lim_{|\pi|\to 0} \sum_\pi u(\xi_i)(t_i - t_{i-1})$. Similarly, the continuity of Au yields $\int_0^t Au(s)\,ds = \lim_{|\pi|\to 0} Au(\xi_i)(t_i - t_{i-1})$. The fact that A is closed implies that $\int_0^t u(s)\,ds \in D(A)$ and

$$A\int_0^t u(s)\,ds = \int_0^t Au(s)\,ds = \int_0^t u'(s)\,ds = u(t) - u(0). \tag{2.10}$$

We shall now briefly recall some common definitions in the spectral theory of closed linear operators. For a closed operator A, the set

$$\rho(A) := \{\lambda \in \mathbb{C} \,|\, \lambda I - A \text{ is bijective}\}$$

is called the *resolvent set* of A. By the closed graph theorem (which states that a closed, everywhere-defined linear operator is bounded), the *resolvent*

$$R(\lambda, A) := (\lambda I - A)^{-1} : X \to D(A)$$

is a bounded linear operator on X. It is easy to show that resolvents satisfy the so-called *resolvent equation*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \text{ for all } \lambda, \mu \in \rho(A). \tag{2.11}$$

The following proposition makes clear the connection between Laplace transform theory and semigroup theory.

Proposition 2.2 (Laplace Transforms and Semigroups). *Let $T(t)$ $(t \geq 0)$ be a strongly continuous semigroup of type (M, ω) on a Banach space X with generator A. Then the Laplace transform $\int_0^\infty e^{-\lambda t}T(t)f\,dt$ of $t \to T(t)f$ exists for all $f \in X$ and for all complex numbers λ in the right half-plane*

$$H_\omega := \{\lambda \in \mathbb{C} : Re(\lambda) > \omega\}.$$

Moreover, if the Laplace transform of the semigroup exists for some $\lambda \in \mathbb{C}$, then $\lambda \in \rho(A)$, and, for all $f \in X$,

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t}T(t)f\,dt. \tag{2.12}$$

Furthermore, $||R(\lambda, A)^n|| \leq \frac{M}{(\lambda-\omega)^n}$ for all $\lambda > \omega$ and $n \in \mathbb{N}$ and, for $\xi > \omega$,

$$T(t)f = \frac{1}{2\pi i}\int_{\xi+i\mathbb{R}} e^{\lambda t}R(\lambda, A)f\,d\lambda \tag{2.13}$$

for all $f \in D(A)$ *and* $t \geq 0$.

Proof. Since $||T(t)|| \leq Me^{\omega t}$ for some $M \geq 1$ and $\omega \in \mathbb{R}$, it follows that the Laplace transform

$$R(\lambda) : f \mapsto \int_0^\infty e^{-\lambda s} T(s) f \, ds$$

of $t \to T(t)f$ exists for all $f \in X$ and $\lambda \in H_\omega$. We show next that the operator $R(\lambda)$ is the inverse of $\lambda I - A$; i.e., $R(\lambda)(\lambda I - A)f = f$ for all $f \in D(A)$ and $(\lambda I - A)R(\lambda)f = f$ for all $f \in X$. For $f \in D(A)$, integration by parts yields

$$\begin{aligned} \lambda R(\lambda)f &= \lambda \int_0^\infty e^{-\lambda s} T(s) f \, ds \\ &= -e^{-\lambda s} T(s) f|_0^\infty + \int_0^\infty e^{-\lambda s} \frac{d}{ds} T(s) f \, ds \\ &= f + \int_0^\infty e^{-\lambda s} T(s) Af \, ds = f + R(\lambda) Af. \end{aligned}$$

On the other hand, for $h > 0$ and $f \in X$, we have

$$\begin{aligned} \frac{1}{h}(T(h) - I)R(\lambda)f &= \frac{1}{h}\left(\int_0^\infty e^{-\lambda s} T(s+h) f \, ds - \frac{1}{h} R(\lambda) f\right) \\ &= \frac{1}{h}\left(\int_0^\infty e^{-\lambda s} T(s) f \, ds - \int_h^\infty e^{-\lambda s} T(s) f \, ds - \int_0^h e^{-\lambda s} T(s) f \, ds\right) \\ &= \frac{e^{\lambda h} - 1}{h} \int_h^\infty e^{-\lambda s} T(s) f \, ds - \frac{1}{h} \int_0^h e^{-\lambda s} T(s) f \, ds. \end{aligned}$$

