Evolution Semigroups for Well-Posed, Non-Autonomous Evolution Families

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Abstract

The goal of this dissertation is to expand Berhard Koopman's operator theoretic global linearization approach to the study of nonautonomous flows. Given a system with states x in a set Ω (the *state space*), a map $t \to \gamma(t, s, x)$ $(t \geq s \geq 0)$ is called a global flow if it describes the time evolution of a system with the initial state $x \in \Omega$ at time $t \geq s \geq 0$. Koopman's approach to the study of flows is to look at the dynamics of the observables of the states instead of studying the dynamics of the states directly. To do so, one considers a vector space Z containing observables (measurements) and a vector space $\mathcal{M} := \mathcal{F}([0, \infty) \times \Omega, Z)$ of functions containing observations $g : [0, \infty) \times \Omega \to Z$. Then every global flow γ induces a family $T(t)(t \geq 0)$ of *linear* maps on $\mathcal{M}$, where

$$T(t)g : (s, x) \mapsto g(t + s, \gamma(t + s, s, x)). \tag{1}$$

Since every global flow γ satisfies $\gamma(s, s, x) = x$ and $\gamma(t, r, \gamma(r, s, x)) = \gamma(t, s, x)$ for $t \geq r \geq s \geq 0$ and $x \in \Omega$, the linear maps $T(t)(t \geq 0)$ define an operator semigroup on $\mathcal{M}$; that is, $T(0) = I$ and $T(t + s) = T(t)T(s)$ for $t, s \geq 0$. Following Koopman's approach, in pursuit of understanding the flow γ, we investigate the linear flow semigroup $T(t)(t \geq 0)$ on $\mathcal{M}$ given by (1), and if $\gamma(t, s, x) = U(t, s)x$ for some linear evolution family $U(t, s)$, an associated special evolution semigroups on a subspace of $\mathcal{M}$ given by $S(t)f : s \mapsto f(t + s)U(t + s, s)$. Of primary concern are continuity properties of the associated linear evolution semigroups on different function spaces (Chapters 1-3). The Lie generator of the flow and a collection of open problems concerning general flow semigroups (1), asymptotics and/or finite time blow-up, and Lie-Totter type approximations are described in Chapter 4.

Chapter 1
Introduction

1.1 Flow Semigroups

This dissertation is based upon the observation that deterministic evolutionary processes (systems that changes in time) give rise to local or global semigroups of *linear* operators $T(t)(t \geq 0)$ on some vector space $\mathcal{M}$. A semigroup is called global if for all $t \geq 0$, $T(t)$ is a linear operator from $\mathcal{M}$ into $\mathcal{M}$, and

$$\begin{aligned} &\text{(i)} \quad T(0) = I, \\ &\text{(ii)} \quad T(t+s) = T(t)T(s) \text{ for all } t, s \geq 0. \end{aligned} \tag{1.1}$$

Let Ω represent the state space of an evolutionary process; that is, Ω contains all possible states the process can be at any time point $t \geq s \geq 0$, where s defines the point in time when the process began. Let

$$H := \{(t, s) \in \mathbb{R}^2 : t \geq s \geq 0\}$$

and assume that the evolving process (flow) can be described by the map

$$\gamma : H \times \Omega \supset D(\gamma) \to \Omega, \text{ where}$$

$$D(\gamma) := \{(t, s, x) \in H \times \Omega : \gamma(t, s, x) \text{ exists } \}$$

denotes the domain of the flow γ and $\gamma(t, s, x) \in \Omega$ is the state of the process at time $t \geq s \geq 0$, assuming it was at initial state $x \in \Omega$ at initial time $s \geq 0$. In fact, we define $\gamma(s, s, x) = x$ for all $s \geq 0$ and all $x \in \Omega$. Thus, the map $\gamma : t \mapsto \gamma(t, s, x)$ describes the time propagation of $x \in \Omega$ starting at time $s \geq 0$ and is called a flow or orbit with initial value $\gamma(s, s, x) = x$. Let $(s, x) \in [0, \infty) \times \Omega$. Then

$$m(s, x) := \sup\{T : \gamma(t + s, s, x) \text{ exists for all } 0 \leq t \leq T\} \tag{1.2}$$

is called the stopping time of the flow $t \mapsto \gamma(t,s,x)$. A flow is called global if $m(s,x) = \infty$ for all $(s,s,x) \in D(\gamma)$. It is called local if there exists $(s,s,x) \in D(\gamma)$ such that $m(s,x) < \infty$. Every flow γ satisfies "Huygens principle of scientific determinism". That is, if $m(s,x) > 0$, then

$$\gamma(t,r,\gamma(r,s,x)) = \gamma(t,s,x)$$

for $0 \leq s \leq r \leq t < s + m(s,x)$ and $x \in \Omega$ (see Figure 1).

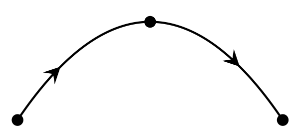

Figure 1

This equality was first formulated by Jacques Hadamard in his 1923 treatise "Lectures on Cauchy's Problem" where he writes:

> *The action of phenomena produced at the instant $t = 0$ on the state of matter at the later time $t = t_0$ takes place by the meditation of every intermediate instant $t = t'$ i.e. (assuming $0 < t' < t_0$) in order to find out what takes place for $t = t_0$, we can deduce from the state at $t = 0$ the state at $t = t'$ and, from the latter, the required state at $t = t_0$. [Huygens' principle] is what philosophers (...) call one of the "laws of thought": that is, an unavoidable law of our reason, which we could by no means conceive as not existing and without which we could not think. If today we discover Assyrian inscriptions, we cannot dream of supposing that, at any instant between the time when they were made and the time of their discovery, those inscriptions could have ceased to exist and all trace of them have disappeared. [Huygens' principle] must therefore be considered as a truism, which does not mean that it cannot interest us; for the geometer does not dislike truisms.*

As illustrated in Figure 1, any evolutionary process with initial state $x \in \Omega$ at initial time $s \geq 0$ with $m(s,x) > 0$ induces a flow map $t \mapsto \gamma(t,s,x) \in \Omega$ that satisfies

$$\begin{aligned} &\text{(i)} \quad \gamma(s,s,x) = x, \text{ and} \\ &\text{(ii)} \quad \gamma(t,r,\gamma(r,s,x)) = \gamma(t,s,x) \end{aligned} \tag{1.3}$$

for $0 \le s \le r \le t < m(s,x)$. Moreover if $m(s,x) > 0$, then it follows from Figure 1 that

$$m(t+s, \gamma(t+s,s,x)) = m(s,x) - t \tag{1.4}$$

for $0 \le s \le t < m(s,x)$. Define

$$\Omega_\gamma := \{(s,x) \in [0,\infty) \times \Omega : m(s,x) > 0\}$$

to be the time-state domain of the flow γ. We assume first that the flow γ is global. Following Berhard Koopman's approach (see [16]), to observe, study, or describe an evolutionary process, we need to assign to the states $x \in \Omega$ attributes or properties which we will call observables (measurements). Since the way and how we observe may also change over time, we will suppose that observations depend on time $t \ge s \ge 0$ and the state $x \in \Omega$. Thus, given a vector space Z containing all possible observables (for simplicity, all $z \in Z$ will be called observables), consider a function space

$$\mathcal{M} := \mathcal{F}(\Omega_\gamma, Z)$$

such that $g \in \mathcal{M}$ assigns to a time-state $(t,x) \in \Omega_\gamma$ the observable $z \in Z$ by $g(t,x) = z$. Since Z is a vector space, the function space $\mathcal{M}$ is a vector space of functions, although Ω and Ω_γ may have no mathematical structure at all. In cases where Ω_γ is a metric space, a natural choice for $\mathcal{M}$ is $C_b(\Omega_\gamma) := C_b(\Omega_\gamma, \mathbb{C})$, the Banach space of all bounded countinuous functions from Ω_γ into $\mathbb{C}$. Since γ is assumed to be global, for all $t \ge 0$, we may define a linear map $T(t) : \mathcal{M} \to \mathcal{M}$ by

$$T(t)g : (s,x) \mapsto g(t+s, \gamma(t+s,s,x)). \tag{1.5}$$

That is, $T(t)g$ is the time-state observation under g starting at (s,x) after time t has passed. Observe that $(t+s, \gamma(t+s,s,x)) \in \Omega_\gamma$ since $m(t+s, \gamma(t+s,s,x)) = m(s,x) - t = \infty$ (by (1.4)). The family of operators $T(t)(t \geq 0)$ has the following properties. Let $g \in \mathcal{M}$, then $T(0)g(s,x) = g(s,x)$. Define $f(s,x) := T(r)g(s,x) = g(s+r, \gamma(s+r,s,x))$. Then

$$\begin{aligned} T(t)T(r)g(s,x) &= T(t)f(s,x) = f(t+s, \gamma(t+s,s,x)) \\ &= g(r+t+s, \gamma(r+t+s, t+s, \gamma(t+s,s,x))) \\ &= g(t+r+s, \gamma(t+r+s,s,x)) \\ &= T(t+r)g(s,x). \end{aligned}$$

That is, the family $T(t)(t \geq 0)$ satisfies the semigroup properties (1.1). Thus, every global flow γ leads to a global semigroup of linear operators on a vector space of functions $\mathcal{M}$ given by (1.5). The semigroup (1.5) is called the global flow semigroup or the general evolution semigroup induced by γ. Its (formal) infinitesimal generator

$$\mathcal{A} = T'(0)$$

is called the Lie generator of γ. It is given by

$$\begin{aligned} (\mathcal{A}g)(s,x) :&= \lim_{t \searrow 0} \frac{g(t+s, \gamma(t+s,s,x)) - g(s,x)}{t} \\ &= g_s(s,x) + g_x(s,x)\gamma'(s,s,x) \end{aligned} \tag{1.6}$$

with domain $D(\mathcal{A}) = \{g \in \mathcal{M} : \mathcal{A}g \in \mathcal{M}\}$. In semigroup theory, the class of strongly continuous semigroups on Banach spaces has a rich theory connecting the semigroup $T(t)(t \geq 0)$ with its generator $\mathcal{A}$ and the resolvent

$$R(\lambda, \mathcal{A}) := (\lambda I - \mathcal{A})^{-1}.$$

This theory can be extended to a large degree to "bi-continuous semigroups on bi-admissible Banach spaces". As shown by F. Kühnemund [20] and B. Farkas [12], this class of semigroups encompases the semigroups studied by J.R. Dorroh and J.W. Neuberger in [7] and [8] and are of particular importance to the study of global flow semigroups (1.5) and their Lie generators $\mathcal{A}$ given by (1.6).

A local flow γ is called ε-local if

$$0 < \varepsilon := \inf\{m(s,x) : (s,x) \in \Omega_\gamma\} < \infty.$$

Now let $g \in \mathcal{M}$ and $0 \le t < \varepsilon$. For $(s,x) \in \Omega_\gamma$ define

$$T(t)g(s,x) := g(t+s, \gamma(t+s,s,x)).$$

Since $m(t+s, \gamma(t+s,s,x)) = m(s,x) - t > 0$, it follows that $(t+s, \gamma(t+s,s,x)) \in \Omega_\gamma$. Thus, $T(t)g$ is well defined and $T(0)g(s,x) = g(s,x)$. For $0 \le r < \varepsilon$ and $(s,x) \in \Omega_\gamma$ define

$$f(s,x) := T(r)g(s,x) = g(r+s, \gamma(r+s,s,x)).$$

Now, if $0 \le t < \varepsilon$ and $t + r < \varepsilon$, then

$$\begin{aligned} T(t)T(r)g(s,x) &= T(t)f(s,x) \\ &= f(t+s, \gamma(t+s,s,x)) \\ &= g(t+r+s, \gamma(t+r+s, r+s, \gamma(r+s,s,x))) \\ &= g(t+r+s,s,x) = T(t+r)g(s,x) \end{aligned} .$$

That is, $T(t) : \mathcal{M} \to \mathcal{M}$ is well-defined, linear, and

$$\begin{aligned} &\text{(i)} \quad T(0)g = g, \\ &\text{(ii)} \quad T(t+r)g = T(t)T(r)g \end{aligned}$$

for all $0 \le t, r < \varepsilon$ such that $t + r < \varepsilon$. We say that $T(t)(0 \le t < \varepsilon)$ is an ε-local semigroup on $\mathcal{M}$ with formal generator $\mathcal{A} = T'(0)$ given by (1.6) (see also [17]).

It is important to notice that the Lie generator

$$(\mathcal{A}g)(s,x) := \lim_{t \searrow 0} \frac{g(t+s, \gamma(t+s,s,x)) - g(s,x)}{t}$$

with $D(\mathcal{A}) := \{g \in \mathcal{M} : \mathcal{A}g \in \mathcal{M}\}$ is well-defined on Ω_γ for all flows since $m(s,x) > 0$ for all $(s,x) \in \Omega_\gamma$. That is, a flow γ has a Lie generator $\mathcal{A}$ defined by (1.6) even if the flow γ does not define a global or ε-local flow semigroup $T(t)(t \geq 0)$ on $\mathcal{F}(\Omega_\gamma, Z)$.

Finally, a flow γ is said to be time-autonomous if

$$\text{(i) } D(\gamma) = [0,\infty) \times \Omega, \text{ and}$$

$$\text{(ii) } \gamma(t+s,s,x) = \gamma(t,0,x)$$

for all $0 \leq s$, $x \in \Omega$ and $0 \leq t < m(s,x)$. Observe that (i) and (ii) imply that $0 < m(s,x) = m(0,x)$ for all $s \geq 0$ and $x \in \Omega$. Define

$$\phi(t,x) := \gamma(t,0,x)$$

for $x \in \Omega$ and $0 \leq t < m(s,x)$. Then $\phi(0,x) = x$ for all $x \in \Omega$. Moreover, if $x \in \Omega$ and if $0 \leq t, r, t+r < m(0,x)$, then

$$\begin{aligned}\phi(t,\phi(r,x)) &= \gamma(t,0,\phi(r,x)) = \gamma(t+r,r,\phi(r,x)) \\ &= \gamma(t+r,r,\gamma(r,0,x)) = \gamma(t+r,r,x) \\ &= \gamma(t+r,0,x) = \phi(t+r,x).\end{aligned}$$

If a flow is autonomous, then we consider a vector space

$$\mathcal{M} = \mathcal{F}(\Omega, Z)$$

from the state space Ω into the observation vector space Z. If the flow is global (that is, if $m(0,x) = \infty$ for all $x \in \Omega$), then we define the flow semigroup $T(t)(T \geq 0)$ on $\mathcal{M}$ by

$$T(t)g(x) := g(\phi(t,x)) \tag{1.7}$$

for $g \in \mathcal{M}$ and $x \in \Omega$. If the flow is ε- local (that is if $0 < \varepsilon = \inf\{m(0,x) : x \in \Omega\}$), then (1.7) defines a local semigroup on $\mathcal{M}$. Moreover, since $m(0,x) > 0$ for all $x \in \Omega$ (by definition of an autonomous flow), the Lie generator

$$\mathcal{A}g(x) := \lim_{t \searrow 0} \frac{g(\phi(t,x)) - g(x)}{t}$$

with $D(\mathcal{A}) := \{g \in \mathcal{M} : \mathcal{A}g \in \mathcal{M}\}$ is always well-defined, even if the flow semigroup (1.7) is not. Observe that formally

$$\mathcal{A}g(x) = g'(x)\phi'(0,x).$$

The idea to investigate autonomous flows in terms of their Lie generators $\mathcal{A}$ goes back to the works of J.R. Dorroh and J.W. Neuberger from the late 60's and early 70's ([5], [6], [22], [23], [32]). They succeeded in 1996 [8] to give a complete characterization of Lie generator of global, autonomous flows and further expanded their results in [9], [24]. Moreover, in a series of papers starting around 2010, J.W. Neuberger and his co-authors investigated how the Lie generators of an autonomous flow can be used to distinguish between local and global flows ([25], [26], [27]). In this dissertation we extend the Dorroh-Neuberger program to include non-autonomous flows. Although the step from non-autonomous flows to autonomous flows with time-space states $\widetilde{x} := (s,x) \in [0,\infty) \times \Omega =: \widetilde{\Omega}$ is standard fare in the mathematical tool-box, it appears that this has never been investigated in detail before.

The purpose of the Chapters 3 and 4 is to investigate continuity properties of flow semigroups on different function spaces $\mathcal{M}$, properties of the Lie generator of flows, asymptotics and/or finite time blow-up, and Lie-Trotter approximation theorems. To do so, we need to review some basic properties of strongly continuous semigroups in Sections 1.2 - 1.4 and bi-continuous semigroups (Chapter 2).

