Minor-equivalence for infinite graphs

Bogdan Oporowski

Louisiana State University

Follow this and additional works at: https://repository.lsu.edu/mathematics_pubs

Recommended Citation


This Article is brought to you for free and open access by the Department of Mathematics at LSU Scholarly Repository. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Scholarly Repository. For more information, please contact ir@lsu.edu.
Minor-equivalence for infinite graphs

Bogdan Oporowski*

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

Received 18 March 1996; received 18 August 1997; accepted 16 February 1998

Abstract

Two graphs are minor-equivalent if each is isomorphic to a minor of the other. In this paper, we give structural characterizations of the minor equivalence classes of the infinite full grid $G_{\mathbb{Z} \times \mathbb{Z}}$ and of the infinite half-grid $G_{\mathbb{Z} \times \mathbb{N}}$. A corollary of these results states that every minor of $G_{\mathbb{Z} \times \mathbb{Z}}$ that has a minor isomorphic to $G_{\mathbb{Z} \times \mathbb{N}}$ is minor-equivalent to one of $G_{\mathbb{Z} \times \mathbb{Z}}$ or $G_{\mathbb{Z} \times \mathbb{N}}$.

© 1999 Elsevier Science B.V. All rights reserved

1. Introduction

Graphs in this paper may be finite or infinite. To simplify the notation, we shall consider only graphs with no loops and no multiple edges. However, all the results presented here can be easily extended to graphs in which loops and multiple edges are allowed.

Let $G$ be a graph. We define a minor of $G$ as follows. Suppose $W$ is a set such that every element is a non-null connected (possibly infinite) subgraph of $G$, and no two such subgraphs meet. Moreover, suppose that $F$ is a set every element of which is a connected (possibly infinite) subgraph of $G$ meeting exactly two elements of $W$, and such that the intersection of every two distinct elements of $F$ lies in some element of $W$. Then the graph with vertex set $W$, edge set $F$, and the obvious incidence relation is a minor of $G$. Conversely, if $J$ is a minor of $G$, then the subgraph of $G$ that is the union of all vertices and edges of $J$ is the expansion of $J$ in $G$, and is denoted by $\Sigma(J)$. We write $H \leq_m G$ if $H$ is isomorphic to a minor of $G$. If $H$ is isomorphic to a minor of $G$, but $G$ is not isomorphic to any minor of $H$, then we write $H <_m G$.

Two graphs $G$ and $H$ are minor-equivalent, written $G \cong_m H$, if $G \leq_m H$ and $H \leq_m G$. We remark that it is easy to check that a graph $H$ is isomorphic to a minor of a graph $G$ if and only if $H$ can be obtained from a subgraph of $G$ by contracting some of

* E-mail: bogdan@math.lsu.edu.
its connected subgraphs where we note that these subgraph may be infinite. Thus our definition of a minor is equivalent to the one usually found in literature.

It is clear that minor-equivalence is indeed an equivalence relation. The equivalence class that contains a graph $G$ will be denoted by $[G]_m$. Obviously, if $G$ is a finite graph, then $[G]_m$ is equal to the isomorphism class containing $G$. In contrast, if $G$ is infinite, then $[G]_m$ may contain graphs from more than one isomorphism class. For example, if $G$ is an infinite clique and $H$ consists of $G$ and a single isolated vertex, then $G \cong_m H$, and thus $[G]_m = [H]_m$, even though $G$ and $H$ are not isomorphic. On the other hand, if $G$ is a two-way-infinite path, then $[G]_m$ is equal to the isomorphism class of $G$. Another interesting example of a graph whose minor-equivalence class coincides with its isomorphism class is a graph that is not isomorphic to any of its proper minors. The existence of such a graph has been shown in [2].

A graph $G$ covers another graph $H$ if $H \prec_m G$ and, for every graph $K$ such that $H \preceq_m K \preceq_m G$, either $K \cong_m H$ or $K \cong_m G$. The following is an immediate consequence of the definitions.

**Proposition 1.1.** If $G$ covers $H$, then $G$ covers every graph that is minor-equivalent to $H$, and, similarly, $H$ is covered by every graph that is minor-equivalent to $G$.

If $G$ is a finite non-null graph, then $G$ covers some graphs, namely those that can be obtained from $G$ by deleting a single edge, contracting a single edge, or deleting a single isolated vertex. Similarly, every finite graph is covered by another graph. However, if $G$ is infinite, then $G$ may or may not cover other graphs. It is easy to verify the following:

**Proposition 1.2.** A graph that has a countably infinite vertex set and no edges covers no graphs.

**Proposition 1.3.** A two-way-infinite path covers the disjoint union of two one-way-infinite paths.

A graph is planar if it has no minor isomorphic to $K_5$ or to $K_{3,3}$. We shall focus our attention on two infinite planar graphs: the full grid and the half-grid, defined as follows. The full grid, denoted by $G_{\mathbb{Z} \times \mathbb{Z}}$, has the set $\mathbb{Z} \times \mathbb{Z}$ as its vertex set, and two of its vertices $(i,j),(i',j')$ are joined by an edge if and only if $|i - i'| + |j - j'| = 1$. The half-grid, denoted by $G_{\mathbb{Z} \times \mathbb{N}}$, is the graph obtained from $G_{\mathbb{Z} \times \mathbb{Z}}$ by deleting all the vertices whose second coordinate is negative. Two results of the paper, stated later as Theorems 4.1 and 5.1, give sufficient structural conditions for a graph to be minor-equivalent to, respectively, the infinite half-grid and the infinite full grid. As a corollary of these results, we prove the following:

**Theorem 1.4.** The full grid covers the half-grid.

Two of the many open problems that are related to Theorem 1.4 are as follows:
Question 1.5. Does the half-grid cover any graphs?

Question 1.6. Does the countably infinite clique cover any graphs?

Several other interesting questions arise by using the relation of topological embedding in place of the minor relation in such covering problems. Let $G$ be a graph and let $G \times 2$ be the disjoint union of two copies of $G$. The graph $G$ is clonable if $G \times 2 \leq_m G$. It is easy to show the following:

Proposition 1.7. If each of the graphs $H$ and $H'$ is covered by a clonable connected graph $G$, then $H \cong_m H'$.

Since both the half-grid and the countably infinite clique are clonable, it follows from Proposition 1.7 that, up to minor-equivalence, there is at most one graph that is covered by the half-grid, and at most one graph that is covered by the countably infinite clique.

This section continues with some more terminology and notation, and then concludes with an outline of the remainder of the paper.

If $E$ is a subset of the edge set of $G$, then $G \setminus E$ denotes the subgraph of $G$ induced by the edges not in $E$. Similarly, if $V$ is a subset of the vertex set of $G$, then $G - V$ denotes the subgraph of $G$ induced by the vertices not in $V$. If $H$ is a subgraph of $G$, then $G \setminus E(H)$ may be abbreviated as $G \setminus H$, and $G - V(H)$ may be abbreviated as $G - H$.

