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Reflection positivity and conformal symmetry

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Abstract

A reflection positive Hilbert space is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where \mathcal{E} is a Hilbert space, θ a unitary involution and \mathcal{E}_+ a closed subspace on which the hermitian form $\langle v, w \rangle_\theta := \langle \theta v, w \rangle$ is positive semidefinite. For a triple (G, τ, S) , where G is a Lie group, τ an involutive automorphism of G and S a subsemigroup invariant under the involution $s \mapsto s^\sharp = \tau(s)^{-1}$, a unitary representation π of G on $(\mathcal{E}, \mathcal{E}_+, \theta)$ is called reflection positive if $\theta\pi(g)\theta = \pi(\tau(g))$ and $\pi(S)\mathcal{E}_+ \subseteq \mathcal{E}_+$. This is the first in a series of papers in which we develop a new and systematic approach to reflection positive representations based on reflection positive distributions and reflection positive distribution vectors. This approach is most natural to obtain classification results, in particular in the abelian case. Among the tools we develop is a generalization of the Bochner–Schwartz Theorem to positive definite distributions on open convex cones. We further illustrate our techniques with a non-abelian example by constructing reflection positive distribution vectors for complementary series representations of the conformal group $O_{1,n+1}^+(\mathbb{R})$ of the sphere \mathbb{S}^n .

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Keywords: Lie group; Unitary representation; Reflection positivity; Conformal group

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0. Introduction

The concept of reflection positivity has its origins in the work of Osterwalder and Schrader [45,46] on constructive quantum field theory and duality between unitary representations of the Euclidean motion group $E_n = O_n(\mathbb{R}) \ltimes \mathbb{R}^n$ and the Poincaré group $P_n = O_{1,n-1}^+(\mathbb{R}) \ltimes \mathbb{R}^{1,n-1}$ of affine isometries of n -dimensional Minkowski space. Here $O_{1,n-1}^+(\mathbb{R})$ is the group preserving the Lorentz form $(t, x) \mapsto t^2 - \|x\|^2$ and mapping the forward light cone

$$\Omega = \{(t, x) \mid t^2 - \|x\|^2 > 0, t > 0\}$$

onto itself. Multiplying the time coordinate t by $i = \sqrt{-1}$ transforms the Lorentz form into $-t^2 - \|x\|^2 = -\|(t, x)\|^2$ and this sets up a duality between the groups P_n and E_n .

On the mathematical side this duality can be made precise as follows. If \mathfrak{g} is a Lie algebra with an involutive automorphism τ , then we have the τ -eigenspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \ker(\tau - \mathbf{1}) \oplus \ker(\tau + \mathbf{1})$ and the subspace $\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of \mathfrak{g} is another real form. We thus obtain a duality between the pairs (\mathfrak{g}, τ) and (\mathfrak{g}^c, τ) . At the core of the notion of reflection positivity is the idea that this duality can sometimes be implemented on the level of unitary representations. This is quite simple on the Lie algebra level: Let \mathcal{E}^0 be a pre-Hilbert space and π be a representation of \mathfrak{g} on \mathcal{E}^0 by skew-symmetric operator. We also assume that there exists a unitary operator θ on \mathcal{E}^0 with $\theta\pi(x)\theta = \pi(\tau x)$ for $x \in \mathfrak{g}$, and a \mathfrak{g} -invariant

subspace \mathcal{E}_+^0 which is θ -positive in the sense that the hermitian form $\langle v, w \rangle_\theta := \langle \theta v, w \rangle$ is positive semidefinite on \mathcal{E}_+^0 . Then complex linear extension leads to a representation of \mathfrak{g}^c on \mathcal{E}_+^0 by operators which are skew-symmetric with respect to $\langle \cdot, \cdot \rangle_\theta$, so that we obtain an infinitesimally unitary representation of \mathfrak{g}^c on the pre-Hilbert space $\widehat{\mathcal{E}}^0 := \mathcal{E}_+^0 / \{v \in \mathcal{E}_+^0 : \langle \theta v, v \rangle = 0\}$ whose completion is denoted $\widehat{\mathcal{E}}$ below. This is the basic idea behind the reflection positivity correspondence between infinitesimally unitary representations of \mathfrak{g} and \mathfrak{g}^c . What this simple picture completely ignores are issues of integrability and essential selfadjointness of operators. There are various natural ways to address these problems. Important first steps in this direction have been undertaken by Klein and Landau in [29,30], and Fröhlich, Osterwalder and Seiler introduced in [11] the concept of a virtual representation, which was developed in greater generality by Jorgensen in [23,24].

This is the first in series of papers in which we develop a systematic approach to the representation theoretic aspects of reflection positivity. Our main new point is to describe representations in terms of cyclic distribution vectors satisfying a reflection positivity condition. The positivity we study in the present paper is expressed in terms of open subsemigroups of Lie groups.

To be more specific, we call a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, consisting of a Hilbert space \mathcal{E} , a unitary involution θ on \mathcal{E} and a closed θ -positive subspace \mathcal{E}_+ a *reflection positive Hilbert space*. For a triple (G, τ, S) of a Lie group G , an involutive automorphism τ of G and an open subsemigroup $S \subseteq G$ which is invariant under the involution $s^\sharp = \tau(s)^{-1}$, a unitary representation π of G on $(\mathcal{E}, \mathcal{E}_+, \theta)$ is called *reflection positive* if $\theta\pi(g)\theta = \pi(\tau(g))$ and $\pi(S)\mathcal{E}_+ \subseteq \mathcal{E}_+$. This data leads to a \sharp -representation of S by contractions on the associated Hilbert space $\widehat{\mathcal{E}}$, and there are powerful tools based on the Lüscher–Mack Theorem to derive from such representations unitary representations of a Lie group G^c with Lie algebra \mathfrak{g}^c (cf. [34], [15, Sect. 9.5] and [36] for a generalization to Banach–Lie groups). We therefore focus on the triple (G, τ, S) and unitary representations as above. In any case, one obtains unitary representations π^c of G^c for which the operators $-i\partial\pi^c(x)$ for $x \in i\mathfrak{q}$ and $\exp(\mathbb{R}_+ix) \subseteq S$ have positive spectrum. This condition imposes serious restrictions on the unitary representations of G^c that one can obtain in this context. We take up this point in the subsequent paper [42], where we take a systematic look at the passage from reflection positive representations of G on \mathcal{E} to unitary representations of the dual group G^c . In particular, we show how to obtain specific representations of G^c and apply our results to the representations underlying generalized free fields (cf. [12]).

One of the basic requirements of quantum theory is that physical states form a Hilbert space \mathcal{H} with a unitary positive energy representation π of the Poincaré group, i.e., the spectral measure of the translation group is supported by the future light cone (cf. [12]). Here the involution τ corresponds to time reversal which implements the duality between the Lie algebras of E_n and P_n . Therefore one is led to subsemigroups $S \subseteq E_n$ containing the ray $\{(x_0, 0) \in \mathbb{R}^n : x_0 > 0\}$ which under c -duality corresponds to time translations. A natural enlargement is the semigroup

$$S = O_{n-1}(\mathbb{R}) \times \{(t, x) \mid t > 0, x \in \mathbb{R}^{n-1}\},$$

but here the semigroup approach is problematic because there is no proper semigroup with interior points in E_n . This makes the passage from unitary representations of E_n to unitary representation of P_n a harder problem which can be addressed with the methods developed in [23,24]. The approach to reflection positivity in terms of c -duality of Lie algebras and contractive representations of semigroups was already taken up in the work by Schrader in [48], where he used reflection positivity to construct from a complementary series representation of $SL_{2n}(\mathbb{C})$ a unitary representation of the c -dual group $SU_{n,n}(\mathbb{C}) \times SU_{n,n}(\mathbb{C})$ but without identifying the

resulting representation. This approach was developed more systematically in [25,26] where duality between semisimple causal symmetric spaces, the theory of compression semigroups and the Lüscher–Mack Theorem were used to construct from a generalized complementary series representation related to an ordered symmetric space G/H an irreducible unitary highest weight (= positive energy) representation of the dual group G^c . The construction was carried out for the case where G is locally isomorphic to the automorphism group of a tube domain $T_\Omega := \mathbb{R}^k + i\Omega$, where $\Omega \subseteq \mathbb{R}^k$ is a symmetric cone and $H = \{g \in G \mid g(\Omega) = \Omega\}$ is the subgroup preserving the totally real submanifold Ω of T_Ω . In this case $\mathfrak{g}^c \cong \mathfrak{g}$, so that one obtains a correspondence between unitary representations of the same group.

For further reference on the physical side of reflection positivity we would like to point out the work of A. Klein [27,28], as well as the more recent work by Jaffe and coauthors [21,20,22,19]. An excellent introduction can be found in the overview article [18], in particular in Section VII.

In the present article we first propose a new approach to a systematic treatment of reflection positive representations of triples (G, τ, S) based on *reflection positive distributions* and *reflection positive distribution vectors* of a unitary representation of G . Here the key tool is a refinement of the well-known GNS construction for positive definite functions and distributions to the reflection positive setting. To understand the nature of reflection positive distributions, it is indispensable to have a complete picture of the abelian case because this leads to suitable spectral measures. To this end we obtain integral representations of reflection positive functions on the real line (Proposition 3.1) and generalize the Bochner–Schwartz Theorem to positive definite distributions on open convex cones (Theorem 4.11). These results will also be crucial in the sequel [43], where we encounter situations with a natural half line or a cone to which the integral representations apply and provide a departure point for constructing reflection positive representations of non-abelian groups. For the verification of positive definiteness of holomorphic kernels, we provide in Appendix A a general theorem asserting that it suffices to verify positive definiteness on open subsets or on totally real submanifolds (Theorem A.1). Finally we apply these techniques to exhibit cyclic reflection positive distribution vectors for some of the complementary series representations of the conformal group $O_{1,n+1}^+(\mathbb{R})$ of the sphere S^n .

We have already mentioned that there are spectral restrictions on the representations of G^c that can result from reflection positivity and this also restricts the class of groups for which this process can possibly apply. For semisimple Lie groups, an inspection of parameters of those representations, in particular the infinitesimal character, indicates that the representations of G to start with should be generalized complementary series representations. This is why we, as well as [48] and [25,26], start with this class of representations to demonstrate the naturality of our approach by reflection positive cyclic distribution vectors.

This article is organized as follows. In Section 1 we introduce reflection positive unitary representations for triples (G, τ, S) on reflection positive Hilbert spaces $(\mathcal{E}, \mathcal{E}_+, \theta)$. To develop a systematic approach to reflection positivity for unitary representations, we first introduce a concept of a *reflection positive operator-valued function* on G and explain how they correspond to reflection positive representations (π, \mathcal{E}) that are generated in a very controlled fashion by a subspace \mathcal{F} of θ -fixed vectors. This reflection positive variant of the GNS construction (Proposition 1.11) is the main point of Section 1. In many interesting situations (see Section 6 and [43]) it turns out that the generating vectors cannot be found in the Hilbert space itself but have to be replaced by distribution vectors in the larger space $\mathcal{E}^{-\infty}$. This leads us in Section 2 to the notion of a *reflection positive distribution*. On the representation side, they correspond to *reflec-*

tion positive distribution cyclic representations, which are triples $(\pi, \mathcal{E}, \alpha)$, where π is a unitary representation of G satisfying $\theta\pi(g)\theta = \pi(\tau(g))$ for a unitary involution θ on \mathcal{E} , $\alpha \in \widehat{\mathcal{E}}^{-\infty}$ is a θ -invariant cyclic distribution vector, and the closed subspace \mathcal{E}_+ generated by $\pi^{-\infty}(\mathcal{D}(S))\alpha$ is θ -positive. Here $\mathcal{D}(S) = C_c^\infty(S)$ denotes the space of test functions on S . In [Proposition 2.12](#), the main result of Section 2, we show that distribution cyclic representations are in one-to-one correspondence with reflection positive distributions on G .

Sections 3 and 4 are devoted to classifying reflection positive functions in the finite-dimensional abelian case. For the triple $(\mathbb{R}, -\text{id}, \mathbb{R}_+)$, we obtain complete information on reflection positive functions in terms of an integral representation, in which the building blocks are the functions $\varphi(x) = e^{-\lambda|x|}$, $\lambda \geq 0$, for which the associated \mathbb{R}_+ -representation on $\widehat{\mathcal{E}}$ is one-dimensional ([Proposition 3.1](#)). This generalizes results of A. Klein [27] obtained in the context of (OS)-positive covariance functions. We connect this case to [26] and the reflection positivity on the group $\text{SL}_2(\mathbb{R})$ by considering natural abelian subgroups. One should also note that the $ax + b$ -group, i.e., the affine group of the real line, is a subgroup of $\text{SL}_2(\mathbb{R})$, so that [26] also gives rise to reflection positivity for this group.

The classification of reflection positive distributions is rather subtle. Here one can only hope to get hold of the restrictions to S and, already for the open half line $S = \mathbb{R}_+ \subseteq \mathbb{R} = G$, the classification of reflection positive extensions of positive definite distributions on S to G seems to be a hopeless task. On the other hand, the restriction to S carries all information required for the representation of G^c , so that one is rather interested in “natural” extensions of distributions from S to G and not in all of them (cf. [43] for more examples). This motivates the main result of Section 4 which is a generalization of the Bochner–Schwartz Theorem ([Theorem 4.11](#)). For the case where $G = V$ is a vector space and $S = \Omega$ is an open \sharp -invariant convex cone, it provides an integral representation of positive definite distributions on S that lead to contraction representations of S . This is precisely the class of distributions showing up for the representations on the spaces $\widehat{\mathcal{E}}$. For $\tau = -\text{id}_V$ and open cones not containing affine lines we obtain the considerably stronger result that positive definite distributions are actually analytic functions ([Theorem 4.13](#)).

In the last two sections we construct reflection positive distribution vectors for some of the complementary series representations of the conformal group of \mathbb{R}^n , resp., its conformal compactification \mathbb{S}^n . This leads to a much simpler and clearer picture of these representations than the presentation in [25,26] which is based on more elaborate results on representations of semisimple Lie groups. Here we start from canonical kernel functions on \mathbb{R}^n and \mathbb{S}^n and relate them via the conformal compactification $\mathbb{R}^n \hookrightarrow \mathbb{S}^n$. This connects the kernel $Q(x, y) = (1 - \langle x, y \rangle)^{-s/2}$ on \mathbb{S}^n to the well-known kernel $\|x - y\|^{-s}$ on \mathbb{R}^n . We show that, for $\max(0, n - 2) \leq s < n$ and $s = 0$, this leads to a reflection positive distribution cyclic representation $(\pi_s, \mathcal{E}_s, \alpha)$. Here α can be represented by a point measure δ_x on the equator corresponding to the unit sphere \mathbb{S}^{n-1} in \mathbb{R}^n , the involution τ corresponds to the reflection in this sphere \mathbb{S}^{n-1} and S is the compression semigroup of the unit ball. A Cayley transform translates this into the time reflection $(x_0, \dots, x_{n-1}) \mapsto (-x_0, x_1, \dots, x_{n-1})$. It transforms the open unit ball into the open half space and the semigroup into the conformal compression semigroup of the half space.

In [8, Lemma 2.1] and [9, Lemma 3.1], Frank and Lieb give two different proofs of the reflection positivity of the distribution $\|x\|^{-s}$, $\max(0, n - 2) \leq s < n$ on \mathbb{R}^n with respect to reflections in a half space. They also use conformal invariance of the corresponding kernel to obtain results similar to our [Theorem 6.7](#). However, they do not consider representation theoretic aspects of their results, whereas our main point is that these distributions lead to H -invariant reflection positive distribution vectors for the complementary series.

It is worth pointing out that we have a tower of groups

$$\begin{array}{ccc}
 \mathrm{O}_{1,n+1}^+ & \longleftrightarrow & \mathrm{O}_{2,n} \\
 \cup & & \cup \\
 E_n & \longleftrightarrow & P_n \\
 \cup & & \cup \\
 \mathbb{R}^n & \longleftrightarrow & \mathbb{R}^{1,n-1}
 \end{array}$$

where the horizontal arrows stand for c -duality and the vertical \cup stands for inclusions. Hence the reflection positivity on the top line results in reflection positivity on each of the other levels. Reflection positivity at the top level is quite rare as our restriction of the parameter, $s = 0$ and $n - 2 \leq s < n$, shows. In particular, our construction does not allow for reflection positivity on the direct limit group $\mathrm{O}_{1,\infty}^+$. On the other hand, reflection positivity on the bottom line is quite common. Much less is known about the physically interesting part in the middle (cf. [30, 23,24]).

We would also like to point out that neither we nor the previous articles [25,26] discuss the interesting case of vector-valued complementary series representations. Our results on vector-valued kernels should provide some of the techniques necessary to treat this case.

0.1. Notation

We write $\mathbb{R}_+ :=]0, \infty[$ for the positive open half line. Elements of \mathbb{R}^n , $n \in \mathbb{N}$, are written $x = (x_0, x_1, \dots, x_{n-1})$, and $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_0 > 0\}$ is the open half space. The Euclidean inner product on \mathbb{R}^n is denoted $\langle x, y \rangle = \sum_{j=0}^{n-1} x_j y_j$, and

$$[x, y] := x_0 y_0 - x_1 y_1 - \dots - x_{n-1} y_{n-1} \tag{1}$$

is the canonical Lorentzian form on \mathbb{R}^n , turning it into the n -dimensional Minkowski space.

For a function $\varphi: G \rightarrow \mathbb{C}$ on a Lie group G , we use the notation

$$\tilde{\varphi}(g) := \overline{\varphi(g^{-1})} \quad \text{and} \quad \varphi^*(g) := \overline{\varphi(g^{-1})} \Delta_G(g)^{-1},$$

where Δ_G is the modular function of G , defined by

$$\Delta_G(y) \int_G f(xy) d\mu_G(x) = \int_G f(x) d\mu_G(x) \quad \text{for } f \in C_c(G), y \in G. \tag{2}$$

If τ is an involutive automorphism of G , then we also put

$$g^\sharp := \tau(g)^{-1} \quad \text{and} \quad \varphi^\sharp := \varphi^* \circ \tau.$$

For the Fourier transform of a measure μ on the dual V^* of a finite-dimensional real vector space V , we write

$$\widehat{\mu}(x) := \int_{V^*} e^{-i\alpha(x)} d\mu(\alpha). \tag{3}$$

The Fourier transform of an L^1 -function f on \mathbb{R}^n is defined by

$$\widehat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-i\langle \xi, x \rangle} dx. \tag{4}$$

For an involution τ on \mathbb{R}^n , we write elements of the dual space as (α_+, α_-) with $\alpha_{\pm} \circ \tau = \pm \alpha_{\pm}$ and define the corresponding Fourier–Laplace transform of a measure μ on \mathbb{R}^n by

$$\mathcal{FL}(\mu)(x) := \int_{\mathbb{R}^n} e^{-i\alpha_+(x)} e^{-\alpha_-(x)} d\mu(\alpha_+, \alpha_-). \tag{5}$$

For $\tau = \text{id}$ this is the Fourier transform $\mathcal{F}(\mu) = \widehat{\mu}$, and for $\tau = -\text{id}$, this is the Laplace transform $\mathcal{L}(\mu)$.

If M is a smooth manifold and $\mathcal{D}(M) = C_c^\infty(M, \mathbb{C})$ is the space of smooth compactly supported functions on M , endowed with its natural LF topology [52], then we write $\mathcal{D}'(M)$ for the space of continuous antilinear functionals on $\mathcal{D}(M)$; the distributions on M . We endow this space with the strong dual topology, i.e., the topology of uniform convergence on bounded subsets of $\mathcal{D}(M)$.

1. An abstract approach to reflection positivity

In this section we introduce a general concept of a reflection positive representation for a triple (G, τ, S) consisting of a group G , an involution τ on G and a subsemigroup $S \subseteq G$. To study reflection positive representations, it is crucial to control the way they are generated by simpler data. This leads to the concept of a reflection positive function.

1.1. Reflection positive representations

A symmetric group is a pair (G, τ) , consisting of a group G and an involutive automorphism τ of G , which is also allowed to be trivial, i.e., $\tau = \text{id}_G$. Let (G, τ) be a symmetric group and $S \subseteq G$ be a subsemigroup which is invariant under the involution $g \mapsto g^\sharp$. Then (S, \sharp) is an involutive semigroup, i.e.,

$$(s^\sharp)^\sharp = s \quad \text{and} \quad (st)^\sharp = t^\sharp s^\sharp \quad \text{for } s, t \in S.$$

In the following we will write $G_\tau := G \rtimes \{\mathbf{1}, \tau\}$.

Example 1.1. (a) Let (G, τ) be a symmetric finite-dimensional Lie group, H an open subgroup of $G^\tau := \{g \in G: \tau(g) = g\}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the decomposition of $\mathbf{L}(G)$ = \mathfrak{g} into $\mathbf{L}(\tau)$ -eigenspaces,

$$\mathfrak{h} = \mathfrak{g}^\tau = \ker(\mathbf{L}(\tau) - \mathbf{1}), \quad \mathfrak{q} = \mathfrak{g}^{-\tau} = \ker(\mathbf{L}(\tau) + \mathbf{1}).$$

If $S \subseteq G$ is a subsemigroup of the form $S = H \exp(C)$, where $C \subseteq \mathfrak{q}$ is an $\text{Ad}(H)$ -invariant convex cone (cf. [17]), then the invariance of S under \sharp follows from $(h \exp x)^\sharp = \exp x h^{-1} = h^{-1} \exp(\text{Ad}(h)x)$ for $h \in H, x \in \mathfrak{q}$.

(b) If G is an abelian group and $\tau(g) = g^{-1}$, then every subsemigroup of G is invariant under the involution $\sharp = \text{id}$.

Below we shall also encounter finite-dimensional vector spaces $G = (V, +)$ with an involution τ and open convex cones $\Omega \subseteq V$ invariant under \sharp .

