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A decomposition of locally finite graphs

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Abstract

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We prove that every infinite, connected, locally finite graph G can be expressed as an edge-disjoint union of a leafless tree T, rooted at an arbitrarily chosen vertex of G, and a collection of finite graphs $H_1, H_2, H_3, \ldots$ such that, for all i less than j, the vertices common to H_i and H_j lie in T, and no vertex of H_j lies on T between a vertex of $H_i \cap T$ and the root.

1. Introduction

In what follows, graphs may be finite or infinite, but have no loops or multiple edges. A *rooted graph* (G, r) is a graph G with a distinguished vertex r called the *root*. The vertex set of a graph G will be denoted by $V(G)$, and the edge set of G by $E(G)$. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, is the cardinality of the set of vertices of G joined to v by an edge. A graph is called *locally finite* if the degree of each of its vertices is finite. By a *path* we will mean a simple open path. By $d_G(x, y)$ we will denote the distance in G between two of its vertices x and y. Similarly, if H is a subgraph of G, then $d_G(x, H)$ denotes the distance in G between x and H, that is, the minimal distance between x and y over all vertices y of H. The subgraph of G induced by those vertices of G whose distance from r does not exceed n will be denoted by $B_G(r, n)$.

We will follow the standard notation for ordinals, in particular, ω will denote the first infinite ordinal, and ω_1 will denote the first uncountable ordinal.

Suppose (T, r) is a rooted tree. The vertex set of T is partially ordered by the following relation:

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(1) $x \leqslant_T y$ if and only if x lies on the path joining r to y in T.
We also define the relation $<_T$ as follows:
(2) $x <_T y$ if and only if $x \leqslant_T y$ and $x \neq y$.
A *leaf* is a vertex of T that is maximal with respect to $\leqslant_T$. A rooted tree (S, q) is a *subtree* of (T, r) if S is a subtree of T and the relation $\leqslant_S$ is the restriction of $\leqslant_T$ to $V(S)$. If (T, r) is leafless and the relation $\leqslant_T$ is a linear order on the vertex set of T, then (T, r) is called a *ray rooted at r*.

Let J be a subgraph of G. A *vertex of attachment* of J in G is a vertex of J that is incident with an edge of G which is not an edge of J. A subgraph H is said to be *J-detached* if all vertices of attachment of H in G are in J. A *bridge B* of J in G is a subgraph of G satisfying the following three conditions:
(1) B is not a subgraph of J.
(2) B is J-detached in G.
(3) No proper subgraph of B satisfies both (1) and (2).

We will investigate countable graphs in search of infinite subtrees whose bridges have their vertices of attachment arranged in a particular way. One of the theorems of this type may be stated as follows.

Theorem 1.1. *Let G be a connected countable graph rooted at r. Then there is a subtree T of G, also rooted at r, such that, for every bridge H of T in G, the following two conditions hold.*
(1) *H is finite.*
(2) *The set $V(H \cap T)$ is linearly ordered by $\leqslant_T$.*
If, additionally, G is assumed to be locally finite, then T may be chosen so that it is leafless.

Theorem 1.1 follows from a result of Jung [2]. Its consequences reach into the subject of end-faithful spanning trees addressed by Halin [1]. We will discuss an arrangement of bridges of T in G which is somewhat different from that stated in Theorem 1.1 and which will lead to a decomposition of locally finite graphs.

2. Finite bundles

Let us consider rooted, leafless subtrees of an infinite, connected, locally finite graph (G, r) ordered by the subgraph relation. Zorn's lemma guarantees that every leafless subtree (T, r) of (G, r) is contained in a maximal leafless subtree, which will be denoted by $(\bar{T}, r)$.

Lemma 2.1. *If (T, r) is a maximal leafless subtree of an infinite, connected, locally finite graph (G, r), then all bridges of T in G are finite.*

Proof. Suppose H is an infinite bridge of T. Let u be a vertex common to T and H. Then, since H is infinite and G is locally finite, König's lemma implies that there is a ray P rooted at u and contained in H. But then $T \cup P$ is a tree that properly contains T, thus contradicting the maximality of T. □

For H_1 and H_2, two bridges of T in G, we say that H_1 *reaches above* H_2 if there are vertices v_1 and v_2 of $H_1 \cap T$ and $H_2 \cap T$, respectively, such that $v_2 <_T v_1$. Consider the binary relation on the set of bridges of T in G that is the transitive and reflexive closure of 'reaching above'. A *bundle generated by a bridge* H_0 of T is the set consisting of all bridges H of T in G such that (H_0, H) is in this binary relation.