Since the limit for $h \to 0$ exists on the right hand side and equals $\lambda R(\lambda)f - f$, it follows that $R(\lambda)f \in D(A)$ and $AR(\lambda)f = \lambda R(\lambda)f - f$. Hence, $R(\lambda) = R(\lambda, A)$. Moreover, for all $n \in \mathbb{N}_0$ and $f \in X$,

$$\begin{aligned} \left\| \frac{(-1)^n}{n!} R^{(n)}(\lambda, A) f \right\| &= \left\| \frac{(-1)^n}{n!} \int_0^\infty e^{-\lambda s} (-s)^n T(s) f \, ds \right\| \\ &\leq M||f|| \int_0^\infty e^{-(\lambda - \omega)s} \frac{s^n}{n!} \, ds = \frac{M}{(\lambda - \omega)^{n+1}} ||f||. \end{aligned}$$

It follows that

$$\left\| \frac{(-1)}{n!} R^{(n)}(\lambda, A) \right\| \le \frac{M}{(\lambda - \omega)^{n+1}} \text{ for all } \lambda > \omega, n \in \mathbb{N}_0.$$

Using the resolvent equation (2.11), a straightforward inductive argument shows that

$$R(\lambda, A)^{n+1} = \frac{(-1)^n}{n!} R^{(n)}(\lambda, A) \text{ for all } n \in \mathbb{N}_0.$$

The last statement (2.13) is an application of the complex inversion of the Laplace transform (for a proof, see [2], Proposition 3.12.1 and Theorem 2.3.4). □

We have seen in the Fundamental Example 2.1 that for suitable functions $a(\cdot)$, the operators

$$Af : s \to f'(s) + a(s)f(s)$$

generate strongly continuous semigroups of the form

$$T(t)f : s \to f(t+s)e^{\int_s^{t+s} a(r)\,dr}$$

on $C_0([0,\infty), \mathbb{C})$. In particular, by taking $t \to a(t) = 0$ for $t \ge 0$, the operator

$$A_1 f : s \to f'(s)$$

generates the strongly continuous (left) shift semigroup

$$T_1(t)f : s \to f(t+s)$$

on $C_0([0,\infty), \mathbb{C})$. We will discuss next the semigroups $T_2(t)f(s) = e^{ta(s)}f(s)$ generated by operators A_2 of the form $A_2 f : s \to a(s)f(s)$ on $C_0([0,\infty), \mathbb{C})$ and will then, at the end of this section, discuss semigroups $T(t)$ whose generator $A = A_1 + A_2$

is the sum of two generators A_1 and A_2.

Example 2.2 (Multiplication Semigroups)**.** Let $X = C_0([0,\infty),\mathbb{C})$. If $a : [0,\infty) \to \mathbb{C}$ is continuous and satisfies the growth restriction $\omega := \sup_{s\geq 0}\{\operatorname{Re} a(s)\} < \infty$, then the multiplication operator A defined by $Af : s \mapsto a(s)f(s)$ on $D(A) = \{f \in X; af \in X\}$ is the generator of a strongly continuous (multiplication) semigroup $T(t)$ $(t \geq 0)$ given by $T(t)f : s \mapsto e^{ta(s)}f(s)$.

It can be easily seen that the operators $T(t)$ are in $\mathcal{L}(X)$ and form a semigroup with $\|T(t)\| \leq e^{\omega t}$ $(t \geq 0)$. Moreover,

$$\begin{aligned}\|T(t)f - f\| &= \sup_{s\geq 0} |e^{ta(s)}f(s) - f(s)| \\ &\leq \sup_{s\in[0,N]} |e^{ta(s)} - 1||f(s)| + \sup_{s\geq N}(e^{wt} + 1)|f(s)|\end{aligned}$$

for all $t \geq 0$ and $f \in C_0([0,\infty),\mathbb{C})$. Let $\varepsilon > 0$ and choose $N > 0$ large enough such that the second term is less than $\varepsilon/2$ for all $t \in [0,1]$. Then

$$\begin{aligned}\|T(t)f - f\| &\leq \sup_{s\in[0,N]} \left| a(s)\int_0^t e^{ra(s)}\,dr \right| |f(s)| + \varepsilon/2 \\ &\leq \left(\sup_{s\in[0,N]} |a(s)|\right)\left(\int_0^t e^{wr}\,dr\right)\|f\|_\infty + \varepsilon/2.\end{aligned}$$

Now choose $0 < \delta < 1$ such that $\|T(t)f - f\| \leq \varepsilon$ for all $t \in [0,\delta]$. This shows that the semigroup $T(t)$ is strongly continuous.