Definition 1.2. Let \mathcal{E} be a Hilbert space and $\theta \in U(\mathcal{E})$ an involution. We call a closed subspace $\mathcal{E}_+ \subseteq \mathcal{E}$ θ -positive if $\langle v, v \rangle_\theta = \langle \theta v, v \rangle \geq 0$ for $v \in \mathcal{E}_+$. We then say that the triple $(\mathcal{E}, \mathcal{E}_+, \theta)$ is a reflection positive Hilbert space. In this case we write

$$\mathcal{N} := \{v \in \mathcal{E}_+ : \langle \theta v, v \rangle = 0\},$$

$q: \mathcal{E}_+ \rightarrow \mathcal{E}_+/\mathcal{N}, v \mapsto [v] = q(v)$ for the quotient map and $\widehat{\mathcal{E}}$ for the Hilbert completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the norm $\|[v]\| := \sqrt{\langle \theta v, v \rangle}$.

Definition 1.3. Let $(\mathcal{E}, \mathcal{E}_+, \theta)$ be a reflection positive Hilbert space. A unitary representation (π, \mathcal{E}) of G is said to be reflection positive with respect to the triple (G, τ, S) , if it extends to a unitary representation π of G_τ by $\pi(\tau) = \theta$, and $\pi(S)\mathcal{E}_+ \subseteq \mathcal{E}_+$. Note that the extendibility of a unitary representation of G to G_τ is equivalent to the existence of a unitary involution θ on \mathcal{E} satisfying $\theta\pi(g)\theta = \pi(\tau(g))$ for $g \in G$.

Lemma 1.4. If (π, \mathcal{H}) is a reflection positive representation of G on $(\mathcal{E}, \mathcal{E}_+, \theta)$, then $\widehat{\pi}(s)[v] := [\pi(s)v]$ defines a representation $(\widehat{\pi}, \widehat{\mathcal{E}})$ of the involutive semigroup (S, \sharp) by contractions.

Proof. (Cf. [26].) For $s \in S$ and $v, w \in \mathcal{E}_+$ we have

$$\langle \pi(s)v, w \rangle_\theta = \langle \theta\pi(s)v, w \rangle = \langle v, \pi(s^{-1})\theta w \rangle = \langle v, \theta\pi(s^\sharp)w \rangle = \langle v, \pi(s^\sharp)w \rangle_\theta,$$

so that the representation of S on \mathcal{E}_+ is involutive with respect to the hermitian form $\langle \cdot, \cdot \rangle_\theta$. Further,

$$\langle \pi(s)v, \pi(s)v \rangle_\theta = \langle \theta\pi(s)v, \pi(s)v \rangle \leq \|v\|$$

holds for every $s \in S$, so that [38, Lemma II.3.8] implies that the action of S on \mathcal{E}_+ induces a contraction representation $(\widehat{\pi}, \widehat{\mathcal{E}})$ of (S, \sharp) on the Hilbert space $\widehat{\mathcal{E}}$. \square

1.2. Operator-valued positive definite functions

A serious problem of the concept of a reflection positive representation is that it does not behave well under direct product decompositions in the sense that if a reflection positive representation π decomposes as $\pi_1 \oplus \pi_2$, then π_1 and π_2 need not be reflection positive because there may be no subspace \mathcal{E}_+ which is adapted to this decomposition. However, every non-zero element $v \in \mathcal{E}_+$ generates a cyclic reflection positive representation, and since it is natural to study only reflection positive representations generated by \mathcal{E}_+ , we first focus on reflection positive representations generated in a rather controlled fashion by a subspace $\mathcal{F} \subseteq \mathcal{E}_+$ (cf. Definition 1.9). First we recall some facts and basic concepts on positive definite kernels and functions.

Definition 1.5. Let X be a set and \mathcal{F} be a complex Hilbert space.

(a) A function $K: X \times X \rightarrow B(\mathcal{F})$ is called a $B(\mathcal{F})$ -valued kernel. A $B(\mathcal{F})$ -valued kernel K on X is said to be *positive definite* if, for every finite sequence $(x_1, v_1), \dots, (x_n, v_n)$ in $X \times \mathcal{F}$,

$$\sum_{j,k=1}^n \langle K(x_j, x_k)v_k, v_j \rangle \geq 0.$$

(b) If $(S, *)$ is an involutive semigroup, then a function $\varphi: S \rightarrow B(\mathcal{F})$ is called *positive definite* if the kernel $K_\varphi(s, t) := \varphi(st^*)$ is positive definite.

Positive definite kernels can be characterized as those for which there exists a Hilbert space \mathcal{H} and a function $\gamma: X \rightarrow B(\mathcal{H}, \mathcal{F})$ such that

$$K(x, y) = \gamma(x)\gamma(y)^* \quad \text{for } x, y \in X \tag{6}$$

(cf. [38, Thm. I.1.4]). Here one may assume that the vectors $\gamma(x)^*v, x \in X, v \in \mathcal{F}$, span a dense subspace of \mathcal{H} . If this is the case, then the pair (γ, \mathcal{H}) is called a *realization of K* . The map $\Phi: \mathcal{H} \rightarrow \mathcal{F}^X, \Phi(v)(x) := \gamma(x)v$, then realizes \mathcal{H} as a Hilbert subspace of \mathcal{F}^X with continuous point evaluations $\text{ev}_x: \mathcal{H} \rightarrow \mathcal{F}, f \mapsto f(x)$. Then $\Phi(\mathcal{H})$ is the unique Hilbert space in \mathcal{F}^X with continuous point evaluations ev_x , for which $K(x, y) = \text{ev}_x \text{ev}_y^*$ for $x, y \in X$. We write $\mathcal{H}_K \subseteq \mathcal{F}^X$ for this subspace and call it the *reproducing kernel Hilbert space with kernel K* .

Example 1.6 (Vector-valued GNS construction). (Cf. [38, Sect. 3.1].) Let (π, \mathcal{H}) be a representation of the unital involutive semigroup $(S, *)$, $\mathcal{F} \subseteq \mathcal{H}$ be a closed subspace for which $\pi(S)\mathcal{F}$ is total in \mathcal{H} and $P: \mathcal{H} \rightarrow \mathcal{F}$ denote the orthogonal projection. Then $\varphi(s) := P\pi(s)P^*$ is a $B(\mathcal{F})$ -valued positive definite function on S with $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{F}}$ because $\gamma(s) := P\pi(s) \in B(\mathcal{H}, \mathcal{F})$ satisfies

$$\gamma(s)\gamma(t)^* = P\pi(st^*)P^* = \varphi(st^*).$$

The map

$$\Phi: \mathcal{H} \rightarrow \mathcal{F}^S, \quad \Phi(v)(s) = \gamma(s)v = P\pi(s)v$$

is an S -equivariant realization of \mathcal{H} as the reproducing kernel space $\mathcal{H}_\varphi \subseteq \mathcal{F}^S$, on which S acts by right translation, i.e., $(\pi_\varphi(s)f)(t) = f(ts)$.

Conversely, let S be a unital involutive semigroup and $\varphi: S \rightarrow B(\mathcal{F})$ be a positive definite function with $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{F}}$. Write $\mathcal{H}_\varphi \subseteq \mathcal{F}^S$ for the corresponding reproducing kernel space and \mathcal{H}_φ^0 for the dense subspace spanned by $\text{ev}_s^*v, s \in S, v \in \mathcal{F}$. Then $(\pi_\varphi(s)f)(t) := f(ts)$ defines a $*$ -representation of S on \mathcal{H}_φ^0 . We say that φ is *exponentially bounded* if all operators $\pi_\varphi(s)$ are bounded, so that we actually obtain a representation of S by bounded operators on \mathcal{H}_φ . As $\mathbf{1}_{\mathcal{F}} = \varphi(\mathbf{1}) = \text{ev}_\mathbf{1} \text{ev}_\mathbf{1}^*$, the map $\text{ev}_\mathbf{1}^*: \mathcal{F} \rightarrow \mathcal{H}$ is an isometric inclusion, so that we may identify \mathcal{F} with a subspace of \mathcal{H} . Then $\text{ev}_\mathbf{1}: \mathcal{H} \rightarrow \mathcal{F}$ corresponds to the orthogonal projection onto \mathcal{F} and $\text{ev}_\mathbf{1} \circ \pi_\varphi(s) = \text{ev}_s$ leads to

$$\varphi(s) = \text{ev}_s \text{ev}_\mathbf{1}^* = \text{ev}_\mathbf{1} \pi_\varphi(s) \text{ev}_\mathbf{1}^*. \tag{7}$$

If $S = G$ is a group with $s^* = s^{-1}$, then φ is always exponentially bounded and the representation $(\pi_\varphi, \mathcal{H}_\varphi)$ is unitary.

Example 1.7. (a) Let (G, τ, S) be as above and X a set on which G_τ acts. We assume that $K: X \times X \rightarrow \mathbb{C}$ is a positive definite kernel for which G_τ acts unitarily on the corresponding reproducing kernel space $\mathcal{E} := \mathcal{H}_K \subseteq \mathbb{C}^X$ by

$$(\pi(g)f)(x) = J_g(x)f(g^{-1}.x),$$

where $J_g: X \rightarrow \mathbb{C}^\times$ is a function with $J_\tau = 1$. For the functions $K_x(y) := K(y, x)$ in \mathcal{E} we then have

$$\pi(g)K_x = \overline{J_{g^{-1}}(x)}K_{g.x} \quad \text{for } g \in G, x \in X \tag{8}$$

(cf. [38, Lemma II.4.1]).

Suppose that $\mathcal{D} \subseteq X$ is an S -invariant subset with the property that the kernel $K^\tau(x, y) := K(x, \tau y)$ is positive definite on \mathcal{D} . Then we consider the closed subspace

$$\mathcal{E}_+ := \overline{\text{span}\{K_x : x \in \mathcal{D}\}}$$

and observe that \mathcal{E}_+ is invariant under S by (8). From the relation

$$\langle \pi(\tau)K_y, K_x \rangle = \langle K_{\tau y}, K_x \rangle = K(x, \tau y) = K^\tau(x, y)$$

it now follows that \mathcal{E}_+ is a θ -positive subspace for the involution $\theta := \pi(\tau)$.

(b) In some cases the semigroup S can be obtained from \mathcal{D} . Suppose that X is a topological space, that G_τ acts continuously and that $\mathcal{D} \subseteq X$ is open with $\tau(\mathcal{D}) = (X \setminus \mathcal{D})^0$. Then the semigroup

$$S_{\mathcal{D}} := \{g \in G : g.\mathcal{D} \subseteq \mathcal{D}\}$$

is invariant under \sharp . In fact, $g\mathcal{D} \subseteq \mathcal{D}$ implies that $\tau(g)\tau\mathcal{D} \subseteq \tau\mathcal{D}$ and hence $g^\sharp(\tau\mathcal{D}) \supseteq \tau\mathcal{D}$. But as $\mathcal{D} = (X \setminus \tau\mathcal{D})^0$, this leads to $g^\sharp.\mathcal{D} \subseteq \mathcal{D}$.

1.3. Reflection positive functions

After these preparations, we now turn to a suitable concept of positive definite functions compatible with reflection positivity.

Definition 1.8. Let (G, τ) be a symmetric group, $S \subseteq G$ a subsemigroup invariant under $s^\sharp = \tau(s)^{-1}$ and \mathcal{F} be a Hilbert space. We call a function $\varphi: G \rightarrow B(\mathcal{F})$ *reflection positive* (with respect to S) if the following conditions are satisfied:

- (RP1) φ is positive definite,
- (RP2) $\varphi \circ \tau = \varphi$, and
- (RP3) $\varphi|_S$ is positive definite as a function on the involutive semigroup (S, \sharp) .

Definition 1.9. A triple $(\pi, \mathcal{E}, \mathcal{F})$, where (π, \mathcal{E}) is a unitary representation of G_τ and $\mathcal{F} \subseteq \mathcal{E}$ is a G -cyclic subspace fixed pointwise by $\theta = \pi(\tau)$ is said to be a *reflection positive \mathcal{F} -cyclic representation* if the closed subspace $\mathcal{E}_+ := \overline{\text{span} \pi(S)\mathcal{F}}$ is θ -positive.

If, in addition, $\mathcal{F} = \mathbb{C}v_o$ is one-dimensional, then we call the triple (π, \mathcal{E}, v_o) a *reflection positive cyclic representation*.

Remark 1.10. Suppose that G is a topological group and $S \subseteq G$ is such that $\mathbf{1} \in \bar{S}$. Let (π, \mathcal{E}) be a continuous unitary representation of G_τ , $\mathcal{F} \subseteq \mathcal{E}$ be a closed θ -invariant subspace and $\mathcal{E}_+ := \overline{\text{span} \pi(S)\mathcal{F}}$. Then $\pi(S)\mathcal{F} \subseteq \mathcal{E}_+$ implies that $\mathcal{F} \subseteq \mathcal{E}_+$. As \mathcal{F} is θ -invariant, each element $v \in \mathcal{F}$ can be written as $v = v_+ + v_-$ with $\theta v = v_+ - v_-$. If \mathcal{E}_+ is θ -positive, then \mathcal{F} is also θ -positive and we obtain $\theta v = v$ for each $v \in \mathcal{F}$. Therefore, in this context, the assumption that θ fixes every element of \mathcal{F} is equivalent to the θ -invariance of \mathcal{F} .

Proposition 1.11 (Reflection positive GNS construction). *Let (G, τ) be a symmetric group and $S \subseteq G$ be a \sharp -invariant subsemigroup.*

- (i) *If $(\pi, \mathcal{E}, \mathcal{F})$ is an \mathcal{F} -cyclic reflection positive representation of G_τ and $P: \mathcal{E} \rightarrow \mathcal{F}$ the orthogonal projection, then $\varphi(g) := P\pi(g)P^*$ is a reflection positive function on G with $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{F}}$.*
- (ii) *Let $\varphi: G \rightarrow B(\mathcal{F})$ is a reflection positive function on G with $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{F}}$ and $\mathcal{H}_\varphi \subseteq \mathcal{F}^G$ be the Hilbert subspace with reproducing kernel $K(x, y) := \varphi(xy^{-1})$ on which G acts by $(\pi_\varphi(g)f)(x) := f(xg)$ and τ by $(\tau f)(x) := f(\tau(x))$. We identify \mathcal{F} with the subspace $\text{ev}_1^* \mathcal{F} \subseteq \mathcal{H}_\varphi$. Then $(\pi_\varphi, \mathcal{H}_\varphi, \mathcal{F})$ is an \mathcal{F} -cyclic reflection positive representation and we have an S -equivariant unitary map*

$$\Gamma: \widehat{\mathcal{E}} \rightarrow \mathcal{H}_{\varphi|_S}, \quad \Gamma([f]) = f|_S.$$

Proof. (i) Clearly, φ is positive definite with $\varphi(\mathbf{1}) = \mathbf{1}$. That φ is τ -invariant follows from $\theta|_{\mathcal{F}} = \text{id}_{\mathcal{F}}$, which leads to $\theta P^* = P^*$ and thus to $P = P\theta$:

$$\varphi(\tau g \tau) = P\theta\pi(g)\theta P^* = P\pi(g)P = \varphi(g).$$

For $s \in S$ we have

$$\varphi(st^\sharp) = P\pi(s)\theta\pi(t^{-1})\theta P^* = P\pi(s)\theta \cdot \theta\pi(t)^* P^*,$$

so that

$$\varphi(st^\sharp) = \gamma(s)\gamma(t)^* \quad \text{for } \gamma(s) = P\pi(s)\theta.$$

Therefore φ is positive definite.

(ii) For the evaluation maps $\text{ev}_x: \mathcal{H}_\varphi \rightarrow \mathcal{F}$, $f \mapsto f(x)$ we have $\text{ev}_x \pi_\varphi(g) = \text{ev}_{xg}$. Therefore $\pi_\varphi(g)\mathcal{F} = \pi_\varphi(g)\text{ev}_1^* \mathcal{F} = \text{ev}_{g^{-1}}^* \mathcal{F}$ shows that \mathcal{F} is G -cyclic in \mathcal{H}_φ .

For $v \in \mathcal{F}$, the corresponding element $f \in \mathcal{H}_\varphi$ is the function $f(g) = \text{ev}_g \text{ev}_1^* v = \varphi(g)v$. Then

$$(\theta f)(g) = f(\tau(g)) = \text{ev}_{\tau(g)} \text{ev}_1^* v = \varphi(\tau(g))v = \varphi(g)v = f(g)$$

shows that $\theta|_{\mathcal{F}} = \text{id}_{\mathcal{F}}$.

To see that $\mathcal{E}_+ := \overline{\text{span } \pi_\varphi(S)\mathcal{F}}$ is θ -positive, we note that, for $v, w \in \mathcal{F}$ and $s, t \in S$, we have

$$\begin{aligned} \langle \theta \pi_\varphi(s) \text{ev}_1^* v, \pi_\varphi(t) \text{ev}_1^* w \rangle &= \langle \pi_\varphi(t)^* \theta \pi_\varphi(s) \text{ev}_1^* v, \text{ev}_1^* w \rangle = \langle \pi_\varphi(t^\sharp s) \text{ev}_1^* v, \theta \text{ev}_1^* w \rangle \\ &= \langle \text{ev}_{s^{-1}\tau(t)}^* v, \text{ev}_1^* w \rangle = \langle \text{ev}_1 \text{ev}_{s^{-1}\tau(t)}^* v, w \rangle = \langle \varphi(t^\sharp s) v, w \rangle. \end{aligned}$$

As $\varphi|_S$ is positive definite on (S, \sharp) , it follows that \mathcal{E}_+ is θ -positive.

Since $\theta|_{\mathcal{F}} = \text{id}_{\mathcal{F}}$, the hermitian form $\langle \cdot, \cdot \rangle_\theta$ restricts on \mathcal{F} to the original scalar product, so that we obtain an isometric inclusion $\iota: \mathcal{F} \rightarrow \widehat{\mathcal{E}}$. As $\widehat{\pi}_\varphi(S)\iota(\mathcal{F}) = [\pi_\varphi(S)\mathcal{F}]$, the subspace $\iota(\mathcal{F})$ is S -cyclic in $\widehat{\mathcal{E}}$, and for $s \in S$ and $v, w \in \mathcal{F}$ we have

$$\langle \iota^* \widehat{\pi}_\varphi(s) \iota(v), w \rangle = \langle \widehat{\pi}_\varphi(s) \iota(v), \iota(w) \rangle = \langle \theta \pi_\varphi(s) v, w \rangle = \langle \pi_\varphi(s) v, w \rangle,$$

so that

$$\iota^* \widehat{\pi}_\varphi(s) \iota = \text{ev}_1 \pi_\varphi(s) \text{ev}_1^* = \text{ev}_s \text{ev}_1^* = \varphi(s).$$

This proves that the map

$$\Gamma: \widehat{\mathcal{E}} \rightarrow \mathcal{F}^S, \quad \Gamma(v)(s) := \iota^* \widehat{\pi}_\varphi(s) v$$

defines an S -equivariant isomorphism $\widehat{\mathcal{E}} \rightarrow \mathcal{H}_{\varphi|_S}$.

For $v \in \mathcal{F}$ and $s \in S$ we further have

$$\Gamma([\pi_\varphi(s)v])(t) = \iota^* \widehat{\pi}_\varphi(t) [\pi_\varphi(s)v] = \iota^* [\pi_\varphi(ts)v] = \text{ev}_1 \pi_\varphi(ts) v = \text{ev}_t \pi_\varphi(s) v,$$

which implies that

$$\Gamma([f])(t) = f(t) \quad \text{for } f \in \mathcal{E}_+, \quad t \in S.$$

Hence the natural map $\mathcal{H}_\varphi \supseteq \mathcal{E}_+ \rightarrow \mathcal{H}_{\varphi|_S}$ is simply given by restriction to S . \square

Corollary 1.12. *Let (G, τ) be a symmetric group and $S \subseteq G$ be a \sharp -invariant subsemigroup.*

- (i) *If (π, \mathcal{E}, ν) is a cyclic reflection positive representation of G_τ , then $\pi^\nu(g) := \langle \pi(g)v, \nu \rangle$ is a reflection positive function on G .*
- (ii) *If φ is a reflection positive function on G , then $(\pi_\varphi, \mathcal{H}_\varphi, \varphi)$ is a cyclic reflection positive representation.*

2. Reflection positive distributions

As we shall see below and in the sequels [42,43], in some cases we have to pass from reflection positive functions to the more general class of reflection positive distributions. This applies in particular to representations that are most naturally realized in spaces of distributions on a homogeneous space G/H . For non-compact subgroups H , we are thus forced to consider unitary representations with H -invariant distribution vectors. Accordingly, we have to study reflection positive distributions in addition to reflection positive functions.

2.1. Positive definite distributions

In the following we write $\mathcal{D}(M) = C_c^\infty(M, \mathbb{C})$ for the space of compactly supported smooth functions on a manifold M and endow this space with the usual LF topology, i.e., the locally convex direct limit of the Fréchet spaces $\mathcal{D}_X(M)$ of test functions supported in the compact subset $X \subseteq M$ (cf. [52]). Its antidual, i.e., the space of continuous antilinear functionals on $\mathcal{D}(M)$ is the space $\mathcal{D}'(M)$ of *distributions on M* .