1.2 Strongly Continuous Semigroups

In order to properly define the derivative $A = T'(0)$ of a semigroup $T(t)(t \geq 0)$, we have to assume that the spaces Z (of observables) and $\mathcal{M}$ (of observations) are topological vector spaces. To begin with, we assume that Z and $\mathcal{M}$ are both Banach spaces and recall the following basic definitions and facts.

A linear map $T : X \to Y$ from a normed vector space X to a Banach space Y is a bounded linear operator if there is a $C > 0$ such that

$$\|Tf\|_Y \leq C\|f\|_X$$

for all $f \in X$. The space $\mathcal{L}(X, Y)$ of all bounded linear operators from X into Y is again a Banach space with norm

$$\|T\| := \sup_{\|f\|_X \leq 1} \|Tf\|_Y.$$

If X is a Banach space, then we denote $\mathcal{L}(X, X)$ by $\mathcal{L}(X)$. A family $T(t)(t \geq 0) \subset \mathcal{L}(X)$ is called a strongly continuous semigroup if (1.1) holds and

$$t \mapsto T(t)x \text{ is continuous on } [0, \infty) \text{ for all } x \in X.$$

Since $\|T(t + t_0)f - T(t_0)f\| \leq \|T(t_0)\|\|T(t)f - f\|$, it follows that a semigroup $T(t)(t \geq 0)$ is strongly continuous if and only if

$$\lim_{t \to 0^+} T(t)f = f \text{ for all } f \in X. \tag{1.8}$$

If $A \in \mathcal{L}(X)$, then

$$T(t) := e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

is well defined for every $t \in \mathbb{C}$, and the map $T : \mathbb{C} \to \mathcal{L}(X)$ defined by

$$t \mapsto T(t) := e^{tA}$$

is an entire function and the semigroup properties (1.1) hold. In fact, every semigroup that is norm-continuous at 0 (that is, $T(t) \to I$ in $\mathcal{L}(X)$ as $t \to 0^+$) is an entire function with $A := T'(0) \in \mathcal{L}(X)$. If a semigroup $T(t)(t \geq 0)$ is strongly continuous, then one can show that $T(t)(t \geq 0)$ is of type (M, ω) for some $M \geq 1$ and $\omega \in \mathbb{R}$; that is,

$$\|T(t)\| \leq Me^{\omega t}$$

for all $t \geq 0$ (see [10] p. 38). The infinitesimal generator of a strongly continuous semigroup $T(t)$ is given by

$$Af := \lim_{h \to 0^+} \frac{T(h)f - f}{h},$$

where $f \in X$ is in the domain $D(A)$ of the linear operator A, given by

$$D(A) := \left\{ f \in X : \lim_{h \to 0^+} \frac{T(h)f - f}{h} \text{ exists} \right\}.$$

The proof of the following proposition can be found in every introductory textbook on semigroup theory (e.g., [1], [10], [13]).

Proposition 1.2.1. *Let A be the generator of a strongly continuous semigroup $T(t)(t \geq 0)$ of type (M, ω) on a Banach space X. Then the following statements hold.*

(i) The semigroup $T(t)(t \geq 0)$ commutes with A on $D(A)$ and, for all $t \geq 0$,

$$T(t)f - f = \begin{cases} A\int_0^t T(s)f\,ds \text{ if } f \in X, \\ \int_0^t T(s)Af\,ds \text{ if } f \in D(A). \end{cases}$$

In particular, if $f \in D(A)$, then $u(t) := T(t)f$ solves the abstract Cauchy problem

$$u'(t) = Au(t), \ u(0) = f.$$

(ii) The domain $D(A)$ is dense in X, A is closed, and $R(\lambda, A) \in \mathcal{L}(X)$ for all $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$, where for all $f \in X$ and $Re\lambda > \omega$,

$$R(\lambda, A)f := (\lambda I - A)^{-1}f = \int_0^\infty e^{-\lambda t}T(t)f\,dt.$$

(iii) For all $Re\lambda > \omega$ and $n \in \mathbb{N}$,

$$\|R(\lambda, A)^n\| \leq \frac{M}{(Re\,\lambda - \omega)^n}.$$

(iv) For $\zeta > \omega$ and all $f \in D(A)$,

$$T(t)f = \frac{1}{2\pi i} \int_{\zeta + i\mathbb{R}} e^{\lambda t} R(\lambda, A)f\,d\lambda.$$

(v) For all $f \in X$,

$$T(t)f = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} f = \lim_{n\to\infty} \left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^n f.$$

Moreover, if A is a densely defined linear operator for which (iii) holds, then A generates a strongly continuous semigroup $T(t)(t \geq 0)$ given by (v). □

Before reviewing the Lie-Trotter product formula for semigroups in Section 1.4 and the theory of bi-continuous semigroups on bi-admissible Banach spaces in Chapter 2, we collect in the next section some basic, preliminary examples of flow semigroups and their Lie generators.

1.3 Flow Semigroups Induced by First Order ODE's

As a first class of examples, we investigate general evolution semigroups of the form (1.5)

$$T(t)g : (s, x) \mapsto g(t + s, \gamma(t + s, s, x)),$$

where the flow $t \mapsto \gamma(t,s,x)$ is defined by the solution of the separable linear equation

$$x'(t) = a(t)x(t), \;\; x(s) = x \in \mathbb{C} = \Omega.$$

Assuming that $a(\cdot)$ is integrable, γ is given by

$$\gamma(t,s,x) = e^{\int_s^t a(r)dr}x. \tag{1.9}$$

In this case, the general evolution semigroup (1.5) induced by γ is given by

$$T(t)g(s,x) := g(t+s, e^{\int_s^{t+s} a(r)\,dr}x). \tag{1.10}$$

If one assumes that $Re\, a(r) \leq \omega$ for almost all $r \geq 0$, then we may choose the observations g to be in $\mathcal{M} := C_0([0,\infty)\times\mathbb{C},\mathbb{C})$, the Banach space of all continuous functions $g : [0,\infty) \times \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{|(s,x)|\to\infty} g(s,x) = 0$$

and

$$\|g\|_\infty := \sup_{(s,x)} \|g(s,x)\| < \infty.$$

Then $T(t)(t \geq 0)$ is strongly continuous and its generator $\mathcal{A}$ is given by

$$\mathcal{A}f(s,x) := f_s(s,x) + f_x(s,x)a(s)x \tag{1.11}$$

for those $f \in D(\mathcal{A})$ for which f_s and f_x exist and $\mathcal{A}f \in \mathcal{M}$. Following [7] and [8], the operator $\mathcal{A}$ is also called the Lie generator of the flow γ given in (1.9). To see that $T(t)(t \geq 0)$ is well-defined, observe that the continuity of $(s,x) \mapsto T(t)g(s,x)$ follows from the continuity of $(s,x) \mapsto g(s,x)$ and $s \mapsto e^{\int_s^{t+s} a(r)\,dr}x$. Furthermore,

$$\lim_{|(s.x)|\to\infty} T(t)g(s,x) = 0$$

since $|(t+s, e^{\int_s^{t+s} a(r)\,dr} x)| \to \infty$ as $|(s,x)| \to \infty$. Therefore, $T(t)g \in \mathcal{M}$ for all $g \in \mathcal{M}$. Obviously $\|T(t)g\|_\infty \leq \|g\|_\infty < \infty$. For the strong continuity, let $\varepsilon > 0$ be given and g be fixed. Now consider, for $0 \leq t \leq 1$,

$$\|T(t)g - g\|_\infty = \sup_{(s,x)} |T(t)g(s,x) - g(s,x)| = \sup_{(s,x)} |g(t+s, e^{\int_s^{t+s} a(r)\,dr} x) - g(s,x)|.$$

Clearly, there is a constant N such that

$$\sup_{|(s,x)| \geq N} |T(t)g(s,x) - g(s,x)| \leq \sup_{|(s,x)| \geq N} |T(t)g(s,x)| + \sup_{|(s,x)| \geq N} |g(s,x)| < \frac{\varepsilon}{2}$$

for all $0 \leq t \leq 1$. Since $z \to \frac{e^z - 1}{z}$ is entire, it follows that there exists $C > 0$ such that $|e^z - 1| \leq C|z|$ for all $|z| \leq 1$. Choose $0 < \delta < 1$ small enough such that $|\int_s^{t+s} a(r)\,dr| \leq 1$ for all $0 \leq s \leq N$ and $0 \leq t < \delta$. Let $\widetilde{y} := (t+s, e^{\int_s^{t+s} a(r)\,dr} x)$, $\widetilde{x} := (s,x)$, and $A(t) := \int_0^t a(r)\,dr$. Then

$$|\widetilde{y} - \widetilde{x}| = |(t, (e^{\int_s^{t+s} a(r)\,dr} - 1)x)| \leq t + CN|A(t+s) - A(s)|$$

for all $0 \leq t \leq \delta < 1$ and all (s,x) with $|(s,x)| \leq N$. Since $A(\cdot)$ is uniformly continuous on compact sets, it follows that for all $\delta^* > 0$ there exists $\delta' > 0$ such that $|\widetilde{y} - \widetilde{x}| < \delta^*$ if $0 \leq t < \delta' \leq \delta$ and $|(s,x)| \leq N$. Furthermore, since g is uniformly continuous on compact sets,

$$\sup_{|(s,x)| \leq N} |g(t+s, e^{\int_s^{t+s} a(r)\,dr} x) - g(s,x)| \leq \frac{\varepsilon}{2}$$

for all $0 \leq t$ sufficiently small. Thus $t \mapsto T(t)g$ is continuous for all $g \in \mathcal{M}$.

If the flow is linear in X (as in (1.9)), and if one chooses observations $g : [0,\infty) \times \mathbb{C} \to \mathbb{C}$ of the form $g(s,x) = f(s)x$ for some $f : [0,\infty) \to \mathbb{C}$, then the general evolution semigroup (1.5) becomes

$$\begin{aligned} T(t)f(s)x &= T(t)g(s,x) = g(t+s, \gamma(t+s,s,x)) \\ &= f(t+s)\gamma(t+s,s,x) = f(t+s)e^{\int_s^{t+s} a(r)dr} x \end{aligned}$$

for all $x \in \mathbb{C}$. That is, taking advantage of the $x-$linearity of γ, a special evolution semigroup induced by the flow $t \mapsto \gamma(t, s, x)$ is given by

$$T(t)f(s) = f(t+s)e^{\int_s^{t+s} a(r)\,dr} \tag{1.12}$$

for functions $f : [0, \infty) \to \mathbb{C}$. If $a : [0, \infty) \to \mathbb{C}$ is locally integrable and $Re\, a(t) \leq \omega$ for almost all $t \geq 0$ for some $\omega \in \mathbb{R}$, then the special evolution semigroup (1.12) is a strongly continuous semigroup on $C_0[0, \infty) := C_0([0, \infty), \mathbb{C})$, the space of continuous functions from $[0, \infty)$ into $\mathbb{C}$ that vanish at infinity with the sup-norm

$$\|f\| := \sup_{s \geq 0} |f(s)|.$$

The straight-forward proof uses heavily the uniform continuity of f on compact intervals and the fact that f vanishes at infinity (see, for example [30]). If f is merely assumed to be continuous and bounded (that is, $f \in C_b([0, \infty), \mathbb{C})$), then $t \mapsto T(t)f$ is, in general, no longer continuous. This can be seen, for example, by taking $a(\cdot) = 0$ and $f(t) = e^{it^2}$. Then

$$\|T(t+h)f - T(t)f\|_\infty = \sup_{s \geq 0} |e^{i(t+h+s)^2} - e^{i(t+s)^2}| = 2$$

for all $t \geq 0$ and $h > 0$. Moreover, if f' exists and $f' + af \in C_0[0, \infty)$, then f is in the domain $D(A)$ of the generator A of the special evolution semigroup (1.12) and

$$Af(s) := f'(s) + a(s)f(s). \tag{1.13}$$

The strongly continuous, special evolution semigroup $T(t)(t \geq 0)$ (given in (1.12)) has interesting stability properties on X. For example, if $a(t) = -\alpha t^{\alpha-1}$ for some $\alpha > 1$, then $Re\, a(t) \leq 0$ for all $t \geq 0$ and the semigroup

$$T(t)f(s) = e^{-(t+s)^\alpha + s^\alpha} f(t+s)$$

is superstable (i.e., $\|T(t)\| = e^{-t^\alpha}$) but not nilpotent (i.e., $T(t) \neq 0$ for all $t \geq 0$). Thus by (1.12), the initial value problem

$$u_t(t,s) = u_s(t,s) - \alpha s^{\alpha-1} u(t,s),\ u(0,s) = f(s)$$

for $t, s \geq 0$, $f \in X$, $\alpha > 1$ provides a physical example that has superstable solutions

$$u(t,s) = T(t)f(s) = e^{-(t+s)^\alpha + s^\alpha} f(t+s)$$

that are not extinct-in-finite-time. This answers the following question of A.V. Balakrishnan from 2005 (see [2]).

> *As we can see, superstability in terms of semigroups simply means that the semigroup is quasi-nilpotent. There are many chacterizations (necessary and sufficient conditions) of quasi-nilpotent semigroups.... However none of them would appear to be constructive ... This raises a currently open question: Are there physical examples of superstability which are not of the type of extinction-in-finite-time?*

If $a(t) = \alpha t^{\alpha-1}$ (instead of $a(t) = -\alpha t^{\alpha-1}$ as in the previous example), then the special evolution semigroup

$$T(t)f(s) = e^{(t+s)^\alpha - s^\alpha} f(t+s) \tag{1.14}$$

does not map $C_0[0,\infty)$ into itself. In this case, one can pursue several alternate directions.

Example 1.3.1. Let $\alpha > 1$ and

$$C_{0,\alpha}[0,\infty) := \{f \in C_0[0,\infty) : e^{s^\alpha}|f(s)| \to 0 \text{ as } s \to \infty\}$$

with the norm

$$\|f\|_\alpha = \sup_{s\geq 0} |e^{s^\alpha} f(s)|.$$

Then $C_{0,\alpha}[0,\infty)$ is a Banach space and the special evolution semigroup (1.14) defines a stronlgy continuous semigroup with $\|T(t)\| = 1$.

Proof. Clearly $T(t)(t \geq 0)$ satisfies the semigroup properties (1.1). First we need to show that $T(t) : X \to X$ is well-defined. We know for any fixed $t \geq 0$, $s \mapsto T(t)f(s)$ is continuous and for any $t, s \geq 0$,

$$|T(t)f(s)| = |e^{(t+s)^\alpha - s^\alpha} f(t+s)| \leq |e^{(t+s)^\alpha} f(t+s)|.$$

This implies $T(t)f(s) \to 0$ as $s \to \infty$. Thus $T(t)(t \geq 0)$ is well defined. Furthermore,

$$\|T(t)f\|_\alpha = \sup_{s \geq 0} |e^{(t+s)^\alpha} f(t+s)| = \sup_{s \geq t} |e^{s^\alpha} f(s)| \leq \|f\|_\alpha$$

implies $\|T(t)\| \leq 1$. Now pick $f_t \in X$ such that

$$\|f_t\|_\alpha = \sup_{s \geq 0} |e^{s^\alpha} f_t(s)| = |e^{t^\alpha} f_t(t)| = 1.$$

Then it follows that

$$1 = |e^{t^\alpha} f_t(t)| \leq \sup_{s \geq t} |e^{s^\alpha} f_t(s)| \leq \sup_{s \geq 0} |e^{(t+s)^\alpha} f_t(t+s)| = \|T(t)f_t\|_\alpha \leq \|T(t)\| \leq 1.$$

Thus $\|T(t)\| = 1$. To see the strong continuity of $T(t)(t \geq 0)$, observe that

$$\|T(t)f - f\|_\alpha = \sup_{s \geq 0} |e^{s^\alpha} (f(t+s) e^{(t+s)^\alpha - s^\alpha} - f(s))|$$

$$\leq \underbrace{\sup_{s \geq 0} |f(t+s) e^{(t+s)^\alpha} - f(t+s) e^{s^\alpha}|}_{(E1)} + \underbrace{\sup_{s \geq 0} |f(t+s) e^{s^\alpha} - f(s) e^{s^\alpha}|}_{(E2)}.$$

Using the facts that f is uniformly continuous on compact intervals and that $e^{s^\alpha} f(s)$ is small for $s \geq N$ (N large), one can easily see that $(E1)$ and $(E2)$ approach 0 as $t \to 0$. Therefore $T(t)(t \geq 0)$ is strongly continuous. □

Another suitable set-up for the special evolution semigroup (1.14) is to consider the semigroup

$$T(t)f(s) = \begin{cases} e^{(t+s)^\alpha - s^\alpha} f(t+s) & t+s < N \\ 0 & \text{otherwise} \end{cases}$$

on the truncated space $C_0([0,N],\mathbb{C})$ of continuous functions $f:[0,N]\to\mathbb{C}$ such that $f(N)=0$ and $\|f\| = \sup_{0\le s\le N}|f(s)|$. Then the special evolution semigroup (1.14) is again well-defined and strongly continuous with $e^{N^2}=\inf\{M:\|T(t)\|\le M \text{ for all } t\ge 0\}$.