The following are more or less obvious statements concerning the multiplication semigroups $T(t)f : s \mapsto e^{ta(s)}f(s)$ and their generators $Af : s \mapsto a(s)f(s)$ on $X = C_0([0,\infty),\mathbb{C})$ where $a : [0,\infty) \to \mathbb{C}$ is assumed to be continuous.

(a) The spectrum of $\sigma(A) := \{\lambda \in \mathbb{C} : \lambda \notin \rho(A)\}$ coincides with the closure of the range of the function $a(\cdot)$ in $\mathbb{C}$ and

$$R(\lambda, A)f : s \to \frac{1}{\lambda - a(s)} f(s)$$

for $\lambda \notin \sigma(A)$ and $f \in X$.

(b) A is a bounded linear operator if and only if $\sigma(A)$ is bounded if and only if the range of a is bounded. Moreover, if A is bounded, then $t \to T(t)$ is an entire function from $\mathbb{C}$ into $\mathcal{L}(X)$.

(c) If $a(r) = -r(r \geq 0)$, then the semigroup is strongly continuous and $t \to T(t)$ is an analytic function from $H_0 := \{t \in \mathbb{C} : \mathrm{Re}\, t > 0\}$ into $\mathcal{L}(X)$.

(d) If $a(r) = -r + ir(r \geq 0)$, then the semigroup is strongly continuous and $t \to T(t)$ is an analytic function from $H_{\pi/4} := \{t \in \mathbb{C} : |\arg(t)| < \pi/4\}$ into $\mathcal{L}(X)$. More generally, if the spectrum of A is contained in an angular region $\Sigma_\beta := \{\lambda \in \mathbb{C} : \frac{\pi}{2} + \beta \leq \arg \lambda \leq \frac{3\pi}{2} - \beta\}$ for some $0 < \beta \leq \frac{\pi}{2}$, then $t \to T(t)$ is analytic for $t \in \mathbb{C}$ with $-\beta < \arg(t) < \beta$.

(e) If $a(r) = -r + ir^2$ $(r \geq 0)$, then the semigroup is strongly continuous, $t \to T(t)$ is continuous on $(0, \infty)$ but $t \to T(t)$ is not analytic in any open sector containing $(0, \infty)$. However, $t \to T(t)f$ is infinitely often differentiable on $(0, \infty)$ for all $f \in X$.

(f) If $a(r) = -r + ie^r$ $(r \geq 0)$, then the semigroup is strongly continuous, $t \to T(t)$ is continuous on $(0, \infty)$ but $t \to T(t)$ is not analytic in any open sector containing $(0, \infty)$. However, $t \to T(t)f$ is n-times continuously differentiable on (n, ∞) for all $f \in X$.

(g) If $a(r) = -r + ie^r$ $(r \geq 0)$, then the semigroup is strongly continuous, $t \to T(t)$ is continuous on $(0, \infty)$ and $t \to T(t)f$ is n-times continuously differentiable on (n, ∞) for all $f \in X$.

(h) If $a(r) = -r + ie^{r^2}$ $(r \geq 0)$, then the semigroup is strongly continuous, $t \to T(t)$ is continuous on $(0, \infty)$, but $t \to T(t)f$ is, in general, not differentiable.

(i) If $a(r) = ir$ $(r \geq 0)$, then the semigroup is strongly continuous, $t \to T(t)$ is not continuous on $(0, \infty)$, and $t \to T(t)f$ is, in general, not differentiable.

The following fundamental perturbation result is due to Dyson and Phillips (from [10], pg. 163) and applies to the operator

$$Af : s \to f'(s) + a(s)f(s)$$

whenever the range of the continuous function $a(\cdot)$ is bounded.

Theorem 2.1 (Dyson-Phillips Perturbation Theorem)**.** *Let $T_1(t)$ $(t \geq 0)$ be a strongly continuous semigroup of type (M, ω) on a Banach space X with generator A_1, and let $A_2 \in \mathcal{L}(X)$ be the generator of the entire semigroup $t \to T_2(t) = e^{tA_2}$. Then $A = A_1 + A_2$ with domain $D(A) = D(A_1)$ generates a strongly continuous semigroup $T(t)$ $(t \geq 0)$ that can be obtained as*

$$T(t) = \sum_{n=0}^{\infty} T_n(t), \tag{2.14}$$

where $T_0(t) := T_1(t)$ and

$$T_{n+1}(t) := VT_n(t) = \int_0^t T_1(t-s)A_2T_n(s)\,ds. \tag{2.15}$$

Finally, if $A = A_1 + A_2$, where A_2 is not bounded, but both A_1 and A_2 generate "stable", strongly continuous semigroups, the following crucial result applies (for a proof, see [10], p. 227).