Definition 2.1. (a) If G is a Lie group, then $\mathcal{D}(G)$ is an involutive algebra with respect to the convolution product and $\varphi^*(g) := \overline{\varphi(g^{-1})} \Delta_G(g^{-1})$, where Δ_G is the modular function from (2). Accordingly, we call a distribution $D \in \mathcal{D}'(G)$ *positive definite*, if it is a positive functional on this algebra, i.e.,

$$D(\varphi^* * \varphi) \geq 0 \quad \text{for } \varphi \in \mathcal{D}(G). \tag{9}$$

(b) If τ is an involution on G and $S \subseteq G$ an open \sharp -invariant subsemigroup, then $\mathcal{D}(S)$ is a $*$ -algebra with respect to the convolution product and the $*$ -operation $\varphi^\sharp = \varphi^* \circ \tau$. Accordingly, we call a distribution $D \in \mathcal{D}'(S)$ *positive definite* if

$$D(\varphi^\sharp * \varphi) \geq 0 \quad \text{for } \varphi \in \mathcal{D}(S). \tag{10}$$

Remark 2.2. If $D_h(\varphi) = \int_G \overline{\varphi(g)} h(g) d\mu_G(g)$ holds for a locally integrable function h on G , then

$$\begin{aligned} D_h(\varphi^* * \varphi) &= \int_G \int_G \varphi(x^{-1}) \Delta_G(x)^{-1} \overline{\varphi(x^{-1}y)} h(y) d\mu_G(x) d\mu_G(y) \\ &= \int_G \int_G \varphi(x) \overline{\varphi(xy)} h(y) d\mu_G(x) d\mu_G(y) \\ &= \int_G \int_G \varphi(x) \overline{\varphi(y)} h(x^{-1}y) d\mu_G(x) d\mu_G(y). \end{aligned}$$

If h is continuous and positive definite, this formula implies that D_h is a positive definite distribution. One can easily see that, conversely, h is positive definite if D_h is.

Definition 2.3 (*Distribution vectors*). Let (π, \mathcal{H}) be a continuous unitary representation of the Lie group G on the Hilbert space \mathcal{H} . We write \mathcal{H}^∞ for the linear subspace of smooth vectors, i.e., of all elements $v \in \mathcal{H}$ for which the orbit map $\pi^v: G \rightarrow \mathcal{H}, g \mapsto \pi(g)v$ is smooth. Identifying \mathcal{H}^∞ with the closed subspace of equivariant maps in the Fréchet space $C^\infty(G, \mathcal{H})$, we obtain a natural Fréchet space structure on \mathcal{H}^∞ for which the G -action on this space is smooth and the inclusion $\mathcal{H}^\infty \rightarrow \mathcal{H}$ (corresponding to evaluation in $\mathbf{1} \in G$) is a continuous linear map (cf. [35,39]).

We write $\mathcal{H}^{-\infty}$ for the space of continuous antilinear functionals on \mathcal{H}^∞ , the space of *distribution vectors*, and note that we have a natural linear embedding $\mathcal{H} \hookrightarrow \mathcal{H}^{-\infty}, v \mapsto \langle v, \cdot \rangle$. Accordingly, we also write $\langle \alpha, v \rangle = \overline{\langle v, \alpha \rangle}$ for $\alpha(v)$ for $\alpha \in \mathcal{H}^{-\infty}$ and $v \in \mathcal{H}^\infty$. The group G acts naturally on $\mathcal{H}^{-\infty}$ by

$$(\pi^{-\infty}(g)\alpha)(v) := \alpha(\pi(g)^{-1}v),$$

so that we obtain a G -equivariant chain of continuous inclusions

$$\mathcal{H}^\infty \subseteq \mathcal{H} \subseteq \mathcal{H}^{-\infty} \tag{11}$$

(cf. [5, Sect. 8.2]). It is $\mathcal{D}(G)$ -equivariant, if we define the representation of $\mathcal{D}(G)$ on $\mathcal{H}^{-\infty}$ by

$$(\pi^{-\infty}(\varphi)\alpha)(v) := \int_G \varphi(g)\alpha(\pi(g)^{-1}v) d\mu_G(g) = \alpha(\pi(\varphi^*)v).$$

Remark 2.4. An important point is that, for any $\varphi \in \mathcal{D}(G)$ and $\alpha \in \mathcal{H}^{-\infty}$, we have $\pi^{-\infty}(\varphi)\alpha \in \mathcal{H}^\infty$ in the sense of (11). To see this, we first note that $\pi(\varphi)$ defines a continuous linear map $\mathcal{H} \rightarrow \mathcal{H}^\infty$, so that its adjoint maps $\mathcal{H}^{-\infty}$ into \mathcal{H} and then apply the Dixmier–Malliavin Theorem [6, Thm. 3.1], asserting that

$$\mathcal{D}(G) = \mathcal{D}(G) * \mathcal{D}(G).$$

Further, the map

$$\gamma_\alpha: \mathcal{D}(G) \rightarrow \mathcal{H}^\infty, \quad \varphi \mapsto \pi^{-\infty}(\varphi)\alpha$$

is continuous because it is continuous on each Fréchet space $\mathcal{D}_X(G)$, $X \subseteq G$ compact, on which it follows from the Closed Graph Theorem and the continuity of the orbit maps $\mathcal{D}(G) \rightarrow \mathcal{H}$. Therefore

$$\pi^\alpha(\varphi) := \langle \alpha, \pi^{-\infty}(\varphi)\alpha \rangle \tag{12}$$

defines a distribution on G . That this distribution is positive definite follows from

$$\pi^\alpha(\varphi^* * \varphi) = \langle \alpha, \pi^{-\infty}(\varphi^* * \varphi)\alpha \rangle = \langle \pi^{-\infty}(\varphi)\alpha, \pi^{-\infty}(\varphi)\alpha \rangle \geq 0. \tag{13}$$

Definition 2.5. We say that $\alpha \in \mathcal{H}^{-\infty}$ is *cyclic* if $\pi^{-\infty}(\mathcal{D}(G))\alpha$ is a dense subspace of \mathcal{H} .

Theorem 2.6. (See [5, Thm. 8.2.1].) *Let (π, \mathcal{H}) be a continuous unitary representation of the Lie group G .*

- (i) *For each distribution vector $\alpha \in \mathcal{H}^{-\infty}$, there exists a G -equivariant continuous linear map*

$$\eta_\alpha: \mathcal{H} \rightarrow \mathcal{D}'(G), \quad \eta_\alpha(v)(\varphi) := \langle v, \pi^{-\infty}(\varphi)\alpha \rangle = \langle \pi(\varphi^*)v, \alpha \rangle,$$

which is injective if and only if α is cyclic. This establishes a one-to-one correspondence between distribution vectors and G -equivariant continuous linear maps $\mathcal{H} \rightarrow \mathcal{D}'(G)$.

(ii) *The assignment*

$$\alpha \mapsto \eta_\alpha^*, \quad \eta_\alpha^*(\varphi) := \pi^{-\infty}(\varphi)\alpha$$

establishes a one-to-one correspondence between distribution vectors and G -equivariant continuous linear maps $\mathcal{D}(G) \rightarrow \mathcal{H}$.

(iii) *The map η_α extends to*

$$\tilde{\eta}_\alpha: \mathcal{H}^{-\infty} \rightarrow \mathcal{D}'(G), \quad \tilde{\eta}_\alpha(\beta)(\varphi) := \langle \beta, \pi^{-\infty}(\varphi)\alpha \rangle.$$

We will now discuss how every positive definite distribution $D \in \mathcal{D}'(G)$ leads to a unitary representation (π, \mathcal{H}) with cyclic distribution vector $\alpha \in \mathcal{H}^{-\infty}$ which defines an embedding $\mathcal{H} \hookrightarrow \mathcal{D}'(G)$. First we recall some functional analytic facts from [52].

Remark 2.7. LF spaces are *barreled*, i.e., all closed absolutely convex absorbing subsets (the barrels) are 0-neighborhoods. This applies in particular to $\mathcal{D}(M)$ for any σ -compact manifold M [52, p. 347]. It follows from [52, Prop. 34.4] that $\mathcal{D}(M)$ is a *Montel space*, which means that it is barreled and every bounded closed subset of $\mathcal{D}(M)$ is compact. This implies that $\mathcal{D}(M)$ is reflexive [52, p. 376] and that every weakly (= weak-*) convergent sequence in $\mathcal{D}'(M)$ converges [52, p. 358].

For the following proposition we recall that the vector-valued GNS construction (Example 1.6) yields for every positive definite distribution $D \in \mathcal{D}'(G)$ a corresponding Hilbert space \mathcal{H}_D which is contained in the space $\mathcal{D}(G)^*$ of all antilinear functionals on $\mathcal{D}(G)$.

Proposition 2.8. *Let $D \in \mathcal{D}'(G)$ be a positive definite distribution on the Lie group G and \mathcal{H}_D be the corresponding reproducing kernel Hilbert space with kernel $K(\varphi, \psi) := D(\psi^* * \varphi)$ obtained by completing $\mathcal{D}(G) * D$ with respect to the scalar product $\langle \psi * D, \varphi * D \rangle = D(\psi^* * \varphi)$. Then the following assertions hold:*

- (i) $\mathcal{H}_D \subseteq \mathcal{D}'(G)$ and the inclusion $\gamma_D: \mathcal{H}_D \rightarrow \mathcal{D}'(G)$ is continuous.
- (ii) We have a unitary representation (π_D, \mathcal{H}_D) of G by $\pi_D(g)E = g_*E$, where $(g_*E)(\varphi) := E(\varphi \circ \lambda_g)$ and the integrated representation of $\mathcal{D}(G)$ on \mathcal{H}_D is given by $\pi_D(\varphi)E = \varphi * E$.
- (iii) There exists a unique distribution vector $\alpha_D \in \mathcal{H}_D^{-\infty}$ with $\alpha_D(\varphi * D) = D(\varphi)$ and $\pi^{-\infty}(\varphi)\alpha_D = \varphi * D$ for $\varphi \in \mathcal{D}(G)$. It satisfies $\pi^{\alpha_D} = D$.
- (iv) γ_D extends to a $\mathcal{D}(G)$ -equivariant injection $\mathcal{H}_D^{-\infty} \hookrightarrow \mathcal{D}'(G)$ mapping α_D to D .

Proof. (i) The relation $K(\psi, \varphi) = \overline{K(\varphi, \psi)}$ implies $D(\varphi^*) = \overline{D(\varphi)}$ for $\varphi \in \mathcal{D}(G)$. The functionals $\psi * D, \psi \in \mathcal{D}(G)$, span a pre-Hilbert space $\mathcal{H}_D^0 \subseteq \mathcal{D}'(G)$ with inner product $\langle \psi * D, \varphi * D \rangle = K(\varphi, \psi)$ on which we have a $*$ -representation of $\mathcal{D}(G)$, given by

$$\pi_D(\varphi)E := \varphi * E.$$

We claim that the corresponding inclusion map $\gamma_D: \mathcal{H}_D^0 \hookrightarrow \mathcal{D}'(G)$ is continuous with respect to the pre-Hilbert structure on \mathcal{H}_D^0 . Since \mathcal{H}_D^0 is metrizable, it suffices to show that γ_D is sequentially continuous, so that Remark 2.7 further implies that it suffices to verify weak continuity. This follows immediately from

$$\gamma_D(\psi * D)(\varphi) = (\psi * D)(\varphi) = D(\psi^* * \varphi) \tag{14}$$

because the multiplication on $\mathcal{D}(G)$ is separately continuous.³ Since $\mathcal{D}'(G)$ is complete (it is the strong dual of an LF space; [52, p. 344]), γ_D extends to a continuous linear map $\gamma_D: \mathcal{H}_D \rightarrow \mathcal{D}'(G)$ on the Hilbert space completion \mathcal{H}_D of \mathcal{H}_D^0 .

(ii) In view of

$$(\varphi * T)(\psi) = T(\varphi^* * \psi) = \int_G \varphi(g)T(\psi \circ \lambda_g) d\mu_G(g),$$

the representation of $\mathcal{D}(G)$ on \mathcal{H}_D^0 corresponds to the unitary G -representation defined by

$$(\pi_D(g)T)(\psi) := T(\psi \circ \lambda_g),$$

which is the dual of the left regular representation of G on $\mathcal{D}(G)$. From the unitarity of the G -representation on \mathcal{H}_D^0 it follows that it extends to a unitary representation (π_D, \mathcal{H}_D) whose continuity follows from the continuity of the G -orbit maps in $\mathcal{D}(G)$.

(iii), (iv) Clearly, the inclusion γ_D is G -equivariant, so that [Theorem 2.6](#) implies the existence of a uniquely determined cyclic distribution vector $\alpha_D \in \mathcal{H}_D^{-\infty}$ satisfying

$$\gamma_D(E)(\varphi) = \langle E, \pi^{-\infty}(\varphi)\alpha_D \rangle = \langle \varphi^* * E, \alpha_D \rangle.$$

For $E = \psi * D$, this leads to

$$D(\psi^* * \varphi) \stackrel{(14)}{=} \gamma_D(\psi * D)(\varphi) = \langle \varphi^* * \psi * D, \alpha_D \rangle.$$

In view of the Dixmier–Malliavin Theorem [6, [Thm. 3.1](#)], we further get

$$\alpha_D(\varphi * D) = \overline{D(\varphi^*)} = D(\varphi). \tag{15}$$

We also obtain

$$\langle \psi * D, \varphi * D \rangle = D(\psi^* * \varphi) = \langle \varphi^* * \psi * D, \alpha_D \rangle = \langle \psi * D, \pi^{-\infty}(\varphi)\alpha_D \rangle,$$

which leads to

$$\pi^{-\infty}(\varphi)\alpha_D = \varphi * D. \tag{16}$$

The map $\beta_D: \mathcal{D}(G) \rightarrow \mathcal{H}_D, \varphi \mapsto \varphi * D$ is $\mathcal{D}(G)$ -equivariant and continuous⁴ and (15) means that

$$\alpha_D \circ \beta_D = D. \tag{17}$$

³ That the convolution product on $\mathcal{D}(G)$ is jointly continuous for every Lie group G with at most finitely many connected components has been shown recently by Birth and Glöckner in [2].

⁴ It suffices to verify continuity on the Fréchet subspaces $\mathcal{D}_X(G)$, where $X \subseteq G$ is a compact subset. On these subspaces, the continuity follows from the Closed Graph Theorem and the continuity of the maps $\varphi \mapsto \langle \beta_D(\varphi), \psi * D \rangle = \langle \varphi * D, \psi * D \rangle = D(\varphi^* * \psi)$.

Actually β_D defines a continuous linear map $\mathcal{D}(G) \rightarrow \mathcal{H}_D^\infty$ with dense range whose adjoint yields an inclusion

$$\beta_D^*: \mathcal{H}_D^{-\infty} \hookrightarrow \mathcal{D}'(G),$$

i.e., the inclusion $\gamma_D: \mathcal{H}_D \hookrightarrow \mathcal{D}'(G)$ even extends to the larger space of distribution vectors. Here (17) means that $\beta_D^* \alpha_D = D$. Finally, (15) implies

$$\pi^{\alpha_D}(\varphi) = \langle \alpha_D, \pi^{-\infty}(\varphi) \alpha_D \rangle = \langle \alpha_D, \varphi * D \rangle \stackrel{(15)}{=} D(\varphi),$$

i.e., $\pi^{\alpha_D} = D$. \square

Example 2.9. If $G = V$ is a real vector group (isomorphic to \mathbb{R}^n), then the Bochner–Schwartz Theorem [49, Thm. XVIII, §VII.9] asserts that a distribution $D \in \mathcal{D}'(V)$ is positive definite if and only if there exists a tempered positive measure μ on V^* with $D = \widehat{\mu}$ (the Fourier transform), i.e.,

$$D(\varphi) = \int_{V^*} \widehat{\varphi}(\alpha) d\mu(\alpha) \quad \text{and} \quad D(\psi^* * \varphi) = \langle \widehat{\psi}, \widehat{\varphi} \rangle_{L^2(V^*, \mu)}.$$

From this it follows that $\widehat{\psi} \mapsto \psi * D$ extends to a unitary map $L^2(V^*, \mu) \rightarrow \mathcal{H}_D$. It intertwines the unitary representation of V on $L^2(V^*, \mu)$ given by

$$(U_v f)(\alpha) := e^{-i\alpha(v)} f(\alpha), \quad f \in L^2(V^*, \mu), \quad v \in V, \quad \alpha \in V^*$$

with the translation action of V on $\mathcal{D}'(V)$.

Since $D = \widehat{\mu} \in \mathcal{S}'(V)$ is a tempered distribution, the construction in Proposition 2.8 actually leads to a continuous inclusion $\gamma_D: \mathcal{H}_D \hookrightarrow \mathcal{S}'(V)$.

2.2. Reflection positivity for distributions

After this brief discussion of representations on Hilbert subspaces of $\mathcal{D}'(G)$, we now turn to reflection positive distributions on Lie groups and the corresponding unitary representations.

Definition 2.10. If (G, τ) is a symmetric Lie group and S is open and \sharp -invariant, then we call a distribution $D \in \mathcal{D}'(G)$ *reflection positive* for (G, τ, S) if the following conditions are satisfied:

- (RP1) D is positive definite, i.e., $D(\varphi^* * \varphi) \geq 0$ for $\varphi \in \mathcal{D}(G)$.
- (RP2) $\tau D = D$, i.e., $D(\varphi \circ \tau) = D(\varphi)$ for $\varphi \in \mathcal{D}(G)$, and
- (RP3) $D|_S$ is positive definite as a distribution on the involutive semigroup (S, \sharp) , i.e., $D(\varphi^\sharp * \varphi) \geq 0$ for $\varphi \in \mathcal{D}(S)$.

Definition 2.11. A triple $(\pi, \mathcal{E}, \alpha)$, where (π, \mathcal{E}) is a unitary representation of G_τ and $\alpha_o \in \mathcal{E}^{-\infty}$ a cyclic distribution vector fixed under $\theta := \pi(\tau)$, is said to be a *reflection positive distribution* or *cyclic representation* if the closed subspace $\mathcal{E}_+ := \text{span } \pi^{-\infty}(\mathcal{D}(S))\alpha_o$ is θ -positive.

Proposition 2.12. For (G, τ, S) as above, the following assertions hold:

- (i) If $(\pi, \mathcal{E}, \alpha)$ is a distribution cyclic reflection positive representation of G_τ , then $\pi^\alpha(\varphi) := \alpha(\pi^{-\infty}(\varphi)\alpha)$ is a reflection positive distribution on G .
- (ii) If D is a reflection positive distribution on G , then $(\pi_D, \mathcal{H}_D, D)$ is a reflection positive distribution cyclic representation, where τ acts on $\mathcal{D}'(G)$ by $(\tau E)(\varphi) := E(\varphi \circ \tau)$. We further have an S -equivariant unitary map

$$\Gamma: \widehat{\mathcal{E}} \rightarrow \mathcal{H}_{D|_S} \subseteq \mathcal{D}'(S), \quad \Gamma([\alpha]) = \alpha|_S.$$

Proof. (i) We have already seen in (13) that the distribution π^α is positive definite. That π^α is τ -invariant follows from the θ -invariance of α :

$$\begin{aligned} \pi^\alpha(\varphi \circ \tau) &= \langle \alpha, \pi^{-\infty}(\varphi \circ \tau)\alpha \rangle = \langle \alpha, \pi(\tau)\pi^{-\infty}(\varphi)\pi^{-\infty}(\tau)\alpha \rangle \\ &= \langle \alpha, \pi^{-\infty}(\varphi)\alpha \rangle = \pi^\alpha(\varphi). \end{aligned}$$

For $\varphi \in \mathcal{D}(S)$ we further have

$$\begin{aligned} \pi^\alpha(\varphi^\sharp * \varphi) &= \langle \alpha, \pi^{-\infty}(\varphi^\sharp)\pi^{-\infty}(\varphi)\alpha \rangle = \langle \pi^{-\infty}(\varphi \circ \tau)\alpha, \pi^{-\infty}(\varphi)\alpha \rangle \\ &= \langle \theta\pi^{-\infty}(\varphi)\theta\alpha, \pi^{-\infty}(\varphi)\alpha \rangle = \langle \pi^{-\infty}(\varphi)\alpha, \pi^{-\infty}(\varphi)\alpha \rangle_\theta \geq 0, \end{aligned}$$

so that π^α is reflection positive.

(ii) The τ -invariance of D implies the τ -invariance of the kernel K , so that $(\theta E)(\varphi) := E(\varphi \circ \tau)$ defines a unitary operator on \mathcal{H}_D (cf. [38, Rem. II.4.5]), and we thus obtain a unitary representation π_D of G_τ on \mathcal{H}_D with $\pi_D(\tau) = \theta$.

In Proposition 2.8 we already observed that $D \in \mathcal{H}_D^{-\infty}$ is a cyclic distribution vector, and its θ -invariance follows from the τ -invariance of D . Finally, we note that any $\varphi \in \mathcal{D}(S)$ satisfies

$$\langle \theta K_\varphi, K_\varphi \rangle = \langle K_{\varphi \circ \tau}, K_\varphi \rangle = K(\varphi, \varphi \circ \tau) = D(\varphi^\sharp * \varphi) \geq 0.$$

Therefore the positive definiteness of $D|_S$ implies that the closed subspace \mathcal{E}_+ generated by the elements $K_\varphi, \varphi \in \mathcal{D}(S)$, is θ -positive. The existence of Γ follows as in Proposition 1.11(ii). \square

Lemma 2.13. For $s \in]-\infty, n[$, the function $\|x\|^{-s}$ on \mathbb{R}^n is locally integrable, hence defines a distribution $D_s \in \mathcal{D}'(\mathbb{R}^n)$. It is positive definite if and only if $s \geq 0$.

Proof. Using polar coordinates, one immediately sees that, for $r := \|x\|$, the function r^{-s} is locally integrable if and only if $s < n$.

For, $0 < s < n$, it follows from [49, Ex. VII.7.13] that the Fourier transform of r^{-s} is given by

$$\mathcal{F}(r^{-s}) = \pi^{s-n/2} \frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2})} r^{s-n}.$$

Since $0 < s < n$ implies $-n < s - n < 0$ and the Γ -factors are positive in this range, this formula shows that $\mathcal{F}(r^{-s})$ is a positive tempered measure, hence that r^{-s} is positive definite.