The goal of this section is to show that every rooted, connected, locally finite graph (G, r) contains as a subgraph a maximal leafless tree (T, r) all of whose bridges generate finite bundles. For the remainder of this section, (T, r) will denote a maximal, rooted, leafless subtree of the rooted, connected, locally finite graph (G, r). Moreover, all the bridges considered will be bridges in G.

Lemma 2.2. *Every bridge H of T reaches above at most finitely many other bridges.*

Proof. The proof follows immediately from the finiteness of H and the local finiteness of G. □

Lemma 2.3. *If a bridge H_0 of T generates an infinite bundle, then there is a sequence $H_0, H_1, H_2, \ldots$ of bridges of T such that H_i reaches above H_{i+1} for $i \in \omega$.*

Proof. Let the set of bridges of T be the vertex set of a directed graph Γ in which two bridges H' and H'' form a directed edge (H', H'') if and only if H' reaches above H''. The assumption that the bundle generated by H_0 is infinite implies that infinitely many vertices are reachable from H_0 by a directed path, and Lemma 2.2 says that every vertex of Γ has finite outdegree. Thus, by König's lemma, there is an infinite directed path in Γ starting at H_0. The consecutive vertices of this path give the desired sequence of bridges. □

Lemma 2.4. *If $H_0, H_1, H_2, \ldots$ is a sequence of bridges of T such that H_i reaches above H_{i+1} for all $i \in \omega$, then there is a sequence $i_0 < i_1 < i_2 < \ldots$ of integers such that*

(1) $d_T(r, H_{i_0}) < d_T(r, H_{i_m})$ *for* $m \geqslant 1$,

(2) H_{i_m} *reaches above* $H_{i_{m+1}}$ *for* $m \in \omega$, *and*

(3) H_{i_m} *does not reach above* $H_{i_{m+n}}$ *for* $m \in \omega$, $n > 1$.

Proof. Let $d = \min\{d_G(r, H_m) \mid m \in \omega\}$. Since G is locally finite, $d_G(r, H_m) = d$ for finitely many m. Let $i_0 = \max\{m \mid d_G(r, H_m) = d\}$. Inductively, assume that H_{i_k} has been defined. By assumption, H_{i_k} reaches above $H_{i_{k+1}}$. Moreover, by Lemma 2.2, H_{i_k} reaches above only finitely many other bridges. Now put $i_{k+1} = \max\{m \mid H_{i_k}$ reaches above $H_m\}$. The conclusion follows. □

If some bridge H of T generates an infinite bundle, we shall describe a construction of another tree (T^+, r), an 'improvement' of (T, r). Now, to formalize this improvement, suppose that (S, r) is a leafless rooted subtree, not necessarily maximal, of (G, r) and define

$$a_n(S) = \sum \{\deg_S(t) \mid t \in V(S) \text{ and } d_S(r, t) = n\}$$

and

$$a(S) = (a_0(S), a_1(S), a_2(S), \dots).$$

Let $\leqslant$ denote the lexicographical order on the set of sequences $a(S)$. It is easy to verify the following lemma.

Lemma 2.5. *Let S_1 and S_2 be leafless subtrees of (G, r) such that $B_{S_1}(r, n) \subseteq S_2$. Then $a_m(S_1) \leqslant a_m(S_2)$ for all $m < n$.*

The next lemma describes the construction of the 'improved' tree (T^+, r). It is the cornerstone of the proof of the main theorem.

Lemma 2.6. *If some bridge of T generates an infinite bundle, then there is a rooted, maximal, leafless subtree (T^+, r) of (G, r) and a number $n \in \omega$ such that*

(1) $T^+ \supseteq B_T(r, n)$,
(2) $a_n(T^+) > a_n(T)$, *and*
(3) $a(T^+) > a(T)$.