Theorem 2.2 (Lie-Trotter Product Formula)**.** *Let $T_1(t)$ and $T_2(t)$ $(t \geq 0)$ be strongly continuous semigroups on a Banach space X satisfying the stability condition*

$$\left\| \left[T_1\left(\frac{t}{n}\right) T_2\left(\frac{t}{n}\right) \right]^n \right\| \leq M e^{\omega t} \quad \text{for all } t \geq 0, n \in \mathbb{N}, \tag{2.16}$$

for some constants $M \geq 1$, $\omega \in \mathbb{R}$. Consider the sum $A_1 + A_2$ on $D := D(A_1) \cap D(A_2)$ of the generators A_i of $T_i(t)$ $(t \geq 0)$ and assume that D and $(\lambda_0 I - A_1 - A_2)D$ are dense in X for some $\lambda_0 > \omega$. Then the closure $A := \overline{A_1 + A_2}$ of the sum $A_1 + A_2$ generates a strongly continuous semigroup $T(t)$ $(t \geq 0)$ given by the Lie-Trotter product formula

$$T(t)f = \lim_{n\to\infty} \left(T_1\left(\frac{t}{n}\right) T_2\left(\frac{t}{n}\right) \right)^n f \tag{2.17}$$

for all $f \in X$ with uniform convergence for t in compact intervals.

Example 2.3. Let $X = C_0([0,\infty), \mathbb{C})$ and assume that $a : [0,\infty) \to \mathbb{C}$ is continuous with $\omega := \sup_{s\geq 0}\{\operatorname{Re} a(s)\} < \infty$. To apply the Lie-Trotter formula to the operator $Af : s \to f'(s) + a(s)f(s)$ generating the strongly continuous semigroup $T(t)f : s \to f(t+s)e^{\int_s^{t+s} a(r)\,dr}$, we choose $A = A_1 + A_2$, where $A_1 f : s \to f'(s)$ generates the strongly continuous (left) shift semigroup $T_1(t)f : s \to f(t+s)$ and $A_2 f : s \to a(s)f(s)$ generates the multiplication semigroup $T_2(t)f(s) = e^{ta(s)}f(s)$ on $C_0([0,\infty), \mathbb{C})$. Then it can be easily seen that the Lie-Trotter approximations

$$\left[T_1\left(\frac{t}{n}\right) T_2\left(\frac{t}{n}\right) \right]^n f(s) = e^{\frac{t}{n}\sum_{j=1}^{n} a(\frac{jt}{n}+s)} f(t+s)$$

are bounded by $e^{\omega t}\|f\|_\infty$ for $n \in \mathbb{N}$. Thus, by the previous theorem, the Lie-Trotter approximations converge to $T(t)f(s)$ uniformly for s in compact subintervals of $[0,\infty)$.

Chapter 3

Stability and Superstability

A strongly continuous semigroup $T(t)$ is called *exponentially stable* with maximal decay rate $\omega_0 > 0$ if

$$\limsup_{t\to\infty} \frac{\ln ||T(t)||}{t} = -w_0.$$

It is called *superstable* if

$$\lim_{t\to\infty} \frac{\ln ||T(t)||}{t} = -\infty,$$

and *nilpotent* or *extinct-in-finite-time* if there exists $t_0 > 0$ such that

$$T(t) = 0 \text{ for all } t \geq t_0.$$

If a strongly continuous semigroup is exponentially stable with maximal decay rate $\omega_0 > 0$, then it can be easily seen that for every $0 < \omega < \omega_0$ there exists $M \geq 1$ such that

$$\|T(t)\| \leq Me^{-\omega t}$$

for all $t \geq 0$. Clearly, a nilpotent semigroup is automatically superstable. We show first that superstability and extinction-in-finite-time are true infinite-dimensional phenomena that can only happen if the generator A of the semigroup $T(t)$ is unbounded. To see this, we start with the *geometric series* for bounded linear operators.

Let $A \in \mathcal{L}(X)$ and $\|A\| < 1$. Then one can show as in the numerical case that $I - A$ is invertible and $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n \in \mathcal{L}(X)$.