For $s \leq 0$, the function r^{-s} is continuous and real-valued. If it is positive definite, it has a maximal value at $x = 0$, which is only the case for $s = 0$. Then $r^s = 1$, which is obviously positive definite. \square

Example 2.14. The preceding lemma and Proposition 2.8 imply that we have a map $\beta: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ given by

$$\beta(\varphi)(\psi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\varphi(x)\overline{\psi(y)}}{\|x - y\|^s} dx dy$$

(cf. Remark 2.2). The completion of $\beta(\mathcal{D}(\mathbb{R}^n))$ is a translation invariant Hilbert subspace $\mathcal{H}_{D_s} \subseteq \mathcal{D}'(\mathbb{R}^n)$ and we even obtain an equivariant embedding $\mathcal{H}_{D_s}^{-\infty} \subseteq \mathcal{D}'(\mathbb{R}^n)$.

To connect this with our discussion of reflection positivity, let τ be the “time” reflection $\tau(x_0, x_1, \dots, x_{n-1}) = (-x_0, x_1, \dots, x_{n-1})$ mapping the open half space

$$\mathbb{R}_+^n = \{(x_0, y) : x_0 > 0, y \in \mathbb{R}^{n-1}\}$$

onto $-\mathbb{R}_+^n$, keeping the bounding hyperplane pointwise fixed. The open half space $S = \mathbb{R}_+^n$ is an involutive semigroup with respect to the involution $(x_0, y)^\# = (x_0, -y)$. Then $D_s \in \mathcal{H}_{D_s}^{-\infty}$ represents a cyclic distribution vector which is fixed under θ because $\|\tau(x)\| = \|x\|$ for $x \in \mathbb{R}^n$. We shall see below that D_s is reflection positive for $s = 0$ and $\max(0, n - 2) \leq s < n$ (cf. Proposition 6.1).

3. Reflection positivity on the real line

A particular special case, where it is interesting to study reflection positivity is the subsemigroup $S =]0, \infty[= \mathbb{R}_+$ of the real line $G = \mathbb{R}$. Here we consider the involution $\tau(x) = -x$, so that S is invariant under the involution $s^\# = s$. In this case we obtain a complete description of the reflection positive functions in terms of an integral representation.

3.1. Reflection positive operator-valued functions on \mathbb{R}

We start with a characterization of continuous reflection positive functions for $(G, \tau, S) = (\mathbb{R}, -\text{id}_{\mathbb{R}}, \mathbb{R}_+)$.

Proposition 3.1. *Let \mathcal{F} be a Hilbert space and $\varphi: \mathbb{R} \rightarrow B(\mathcal{F})$ be positive definite and strongly continuous. Then φ is reflection positive if and only if there exists a finite $\text{Herm}(\mathcal{F})_+$ -valued Borel measure Q on $[0, \infty[$ such that*

$$\varphi(x) = \int_0^\infty e^{-\lambda|x|} dQ(\lambda). \tag{18}$$

Proof. Suppose first that φ is reflection positive for $(\mathbb{R}, -\text{id}, \mathbb{R}_+)$ and put $S := (\mathbb{R}_+, +)$. Then $\varphi_S := \varphi|_S$ is positive definite with respect to the trivial involution and corresponds to a contraction representation of S because $|\langle \varphi(s)v, v \rangle| \leq \langle \varphi(\mathbf{1})v, v \rangle$ holds for the positive definite functions $x \mapsto \langle \varphi(x)v, v \rangle$, $v \in \mathcal{F}$, on \mathbb{R} [38, Cor. III.1.20(ii)]. Hence there exists a unique finite $\text{Herm}(\mathcal{F})_+$ -valued Borel measure Q on $[0, \infty[$ such that

$$\varphi(x) = \int_0^\infty e^{-\lambda x} dQ(\lambda) \quad \text{for } x > 0$$

[37, Cor. IV.4]. The Dominated Convergence Theorem and the (strong) continuity of φ then imply

$$\varphi(0) = Q([0, \infty[) = \int_0^\infty dQ(\lambda).$$

Now (18) follows from the fact that $\varphi(-x) = \varphi(x)^* = \varphi(x)$ holds for $x \geq 0$.

For the converse, we assume that φ has an integral representation as in (18). This immediately implies that $\varphi|_S$ is positive definite on S for the involution $s^\sharp = s$ and that φ is continuous [37, Prop. II.11]. We have to show that φ is positive definite. Since the cone of positive definite functions is closed under pointwise limits, it suffices to show that, for $\lambda > 0$ and $A \in \text{Herm}(\mathcal{F})_+$, the function $\varphi_\lambda(x) := e^{-\lambda|x|}A$ is positive definite on \mathbb{R} . For $\lambda = 0$ this is trivial, and for $\lambda > 0$ it follows from

$$\varphi_\lambda(x) = e^{-\lambda|x|}A = \int_{\mathbb{R}} e^{ixy} A d\mu_\lambda(y), \quad \text{where } d\mu_\lambda(y) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + y^2} dy \tag{19}$$

is the Cauchy distribution [1, §47]. \square

Remark 3.2. (a) For the special case where $\varphi(0) = \mathbf{1}$, strongly continuous reflection positive functions are called (OS)-positive covariance functions in [27], and the preceding result specializes to [27, Rem. 2.7] in the following sense.

Using (7) to write $\varphi(s) = \text{ev}_1 \circ \pi_\varphi(s) \circ \text{ev}_1^*$ and representing π_φ by a spectral measure P on $[0, \infty[$ as

$$\pi_\varphi(s) = \int_0^\infty e^{-\lambda s} dP(\lambda),$$

we obtain the integral representation of φ with $Q(E) := \text{ev}_1 \circ P(E) \circ \text{ev}_1^*$ for any Borel subset $E \subseteq [0, \infty[$.

(b) The proof above implies in particular that every bounded strongly continuous $B(\mathcal{F})$ -valued positive definite function $\varphi: \mathbb{R}_+ \rightarrow B(\mathcal{F})$ extends by $\tilde{\varphi}(x) := \varphi(|x|)$ to a positive definite function on \mathbb{R} . Clearly, $\tilde{\varphi}$ is reflection positive. This observation can also be found in [51, §I.8.2, p. 30]. If φ is a representation, i.e., a one-parameter semigroup of contractions, then the unitary representation $\pi_{\tilde{\varphi}}$ of \mathbb{R} on the reproducing kernel Hilbert space $\mathcal{H}_{\tilde{\varphi}}$ is called the *minimal unitary dilation* of φ .

If $\varphi(0) = \mathbf{1}$, then we can identify \mathcal{F} with a subspace of $\mathcal{H}_{\tilde{\varphi}}$ and $\varphi(s) = P\pi_{\tilde{\varphi}}(s)P^*$ holds for $s \geq 0$ and the orthogonal projection $P: \mathcal{H}_{\tilde{\varphi}} \rightarrow \mathcal{F}$.

Specializing the preceding result to $\mathcal{F} = \mathbb{C}$, we obtain the following integral representation.

Corollary 3.3. *A continuous function $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ is reflection positive if and only if it has an integral representation of the form*

$$\varphi(x) = \int_0^\infty e^{-\lambda|x|} d\nu(\lambda), \tag{20}$$

where ν is a finite positive Borel measure on $[0, \infty[$.

A version of the preceding corollary for reflection positivity on the group \mathbb{Z} can be found in [10, Prop. 3.2].

Remark 3.4. (a) Since the function $\varphi_\lambda(x) = e^{-\lambda|x|}$, $\lambda \geq 0$, is real-valued, it can also be written as

$$\varphi_\lambda(x) = e^{-\lambda|x|} = \frac{2\lambda}{\pi} \int_0^\infty \cos(xy) \frac{dy}{\lambda^2 + y^2}.$$

(b) Using the isomorphism $\exp: (\mathbb{R}, +) \rightarrow (\mathbb{R}_+, \cdot)$, we also get a description of the reflection positive functions on \mathbb{R}_+ with respect to the involution $\tau(a) = a^{-1}$ and the subsemigroup $S =]0, 1[$. They are given by

$$\varphi_\lambda(a) = e^{-\lambda|\log a|} = \begin{cases} a^{-\lambda} & \text{for } a \geq 1, \\ a^\lambda & \text{for } a \leq 1. \end{cases}$$

Example 3.5. We take a closer look at the representation associated to the positive definite function $\varphi(x) = e^{-\lambda|x|}$ on \mathbb{R} , where $\lambda > 0$.

(a) Let $d\mu(y) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + y^2} dy$ be the Cauchy distribution from (19), so that $\varphi = \widehat{\mu}$ is the Fourier transform of μ . Then we may also realize \mathcal{H}_φ for $\varphi(x) = e^{-\lambda|x|}$ as $L^2(\mathbb{R}, \mu)$, where the unitary isomorphism is given by

$$\Gamma: L^2(\mathbb{R}, \mu) \rightarrow \mathcal{H}_\varphi, \quad \Gamma(f)(x) = \langle f, \pi(x)1 \rangle = \int_{\mathbb{R}} f(y) e^{-ixy} d\mu(y) = (f\mu)^\wedge(x)$$

and τ acts on $L^2(\mathbb{R}, \mu)$ by $\theta(f)(x) = f(-x)$ (cf. Example 2.9). Note that $\Gamma(1) = \varphi$.

We consider the closed subspace $\mathcal{E}_+ \subseteq \mathcal{H}_\varphi$ generated by $\pi_\varphi(S)\varphi$ and note that

$$\langle \theta\pi_\varphi(s)\varphi, \pi_\varphi(t)\varphi \rangle = \langle \pi(-t-s)\varphi, \varphi \rangle = \varphi(-t-s) = \varphi(t+s) = e^{-\lambda(t+s)}.$$

Hence the corresponding reproducing kernel space on S is one-dimensional, generated by the character e_λ with $e_\lambda(x) = e^{-\lambda x}$. Moreover, $(\pi_\varphi, \mathcal{H}_\varphi)$ is the minimal unitary dilation of e_λ (cf. Remark 3.2(b)).

(b) A slightly different realization of \mathcal{H}_φ , which makes the scalar product on this space more explicit, is to consider it as a completion of the space $C_c(\mathbb{R}, \mathbb{C})$ of compactly supported continuous function, endowed with the hermitian form

$$\begin{aligned} (f, g) &\mapsto \langle \pi_\varphi(f)\varphi, \pi_\varphi(g)\varphi \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \overline{g(y)} \langle \pi_\varphi(x)\varphi, \pi_\varphi(y)\varphi \rangle dx dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \overline{g(y)} \varphi(x - y) dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \overline{g(y)} e^{-\lambda|y-x|} dx dy \end{aligned}$$

(cf. Remark 2.2). In this picture, \mathcal{E}_+ corresponds to the subspace generated by those functions $f \in C_c(\mathbb{R}, \mathbb{C})$, which are supported by $[0, \infty[$. For two such functions f, g , we obtain

$$\begin{aligned} \langle \theta f, g \rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(-x) \overline{g(y)} e^{-\lambda|y-x|} dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \overline{g(y)} e^{-\lambda|y+x|} dx dy \\ &= \int_0^\infty \int_0^\infty f(x) \overline{g(y)} e^{-\lambda(y+x)} dx dy = \int_0^\infty f(x) e^{-\lambda x} dx \cdot \overline{\int_0^\infty g(y) e^{-\lambda y} dy}. \end{aligned}$$

This formula reflects the fact that the space $\widehat{\mathcal{E}}$ is one-dimensional and that the natural map $\mathcal{E}_+ \rightarrow \widehat{\mathcal{E}}$ can be realized by $f \mapsto \int_0^\infty f(x) e^{-\lambda x} dx$.

3.2. Reflection positive distributions on \mathbb{R}

In this subsection we discuss two interesting families of reflection positive distributions for the triple $(\mathbb{R}, -\text{id}, \mathbb{R}_+)$. These distributions occur naturally by restriction of the complementary series representations of $\text{SL}_2(\mathbb{R})$ (which is locally isomorphic to $\text{O}_{1,2}(\mathbb{R})_0$) to the subgroups N and A in the Iwasawa decomposition (cf. [26]). These representations are discussed in a more general context in Sections 5 and 6 below.

Example 3.6. (a) For $0 < s < 1$ the locally integrable function $|x|^{-s}$ on \mathbb{R} defines a positive definite measure, resp., distribution (Lemma 2.13). The corresponding Hilbert space \mathcal{H}_s is the completion of $C_c(\mathbb{R})$ with respect to the scalar product

$$\langle f, g \rangle := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \overline{g(y)} |x - y|^{-s} dx dy \tag{21}$$

(cf. Proposition 2.8).

The distribution $|x|^{-s}$ restricts to the function x^{-s} on \mathbb{R}_+ which is positive definite with respect to the trivial involution because, for any $\alpha > 0$,

$$\frac{1}{\Gamma(\alpha)} \mathcal{L}(y^{\alpha-1} dy)(x) = x^{-\alpha} \quad \text{for } x > 0. \tag{22}$$

We conclude that the distribution $|x|^{-s}$ is reflection positive for the triple $(\mathbb{R}, -\text{id}_{\mathbb{R}}, \mathbb{R}_+)$. This observation will also be used in Proposition 6.1 below.

(b) On the Hilbert space \mathcal{H}_s we also have a unitary representation of the multiplicative group $A := \mathbb{R}$, given by

$$(\pi(a)f)(x) := |a|^{\frac{s}{2}-1} f(a^{-1}x)$$

and an involution

$$(\theta f)(x) := |x|^{s-2} f(x^{-1})$$

satisfying $\theta\pi(a)\theta = \pi(a^{-1})$, so that we obtain for $\tau(a) := a^{-1}$ a representation of A_τ on \mathcal{H}_s .

Let $\mathcal{E}_+ \subseteq \mathcal{H}_s$ be the closed subspace generated by functions supported in $] -1, 1[$ and note that it is invariant under the action of the subsemigroup $S := \{a \in A : |a| < 1\}$. For $f \in C_c(]-1, 1[, \mathbb{C})$, we have

$$\begin{aligned} \langle \theta f, f \rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}} |x|^{s-2} f(x^{-1}) \overline{f(y)} |x - y|^{-s} dx dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |x^{-1}|^{s-1} f(x) \overline{f(y)} |x^{-1} - y|^{-s} \frac{dx}{|x|} dy \\ &= \int_{-1}^1 \int_{-1}^1 f(x) \overline{f(y)} |x|^{-s} |x^{-1} - y|^{-s} dx dy \\ &= \int_{-1}^1 \int_{-1}^1 f(x) \overline{f(y)} (1 - xy)^{-s} dx dy \geq 0 \end{aligned}$$

because the kernel $(1 - xy)^{-s} = \sum_{n=0}^\infty \binom{n-1+s}{n} (xy)^n$ on $] -1, 1[$ is positive definite, which follows from the non-negativity of the binomial coefficients. Therefore the unitary representation (π, \mathcal{H}_s) of A is reflection positive.

We now describe a cyclic distribution vector and the corresponding distribution on A . For the integrated representation we find

$$(\pi(\varphi)f)(x) = \int_{\mathbb{R}^\times} \varphi(a) |a|^{\frac{s}{2}-1} f(a^{-1}x) \frac{da}{|a|} = \int_{\mathbb{R}^\times} \varphi(a) |a|^{\frac{s}{2}-2} f(a^{-1}x) da.$$

The antilinear functional α on $C_c(\mathbb{R}^\times) \subseteq \mathcal{H}_s$, given by

$$\alpha(f) := \int_{\mathbb{R}^\times} \overline{f(x)} |x - 1|^{-s} dx, \tag{23}$$

is formally the scalar product $\langle \delta_1, f \rangle$ with the δ -function in 1. We then have

$$\begin{aligned} \alpha(\pi(\varphi^*)f) &= \int_{\mathbb{R}^\times} \int_A \varphi(a^{-1}) |a|^{s/2-1} \overline{f(a^{-1}x)} \frac{da}{|a|} |x - 1|^{-s} dx \\ &= \int_A \varphi(a^{-1}) |a|^{s/2-1} \int_{\mathbb{R}^\times} \overline{f(a^{-1}x)} |x - 1|^{-s} dx \frac{da}{|a|} \end{aligned}$$

$$\begin{aligned}
 &= \int_A \varphi(a^{-1})|a|^{s/2} \int_{\mathbb{R}^\times} \overline{f(x)}|ax - 1|^{-s} dx \frac{da}{|a|} \\
 &= \int_A \varphi(a)|a|^{-s/2} \int_{\mathbb{R}^\times} \overline{f(x)}|a^{-1}x - 1|^{-s} dx \frac{da}{|a|} \\
 &= \int_A \varphi(a)|a|^{s/2} \int_{\mathbb{R}^\times} \overline{f(x)}|x - a|^{-s} dx \frac{da}{|a|} \\
 &= \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \varphi(a)|a|^{s/2-1} \overline{f(x)}|x - a|^{-s} dx da.
 \end{aligned}$$

Therefore $\alpha \circ \pi(\varphi^*)$ can be identified with the element of \mathcal{H}_s given by

$$(\alpha \circ \pi(\varphi^*))(x) = \varphi(x)|x|^{\frac{s}{2}-1}.$$

Since the map

$$\mathcal{D}(A) \rightarrow \mathcal{H}_s, \quad \varphi \mapsto (x \mapsto \varphi(x)|x|^{\frac{s}{2}-1})$$

is continuous, α defines a distribution vector in $\mathcal{H}_s^{-\infty}$ with $\pi^{-\infty}(\varphi)\alpha = \alpha \circ \pi(\varphi^*)$ for every $\varphi \in \mathcal{D}(A)$ (cf. Proposition 2.8). As $C_c(\mathbb{R}^\times)$ is dense in \mathcal{H}_s , this distribution vector is cyclic for the representation of A on \mathcal{H}_s . From (23) we further derive that $\theta\alpha = \alpha$:

$$\begin{aligned}
 \langle \alpha, \theta f \rangle &= \int_{\mathbb{R}} |x|^{s-2} \overline{f(x^{-1})}|x - 1|^{-s} dx = \int_{\mathbb{R}} |x|^{s-1} \overline{f(x^{-1})}|x - 1|^{-s} \frac{dx}{|x|} \\
 &= \int_{\mathbb{R}} |x|^{1-s} \overline{f(x)}|x^{-1} - 1|^{-s} \frac{dx}{|x|} = \int_{\mathbb{R}} \overline{f(x)}|1 - x|^{-s} dx = \langle \alpha, f \rangle.
 \end{aligned}$$

The corresponding distribution on A is given by

$$\pi^\alpha(\varphi) = \langle \alpha, \pi^{-\infty}(\varphi)\alpha \rangle = \int_{\mathbb{R}} \overline{\varphi(x)}|x|^{\frac{s}{2}-1}|x - 1|^{-s} dx = \int_{\mathbb{R}} \overline{\varphi(x)}|x|^{\frac{s}{2}}|x - 1|^{-s} \frac{dx}{|x|},$$

hence represented by the locally integrable function

$$\gamma(x) := |x|^{\frac{s}{2}}|x - 1|^{-s} \quad \text{on } A = \mathbb{R}^\times,$$

which is τ -invariant. On the open subsemigroup $S = \{a \in A : |a| < 1\}$ we have the expansion

$$|x|^{s/2}(1 - x)^{-s} = \sum_{n=0}^{\infty} \binom{n - 1 + s}{n} |x|^{s/2} x^n.$$

Accordingly, the corresponding representation of S on $\widehat{\mathcal{E}}$ decomposes as a direct sum of one-dimensional representations corresponding to the characters $|x|^{s/2}x^n, n \in \mathbb{N}_0$.

Since \mathcal{H}_s is a space defined by the positive definite distribution $D = |x|^{-s}$ on \mathbb{R} , it also has a natural realization $\Phi: \mathcal{H}_s \rightarrow \mathcal{D}'(\mathbb{R})$ as a space \mathcal{H}_D of distributions. This map is given by

$$\Phi(f)(\varphi) = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\varphi(x)} f(y) |x - y|^{-s} dx dy$$

(cf. [Example 3.6\(a\)](#)). In particular, the distribution $\Phi(f)$ is represented by a continuous function $\Gamma(f)$. In this sense, we have

$$\begin{aligned} \Phi(\pi(a)f)(x) &= |a|^{s/2-1} \int_{\mathbb{R}} f(a^{-1}y) |x - y|^{-s} dy = |a|^{s/2} \int_{\mathbb{R}} f(y) |x - ay|^{-s} dy \\ &= |a|^{-s/2} \int_{\mathbb{R}} f(y) \left| \frac{x}{a} - y \right|^{-s} dy = |a|^{-s/2} \Gamma(f)(x/a), \end{aligned}$$

so that the corresponding representation of A on $\Phi(C_c(G))$ is given by

$$(\widehat{\pi}(a)\Gamma(f))(x) := |a|^{-s/2} \Gamma(f)(x/a).$$

3.3. Extending distributions

In this subsection we briefly discuss the existence of reflection positive extensions of positive definite distributions on the open subsemigroup $S = (\mathbb{R}_+, +)$ of \mathbb{R} to the whole real line.

For each real $\alpha > -1$, we obtain a measure $\mu_\alpha := x^\alpha dx$ on $[0, \infty[$ whose Laplace transform exists and satisfies $\mathcal{L}(\mu_\alpha) = c_\alpha x^{-1-\alpha}$. This shows that the functions $x^{-s}, s > 0$, on S are positive definite (cf. [\(22\)](#) in [Example 3.6](#)).

We write $r^{-s} = |x|^{-s}$ for their symmetric extensions to \mathbb{R}^\times . For $s < 1$, r^{-s} is locally integrable, so that it defines a distribution on \mathbb{R} , and by [Lemma 2.13](#), this distribution is positive definite.