Suppose that $H$, $K$, and $L$ are subgraphs of a graph $G$ such that $H \cap K = \emptyset$. Assume also that $L$ contains exactly one path that has one endvertex in each of $H$ and $K$ and is internally disjoint from $H \cup K$. Then this path will be denoted by $[H,K]_L$. If $H$ has only one vertex $h$, then $[H,K]_L$ will be denoted by $[h,K]_L$. A similar convention applies when $K$ has just one vertex.

A tree is a connected (possibly infinite) graph without cycles. A rooted tree is a pair $(T,r)$ where $T$ is a tree and $r$ is one of its vertices. A ray is a tree that is a one-way-infinite path. A ray will always be considered to be rooted at its unique vertex of degree one. For two vertices $u$ and $v$ of a rooted tree $(T,r)$, we shall write $u \leq_{(T,r)} v$, or simply $u \leq r v$, if $u$ is a vertex of $[r,v]_T$. If $u \leq r v$ and, additionally, $u$ and $v$ are distinct, then we may write $u < r v$. If $\rho$ is a ray in $G$ and $K$ is a finite subgraph of $G$ meeting $\rho$, then $K_\rho$ denotes the maximal subpath of $\rho$ that has both endvertices in $K$.

The tree that plays a special role in this paper is the infinite binary tree. Before formally introducing this tree, we need a few definitions on binary sequences.

A sequence is binary if all of its elements are in the set $\{0,1\}$. For a non-negative integer $n$, the set of all binary sequences of length $n$ will be denoted by $[2^n]$. The set of all finite binary sequences will be denoted by $[2^{<\omega}]$. If $\alpha$ and $\beta$ are elements of $[2^{<\omega}]$, then $\alpha + \beta$ denotes the sequence that is the concatenation of $\alpha$ and $\beta$. In particular, $\alpha + 0$ is the sequence obtained by adjoining a single zero to the end of the sequence $\alpha$. For a finite binary sequence $\alpha$ of positive length, the symbol $\alpha'$ will denote the sequence obtained from $\alpha$ by deleting its last element. For two binary sequences
\(\alpha\) and \(\beta\), we write \(\alpha \prec \beta\) if the length of \(\alpha\) is less than the length of \(\beta\), or if their lengths are equal and \(\alpha\) precedes \(\beta\) lexicographically. If \(\alpha \prec \beta\) or \(\alpha = \beta\), we shall write \(\alpha \preceq \beta\). Clearly, the relation \(\preceq\) is a linear order on the set of finite binary sequences. For a binary sequence \(\alpha\), let \(\alpha^+\) denote the successor of \(\alpha\) in this relation.

The two binary sequences of length \(n\) that will be most frequently referred to in this paper are the one consisting of all zeros and the one consisting of all ones. These will be denoted by \(0_n\) and \(1_n\), respectively. If the value of \(n\) can be inferred from the context, then \(0_n\) and \(1_n\) will be abbreviated as \(0\) and \(1\). The binary sequence of length zero will be denoted by \(0\).

The infinite binary tree \(T^\omega\) is the graph with vertex set \([2^{<\omega}]\) in which all sets of the form \(\{\alpha, \alpha + 0\}\) or \(\{\alpha, \alpha + 1\}\) are edges. The infinite binary tree will be always considered to be rooted at the vertex \(0\).

Two rays \(\rho\) and \(\sigma\), which are subgraphs of the same graph \(G\), are equivalent if, for every finite subgraph \(H\) of \(G\), the infinite parts of \(\rho - H\) and \(\sigma - H\) lie in the same connected component of \(G - H\). Halin [1] proved the following:

**Theorem 1.8.** Two rays are equivalent if and only if there is another ray that meets both of them infinitely often.

It is easy to verify that the above relation is an equivalence relation on rays which are subgraphs of a fixed infinite graph \(G\). The equivalence classes of this relation are called the ends of \(G\). An end is thick if it contains infinitely many pairwise-disjoint rays.

Suppose \(G\) is an infinite graph and \(\mathcal{R}\) is a finite set of rays in \(G\). A path \(P\) (or a cycle \(C\)) of \(G\) is reduced with respect to a ray \(\rho\) if \(P\) (or \(C\)) either intersects \(\rho\) along a path (perhaps consisting of one vertex only) or does not intersect it at all. A path \(P\) (or cycle \(C\)) collates a set \(\mathcal{R}\) of rays of \(G\) if it meets all elements of \(\mathcal{R}\) and is reduced with respect to every element of \(\mathcal{R}\). A graph \(G\) is round if it satisfies conditions (C1)-(C3) below.

(C1) \(G\) is planar, connected, locally finite, and has exactly one end.

(C2) The end of \(G\) is thick.

(C3) For every finite subgraph \(H\) of \(G\) and every set \(\mathcal{R}\) of pairwise disjoint rays of \(G\) such that \(3 \leq |\mathcal{R}| < \infty\), there is a cycle in \(G - H\) that collates \(\mathcal{R}\).

The graph \(G\) is flat if it satisfies (C1) and (C2), but fails (C3).

It is straightforward to verify the following propositions.

**Proposition 1.9.** The full grid is round.

**Proposition 1.10.** The half-grid is flat.

Let \(J\) be a subgraph of \(G\). A vertex of attachment of \(J\) in \(G\) is a vertex of \(J\) that is incident with an edge of \(G\) which is not an edge of \(J\). A subgraph \(H\) is said to be \(J\)-detached if all vertices of attachment of \(H\) in \(G\) are in \(J\). A bridge \(B\) of \(J\) in \(G\) is a subgraph of \(G\) satisfying the following three conditions.
(B1) $B$ is not a subgraph of $J$.
(B2) $B$ is $J$-detached in $G$.
(B3) No proper subgraph of $B$ satisfies both (B1) and (B2).

The remainder of this paper is organized as follows. In Section 2, we investigate paths and cycles collating finite sets of pairwise-disjoint sets of rays. Section 3 contains two technical refinements of the well-known result of Robertson et al. stating that every finite planar graph is isomorphic to a minor of a sufficiently large grid. In Sections 4 and 5, we use the results of Sections 2 and 3 to prove theorems that may be viewed as inverses of Propositions 1.9 and 1.10. These results state that every flat graph is minor-equivalent to the half-grid, and every round graph is minor-equivalent to the full grid. As a consequence of these results, flat and round graphs satisfy Seymour’s self-minor conjecture. More precisely, we have the following:

**Corollary 1.11.** If $G$ is flat or round, then $G$ is isomorphic to a proper minor of itself.

In Section 6, we use the results of the previous two sections to investigate subgraphs of $G_{\mathbb{Z} \times \mathbb{Z}}$. In Section 7, we employ the results of the previous section and the concept of planar duals to conclude the proof of Theorem 1.4.