An immediate consequence of the geometric series result is that the set of invertible operators is open in $\mathcal{L}(X)$. To see this, let $B \in \mathcal{L}(X)$ be invertible and let $C \in \mathcal{L}(X)$ with $\|B - C\| < \frac{1}{2\|B^{-1}\|}$. Then $\|B^{-1}(B-C)\| \le 1/2$ and therefore $I - B^{-1}(B-C)$ is invertible. Thus, $C = B - (B-C) = B[I - B^{-1}(B-C)]$ is invertible.

Let $A \in \mathcal{L}(X)$. Then the *spectrum*

$$\sigma(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible }\}$$

is closed and bounded in $\mathbb{C}$ and its complement, the *resolvent set*

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}^1, \text{ is open and nonempty.} \tag{3.1}$$

To see this, let $\lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$. Then $I - \frac{A}{\lambda}$ is invertible. Thus, $\lambda I - A$ is invertible and therefore $\lambda \in \rho(A)$. This shows that $\rho(A)$ is nonempty and that $\sigma(A)$ is contained in the closed sphere of radius $\|A\|$.
Now let $\lambda_0 \in \rho(A)$ and $|\lambda - \lambda_0| < \frac{1}{2\|(\lambda_0 I - A)^{-1}\|}$. Then $\|(\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}\| \le 1/2$. Then $\lambda I - A = (\lambda_0 I - A)[I - (\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}]$ is invertible and

$$(\lambda I - A)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n (\lambda_0 I - A)^{-(n+1)}.$$

This shows that $\rho(A)$ is open. Therefore, the complement $\sigma(A)$ is closed. The

[1] It follows from the open mapping theorem (see [11]) that the inverse of a bounded linear operator is again bounded.

following result (with a different proof) is due to A.V. Balakrishnan [5].

Proposition 3.1. *Let $T(t)$ $(t \geq 0)$ be a superstable semigroup on a Banach space X. Then the spectrum of the generator A is empty and the resolvent $\lambda \to R(\lambda, A)$ is an entire function given by*

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt$$

for all $f \in X$ and $\lambda \in \mathbb{C}$. In particular, if $A \in \mathcal{L}(X)$, then the semigroup $T(t) = e^{tA}$ cannot be superstable.

Proof. If a strongly continuous semigroup $T(t)$ $(t \geq 0)$ is superstable, then for all $\omega > 0$ there exists $M \geq 1$ such that $\|T(t)\| \leq Me^{-\omega t}$ for all $t \geq 0$. It follows that $R(\lambda)f := \int_0^\infty e^{-\lambda t} T(t) f \, dt$ exists for all $f \in X$ and $\lambda \in \mathbb{C}$. It follows from Proposition 2.2 that $R(\lambda) = R(\lambda, A)$. Since the spectrum of a bounded linear operator $A \in \mathcal{L}(X)$ cannot be empty, it follows that superstability (and extinction-in-finite-time) cannot happen if the generator is bounded. □

We now present some examples of stability.

Proposition 3.2 (Exponential Stability)**.** *Let $A \in \mathcal{L}(\mathbb{C}^n)$. Then*

$$\limsup_{t \to \infty} \frac{\ln ||T(t)||}{t} = \sup\{Re\ \lambda, \lambda \in \sigma(A)\}.$$

Example 3.1 (Bounded, Nilpotent Semigroups with $M > 1$)**.** Let $a : [0, \infty) \to \mathbb{C}$ be integrable with unbounded range $\{a(s), s \geq 0\}$ and let $X = C_0([0, N], \mathbb{C})$ be the space of all continuous paths $f : [0, N] \to \mathbb{C}$ with $f(N) = 0$. Then

$$T(t)f(s) := \begin{cases} f(t+s)e^{\int_s^{t+s} a(r)\, dr} & \text{if } 0 \leq t+s \leq N, \\ 0 & \text{if } t+s \geq N \end{cases} \tag{3.2}$$

is a nilpotent (i.e., $T(t) = 0$ for $t \geq N$), strongly continuous semigroup with

$$||T(t)|| = \sup_{s \in [0,N]} |e^{\int_s^{t+s} a(r)\,dr}|. \tag{3.3}$$

In particular, if $a(r) = 2r$, then $||T(t)|| = e^{N^2} > 1$.

The Fundamental Example in Chapter 2 is an example of a superstable semigroup that is not extinct-in-finite-time.

Theorem 3.1 (Doetsch)**.** *Let $T(t)$ $(t \geq 0)$ be a strongly continuous semigroup on a Banach space X with generator A. Then the following are equivalent.*

(i) $T(t) = 0$ for $t \geq h > 0$ (i.e., $T(t)$ is nilpotent or extinct-in-finite-time).