Yet another way to study the special evolution semigroup (1.14) is to consider it on the Fréchet space $C([0,\infty),\mathbb{C})$ of continuous functions with the topology of uniform convergence on compact subsets where the special evolution semigroup defines a locally equicontinuous semigroup. This case is considered in Section 4.2.

Example 1.3.2. Another interesting case of a special evolution semigroup (1.12) results if one chooses

$$a(t)=\frac{-\alpha}{t}\ (t>0)$$

(see also [1], p. 196). Since $a(\cdot)$ is not locally integrable on $[0,\infty)$, the results of the beginning of the section do not apply. In this case, the special evolution semigroup is given by

$$T(t)f(s)=\frac{s^\alpha}{(t+s)^\alpha}f(t+s),$$

where $T(0)f(s)=f(s)$ for all $s\ge 0$. Clearly the operators $T(t)(t\ge 0)$ define bounded linear operators on $X=C_0([0,\infty),\mathbb{C})$. Since

$$\|T(t)f-f\|=\sup_{s\ge 0}\left|\frac{s^\alpha}{(t+s)^\alpha}f(t+s)-f(s)\right|\ge |f(0)|$$

for all $t>0$, it follows that $t\mapsto T(t)f(s)$ is not continuous in $t_0=0$ if $f(0)\ne 0$. However, it can be easily seen that $t\mapsto T(t)f(s)$ is continuous on $(0,\infty)$ for all $f\in X$. It should be mentioned that $T(t)(t\ge 0)$ is not of type C_1 or type A as defined be Hille and Phillips (see [14]), but does satisfy the stability condition

$$\int_0^1 \|T(t)\|\,dt\le 1.$$

□

1.4 Lie-Trotter Product Formula

In later chapters, the following observation will prove to be useful. The formal generators of the general and special evolution semigroup (1.10) and (1.12) are of the form

$$\mathcal{A}f(s,x) = f_s(s,x) + f_x(s,x)a(s)x = A_1 f(s,x) + A_s f(s,x) \text{ or}$$
$$\mathcal{A}f(s) = f'(s) + a(s)f(s) = A_1 f(s) + A_2 f(s),$$

This leads to the consideration of the Lie-Trotter Product Formula, a key result of semigroup theory (for a proof see [10] p. 227).

Theorem 1.4.1. *Let $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$ be strongly continuous semigroups on a Banach space X with generators $(A_1, D(A_1))$ and $(A_2, D(A_2))$ respectively. Suppose $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$ satisfy the stability condition*

$$\left\| \left[T_1\left(\frac{t}{n}\right) T_2\left(\frac{t}{n}\right) \right]^n \right\| \leq M e^{\omega t} \text{ for all } t \geq 0, n \in \mathbb{N} \tag{1.15}$$

for some constants $M \geq 1$ and $\omega \in \mathbb{R}$. Furthermore, Let $D := D(A_1) \cap D(A_2)$ and suppose D and $(\lambda_0 I - A_1 - A_2)D$ are dense in X for some $\lambda_0 > \omega$. Then the closure

$$A := \overline{A_1 + A_2}$$

generates a strongly continuous semigroup $T(t)(t \geq 0)$ given by the Lie-Trotter Product Formula

$$T(t)f = \lim_{n\to\infty} \left(T_1\left(\frac{t}{n}\right) T_2\left(\frac{t}{n}\right) \right)^n f \tag{1.16}$$

for all $f \in X$. □

Example 1.4.2. Let $X = C_0([0,\infty),\mathbb{C})$ and let $a : [0,\infty) \to \mathbb{C}$ be continuous with $Re\, a(t) \le \omega \in \mathbb{R}$ for all $t \ge 0$. Then $A_1 f = f'$ with $D(A_1) := \{f \in X : f' \in X\}$ and $A_2 f = af$ with $D(A_2) := \{f \in X : af \in X\}$ generate the strongly continuous semigroups $T_1(t)f(s) := f(t+s)$ and $T_2(t)f(s) := e^{ta(s)}f(s)$, respectively. Since $\|T_1(t)\| = 1$ and $\|T_2(t)\| \le e^{\omega t}$, it follows that

$$\left\| \left[T_1\left(\frac{t}{n}\right) T_2\left(\frac{t}{n}\right)\right]^n \right\| \le \left\| T_1\left(\frac{t}{n}\right)\right\|^n \left\| T_2\left(\frac{t}{n}\right)\right\|^n \le Me^{\omega t}$$

for all $t \ge 0$ and $n \in \mathbb{N}$. Let D_0 denote the set of all smooth functions with compact support. Then $D_0 \subset D := D(A_1) \cap D(A_2)$. We know from (1.12) and Proposition 1.2.1 that

$$\begin{aligned} R(\lambda, A)f(s) &= (\lambda - A_1 - A_2)^{-1} f(s) \\ = \int_0^\infty e^{-\lambda t} T(t) f(s)\, dt &= \int_0^\infty e^{-\lambda t} e^{\int_s^{t+s} a(r)\, dr} f(t+s)\, dt. \end{aligned}$$

Now let $f \in D_0$, then $g := R(\lambda, A)f \in D_0$ and $(\lambda - A_1 - A_2)f = g$. Thus $D_0 \subset (\lambda - A_1 - A_2)D$. Since D_0 is dense in X, it follows that D and $(\lambda - A_1 - A_2)D$ are dense in X. Thus, by Theorem 1.4.1, $A = \overline{A_1 + A_2}$ generates the strongly continuous semigroup

$$\begin{aligned} T(t)f(s) &= f(t+s) e^{\int_s^{t+s} a(r)\, dr} \\ &= \lim_{n\to\infty} \left[T_1\left(\frac{t}{n}\right) T_2\left(\frac{t}{n}\right)\right]^n f(s) \\ &= \lim_{n\to\infty} f(t+s) e^{\frac{t}{n} \sum_{i=1}^{n} a\left(\frac{it}{n} + s\right)} \end{aligned}$$

where the limit is uniform for $f \in X$, $s \ge 0$, and t in compact intervals. □

A natural extension of Example 1.3.1 arises from the consideration of nonautonomous, linear Cauchy problems

$$u'(t) = A(t)u(t), \quad u(s) = x \tag{1.17}$$

on a Banach space X whose solutions $t \to u(t)$ are given by evolution families $U(t,s)(t \geq s \geq 0)$; that is, $u(t) := U(t,s)x$ for $t \geq s \geq 0$.

Definition 1.4.3. *A family $U(t,s) \subset \mathcal{L}(X)(t \geq s \geq 0)$ of bounded linear operators on a normed vector space X is called an evolution family if it satisfies*

$$\begin{aligned} &(i) \quad U(s,s) = I \text{ for all } s \geq 0, \text{ and} \\ &(ii) \quad U(t,r)U(r,s) = U(t,s) \text{ for all } t \geq r \geq s \geq 0. \end{aligned} \tag{1.18}$$

If the mapping

$$(t,s) \to U(t,s)x$$

is continuous for all $x \in X$, then the evolution family is said to be strongly continuous. Furthermore, the evolution family is called exponentially bounded if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that,

$$\|U(t,s)\| \leq Me^{\omega(t-s)}$$

for all $t \geq s \geq 0$.

Contrary to strongly continuous semigroups, it is important to note that strongly continuous evolution families are not automatically exponentially bounded. An easy one-dimensional example is the evolution family

$$U(t,s)x := e^{\int_s^t a(r)dr}x$$

that solves the ordinary differential equation $u'(t) = a(t)u(t)$ for some integrable function $a(\cdot)$ where $t \geq s \geq 0$ and $u(s) = x \in \mathbb{C}$. In this case, $U(t,s)(t \geq s \geq 0)$ is in general not exponentially bounded (for example, if $a(t) = t$).

To connect evolution families and flow maps, observe that to any linear nonautonomous evolution family $U(t,s)(t \geq s \geq 0)$ on a Banach space X, we can define a global flow map γ by

$$\gamma(t,s,x) := U(t,s)x$$

for $t \geq s \geq 0$ and $x \in X$. From (1.5) we can associate an autonomous linear semigroup $T(t)(t \geq 0)$ induced by the flow map as

$$T(t)g(s,x) := g(t+s, \gamma(t+s,s,x)) = g(t+s, U(t+s,s)x). \tag{1.19}$$

Because of the linearity of γ in $x \in X$ we may choose $g : [0,\infty) \times X \to Y$ (Y some Banach space) of the form

$$g(s,x) := f(s)x \tag{1.20}$$

for some $f \in C_0([0,\infty), \mathcal{L}(X,Y))$. By choosing observations of the form (1.20), the flow semigroup $T(t)(t \geq 0)$ defined in (1.19) becomes

$$T(t)f(s)x = T(t)g(s,x) = g(t+s, U(t+s,s)x) = f(t+s)U(t+s,s)x. \tag{1.21}$$

That is, the special evolution semigroup associated to the linear evolution family $U(t,s) \in \mathcal{L}(X)$ for $t \geq s \geq 0$ is given by

$$T(t)f(s) := f(t+s)U(t+s,s) \tag{1.22}$$

for $f \in C_0([0,\infty), \mathcal{L}(X,Y))$. In particular, if one chooses $Y = \mathbb{C}$, then $\mathcal{L}(X,Y) = X^*$ and $T(t)f(s) = f(t+s)U(t+s,s) \in X^*$ is defined by

$$\langle x, T(t)f(s)\rangle := \langle U(t+s,s)x, f(t+s)\rangle = \langle x, U^*(t+s,s)f(t+s)\rangle$$

for all $x \in X$.

Furthermore, $T(t)(t \geq 0)$ acting on $C_0([0,\infty), X^*)$ is in a certain sense "adjoint-like" to the well-established evolution semigroup

$$V(t)g(s) := U(s,s-t)g(s-t) \ (t,s \in \mathbb{R}) \tag{1.23}$$

acting on functions $g \in C_0(\mathbb{R}, X)$, the space of X-valued continuous functions on $\mathbb{R}$ that vanish at infinity. First proposed by Howland [15] in 1974, the evolution semigroup $V(t)(t \geq 0)$ was studied in detail by Chicone and Latushkin [4], as well as Nagel and Nickel (see [10], [28], and [29]) for the exponentially bounded case.

To see how the two evolution semigroups (1.22) and (1.23) are connected, let g and f be functions with values in X and $X,^*$ respectively. Then, formally, the right shift semigroup $S_r(t)f := f(t+\cdot)$ is the adjoint of the left shift semigroup $S_l(t)g := g(\cdot - t)$ and therefore

$$\begin{aligned} \langle V(t)g, f\rangle &= \langle U(\cdot, \cdot - t)g(\cdot - t), f(\cdot)\rangle = \langle S_l(t)U(t+\cdot, \cdot)g(\cdot), f(\cdot)\rangle \\ &= \langle U(t+\cdot, \cdot)g(\cdot), S_r f(\cdot)\rangle = \langle U(t+\cdot, \cdot)g(\cdot), f(t+\cdot)\rangle \\ &= \langle g, T(t)f\rangle. \end{aligned}$$

That is, the special evolution semigroup $T(t)f : s \mapsto f(t+s)U(t+s, s)$ acting on $C_0([0,\infty), X^*)$ is adjoint-like to the Howland evolution semigroup $V(t)g : s \mapsto U(s, s-t)g(s-t)$ acting on functions $g : \mathbb{R} \to X$.

We know if $U(t,s)(t \geq s \geq 0)$ is an exponentially bounded, strongly continuous evolution family, then the Howland semigroup $V(t)(t \geq 0)$ is strongly continuous for $t \geq 0$; that is,

$$t \mapsto V(t)g$$

is continuous for all $g \in C_0(\mathbb{R}, X)$ (see, for example [10] and [28]). However, if $U(t,s)(t \geq s \geq 0)$ is an exponentially bounded, strongly continuous evolution family, then it is easy to see that the special evolution semigroup

$$T(t)f(s) := f(t+s)U(t+s, s) \tag{1.24}$$

is, in general, not strongly continuous on $C_0([0,\infty), X^*)$. However, as we will show in Chapter 3, if $U(t,s)(t \geq s \geq 0)$ is an exponentially bounded, strongly continuous evolution family, then (1.24) defines a bi-continuous semigroup (as defined in

[11] and [20]) on $C_b([0,\infty), X^*_{\omega^*})$, where $X^*_{\omega^*}$ denotes X^* with the weak-* topology. Moreover, if $U(t,s)(t \geq s \geq 0)$ is strongly continuous (but not necessarily exponentially bounded), then

$$T(t)f(s) := \begin{cases} f(t+s)U(t+s,s) & t+s < N \\ 0 & \text{otherwise} \end{cases}$$

is bi-continuous on $C_0([0,N], X^*_{\omega^*})$. In Chapter 4 we will look at blueprints for approximation theorems for (1.19) and (1.24) similar to Example 1.4.2. To conclude we will also consider introductory cases where a flow $t \mapsto \gamma(t,s,x)$ that solves a nonlinear, nonautonomous Cauchy problem

$$u'(t) = A(t, u(t)),\ u(s) = x,$$

defines a bi-continuous linear evolution semigroup on a suitable function space $\mathcal{M} := \mathcal{F}([0,\infty) \times X, Z)$ by $T(t)g(s,x) := g(t+s, \gamma(t+s,s,x))$ and study its infinitesimal Lie generator $\mathcal{A}$ defined by $\mathcal{A}f(s,x) := g_s(s,x) + g_x(s,x)A(s,x)$.

Chapter 2
Bi-Continuous Semigroups

2.1 Basic Definitions and Examples

The framework of bi-continuous semigroups was developed in the dissertations of Franziska Kühnemund and Balint Farkas at the Functional Analysis group at the University of Tübingen around 2001-2003 (see [10], [12], [20], and [29]). This class of semigroups provides a suitable framework for many classes of semigroups $T(t) \in \mathcal{L}(X)(t \geq 0, X$ Banach space) that are not strongly continuous. Bi-continuous semigroups have continuity properties with respect to the norm topology (that is, $T(t) : X \to X$ is a continuous linear operator with respect to the norm topology on X), while the map $t \mapsto T(t)f$ from $[0, \infty) \to X$ is continuous with respect to a weaker, but suitable, locally convex topology on X. In this section we will consider the conditions we need on X to begin studying bi-continuous semigroups and look at several examples of such spaces. When a locally convex topology is suitable, we will call the Banach space X, in conjunction with its two topologies, a bi-admissible Banach space.

Definition 2.1.1. *Let $(X, \|\cdot\|)$ be a Banach space and let σ be a topology on X. We say $X = (X, \|\cdot\|, \sigma)$ is a bi-admissible Banach space if the following conditions hold.*

(i) The topology σ is locally convex; that is, the origin has a local base of convex, balanced, and absorbing sets (see [31]).

(ii) The space (X, σ) is sequentially complete on $\| \cdot \|$- bounded sets; that is, every $\| \cdot \|$-bounded σ-Cauchy sequence converges in (X, σ).

(iii) The topology σ is Hausdorff and coarser than the $\| \cdot \|$-topology; that is, for every distinct pair x_1, $x_2 \in X$, there are disjoint open sets U_1, $U_2 \subset (X, \sigma)$ with $x_1 \in U_1$ and $x_2 \in U_2$ and every open set $U \subset (X, \sigma)$ is open in $(X, \| \cdot \|)$.

(iv) The space $(X, \sigma)^*$ is norming for $(X, \| \cdot \|)$; that is,

$$\|x\| = \sup_{\|\varphi\| \leq 1} \{ | < x, \varphi > | : \varphi \in (X, \sigma)^* \}.$$

In 1972, F. D. Sentilles [32], a Ph.D. student of J. R. Dorroh, studied three "strict topologies" β_0, β, and β_1 on the space $C_b(\Omega)$ of bounded real-valued continuous functions on a regular Hausdorff space Ω and proved that the three topologies coincide if Ω is a complete, separable, metric space. J. R. Dorroh and J. W. Neuberger used this and several other results from Sentilles' paper [32] to help characterize semigroups in terms of their Lie-generators [8]. Later Kühnemund [20] showed that the Dorroh-Neuberger characterization fits into the framework of bi-continuous semigroups. The following results from [32] are essential to our discussion of bi-continuous semigroups arising from non-autonomous flows in Chapter 4.