## 2. Collating sets of rays

In this section, we shall study ways in which a path or a cycle can intersect a finite set of rays of a graph.

**Lemma 2.1.** Let $G$ be a locally finite planar graph with exactly one end, and let $\mathcal{R}$ be a finite set of pairwise-disjoint rays. Let $H$ be a finite connected subgraph of $G$ that meets all elements of $\mathcal{R}$. If $P$ is a path meeting all elements of $\mathcal{R}$, then there is a path $Q$ that is contained in $P \cup (\bigcup_{p \in \mathcal{R}} P_p)$ and collates $\mathcal{R}$.

**Proof.** Without loss of generality, we may assume that no proper subpath of $P$ meets all elements of $\mathcal{R}$. Suppose the vertices of $P$ are $p_0, p_1, \ldots, p_m$ in the order listed. We define the path $Q$ as follows. Let $s_0 = t_0 = p_0$. Assume that $s_{i-1}$ and $t_{i-1}$ have been defined as two vertices of $P$ that lie on some ray $\rho$ of $\mathcal{R}$ such that $t_{i-1} \neq p_m$, and $[t_{i-1}, p_m]_\rho - \{t_{i-1}\}$ avoids $\rho$. Let $s_i$ be the vertex $p_k$ of $[t_{i-1}, p_m]_\rho - \{t_{i-1}\}$ that meets some ray $\rho'$ of $\mathcal{R}$ and whose index is as small as possible. Let $t_i$ be the vertex $p_k$ of $[s_i, p_m]_\rho$ that lies on $\rho$ and whose index is as large as possible. Let $h$ be the number such that $t_h = p_m$. For each $i \in \{0, 1, \ldots, h\}$, let $\rho_i$ denote the element of $\mathcal{R}$ that contains both $s_i$ and $t_i$. Finally, let

$$Q = \bigcup_{1 \leq i \leq h} ([t_{i-1}, s_i]_{\rho_i} \cup [s_i, t_i]_{\rho_i}).$$
The process of constructing $Q$ is illustrated in Fig. 1. It is clear from the construction that $Q$ is reduced with respect to every element of $\mathcal{R}$ and that it is contained in $P \cup (\bigcup_{p \in \mathcal{R}} P_p)$. Thus, it remains to show that $Q$ meets all elements of $\mathcal{R}$.

Suppose that $Q$ avoids an element $p$ of $\mathcal{R}$. Recall that $P$ meets all elements of $\mathcal{R}$. Hence, there is a vertex $p$ of $P \cap p$. It follows from the construction that there is a number $i \in \{1, 2, \ldots, h - 1\}$ such that $p$ lies on $P$ between $s_i$ and $t_i$. Let $K$ be a finite subgraph of $G$ that contains $H \cup P$ and all subpaths of elements of $\mathcal{R}$ with both end-vertices in $H \cup P$. Since the rays of $\mathcal{R}$ are in the same end of $G$, there is a connected subgraph $L$ of $G - K$ that meets all elements of $\mathcal{R}$. Upon contracting $H$ and $L$, it is easy to see that the graph $H \cup L \cup P \cup P_0 \cup P_h \cup P_i \cup p$ contains a minor isomorphic to $K_{3,3}$; a contradiction. [1]

Let $\mathcal{R}$ be a finite non-empty set of rays in a graph $G$, and let $H$ and $K$ be finite subgraphs of $G$. We shall write $H \prec \mathcal{R} K$ if
(R1) both $H$ and $K$ meet all rays in $\mathcal{R}$;
(R2) $H$ and $K$ are disjoint; and
(R3) for every ray $p \in \mathcal{R}$ and every $h \in V(H \cap p)$ and $k \in V(K \cap p)$, we have $h < k$.

**Lemma 2.2.** Let $G$ be a locally finite planar graph with exactly one end. Let $\mathcal{R}$ be a finite non-empty set of pairwise-disjoint rays in $G$, and let $H$ be a finite subset of $G$ meeting all rays in $\mathcal{R}$. Then there is a path $P$ in $G$ that collates $\mathcal{R}$ and such that $H \prec \mathcal{R} P$.

**Proof.** First we show the following.
(1) For every finite subgraph $K$ of $G$ that meets all rays in $\mathcal{R}$, there is a subgraph $K'$ of $G$ such that $K \prec \mathcal{R} K'$.

For every ray $p$ in $\mathcal{R}$, let $V'(p)$ denote the vertices of $p$ that are not in the infinite component of $p - K$. Let $L$ be the subgraph of $G$ induced by the vertices in $V(K) \cup (\bigcup_{p \in \mathcal{R}} V'(p))$. It is clear that, since the set $\mathcal{R}$ is finite, the graph $L$ is also
finite. Since \( G \) has only one end, for every two rays in \( \mathcal{R} \), there is a path in \( G - L \) that joins these two rays. Thus, as \( \mathcal{R} \) has only finitely many elements, we conclude that (1) holds.

We apply (1) with \( K = H \) to obtain a finite subgraph \( H' \) of \( G \) such that \( H < H' \). We apply (1) again with \( K = H' \) to obtain a subgraph \( H'' \) such that \( H' < H'' \). Let \( T' \) be a minimal connected subgraph of \( H' \) that meets all the rays in \( \mathcal{R} \). Clearly, \( T' \) is a tree. Suppose that \( T' \) is not a path. Then \( T' \) has at least three vertices \( v_1, v_2, \) and \( v_3 \) whose degree in \( T' \) is one. By the minimality of \( T' \), these vertices lie in distinct rays in \( \mathcal{R} \), say \( \rho_1, \rho_2, \) and \( \rho_3 \), respectively. By construction, for each \( i \in \{1,2,3\} \), vertex \( v_i \) lies on \( \rho_i \) between \( H \) and \( H'' \). It is easy to see that the subgraph of \( G \) that is the union of \( \rho_1, \rho_2, \rho_3, H, T' \), and \( H'' \) contains a minor isomorphic to \( K_{3,3} \), contradicting the assumption that \( G \) is planar. Thus \( T' \) is a path and the conclusion follows from Lemma 2.1. \( \square \)

The next few lemmas describe the order in which paths and cycles of a graph \( G \) meet the members of a finite set \( \mathcal{R} \) of pairwise-disjoint rays of \( G \). To formalize the notion of this order, we introduce the following definitions. Two sequences of distinct members of \( \mathcal{R} \) are path-equivalent if they are equal, or one can be obtained from the other by reversing the order. Two such sequences \( (\rho_1, \rho_2, ..., \rho_m) \) and \( (\rho'_1, \rho'_2, ..., \rho'_m) \) are cycle-equivalent if \( m = m' \) and there are integers \( a \) and \( b \) such that, for all \( i \in \{1,2, ..., m\} \), \( \rho'_i = \rho_{a + bi} \) where \( b \in \{-1, 1\} \) and all subscripts are read modulo \( m \). Suppose \( P \) is a path of \( G \) that collates \( \mathcal{R} \) and \( \rho_1, \rho_2, ..., \rho_m \) are the rays of \( \mathcal{R} \) listed in the order they are met by \( P \). Then the \( \mathcal{R} \)-trace of \( P \), written \( \text{tr}_\mathcal{R}(P) \), is the path-equivalence class of \( (\rho_1, \rho_2, ..., \rho_m) \). Similarly, suppose \( C \) is a cycle that collates \( \mathcal{R} \) and \( \sigma_1, \sigma_2, ..., \sigma_m \) are the rays of \( \mathcal{R} \) listed in the order they are met by \( C \). Then the \( \mathcal{R} \)-trace of \( C \), written \( \text{tr}_\mathcal{R}(C) \), is the cycle-equivalence class of \( (\sigma_0, \sigma_1, ..., \sigma_m) \).