Proof. Let $H_{i_0}, H_{i_1}, H_{i_2}, \dots$ be the sequence of bridges found in Lemma 2.4 and relabel this sequence $H_0, H_1, H_2, \dots$. For all i in ω, the bridge H_i reaches above H_{i+1}; so, there are vertices v_i and u_{i+1} of $T \cap H_i$ and $T \cap H_{i+1}$, respectively, such that $u_{i+1} <_T v_i$. Moreover, let u_0 be a vertex of $T \cap H_0$ for which $d_T(r, u_0) = d_T(r, T \cap H_0)$. Observe that, by Lemma 2.4(1),

$$d_T(r, u_0) < d_T(r, u_m) \quad \text{for all } m \geqslant 1. \tag{2.1}$$

Denote by U and V the sets $\{u_i \mid i \in \omega\}$ and $\{v_i \mid i \in \omega\}$, respectively. Suppose w is a vertex of T. A rooted subtree (S, w) of (T, r) is said to be *U-terminal* if it satisfies the following conditions:

(c1) S has no vertices from V, except possibly for w.
(c2) If $u \in U \cap V(S)$, then either $u = w$ or u is a leaf of S.
(c3) All leaves of S lie in U.

By w^U we will denote the maximal U-terminal tree rooted at w. For each $i \in \omega$, let P_i denote a path from u_i to v_i which lies inside H_i and is internally disjoint from T. Define

$$T_{-1} = r^U$$

and, inductively,

$$T_i = T_{i-1} \cup P_i \cup u_i^U \cup v_i^U \quad \text{for all } i \in \omega.$$

Finally, put

$$T^* = \bigcup_{i\in\omega} T_i.$$

Then we claim that T^* is a leafless tree.

To prove this claim, we shall first establish a few properties of the T_i's. First we show that

$$u_{i+1} \in V(T_i) \quad \text{for } i = -1, 0, 1, 2, \ldots \,. \tag{2.2}$$

In order to see this, let w denote a vertex of $T \cap (U \cup V \cup \{r\})$ such that $w <_T u_{i+1}$ and w is a maximal such vertex with respect to $\leqslant_T$. Now suppose that $w = u_j$ or $w = v_j$ for some j exceeding $i+1$. In the first case, $u_j <_T u_{i+1} <_T v_i$; in the second, $v_j <_T u_{i+1} <_T v_i$. In both cases H_i reaches above H_j and Lemma 2.4(3) is contradicted. If $w = v_{i+1}$, then $u_{i+2} <_T v_{i+1} <_T u_{i+1} <_T v_i$. Thus, H_i reaches above H_{i+2}, a contradiction to Lemma 2.4(3). Therefore,

$$w \in \{r\} \cup \bigcup_{j\leqslant i} \{u_j, v_j\},$$

and, hence, w is a vertex of T_i, and w^U contains the vertex u_{i+1}. Thus, (2.2) holds.

It is straightforward to check that the following holds:

(i) *If j exceeds i, then v_j is not a vertex of T_i.*

Next we will show the following statement:

(ii) *If v is a member of V but not of U, then the tree v^U contains an edge.*

Suppose v_i is not in U and the tree v_i^U consists of the vertex v_i and no edges. Then $v_i <_T v_j$ for some $j \in \omega$ such that there is no vertex u in $U \cup V$ for which $v_i <_T u <_T v_j$. If $j < i$, then $u_{i+1} <_T v_i <_T v_j$. This means that H_j reaches above H_{i+1}, which contradicts Lemma 2.4(3). If $i < j$, then $u_{j+1} <_T v_i <_T v_j$ and, so, H_i reaches above H_{j+1}, which is also impossible. Hence, (ii) follows.

Now let us examine inductively the T_i's in order to establish the following properties:

(iii) *The graph T_i is a tree.*

(iv) *All leaves of T_i are contained in U.*

Observe that r^U is a U-terminal tree. Thus, (iii) and (iv) hold for $i = -1$. Inductively, assume that (iii) and (iv) hold for some i. We shall consecutively examine $T_i \cup u_{i+1}^U$, $(T_i \cup u_{i+1}^U) \cup P_{i+1}$, and T_{i+1}. Each of these graphs will be shown to be a tree by proving that it is a union of two trees which have exactly one vertex in common. First, consider $T_i \cap u_{i+1}^U$. It is nonnull as, by (2.2), it contains u_{i+1}. Suppose it contains another vertex v. Then $u_{i+1} <_T v$ and, at the same time, $v \in V(T_i)$. Thus, v is a vertex of P_j for some j not exceeding i, and P_j meets T only in the vertices u_j and v_j. If $v = u_j$, then

$$u_{i+1} <_T u_j <_T v_{j-1}, \tag{2.3}$$

which contradicts Lemma 2.4(3); and if $v=v_j$, then v_j is a vertex of u_{i+1}^U, contrary to (c1). Hence, $T_i \cup u_{i+1}^U$ is a union of two trees intersecting at exactly one vertex and, thus, is a tree itself.