For $s \geq 1$, the situation is more complicated. According to [\[49, Thm. VIII, §VII.4\]](#), the measure $|x|^{-s} dx$ on \mathbb{R}^\times extends to a distribution on \mathbb{R} . To describe such extensions more explicitly, one applies the “finite part” technique to the restrictions to $]0, \infty[$ and $] -\infty, 0[$ (see [\[49, \(II.2;26\)\]](#) for the functions r^m on $]0, \infty[$). One shows that, for $\varphi \in \mathcal{D}(\mathbb{R})$, the function $I(\varphi, \varepsilon) := \int_{|x|>\varepsilon} \varphi(x) |x|^{-s} dx$ has an asymptotic expansion for $\varepsilon > 0$ of the form

$$I(\varphi, \varepsilon) = \sum_{k,\ell} a_{k\ell}(\varphi) \varepsilon^{-k} (\log \varepsilon)^\ell$$

and, for $s \leq -1$, one defines $\text{Pf}(r^{-s}) \in \mathcal{D}'(\mathbb{R})$ by

$$\text{Pf}(r^{-s})(\varphi) := a_{00}(\varphi).$$

If $s \notin -1 - 2\mathbb{N}_0$, then the Fourier transform of $\text{Pf}(r^{-s})$ is a multiple of $\text{Pf}(r^{s-1})$, and for $\text{Pf}(r^{-1-2h}), h \in \mathbb{N}_0$, it is a sum of a multiples of r^{2h} and $r^{2h} \log r$ [\[49, \(VII.7;13\)\]](#). We conclude

that, for each $s \geq 1$, there exists a polynomial P for which $\mathcal{F}(\text{Pf}(r^{-s}))(x) + P(4\pi^2 x^2) \geq 0$. Therefore $\text{Pf}(r^{-s}) + P(-\frac{d^2}{dx^2})\delta_0$ is a reflection positive extension of r^{-s} .

4. A Bochner–Schwartz Theorem for involutive cones

In this section we study the convolution algebra $\mathcal{D}(S) = C_c^\infty(S, \mathbb{C})$ of complex-valued test functions on an open subsemigroup S of a symmetric Lie group (G, τ) . We assume that $\mathbf{1} \in \bar{S}$ and that S is invariant under the involution $s^\sharp = \tau(s)^{-1}$. Endowed with the L^1 -norm, we obtain on $\mathcal{D}(S)$ the structure of a normed $*$ -algebra \mathcal{A} with an approximate identity. We are mainly interested in the case where $G = V$ is a finite-dimensional real vector space and $S = \Omega$ is an open convex cone. In this case we show that the spectrum $\widehat{\mathcal{A}}$ (= the set of non-zero bounded characters) of \mathcal{A} is homeomorphic to a certain closed convex cone $\widehat{\Omega} \subseteq V_{\mathbb{C}}^*$ parametrizing the bounded $*$ -homomorphisms $S \rightarrow \mathbb{C}$. Our main result is a generalization of the Bochner–Schwartz Theorem ([Theorem 4.11](#)) characterizing positive definite distributions $D \in \mathcal{D}'(S)$ corresponding to contraction representations of S as Fourier–Laplace transforms of regular Borel measures on $\widehat{\Omega}$. In [\[43\]](#) this result will be an important tool which is used to describe the abstract Hilbert spaces \mathcal{H}_D as more concrete L^2 -spaces.

4.1. Some generalities on open subsemigroups

Let (G, τ) be a symmetric Lie group and $S \subseteq G$ be an open subsemigroup with $\mathbf{1} \in \bar{S}$ which is invariant under the involution $s^\sharp = \tau(s)^{-1}$. Then $\mathcal{D}(S)$ is a convolution subalgebra of $\mathcal{D}(G)$ with an approximate identity and involution $\varphi^\sharp = \varphi^* \circ \tau$. We define a representation of $\mathcal{D}(S)$ on the space $\mathcal{D}'(S)$ by

$$(\varphi * D)(\psi) = D(\varphi^\sharp * \psi).$$

The following proposition generalizes the classical Dixmier–Malliavin Theorem [\[6, Thm. 3.1\]](#) to open subsemigroups of Lie groups.

Proposition 4.1. *Let $S \subseteq G$ be an open subsemigroup with $\mathbf{1} \in \bar{S}$. Then every test function $\varphi \in \mathcal{D}(S)$ is a sum of products $\alpha * \beta$ with $\alpha, \beta \in \mathcal{D}(S)$.*

Proof. Let $\varphi \in \mathcal{D}(S)$. Then $\text{supp}(\varphi)$ is a compact subset of S , so that there exists a $\mathbf{1}$ -neighborhood $U \subseteq G$ with $U \text{supp}(\varphi) \subseteq S$. Pick $u \in U \cap S^{-1}$ and let $V \subseteq G$ be a $\mathbf{1}$ -neighborhood in G with $u^{-1}V \subseteq S$. Then $\varphi \circ \lambda_u^{-1} = \delta_u * \varphi \in \mathcal{D}(S)$.

According to [\[6, Thm. 3.1\]](#), $\delta_u * \varphi$ is a finite sum of functions of the form $\alpha * \beta$ with $\text{supp}(\alpha) \subseteq V$ and $\text{supp}(\beta) \subseteq \text{supp}(\delta_u * \varphi) = u \text{supp}(\varphi)$. Then φ is a finite sum of functions of the form $\delta_{u^{-1}} * \alpha * \beta$, with

$$\text{supp}(\delta_{u^{-1}} * \alpha) = u^{-1} \text{supp}(\alpha) \subseteq u^{-1}V \subseteq S$$

and $\text{supp}(\beta) \subseteq u \text{supp}(\varphi) \subseteq S$. \square

Lemma 4.2. *For each $D \in \mathcal{D}'(S)$ and $\varphi \in \mathcal{D}(S)$ the distribution $\varphi * D$ is represented by the smooth function*

$$(\varphi * D)(x) = D((\varphi \circ \lambda_{\tau(x)})^\sharp)$$

on S which extends smoothly to an open neighborhood of \bar{S} .

Proof. The first assertion follows easily from [53, Prop. A.2.4.1] if we take into account that we defined distributions as antilinear functionals on $\mathcal{D}(G)$.

For the second assertion we note that the function

$$\Phi: G \rightarrow \mathcal{D}(G), \quad x \mapsto \varphi \circ \lambda_{\tau(x)}$$

is smooth. As $\text{supp}(\Phi(x)) = x^\sharp \text{supp}(\varphi)$ and $\text{supp}(\varphi)$ is a compact subset of S , the set $S_\varphi := \{x \in G: x^\sharp \text{supp}(\varphi) \subseteq S\}$ is an open neighborhood of \bar{S} , and the function

$$\Phi: \{x \in G: x^\sharp \text{supp}(\varphi) \subseteq S\} \rightarrow \mathcal{D}(S),$$

is smooth. This completes the proof. \square

Lemma 4.3. *If $\chi \in \mathcal{D}'(S)$ is a non-zero homomorphism of $*$ -algebras, then χ is represented by a smooth homomorphism $S \rightarrow (\mathbb{C}, \cdot)$ of involutive semigroups. If, in addition, for every 1-neighborhood U in G the set $S \cap U$ generates S , then χ has no zeros.*

Proof. Since χ is non-zero, there exists $\varphi \in \mathcal{D}(S)$ with $\chi(\varphi) \neq 0$. Then

$$(\varphi * \chi)(\psi) = \chi(\varphi^\sharp * \psi) = \chi(\psi) \overline{\chi(\varphi)}$$

implies that

$$\chi = \overline{\chi(\varphi)}^{-1}(\varphi * \chi), \tag{24}$$

and hence that χ is represented by a smooth function which we also denote by χ (Lemma 4.2).

For $\psi \in \mathcal{D}(S)$ the relation

$$\int_S \varphi(s) \overline{\chi(s)} d\mu_G(s) = \overline{\chi(\varphi)} = \chi(\varphi^\sharp) = \int_S \varphi(s^\sharp) \Delta(s^{-1}) \chi(s) d\mu_G(s) = \int_S \varphi(s) \chi(s^\sharp) d\mu_G(s)$$

implies that $\chi(s^\sharp) = \overline{\chi(s)}$.

Using (24), we obtain the relation

$$\overline{\chi(\varphi)} \chi(x) = \chi((\varphi \circ \lambda_{\tau(x)})^\sharp) = \overline{\chi(\varphi \circ \lambda_{\tau(x)})}. \tag{25}$$

Now we let φ_n be a sequence of test functions with total integral 1 such that $\text{supp}(\varphi_n)$ converges to the point $y \in S$. Passing to the limit in (25) now leads to

$$\chi(y^\sharp) \chi(x) = \overline{\chi(y)} \chi(x) = \overline{\chi(x^\sharp y)} = \chi(y^\sharp x).$$

Therefore $\chi: S \rightarrow (\mathbb{C}, \cdot)$ is a homomorphism.

It remains to show that $\chi(x) \neq 0$ for all $x \in S$ if S is generated by every neighborhood of $\mathbf{1}$, intersected with S . Pick $x \in S$ with $\chi(x) \neq 0$. Then the open set xS^{-1} is a neighborhood of $\mathbf{1}$ and for $z \in S \cap xS^{-1}$ the relation $0 \neq \chi(x) = \chi(z)\chi(z^{-1}x)$ implies that $\chi(z) \neq 0$. Since S is generated by $S \cap xS^{-1}$, the assertion follows. \square

Lemma 4.4. *The norm*

$$\|\varphi\|_1 = \int_S |\varphi(x)| d\mu_G(x)$$

on $\mathcal{D}(S)$ is submultiplicative, so that $(\mathcal{D}(S), \|\cdot\|_1)$ is a normed algebra and $\varphi^\sharp = \varphi^* \circ \tau$ defines an isometric involution on $\mathcal{D}(S)$.

Proof. We know already from Definition 2.1 that $(\mathcal{D}(S), *, \sharp)$ is an involutive algebra. The submultiplicativity of $\|\cdot\|_1$ and $\|\varphi^\sharp\|_1 = \|\varphi\|_1$ follow immediately from the corresponding properties of $L^1(G)$ because the involution τ preserves the Haar measure on G . \square

Definition 4.5. We write \widehat{S} for the set of all continuous homomorphisms $\chi: S \rightarrow \mathbb{C}$ satisfying

$$\chi(s^\sharp) = \overline{\chi(s)} \quad \text{and} \quad |\chi(s)| \leq 1 \quad \text{for } s \in S.$$

It is easy to see that any $\chi \in \widehat{S}$ defines a $*$ -homomorphism

$$\mathcal{D}(S) \rightarrow \mathbb{C}, \quad \varphi \mapsto \int_S \varphi(s)\chi(s) d\mu_G(s),$$

hence is smooth by Lemma 4.3.

Remark 4.6. Let $D \in \mathcal{D}'(S)$ be a positive definite distribution and (π_D, \mathcal{H}_D^0) be the corresponding representation, where $\mathcal{H}_D^0 = \mathcal{D}(S) * D \subseteq \mathcal{D}'(S)$ and $\pi_D(\varphi)T = \varphi * T$.

(a) We assume, in addition, that D is 1-bounded in the sense that

$$|D(\varphi * \psi * \varphi^*)| \leq \|\psi\|_1 D(\varphi * \varphi^*) \quad \text{for } \varphi, \psi \in \mathcal{D}(S).$$

Then the representation of $\mathcal{D}(S)$ on \mathcal{H}_D^0 extends to a representation on the reproducing kernel Hilbert space $\mathcal{H}_D \subseteq \mathcal{D}'(S)$ (cf. [38, Th. III.1.19] and Section 2). If $D' \in \mathcal{D}'(S)$ satisfies $\mathcal{D}(S) * D' = \{0\}$, then the existence of an approximate identity in S implies that $D' = 0$. Therefore the representation of $\mathcal{D}(S)$ on \mathcal{H}_D is non-degenerate.

(b) Suppose, in addition, that S is commutative. We claim that $\pi_D(\varphi) = 0$ implies that $\varphi * D = 0$. In fact, $\pi_D(\varphi) = 0$ leads to

$$0 = \pi_D(\varphi)(\psi * D) = \varphi * \psi * D = \psi * \varphi * D \quad \text{for } \psi \in \mathcal{D}(\Omega),$$

so that

$$D(\varphi^\sharp * \psi^\sharp * \xi) = 0 \quad \text{for } \xi, \psi \in \mathcal{D}(\Omega),$$

i.e., $\varphi * D$ vanishes on $\mathcal{D}(S) * \mathcal{D}(S) = \mathcal{D}(S)$ (Proposition 4.1). This means that $\varphi * D = 0$.

4.2. Open convex cones

We now turn to the special case where $G = V$ is a finite-dimensional real vector space and $\tau \in GL(V)$ is an involution. Let $\Omega \subseteq V$ be an open convex cone invariant under the involution $s^\sharp = -\tau(s)$. Then (Ω, \sharp) is an involutive semigroup and the convolution algebra $\mathcal{D}(\Omega)$ of test functions on Ω is an involutive algebra with respect to the involution $\varphi^\sharp(s) = \overline{\varphi(s^\sharp)}$.

Since Ω is generated by any intersection $U \cap \Omega$, where $U \subseteq V$ is a 0-neighborhood, any non-zero $\chi \in \widehat{\Omega}$ (the set of bounded $*$ -homomorphisms into (\mathbb{C}, \cdot)) maps into \mathbb{C}^\times . Hence χ extends to a $*$ -homomorphism $(V, \sharp) \rightarrow \mathbb{C}^\times$, which means that $\chi = e^{-\alpha}$ for a linear functional $\alpha: V \rightarrow \mathbb{C}$ satisfying $\alpha^\sharp := -\bar{\alpha} \circ \tau = \alpha$. The set of all these functionals can be identified with the dual space of

$$V_c := \{x + iy \in V_{\mathbb{C}}: x^\sharp = x, y^\sharp = -y\}.$$

In the following we therefore identify $\widehat{\Omega}$ with the closed convex cone

$$\widehat{\Omega} = \{\alpha \in V_c^*: (\forall x = x^\sharp \in \Omega) \alpha(x) \geq 0\}$$

in the real vector space V_c^* .

Examples 4.7. (a) $\tau = -\text{id}_V$, $\sharp = \text{id}_V$, and Ω is any open convex cone. Then $V_c = V$ and $\widehat{\Omega} = \Omega^*$ is the dual cone.

(b) $\tau = \text{id}_V$ and $V = \Omega$. Then $V_c = iV$ and $\widehat{\Omega} = V_c^* = iV^*$.

(c) $V = \mathbb{R}^n$, $\tau(x_0, x_1, \dots, x_{n-1}) = (-x_0, x_1, \dots, x_{n-1})$ and

$$\Omega := \mathbb{R}_+^n = \{x \in \mathbb{R}^n: x_0 > 0\}$$

an open half space. Then

$$V_c = \mathbb{R}e_0 + i \sum_{j>0} \mathbb{R}e_j \quad \text{and} \quad \widehat{\Omega} = \{\alpha \in V_c^*: \alpha_0 \geq 0\}.$$

Remark 4.8. For the case $\Omega = V$ with $\tau(v) = v$, the Bochner–Schwartz Theorem [49, Thm. XVIII, §VII.9] asserts that $D \in \mathcal{D}'(V)$ is positive definite if and only if it is tempered and its Fourier transform \widehat{D} is a positive measure. We then have an embedding $\mathcal{H}_D \subseteq \mathcal{S}'(V)$ (cf. Example 2.9).

Definition 4.9. (a) Let $\mathcal{A} := C^*(\mathcal{D}(\Omega), \|\cdot\|_1)$ be the enveloping C^* -algebra of the normed involutive algebra $(\mathcal{D}(\Omega), \|\cdot\|_1)$ and $\eta: \mathcal{D}(\Omega) \rightarrow \mathcal{A}$ be the canonical map.

(b) For each $\varphi \in \mathcal{D}(\Omega)$, the Fourier–Laplace transform

$$\widehat{\varphi}: \widehat{\Omega} \rightarrow \mathbb{C}, \quad \widehat{\varphi}(\alpha) := \int_{\Omega} \varphi(x)e^{-\alpha(x)} dx$$

is a continuous function on $\widehat{\Omega}$. It corresponds to evaluation of the character $e^{-\alpha}$ of $\widehat{\Omega}$ on φ . In particular, we have

$$(\varphi * \psi)^\wedge = \widehat{\varphi} \cdot \widehat{\psi} \quad \text{and} \quad \widehat{\varphi^\sharp} = \overline{\widehat{\varphi}} \quad \text{for } \varphi, \psi \in \mathcal{D}(\Omega).$$

Proposition 4.10. *The Fourier–Laplace transform defines a homeomorphism*

$$\Gamma: \widehat{\Omega} \rightarrow \widehat{\mathcal{A}}, \quad \Gamma(\alpha)(\eta(\varphi)) := \widehat{\varphi}(\alpha) = \int_{\Omega} \varphi(s)e^{-\alpha(s)} ds, \quad \varphi \in \mathcal{D}(\Omega),$$

where $\widehat{\Omega}$ carries the topology induced from V_c^* and $\widehat{\mathcal{A}} \subseteq \mathcal{A}'$ the weak- $*$ -topology.

Proof. By definition, the spectrum $\widehat{\mathcal{A}}$ of \mathcal{A} is the set of all non-zero one-dimensional representations of \mathcal{A} , which is the same as the set of all non-zero $*$ -homomorphisms $\chi: \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ with $\|\chi\|_1 \leq 1$. In view of Lemma 4.3, any such character is represented by a $*$ -homomorphism $\beta: \Omega \rightarrow \mathbb{C}^\times$ and the condition $\|\chi\|_1 \leq 1$ corresponds to the boundedness of β . We conclude that Γ is bijective.

For $\varphi \in \mathcal{D}(\Omega)$ its Fourier–Laplace transform $\widehat{\varphi}: \widehat{\Omega} \rightarrow \mathbb{C}$ is continuous and since $\eta(\mathcal{D}(\Omega))$ is dense in \mathcal{A} and $\widehat{\mathcal{A}}$ is bounded, it follows that $\alpha \mapsto \Gamma(\alpha)(A)$ is continuous for every $A \in \mathcal{A}$, i.e., that Γ is continuous. It remains to show that its inverse is also continuous.

To this end, we first observe that the translation action of Ω on $\mathcal{D}(\Omega)$

$$(s \cdot \varphi)(x) := \varphi(x - s)$$

defines multipliers of the C^* -algebra \mathcal{A} , which leads to a homomorphism

$$\eta: \Omega \rightarrow M(\mathcal{A}) \cong C_b(\widehat{\mathcal{A}})$$

[3, Ex. 12.1.1(b)]. More concretely,

$$(s \cdot \varphi)^\wedge(\alpha) = \int_{\Omega} \varphi(x - s)e^{-\alpha(x)} dx = \int_{\Omega - s} \varphi(x)e^{-\alpha(x+s)} dx = e^{-\alpha(s)} \widehat{\varphi}(\alpha).$$

For every $A \in \mathcal{A}$, the map $\Omega \rightarrow \mathcal{A}$, $x \mapsto \eta(x)A$ is continuous, i.e., η is continuous with respect to the so-called strict topology on $M(\mathcal{A})$. Since $\eta(\Omega)$ is bounded and this topology coincides on bounded subsets with the compact open topology [13, Lemma A.1], the map $\eta: \Omega \rightarrow C_b(\widehat{\mathcal{A}})$ is continuous with respect to the compact open topology on $C_b(\widehat{\mathcal{A}})$. This in turn means that the map

$$\Omega \times \widehat{\mathcal{A}} \rightarrow \mathbb{C}, \quad (s, \chi) \mapsto \chi(\eta(s))$$

is continuous [4, Thm. VII.2.4], and hence that the map

$$\widehat{\eta}: \widehat{\mathcal{A}} \rightarrow \widehat{\Omega}, \quad \widehat{\eta}(\chi)(s) := \chi(\eta(s))$$

is continuous if $\widehat{\Omega}$ carries the topology of uniform convergence on compact subsets of Ω . It is easy to see that this topology coincides with the subspace topology inherited from V_c^* . This proves that Γ is a homeomorphism. \square

Theorem 4.11 (Generalized Bochner–Schwartz Theorem). *Let τ be an involution on the finite-dimensional vector space V and $\Omega \subseteq V$ be an open convex cone invariant under the involution $v \mapsto v^\sharp = -\tau(v)$. If $D \in \mathcal{D}'(\Omega)$ is a positive definite 1-bounded distribution, then the following assertions hold:*

(a) There exists a unique positive Radon measure μ on the closed convex cone $\widehat{\Omega}$ with

$$D(\varphi) = \int_{\widehat{\Omega}} \overline{\widehat{\varphi}(\alpha)} d\mu(\alpha) \quad \text{for } \varphi \in \mathcal{D}(\Omega),$$

resp.,

$$D = \int_{\widehat{\Omega}} e_{-\alpha} d\mu(\alpha) \quad \text{for } e_{-\alpha}(x) = e^{-\alpha(x)}, \quad x \in \Omega.$$

(b) There exists a unitary map

$$\Gamma: L^2(\widehat{\Omega}, \mu) \rightarrow \mathcal{H}_D, \quad \Gamma(f)(\varphi) := \int_{\widehat{\Omega}} f \overline{\widehat{\varphi}} d\mu = \langle f, \widehat{\varphi} \rangle_{L^2(\widehat{\Omega}, \mu)}$$

which, for $\psi \in \mathcal{D}(\Omega)$, maps $\widehat{\psi}$ to $\psi * D$ and intertwines the contraction representation of Ω on $\mathcal{H}_D \subseteq \mathcal{D}'(\Omega)$ with the multiplication representation $\pi_\mu(s)f = e_{-s}f$. It extends naturally to the space of those measurable functions f for which $\varphi \mapsto \int_{\widehat{\Omega}} f \overline{\widehat{\varphi}} d\mu$ defines an element of $\mathcal{D}'(\Omega)$, and in this sense we have $\Gamma(1) = D$.