**Lemma 2.3.** Let \( G \) be a locally finite planar graph with exactly one end. Suppose \( \mathcal{R} \) is a finite set of pairwise disjoint rays in \( G \), and \( P_1 \) and \( P_2 \) are paths collating \( \mathcal{R} \) such that \( P_1 < P_2 \) and \( \text{tr}_\mathcal{R}(P_1) = \text{tr}_\mathcal{R}(P_2) \). Then at least one of the following holds:

(i) \( G \) has a cycle \( C \) collating \( \mathcal{R} \) such that \( P_1 < C \); or

(ii) \( G \) has a path \( P_3 \) collating \( \mathcal{R} \) such that \( P_2 < P_3 \) and \( \text{tr}_\mathcal{R}(P_2) = \text{tr}_\mathcal{R}(P_3) \).

**Proof.** Let \( H \) be a subgraph of \( G \) that is the union of \( P_2 \) and all finite components of \( \rho - P_2 \) over all \( \rho \in \mathcal{R} \). Clearly, \( H \) is finite. Upon applying Lemma 2.2, we conclude that \( G \) has a path \( P_3 \) that collates \( \mathcal{R} \) and is such that \( P_2 < P_3 \). Similarly, \( G \) has a path \( P_4 \) that collates \( \mathcal{R} \) and is such that \( P_3 < P_4 \). If \( \text{tr}_\mathcal{R}(P_2) = \text{tr}_\mathcal{R}(P_3) \), then (ii) holds. If \( \text{tr}_\mathcal{R}(P_2) \neq \text{tr}_\mathcal{R}(P_3) \), then (i) holds. Thus, we may assume that \( \text{tr}_\mathcal{R}(P_2) \neq \text{tr}_\mathcal{R}(P_3) \). Let \( \text{tr}_\mathcal{R}(P_2) \) be the equivalence class of the sequence \( (\rho_1, \rho_2, ..., \rho_m) \), and let \( \pi \) be a permutation of \( \{1,2, ..., n\} \) such that \( \text{tr}_\mathcal{R}(P_3) \) is the equivalence class of \( (\rho_{\pi(1)}, \rho_{\pi(2)}, ..., \rho_{\pi(n)}) \). Since the traces of \( P_2 \) and \( P_3 \) differ, there is a number \( k \) in \( \{1,2, ..., n-1\} \) such that \( |\pi(k+1) - \pi(k)| \geq 1 \). By symmetry, we may assume that \( \pi(k+1) > \pi(k) \). If \( \pi(k+1) - \pi(k) = n-1 \), then it is easy to see that (i) holds. Thus, we may assume that \( \pi(k+1) - \pi(k) < n-1 \), which
implies that \( \pi(k + 1) \neq n \) or \( \pi(k) \neq 1 \). By symmetry, we may assume that the latter holds. Then it follows that \( G \) has a minor isomorphic to \( K_{3,3} \), as illustrated in Fig. 2; a contradiction. \( \square \)

Lemma 2.4. Let \( G \) be a locally finite planar graph with exactly one end, and let \( \mathcal{R} \) be a finite set of pairwise disjoint rays. Suppose \( C_1 \) and \( C_2 \) are two cycles of \( G \) such that \( C_1 \prec_{\mathcal{R}} C_2 \) and each collates \( \mathcal{R} \). Then \( tr_{\mathcal{R}}(C_1) = tr_{\mathcal{R}}(C_2) \).

Proof. Suppose the lemma fails. Let \((P_1, P_2, \ldots, P_n)\) be a sequence in the equivalence class \( tr_{\mathcal{R}}(C_1) \). Let \( \pi \) be a permutation of \( \{1, 2, \ldots, n\} \) such that \( tr_{\mathcal{R}}(C_2) \) is the cycle-equivalence class of \((P_{\pi(1)}, P_{\pi(2)}, \ldots, P_{\pi(n)})\). Since the traces of \( C_1 \) and \( C_2 \) differ, there is a number \( k \) in \( \{1, 2, \ldots, n - 1\} \) such that \( 1 < |\pi(k + 1) - \pi(k)| < n - 1 \). Without loss of generality, we may assume that \( \pi(k) < \pi(k + 1) \). Then \( \pi(k) \neq 1 \) or \( \pi(k + 1) \neq n \). By symmetry, we may assume the former. Then the graph \( C_1 \cup C_2 \cup P_1 \cup P_{\pi(k)} \cup P_{\pi(k) + 1} \cup P_{\pi(k + 1)} \) contains a minor isomorphic to \( K_{3,3} \); a contradiction. \( \square \)

Lemma 2.5. Let \( G \) be a planar locally finite graph and let \( \mathcal{R} \) be a finite set of pairwise disjoint rays of \( G \) that are in the same end of \( G \). Let \( H \) be a finite connected subgraph of \( G \) that meets all members of \( \mathcal{R} \), and let \( C \) be a cycle in \( G - H \) that collates at least three members of \( \mathcal{R} \). Then \( C \) meets every ray in \( \mathcal{R} \).