Let Ω be a complete separable metric space, let $C_b(\Omega) := C_b(\Omega, \mathbb{R})$ be the Banach space of bounded real-vauled function on Ω equipped with the supremum norm

$$\|f\|_\infty := \sup_{x \in \Omega} |f(x)|,$$

let σ_c be the topology of compact convergence on $C_b(\Omega)$, and let $B_r := \{f \in C_b(\Omega) : \|f\|_\infty \leq r\}$. The strict topology β on $C_b(\Omega)$ is defined as the strongest locally convex topology that coincides with σ_c on each set B_r. This implies $\sigma_c \leq \beta$ and trivially $\beta \leq \| \cdot \|_\infty$ where $\| \cdot \|_\infty$ denotes the norm topology on $C_b(\Omega)$. In

[32], Sentilles further shows that β is Hausdorff and is sequentially complete on $\|\cdot\|_\infty$-bounded sets since β coincides with σ_c on the sets B_r. Lastly, $(C_b(\Omega), \beta)^*$ is characterized as all real-valued linear functionals ϕ such that $\langle f_n, \phi\rangle \to 0$ for any pointwise decreasing sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_b(\Omega)$ with $f_n(x) \to 0$ for each $x \in \Omega$. Trivially, $C_b(\Omega, \beta)^*$ contains the point evaluations ϕ_x defined by

$$\langle f, \phi_x\rangle := f(x),$$

and thus is norming for $(C_b(\Omega), \|\cdot\|_\infty)$. We summarize these observations in the following proposition.

Proposition 2.1.2. Suppose Ω is a Polish space (that is Ω is a complete, separable, metric space). Then the space $(C_b(\Omega), \|\cdot\|_\infty, \beta)$ is bi-admissible, where $\|\cdot\|_\infty$ defines the supremum norm and β defines the strict topology on $C_b(\Omega)$.

On vector spaces, locally convex topologies and families of seminorms are closely related. Let $\{p_\alpha\}_{\alpha\in I}$ be a family of seminorms on a vector space X, where α is in some index set I. Then $\{p_\alpha\}_{\alpha\in I}$ is called separating if for every $x \in X$ with $x \neq 0$ there is an $\alpha \in I$ such that $p_\alpha(x) > 0$. It is well known (e.g., [31] p. 25) that the following equivalence holds. Every separating family of seminorms $\{p_\alpha\}_{\alpha\in I}$ defines a locally convex Hausdorff topology σ on X by taking as a base $\mathcal{B}$ all finite intersections of the sets

$$B_n^\alpha := \left\{x : p_\alpha(x) < \frac{1}{n}\right\},$$

where $\alpha \in I$ and $n \in \mathbb{N}$. By defining the topology in this way, each $p_\alpha : (X, \sigma) \to \mathbb{R}$ is continuous. Therefore, a set $U \subset X$ is open in σ if U is a union of translates of members of $\mathcal{B}$, or equivalently, for every $\alpha \in I$ and $x \in U$ there is an $n \in \mathbb{N}$ such that $p_\alpha(x - y) < \frac{1}{n}$ implies $y \in U$. For a given separating family of seminorms

$\{p_\alpha\}_{\alpha \in I}$, define

$$X_p := (X, \sigma)$$

to be X equipped with the locally convex Hausdorff topology σ generated by $\{p_\alpha\}_{\alpha \in I}$.

Conversely, for every locally convex Hausdorff topology σ with base $\mathcal{B}$ on X, we can construct a separating family of continuous seminorms $\{\mu_V\}_{V \in \mathcal{B}}$ by considering the *Minkowski functional* μ_V for every $V \in \mathcal{B}$. The topology generated by $\{\mu_V\}_{V \in \mathcal{B}}$ and σ coincide. For a more detailed description of this relationship, see [31], Theorem 1.36, Theorem 1.37, and Remark 1.38.

From this relationship between locally convex topologies and seminorms, we can redefine a bi-admissible Banach space in terms of the seminorms generating the topology σ.

Proposition 2.1.3. *Let $(X, \|\cdot\|)$ be a Banach space and let σ be the topology generated by a family of seminorms $\{p_\alpha\}_{\alpha \in I}$ on X. Then $X = (X, \|\cdot\|, \sigma)$ is a bi-admissible Banach space if p_α satisfies the following conditions:*

(i) $\|x\| = \sup\limits_{\alpha \in I} p_\alpha(x)$ *for all* $x \in X$

(ii) Every norm-bounded, p-Cauchy sequence is p-convergent in X.

(iii) $x = 0$ *if and only if* $p_\alpha(x) = 0$ *for all* $\alpha \in I$ *(separating).*

(iv) $(X, \sigma)^*$ *is norming for* $(X, \|\cdot\|)$.

Proof. Since $\{p_\alpha\}_{\alpha \in I}$ is a separating family of seminorms on X, we know from [31] that σ is a locally convex Hausdorff topology. By assumption (ii), (X, σ) is sequentially complete on $\|\cdot\|$-bounded sets. By assumption (i), $p_\alpha(x) \leq \|x\|$ for all $\alpha \in I$ and $x \in X$. Thus, the identity map is continuous from $(X, \|\cdot\|)$ into (X, σ) and therefore σ is coarser than the norm topology. □

To give an elementary example, let $(X, \|\cdot\|)$ be some Banach space and consider the space of bounded continuous functions $C_b([0,\infty), X)$ equipped with the sup-norm

$$\|f\|_\infty := \sup_{s\geq 0} \|f(s)\|$$

and the locally convex topology of compact convergence σ_c that is generated by the seminorms $\{p_n\}_{n\in\mathbb{N}}$, where

$$p_n(f) := \sup_{0\leq s\leq n} \|f(s)\|.$$

Proposition 2.1.4. *$(C_b([0,\infty), X), \|\cdot\|_\infty, \sigma_c)$ is bi-admissible.*

Proof. Clearly, the seminorms p_n satisfy the conditions (i) - (iii) of Proposition 2.1.3. To show (iv), observe that for all $s \geq 0$ and $x^* \in X^*$ with $\|x^*\| \leq 1$, the point evaluations ϕ_{s,x^*} defined by

$$\langle f, \phi_{s,x^*}\rangle := \langle f(s), x^*\rangle$$

are contained in $(C_b([0,\infty), X), \sigma)^*$ and $\|\phi_{s,x^*}\| \leq 1$. Moreover, as a consequence of the Hahn-Banach Theorem, for all $x \in X$, there is an $x^* \in X^*$ with $\|x^*\| = 1$ and $\|x\| = |\langle x, x^*\rangle|$. Now, for all $f \in C_b([0,\infty), X)$ there is a sequence $s_n \geq 0$ and $x_n^* \in X^*$ with $\|x_n^*\| = 1$ such that

$$\|f\| = \sup_n \|f(s_n)\| = \sup_n |\langle f(s_n), x_n^*\rangle| = \sup_n |\langle f, \phi_{s_n,x_n^*}\rangle|$$

$$\leq \sup_{\substack{\phi\in(C_b([0,\infty),X),\sigma)^* \\ \|\phi\|\leq 1}} |\langle f, \phi\rangle| \leq \sup_{\substack{\phi\in(C_b([0,\infty),X),\|\cdot\|_\infty)^* \\ \|\phi\|\leq 1}} |\langle f, \phi\rangle| \leq \|f\|.$$

Therefore, $(C_b([0,\infty), X), \|\cdot\|_\infty, \sigma_c)$ is bi-admissible. □

Let $X = (X, \|\cdot\|, \sigma_p)$ be a bi-admissible Banach space with seminorms $\{p_\alpha\}_{\alpha\in I}$ generating the topology σ_p and let $X_p = (X, \sigma_p)$. Define

$$C_{b,p} := C_b([0,\infty), X_p)$$

to be the space of continuous functions $f : [0, \infty) \to X_p$ with $\|f\|_\infty < \infty$. On $C_{b,p}$ define a family of seminorms by

$$\widetilde{p}_{\alpha,n}(f) := \sup_{0 \le s \le n} p_\alpha(f(s))$$

and let $\widetilde{\sigma}_p$ denote the topology on $C_{b,p}$ generated by $\widetilde{p}_{\alpha,n}$.

Theorem 2.1.5. *$C_{b,p}$ is a Banach space and $(C_{b,p}, \|\cdot\|_\infty, \widetilde{\sigma}_p)$ is bi-admissible.*

Proof. (i) To show that $C_{b,p}$ with $\|\cdot\|_\infty$ is a Banach space, let $\{f_k\}_{k\in\mathbb{N}}$ be a $\|\cdot\|_\infty$-Cauchy sequence in $C_{b,p}$. Let $\epsilon > 0$ be given and pick $k_0 \in \mathbb{N}$ such that $\|f_k - f_j\|_\infty < \frac{\epsilon}{2}$ for all $k, j \ge k_0$. Now, for all $s \ge 0$,

$$\|f_k(s) - f_j(s)\| < \frac{\epsilon}{2}.$$

Thus $\{f_k(s)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $(X, \|\cdot\|)$ for each $s \ge 0$. Thus $f : [0, \infty) \to X$ defined by

$$f(s) := \lim_{k\to\infty} f_k(s)$$

is well defined for all $s \ge 0$ and

$$\|f_k(s) - f(s)\| < \epsilon$$

for all $k \ge k_0$ and $s \ge 0$. This implies that $\|f_k - f\|_\infty \le \epsilon$ for $k \ge k_0$ and furthermore that $\|f\|_\infty < \infty$. Lastly, we need to show that $f : [0, \infty) \to X_p$ is continuous. Let $\alpha \in I$ and $s_0 \ge 0$. For $\epsilon > 0$ we may pick f_K such that

$$p_\alpha(f_K(s) - f_K(s_0)) < \frac{\epsilon}{2}$$

for $|s - s_0| < \delta$ for some $\delta > 0$ and

$$p_\alpha(f(s) - f_K(s)) + p_\alpha(f_K(s_0) - f(s_0)) < \frac{\epsilon}{2}.$$

Then

$$p_\alpha(f(s) - f(s_0)) \leq p_\alpha(f(s) - f_K(s)) + p_\alpha(f_K(s) - f_K(s_0))$$
$$+ p_\alpha(f_K(s_0) - f(s_0)) < \epsilon.$$

(ii) $(C_{b,p}, \|\cdot\|_\infty, \widetilde{\sigma}_p)$ is bi-admissible if the seminorms $\widetilde{p}_{\alpha,n}$ satisfy the conditions (i)-(iv) of Proposition 2.1.3. To show condition (i) of Proposition 2.1.3 observe that $p_\alpha(f(s)) \leq \|f(s)\|$ for all $s \geq 0$, $\alpha \in I$. Therefore

$$\widetilde{p}_{\alpha,n}(f) = \sup_{0 \leq s \leq n} p_\alpha(f(s)) \leq \sup_{0 \leq s \leq n} \|f(s)\| \leq \|f\|_\infty$$

for all $\alpha \in I$ and $n \in \mathbb{N}$. For the reverse inequality, observe that for every $f \in C_{b,p}$ and for every $\epsilon > 0$ there is an $s_0 \in \mathbb{R}_+$ such that $\|f\|_\infty < \|f(s_0)\| + \epsilon$. Similarly, there is an $\alpha \in I$ such that $\|f(s_0)\| < p_\alpha(f(s_0)) + \epsilon$. Thus,

$$\|f\|_\infty < \|f(s_0)\| + \epsilon < p_\alpha(f(s_0)) + 2\epsilon \leq \widetilde{p}_{\alpha,n}(f(s)) + 2\epsilon$$

for $n \geq s_0$. Thus, for every $\epsilon > 0$ there is $\widetilde{p}_{\alpha,n}$ such that $\|f\|_\infty < \widetilde{p}_{\alpha,n}(f) + 2\epsilon$. Thus, $\sup_{\alpha,n} \widetilde{p}_{\alpha,n}(f) = \|f\|_\infty$ and Proposition 2.1.3 (i) is satisfied.

For Proposition 2.1.3 (ii), let $\{f_m\}_{m\in\mathbb{N}}$ be a $\|\cdot\|_\infty$-bounded $\widetilde{p}$-Cauchy sequence in $C_{b,p}$. By construction, $\widetilde{p}_{\alpha,n}(f) \geq p_\alpha(f(s))$ for all $0 \leq s \leq n$. If $\{f_m\}_{m\in\mathbb{N}}$ is a $\widetilde{p}$-Cauchy sequence, then $\{f_m(s)\}_{m\in\mathbb{N}}$ is a p-Cauchy sequence for each $s \geq 0$. Since $X = (X, \|\cdot\|, \sigma)$ is bi-admissible, it follows that

$$f(s) := p - \lim_{m\to\infty} f_m(s)$$

is well defined for every $s \geq 0$. Furthermore, since $\{f_m\}_{m\in\mathbb{N}}$ is a $\widetilde{p}$-Cauchy sequence, if $\widetilde{p}_{\alpha,N}(f_m - f_j) < \epsilon$ for $m, j \geq m_0 \in \mathbb{N}$, then $p_\alpha(f_m(s) - f(s)) < 2\epsilon$ for every $s \in [0, N]$ and $m \geq m_0$. This implies that $\widetilde{p}_{\alpha,N}(f_m - f) < 2\epsilon$ for $m \geq m_0$ and thus f is the $\widetilde{p}$-limit of $\{f_m\}_{m\in\mathbb{N}}$. Now, for any $s \in \mathbb{R}_+$ and $\alpha \in I$,

$$p_\alpha(f(s)) < p_\alpha(f(s) - f_m(s)) + p_\alpha(f_m(s))$$

for some $m \in \mathbb{N}$ with $p_\alpha(f(s) - f_m(s)) < 1$. However, $\{f_m\}_{m\in\mathbb{N}}$ is $\|\cdot\|_\infty$-bounded, so there is an $M > 0$ such that $\|f_m(s)\| \leq M$ for all $s \in \mathbb{R}_+$ and $m \in \mathbb{N}$. Thus,

$$p_\alpha(f(s)) < p_\alpha(f(s) - f_m(s)) + p_\alpha(f_m(s)) < 1 + M.$$

Since this holds for all $\alpha \in I$ and $s \in \mathbb{R}_+$, it follows that

$$\|f\|_\infty = \sup_{s\geq 0} \|f(s)\| = \sup_{s\geq 0} \left(\sup_{\alpha\in I} p_\alpha(f(s)) \right) \leq 1 + M.$$

Finally, fix $\alpha \in I$, $s_0 \in R_+$, and pick $f_{m_1} \in \{f_m\}_{m\in\mathbb{N}}$ such that $p_\alpha(f_{m_1}(s) - f_{m_1}(s_0)) < \frac{\epsilon}{2}$ for $|s - s_0| < \delta$ for some $\delta > 0$ and

$$p_\alpha(f(s) - f_{m_1}(s)) + p_\alpha(f_{m_1}(s_0) - f(s_0)) < \frac{\epsilon}{2}.$$

Now we have

$$p_\alpha(f(s) - f(s_0))$$

$$\leq p_\alpha(f(s) - f_{m_1}(s)) + p_\alpha(f_{m_1}(s) - f_{m_1}(s_0)) + p_\alpha(f_{m_1}(s_0) - f(s_0)) < \epsilon.$$

Thus, $s \to f(s)$ is p_α-continuous and Proposition 2.1.3 (ii) is satisfied. For condition (iii) we only need to consider the case $\widetilde{p}_{\alpha,N}(f) = 0$ for all $\alpha \in I$ and $N \in \mathbb{N}$. This would imply that for each $s \in \mathbb{R}_+$, $p_\alpha(f(s)) = 0$ for all $\alpha \in I$. Since $X = (X, \|\cdot\|, \sigma)$ is bi-admissible, this would imply $f(s) = 0$. Therefore $f = 0$ and condition (iii) is satisfied. To show condition (iv) we want to consider point evaluations similar to Proposition 2.1.4. Since $X = (X, \|\cdot\|, \sigma)$ is assumed to be bi-admissible, for every $f \in C_{b,p}$ and $s \in \mathbb{R}_+$,

$$\|f(s)\| = \sup_{\substack{\phi\in(X,\sigma)^* \\ \|\phi\|\leq 1}} |\langle f(s), \phi\rangle|.$$

So, for every $\phi \in (X, \sigma)^*$ with $\|\phi\| \leq 1$ and $s \in \mathbb{R}_+$, we can define a linear functional $\varphi_{s,\phi} : C_{b,p} \to \mathbb{R}$ by

$$\langle f, \varphi_{s,\phi}\rangle := \langle f(s), \phi\rangle.$$

First observe that

$$|\langle f, \varphi_{s,\phi}\rangle| = |\langle f(s), \phi\rangle| \le |f(s)|$$

which gives $\|\varphi_{s,\phi}\| \le 1$. Now let $\{f_l\}_{l\in\mathbb{N}} \subset C_{b,p}$ be a null-$\widetilde{p}$-sequence; that is, $\widetilde{p} - \lim_{l\to\infty} f_l = 0$. Consider

$$\lim_{l\to\infty} |\langle f_l, \varphi_{s,\phi}\rangle| = \lim_{l\to\infty} |\langle f_l(s), \phi\rangle|.$$

Since $\{f_l\}_{l\in\mathbb{N}}$ is a $\widetilde{p}$-null sequence, $\{f_l(s)\}_{l\in\mathbb{N}}$ is a p-null sequence for each $s \in R_+$. This implies $\lim_{l\to\infty} |\langle f_l(s), \phi\rangle| = 0$ and thus $\lim_{l\to\infty} |\langle f_l, \varphi_{s,\phi}\rangle| = 0$ and $\varphi_{s,\phi} \in (C_{b,p}, \widetilde{\sigma})^*$. For every $f \in C_{b,p}$ there is a sequence $\{s_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{i\to\infty} \|f(s_i)\| = \|f\|$. This yields

$$\begin{aligned} \|f\| = \lim_{i\to\infty} \|f(s_i)\| &= \lim_{i\to\infty} \sup_{\substack{\phi\in(X,\sigma)^* \\ \|\phi\|\le 1}} |\langle f(s_i), \phi\rangle| \\ &= \lim_{i\to\infty} \sup_{\substack{\phi\in(X,\sigma)^* \\ \|\phi\|\le 1}} |\langle f, \varphi_{s_i,\phi}\rangle| \le \sup_{\substack{\varphi\in(C_{b,p},\widetilde{\sigma})^* \\ \|\varphi\|\le 1}} |\langle f, \varphi\rangle| \le \|f\|. \end{aligned}$$

Therefore, $(C_{b,p}, \widetilde{\sigma})^*$ is norming for $(C_{b,p}, \|\cdot\|_\infty)$ and $(C_{b,p}, \|\cdot\|_\infty, \widetilde{\sigma}_p)$ is bi-admissible.

