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A Note on Intertwines of Infinite Graphs

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We present a construction of two infinite graphs G_1, G_2 and of an infinite set $\mathscr{F}$ of graphs such that $\mathscr{F}$ is an antichain with respect to the minor relation and, for every graph G in $\mathscr{F}$, both G_1 and G_2 are subgraphs of G but no graph obtained from G by deletion or contraction of an edge has both G_1 and G_2 as minors. These graphs show that the extension to infinite graphs of the intertwining conjecture of Lovász, Milgram, and Ungar fails.

1. INTRODUCTION

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$, the set of vertices, is an arbitrary (possibly infinite) set and $E(G)$, the set of edges, is a subset of the set of two-element subsets of $V(G)$. The class of finite graphs will be denoted $\mathscr{G}_{<\infty}$ and the class of graphs whose vertex set has the same cardinality as the set of real numbers will be denoted $\mathscr{G}_{\infty}$. A *minor* J of a graph G is a graph whose vertex set $V(J)$ is a set of pairwise disjoint non-null connected subgraphs of G; every edge $\{H, K\}$ of J corresponds to a path P in G having one endvertex in H and the other in K such that, except for the endvertices, P is disjoint from the vertices of J and from the paths of G corresponding to the other edges of J. A *topological minor* J of G is a minor of G in which every vertex is a single-vertex subgraph of G. For graphs G and H, we will write $H \leqslant_m G$ if H is isomorphic to a minor of G, and $H \leqslant_e G$ if H is isomorphic to a topological minor of G. An easy lemma about the minor relation can be stated as follows.

(1.1) LEMMA. *Let f be an isomorphism from H to a minor J of G and let C be a block of H. Then there is a block D of G such that the mapping defined, for all vertices v of C, by $f'(v) = f(v) \cap D$ is an isomorphism from C to a minor K of D.*

A proof of the above lemma may be found, for example, in [1].

A pair $(\mathscr{G}, \leqslant)$, consisting of a class of graphs $\mathscr{G}$ and a binary relation $\leqslant$ on $\mathscr{G}$, is a *quasi-order* if the relation $\leqslant$ is reflexive and transitive. A quasi-

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order $(\mathscr{G}, \leqslant)$ is a *well-quasi-order* if it contains no infinite antichains and no infinite descending chains. Suppose $(\mathscr{G}, \leqslant)$ is a quasi-order and G_1, G_2 are two elements of $\mathscr{G}$. The *class* $\mathscr{I}(G_1, G_2)$ *of intertwines of* G_1 *and* G_2 is defined as the subclass of $\mathscr{G}$ consisting of elements G that satisfy the following conditions:

(1.2) $G_1 \leqslant G$ and $G_2 \leqslant G$.

(1.3) If $G' \leqslant G$ but $G \nleqslant G'$, then $G_1 \nleqslant G'$ or $G_2 \nleqslant G'$.

The elements of $\mathscr{I}(G_1, G_2)$ are *intertwines* of G_1 and G_2. A quasi-order $(\mathscr{G}, \leqslant)$ satisfies the *finite intertwine property* if, for every pair G_1, G_2 of elements of $\mathscr{G}$, class of intertwines $\mathscr{I}(G_1, G_2)$ contains no infinite antichains. Robertson and Seymour proved in [2] that

(1.4) THEOREM. $(\mathscr{G}_{<\infty}, \leqslant_{\mathrm{m}})$ *is a well-quasi-order.*

It is clear that every well-quasi-order satisfies the finite intertwine property since it contains no infinite antichains. Therefore it also follows from [2] that

(1.5) THEOREM. $(\mathscr{G}_{<\infty}, \leqslant_{\mathrm{m}})$ *satisfies the finite intertwine property.*

It is not true, however, that every class of graphs that satisfies the finite intertwine property is a well-quasi-order. For example, $(\mathscr{G}_{<\infty}, \leqslant_{\mathrm{e}})$ also satisfies the finite intertwine property, as shown in [2]. Yet $(\mathscr{G}_{<\infty}, \leqslant_{\mathrm{e}})$ is not a well-quasi-order since it contains an infinite antichain consisting of the graphs $\{D_n\}_{n \geqslant 3}$, where each D_n is obtained from a circuit on n vertices by joining each pair of adjacent vertices of the circuit by a path of length two.