Proof. Let \( \{\rho_1, \rho_2, \rho_3\} \) be a 3-element subset of \( \mathcal{R} \) that is collated by \( C \). Suppose that \( \rho \) is a ray in \( \mathcal{R} \) that avoids \( C \). Let \( K = H \cup C \cup [H, C]_{\rho_1} \cup [H, C]_{\rho_2} \cup [H, C]_{\rho_3} \). Then, as all rays in \( \mathcal{R} \) are in the same end of \( G \), there is a connected subgraph \( L \) in \( G - K \) that meets all of \( \rho, \rho_1, \rho_2, \) and \( \rho_3 \). Upon contracting all edges in \( H \cup L \), it becomes clear that the graph \( H \cup L \cup C \cup [H, C]_{\rho_1} \cup [H, C]_{\rho_2} \cup [H, C]_{\rho_3} \) has a minor isomorphic to \( K_5 \); a contradiction. \( \square \)

3. Grids and cylinders

Let \( n \) be a positive integer. The \((n \times n)\)-grid is the subgraph of \( G_{\mathbb{Z} \times \mathbb{Z}} \) induced by the subset \( \{0, 1, \ldots, n - 1\} \times \{0, 1, \ldots, n - 1\} \) of its vertex set. The \((n \times n)\)-cylinder is the
graph obtained from the \( n \times n \)-grid by adding edges joining vertices \((i,0)\) to \((i,n-1)\), for all \(i\) in \(\{0,1,\ldots,n-1\}\). It turns out in our studies, however, that finite grids and cylinders are easier to handle if their vertices are pairs of binary sequences, rather than pairs of integers. Thus, we shall denote by \( \Gamma_n \) the graph with vertex set \([2^n] \times [2^n]\) that is isomorphic to the \((2^n \times 2^n)\)-grid, so that the vertex \((i,j)\) of \( \Gamma_n \) corresponds to the vertex \((x_i, x_j)\) of \( \Gamma_n \), where \(x_i\) and \(x_j\) are the binary expansions of \(i\) and \(j\) with the appropriate numbers of leading zeros. Using the same correspondence as above, we define \( \Theta_n \) to be the graph isomorphic to the \((2^n \times 2^n)\)-cylinder.

The following well-known theorem of Robertson et al. \cite{5} states that every finite planar graph is a minor of a sufficiently large grid.

**Theorem 3.1.** For every planar graph \(G\), there is a positive integer \(N\) such that \(G\) is isomorphic to a minor \(\Gamma_N\).

The remainder of this section contains two results, which are technical modifications of Theorem 3.1, and which will be useful for proving the main theorems of this paper. The first of these modifications strengthens Theorem 3.1 by specifying that the isomorphism in Theorem 3.1 can be chosen so that some parts of the boundary of the infinite face of \(G\) are mapped to the appropriate parts of the boundary of the grid. The second of these modifications pertains to cylinders, rather than grids. Before formally stating these results, we need some preparation.

Let \(m\) be a positive integer and let \(A_m = \{(0_m, \beta) : \beta \in [2^n]\}\), and \(B_m = \{(1_m, \beta) : \beta \in [2^n]\}\). Let \(\Gamma_n^0\) and \(\Gamma_n^1\) denote the subgraphs of \(\Gamma_n\) that are induced by the sets \(A_m\) and \(B_m\), respectively. Similarly, let \(\Theta_n^0\) and \(\Theta_n^1\) denote the subgraphs of \(\Theta_n\) that are induced by \(A_m\) and \(B_m\). Suppose that \(G\) is a plane graph and \(\varphi\) is one of its faces. A **flat** \(m\)-attachment of \(G\) is a pair \((P, h)\) where \(P\) is the null graph or a path in the boundary of \(\varphi\), and \(h\) is an isomorphism from \(P\) to a minor of \(\Gamma_n^0\). Suppose now that \(C\) is the null graph or the cycle that forms the boundary of \(\varphi\), and \(k\) is an isomorphism from \(C\) to \(\Theta_n^0\). The pair \((C, k)\) will be called a **round** \(m\)-attachment of \(G\).

Let \((H, h)\) be either a flat \(m\)-attachment of \(G\) or a round \(m\)-attachment of \(G\), and let \(n\) be an integer with \(n \geq m\). Suppose that \(k\) is an isomorphism from \(H\) to, respectively, \(\Gamma_n^0\) or \(\Theta_n^0\). Then \(k\) agrees with \(h\) if, for every vertex \(v\) of \(H\), there are vertices \((0_m, \alpha) \in h(v)\) and \((0_n, \beta) \in k(v)\) such that \(\alpha\) is an initial segment of \(\beta\).

A **flat** \(n\)-segment is a triple \((G, (P, h), Q)\) consisting of a graph \(G\), a flat \(n\)-attachment \((P, h)\), and a path \(Q\) that lies in the boundary of \(\varphi\) and is disjoint from \(P\). A **round** \(n\)-segment is a triple \((G, (C, h), D)\) consisting of the graph \(G\), a round \(n\)-attachment \((C, h)\), and a cycle \(D\) that bounds a face of \(G\) and is disjoint from \(C\). Let \((G, (H, h), K)\) be a flat \(n\)-segment or a round \(n\)-segment. An isomorphism \(f\) from \(G\) to a minor of \(\Gamma_n\) or to a minor of \(\Theta_n\) agrees with \((G, (H, h), K)\) if \(f\) restricted to \(P\) agrees with \(h\), and, for every vertex \(v\) of \(K\), the set \(V(f(v))\) contains an element \((1_n, \beta)\) for some \(\beta \in [2^n]\).

The two modifications of Theorem 3.1 referred to earlier in this section are stated below.
Theorem 3.2. For every positive integer \( n \) and every flat \( n \)-segment \((G, (P, h), Q)\), there is an integer \( N \) exceeding \( n \) and an isomorphism \( f \) from \( G \) to a minor of \( I_N \) that agrees with \((G, (P, h), Q)\).

Theorem 3.3. For every positive integer and every round \( n \)-segment \((G, (C, k), D)\), there is an integer \( N \) exceeding \( n \) and an isomorphism from \( G \) to a minor of \( \Theta_N \) that agrees with \((G, (C, k), D)\).

The remainder of this section will be devoted to proving Theorems 3.2 and 3.3. Before presenting formal proofs, however, we need some more terminology and some auxiliary results.

For a binary sequence \( \alpha \), let \( \alpha^\# \) denote the number whose binary expansion consists of a zero followed by the point, followed by the elements of \( \alpha \). Observe that the graph \( I_n \) is planar with the obvious plane embedding that maps every vertex \((\alpha, \beta)\) of \( I_n \) to the point \((\alpha^\#, \beta^\#)\) of the plane with the edges of \( I_n \) mapping to the appropriate line segments. In what follows, we shall identify the graph \( I_n \) with the plane graph induced by this embedding, and we shall, for instance, refer to the faces of \( I_n \).

Suppose \( C \) is a cycle in \( I_n \). Then \( C \) induces a decomposition of \( I_n \) as \( C \subseteq C_{in} \cup C_{out} \), where \( C_{in} \) and \( C_{out} \) lie, respectively, inside and outside \( C \) in the above plane embedding of \( I_n \), and are such that \( C_{in} \cap C_{out} \) is edgeless and contained in \( C \).

Suppose now that \( J \) is a minor of \( I_n \) that is isomorphic to a cycle. An \textit{innermost cycle} of \( J \) is a cycle \( M \) in the expansion \( \Sigma(J) \) satisfying the following conditions.

(i) The intersection of every edge of \( J \) with \( M \) is a path.
(ii) If \( L \) is a cycle in \( \Sigma(J) \) whose intersection with every edge of \( J \) is a path, then \( M \subseteq C \subseteq L \).