(ii) The spectrum of A is empty and

(a) $||R(x+iy, A)|| \leq C$

(b) $||R(-x+iy, A)|| \leq Ce^{hx}$

for all $x \geq 0$ and $y \in \mathbb{R}$.

Proof. If $T(t) = 0$ for $t \geq h > 0$, then it follows from Proposition 3.1 that

$$R(\lambda, A)f = \int_0^h e^{-\lambda t} T(t) f\,dt \tag{3.4}$$

for all $\lambda \in \mathbb{C}$ and $f \in X$. Since $||T(t)|| \leq M$ $(0 \leq t \leq h)$, it follows that

$$\|R(\lambda, A)\| \leq h \cdot M \qquad \text{if } \lambda = x + iy,$$
$$\|R(\lambda, A)\| \leq h \cdot Me^{hx} \qquad \text{if } \lambda = -x + iy$$

for $x \geq 0$ and $y \in \mathbb{R}$. Setting $C := h \cdot M$ yields statement (a).

Conversely, assume that (ii) holds. Let $T(t)$ be a semigroup of type (M, ω) and assume that $\xi > \max(\omega, 0)$. From Proposition 2.2, one knows that

$$T(t)f = \frac{1}{2\pi i} \int_{\xi + i\mathbb{R}} e^{\lambda t} R(\lambda, A) f \, d\lambda$$

for all $f \in D(A)$ and $t \geq 0$. Since $D(A)$ is dense in X it suffices to show that $T(t)f = 0$ for all $f \in D(A)$ and $t \geq k$. For $N > 0$, we construct the following path (see Doetsch [8] for a diagram):

$\Gamma^N = \Gamma_1^N \cup \Gamma_2^N \cup \Gamma_3^N \cup \Gamma_4^N$, where

$$\begin{aligned}
\Gamma_1^N &= \{\lambda = \xi + ir, \quad -N \leq r \leq N\} \\
\Gamma_2^N &= \{\lambda = \xi - r + iN, \quad 0 \leq r \leq \xi\} \\
\Gamma_3^N &= \{\lambda = Ne^{ir}, \quad \frac{\pi}{2} \leq r \leq \frac{3\pi}{2}\} = \{\lambda = N\cos r + iN\sin r, \quad \frac{\pi}{2} \leq r \leq \frac{3\pi}{2}\} \\
\Gamma_4^N &= \{\lambda = r + iN, \quad 0 \leq r \leq \xi\}.
\end{aligned}$$

Then, by Cauchy's Theorem, $\int_{\Gamma^N} e^{\lambda t} R(\lambda, A) f \, d\lambda = 0$. Thus

$$T(t)f = -\lim_{N \to \infty} \frac{1}{2\pi i} \int_{\Gamma_2^N \cup \Gamma_3^N \cup \Gamma_4^N} e^{\lambda t} R(\lambda, A) f \, d\lambda$$

for all $f \in D(A)$ and $t \geq 0$. Now it follows from the estimates below that $T(t)f = 0$ for all $t \geq 0$ and $f \in D(A)$.

(a) Along Γ_2^N: It follows from $R(\lambda, A)(\lambda I - A)f = f$ that $R(\lambda, A)f = \frac{1}{\lambda}[R(\lambda, A)Af - f]$ for all $f \in D(A)$ and $\lambda \in \mathbb{C}$. Thus

$$\frac{1}{2\pi i} \int_{\Gamma_2^N} e^{\lambda t} R(\lambda, A) f \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_2^N} \frac{e^{\lambda t}}{\lambda} H(\lambda) \, d\lambda$$

where $H(\lambda) = R(\lambda, A)Af - f$ is an entire function that is bounded by some constant $\widetilde{C} > 0$ for $\operatorname{Re} \lambda \geq 0$. Thus

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_2^N} e^{\lambda t} R(\lambda, A) f \, d\lambda \right\| \leq \frac{\widetilde{C}}{2\pi} \int_0^{\xi} \frac{e^{(\xi - r)t}}{|\xi - r + iN|} \, dr$$

$$\leq \frac{\widetilde{C}}{2\pi} \frac{1}{N} \left[\frac{e^{\xi t} - 1}{t} \right] \to 0 \text{ as } N \to \infty.$$

The same estimate works with obvious modification along Γ_4^N.