It has been shown by Thomas in [3] that $(\mathscr{G}_{\infty}, \leqslant_{\mathrm{m}})$ is not a well-quasi-order. In this paper we strengthen his result by showing that $(\mathscr{G}_{\infty}, \leqslant_{\mathrm{m}})$ does not satisfy the finite intertwine property. In particular, we shall construct two infinite graphs G_1, G_2 and an infinite antichain $\mathscr{F}$ in $(\mathscr{G}_{\infty}, \leqslant_{\mathrm{m}})$ such that, for every element G of $\mathscr{F}$:

(1.6) G is connected;

(1.7) both G_1 and G_2 are subgraphs of G; and

(1.8) if G' is obtained from G by deleting or contracting an edge, then $G_1 \nleqslant_{\mathrm{m}} G'$ or $G_2 \nleqslant_{\mathrm{m}} G'$.

Before describing this construction, we observe that (1.7) implies that both G_1 and G_2 are topological minors of G and hence both are minors of G. Moreover, if G is connected and G'' is such that $G'' \leqslant G$ but $G \nleqslant G''$, where $\leqslant$ is $\leqslant_{\mathrm{m}}$ or $\leqslant_{\mathrm{e}}$, then there is a graph G', which is obtained from G by deleting or contracting an edge, such that $G'' \leqslant G'$. Therefore the existence of infinite graphs G_1, G_2, and a class $\mathscr{F}$ as described in (1.6)–(1.8) implies the following two statements.

(1.9) THEOREM. *$(\mathscr{G}_\infty, \leqslant_m)$ does not satisfy the finite intertwine property.*

(1.10) THEOREM. *$(\mathscr{G}_\infty, \leqslant_e)$ does not satisfy the finite intertwine property.*

2. THE CONSTRUCTION

It has been shown in [1] that there is a graph G such that no graph obtained from G by deleting or contracting an edge contains a minor isomorphic to G. This graph will play a crucial role in the present paper and should be described in somewhat more detail. The construction presented in [1] starts with any infinite antichain of graphs in $(\mathscr{G}_\infty, \leqslant_m)$ such that every graph in this antichain has a countable subset of its vertex set that meets all the edges of the graph. The existence of such an antichain has been shown by Thomas in [3]. The graphs in the antichain are modified so that, in addition to forming an antichain $\mathscr{A}$ in $(\mathscr{G}_\infty, \leqslant_m)$, they are non-separable. Finally these modified graphs are assembled together so that the set of blocks of the resulting graph is exactly the set of graphs of the modified antichain.

We start the construction of G_1 and G_2 by partitioning the infinite antichain $\mathscr{A}$ of non-separable graphs into infinite antichains $\mathscr{A}_i$, where i runs through the set of all integers. Then, for each integer i, we follow the construction presented in [1] to construct a graph B_i. Then the following hold:

(2.1) The graph B_i is connected.

(2.2) The set of blocks of B_i is $\mathscr{A}_i$.

(2.3) No graph obtained from B_i by deleting or contracting an edge contains a minor isomorphic to B_i.

Oberve that by (1.1), if i and j are distinct integers, then $B_i \nleqslant_m B_j$, since no block of B_i is isomorphic to a minor of a block of B_j. Hence the set $\{B_i\}_{i\in\mathbb{Z}}$ is an antichain in $(\mathscr{G}_\infty, \leqslant_m)$. In every graph B_i distinguish one vertex and call it v_i. Construct G_1 by taking the disjoint union of the graphs B_i, over all odd integers i, and adding new edges $\{v_i, v_{i+2}\}$, for all odd integers i. Similarly, construct G_2 by taking the disjoint union of the graphs B_i, over all even integers i, and then adding new edges $\{v_i, v_{i+2}\}$, for all even integers i. Observe that

(2.4) if B is a block of G_1, then there is an odd integer i such that either B is in $\mathscr{A}_i$, or B consists of the vertices v_i and v_{i+2} and the edge joining them; and

(2.5) if B is a block of G_2, then there is an even integer i such that either B is in $\mathscr{A}_i$, or B consists of the vertices v_i and v_{i+2} and the edge joining them.

Now let j be an integer. We shall describe the construction of an intertwine F_j of G_1 and G_2. Begin by taking the disjoint union of G_1 with all of the graphs B_i for which i is even. Next, for each odd integer i, identify the vertex v_i of G_1 with the vertex v_{i+2j+1} of B_{i+2j+1}. Let F_j be the resulting graph. It is clear from (2.1) and from the construction that

(2.6) F_j is connected.

Define $\mathscr{F}$ as the set $\{F_j\}_{j\in\mathbb{Z}}$. We claim that the graphs G_1 and G_2 and the set of graphs $\mathscr{F}$ are such that $\mathscr{F}$ is an antichain in $(\mathscr{G}_{<\infty}, \leqslant_m)$ and, for every G in $\mathscr{F}$, conditions (1.6)–(1.8) hold. To prove this, first suppose that G is a member of $\mathscr{F}$. Then $G = F_j$ for some integer j. By the construction of F_j, it is clear that G_1, G_2, and G satisfy(1.7).