□

Now we will begin to explore the space $C_b([0,\infty), \mathcal{L}_s(X,Y))$, where $(X, \|\cdot\|_X)$ is a normed vector space and $(Y, \|\cdot\|_Y)$ is a Banach space. Define

$$B_r := \{x \in X : \|x\| \le r\}.$$

Then the strong operator topology σ_s is generated by the family of seminorms $\{p_x\}_{x\in B_1}$ on $\mathcal{L}(X,Y)$ defined by

$$p_x(T) := \|T(x)\|_Y$$

for every $T \in \mathcal{L}(X,Y)$. First we will show that $(\mathcal{L}(X,Y), \|\cdot\|, \sigma_s)$ is bi-admissible which will imply that $(C_b([0,\infty), \mathcal{L}_s(X,Y)), \|\cdot\|_\infty, \widetilde{\sigma}_s)$ is bi-admissible by Proposition 2.1.5.

Theorem 2.1.6. *Let X be a normed vector space and Y a Banach space. Then $(\mathcal{L}(X,Y), \|\cdot\|, \sigma_s)$ is bi-admissible*

Proof. We need to show that the seminorms $\{p_x\}_{x\in B_1}$ that generate the topology σ_s on $\mathcal{L}(X,Y)$ satisfy the conditions of Proposition 2.1.3. Condition (i) is satisfied by the definition of the operator norm on $\mathcal{L}(X,Y)$; namely

$$\|T\| := \sup_{x\in B_1} \|T(x)\|_Y = \sup_{x\in B_1} p_x(T).$$

For condition (ii), let $\{T_n\}_{n\in\mathbb{N}}$ be a $\|\cdot\|$-bounded p-Cauchy sequence. Then, for every $x \in B_1$, $\{T_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Since Y is a Banach space, we can define $T : B_1 \to Y$ by

$$T(x) := \lim_{n\to\infty} T_n(x).$$

Thus, T is the p-limit of $\{T_n\}_{n\in\mathbb{N}}$. From the linearity of $\{T_n\}_{n\in\mathbb{N}}$, we can extend the domain of T to X and, for any constant $c \in \mathbb{C}$,

$$T(cx_1 + x_2) = \lim_{n\to\infty} T_n(cx_1 + x_2) = \lim_{n\to\infty} (cT_n(x_1) + T_n(x_2))$$

$$= cT(x_1) + T(x_2)$$

for all $x_1, x_2 \in X$. Furthermore, since $\{T_n\}_{n\in\mathbb{N}}$ is $\|\cdot\|$-bounded, it follows that

$$\|T(x)\|_Y \leq \sup_{n\in\mathbb{N}} \|T_n(x)\|_Y < \infty$$

for every $x \in B_1$. Thus,

$$T_n \xrightarrow{p_x} T \in \mathcal{L}(X,Y)$$

and condition (ii) is satisfied. The separating condition (iii), is satisfied trivially since $p_x(T) = 0$ for all $x \in B_1$ implies that $\|T\| = 0$. Condition (iv) is satisfied by constructing point evaluations $\varphi_n : \mathcal{L}(X,Y) \to \mathbb{C}$ similar to Proposition 2.1.4.

Pick a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n^*\}_{n\in\mathbb{N}} \subset Y^*$ with $\|x_n\| \le 1$ and $\|y_n^*\| \le 1$ such that

$$\sup_{n\in\mathbb{N}} \|T(x_n)\|_Y = \|T\| \text{ and } \|T(x_n)\|_Y = |\langle T(x_n), y_n^*\rangle|.$$

Define $\varphi_n \in (\mathcal{L}(X,Y), \sigma_s)^*$ by

$$\varphi_n(T) := \langle T(x_n), y_n^*\rangle.$$

Then $\|\varphi\| \le 1$ and we obtain the inequality

$$\|T\| = \sup_n \|T(x_n)\|_Y = \sup_n |\langle T(x_n), y_n^*\rangle|$$

$$= \sup_n |\langle T, \varphi_n\rangle| \le \sup_{\substack{\varphi\in(\mathcal{L}(X,Y),\sigma_s)^* \\ \|\varphi\|\le 1}} |\langle T, \varphi\rangle| \le \|T\|.$$

Thus $(\mathcal{L}(X,Y), \sigma_s)^*$ is norming for $(\mathcal{L}(X,Y), \|\cdot\|)$ and $(\mathcal{L}(X,Y), \|\cdot\|, \sigma_s)$ is bi-admissible. □

Moreover, if X is a normed vector space and $Y = \mathbb{C}$, then $(\mathcal{L}(X,\mathbb{C}), \|\cdot\|, \sigma_s) = (X^*, \|\cdot\|, \omega^*)$, where ω^* denotes the weak*-topology on X^*. This yields the following corollary.

Corollary 2.1.7. *Let X be a normed vector space. Then $(X^*, \|\cdot\|, \omega^*)$ is bi-admissible.*

Theorems 2.1.5 and 2.1.6 show that the spaces

$$(C_b([0,\infty), \mathcal{L}_s(X,Y)), \|\cdot\|_\infty, \widetilde{\sigma}_s) \text{ and } (C_b([0,\infty), X^*_{\omega^*}), \|\cdot\|_\infty, \widetilde{\omega}^*)$$

are bi-admissible Banach spaces. Furthermore, any theorems shown for the space $C_b([0,\infty), \mathcal{L}_s(X,Y))$ in the following chapters can be applied directly to the space $C_b([0,\infty), X^*_{\omega^*})$ by setting $Y = \mathbb{C}$. Another important observation from Proposition

2.1.6 is that the truncated space $(C_b([0,N],\mathcal{L}_s(X,Y)),\|\cdot\|_N,\widetilde{\sigma}_s)$ is bi-admissible with

$$\|f\|_N := \sup_{0\le s\le N} \|f(s)\|_Y.$$

For the reader: It will be important to note the differences between the two compact topologies σ_c and $\widetilde{\sigma}_p$. The topology of compact convergence σ_c does not depend on a bi-admissible Banach space X, while the compact $\widetilde{\sigma}_p$ topology depends on a bi-admissible Banach space $(X,\|\cdot\|,\sigma_p)$ and the topology σ_p.

2.2 Bi-Continuous Semigroups

Now that we have established the framework of bi-admissible Banach spaces, we can collect the main properties of bi-continuous semigroups (closely following Kühnemund's dissertation [20]), look at a basic example, and discuss generators as well as a Lie-Trotter product formula for bi-continuous semigroups.

Definition 2.2.1. *Let $(\mathcal{M},\|\cdot\|,\sigma)$ be a bi-admissible Banach space. An operator family $T(t)(t\ge 0)\subset\mathcal{L}(\mathcal{M})$, the Banach space of all bounded linear operators on $\mathcal{M}$, is called a bi-continuous semigroup of type ω if the following conditions hold.*

(i) The family of operators $T(t)(t\ge 0)$ satisfies the semigroup properties: $T(0)=Id$ and $T(t+s)=T(t)T(s)$ for all $s,t\ge 0$.

(ii) The operators $T(t)$ are exponentially bounded. That is, $\|T(t)\|\le Me^{\omega t}$ for all $t\ge 0$ and some constants $M\ge 1$ and $\omega\in\mathbb{R}$.

(iii) The map $t\to T(t)f$ $(t\ge 0)$ is strongly σ-continuous for each $f\in\mathcal{M}$.

(iv) $T(t)(t\ge 0)$ is locally bi-equicontinuous. That is, for every σ-convergent null sequence $\{f_n\}\subset\mathcal{M}$,

$$\sigma\text{-}\lim_{n\to\infty} T(t)f_n = 0$$

uniformly for t in compact intervals of $\mathbb{R}_+$.

Remark 2.2.2. *If (iv) holds, then for every $\|\cdot\|$-bounded sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ which is σ convergent to f, we have that*

$$\sigma\text{-}\lim_{n\to\infty}(T(t)(f_n - f)) = 0$$

uniformly for t in compact intervals of R_+.

Remark 2.2.3. *If $T(t)(t \geq 0)$ is locally bi-equicontinuous and $\sigma\text{-}\lim_{t\searrow 0} T(t)f = f$ for every $f \in \mathcal{M}$, then $T(t)(t \geq 0)$ is strongly σ-continuous. This can be seen by defining for every $f \in \mathcal{M}$ a σ-null sequence $T(\frac{1}{n})f - f$ and with the local bi-equicontinuity*

$$T(t)\left(T\left(\frac{1}{n}\right)f - f\right) = T\left(t + \frac{1}{n}\right)f - T(t)f \xrightarrow{\sigma} 0$$

uniformly for t in compact intervals.

An easy example of a bi-continuous semigroup that is not strongly continuous is given by the left shift semigroup $T(t)(t \geq 0)$ on the bi-admissible Banach space $(C_b([0,\infty), X), \|\cdot\|_\infty, \sigma_c)$ (recall Proposition 2.1.4).

Example 2.2.4. *The left shift semigroup $T(t)(t \geq 0)$ defined by*

$$T(t)f(s) = f(t+s),$$

is bi-continuous on $(C_b([0,\infty), X), \|\cdot\|_\infty, \sigma_c)$.

Proof. Clearly conditions (i)-(ii) of Definition 2.2.1 are satisfied. We will show $t \to T(t)f$ is σ_c-continuous at $t = 0$ and that $(T(t)(t \geq 0)$ is locally bi-equicontinuous. First let $f \in C_b([0,\infty), X)$, $\varepsilon > 0$, and $n \in \mathbb{N}$. Then

$$p_n(T(t)f - f) = \sup_{0\leq s\leq n} \|T(t)f(s) - f(s)| = \sup_{0\leq s\leq n} \|f(t+s) - f(s)\|.$$

Since $f(\cdot)$ is uniformly continuous on compact intervals, there is a $\delta > 0$ such that $0 \leq t < \delta < 1$ implies

$$\sup_{0 \leq s \leq n} \|f(t+s) - f(s)\| < \varepsilon.$$

Thus $t \to T(t)f$ is σ_c-continuous at $t = 0$. Now let $\{f_k\}_{k \in \mathbb{N}} \subset C_b([0,\infty), X)$ be a σ_c-null sequence, $t \in [0, n_0]$. Then

$$\begin{aligned} p_n(T(t)f_k) &= \sup_{0 \leq s \leq n} \|T(t)f_k(s)| = \sup_{0 \leq s \leq n} \|f_k(t+s)\| \\ &\leq \sup_{0 \leq s \leq n+n_0} \|f(s)\| = p_{n+n_0}(f_k). \end{aligned}$$

Since $\{f_k\}_{k \in \mathbb{N}}$ is a σ_c-null sequence, there is a $K \in \mathbb{N}$ such that $k \geq K$ implies $p_{n+n_0}(f_k) < \varepsilon$. Therefore, $T(t)(t \geq 0)$ is locally bi-equicontinuous and by Remark 2.2.3, strongly σ_c-continuous. Thus, $T(t)(t \geq 0)$ is a bi-continuous semigroup on $(C_b([0,\infty), X), \|\cdot\|_\infty, \sigma_c)$. □

The generator of a bi-continuous semigroups is defined as follows.

Definition 2.2.5. *Let $T(t)(t \geq 0)$ be a bi-continuous semigroup on a bi-admissible Banach space $(\mathcal{M}, \|\cdot\|, \sigma)$. The generator $\mathcal{A}$ of $T(t)(t \geq 0)$ is defined as*

$$\mathcal{A}f = \sigma - \lim_{t \searrow 0} \frac{T(t)f - f}{t}$$

for $f \in \mathcal{M}$ such that $\mathcal{A}f$ exists and $\mathcal{A}f \in \mathcal{M}$.

The following proposition contains some of the main properties of bi-continuous semigroups. Proofs and further explanations can be found in [20].

Theorem 2.2.6. *Let $T(t)(t \geq 0)$ be a bi-continuous semigroup on a bi-admissible Banach space $(\mathcal{M}, \|\cdot\|, \sigma)$ with growth bound ω and generator $(\mathcal{A}, D(\mathcal{A}))$. Then the following hold.*

(i) For all $f \in \mathcal{M}$ and $Re\,\lambda > \omega$, $\sigma - \lim_{\lambda \to \infty} \lambda R(\lambda, \mathcal{A})f = f$ and

$$\|R(\lambda, \mathcal{A})\|^n \leq \frac{M}{(Re\,\lambda - \omega)^n} \text{ for all } n \in \mathbb{N} \text{ and } Re\,\lambda > \omega.$$

(ii) *If* $f \in D(\mathcal{A})$*, then* $T(t)f \in D(\mathcal{A})$ *for all* $t \geq 0$*,* $T(t)f$ *is continuously* σ*-differentiable and* $\sigma - \frac{d}{dt}T(t)f = \mathcal{A}T(t)f = T(t)\mathcal{A}f$ *for all* $t \geq 0$*, and* $\mathcal{A}f = g$ *if and only if*

$$T(t)f - f = \int_0^t T(s)g\,ds$$

for all $t \geq 0$*.*

(iii) $(\mathcal{A}, D(\mathcal{A}))$ *is bi-closed; that is, for all sequences* $\{f_n\}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ *with* $\{f_n\}_{n\in\mathbb{N}}$ *and* $\{\mathcal{A}f_n\}_{n\in\mathbb{N}}$ $\|\cdot\|$ *- bounded with* $f_n \xrightarrow{\sigma} f$ *and* $\mathcal{A}f_n \xrightarrow{\sigma} g \in \mathcal{M}$*, we have* $f \in D(\mathcal{A})$ *and* $\mathcal{A}f = g$*.*

(iv) $D(\mathcal{A})$ *is bi-dense; that is, for all* $f \in \mathcal{M}$ *there is a* $\|\cdot\|$*- bounded sequence* $\{f_n\}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ *such that* $f_n \xrightarrow{\sigma} f$*.*

(v) *If* $D \subset D(\mathcal{A})$ *is a bi-dense subset in* $\mathcal{M}$*, then* $R(\lambda, \mathcal{A})D$ *is bi-dense in* $D(\mathcal{A})$*.*

(vi) $D_0 := \overline{D(\mathcal{A})}^{\|\cdot\|} \subset \mathcal{M}$ *is* $T(t)$ *invariant and* $T(t)|_{D_0}$ *is the strongly continuous semigroup generated by* $\mathcal{A}_0 f := \mathcal{A}|_{D_0} f$ *with* $D(\mathcal{A}_0) := \{D(\mathcal{A}) \cap D_0 : \mathcal{A}f \in D_0\}$*. Moreover, if* $\mathcal{A}$ *is a bi-closed linear operator on* $\mathcal{M}$ *for which* $D(\mathcal{A})$ *is bi-dense and for which (i) holds, then* $\mathcal{A}$ *is the generator of a bi-continuous semigroup* $T(t)(t \geq 0)$ *given by*

$$T(t)f := \sigma - \lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, \mathcal{A}\right)\right)^n f.$$

□

Finally, similar to strongly continuous semigroups, there are approximation formulas for bi-continuous semigroups. In particular, we have a Lie-Trotter product formula for bi-continuous semigroups (for a proof see [20] p. 49).