(b) Along Γ_3^N we have that $||R(\lambda, A)f|| \leq Ce^{h|r|}||f||$ for $\lambda = r + iy$ $(r \leq 0)$. Thus

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_3^N} e^{\lambda t} R(\lambda, A) f \, d\lambda \right\| \leq \frac{C||f||}{2\pi} \int_{\pi/2}^{3\pi/2} e^{N\cos(r)t} e^{hN|\cos r|} | - N \sin r + iN \cos r| \, dr$$

$$\leq \frac{C||f||}{2\pi} \int_{\pi/2}^{3\pi/2} N e^{N|\cos r|(h-t)} \, dr \to 0 \text{ as } N \to \infty$$

if $t > k$. Thus $T(t)f = 0$ if $t > h$ (for $f \in D(A)$) and, by strong continuity, $T(t)f = 0$ if $t \geq k$ and $f \in D(A)$.

□

G. Doetsch's characterization of functions (semigroups) with finite-time extinction is often attributed to the monograph [12] by I.C. Gohberg and M.G. Krein from 1970 (see [5], [18]). It seems that the elementary proof by G. Doetsch from 1951 was largely unnoticed by the semigroup community. The following necessary condition for superstability is taken from unpublished notes of Günter Lumer; the proof given here seems to be new. We expect these conditions to also be sufficient, but we cannot yet prove it.

Proposition 3.3 (Lumer). *Let $T(t)$ $(t \geq 0)$ be a strongly continuous semigroup on a Banach space X with generator A. If there exist $\omega, \beta > 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{-\omega t^{1+\beta}}$$

for all $t \geq 0$, then the spectrum of A is empty and for all $0 < c < \omega$ there exists $\widetilde{M} > 0$ such that

(a) $||R(x+iy, A)|| \leq \widetilde{M}$

(b) $||R(-x+iy, A)|| \leq \widetilde{M} e^{h\, x^{1+\frac{1}{\beta}}}$

for all $x \geq 0$ and $y \in \mathbb{R}$, where $h := c^{-\frac{1}{\beta}}$.

Proof. Since $T(t)$ is superstable with $\|T(t)\| \leq Me^{-\omega t^{1+\beta}}$ for all $t \geq 0$, it follows that the spectrum of A is empty and

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt$$

for all $\lambda \in \mathbb{C}$ and $f \in X$. In particular, if $\lambda = x + iy$ for $x, y \in \mathbb{R}$, then

$$\|R(x+iy, A)\| \leq M \int_0^\infty e^{-xt-\omega t^{1+\beta}} \, dt.$$

If $x \geq 0$, then there exists a constant $M' > 0$ such that $e^{-xt-\omega t^{1+\beta}} \leq M' e^{-t}$ for all $x \geq 0$ and all $t \geq 0$. Thus, for $x \geq 0$ and $y \in \mathbb{R}$,

$$\|R(x+iy, A)\| \leq M \cdot M' \cdot \int_0^\infty e^{-t} \, dt = M \cdot M' =: \widetilde{M}.$$

If $x \leq 0$ and $0 < c < \omega$, then

$$\begin{aligned}
\|R(x+iy, A)\| &\leq M \int_0^\infty e^{-xt-\omega t^{1+\beta}} \, dt \\
&= M \int_0^\infty e^{|x|t - ct^{1+\beta}} e^{-(\omega - c)t^{1+\beta}} \, dt \\
&\leq \widetilde{M} \max_{t \geq 0} e^{|x|t - ct^{1+\beta}} = \widetilde{M} \cdot \operatorname{Exp}\left(\left(\frac{1}{c}\right)^{\frac{1}{\beta}} \left(\frac{1}{1+\beta}\right)^{\frac{1}{\beta}} \frac{\beta}{1+\beta} |x|^{1+\beta} \right) \\
&\leq \widetilde{M} e^{h\, x^{1+\frac{1}{\beta}}},
\end{aligned}$$

where $\widetilde{M} := M \int_0^\infty e^{-(\omega-c)t^{1+\beta}}\,dt$ and $h := c^{-\frac{1}{\beta}}$.

$\square$

For the sake of completeness, we mention the following result from [6] that expresses stability in terms of integrability conditions of the function $t \to ||T(t)||$.