Proof. (a) The representation (π_D, \mathcal{H}_D^0) of $\mathcal{D}(\Omega)$ leads to a representation $\widetilde{\pi}_D: \mathcal{A} \rightarrow B(\mathcal{H}_D)$ with $\widetilde{\pi}_D \circ \eta = \pi_D$ and $\pi_D(\varphi) = 0$ implies that $\varphi * D = 0$ (Remark 4.6). We conclude that the map

$$\gamma: \mathcal{D}(\Omega) \rightarrow \mathcal{H}_D, \quad \varphi \mapsto \varphi * D$$

factors through a linear map

$$\bar{\gamma}: \pi_D(\mathcal{D}(\Omega)) \rightarrow \mathcal{H}_D, \quad \pi_D(\varphi) \mapsto \varphi * D.$$

Hence

$$K(\pi_D(\varphi), \pi_D(\psi)) := \langle \varphi * D, \psi * D \rangle_{\mathcal{H}_D} = D(\varphi^\sharp * \psi)$$

is a positive definite sesquilinear kernel on the dense subalgebra $\mathcal{B} := \pi_D(\mathcal{D}(\Omega))$ of the C^* -algebra $\widetilde{\pi}_D(\mathcal{A})$.

From Proposition 4.1 we know that $\mathcal{D}(\Omega) * \mathcal{D}(\Omega) = \mathcal{D}(\Omega)$, which also implies that $\mathcal{D}(\Omega) * \mathcal{D}(\Omega) * \mathcal{D}(\Omega) = \mathcal{D}(\Omega)$. Therefore [38, Prop. II.4.13] shows that the kernel K is non-degenerate because all totally degenerate kernels on the $*$ -algebra $\mathcal{D}(\Omega)$ vanish. By assumption, the kernel K is $\|\cdot\|_1$ -bounded. Now the abstract Plancherel Theorem [38, Thm. VI.1.6] implies the existence of a unique positive Borel measure on $\widehat{\mathcal{A}} \cong \widehat{\Omega}$ (Proposition 4.10) with

$$D(\varphi^\sharp * \psi) = \int_{\widehat{\Omega}} \widehat{\varphi}(\alpha) \overline{\widehat{\psi}(\alpha)} d\mu(\alpha) = \int_{\widehat{\Omega}} (\varphi * \psi^\sharp)^\wedge(\alpha) d\mu(\alpha).$$

Applying [Proposition 4.1](#) again, it follows that

$$D(\varphi) = \int_{\widehat{\Omega}} \overline{\widehat{\varphi}(\alpha)} d\mu(\alpha) \quad \text{for } \varphi \in \mathcal{D}(S).$$

(b) The first part of (b) follows from [\[38, Thm. VI.1.6\]](#), and the second part is obvious. \square

For a distribution $D_f(\varphi) = \int_{\Omega} \overline{\varphi(x)} f(x) dx$ given by a positive definite function f on Ω , we apply the integral representation from the preceding theorem to δ -sequences in the point $s \in \Omega$ to obtain

$$f(s) = \int_{\widehat{\Omega}} \overline{e^{-\alpha(s)}} d\mu(\alpha) = \int_{\widehat{\Omega}} e^{-\overline{\alpha}(s)} d\mu(\alpha).$$

This is the integral representation obtained by Shucker in [\[50, Thm. 5\]](#) for continuous positive definite functions $r: \Omega \rightarrow \mathbb{C}$ on convex domains $\Omega \subseteq V$ satisfying $\Omega + V^\tau = \Omega$ for which the kernel

$$K(z, w) = r\left(\frac{z + w^\sharp}{2}\right)$$

is positive definite. For the case of convex cones, the preceding theorem extends Shucker’s Theorem to distributions.

Example 4.12. (Cf. [Examples 4.7.](#)) (a) For an open convex cone $\Omega \subseteq V$ with the involution $\tau = -\text{id}_V$, we have $\widehat{\Omega} = \Omega^*$ and we obtain an integral representation of the form

$$D = \int_{\Omega^*} e_{-\alpha} d\mu(\alpha), \quad e_{-\alpha}(x) = e^{-\alpha(x)}, \quad x \in \Omega.$$

(b) For $\tau = \text{id}_V$ and $V = \Omega$ we have $\widehat{\Omega} = iV^*$ and we recover the classical Bochner–Schwartz Theorem asserting that every positive definite distribution on the group $(V, +)$ is the Fourier transform

$$D = \int_{V^*} e_{i\alpha} d\mu(\alpha)$$

of a measure on V^* (cf. [Remark 4.8](#)).

(c) For $V = \mathbb{R}^n$, $\tau(x_0, x_1, \dots, x_{n-1}) = (-x_0, x_1, \dots, x_{n-1})$ and the open half space $\Omega = \mathbb{R}_+^n$ we obtain an integral representation

$$D(x_0, x') = \int_0^\infty \int_{\partial\Omega} e^{-\lambda x_0 + i\langle y, x' \rangle} d\mu(\lambda, y).$$

In the case where $\tau = -\text{id}_V$, we have the following sharper version of the Bochner–Schwartz Theorem. It implies in particular that positive definite distributions are functions.

Theorem 4.13. *Suppose that $\tau = -\text{id}_V$ and that $\Omega \subseteq V$ is an open convex cone not containing affine lines. Then any 1-bounded positive definite distribution $D \in \mathcal{D}'(\Omega)$ is represented by an analytic function on Ω , also denoted D , and there exists a unique Radon measure μ on the dual cone Ω^* with*

$$D(x) = \mathcal{L}(\mu)(x) = \int_{\Omega^*} e^{-\alpha(x)} d\mu(\alpha).$$

Proof. First we use [Theorem 4.11](#) to obtain an integral representation

$$D = \int_{\Omega^*} e_{-\alpha} d\mu(\alpha),$$

where μ is a positive Radon measure on the dual cone $\Omega^* \subseteq V^*$. For $\varphi \in \mathcal{D}(\Omega)$ this leads to

$$D(\varphi) = \int_{\Omega^*} \langle e_\alpha, \varphi \rangle d\mu(\alpha), \quad \text{where } \langle e_\alpha, \varphi \rangle = \int_{\Omega} e^{-\alpha(x)} \varphi(x) dx =: \mathcal{L}(\varphi)(\alpha)$$

and dx stands for a Haar measure on V . It follows that, for every $\varphi \in \mathcal{D}(\Omega)$, the Laplace transform $\mathcal{L}(\varphi)$, viewed as a function on Ω^* , is integrable with respect to μ .

Let $x \in \Omega$. Since $x - \Omega$ is a 0-neighborhood and $0 \in \overline{\Omega}$, there exists a non-negative $\varphi \in \mathcal{D}(\Omega)$ with $\int_{\Omega} \varphi(x) dx = 1$ and $\text{supp}(\varphi) \subseteq x - \Omega$. Then we have $e^{-\alpha(x)} \leq e^{-\alpha(y)}$ for all $y \in \text{supp}(\varphi)$ and $\alpha \in \Omega^*$, so that we also get $e^{-\alpha(x)} \leq \mathcal{L}(\varphi)(\alpha)$. Therefore the integrability of $\mathcal{L}(\varphi)$ implies the integrability of the function $e_{-x}(\alpha) := e^{-\alpha(x)}$ for all $x \in \Omega$. Now

$$\mathcal{L}(\mu)(x) = \int_{\Omega^*} e^{-\alpha(x)} d\mu(\alpha)$$

exists as a function on Ω . According to [\[38, Prop. V.4.6\]](#), the function $\mathcal{L}(\mu)$ has a holomorphic extension to the tube domain $\Omega + iV \subseteq V_{\mathbb{C}}$, hence is in particular analytic. That D is represented by the function $\mathcal{L}(\mu)$ follows by Fubini’s Theorem which applies because all functions $(x, \alpha) \mapsto e^{-\alpha(x)} \varphi(x)$ are integrable with respect to the product measure $dx \otimes \mu$ on $\Omega \times \Omega^*$. \square

5. The complementary series of the conformal group

In this section we discuss the complementary series $(\pi_s, \mathcal{H}_s)_{0 < s < n}$ of irreducible unitary representations of the group $O_{1,n+1}(\mathbb{R})$ which acts by partially defined conformal transformations on \mathbb{R}^n and by everywhere defined maps on its conformal completion \mathbb{S}^n (cf. [\[5\]](#)). As we shall see in [Section 6](#), these representations are reflection positive for several involutions, resp., semi-groups, and this leads to a remarkable connection of the distributions $\|x\|^{-s}$ on \mathbb{R}^n with positive definite kernels on the open unit ball.

5.1. The conformal group on \mathbb{R}^n

We describe the conformal completion of \mathbb{R}^n by the stereographic map

$$\eta: \mathbb{R}^n \rightarrow \mathbb{S}^n, \quad \eta(x) = \left(\frac{1 - \|x\|^2}{1 + \|x\|^2}, \frac{2x}{1 + \|x\|^2} \right) \tag{26}$$

whose image is the complement of the point $-e_0 \in \mathbb{S}^n$.

To obtain an action of the conformal group on the sphere, we embed \mathbb{S}^n into the set of isotropic vectors of the $(n + 2)$ -dimensional Minkowski space

$$\zeta: \mathbb{S}^n \rightarrow \mathbb{R}^{n+2}, \quad \zeta(y) = (1, y).$$

The image of this map intersects each isotropic line in \mathbb{R}^{n+2} exactly once, so that it leads to a diffeomorphism of \mathbb{S}^n with the projective quadric of isotropic lines in $\mathbb{P}(\mathbb{R}^{n+2})$.

The group $O_{1,n+1}(\mathbb{R})$ acts on \mathbb{R}^{n+2} preserving the Lorentzian form. Writing elements of \mathbb{R}^{n+2} as pairs $z = (z_0, z')$ with $z_0 \in \mathbb{R}, z' \in \mathbb{R}^{n+1}$, we find for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O_{1,n+1}(\mathbb{R})$ the relation

$$gz = (az_0 + \langle b, z' \rangle, cz_0 + dz'). \tag{27}$$

If $z \neq 0$ satisfies $[z, z] = 0$, then $z_0 \neq 0$. This leads to $az_0 + \langle b, z' \rangle \neq 0$. Therefore the induced action of $O_{1,n+1}(\mathbb{R})$ on

$$\mathbb{S}^n \cong \{z = (1, x) : \|x\| = 1\} \cong \{0 \neq z \in \mathbb{R}^{n+2} : [z, z] = 0\} / \mathbb{R}^\times$$

can be written as

$$g.x = (a + \langle b, x \rangle)^{-1}(c + dx), \quad x \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}. \tag{28}$$

In terms of the realization of \mathbb{R}^n as the complement $\eta(\mathbb{R}^n)$ of the point $-e_0$ in \mathbb{S}^n , this action of $O_{1,n+1}(\mathbb{R})$ on \mathbb{S}^n yields an “action” on \mathbb{R}^n by rational maps. For $n > 2$, these maps exhaust all conformal maps on open subsets of \mathbb{R}^n [47, Thm. 1.9].

The stabilizer of the point $-e_0 \in \mathbb{S}^n$ “at infinity”, which is represented by the element $(1, -e_0) \in \mathbb{R}^{n+2}$ is the stabilizer $P \subseteq O_{1,n+1}(\mathbb{R})$ of the isotropic line $\mathbb{R}(1, -e_0) \subseteq \mathbb{R}^{n+2}$. This group is a parabolic subgroup of $O_{1,n+1}(\mathbb{R})$, which acts by affine conformal maps on \mathbb{R}^n . The latter description easily shows that P is isomorphic to

$$\text{Aff}_c(\mathbb{R}^n) := \mathbb{R}^n \rtimes (\mathbb{R}^\times \times O_n(\mathbb{R})), \tag{29}$$

the group of affine conformal isomorphism of \mathbb{R}^n . In the standard notation for parabolic subgroups, we have $P = NAM$ with $N \cong \mathbb{R}^n$, acting by translations, $A \cong \mathbb{R}^\times$ acting by homotheties, and $M \cong O_n(\mathbb{R})$, the centralizer of A in the maximal compact subgroup $K \cong O_{n+1}(\mathbb{R})$ of $O_{1,n+1}(\mathbb{R})$.

Remark 5.1. The group $O_{1,n+1}(\mathbb{R})$ does not act faithfully on \mathbb{S}^n because the element -1 acts trivially. However, the index 2 subgroup

$$O_{1,n+1}^+(\mathbb{R}) := \{g \in O_{1,n+1}(\mathbb{R}) : g_{00} > 0\}$$

of those Lorentz transformations preserving the future light cone

$$\Omega := \{z \in \mathbb{R}^{n+2} : [z, z] > 0, z_0 > 0\} \tag{30}$$

(cf. (1) above), acts faithfully. Note that $g_{00} = [ge_0, e_0] \neq 0$ follows from $[ge_0, ge_0] = [e_0, e_0] = 1$.

5.2. Canonical kernels

In this section we discuss the canonical kernel $1 - \langle x, y \rangle$ on \mathbb{S}^n . This kernel is projectively invariant under $O_{1,n+1}(\mathbb{R})$, and the corresponding cocycle $J : O_{1,n+1}(\mathbb{R}) \rightarrow C^\infty(\mathbb{S}^n, \mathbb{R}^\times)$ is easily described in terms of the conformal structure.

On the light cone $\Omega \subseteq \mathbb{R}^{n+2}$ (cf. (30)), we have $[x, y] > 0$ for $x, y \in \Omega$, which leads to the canonical kernel $[x, y]^{-1}$ invariant under the action of $O_{1,n+1}(\mathbb{R})$. Restricting to the sphere \mathbb{S}^n in the boundary of Ω , we find

$$[(1, x), (1, y)] = 1 - \langle x, y \rangle,$$

which leads to the singular kernel $(1 - \langle x, y \rangle)^{-1}$ on \mathbb{S}^n .

Remark 5.2. (a) For $g \in O_{1,n+1}(\mathbb{R})$ and $x, y \in \mathbb{S}^n$, we have

$$\begin{aligned} 1 - \langle x, y \rangle &= [(1, x), (1, y)] = [g(1, x), g(1, y)] \\ &= [(a + \langle b, x \rangle, c + dx), (a + \langle b, y \rangle, c + dy)] \\ &= (a + \langle b, x \rangle)(a + \langle b, y \rangle)[(1, g.x), (1, g.y)] \\ &= (a + \langle b, x \rangle)(a + \langle b, y \rangle)(1 - \langle g.x, g.y \rangle). \end{aligned}$$

In view of

$$\|x - y\|^2 = 2(1 - \langle x, y \rangle) \quad \text{for } x, y \in \mathbb{S}^n,$$

this in turn leads to

$$\|g.x - g.y\|^2 = (a + \langle b, x \rangle)^{-1}(a + \langle b, y \rangle)^{-1}\|x - y\|^2, \tag{31}$$

and hence to

$$\|\mathfrak{d}g(x)v\| = |a + \langle b, x \rangle|^{-1}\|v\| \quad \text{for } v \in T_x(\mathbb{S}^n).$$

The function

$$J_g(x) = |a + \langle b, x \rangle|^{-1} = \|\mathfrak{d}g(x)\|$$

is called the *conformal factor* of g . For $g \in O_{1,n}^+(\mathbb{R})$, the number $a + \langle b, x \rangle$ is always positive because it is the 0-component of $g(1, x)$ (cf. Remark 5.1). The Chain Rule implies the cocycle relation

$$J_{g_1 g_2}(x) = J_{g_1}(g_2 \cdot x) J_{g_2}(x), \tag{32}$$

and from above we derive for $Q_{-1}(x, y) := 1 - \langle x, y \rangle$ the transformation formula

$$Q_{-1}(g \cdot x, g \cdot y) = J_g(x) Q_{-1}(x, y) J_g(y) \quad \text{for } x, y \in \mathbb{S}^n, \quad g \in O_{1,n}^+(\mathbb{R}). \tag{33}$$

(b) The stereographic map $\eta: \mathbb{R}^n \rightarrow \mathbb{S}^n$ from (26) satisfies

$$1 - \langle \eta(x), \eta(y) \rangle = 1 - \frac{(1 - \|x\|^2)(1 - \|y\|^2) + 4\langle x, y \rangle}{(1 + \|x\|^2)(1 + \|y\|^2)} = 2 \frac{\|x - y\|^2}{(1 + \|x\|^2)(1 + \|y\|^2)},$$

so that the same argument as above, using $\|\eta(x) - \eta(y)\|^2 = 2 - 2\langle \eta(x), \eta(y) \rangle$, implies that its conformal factor, as a map $\mathbb{R}^n \rightarrow \mathbb{S}^n$, is given by

$$J_\eta(x) = \frac{2}{1 + \|x\|^2}. \tag{34}$$

This implies in particular that, up to a normalizing constant,

$$(\eta^{-1})_* \text{d}\mu_{\mathbb{S}} = \frac{2^n}{(1 + \|x\|^2)^n} dx \tag{35}$$

is the pullback of the surface measure $\mu_{\mathbb{S}}$ on \mathbb{S}^n to \mathbb{R}^n .

(c) In view of (b), the pullback of the kernel $Q(x, y) = (1 - \langle x, y \rangle)^{-s/2}$ on \mathbb{S}^n by η is the kernel

$$K(x, y) := 2^{-s/2} (1 + \|x\|^2)^{s/2} \|x - y\|^{-s} (1 + \|y\|^2)^{s/2} \tag{36}$$

on \mathbb{R}^n .

(d) From (31) it follows in particular, that the cross ratio

$$\frac{\|y - a\| \|x - b\|}{\|y - b\| \|x - a\|}$$

is invariant under the action of $O_{1,n+1}(\mathbb{R})$ on the set of pairwise distinct 4-tuples in \mathbb{S}^n .

Remark 5.3 (A conformal Cayley transform). We consider the involution $c \in O_{1,n+1}^+(\mathbb{R})$ defined by

$$ce_0 = e_0, \quad ce_1 = e_2, \quad ce_2 = e_1 \quad \text{and} \quad ce_j = e_j \quad \text{for } 2 < j \leq n + 1.$$

With respect to the action defined in (28), we then have

$$c\eta(x) = \frac{1}{1 + \|x\|^2} (2x_0, 1 - \|x\|^2, 2x_1, \dots, 2x_{n-1}) = \eta(\varphi(x))$$

for the map

$$\varphi(x) = \left(\frac{1 - \|x\|^2}{1 + \|x\|^2 + 2x_0}, \frac{2x_1}{1 + \|x\|^2 + 2x_0}, \dots, \frac{2x_{n-1}}{1 + \|x\|^2 + 2x_0} \right).$$

This formula implies that $x_0 > 0$ is equivalent to $\|\varphi(x)\| < 1$ and $\varphi(-e_0) = \infty$. Therefore φ is an involutive conformal map on \mathbb{R}^n mapping the open half space \mathbb{R}_+^n onto the open unit ball $\mathcal{D} \subseteq \mathbb{R}^n$. For $x_0 = 0$ we have $\varphi(x) \in \mathbb{S}^{n-1}$ and

$$\varphi(0, x) = \left(\frac{1 - \|x\|^2}{1 + \|x\|^2}, \frac{2x_1}{1 + \|x\|^2}, \dots, \frac{2x_{n-1}}{1 + \|x\|^2} \right)$$

is the stereographic map $\mathbb{R}^{n-1} \hookrightarrow \mathbb{S}^{n-1}$ whose image is the complement of $-e_0$.

5.3. A Hilbert space of measures

We now assume that $0 < s < n$, so that $\|x\|^{-s}$ is locally integrable on \mathbb{R}^n . From the positive definiteness of the distribution $\|x - y\|^{-s}$ (Lemma 2.13), it follows that the kernel K in (36) is also positive definite. Accordingly, the kernel $Q(x, y) = (1 - \langle x, y \rangle)^{-s/2}$ on \mathbb{S}^n is positive definite (cf. Remark 5.2(b)).

Let $\mu_{\mathbb{S}}$ be the $O_{n+1}(\mathbb{R})$ -invariant measure on \mathbb{S}^n , which, in stereographic coordinates, is given by

$$d\mu_{\mathbb{S}}(x) = \frac{2^n}{(1 + \|x\|^2)^n} dx.$$

Remark 5.4. Let X be a locally compact space. A Radon measure μ on $X \times X$ is called *positive definite* if

$$\int_{X \times X} f(x) \overline{f(y)} d\mu(x, y) \geq 0 \quad \text{for } f \in C_c(X).$$

Completing $C_c(X)$ with respect to the scalar product

$$\langle f, g \rangle_{\mu} := \int_{X \times X} f(x) \overline{g(y)} d\mu(x, y)$$

then leads to a Hilbert space \mathcal{H}_{μ} (cf. Remark 2.2). If X is compact, then \mathcal{H}_{μ} can most naturally be realized as a subspace of the space $\mathcal{M}(X) := C(X)'$ of Radon measures on X via the inclusion

$$T_{\mu}: \mathcal{H}_{\mu} \hookrightarrow \mathcal{M}(X), \quad T_{\mu}(f)(g) = \langle f, g \rangle_{\mu}, \quad T_{\mu}(f)(E) = \int_{X \times E} f(x) d\mu(x, y).$$

In the special case where μ has a density ρ with respect to a product measure $\mu_X \otimes \mu_X$, the measure $T_\mu(f)$ can be written as $\Gamma_\mu(f)\mu_X$ with

$$\Gamma_\mu(f)(y) = \int_X f(x)\rho(x, y) d\mu_X(x).$$

Lemma 5.5. For $Q(x, y) = (1 - \langle x, y \rangle)^{-s/2}$ and $0 \leq s < n$, the measure

$$d\mu(x, y) := Q(x, y) d\mu_{\mathbb{S}^n}(x) d\mu_{\mathbb{S}^n}(y) \tag{37}$$

on $\mathbb{S}^n \times \mathbb{S}^n$ is a finite positive Radon measure which is positive definite.