Theorem 2.2.7. *Let $T(t)(t \geq 0)$ and $S(t)(t \geq 0)$ be bi-continuous semigroups on $(X, \|\cdot\|, \sigma)$ with generators (A,D(A)) and (B, D(B)), respectively. Assume the following stability conditions.*

(i) $\left\| \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n \right\| \leq Me^{\omega t}$ *for all* $n \in \mathbb{N}$ *and* $t \geq 0$, *and some constants* $M \geq 1, \omega \in \mathbb{R}$.

(ii) The operator family $\left\{ \left(T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right)^n : t \geq 0 \right\}$ *is locally bi-equicontinuous uniformly for* $n \in \mathbb{N}$.

Let $D(A_0) \subset \overline{D(A)}^{\|\cdot\|}$ *and* $D(B_0) \subset \overline{D(B)}^{\|\cdot\|}$ *be contained in the* $\|\cdot\|$ *- closure of* $D(A)$ *and* $D(B)$ *respectively. If there is a subspce* $D \subset D(A_0) \cap D(B_0)$ *such that* D *and* $(\lambda_0 - A - B)D$ *are bi-dense in* X *for some* $\lambda_0 > \omega$, *then the bi-closure of* $A + B$ *on* D *exists and generates a bi-continuous semigroup* $U(t)(t \geq 0)$ *given by the* σ*-limit of* (1.16);

$$U(t)x := \sigma - \lim_{n \to \infty} \left(T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right)^n x \tag{2.1}$$

where the limit exists for all $x \in X$ *and uniformly for* t *in compact intervals of* $[0, \infty)$. □

Chapter 3
Evolution Semigroups

3.1 Special Evolution Semigroups on $\mathbf{C_b}([\mathbf{0},\infty),\mathbf{X}^*_{\omega^*})$

The main result of this section is that the special evolution semigroup

$$T(t)f(s) = f(t+s)U(t+s,s)$$

is bi-continuous on the bi-admissible Banach space $(C_b([0,\infty),X^*_{\omega^*}),\|\cdot\|_\infty,\widetilde{\omega^*})$ (see Theorem 2.1.5 and Corollary 2.1.7) if and only if the evolution family $U(t,s)(t\geq s\geq 0)$ is strongly continuous and exponentially bounded on the normed vector space X.

Theorem 3.1.1. *Let X be a normed vector space, $U(t,s)\in\mathcal{L}(X)$ for $t\geq s\geq 0$ and let $T(t)f(s):=f(t+s)U(t+s,s)$. Then the following are equivalent.*

(i) $T(t)(t\geq 0)$ is a bi-continuous semigroup on $(C_b([0,\infty),X^*_{\omega^*}),\|\cdot\|_\infty,\widetilde{\omega^*})$ with $\|T(t)\|\leq Me^{\omega t}$.

(ii) $U(t,s)(t\geq s\geq 0)$ is a strongly continuous evolution family on X with $\|U(t,s)\|\leq Me^{\omega(t-s)}$.

Proof. If (i) holds, then $T(0)f(s)=f(s)U(s,s)=f(s)$ for all $f\in\mathcal{M}$. In particular, if one takes $f(s):=x'$ for $x'\in X^*$, then it follows that

$$\begin{aligned}\langle x,x'\rangle &= \langle x,f(s)\rangle = f(s)[x] = T(0)f(s)[x]\\ &= f(s)U(s,s)[x] = \langle U(s,s)x,f(s)\rangle = \langle U(s,s)x,x'\rangle\end{aligned}$$

for all $x\in X$ and $x'\in X^*$. Thus, $U(s,s)=I$.

Now let $h(s) := f(r+s)U(r+s,s)$. Then

$$f(t+r+s)U(t+r+s,s) = T(t+r)f(s) = T(t)T(r)f(s) = T(t)h(s)$$

$$= h(t+s)U(t+s,s) = f(t+r+s)U(t+r+s,t+s)U(t+s,s).$$

As above, by defining $f(s) := x'$ we obtain that

$$\langle U(t+r+s,s)x, x'\rangle = \langle U(t+r+s,s)x, f(t+r+s)\rangle$$

$$= \langle U(t+r+s,t+s)U(t+s,s)x, f(t+r+s)\rangle = \langle U(t+r+s,s)U(t+s,s)x, x'\rangle$$

for all $x \in X$ and $x' \in X^*$. Thus $U(t+r+s,t+s)U(t+s,s) = U(t+r+s,s)$ for all $t, r, s \geq 0$. Equivalently, for all $\widetilde{t} \geq \widetilde{r} \geq s \geq 0$ we have that

$$U(\widetilde{t},\widetilde{r})U(\widetilde{r},s) = U(\widetilde{t},s).$$

That is, if $T(t)(t \geq 0)$ is a semigroup, then $U(t,s)$ is an evolution family. Now assume that $\|T(t)\| \leq Me^{\omega t}$. Then

$$\|T(t)f\| = \sup_{s\geq 0}\|f(t+s)U(t+s,s)\| = \sup_{s\geq 0}\left(\sup_{x\in B_1}|\langle U(t+s,s)x, f(t+s)\rangle|\right)$$

for all $f \in \mathcal{M}$. Let $x' \in X^*$ with $\|x'\| \leq 1$ and define $f_{x'}(s) := x'$. Then $f_{x'} \in \mathcal{M}$, $\|f_{x'}\| \leq \|x'\| \leq 1$, and, for $x \in B_1$,

$$\|U(t+s,s)x\| = \sup_{\|x'\|\leq 1}|\langle U(t+s,s)x, x'\rangle| = \sup_{\|x'\|\leq 1}|\langle U(t+s,s)x, f_{x'}(t+s)\rangle|$$

$$\leq \sup_{\|x'\|\leq 1}\|T(t)f_{x'}\| \leq \sup_{\|x'\|\leq 1} Me^{\omega t}\|f_{x'}\| \leq Me^{\omega t}.$$

Thus, for all $x \in X$,

$$\left\|U(t+s,s)\frac{x}{\|x\|}\right\| \leq Me^{\omega t}$$

or $\|U(t+s,s)\| \leq Me^{\omega t}$ or $\|U(\widetilde{t},s)\| \leq Me^{\omega(\widetilde{t}-s)}$ for $\widetilde{t} \geq s \geq 0$.

Finally, if $t \mapsto T(t)$ is bi-continuous, then it follows that $(t,s) \mapsto T(t)f(s)[x]$ is continuous for all $f \in \mathcal{M}$ and all $x \in X$. Let $(t_n, s_n) \to (t,s)$ and $x \in X$. Then, by the Hahn-Banach Theorem, there exists $x'_n \in X^*$ with $\|x'_n\| = 1$ such that

$$\|U(t_n + s_n, s_n)x - U(t+s,s)x\| = |\langle U(t_n + s_n, s_n)x - U(t+s,s)x, x'_n\rangle|.$$

Now choose $f \in C_b([0,\infty), X^*_{\omega^*})$ with $f(t_n + s_n) = x'_n$. Then

$$\|U(t_n + s_n, s_n x - U(t+s,s)x\| = |\langle U(t_n + s_n, s_n)x - U(t+s,s)x, f(t_n + s_n)\rangle|$$

$$\leq |\langle U(t_n + s_n, s_n)x, f(t_n + s_n)\rangle - \langle U(t+s,s)x, f(t+s)\rangle|$$
$$- |\langle U(t+s,s)x, f(t_n + s_n) - f(t+s)\rangle|$$

$$= |T(t_n)f(s_n)[x] - T(t)f(s)x| - |\langle U(t+s,s)x, f(t_n + s_n) - f(t+s)\rangle|.$$

Since both expressions converge to zero as $(t_n, s_n) \to (t,s)$, it follows that $(t,s) \mapsto U(t+s,s)$ is strongly continuous. This concludes the proof of the implication (i) $\to$ (ii).

If (ii) holds, then

$$\|T(t)f\|_\infty = \sup_{s \geq o} \|f(t+s)U(t+s,s)\| \leq Me^{\omega t}\|f\|_\infty.$$

Let $t \geq 0$. Then we have to show that the map $s \to T(t)f(s)$ is ω^*-continuous for all $f \in C_b([0,\infty), X^*_{\omega^*})$. Let $x \in X$. Then

$$|T(t)f(s)[x] - T(t)f(s_0)[x]|$$
$$= |\langle U(t+s,s)x, f(t+s)\rangle - \langle U(t+s_0, s_0)x, f(t+s_0)\rangle|$$

$$\leq |\langle U(t+s,s)x - U(t+s_0,s_0)x, f(t+s)\rangle|$$
$$+ |\langle U(t_0 + s_0, s)x, f(t+s) - f(t+s_0)\rangle|$$

$$\leq \|f\|_\infty \|U(t+s,s)x - U(t+s_0,s_0)x\|$$

$$+ |\langle U(t+s_0,s_0)x, f(t+s) - f(t+s_0)\rangle| \to 0$$

as $s \to s_0$ because of the continuity of $(t,s) \mapsto U(t+s,s)x$ and the weak*-continuity of $s \mapsto f(s)$. Thus $T(t)(t \geq 0)$ is a semigroup of bounded linear operators on $C_b([0,\infty), X^*_{\omega^*})$. To show the local bi-equicontinuity of the semigroup we have to show that $T(t)f_n \xrightarrow{\omega^*} 0$ uniformly in $t \in [0,t_0]$ (for any $t_0 > 0$) if $f_n \xrightarrow{\omega^*} 0$. That is, if $x \in X$ and $s_0 > 0$, then $T(t)f_n(s)[x] \to 0$ uniformly in $t \in [0,t_0]$ and $s \in [0,s_0]$ assuming that $f_n(s)[x] \to 0$ uniformly in $s \in [0,N]$ for all $N > 0$. This follows from

$$|T(t)f_n(s)[x]| = |\langle U(t+s,s)x, f_n(t+s)\rangle|$$

$$= |\langle x, U^*(t+s,s)f_n(t+s)\rangle| \leq \|U^*(t+s,s)\| |f_n(t+s)[x]|$$

$$\leq Me^{\omega t_0} |f_n(t+s)[x]| \to 0$$

uniformly for $t \in [0,t_0]$ since $f_n(s)[x] \to 0$ uniformly for $s \in [0,t_0+s_0]$. By Remark 2.2.3, strong $\widetilde{\omega^*}$-continuity of $t \mapsto T(t)f$ on $[0,\infty)$ follows from the strong $\widetilde{\omega^*}$-continuity of $t \mapsto T(t)f$ at $t = 0$. To see this, observe that

$$\sup_{0 \leq s \leq N} |T(t)f(s)[x] - f(s)[x]| = \sup_{0 \leq s \leq N} |\langle U(t+s,s)x, f(t+s)\rangle - \langle x, f(s)\rangle|$$

$$\leq |\langle U(t+s,s)x - U(s,s)x, f(t+s)\rangle| + \sup_{0 \leq s \leq N} |\langle x, f(t+s) - f(s)\rangle|.$$

This converges to zero as $t \to 0$ because of the uniform continuity of $(t,s) \mapsto U(t+s,s)x$ on compact subsets of $[0,\infty) \times [0,\infty)$ and the uniform continuity of $s \mapsto f(s)[x]$ on compact subsets of $[0,\infty)$. □

If we assume that the evolution family $U(t,s)(t \geq s \geq 0)$ is norm continuous, then we find that the semigroup $T(t)(t \geq 0)$ defined by $T(t)f(s) := f(t+s)U(t+s,s)$

is bi-continuous on $(C_b(0,\infty), \mathcal{L}(X,Y)), \|\cdot\|_\infty, \sigma_c)$ where σ_c is the topology of compact convergence.

Theorem 3.1.2. *Let X be a normed vector space, Y a Banach space, and suppose $U(t,s) : X \to X$ $(t \geq s \geq 0)$ is a norm continuous evolution family with $\|U(t,s)\| \leq Me^{\omega(t-s)}$ for some $M \geq 1$ and $\omega \in \mathbb{R}$. Then $T(t)(t \geq 0)$, defined by*

$$T(t)f(s) := f(t+s)U(t+s,s),$$

is a bi-continuous semigroup on $(C_b[0,\infty), \mathcal{L}(X,Y)), \|\cdot\|_\infty, \sigma_c)$ and a strongly continuous semigroup on $C_0([0,\infty), \mathcal{L}(X,Y))$.

Proof. The semigroup $T(t)(t \geq 0)$ is clearly well-defined and satisfies properties (i) and (ii) of Definition 2.2.1. We need to show that $T(t)(t \geq 0)$ is strongly σ_c-continuous and locally bi-equicontinuous on $C_b(0,\infty), \mathcal{L}(X,Y))$. To show that $t \mapsto T(t)f$ is σ_c-continuous we have, for any $N \in \mathbb{N}$ and $t_0 \geq 0$,

$$\sup_{0\leq s\leq N} \|T(t)f(s) - T(t_0)f(s)\| = \sup_{0\leq s\leq N} \|f(t+s)U(t+s,s) - f(t_0+s)U(t_0+s,s)\|$$

$$\leq \sup_{0\leq s\leq N} \|f(t+s)-f(t_0+s)\|\|U(t+s,s)\| + \sup_{0\leq s\leq N} \|f(t_0+s)\|\|U(t+s,s)-U(t_0+s,s)\|,$$

where both terms converge to zero because of uniform continuity on compact intervals. For the local bi-equicontinuity, let $\{f_n\}_{n\in\mathbb{N}}$ be a σ_c-null sequence, $N \in \mathbb{N}$, and $t_0 \geq 0$ be fixed. Then, for $t \in [0, t_0]$,

$$\sup_{0\leq s\leq N} \|T(t)f_n(s)\| = \sup_{0\leq s\leq N} \|f_n(t+s)U(t+s,s)\|$$

$$\leq \sup_{0\leq s\leq N} \|f_n(t+s)\|\|U(t+s,s)\| \leq \sup_{0\leq s\leq N} \|f_n(t+s)\|Me^{\omega t_0}.$$

$$\leq \sup_{0\leq s\leq t_0+N} \|f_n(s)\|Me^{\omega t_0}$$

which converges to zero since $\{f_n\}_{n\in\mathbb{N}}$ is a σ_c-null sequence.