It is clear that every minor \( J \) of \( I_n \) that is isomorphic to a cycle contains exactly one innermost cycle. The concept of the innermost cycle is illustrated in Fig. 3.

If a binary sequence \( \alpha \) contains at least one zero, then let \( \alpha^* = \{\alpha + 00, \alpha + 01, \alpha + 10\} \); and if \( \alpha \) consists of all ones, then let \( \alpha^* = \{\alpha + 00,\alpha + 01,\alpha + 10,\alpha + 11\} \).

Suppose now that \( n \) is an integer exceeding one and \((\alpha, \beta)\) is a vertex of \( I_n \). Let \( v_{\alpha\beta} \) be the subgraph of \( I_{n+2} \) that is induced by the vertices in \( \alpha^* \times \beta^* \). If, additionally, \( \alpha \neq 1 \), then let \( e_{\alpha\beta} \) be the path in \( I_{n+2} \) that is induced by the vertices in \( \{(\alpha + 10, \beta + 01), (\alpha + 11, \beta + 01), (\alpha^* + 00, \beta + 01)\} \). Similarly, if \( \beta \neq 1 \), then let \( f_{\alpha\beta} \) be the path in \( I_{n+2} \) that is induced by the vertices in \( \{(\alpha + 01, \beta + 10), (\alpha + 01, \beta + 11), (\alpha + 01, \beta^* + 00)\} \).

The minor of \( I_{n+2} \) whose vertex set is \( \{v_{\alpha\beta}: (\alpha, \beta) \in V(I_n)\} \) and whose edge set is \( \{e_{\alpha\beta}: (\alpha, \beta) \in V(I_n), \alpha \neq 1\} \cup \{f_{\alpha\beta}: (\alpha, \beta) \in V(I_n), \beta \neq 1\} \) will be denoted by \( I_n' \). Let \( i_n \) denote the natural isomorphism from \( I_n \) to \( I_n' \) that maps each \((\alpha, \beta)\) to \( v_{\alpha\beta} \). Let \( i \) denote the function defined on \( \bigcup_{n \in \mathbb{N}} I_n \) whose restriction to each \( I_n \) is \( i_n \). It is clear that if \( J \) is a minor of \( I_n \), then the function \( i \) determines in a natural way a minor \( i J \) of \( I_{n+2} \), which is isomorphic to \( J \). More specifically, a vertex \( W \) of \( J \), which is a subgraph of \( I_n \), corresponds to \( (\bigcup_{v \in V(W)} i(v)) \cup (\bigcup_{e \in E(W)} i(e)) \), and, similarly, an edge...
$F$ of $J$ corresponds to $(\bigcup_{v \in \Gamma(F)} \iota(v)) \cup (\bigcup_{e \in E(F)} \iota(e))$. Also, if $f$ is an isomorphism from a graph $G$ to a minor $J$ of $\Gamma_n$, then the composition $\iota f$ is an isomorphism from $G$ to the minor $\iota J$ of $\Gamma_{n+2}$. For an illustration of $\Gamma_2'$ and $\iota_2$ see Fig. 4.

The following lemmas describe the properties of $\iota$ that will be used later in this section. The proofs of these lemmas are routine, if sometimes tedious, and so they are left for the reader.

**Lemma 3.6.** Suppose $(H, h)$ is either a flat $n$-attachment or a round $n$-attachment of a graph $G$. Then the composition $\iota h$ agrees with $h$.

**Lemma 3.7.** Suppose $S = (G, (P, h), Q)$ is either a flat $n$-segment or a round $n$-segment and $f$ is an isomorphism from $G$ to a minor of, respectively, $\Gamma_n$ or $\Theta_n$ that agrees with $S$. Then the composition $\iota f$ also agrees with $S$.

**Lemma 3.8.** If $P$ is a path in $\Gamma_n$ joining two of its subgraphs $H$ and $K$, then $\iota(P)$ contains a path $P'$ that joins $\iota(H)$ to $\iota(K)$ and has more vertices than $P$.

**Lemma 3.9.** Suppose $G$ is a subgraph of $\Gamma_n$ and $C$ is a cycle of $G$ that bounds a finite face. Then the innermost cycle of $\iota(C)$ also bounds a finite face of $\Sigma(\iota(G))$. 
Lemma 3.10. Suppose \( f \) is an isomorphism from a cycle \( C \) to a minor \( J \) of \( \Gamma_n \), and \( u \) and \( v \) are distinct vertices of \( C \). Let \( D \) denote the innermost cycle of \( J \). Then there is a path in \( D^m \) that is internally disjoint from \( \Sigma(\iota(J)) \), and joins \( \iota(f(u)) \) to \( \iota(f(v)) \).

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. First, we shall argue that \( G \) may be assumed to be non-separable. Suppose that \( G \) has more than one block and construct a plane graph \( G_3 \) by following steps (1)–(3) below.

1. If no block of \( G \) contains both \( P \) and \( Q \), then construct \( G_1 \) from \( G \) by adding two edges \( e \) and \( f \) so that the paths \( P \) and \( Q \) together with the edges \( e \) and \( f \) form a cycle in the boundary of the infinite face \( \varphi_1 \) of \( G_1 \). If both \( P \) and \( Q \) are contained in the same block, then let \( G_1 = G \), and \( \varphi_1 = \varphi \).

2. If the boundary of \( \varphi_1 \) contains edges from more than one block of \( G_1 \), then redraw the blocks of \( G_1 \) other than the one containing \( P \) in the faces other than \( \varphi_1 \). Call the resulting graph \( G_2 \) and its infinite face \( \varphi_2 \).

3. Form \( G_3 \) from \( G_2 \) by adding edges, if necessary, so that \( G_3 \) is 2-connected and plane.
It is easy to verify that \((G_3, (P, h), Q)\) is also a flat \(n\)-segment and that an isomorphism from \(G_3\) to \(I_n\) that agrees with \((G_3, (P, h), Q)\) induces an isomorphism from \(G\) to \(I_n\) that agrees with \((G, (P, h), Q)\). Hence \(G\) may be assumed to be non-separable.

Let \(C\) denote the cycle of \(G\) that bounds the infinite face and let \(p\) denote the number of elements in \(E(G) - E(C)\). The following is an easy consequence of a well-known property of non-separable graphs (see [6]).

(4) There is a sequence of plane non-separable graphs \(G_0, G_1, \ldots, G_p\) such that 
\[ G_0 = C, G_p = G, \text{ and, for each } i \in \{1, 2, \ldots, p\}, \text{ the graph } G_i \text{ has been obtained from } G_{i-1} \text{ by adding a path that meets } G_{i-1} \text{ only in its endvertices.} \]

The proof will proceed by induction on \(p\). We shall strengthen the statement of (3.2) by requiring additionally that (5) and (6) below hold. Condition (5) will facilitate the induction; and condition (6), while not needed to prove Theorem 3.2, will be used in proving Theorem 3.3.