Now consider $(T_i \cup u_{i+1}^U) \cup P_{i+1}$. The internal vertices of P_{i+1} are disjoint from $T_i \cup u_{i+1}^U$ since they lie in $H_{i+1} \setminus T$. By (i), the vertex v_{i+1} is not in T_i and, by (c1), it is not in u_{i+1}^U. Thus, the intersection of $T_i \cup u_{i+1}^U$ with P_{i+1} has exactly one vertex u_{i+1}; hence, $T_i \cup u_{i+1}^U \cup P_{i+1}$ is a tree.

Finally, consider T_{i+1}. Obviously, v_{i+1} is a vertex of both $T_i \cup u_{i+1}^U \cup P_{i+1}$ and v_{i+1}^U. Suppose v is another such vertex. Then $v_{i+1} <_T v$. If v is a vertex of T_i, then, for some j not exceeding i, either $v=u_j$ or $v=v_j$. In the first case, (2.3) holds, which contradicts Lemma 2.4(3); in the second case, v_j is a vertex of v_{i+1}^U, which contradicts (c1). Also, by (c1), vertex v_{i+1} is not in u_{i+1}^U, and if v were a vertex of P_{i+1}, then we would have $u_{i+2} <_T v_{i+1} <_T v = u_{i+1} <_T v_i$, which contradicts Lemma 2.4(3). Thus, T_{i+1} is a tree. This completes the inductive step for (iii).

For the inductive step for (iv), note that, by (c3), all the leaves that u_{i+1}^U and v_{i+1}^U contribute to T_{i+1} lie in U. Moreover, the path P_{i+1} contributes at most one leaf to T_{i+1}, and, by (ii), such a leaf is in U.

Now observe that T^* is an ascending union of the T_i's. Since, by (iii), all T_i's are trees, so is T^*. Moreover, if T^* had a leaf, it would have to be a leaf of some T_i and, thus, by (iv), would have to lie in U. Yet for every $i \in \omega$, we have $u_i <_{T_i} v_i$ and, thus, $u_i <_{T^*} v_i$. Therefore, u_i is not a leaf of T^* and, hence, T^* is leafless.

Let $n = d_T(r, u_0)$ and observe that, by (2.1), $n < d_T(r, u_i)$ for all $i \in \omega$, $i \geqslant 1$. By the definition of a U-terminal tree,

$$B_T(r, n) \subseteq r^U. \tag{2.4}$$

Next we show that, for every vertex s of T^* such that $d_T(r, s) = n$,

$$\deg_{T^*}(s) \geqslant \deg_T(s) + \varepsilon, \quad \text{where } \varepsilon = \begin{cases} 0 & \text{if } s \neq u_0, \\ 1 & \text{if } s = u_0. \end{cases} \tag{2.5}$$

First, we remark that the equality actually holds in (2.5), although we shall not need this fact here. Suppose $d_T(r, s) = n$ and $s \neq u_0$. Then, by (2.1), we have $\deg_T(s) = \deg_{r^U}(s) \leqslant \deg_{T^*}(s)$. Now note that every ray of T rooted at u_0 contains an edge of u_0^U. If not, then there is a vertex v_i of V such that $u_0 <_T v_i$ and no vertices of $U \cup V$ lie between u_0 and v_i in T. By Lemma 2.4(2), it follows that $u_{i+1} <_T v_i$; hence, by the choice of v_i, it follows that $u_{i+1} <_T u_0$. This contradicts (2.1). Therefore, the set of edges of T^* that are incident with u_0 contains all the edges of T that are incident with u_0 and, moreover, it contains the first edge of the path P_0. This proves (2.5).

Finally, since T^* need not be maximal, we put $T^+ = \overline{T^*}$. By (2.4), it follows that $B_T(r, n) \subseteq r^U \subseteq T^* \subseteq T^+$, which proves Lemma 2.6(1). By (2.5), we have $a_n(T) < a_n(T^*) \leqslant a_n(T^+)$; so, Lemma 2.6(2) holds. Lemma 2.6(3) follows immediately from Lemmas 2.5, 2.6(1) and 2.6(2). □

Now we are ready to state the main theorem of this section.