Now we show that $T(t)(t \geq 0)$ is a strongly continuous semigroup on $C_0([0,\infty),\mathcal{L}(X,Y))$. First observe that

$$0 \leq \lim_{s\to\infty} \|T(t)f(s)\| = \lim_{s\to\infty} \|f(t+s)U(t+s,s)\|$$

$$\leq \lim_{s\to\infty} \|f(s)\|\|U(t+s,s)\| \leq \lim_{s\to\infty} \|f(s)\|Me^{\omega t} = 0.$$

For the continuity of $s \mapsto T(t)f(s)$, let $s_0 \geq 0$ and observe that

$$\|T(t)f(s) - T(t)f(s_0)\| = \|f(t+s)U(t+s,s) - f(t+s_0)U(t+s_0,s_0)\|$$

$$\leq \|f(t+s)U(t+s,s) - f(t+s)U(t+s_0,s_0)\|$$
$$+ \|f(t+s)U(t+s_0,s_0) - f(t+s_0)U(t+s_0,s_0)\|$$

$$\leq \|f\|_\infty \|U(t+s,s) - U(t+s_0,s_0)\| + \|f(t+s) - f(t+s_0)\|\|U(t+s_0,s_0)\|.$$

Since $(t,s) \mapsto U(t+s,s)$ is norm continuous and $f \in C_0([0,\infty),\mathcal{L}(X,Y))$, both expressions converge to zero as $s \to s_0$. Thus, $s \mapsto T(t)f(s)$ is continuous and $T(t)f$ is well-defined. Now we show that $t \mapsto T(t)f$ is continuous for every $f \in C_0([0,\infty),\mathcal{L}(X,Y))$. Let $\varepsilon > 0$. Observe that, for all $N \geq 0$,

$$\|T(t)f - f\| = \sup_{s\geq 0} \|T(t)f(s) - f(s)\|$$

$$\leq \sup_{s\leq N} \|T(t)f(s) - f(s)\| + \sup_{s\geq N} \|T(t)f(s)\| + \sup_{s\geq N} \|f(s)\|.$$

Since $f \in C_0([0,\infty),\mathcal{L}(X,Y))$ and

$$\|T(t)f(s)\| = \|f(t+s)U(t+s,s)\| \leq \|f(t+s)\|Me^{\omega t},$$

it follows that there exists $N > 0$ such that

$$\|T(t)f - f\| \leq \sup_{0\leq s\leq N} \|T(t)f(s) - f(s)\| + \frac{\varepsilon}{2}$$

for all $0 \leq t \leq 1$. Since

$$\|T(t)f(s) - f(s)\| = \|f(t+s)U(t+s,s) - f(s)U(s,s)\|$$

$$\leq \|f(t+s) - f(s)\|\|U(t+s,s)\| + \|f(s)\|\|U(t+s,s) - U(s,s)\|$$

$$\leq Me^{\omega}\|f(t+s) - f(s)\| + \|f\|_\infty\|U(t+s,s) - U(s,s)\|$$

it follows from the uniform continuity of $s \mapsto f(s)$ and $(t,s) \mapsto U(t,s)$ on compact subsets that there exists $0 < \delta < 1$ such that

$$\sup_{0 \leq s \leq N} \|T(t)f(s) - f(s)\| \leq \frac{\varepsilon}{2}$$

for all $0 \leq t < \delta$. Thus $T(t)f \to f$ as $t \to 0$. □

Theorem 3.1.3. *Let X be a normed vector space, $U(t,s) \in \mathcal{L}(X)$ for $0 \leq s \leq t$, and $T(t)f(s) = f(t+s)U(t+s,s)$. Then the following are equivalent.*

(i) The family $U(t,s)(t \geq s \geq 0)$ is a norm-continuous evolution family on X with $\|U(t,s)\| \leq Me^{\omega(t-s)}$.

(ii) The family $T(t)(t \geq 0)$ is a bi-continuous evolution semigroup on $(C_b([0,\infty), X^*), \|\cdot\|_\infty, \sigma_c)$ with $\|T(t)\| \leq Me^{\omega t}$.

(iii) The family $T(t)(t \geq 0)$ is a strongly continuous evolution semigroup on $C_0([0,\infty), X^*)$ with $\|T(t)\| \leq Me^{\omega t}$.

Proof. The implication (i) → (ii) follows from Theorem 3.1.2. If (ii) holds, then $T(t)f : s \mapsto f(t+s)U(t+s,s)$ defines a bi-continuous semigroup on $C_b([0,\infty), X^*)$ with respect to the norm topology on X^* and the topology of uniform convergence on compact subsets of $\mathbb{R}_+$. Consider

$$\|T(t)f - f\| = \sup_{s \geq 0} \|T(t)f(s) - f(s)\|$$

$$\leq \sup_{s\in[0,N]} \|T(t)f(s) - f(s)\| + \sup_{s\geq N} \|T(t)f(s)\| + \sup_{s\geq N} \|f(s)\|$$

$$= \sup_{s\in[0,N]} \|T(t)f(s) - f(s)\| + \sup_{s\geq N} Me^{\omega t}\|f(s)\| + \sup_{s\geq N} \|f(s)\|.$$

Let $\varepsilon > 0$. If $f \in C_0([0,\infty), X^*)$, then there exists $n > 0$ such that

$$\|T(t)f - f\| \leq \sup_{s\in[0,N]} \|T(t)f(s) - f(s)\| + \frac{\varepsilon}{2}$$

for all $t \in [0,1]$. By assumption $\sup_{s\in[0,N]} \|T(t)f(s) - f(s)\| \to 0$ as $t \to 0$. Thus, $\|T(t)f(s) - f(s)\| \to 0$ as $t \to 0$ for all $f \in C_0([0,\infty), X^*)$. This shows that $T(t)(t \geq 0)$ defines a strongly continuous semigroup on $C_0([0,\infty), X^*)$. Thus (ii) $\to$ (iii).

Assume that (iii) holds. Then, as in Theorem 3.1.1, the semigroup properties of the operators $T(t)(t \geq 0)$ imply the evolution family properties of the operators $U(t,s)(t \geq s \geq 0)$ as well as the estimate $\|U(t,s)\| \leq Me^{\omega(t-s)}$. It remains to be shown that the continuity of $t \mapsto T(t)f$ for $f \in C_0([0,\infty), X^*)$ implies the norm-continuity of $(t,s) \mapsto U(t+s,s)$ for $t \geq s \geq 0$. Let $(t_n, s_n) \to (t,s)$. Then there exists $x_n \in X$ with $\|x_n\| = 1$ and $x'_n \in X^*$ with $\|x'_n\| = 1$ such that

$$0 \leq \|U(t_n+s_n,s_n) - U(t+s,s)\| \leq \|U(t_n+s_n,s_n)x_n - U(t+s,s)x_n\| + \frac{1}{n}$$

$$= |\langle U(t_n+s_n,s_n)x_n - U(t+s,s)x_n, x'_n\rangle| + \frac{1}{n}$$

$$= |\langle U(t_n+s_n,s_n)x_n - U(t+s,s)x_n, f(t_n+s_n)\rangle| + \frac{1}{n}$$

where $f \in C_0([0,\infty), X^*)$ is such that $f(t_n+s_n) = x'_n$ and $\|f\|_\infty \leq 1$. Therefore,

$$0 \leq \|U(t_n+s_n,s_n) - U(t+s,s)\| \leq \|U(t_n+s_n,s_n)x_n - U(t+s,s)x_n\| + \frac{1}{n}$$

$$= |\langle U(t_n+s_n,s_n)x_n - U(t+s,s)x_n, f(t_n+s_n)\rangle$$
$$+ \langle U(t+s,s)x_n, f(t+s)\rangle - \langle U(t+s,s)x_n, f(t+s)\rangle| + \frac{1}{n}$$

$$\leq |\langle x_n, T(t_n)f(s_n)\rangle - \langle x_n, T(t)f(s)\rangle| + |\langle U(t+s,s)x_n, f(t+s) - f(t_n+s_n)\rangle| + \frac{1}{n}$$

$$\leq \|T(t_n)f(s_n) - T(t)f(s)\|\|x_n\| + \|f(t_n+s_n) - f(t+s)\|\|U(t+s,s)\|\|x_n\| + \frac{1}{n}$$

$$\leq \|T(t_n)f(s_n) - T(t)f(s_n)\| + \|T(t)\|\|f(s_n) - f(s)\| + \|f(t_n+s_n) - f(t+s)\|Me^{\omega t} + \frac{1}{n}$$

$$\leq \sup_{r\in[0,s+1]} \|T(t_n)f(r) - T(t)f(r)\| + Me^{\omega t}\|f(s_n) - f(s)\| + \|f(t_n+s_n) - f(t+s)\|Me^{\omega t} + \frac{1}{n} \to 0$$

as $n \to \infty$. □

3.2 Evolution Semigroups for Non-Exponential Evolution Families

Now we consider the case of the truncated space $C_0([0,N], \mathcal{L}(X,Y))$. An advantage of this Banach space is that the evolution family $U(t,s)(t \geq s \geq 0)$ does not necessarily need to be exponentially bounded to ascertain continuity properties of the evolution semigroup.

Theorem 3.2.1. *Let X be a normed vector space. Suppose $U(t,s): X \to X$ is a norm continuous evolution family that is not necessarily exponentially bounded. Then, for all $N > 0$,*

$$T(t)f(s) := \begin{cases} f(t+s)U(t+s,s) & t+s < N \\ 0 & otherwise \end{cases}$$

defines a strongly continuous semigroup on $C_0([0, N), \mathcal{L}(X, Y))$, the set of continuous functions $f : [0, N] \to \mathcal{L}(X, Y)$ with $f(N) = 0$.

Proof. Clearly, $T(t)f = 0$ for every $t \geq N$ and $f \in C_0([0, N], \mathcal{L}(X, Y))$. Furthermore, because of the norm continuity of $(t, s) \to U(t + s, s)$ for $t, s \geq 0$, it follows that

$$\sup_{0 \leq t, s \leq N} \|U(t + s, s)\| =: M < \infty$$

for some $M \geq 1$. Thus, for every $t \geq 0$, $s \in [0, N]$, and $f \in C_0([0, N], \mathcal{L}(X, Y))$, we have $T(t)f(s) \in \mathcal{L}(X, Y)$. Now for $f \in C_0([0, N], \mathcal{L}(X, Y))$, $T(t)f(N) = \widetilde{f}(t + N)U(t + N, s) = 0$ where $\widetilde{f}$ is defined as $\widetilde{f}(s) = f(s)$ for $0 \leq s \leq N$ and $\widetilde{f}(s) = 0$ for $s > N$. Then $\widetilde{f}$ is a bounded, uniformly continuous function on $[0, \infty)$ with $\|\widetilde{f}\|_\infty = \|f\|_\infty$. Let $t \geq 0$ be fixed. Then the map $s \mapsto T(t)f(s) = f(t+s)U(t+s, s)$ is continuous since

$$\|T(t)f(s) - T(t)f(s_0)\| = \|f(t + s)U(t + s, s) - f(t + s_0)U(t + s_0, s_0)\|$$

$$\leq f(t + s)[U(t + s, s) - U(t + s_0, s_0)]\| + \|(f(t + s) - f(t + s_0))[U(t + s_0, s_0)]\|$$

$$\leq \|f\|_\infty \|U(t + s, s) - U(t + s_0, s_0)\| + \|f(t + s) - f(t + s_0)\| M e^{\omega t} \to 0$$

as $s \to s_0$. Thus, $f \mapsto T(t)f$ is well defined. The semigroup properties (1.1) are satisfied trivially, and since $T(t) \equiv 0$ for $t \geq N$ on $C_0([0, N], \mathcal{L}(X, Y))$, it follows that $\|T(t)\| \leq M$ by construction. To show the continuity of $t \mapsto T(t)f$ for $f \in C_0([0, N], \mathcal{L}(X, Y))$, observe that

$$\|T(t)f - f\| = \sup_{0 \leq s \leq N} \|\widetilde{f}(t + s)U(t + s, s) - f(s)\|$$

$$\leq \sup_{0 \leq s \leq N} \|\widetilde{f}(t + s)[U(t + s, s) - U(s, s)]\| + \sup_{0 \leq s \leq N} \|\widetilde{f}(t + s) - \widetilde{f}(s)\|$$

$$\leq \|\widetilde{f}\|_\infty \sup_{0 \leq s \leq N} \|U(t + s, s) - U(s, s)\| + \sup_{0 \leq s \leq N} \|\widetilde{f}(t + s) - \widetilde{f}(s)\|.$$

It follows from the uniform continuity of $s \mapsto \widetilde{f}(s)$ and $(t,s) \mapsto U(t+s,s)$ on compact subsets of R_+ that $\|T(t)f - f\| \to 0$. Thus, $t \mapsto T(t)f$ is continuous and $T(t)(t \geq 0)$ defines a strongly continuous on $C_0([0,N), \mathcal{L}(X,Y))$. $\square$

Theorem 3.2.2. *Let X be a Banach space. Suppose $U(t,s) : X \to X$ $(t \geq s \geq 0)$ is a strongly continuous evolution family that is not necessarily exponentially bounded. For $f \in C_0([0,N), X^*_{\omega^*})$ define $\widetilde{f}$ to be the zero-extension of f onto $[0,\infty)$. Then*

$$T(t)f(s) := \widetilde{f}(t+s)U(t+s,s)$$

defines a bi-continuous semigroup on $(C_0([0,\infty), X^*_{\omega^*}), \|\cdot\|_\infty, \widetilde{\omega^*})$.

Proof. As in Theorem 2.1.5 we can show that $C_0([0,N), X^*_{\omega^*})$ is bi-admissible with the $\|\cdot\|_\infty$-topology and the $\widetilde{\omega^*}$-topology generated by the seminorms

$$p_M, x(f) = \sup_{s\in[0,M]} |\langle x, f(s)\rangle|.$$

Let $x \in X$. Then

$$(t,s) \mapsto U(t+s,s)x$$

is continuous for $0 \leq t, s \leq N$. Thus, there exists a constant $M_x \geq 0$ such that

$$\|U(t+s,s)x\| \leq M_x$$

for all $0 \leq t, s \leq N$. By the Principle of Uniform Boundedness, there exists $M > 0$ such that

$$\|U(t+s,s)\| \leq M$$

for all $0 \leq t, s \leq N$. Thus

$$\|T(t)f\|_\infty \leq M\|f\|_\infty$$

for all $t \geq 0$. Let $t > 0$ and $f \in C_0([0,N), X^*_{\omega^*})$. Then the map $s \mapsto T(t)f(s)$ is ω^*-continuous with the same argument as in the proof of (ii) $\to$ (i) of Theorem

3.1.1. Clearly, $T(t)f(N) = 0$ for all $t \geq 0$. Thus, $T(t)$ is a bounded linear operator on $C_0([0, N), X^*_{\omega^*})$. The bi-continuity of $t \mapsto T(t)$ can be shown exactly as in the proof of the implication (ii) $\to$ (i) of Theorem 3.1.1.

$\square$

Chapter 4
Outlook

4.1 General Flow Semigroups

This chapter contains some preliminary results and illustrative examples as a starting point for future investigations regarding the general evolution semigroup (4.2) and the Lie generator (1.6). That is, we consider the nonlinear initial value problem (Cauchy problem)

$$u'(t) = F(t, u(t)),\ u(s) = x \in \Omega,\ t \geq s \geq 0$$

with solution flow $u(t) = \gamma(t, s, x) \in \Omega$. Then the map $t \to \gamma(t+s, s, x) \in \Omega$ is defined for

$$0 \leq t < m(s, x) = \sup\{T \geq 0 : \gamma(t+s, s, x) \text{ exists for } t \in [0, T]\}.$$

Let $\Omega_\gamma = \{(s, x) \in [0, \infty) \times \Omega : m(s, x) > 0\}$. As explained in Section 1.1 (see (1.5) and (1.6)), the map

$$T(t)g(s, x) := g(t+s, \gamma(t+s, s, x))$$

defines a semigroup of linear operators on $C_b(\Omega_\gamma)$ with formal generator

$$\mathcal{A}g(s, x) = \lim_{t \to 0} \frac{g(t+s, \gamma(t+s, s, x)) - g(s, x)}{t},$$

called the Lie-generator of the flow γ. Let us assume first that

(a) $\gamma(t+s, s, x) \in \Omega$ exists for all times $t \geq 0, s \geq 0$, and states $x \in \Omega$,

(b) $\widetilde{\Omega} := [0, \infty) \times \Omega$ is a Polish space (separable, complete metric space), and

(c) the map $(t, s, x) \to \gamma(t+s, s, x)$ is jointly continuous on $[0, \infty) \times \widetilde{\Omega}$.

Define $\phi : [0, \infty) \times \widetilde{\Omega} \to \widetilde{\Omega}$ by

$$\phi(t, \widetilde{x}) = \phi(t, (s, x)) = (t+s, \gamma(t+s, s, x)) \in \widetilde{\Omega} \tag{4.1}$$

Then $\phi(0, \widetilde{x}) = (s, \gamma(s, s, x)) = (s, x) = \widetilde{x}$ and

$$\phi(t, \phi(r, \widetilde{x})) = \phi(t, \phi(r, (s, x))) = \phi(t, (r+s, \gamma(r+s, s, x)))$$

$$= (t+r+s, \gamma(t+r+s, r+s, \gamma(r+s, s, x))) = (t+r+s, \gamma(t+r+s, s, x))$$

$$= \phi(t+r, (s, x)) = \phi(t+r, \widetilde{x})$$

for all $\widetilde{x} \in \widetilde{\Omega}$. Then $(t, \widetilde{x}) \to \phi(t, \widetilde{x})$ is a jointly continuous, time autonomous flow on $\widetilde{\Omega}$. The following result is an immediate consequence of the Dorroh-Neuberger Theorem concerning jointly continuous, autonomous flows (see [7], [8], [20]).

Theorem 4.1.1. *Let $t \mapsto \gamma(t+s, s, x)$ be a global flow for times $t, s \geq 0$ and states $x \in \Omega$. If*

(a) $\widetilde{\Omega} = [0, \infty) \times \Omega$ is a Polish space (complete, separable metric space), and

(b) $(t, s, x) \mapsto \gamma(t+s, s, x)$ is jointly continuous,

then the general evolution semigroup

$$T(t)g(s, x) := g(t+s, \gamma(t+s, s, x)), \; (t \geq 0) \tag{4.2}$$

defines a bi-continuous semigroup on $(C_b([0, \infty) \times \Omega), \|\cdot\|_\infty, \beta)$.