(5) For every cycle \(K\) of \(G\) that bounds a finite face of \(G\), the innermost cycle of \(f(K)\) in \(I_n\) bounds a finite face of \(\Sigma(f(G))\).

Let \(t_0', t_1', \ldots, t_q'\) be the vertices of \(T'\) listed in the order they appear on \(T'\) where \(t_0' = V(T) \cap V(P)\). Similarly, let \(t_0'', t_1'', \ldots, t_q''\) be the vertices of \(T''\) listed in the order they appear on \(T''\) where \(t_0'' = V(T'') \cap V(P)\).

(6) If \(q' = q''\), then, for every \(i\) in \(\{0, 1, \ldots, q'\}\), there is an \(x \in [2N]\) for which \((x, 0) \in V(f(t_i'))\) and \((x, 1) \in V(f(t_i''))\).

Suppose first that \(p = 0\). Then \(G = C\) and so \(G\) can be expressed as a union of four pairwise edge-disjoint paths \(P, Q, T',\) and \(T''\). Let \(N\) be the smallest integer such that 
\[ 2^N \geq \max\{2^n, |V(Q)|, |V(T)|, |V(T'')|\}. \]
Clearly, there is an isomorphism \(f\) from \(C\) to a minor of the boundary of the infinite face of \(I_{n_0}\) that agrees with \((C, (P, h), Q)\) and satisfies (6). It is also clear that (5) holds for the graph \(C\) and the isomorphism \(f\). Thus the claim holds if \(p = 0\).

Now suppose that \(p\) is a positive integer and the theorem holds for all smaller values of \(p\). By the inductive hypothesis, there is an integer \(M\) and an isomorphism \(g\) from \(G_{p-1}\) to a minor of \(I_M\) that agrees with \((G_{p-1}, (P, h), Q)\). Let \(R\) be a path in \(G_p\) that meets \(G_{p-1}\) only in its endvertices \(u\) and \(v\), and is such that \(G_p = G_{p-1} \cup R\). As \(G_p\) is a plane graph and \(R\) meets \(G_{p-1}\) only in its endvertices, there is a finite face \(\psi\) of \(G_{p-1}\) such that \(R\) splits \(\psi\) into \(\psi_1\) and \(\psi_2\) which form faces of \(G_p\). Let \(D\) denote the cycle in \(G_{p-1}\) that is the boundary of \(\psi_1\), and let \(E\) denote the innermost cycle of \(g(D)\) in \(I_{p-1}\).

Then, by applying (5) to the graph \(G_{p-1}\) and the isomorphism \(g\), we conclude that \(E\) bounds a finite face \(\psi'\) of \(\Sigma(g(G_{p-1}))\). It is clear that each of \(g(u)\) and \(g(v)\) meets \(E\). Thus, upon combining Lemma 3.10 with a multiple application of Lemma 3.8, we conclude that there is a positive integer \(k\) such that \((i^k g(D))^w\) contains a path \(R'\) that joins \(i^k g(u)\) to \(i^k g(v)\) and has at least as many vertices as \(R\). Now let \(N = M + 2k\). It follows from Lemma 3.7 that \(i^k g\) is an isomorphism from \(G_{p-1}\) to \(I_N\) that agrees with \((G_{p-1}, (P, h), Q)\). It is clear that \(i^k g\) can be extended to an isomorphism \(f\) from \(G_p\) to \(I_N\) that maps \(R\) to a minor of \(R'\) and that also agrees with \((G_{p-1}, (P, h), Q)\). It is clear that (5) holds for \(G_p\) and \(f\). Moreover, it follows from the inductive hypothesis and from the definition of \(t\) that \(f\) satisfies (6). \(\square\)
We conclude this section by using Theorem 3.2 to prove Theorem 3.3.

**Proof of Theorem 3.3.** In the first part of the proof, we shall construct a flat n-segment $(H,(P,h),Q)$ from $(G,(C,k),D)$, and then we shall use Theorem 3.2 to derive Theorem 3.3.

Let $\varphi$ and $\psi$ denote the faces of $G$ whose boundaries are, respectively, $C$ and $D$. Let $x_0$ denote the lexicographically smallest element $x$ of $[2^n]$ for which $(0_n,x)$ is a vertex of $k(v_0)$ for some vertex $v_0$ of $C$. We construct $G_1$ from $G$ by adding edges, if necessary, in faces other than $\varphi$ and $\psi$ so that $G_1$ has a path $T$ joining $v_0$ to $D$. Let $t_0, t_1, \ldots, t_s$ denote the vertices of $T$, listed here in the order they appear on $T$ and so that $t_0 \in V(C)$ and $t_s \in V(D)$. Let $T'$ and $T''$ be two copies of $T$ that are disjoint from each other and from $G$. For each $i \in \{0,1,\ldots,s\}$, let $t'_i$ and $t''_i$ be the vertices of $T'$ and $T''$, respectively, that correspond to $t_i$. A T-ladder is a plane graph obtained from the union $T' \cup T''$ by adding edges of the form $\{t'_i, t''_i\}$, called the rungs, for all $i \in \{0,1,\ldots,s\}$. We construct a plane graph $G_2$ from $G_1$ by replacing $T$ in $G_1$ by a T-ladder, as illustrated in Fig. 5.

Let $\varphi_2$ and $\psi_2$ denote the faces of $G_2$ that correspond to, respectively, the faces $\varphi$ and $\psi$ of $G$, and let $C_2$ and $D_2$ be the boundaries of $\varphi_2$ and $\psi_2$. It is easy to see that the isomorphism $k$ from $C$ to a minor of $\Theta^0_{n+2}$ can be extended to $k_2$, which is an isomorphism from $C_2$ to a minor of $\Theta^0_{n+2}$ that agrees with $k$ and is such that $V(k_2(t'_0)) \ni (0_{n+2}, 0_{n+2})$ and $V(k_2(t''_0)) \ni (0_{n+2}, 1_{n+2})$. Let $G_3$ be the graph obtained from $G_2$ by removing the rungs of the T-ladder, and let $C_3$ and $D_3$ denote the resulting subgraphs of $C_2$ and $D_2$, respectively. Let $k_3$ be the restriction of $k_2$ to $C_3$. Then $(G_3,(C_3,k_3),D_3)$ is a flat attachment. Moreover, the boundary of the infinite face of $G_3$ consists of four edge-disjoint paths $C_3, D_3, T'$, and $T''$ that satisfy $|V(T')| = |V(T'')|$. Thus, by Theorem 3.2 and (6) in its proof, there is a number $N$ and an isomorphism $f_3$ from $G_3$ to a minor of $\Gamma_N$ that agrees with $(G_3,(C_3,k_3),D_3)$ such that, for each vertex
of $T$, there is an $x_i \in [2^N]$ so that $(x_i, 0) \in V(f_3(t'_i))$ and $(x_i, 1) \in V(f_3(t''_i))$. Extend $f_3$ to the isomorphism $f_2$ from $G_2$ to a minor of $\Theta_N$ by mapping the rung $\{t'_i, t''_i\}$ of the $T$-ladder to the edge $\{(x_i, 0), (x_i, 1)\}$. It is clear that $f_2$ induces an isomorphism $f$, as desired. □

4. Flat graphs

The main result of this section is the following:

**Theorem 4.1.** Every flat graph is minor-equivalent to $G_{\mathbb{Z} \times N}$.

Let $G'_{\mathbb{Z} \times N}$ be the graph that is formed as follows. First take the disjoint union of all of the graphs $I_n$ for $n \geq 1$, then add new edges that join the vertex $(1, n, \beta)$ to both $(0,n+1, \beta + 0)$ and $(0, n+1, \beta + 1)$ for all positive integers $n$ and all $\beta \in [2^n]$. It is clear that Theorem 4.1 follows from the following three lemmas.