Now suppose that G' has been obtained from G by deletion or contraction of an edge e and both G_1, G_2 are isomorphic to minors J_1 and J_2, respectively, of G'. Let the isomorphism from G_1 to J_1 and from G_2 to J_2 be denoted f_1 and f_2, respectively.

Suppose that e is an edge of B_i for some i. We shall assume that i is odd; the argument in the case when i is even is very similar. Then, by (2.2), e is an edge of some block C of B_i which is a member of $\mathscr{A}_i$. By construction, C is also a block of G_1. Therefore, by (1.1), there is a block D of G' such that the mapping defined, for all vertices v of C, by $f'_1(v) = f_1(v) \cap D$ is an isomorphism from C to a minor of D. Since G' has been obtained from G by deleting or contracting e, either D is a block of G different from C, or D is a minor of C. As C is not isomorphic to a minor of any block of G other than itself, the former is impossible, and hence the isomorphism f_1 restricted to C induces an isomorphism from C to a proper minor of C. Similarly, f_1 restricted to any block C' of B_i different from C induces an isomorphism from C' to a minor of itself. Therefore f_1 restricted to B_i induces an isomorphism from B_i to a minor of the graph obtained from B_i by deleting or contracting e; a contradiction to (2.3).

Now suppose that e does not belong to any B_i. Then, by (2.4), e is an edge of G_1 such that $e = \{v_j, v_{j+2}\}$ for some odd integer j. If G' was obtained from G by deleting e, then G' is disconnected with B_j and B_{j+2} in distinct components. Since G_1 is connected, so is J_1, and hence J_1 is also a minor of a component K of G'. Without loss of generality, we may assume that K does not contain B_j. Let C be a block of B_j. Then C is also a block of G_1 and, by (1.1), the isomorphism f_1 restricted to C induces an isomorphism from C to a minor of a block D of K. But, by construction, either D is an element of $\mathscr{A}$ different from C, or D contains just one edge. In each case we obtain a contradiction, since $\mathscr{A}$ is an antichain in $(\mathscr{G}_\infty, \leqslant_m)$ and C contains infinitely many edges.

Finally, suppose that $e = \{v_j, v_{j+2}\}$, for some odd integer j, and G' has been obtained from G by contracting e. Let v denote the vertex of G' that has been obtained by identifying v_j and v_{j+2}. By (2.2), there are blocks C and D of G_1 such that both are members of $\mathscr{A}$, $v_j \in C$ and $v_{j+2} \in D$. Let C' and D' denote the blocks of G' that correspond to C and D, respectively. Then C' and D' have v as their only common vertex, while C and D are disjoint. Observe that $f_1(v_j)$ and $f_1(v_{j+2})$ are vertices of J_1 and so they are disjoint non-null connected subgraphs of G'. One of these subgraphs, say $f_1(v_j)$, does not contain v. Yet, by (1.1), f_1 restricted to C induces an isomorphism from C to a minor of a block of G'. Now every block of G_1, except the one that consists of the single edge $\{v_j, v_{j+2}\}$ and its endvertices, is also a block of G'. Therefore the restriction of f_1 to B_j induces an isomorphism from B_j to a minor of B_j in which no vertex contains v. This contradicts (2.3) and hence (1.8) follows.

To show that $\mathscr{F}$ is an antichain in $(\mathscr{G}_\infty, \leqslant_m)$, suppose, that F_i is isomorphic to a minor J of F_j, for some distinct integers i and j. By construction, F_i is not isomorphic to F_j, and, by (2.6), F_j is connected, so at least one edge of F_j has been deleted or contracted in obtaining J. But then, by (1.8), $G_1 \nleqslant_m J$ or $G_2 \nleqslant_m J$, while, by (1.7), $G_1 \leqslant_m F_i$ and $G_2 \leqslant_m F_i$, a contradiction.

We conclude that $\{F_j\}_{j \in \mathbb{Z}}$ is an antichain in $(\mathscr{G}_\infty, \leqslant_m)$ that is contained in $\mathscr{I}(G_1, G_2)$ and hence (1.9) and (1.10) follow. We close by remarking that all presently known counterexamples to the extension of Wagner's conjecture to infinite graphs, Seymour's self-minor conjecture, and the extension of the intertwining conjecture to infinite graphs have uncountable sets of vertices. All three conjectures remain open for countable graphs.

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