Proof. The statement follows immediately from Proposition 2.1.2 as well as Theorem 3.4 in [20] by observing that

$$T(t)g(\widetilde{x}) = g(\phi(t, \widetilde{x}))$$

for $\widetilde{x} = (s, x) \in [0, \infty) \times \Omega$ and ϕ defined as in (4.1). □

Example 4.1.2. Consider

$$u'(t) = -tu(t)^2,\ u(s) = x \in \mathbb{R},\ t \geq 0$$

with solution

$$u(t) = \gamma(t, s, x) = \frac{2x}{2 + x(t^2 - s^2)}.$$

Then

$$\gamma(t + s, s, x) = \frac{2x}{2 + x((t + s)^2 - s^2)} = \frac{x}{1 + \frac{xt}{2}(t + 2s)}$$

for $0 \leq t < m(s, x)$, where

$$m(s, x) = \begin{cases} \infty & \text{if } x \geq 0, s \in \mathbb{R} \\ -s + \sqrt{s^2 - \frac{2}{x}} & \text{if } x < 0, s \in \mathbb{R}. \end{cases}$$

In particular, by Theorem 4.1.1 above and since $\gamma(t + s, s, x) \geq 0$ for $x \geq 0$,

$$T(t)g(s, x) = g\left(t + s, \frac{2x}{2 + x((t + s)^2 - s^2)}\right)$$

defines a bi-continuous semigroup on $C_b(\mathbb{R}^2_+)$ with (formal) generator

$$\mathcal{A}g(s, x) = g_s(s, x) - sx^2 g_x(s, x).$$

Observe that the operator $\mathcal{A}_1 g(s, x) = g_s(s, x)$ generates the bi-continuous semigroup

$$T_1(t)g(s, x) = g(t + s, x) \text{ on } C_b(\mathbb{R}^2_+).$$

Consider the semigroup

$$T_2(t)g(s, x) = g(s, \sigma_s(t, x)) \text{ for } g \in C_b(\mathbb{R}^2_+),$$

where $\sigma_s(t, x)$ solves the autonomous differential equation

$$u'(t) = -su(t)^2,\ u(0) = x.$$

That is, $\sigma_s(t,x) = \frac{x}{1+stx}$. It follows from the Dorroh- Neuberger Theorem that

$$T_2(t)g(s,x) = g\left(s, \frac{x}{1+stx}\right)$$

is a bi-continuous contraction semigroup on $C_b(\mathbb{R}_+^2)$ with generator

$$\mathcal{A}_2 g(s,x) = -sx^2 g_x(s,x).$$

Thus, the Lie-Trotter Product Formula yields

$$\left(T_1\left(\frac{t}{n}\right)T_2\left(\frac{t}{n}\right)\right)^n g(s,x) = g\left(t+s, \frac{2x}{2+tx\left(\frac{n+1}{n}t+2s\right)}\right)$$

$$\longrightarrow g\left(t+s, \frac{2x}{2+x((t+s)^2-s^2)}\right) \text{ as } n \to \infty.$$

□

This example shows the reasonability of the following line of investigation.

Consider the non-autonomous initial value problem

$$u'(t) = F(t,u(t))\, u(s) = x \in X \ (X \text{ Banach space})$$

with solution $u(t) = \gamma(t,s,x)$, $(t \geq s)$. Let $g \in C_b([0,\infty) \times X)$. Then

$$g(t+s, \gamma(t+s,s,x)) = \lim_{n\to\infty}\left[T_1\left(\frac{t}{n}\right)T_2\left(\frac{t}{n}\right)\right]^n g(s,x), \tag{4.3}$$

where

$$T_1(t)g(s,x) = g(t+s,x) \text{ and}$$

$$T_2(t)g(s,x) = g(s,\sigma_s(t,x)),$$

where $\sigma_s(t,x)$ is the solution of the autonomous initial value problem

$$u'(t) = F(s,u(t)),\ u(0) = x \in X.$$

This program was already completed in [3], [21], and [29] for the linear Cauchy problem

$$u'(t) = A(t)u(t),\, u(s) = x$$

in cases for which each linear operator $A(s)$ generates a strongly contoinuous semigroup $T_s(t)(t \geq 0)$ under the framework of special evolution semigroups

$$T(t)f(s) = f(t+s)U(t+s,s)$$

that were studied in Chapter 3. These types of approaches can be lifted to general evolution semigroups induced by nonlinear, nonautonomous problems if one considers semigroups of the form (4.2) instead.

Example 4.1.3. An illustrative example showing the potential usefullness of the Lie-Trotter Product Formula is the equation

$$u'(t) = t^2 - u(t)^2,\, u(s) = x \in \mathbb{R}.$$

In this case, the solution $u(t) = \gamma(t,s,x)$ is certainly not easily computable. However, at least formally, the Lie generator of the induced flow semigoup

$$T(t)g(s,x) = g(t+s, \gamma(t+s,s,x))$$

is given by

$$\mathcal{A}g(s,x) = g_s(s,x) + (s^2 - x^2)g_x(s,x).$$

As above, $\mathcal{A}_1 g(s,x) = g_s(s,x)$ generates the bi-continuous contraction semigroup $T_1(t)g(s,x) = g(t+s,x)$ on $C_b(\mathbb{R}^2_+)$. Consider the semigroup

$$T_2(t)g(s,x) = g(s, \sigma_s(t,x)),$$

where

$$\sigma_s(t,x) = s\frac{s(e^{2st}-1) + x(e^{2st}+1)}{s(e^{2st}+1) + x(e^{2st}-1)}$$

is the solution of the autonomous differential equation

$$u'(t) = s^2 - u(t)^2, \; u(0) = x.$$

Then, by the Dorroh-Neuberger Theorem, $T_2(t)$ is a bi-continuous contraction semigroup on $C_b(\mathbb{R}^2_+)$ with generator

$$\mathcal{A}_2 g(s,x) = (s^2 - x^2) g_x(s,x).$$

Thus, $[T_1(\frac{t}{n})T_2(\frac{t}{n})]^n g(s,x)$ is computable. If the condition of the Lie-Trotter Product Formula (Theorem 2.2.7) could be verified, then it would follow that

(a) $m(s,x) = +\infty$ for all $(s,x) \in \mathbb{R}^2_+$, and

(b) (4.2) holds.

In other words, the Lie-Trotter Product Formula might provide us with a tool that allows us to

(a) determine flows γ for which $m(s,x) = \infty$, and

(b) approximate flows γ by (4.2).

□

As pointed out by John Neuberger ([7], [8], [26]), with suitable continuity conditions on the flow γ and its time-stopping function m, an autonomous flow σ is local if and only if its Lie generator has a positive eigenvalue. The argument can be easily be lifted to non-autonomous flows γ and goes as follows. If $(s_0, x_0) \in \Omega_\gamma$ then $0 < m(s_0, x_0) \leq \infty$. Now define $g \in C_b(\Omega_\gamma)$ by

$$g(s,x) := e^{-m(s,x)}.$$

Then, because of the assumed continuity of m, g is continuous and the Lie-generator of γ (see (1.6)) satisfies

$$
\begin{aligned}
\mathcal{A}g(s,x) &= \lim_{t\searrow 0} \frac{g(t+s,\gamma(t+s,s,x)) - g(s,x)}{t} \\
&= \lim_{t\searrow 0} \frac{e^{-m(t+s,\gamma(t+s,s,x))} - e^{-m(s,x)}}{t} \\
&= \lim_{t\searrow 0} \frac{e^{-m(s,x)+t} - e^{-m(s,x)}}{t} \\
&= \lim_{t\searrow 0} \frac{e^{t}-1}{t} e^{-m(s,x)} = e^{-m(s,x)} = g(s,x).
\end{aligned}
\tag{4.4}
$$

Since g is continuous and $g(s_0,x_0) \neq 0$, it follows that $g \neq 0$. Thus $\mathcal{A}$ has eigenvalue 1 with eigenvector g. Conversely, assume that $\mathcal{A}$ generates a flow semigroup $T(t)$ (given by (1.5)) and has a positive eigenvalue λ_0 with eigenvector g_0. Then $T(t)$ is automatically a contraction. If $T(t)$ would be global, then it follows from basic semigroup theory (see below) that all $\lambda \in \mathbb{C}$ with $Re\,\lambda > 0$ are not in the spectrum of $\mathcal{A}$, contradicting that λ_0 is an eigenvalue. Thus $T(t)$ must be local.

Example 4.1.4. Let us consider again $u'(t) = -tu(t)^2$, $u(s) = x$ with solution $u(t) = \gamma(t,s,x) = \frac{2x}{2+x[t^2-s^2]}$ (see Example 4.1.2). Then

$$
m(s,x) = \begin{cases} \infty & \text{if } x \geq 0, s \in \mathbb{R} \\ -s + \sqrt{s^2 - \frac{2}{x}} & \text{if } x < 0, s \in \mathbb{R} \end{cases}
$$

and the Lie generator of γ is given by

$$
\mathcal{A}g(s,x) = \lim_{t\to 0} \frac{g(t+s,\gamma(t+s,s,x)) - g(s,x)}{t} \qquad = g_s(s,x) - sx^2 g_x(s,x).
$$

Let

$$
h(s,x) = e^{-m(s,x)} = \begin{cases} 0 & \text{if } x \geq 0, s \in \mathbb{R} \\ e^{s-\sqrt{s^2-\frac{2}{x}}} & \text{if } x < 0, s \in \mathbb{R}. \end{cases}
$$

Then $h \in C_b(\mathbb{R}^2)$ and $\mathcal{A}h = h$. Unfortunately, there are other functions like

$$
g(s,x) = e^{s}\left[\frac{-1}{x} + \frac{s^2}{2}\right]
$$

that satisfy the equation $\mathcal{A}g = g$. However, observe that $g \notin C_b(\mathbb{R}^2)$ and g is not positive. If one could show that the positive eigenfunction in $C_b(\mathbb{R}^2)$ to the eigenvalue 1 is uniquely determined (up to a constant), then the numerical computation of the eigenfunctions to the eigenvaule 1 could help us to characterize those $(s, x) \in \Omega_\gamma$ for which $m(s, x) < \infty$ and $m(s, x) = \infty$.

□

4.2 Semigroups on Locally Convex Spaces

If an evolution family $U(t, s)$ is not exponentially bounded, another way to consider the special evolution semigroup (1.22) is on locally convex spaces. The case of equicontinuous semigroups on locally convex spaces is developed in parallel with the case of strongly continuous semigroups on Banach spaces (see [18] and [19]).

Theorem 4.2.1. *Let $\mathcal{F} := (C([0,\infty), \mathcal{L}(X, Y)), \sigma_c)$ be the Fréchet space of continuous functions from $[0,\infty)$ to the continuous dual of some Banach space $(X, \|\cdot\|)$ with the topology generated by the seminorms defined after Proposition 2.1.3 and $U(t, s) : X \to X$ be a norm continuous evolution family on X (but not necessarily exponentially bounded), then for $t \geq 0$, $T(t) : \mathcal{F} \to \mathcal{F}$ defined as before by*

$$T(t)f(s) = f(t + s)U(t + s, s)$$

is a locally equicontinuous strongly continuous semigroup, i.e., $T(t)(N \geq t \geq 0)$ is equicontinuous on $\mathcal{F}$ for every $N \in \mathbb{N}$ (but not necessarily exponentially bounded).

Proof. First we need to show $T(t)f$ is well-defined. Let $t \geq 0$ and $s \geq 0$ be fixed and consider

$$\|T(t)f(s) - T(t)f(r)\| = \|f(t + s)U(t + s, s) - f(t + r)U(t + r, r)\|$$

$$\leq \|f(t+s)U(t+s, s)-f(t+r)U(t+s, s)\|+\|f(t+r)U(t+s, s)-f(t+r)U(t+r, r)\|$$

$$\leq \|f(t+s) - f(t+r)\|\|U(t+s,s)\| + \|f(t+r)\|\|U(t+s,s) - U(r+s,s)\|.$$

Since t and s are fixed, $U(t+s,s)$ is a bounded linear operator and since f is continuous, the first expression goes to 0. In the second expression, $f(r+s)$ is a uniformly continuous function on $[0,n]$ for all $n \geq s+1$, where $n \in \mathbb{N}$. Since $U(t,s)$ is a uniformly continuous evolution family, as $r \to s$, the second expression goes to 0. Therefore $T(t)f$ is well-defined. To show that $t \to T(t)f$ is continuous, we have to show that

$$p_k(T(t)f - T(t_0)f) = \sup_{0 \leq s \leq k} \|T(t)f(s) - T(t_0)f(s)\| \to 0$$

for each seminorm p_k as $t \to t_0$. Suppose $t_0 \in [0,\infty)$ is fixed and consider

$$\begin{aligned}
\sup_{0 \leq s \leq k} \|T(t)f(s) - T(t_0)f(s)\| &= \sup_{0 \leq s \leq k} \|f(t+s)U(t+s,s) - f(t_0+s)(U(t_0+s,s)\| \\
&\leq \sup_{0 \leq s \leq k} \|f(t+s)U(t+s,s) - f(t_0+s)U(t+s,s)\| \\
&\quad + \sup_{0 \leq s \leq k} \|f(t_0+s)U(t+s,s) - f(t_0+s)U(t_0+s,s)\| \\
&\leq \sup_{0 \leq s \leq k} \|f(t+s) - f(t_0+s)\| \sup_{0 \leq s \leq k} \|U(t+s,s)\| \\
&\quad + \sup_{0 \leq s \leq k} \|f(t_0+s)\| \sup_{0 \leq s \leq k} \|U(t+s,s) - U(t_0+s,s)\|.
\end{aligned}$$

The first expression goes to 0 since f is uniformly continuous on $[0,k+n]$, where $n \in N$ and $n \geq t, t_0$. The second expression goes to 0 since $U(t,s)$ is a norm continuous evolution family. For the local equicontinuity, observe that

$$p_k(T(t)f) \leq p_{k+t_0}(f)M_{k+t_0}$$

for all $t \in [0,t_0]$ and $f \in \mathcal{F}$, where $M_{k+t_0} = \sup\{\|U(t+s,s)\| : t \in [0,t_0] \text{ and } s \in [0,k]\}$. □

Another suitable set-up for the non-exponential special evolution semigroup (1.22) is on the locally convex space $(C([0,\infty), X^*_{\omega^*}), \widetilde{\omega^*}_s)$ of continuous X^*-valued functions (in the weak*-topology).

Theorem 4.2.2. *Let $\mathcal{M} := (C([0,\infty), X^*_{\omega^*}), \widetilde{\omega^*}_s)$ be the space of continuous functions $f : [0,\infty) \to X^*_{\omega^*}$ and $U(t,s) : X \to X$ be a strongly continuous evolution family for $0 \le s \le t$, then for $t \ge 0$, $T(t) : \mathcal{M} \to \mathcal{M}$ defined as before by*

$$T(t)f(s)[x] = \langle U(t+s,s)x, f(t+s)\rangle$$

is a locally equicontinuous strongly continuous semigroup.

Proof. First we need to note that $\mathcal{M}$ is a locally-convex, Hausdorff, $\widetilde{\omega^*}_s$-complete topological vector space. Furthermore, for t and r fixed, $T(t)f(s) \to T(t)f(r)$ if and only if $T(t)f(s)[x] \to T(t)f(r)[x]$ for all $x \in X$. This holds because of the strong continuity of $U(t,s)$. Thus $T(t)$ is well-defined for each $t \ge 0$. The continuity of $t \to T(t)f$ follows as in Theorem 4.2.1 but with the seminorm $p_{k,x}$. For the local equicontinuity, let $0 \le t \le t_0$ for some $t_0 \ge 0$ and consider, for any seminorm $p_{k,x}$,

$$\begin{aligned} p_{k,x}(T(t)f) &= \sup_{0\le s\le k} |\langle U(t+s,s)x, f(t+s)\rangle| \\ &\le \sup_{0\le s\le k} (\|f(t+s)\|_{\mathcal{L}} \|U(t+s,s)x\|_X) \\ &\le \sup_{0\le s\le k+t_0} \|f(s)\|_{\mathcal{L}} M_{k+t_0,x} \end{aligned}$$

where $M_{k+t_0,x} := \sup\{\|U(t+s,s)x\|_X : t \in [0,t_0] \text{ and } s \in [0,k]\}$. □

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