Theorem 2.7. *Every infinite, connected, locally finite graph rooted at a vertex r contains a maximal, leafless subtree for which the root is the same and all bridges generate finite bundles.*

Proof. Suppose the rooted graph (G,r) is a counterexample to the theorem. We will proceed by induction on ordinals to construct maximal leafless subtrees (T_α, r) for $\alpha \in \omega_1$ such that

$$a(T_\alpha) < a(T_\beta) \quad \text{for } \alpha < \beta.$$

Starting with any rooted, maximal, leafless subtree (T_0, r) of (G,r), let

$$T_{\alpha+1} = T_\alpha^+ .$$

Then, by Lemma 2.6(3), we have $a(T_\alpha) < a(T_{\alpha+1})$. To define the limit step, assume that $\alpha_k \to \alpha$ as $k \to \omega$. Let n_{α_k} be a number for which

$$T_{\alpha_k}^+ \supseteq B_{T_{\alpha_k}}(r, n_{\alpha_k})$$

and

$$a_{n_{\alpha_k}}(T_{\alpha_k}^+) > a_{n_{\alpha_k}}(T_{\alpha_k}).$$

The existence of such an n_{α_k} is guaranteed by Lemma 2.6. Observe that $a(T_{\alpha_k})$ strictly increases, while each $a_n(T_{\alpha_k})$ is bounded for a fixed n, due to the local finiteness of G. Thus, $n_{\alpha_k} \to \omega$ as $k \to \omega$. Hence, for every $s \in \omega$, there is an element p of ω such that $n_{\alpha_k} \geqslant s$ for all $k \geqslant p$ and, thus,

$$B_{T_{\alpha_k}}(r,s) = B_{T_{\alpha_{k+1}}}(r,s). \tag{2.6}$$

Define S_α to be the union of the graphs $B_{T_{\alpha_p}}(r,s)$ over all s in ω, where p is as described above. From (2.6) it follows that $(B_{T_{\alpha_p}}(r,s))_{s\in\omega}$ forms an ascending sequence of trees, and every tree in this sequence has its leaves a distance s from r in S_α. Thus, S_α itself is a rooted leafless subtree of (G,r). Finally, put $T_\alpha = \overline{S_\alpha}$ in the limit-step case of the transfinite definition of the T_α's.

Since $B_{T_{\alpha_p}}(r,s) \subseteq T_\alpha$, we have $a(T_\alpha) > a(T_{\alpha_p})$ for all $p \in \omega$. Therefore, the transfinite sequence $(a(T_\alpha))_{\alpha\in\omega_1}$ is strictly increasing, and yet it is bounded from above, due to the local finiteness of G. This is impossible. □

3. Decomposition

We are now ready to derive the decomposition described in the abstract.

Theorem 3.1. *Suppose that G is an infinite, connected, locally finite graph rooted at a vertex r. Then there is a leafless tree T rooted at r, and a sequence of finite graphs* $H_0, H_1, H_2, \ldots$ *such that the following conditions hold:*

(1) *G is an edge-disjoint union of* $H_0, H_1, H_2, \ldots$ *and T.*

(2) *If u is a vertex of* $H_i \cap H_j$ *for some distinct i and j, then u is also a vertex of T.*
(3) *If u is a vertex of* $H_i \cap T$ *and v is a vertex of* $H_j \cap T$ *such that* $u <_T v$, *then* $i \leqslant j$.

Proof. From Theorem 2.7 it follows that G contains a maximal, leafless, rooted subtree (T, r), all of whose bridges generate finite bundles. Suppose first that there are only finitely many bridges of T in G. Then let H_0 be the union of these bridges (or the graph consisting of the vertex r alone, if there are no bridges of T at all). Note that H_0 is finite, as by Lemma 2.1, all bridges of T are finite. Let H_1 be the graph consisting of a single vertex v_1 of T such that no vertex u of $H_0 \cap T$ satisfies $v_1 <_T u$. Let $v_2, v_3, \ldots$ denote the consecutive vertices of the ray of T rooted at v_1, and define H_i, for $i = 2, 3, \ldots$, to be the graph consisting of the single vertex v_i. It is easy to verify that the theorem holds.

We may now assume that there are infinitely many bridges of T in G. Since G is connected and locally finite, it follows that it is countable and, thus, that there are countably many bridges of T in G. Let $\{B_i\}_{i \in \omega}$ be the collection of all bridges of T in G. Put H_0 equal to the union of all bridges in the bundle generated by B_0. Inductively, assume that H_n has been defined. Let i_n be the smallest integer such that B_{i_n} is contained in none of $H_0, H_1, \ldots, H_n$. Let H_{n+1} be the union of all the bridges that are in the bundle generated by B_{i_n} but that are contained in none of $H_0, H_1, \ldots, H_n$. Verification of the theorem is straightforward and the result follows. □

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