Proposition 3.4 (Pazy-Type Characterizations of Stability)**.** *Let $T(t)$ be a strongly continuous semigroup on a Banach space X and let $a \geq 0$. Then*

(a) If $\displaystyle\int_a^\infty ||T(t)||^p\,dt < \infty$ for some $0 < p < \infty$, then $T(t)$ is exponentially stable;

(b) If $\displaystyle\int_a^\infty |\log ||T(t)|||^{-p}\,dt < \infty$ for some $1 < p < \infty$, then $T(t)$ is exponentially stable;

(c) If $\displaystyle\int_a^\infty |\log ||T(t)|||^{-1}\,dt < \infty$, then $T(t)$ is superstable;

(d) If $\displaystyle\lim_{p\searrow 0}\int_a^\infty |\log ||T(t)|||^{-p}\,dt < \infty$, then $T(t)$ has finite time extinction.

Proposition 3.5. *Let $T(t)$ be a strongly continuous semigroup on a Banach space X. Define the relative entry time for each $n \in \mathbb{N}$ as*

$$u_r := \sup\{t_{r+1}(f) - t_r(f) : ||f|| \leq 1\},$$

where $t_r(f) = \inf\{t \geq 0 : ||T(t')f|| \leq e^{-r}$ for all $t' \geq t$ and $r = n!\}$. Then

(a) $T(t)$ is exponentially stable if and only if $\limsup_{r\to\infty} u_r < \infty$;

(b) $T(t)$ is superstable if and only if $\lim u_r = 0$;

(c) $T(t)$ has finite time extinction if and only if $\sum_r u_r < \infty$.

Bibliography

[1] Apostol, T.M., *Mathematical Analysis*, Addison-Wesley, 1974.

[2] Arendt, W., Batty, C. J., Hieber, M., and Neubrander, F., *Vector-valued Laplace transforms and Cauchy problems* (Vol. 96), Springer Science & Business Media, 2nd ed., 2011

[3] Balakrishnan, A. V., *On Superstability of Semigroups*, presented at the 18th IFIP TC (1997).

[4] Balakrishnan, A. V., *On superstability of semigroups, systems modelling and optimization*, in Proceedings of the 18th IFIP Conference on System Modelling and Optimization (eds. M. P. Polis, et al.), CRC Research Notes in Mathematics, Chapman and Hall, 1999, 12-19.

[5] Balakrishnan, A. V., *Superstability of Systems*, Applied Mathematics and Computation 164 (2005), 321-326.

[6] Creutz, D., Mazo Jr, M., and Preda, C., *Superstability and Finite Time Extinction For C_0-Semigroups*, arXiv preprint arXiv:0907.4812 (2009).

[7] Doetsch, G., *Handbuch der Laplace-Transformation*, Vol. III., Birkhäuser, 1956, p. 225.

[8] Doetsch, G., *Über die endliche Laplace-Transformation*, Mathematische Annalen 123 (1951).

[9] Dorroh, J.R. and Neuberger, J.W., *A Theory of Strongly Continuous Semigroups in Terms of Lie Generators*, Journal of Functional Analysis (Impact Factor: 1.32). 02/1996; 136(1):114-126. DOI: 10.1006/jfan.1996.0023.

[10] Engel, K. J., Nagel, R., *One-parameter semigroups for linear evolution equations* (Vol. 194). Springer Science & Business Media, 2000.

[11] Haase, M., *Functional Analysis: An Elementary Introduction*, American Mathematical Society, Graduate Studies in Mathematics 156, 2014.

[12] Krein, M.G., Gohberg, I.C., *Theory and Applications of Volterra Operators in Hilbert Space*, American Math Society, 1970.

[13] Limaye, B.V., *Functional Analysis*, New Age International Limited, Publishers, New Delhi, 3rd edition, 2014.

[14] Lumer, G., *On the Growth Orders of the Resolvents for an Explicit Class of Superstable Semigroups*, Ulmer Seminare über Funktionalanalysis und Differentialgleichungen, Volume 6, 2001.

[15] Lumer, G., *Uniform Aspects of Pointwise Laplace Estimates, Stability and One-Sided Paley Wiener Type Theorems*, Ulmer Seminare über Funktionalanalysis und Differentialgleichungen, Volume 7 (2002), 346352.

[16] Müller, V. and Vrŝovský, J., *Orbits of Linear Operators Tending to Infinity*, Rocky Mountain J. Math. 39 (2009), 219230.

[17] Sokal, A.D., *A Really Simple Elementary Proof of the Uniform Boundedness Theorem*, The American Mathematical Monthly, 118 (2011), 450-452.

[18] Udwadia, Firdaus E., *Boundary control, quiet boundaries, super-stability and super-instability*, Applied Mathematics and Computation 164 (2005), 327-349.