Proof. First we observe that the $O_{n+1}(\mathbb{R})$ -invariance of the measure $\mu_{\mathbb{S}^n}$ and the kernel Q implies that the function $F(y) := \int_{\mathbb{S}^n} Q(x, y) d\mu_{\mathbb{S}^n}(x)$ is constant. For $y = e_0$, we find with [5, Prop. 7.3.11] and some $c > 0$:

$$\begin{aligned} F(e_0) &= \int_{\mathbb{S}^n} Q(x, e_0) d\mu_{\mathbb{S}^n}(x) = c \int_{-1}^1 (1-t)^{-s/2} (1-t^2)^{(n-2)/2} dt \\ &= c \int_{-1}^1 (1+t)^{(n-2)/2} (1-t)^{(n-2-s)/2} dt \end{aligned} \tag{38}$$

and this integral is finite if and only if $s < n$. This implies that

$$\mu(\mathbb{S}^n \times \mathbb{S}^n) = \int_{\mathbb{S}^n} F(y) d\mu_{\mathbb{S}^n}(y) < \infty.$$

To see that μ is positive definite, we use the conformal map $\eta: \mathbb{R}^n \rightarrow \mathbb{S}^n$ and Remark 5.2(b), (c) to obtain

$$\int_{\mathbb{S}^n \times \mathbb{S}^n} f(x)\overline{f(y)} d\mu(x, y) = c \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(\eta(x))}{(1 + \|x\|^2)^{n+s/2}} \frac{\overline{f(\eta(y))}}{(1 + \|y\|^2)^{n+s/2}} \frac{dx dy}{\|x - y\|^s} \geq 0,$$

where we have used that the kernel $\|x - y\|^{-s}$ is positive definite for $0 < s < n$ (Lemma 2.13). \square

Lemma 5.6. For $f \in C(\mathbb{S}^n)$, the function

$$\Gamma(f): \mathbb{S}^n \rightarrow \mathbb{C}, \quad \Gamma(f)(y) := \int_{\mathbb{S}^n} f(x)Q(x, y) d\mu_{\mathbb{S}^n}(x)$$

is continuous.

Proof. It is enough to show that $F(g) := \int_{\mathbb{S}^n} f(x)Q(x, g.e_0) d\mu_{\mathbb{S}}(x)$ is a continuous function on $O_{n+1}(\mathbb{R})$. In view of $Q(x, g.y) = Q(g^{-1}.x, y)$, we have

$$F(g) = \int_{\mathbb{S}^n} f(g.x)(1 - \langle x, e_0 \rangle)^{-s/2} d\mu_{\mathbb{S}}(x).$$

The claim now follows by $|f(g.x)(1 - \langle x, e_0 \rangle)^{-s/2}| \leq \|f\|_{\infty}(1 - \langle x, e_0 \rangle)^{-s/2}$ and this function is integrable by (38) and $g \mapsto |f(g.x)|$ is continuous. \square

Definition 5.7 (The Hilbert spaces $\mathcal{H}_s, 0 < s < n$). Lemma 5.5 implies that μ is positive definite, so that we obtain with Remark 5.4 a corresponding Hilbert space $\mathcal{H}_s := \mathcal{H}_{\mu} \subseteq \mathcal{M}(\mathbb{S}^n)$ of measures. It is the completion of the range of the map

$$T_{\mu}: C(\mathbb{S}^n) \rightarrow \mathcal{M}(\mathbb{S}^n), \quad f \mapsto \Gamma(f)\mu_{\mathbb{S}},$$

with respect to the inner product

$$\langle \Gamma(f_1)\mu_{\mathbb{S}}, \Gamma(f_2)\mu_{\mathbb{S}} \rangle = \langle f_1, f_2 \rangle_{\mu}. \tag{39}$$

Lemma 5.8. The prescription

$$\pi_s(g)v := J_{g^{-1}}^{s/2-n} \cdot g_*v$$

defines a unitary representation of $G = O_{1,n+1}^+(\mathbb{R})$ on the Hilbert subspace $\mathcal{H}_s \subseteq \mathcal{M}(\mathbb{S}^n)$.

Proof. For $g \in G$ we recall from Remark 5.2(a) that

$$Q(g.x, g.y) = J_g^{-s/2}(x)J_g^{-s/2}(y)Q(x, y). \tag{40}$$

The measure $\mu_{\mathbb{S}}$ transforms according to

$$g_*\mu_{\mathbb{S}} = J_{g^{-1}}^n \cdot \mu_{\mathbb{S}} \quad \text{for } g \in G. \tag{41}$$

Combining (40) with (41) leads to

$$d(g_*\mu)(x, y) = J_{g^{-1}}(x)^{n-s/2}J_{g^{-1}}(y)^{n-s/2} d\mu(x, y). \tag{42}$$

Therefore the action of G on $C(\mathbb{S}^n)$ defined by

$$g.f := J_{g^{-1}}^{n-s/2} \cdot g_*f = g_*(J_g^{s/2-n} \cdot f) \tag{43}$$

satisfies

$$\begin{aligned} \int_{\mathbb{S}^n \times \mathbb{S}^n} (g \cdot f_1)(x) \cdot \overline{g \cdot f_2(y)} d\mu(x, y) &= \int_{\mathbb{S}^n \times \mathbb{S}^n} (J_g^{s/2-n} f_1)(x) (J_g^{s/2-n} f_2)(y) d(g_*^{-1} \mu)(x, y) \\ &= \int_{\mathbb{S}^n \times \mathbb{S}^n} f_1(x) \overline{f_2(x)} d\mu(x, y), \end{aligned}$$

i.e., the sesquilinear form defined by the measure μ on $C(\mathbb{S}^n)$ is invariant under the G -action. This in turn implies that T_μ is equivariant with respect to the action on $\mathcal{H}_s \subseteq \mathcal{M}(\mathbb{S}^n)$ given by

$$\begin{aligned} (g \cdot v)(f) &:= v(g^{-1} \cdot f) = v((g^{-1})_* (J_{g^{-1}}^{s/2-n} \cdot f)) = (g_* v)(J_{g^{-1}}^{s/2-n} \cdot f) \\ &= (J_{g^{-1}}^{s/2-n} \cdot g_* v)(f). \quad \square \end{aligned}$$

Remark 5.9. The preceding proof shows that the map

$$T_\mu : C(\mathbb{S}^n) \rightarrow \mathcal{H}_s, \quad f \mapsto \Gamma(f) \mu_{\mathbb{S}}$$

is G -equivariant with respect to the action on $C(\mathbb{S}^n)$ given by (43): $g \cdot f := J_{g^{-1}}^{n-s/2} \cdot g_* f$. As $\pi_s(g)(\Gamma(f) \mu_{\mathbb{S}}) = J_{g^{-1}}^{s/2} \cdot g_* \Gamma(f) \mu_{\mathbb{S}}$, it follows that

$$\Gamma(g \cdot f) = \Gamma(J_{g^{-1}}^{n-s/2} \cdot g_* f) = J_{g^{-1}}^{s/2} \cdot g_* \Gamma(f). \tag{44}$$

Remark 5.10. For $n = 1$, every diffeomorphism of \mathbb{S}^1 is conformal, so that one may expect that the representation π_s on \mathcal{H}_s can be extended from $O_{1,2}(\mathbb{R})$ to the group of all diffeomorphisms of \mathbb{S}^1 . Clearly, the conformal cocycle $J_g(x) = |\mathrm{d}g(x)|$ is well defined, but we also need the projective invariance of the kernel Q , i.e., the invariance up to a cocycle. In view of Remark 5.2(d), the projective invariance of Q under some $g \in \mathrm{Diff}(\mathbb{S}^1)$ implies the preservation of the absolute value of the cross ratio. This implies that g is fractional linear. In fact, composing with a suitable fractional linear map, we may assume that g fixes $0, 1$ and ∞ . Then the invariance of the absolute value of the cross ratio leads to $|g(x)| = |x|$ for $x \in \mathbb{R}$ (in stereographic coordinates). Now $g(1) = 1$ implies that $g = \mathrm{id}_{\mathbb{S}^1}$.

The following abstract lemma is useful to identify smooth and distribution vectors for the representation (π_s, \mathcal{H}_s) .

Lemma 5.11. *Let X be a topological vector space and $F_1 \subseteq F_2 \subseteq X$ be two linear subspaces, both carrying Fréchet topologies for which the inclusions $F_j \rightarrow X$, $j = 1, 2$, are continuous. Then the inclusion $\iota : F_1 \rightarrow F_2$ is also continuous.*

Proof. Since F_1 and F_2 are Fréchet spaces, it suffices to verify that the graph of ι is closed. So let $(v_n, \iota(v_n)) \rightarrow (v, w)$ in $F_1 \times F_2$. Then $v_n \rightarrow v$ in F_1 implies $v_n \rightarrow v$ in X . We also have $v_n \rightarrow w$ because the inclusion $F_2 \rightarrow X$ is continuous, and this leads to $w = v$. Hence the graph of ι is closed. \square

Lemma 5.12. For each $f \in C^\infty(\mathbb{S}^n)$, the element $\Gamma(f)\mu_{\mathbb{S}} \in \mathcal{H}_s$ is a smooth vector. Conversely, every smooth vector $v \in \mathcal{H}_s^\infty$ is of the form $v_f = f\mu_{\mathbb{S}}$ for a smooth function $f \in C^\infty(\mathbb{S}^n)$ and the so obtained map $\mathcal{H}_s^\infty \rightarrow C^\infty(\mathbb{S}^n)$, $v_f \mapsto f$ is continuous. In particular, its adjoint defines a linear map

$$\Psi: \mathcal{D}'(\mathbb{S}^n) \rightarrow \mathcal{H}_s^{-\infty}, \quad \Psi(D)(v_f) := D(f).$$

Proof. In view of $\|\Gamma(f)\mu_{\mathbb{S}}\|^2 = \langle f, f \rangle_\mu$ (cf. (39)), the map

$$C(\mathbb{S}^n) \rightarrow \mathcal{H}_s, \quad f \mapsto \Gamma(f)\mu_{\mathbb{S}}$$

is continuous. We have seen in (44) that it is G -equivariant with respect to the action on $C(\mathbb{S}^n)$ by $g.f = J_{g^{-1}}^{n-s/2} g_* f$. As the cocycle J is smooth, smooth functions f on \mathbb{S}^n have smooth orbit maps for this action, so that $\Gamma(f)\mu_{\mathbb{S}} \in \mathcal{H}_s^\infty$.

From [6, Thm. 3.3] we know that \mathcal{H}_s^∞ is spanned by $\pi_s(\mathcal{D}(G))\mathcal{H}_s$. Accordingly, we obtain for the maximal compact subgroup $K \cong \text{O}_{n+1}(\mathbb{R})$ of $G = \text{O}_{1,n+1}^+(\mathbb{R})$ that the corresponding space $\mathcal{H}_s^\infty(K)$ of smooth vectors for K is spanned by $\pi_s(\mathcal{D}(K))\mathcal{H}_s$. Since K acts on \mathbb{S}^n by isometries, $J_k = 1$ for every $k \in K$, so that $\pi_s(k)v = k_* v$. As \mathcal{H}_s is realized in $\mathcal{M}(\mathbb{S}^n)$, which can be identified with a subspace of $\mathcal{M}(K)$ because K acts transitively on \mathbb{S}^n , it follows that

$$\mathcal{H}_s^\infty \subseteq \text{span}(\pi_s(\mathcal{D}(K))\mathcal{H}_s) \subseteq \mathcal{D}(K) * \mathcal{M}(\mathbb{S}^n) \subseteq C^\infty(\mathbb{S}^n) \cdot \mu_{\mathbb{S}}$$

(cf. [53, Prop. A.2.4.1]). From the continuity of the inclusions $\mathcal{H}_s^\infty \rightarrow \mathcal{H}_s^\infty(K) \rightarrow \mathcal{M}(\mathbb{S}^n)$ and Lemma 5.11, it now follows that the linear map $\mathcal{H}_s^\infty \hookrightarrow C^\infty(\mathbb{S}^n)$ is also continuous. Its adjoint therefore defines a continuous linear map $\mathcal{D}'(\mathbb{S}^n) \rightarrow \mathcal{H}_s^{-\infty}$. \square

Remark 5.13. The representations $(\pi_s, \mathcal{H}_s)_{0 < s < n}$ are complementary series representations. Those representations are well known and arise from intertwining operators, usually given by singular integral operators [31]. The reader finds a detailed discussion in Section 7.5 of [5].

6. Reflection positivity for π_s

In this section we construct explicit reflection positive distribution vectors for the complementary series representations $(\pi_s, \mathcal{H}_s)_{0 < s < n}$ introduced in the preceding section. It turns out that these representations of the conformal group $\text{O}_{1,n+1}(\mathbb{R})$ on \mathbb{R}^n provide a natural context for relating reflection positivity, resp., positive definiteness of kernels of the form $K(x, \tau y)$ (cf. Example 1.7) on open half spaces and open balls. In the following we write $G = \text{O}_{1,n+1}^+(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{o}_{1,n+1}(\mathbb{R})$ for the Lie algebra of G .

6.1. The half space picture

Formulas for kernels and questions of positive definiteness turn out to be simpler for the open half space \mathbb{R}_+^n compared to the open unit ball \mathcal{D} . In particular, this connects much better with involutive semigroups and integral representations of positive definite functions thereon. We therefore discuss this case first.

As before, let

$$\Omega = \{z \in \mathbb{R}^n : z_0 > 0, [z, z] > 0\} \subseteq \mathbb{R}^n$$

be the forward light cone and $T_\Omega = \Omega + i\mathbb{R}^n \subseteq \mathbb{C}^n$ be the corresponding tube domain. Then it is easy to show that the complex bilinear extension of $[\cdot, \cdot]$ to \mathbb{C}^n satisfies

$$\Delta(z) := [z, z] \neq 0 \quad \text{for } z \in T_\Omega.$$

Since T_Ω is simply connected, there exists a holomorphic function $\log[z, z]$ taking the value 0 in e_0 . We thus obtain on this domain for each $\lambda \in \mathbb{R}$ a holomorphic function

$$\Delta^{-\lambda}: T_\Omega \rightarrow \mathbb{C}^\times, \quad z \mapsto e^{-\lambda \log[z, z]}.$$

Since $(x + iy)^* = x - iy$ defines on T_Ω the structure of a complex involutive semigroup, the function Δ^λ defines a kernel $(z, w) \mapsto \Delta^{-\lambda}(z + w^*)$ on T_Ω . From [7, Thm. XIII.2.7], we know that this kernel is positive definite, i.e., $\Delta^{-\lambda}$ is a positive definite function on T_Ω , if and only if

$$\lambda \in \mathcal{W} := \left\{ 0, \frac{n-2}{2} \right\} \cup \left] \frac{n-2}{2}, \infty \right[. \tag{45}$$

From [7, Lemma X.4.4 and p. 262] it now follows that this is equivalent to the positive definiteness of the function $\Delta^{-\lambda}$ on the light cone Ω .

The domain

$$\Omega^c = \{z \in T_\Omega : z_0 \in \mathbb{R}, z_1, \dots, z_{n-1} \in i\mathbb{R}\} = \mathbb{R}_+ \times i\mathbb{R}^n$$

is a totally real submanifold and also an involutive subsemigroup of T_Ω . Therefore Theorem A.1 implies that the restriction of $\Delta^{-\lambda}$ to Ω^c is a positive definite function if and only if (45) holds.

Define $\iota: \mathbb{R}^n \rightarrow \mathbb{C}^n, x \mapsto (x_0, ix_1, \dots, ix_{n-1})$. Then ι restricts to an isomorphism of involutive semigroups $\mathbb{R}_+^n \rightarrow \Omega^c$, where the involution on \mathbb{R}_+^n is given by

$$(x_0, x_1, \dots, x_{n-1})^\sharp = (x_0, -x_1, \dots, -x_{n-1}).$$

In view of $\Delta(\iota(x)) = [\iota(x), \iota(x)] = \|x\|^2$, we have for $\lambda = s/2$ the relation $(\iota^* \Delta^{-\lambda})(x) = \|x\|^{-s}$. This leads to:

Proposition 6.1. *The function $\|x\|^{-s}$ is positive definite on the involutive semigroup (\mathbb{R}_+^n, \sharp) if and only if $s = 0$ or $s \geq \max\{0, n - 2\}$.*

Proof. For $n \geq 2$ this follows from the preceding discussion, and for $n = 1$ we have already seen in Example 3.6 that the function x^{-s} is positive definite on \mathbb{R}_+ for $s \geq 0$. \square

6.2. The ball picture

Proposition 6.2. *The kernel*

$$R(x, y) := (1 - 2\langle x, y \rangle + \|x\|^2 \|y\|^2)^{-s/2}$$

on the open unit ball $\mathcal{D} \subseteq \mathbb{R}^n$ is positive definite if and only if

$$s = 0 \quad \text{or} \quad s \geq \max(0, n - 2).$$

Proof. Open half spaces and open balls in \mathbb{R}^n are conformally equivalent. The equivalence is obtained by rotation a lower hemisphere (open ball) in the conformal compactification into a right hemisphere, which corresponds to a half space. Let $c \in O_{n+1}(\mathbb{R})$ be an involution exchanging \mathbb{R}_+^n with the unit ball \mathcal{D} and \tilde{c} be the corresponding map on \mathbb{R}^n defined by $\eta \circ \tilde{c} = c \circ \eta$ (Remark 5.3). Then $J_c = 1$, so that c leaves the kernel Q invariant.

Let $\tau \in G$ be the element inducing on \mathbb{R}^n the reflection $\tilde{\tau}$ in $\partial\mathbb{R}_+^n \cong \mathbb{R}^{n-1}$, and σ be the reflection in $\partial\mathcal{D}$. Then $\tau = c\sigma c$. Now the invariance of the kernel Q on \mathbb{S}^n under c leads to

$$\begin{aligned} K(\tilde{\tau}(x), y) &= Q(\eta(\tilde{\tau}(x)), \eta(y)) = Q(\tau\eta(x), \eta(y)) = Q(\sigma c\eta(x), c\eta(y)) \\ &= Q(\eta(\tilde{\sigma}\tilde{c}(x)), \eta(\tilde{c}(y))) = K(\tilde{\sigma}(\tilde{c}(x)), \tilde{c}(y)). \end{aligned}$$

Clearly, this kernel is positive definite on \mathbb{R}_+^n if and only if $K(\tilde{\sigma}(x), y)$ is positive definite on \mathcal{D} .

From $\tilde{\sigma}(x) = \frac{x}{\|x\|^2}$ we obtain

$$\|x\|^2 \|\tilde{\sigma}(x) - y\|^2 = \left\| \frac{x}{\|x\|} - \|x\|y \right\|^2 = 1 - 2\langle x, y \rangle + \|x\|^2 \|y\|^2.$$

Therefore the pullback of the kernel $Q(\sigma(x), y)$ under η is given by

$$\begin{aligned} K(\tilde{\sigma}(x), y) &= (1 + \|\sigma(x)\|^2)^{s/2} \|\sigma(x) - y\|^{-s} (1 + \|y\|^2)^{s/2} \\ &= (1 + \|x\|^2)^{s/2} \|x\|^{-s} \|\sigma(x) - y\|^{-s} (1 + \|y\|^2)^{s/2} \\ &= (1 + \|x\|^2)^{s/2} (1 - 2\langle x, y \rangle + \|x\|^2 \|y\|^2)^{-s/2} (1 + \|y\|^2)^{s/2}. \end{aligned} \tag{46}$$

We conclude that the kernel $K(\tilde{\tau}(x), y)$ is positive definite if and only if R is positive definite.

Next we observe that

$$\begin{aligned} K(\tilde{\tau}(x), y) &= (1 + \|\tilde{\tau}(x)\|^2)^{s/2} \|\tilde{\tau}(x) - y\|^{-s} (1 + \|y\|^2)^{s/2} \\ &= (1 + \|x\|^2)^{s/2} ((x_0 + y_0)^2 + \|x' - y'\|^2)^{-s/2} (1 + \|y\|^2)^{s/2} \end{aligned}$$

is positive definite if and only if the kernel

$$(x, y) \mapsto ((x_0 + y_0)^2 + \|x' - y'\|^2)^{-s/2}$$

is positive definite on \mathbb{R}_+^n . This means that the function $\Delta(x) = \|x\|^{-s}$ is positive definite on the involutive semigroup $(\mathbb{R}_+^n, -\tau)$. In view of Proposition 6.1, this is equivalent to $s = 0$ or $s \geq \max(0, n - 2)$. \square

Remark 6.3. For an alternative proof of the preceding proposition, one can observe that the unit ball $\mathcal{D} \subseteq \mathbb{R}^n$ is a totally real submanifold of the Lie ball $\mathcal{D}_{\mathbb{C}} \subseteq \mathbb{C}^n$. Since R is a power of the Bergman kernel on the Lie ball which is biholomorphic to the tube domain T_{Ω} , one can also derive our result from [7, Thm. XIII.2.7] by using Theorem A.1 and $\mathcal{D} = \mathcal{D}_{\mathbb{C}} \cap \mathbb{R}^n$.

Example 6.4. (a) For $n = 1$ we have $R(x, y) = (1 - 2xy + x^2y^2)^{-s/2} = (1 - xy)^{-s}$, which is positive definite for each $s \geq 0$.

(b) For $n = 2$ the kernel R is also positive definite for $s \geq 0$. This is basically due to the fact that it corresponds to the positive definiteness of the function $\Delta^{-s/2}(x) = [x, x]^{-s/2}$ on the open light cone Ω (cf. Section 6.1). From

$$[x, x] = x_0^2 - x_1^2 = (x_0 - x_1)(x_0 + x_1)$$

we obtain a factorization of this function, so that the positive definiteness of the functions $(x_0 \pm x_1)^{-s/2}$ on Ω for $s \geq 0$ implies that R is positive definite.

6.3. Reflection positivity on \mathcal{H}_s

Let us fix some notation used in this section. Let $\tilde{\mathcal{D}} \subseteq \mathbb{R}^n$ be the interior of the unit ball and $\mathcal{D} = \eta(\tilde{\mathcal{D}})$ be its image in \mathbb{S}^n . We consider the corresponding compression semigroup

$$S_{\mathcal{D}} = \{g \in G : g\mathcal{D} \subseteq \mathcal{D}\}$$

in G . Let $\sigma \in G$ be the element implementing the conformal reflection $\tilde{\sigma}(x) = x/\|x\|^2$ in $\mathbb{S}^{n-1} = \partial\tilde{\mathcal{D}}$. On \mathbb{S}^n , it acts by

$$(x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n).$$

According to (28), it is realized by the diagonal matrix

$$\sigma = \text{diag}(1, -1, 1, \dots, 1) \in O_{1,n+1}(\mathbb{R}).$$

The corresponding involution on $O_{1,n+1}(\mathbb{R})$ is given by the conjugation by the same matrix. $\text{Ad}(\sigma)$ defines an involution on \mathfrak{g} , so that \mathfrak{g} decomposes into ± 1 -eigenspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{h} = \text{Fix}(\text{Ad}(\sigma))$ and $\mathfrak{q} = \text{Fix}(-\text{Ad}(\sigma))$. The Lie algebra \mathfrak{h} is isomorphic to $\mathfrak{o}_{1,n}(\mathbb{R})$.

Proposition 6.5. *Let $\sigma \in G$ be the conformal reflection in $\partial\mathcal{D}$. Then*

$$S_{\mathcal{D}} = H \exp(C), \quad \text{where } H = O_{1,n}^+(\mathbb{R}) \subseteq G^\sigma,$$

and $C \subseteq \mathfrak{q}$ is a closed convex $\text{Ad}(H)$ -invariant cone.

Proof. Let \tilde{G}_0 be the simply connected covering group of the identity component $G_0 = O_{1,n+1}(\mathbb{R})_0 = \text{SO}_{1,n+1}^+(\mathbb{R})$ and $\hat{\sigma}$ denote the lift of the involution defined by conjugation with σ on G_0 to its universal covering group \tilde{G}_0 . Then the group $(\tilde{G}_0)^{\hat{\sigma}}$ is connected [33, Thm. IV.3.4]. From [16, Thm. VI.11, Rem. VI.12] we thus derive that

$$S_{\mathcal{D}} \cap G_0 = \text{SO}_{1,n}^+(\mathbb{R}) \exp(C)$$

for a proper closed convex cone $C \subseteq \mathfrak{q}$ which is invariant under $\text{Ad}(\text{SO}_{1,n}^+(\mathbb{R}))$.

Since the group G has two connected components determined by the values of the determinant, and

$$H = O_{1,n}^+(\mathbb{R}) \subseteq G^\sigma \cong O_1(\mathbb{R}) \times O_{1,n}^+(\mathbb{R}) = \{\mathbf{1}, \sigma\} \times O_{1,n}^+(\mathbb{R})$$

likewise does, it follows that $G = HG_0$. Clearly, G^σ preserves the fixed point set $\partial\mathcal{D}$ of σ in \mathbb{S}^n and σ exchanges the two connected components of its complement, which are both preserved by H . This shows that $H \subseteq S_{\mathcal{D}}$.

Pick $h_0 \in H$ with $\det(h_0) = -1$. If $s \in S_{\mathcal{D}} \setminus G_0$, then $h_0^{-1}s \in S_{\mathcal{D}} \cap G_0$, so that $S_{\mathcal{D}} = H(S_{\mathcal{D}} \cap G_0) = H \exp(C)$. \square

Let $K = O_n(\mathbb{R}) \subset O_{1,n+1}^+(\mathbb{R})$ and $H_K := H \cap K \simeq O_{n-1}(\mathbb{R})$. On the subgroup $A \cong (\mathbb{R}_+, \cdot) \subseteq G$, acting on \mathbb{R}^n by multiplication with positive scalars (cf. (29)), the involution σ acts by inversion, so that there exists a Lie algebra element x_ρ such that $A \cap S = \exp(\mathbb{R}_+x_\rho)$ is a one-parameter semigroup acting as $(]0, 1[, \cdot)$ on \mathbb{R}^n which commutes with H_K . Moreover, we have $C = \text{Ad}(H)\mathbb{R}_+x_\rho$ (cf. [17, Sect. 4.4]).

On the Hilbert space $\mathcal{H}_s \subseteq \mathcal{M}(\mathbb{S}^n)$, we consider the involution $\theta = \pi_s(\sigma)$, where $\sigma \in G = O_{1,n+1}^+(\mathbb{R})$ is as above.

Lemma 6.6. *If $s = 0$ or $s \geq n - 2$, then the closed subspace*

$$\tilde{\mathcal{E}}_+ := \overline{\{\Gamma(f)\mu_{\mathbb{S}} : \text{supp}(f) \subseteq \mathcal{D}\}} \subseteq \mathcal{H}_s$$

is θ -positive.

Proof. As σ acts isometrically on \mathbb{S}^n , we have $J_\sigma = 1$, so that

$$\theta v = \pi_s(\sigma)v = \sigma_*v.$$

For $f \in C_c(\mathcal{D})$ we thus find

$$\begin{aligned} \langle \theta \Gamma(f)\mu_{\mathbb{S}}, \Gamma(f)\mu_{\mathbb{S}} \rangle &= \langle \Gamma(\sigma_*f)\mu_{\mathbb{S}}, \Gamma(f)\mu_{\mathbb{S}} \rangle \\ &= \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f(\sigma(x)) \overline{f(y)} Q(x, y) d\mu_{\mathbb{S}}(x) d\mu_{\mathbb{S}}(y) \\ &= \int_{\mathcal{D}} \int_{\mathcal{D}} f(x) \overline{f(y)} Q(\sigma(x), y) d\mu_{\mathbb{S}}(x) d\mu_{\mathbb{S}}(y). \end{aligned}$$

To evaluate this integral in stereographic coordinates, we recall from (46) the proof of Proposition 6.2 that the pullback of the kernel $Q(\sigma(x), y)$ under η is

$$(1 + \|x\|^2)^{s/2} (1 - 2\langle x, y \rangle + \|x\|^2\|y\|^2)^{-s/2} (1 + \|y\|^2)^{s/2}.$$

We thus obtain with a positive constant c' :

$$\langle \theta v, v \rangle = c' \int_{\mathcal{D}} \int_{\mathcal{D}} \frac{f(x) \overline{f(y)} (1 + \|x\|^2)^{s/2} (1 + \|y\|^2)^{s/2}}{(1 - 2\langle x, y \rangle + \|x\|^2\|y\|^2)^{s/2}} \frac{dx}{(1 + \|x\|^2)^n} \frac{dy}{(1 + \|y\|^2)^n}.$$

That this expression is non-negative for $f \in C_c(\mathcal{D})$ follows from Proposition 6.2. \square

For the following theorem we recall the involution $\tau \in G$ inducing on \mathbb{R}^n the reflection $\tilde{\tau}$ in $\partial\mathbb{R}_+^n \cong \mathbb{R}^{n-1}$.

Theorem 6.7. *For $x \in \mathbb{S}^{n-1} = (\mathbb{S}^n)^\sigma$ let $\delta_x \in \mathcal{H}_s^{-\infty}$ be the delta measure in x . The triple $(\pi_s, \mathcal{H}_s, \delta_x)$ is a reflection positive distribution cyclic representation for (G, τ, S_D^0) if $s = 0$ or $n - 2 \leq s < n$.*

Proof. If $\varphi \in \mathcal{D}(G)$ then $\pi_s^{-\infty}(\varphi)\delta_x \in \mathcal{H}_s^\infty \subseteq \mathcal{D}(\mathbb{S}^n)\mu_{\mathbb{S}}$. Thus $\pi_s^{-\infty}(\varphi)\delta_x = \varphi^b \mu_{\mathbb{S}}$ for some uniquely determined $\varphi^b \in \mathcal{D}(\mathbb{S}^n)$.

To determine this function, we first have to identify the distribution vector δ_x as a measure on \mathbb{S}^n . Since $\langle \delta_x, \Gamma(f)\mu_{\mathbb{S}} \rangle = \overline{\Gamma(f)(x)}$ for $f \in C^\infty(\mathbb{S}^n)$, the measure corresponding to δ_x is

$$\Gamma^*(\delta_x) = Q_x \cdot \mu_{\mathbb{S}}$$

and therefore $\pi_s^{-\infty}(\varphi)\delta_x$ corresponds to the measure

$$\int_G \varphi(g)\pi_s(g)(Q_x \cdot \mu_{\mathbb{S}}) d\mu_G(g) = \left(\int_G \varphi(g)J_g^{s/2}g_*Q_x d\mu_G(g) \right) \cdot \mu_{\mathbb{S}},$$

which means that

$$\varphi^b = \int_G \varphi(g)J_g^{s/2}g_*Q_x d\mu_G(g).$$

Let G_x be the stabilizer of x in G . Then $G/G_x \simeq \mathbb{S}^n$ via $gG_x \mapsto g.x$. The quasi-invariant measure $\mu_{\mathbb{S}}$ on \mathbb{S}^n and the left invariant measure on G are related by⁵

$$\int_G f(g)J_g^n(x) d\mu_G(g) = \int_{\mathbb{S}^n} \int_{G_x} f(kp) d\mu_{G_x}(p) d\mu_{\mathbb{S}}(k.x).$$

From (40) we have $Q(g.x, g.y) = J_g^{-s/2}(x)J_g^{-s/2}(y)Q(x, y)$ and hence

$$\begin{aligned} (g_*Q_x)(y) &= Q(g^{-1}.y, x) = Q(y, g.x)J_g^{s/2}(x)J_g^{s/2}(g^{-1}.y) \\ &= Q(y, g.x)J_g^{s/2}(x)J_{g^{-1}}^{-s/2}(y). \end{aligned}$$

This leads to

$$\varphi^b = \int_G \varphi(g)J_g^{s/2}(x)Q_{g.x} d\mu_G(g) = \int_{\mathbb{S}^n} \int_{G_x} J_{gp}^{s/2-n}(x)\varphi(gp) d\mu_{G_x}(p)Q_{g.x} d\mu_{\mathbb{S}}(g.x).$$

⁵ This formula is most easily verified by showing that $f \mapsto \int_{\mathbb{S}^n} \int_{G_x} f(kp)J_{kp}^{-n}(x) d\mu_{G_x}(p) d\mu_{\mathbb{S}}(k.x)$ defines a left invariant integral on G .

This means that

$$\varphi^b = \Gamma(\Phi) \quad \text{with } \Phi(g.x) = \int_{G_x} J_{gp}^{s/2-n}(x)\varphi(gp) d\mu_{G_x}(p). \tag{47}$$

As $J_g(x) > 0$ for all g it follows that the map $\mathcal{D}(G) \rightarrow \mathcal{D}(\mathbb{S}^n)$, $\varphi \mapsto \Phi$, is surjective and hence that $\pi_s^{-\infty}(\mathcal{D}(G))\delta_x$ is dense in \mathcal{H}_s . Thus δ_x is a cyclic distribution vector.

Suppose that $\text{supp}(\varphi) \subset S_D^0$ and assume that $\Phi(g.x) \neq 0$. Then there exists $p \in G_x$ such that $gp \in \text{supp}(\varphi) \subseteq S_D^0$. Hence $g.x \in S_D^0.x \subset \mathcal{D}$. Thus $\text{supp } \Phi \subset \mathcal{D}$, and hence

$$\pi_s^{-\infty}(\mathcal{D}(S))\delta_x \subset \widetilde{\mathcal{E}}_+ \tag{48}$$

(cf. [Lemma 6.6](#)). The claim now follows from [Lemma 6.6](#). \square

Remark 6.8. As $O_n(\mathbb{R})$ acts transitively on \mathbb{S}^n we can in [\(47\)](#) assume that $g \in O_n(\mathbb{R})$. Then $J_g(x) = 1$ and $J_{gp}(x) = J_g(p.x)J_p(x) = J_p(x)$. Thus

$$\Phi(g.x) = \int_{G_x} J_p(x)^{s/2-n}(x)\varphi(gp) d\mu_{G_x}(p)$$

for $g \in O_n(\mathbb{R})$.

Remark 6.9. If α is a positive linear combination, or even an integral with respect to a positive Borel measure, of δ -distributions supported on \mathbb{S}^{n-1} , then [\(48\)](#) shows that $\pi_s^{-\infty}(\mathcal{D}(S))\alpha \subset \widetilde{\mathcal{E}}_+$. According to [Remark 5.13](#) and [\[5, p. 119\]](#), the representation (π_s, \mathcal{H}_s) is irreducible. Thus every non-zero distribution vector is cyclic. It then follows that $(\pi_s, \mathcal{H}_s, \alpha)$ is a reflection positive distribution cyclic representation. In particular this holds for the measure $\mu_{\mathbb{S}^{n-1}}$. We have

$$\Gamma^*(\mu_{\mathbb{S}^{n-1}}) = \int_{\mathbb{S}^{n-1}} Q_x d\mu_{\mathbb{S}^{n-1}}(x) \mu_{\mathbb{S}}.$$

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra corresponding to the involution τ on G and $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$ the c -dual symmetric Lie algebra. Let G^c denote the simply connected Lie group with Lie algebra \mathfrak{g}^c . Then τ defines an involution on G^c and $(G^c)^\tau$ is connected ([\[33, Thm. IV.3.4\]](#) or [\[14, Thm. 8.2, p. 320\]](#)). The symmetric space $G^c/(G^c)^\tau$ is the c -dual of G/H . As explained in [\[26, Sects. 6 and 10\]](#), see also [\[15,25\]](#), the reflection positivity and the Lüscher–Mack Theorem now gives an irreducible highest weight (or positive energy) representation of the c -dual group G^c on $\widehat{\mathcal{E}}$. Multiplying by i in the second coordinate transforms the Lorentz form $[x, y] = x_0y_0 - x_1y_1 - \dots - x_{n+1}y_{n+1}$ into the form

$$[x, y]_2 = x_0y_0 + x_1y_1 - x_2y_2 - \dots - x_{n+1}y_{n+1}.$$

Hence the group G^c , c -dual to $O_{1,n+1}^+(\mathbb{R})$, is locally isomorphic to $O_{2,n}(\mathbb{R})$. We point out that the condition that $s = 0$ or $n - 2 \leq s < n$ indicates that this construction does not carry over to infinite dimension, or the duality between $O_{1,\infty}^+(\mathbb{R})$ and $O_{2,\infty}(\mathbb{R})$. This is also reflected in

the fact, that the group $O_{2,\infty}(\mathbb{R})$ does not have any unitary highest weight representations ([44, pp. 276–277], [41, Thm. 7.5]).

Remark 6.10. We have seen above that the open cones Ω and \mathbb{R}_+^n , endowed with their natural semigroup involution, can be viewed as real forms of the tube $T_\Omega = \Omega + i\mathbb{R}^n$, endowed with its natural involution given by conjugation. Every positive definite function $\varphi: \Omega \rightarrow \mathbb{C}$ extends to T_Ω [50, Cor. to Thm. 4] and hence restricts to \mathbb{R}_+^n but the converse is not true. For bounded positive definite functions on Ω , we know that they are precisely the Laplace transforms $\varphi = \mathcal{L}(\mu)$ of measures μ on the dual cone $\widehat{\Omega} = \Omega^*$ whose Laplace transform is defined on Ω . Likewise bounded positive definite functions on \mathbb{R}_+^n are Fourier–Laplace transforms of measures on the closed half space $\mathbb{R}_+^n \supseteq \Omega^*$ and not all of them extend to holomorphic positive definite functions on T_Ω .

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Appendix A. Propagation of positive definiteness

In this appendix we discuss some useful results providing criteria for kernels on complex manifolds to be positive definite.

Let M be a connected complex Fréchet manifold, \overline{M} its complex conjugate, and $K: M \times \overline{M} \rightarrow \mathbb{C}$ be a holomorphic function. We call such functions *holomorphic kernels on M* . A submanifold $\Sigma \subseteq M$ is called *totally real* if, for each point $s \in \Sigma$, there exists a holomorphic chart $\varphi: U \rightarrow V_\mathbb{C}$, where U is an open neighborhood of s in M and $V_\mathbb{C}$ is a complexification of the real locally convex space V , such that $\varphi(U \cap \Sigma) = \varphi(U) \cap V$.

The following theorem generalizes [32, §1.8] from domains in \mathbb{C} to complex Fréchet manifolds.

Theorem A.1. *For a holomorphic kernel K on the connected complex Fréchet manifold M the following conditions are sufficient for K to be positive definite:*

- (i) K is positive definite on a non-empty open subset.
- (ii) K is positive definite on a non-empty totally real submanifold.

Proof. (i) **Step 1:** Let $\emptyset \neq U \subseteq M$ be an open subset on which K is positive definite and $\mathcal{H} := \mathcal{H}_{K|_{U \times U}} \subseteq \mathcal{O}(U, \mathbb{C})$ be the corresponding reproducing kernel Hilbert space. We want to show that the corresponding realization map

$$\gamma: U \rightarrow \mathcal{H}, \quad \gamma(s) = K_s, \quad K_s(t) = K(t, s) = \langle K_s, K_t \rangle$$

extends to an antiholomorphic map $\gamma: M \rightarrow \mathcal{H}$.

If this is the case, then, for each $s \in U$, the map $M \rightarrow \mathbb{C}, m \mapsto \langle K_s, \gamma(m) \rangle = \overline{\gamma(m)(s)}$ is holomorphic. For $m \in M$ we have

$$\overline{\gamma(m)(s)} = \overline{K_m(s)} = \overline{K(s, m)} = K(m, s),$$

so that the uniqueness of analytic continuation implies that $\gamma(m)(s) = K_m(s)$, i.e., $\gamma(m) = K_m|_U$. We conclude in particular that, whenever γ exists on some connected open subset $N \subseteq M$ intersecting U , we necessarily have $\gamma(n) = K_n|_U$ for each $n \in N$.

Next we note that the function

$$M \times \overline{M} \rightarrow \mathbb{C}, \quad (m, n) \mapsto \langle \gamma(n), \gamma(m) \rangle$$

is holomorphic and coincides for $(m, n) \in U \times U$ with $K(m, n)$. By uniqueness of analytic continuation, we thus obtain

$$K(m, n) = \langle \gamma(n), \gamma(m) \rangle \quad \text{for } m, n \in M,$$

and hence that K is positive definite on $N \times N$.

Step 2: Suppose that N_1 and N_2 are two open connected subsets of M containing U on which antiholomorphic extensions $\gamma^{N_1}: N_1 \rightarrow \mathcal{H}$ and $\gamma^{N_2}: N_2 \rightarrow \mathcal{H}$ exist. For each $n \in N_1 \cap N_2$ Step 1 implies that $\gamma^{N_1}(n) = K_n|_U = \gamma^{N_2}(n)$, so that $\gamma^{N_1}|_{N_1 \cap N_2} = \gamma^{N_2}|_{N_1 \cap N_2}$. Therefore these two maps combine to a holomorphic map

$$\gamma^{N_1 \cup N_2}: N_1 \cup N_2 \rightarrow \mathcal{H}.$$

Let $N \subseteq M$ be the union of all open connected subsets of M containing U on which an antiholomorphic extension of γ exists. Then γ extends to a holomorphic map on N , and, in view of Step 1, it only remains to show that $N = M$.

Step 3: $N = M$. We argue by contradiction. Suppose that $m_0 \in M$ is a boundary point of N . For each open connected neighborhood U of m_0 the intersection $U \cap N$ is non-empty. Since K is in particular continuous, we may choose U so small that K is bounded on $U \times U$. Fixing a local chart around m_0 , we may further assume that U is biholomorphic to an open convex subset of a complex locally convex space X . In the following we therefore consider U as such an open subset.

Then the arguments in the proof of [40, Thm. 5.1] imply the existence of an open 0-neighborhood $U_1 \subseteq X$ (only depending on the bound for K on $U \times U$) such that for every point $m \in N \cap U$, the Taylor series of γ in m converges in $m + U_1$ to a holomorphic function which coincides with γ in $(m + U_1) \cap N$. Since N intersects $m_0 - U_1$, there exists an $m \in N$ with $m_0 \in m + U_1$. Therefore the holomorphic function $\gamma|_{(m+U_1) \cap N} \rightarrow \mathcal{H}$ extends to a holomorphic function $\widehat{\gamma}: m + U_1 \rightarrow \mathcal{H}$. Step 2 now implies that $m + U_1 \subseteq N$, contradicting $m_0 \in \partial N$. This proves that $M = N$.

(ii) Let $\emptyset \neq \Sigma \subseteq M$ be a totally real submanifold on which $K^\Sigma := K|_{\Sigma \times \Sigma}$ is positive definite. Then we obtain a reproducing kernel Hilbert space $\mathcal{H} := \mathcal{H}_{K^\Sigma}$ of functions on Σ . Again we want to show that the corresponding realization map

$$\gamma: \Sigma \rightarrow \mathcal{H}, \quad \gamma(s) = K_s^\Sigma, \quad K_s^\Sigma(t) = K^\Sigma(t, s) = \langle K_s^\Sigma, K_t^\Sigma \rangle$$

extends to an antiholomorphic map $\gamma: M \rightarrow \mathcal{H}$.

As in (i), any holomorphic extension $\gamma: U \rightarrow \mathcal{H}$ to an open connected subset intersecting Σ satisfies $\gamma(u)|_\Sigma = K_u|_\Sigma$ for $u \in U$ and

$$\langle \gamma(u), \gamma(v) \rangle = K(v, u) \quad \text{for } u, v, \in U. \tag{49}$$

Since K is real analytic on $\Sigma \times \Sigma$, it follows from [40, Thm. 5.1] that the map $\gamma: \Sigma \rightarrow \mathcal{H}$ is analytic. By definition, each point $s \in \Sigma$ has a connected neighborhood to which γ extends holomorphically. The preceding argument shows that all these extensions can be patched together, so that γ extends holomorphically to an open neighborhood U of Σ . Now (49) implies that K is positive definite on U . Therefore (ii) follows from (i). \square

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