**Lemma 4.2.** If $G$ is flat, then $G'_{\mathbb{Z} \times N}$ is isomorphic to a minor of $G$.

**Lemma 4.3.** If $G$ is flat, then $G$ is isomorphic to a minor of $G'_{\mathbb{Z} \times N}$.

**Lemma 4.4.** The graph $G'_{\mathbb{Z} \times N}$ is isomorphic to a minor of $G_{\mathbb{Z} \times N}$.

Observe that upon combining Lemmas 4.3 and 4.4, we conclude that every flat graph is isomorphic to a minor of $G_{\mathbb{Z} \times N}$. A direct proof of this fact, however, is very messy. Introducing the graph $G'_{\mathbb{Z} \times N}$ and dealing with Lemmas 4.3 and 4.4 instead, makes for a much cleaner argument. The proof of Lemma 4.4 is straightforward but somewhat tedious to write out in detail; we leave a formal proof to the reader. The remainder of this section will be concerned with proving Lemmas 4.2 and 4.3. Lemma 4.2 will be derived from a result of [4], but before this result can be stated, we need some preparation.

Let $G$ be a graph and let $\Theta$ be the pair $(T, (X_t)_{t \in T})$ which consists of a tree $T$ and a multiset whose elements $X_t$, indexed by the vertices of $T$, are finite subsets of $V(G)$. For a vertex $v$ of $G$, we denote by $T_v$ the subgraph of $T$ induced by those vertices $t$ of $T$ for which $X_t$ contains $v$. For a subgraph $H$ of $G$, we let $T_H = \bigcup_{t \in V(H)} T_t$. Then $\Theta$ is called a finite tree-decomposition of $G$ if it satisfies conditions (T1)–(T3) below.

(T1) The union, over all vertices $t$ of $T$, of the subgraphs of $G$ induced by $X_t$ equals $G$.

(T2) For every vertex $v$ of $G$, the subgraph $T_v$ of $T$ is a tree.

(T3) For every ray $\rho$ of $T$, there is an integer $n_\rho$ such that for $t'$ and $t''$ are two adjacent vertices of $\rho$, then $|X_{t'} \cap X_{t''}| \leq n_\rho$.

The following lemma is an easy consequence of (T2) and (T3) — we omit the proof.

**Lemma 4.5.** If $(T, (X_t)_{t \in V(T)})$ is a finite tree-decomposition of a graph $G$, and $H$ is a connected subgraph of $G$, then $T_H$ is a tree.
One of the results of [4] is the following:

**Theorem 4.6.** A graph $G$ has a minor isomorphic to $G_{\mathbb{Z} \times \mathbb{N}}$ if and only if there is no finite tree-decomposition of $G$.

We deduce from Theorem 4.6 that Lemma 4.2 is a consequence of the following:

**Lemma 4.7.** No graph with a thick end has a finite tree-decomposition.

**Proof.** Suppose the lemma fails and $G$ is a counterexample. Let $\varepsilon$ be a thick end of $G$, and let $(T,(X_t)_{t \in V(T)})$ be a finite tree-decomposition of $G$. Let $\rho$ be a ray from $\varepsilon$. From Lemma 4.5 and the fact that each set $X_t$ is finite, we conclude that, for every vertex $t$ of $T_\rho$, all but finitely many vertices of $\rho$ are in the same connected component of the forest obtained from $T$ by deleting $t$. As $\rho$ is infinite, this immediately implies that

1. the graph $T_\rho$ has exactly one ray.

For a ray $\rho$ of $G$, let $\rho'$ denote the unique ray of $T_\rho$. Suppose now that $\rho$ and $\sigma$ are two distinct rays in $\varepsilon$. We shall show that

2. $\rho'$ and $\sigma'$ are in the same end of $T$.

Suppose not. Then there is a vertex $t$ of $T$ such that the infinite connected components of $\rho' - \{t\}$ and $\sigma' - \{t\}$ are in distinct connected components of $T - \{t\}$. Since $\rho$ and $\sigma$ are both in $\varepsilon$, there are infinitely many pairwise-disjoint paths in $G$ each of which joins the infinite connected component of $\rho - \{t\}$ to the infinite connected component of $\sigma - \{t\}$. By Lemma 4.5, for every such path $P$, the graph $T_\rho$ is connected, and hence its vertex set contains $t$. Thus, every such path has at least one of its vertices in $X_t$; a contradiction to the finiteness of $X_t$. This proves (2).

Now it follows from (2) that there is a ray $\tau$ of $T$ such that, for every ray $\rho$ from $\varepsilon$, the unique ray $\rho'$ of $T_\rho$ is equivalent to $\tau$. From (T3), there is a number $n$, such that if $t'$ and $t''$ are two adjacent vertices of $\tau$, then $|X_{t'} \cap X_{t''}| \leq n$. Since $\varepsilon$ is thick, there a finite set $\mathcal{R}$ of pairwise-disjoint rays from $\varepsilon$ that has more than $n$ elements. Since $\mathcal{R}$ is finite, and, for each of its elements $\rho$, the ray $\rho'$ is equivalent to $\tau$, there is a vertex $s$ of $\tau$ that meets $\rho'$ for all $\rho \in \mathcal{R}$. But this is impossible, since the rays in $\mathcal{R}$ are pairwise disjoint and the cardinality of $\mathcal{R}$ exceeds the cardinality of $X_\varepsilon$. $\square$

To conclude this section, it remains to show Lemma 4.3.

**Proof of Lemma 4.3.** For a finite subgraph $A$ of $G$, let $\bar{A}$ denote the subgraph of $G$ that is the union of $A$ and all finite bridges of $A$ in $G$. Clearly, as $A$ is finite and $G$ is locally finite, $\bar{A}$ is also finite.

Without loss of generality, we may assume that $G$ is a plane graph. Since $G$ is flat, it has a finite subgraph $H$ and a set $\mathcal{R}$ of pairwise-disjoint rays such that $|\mathcal{R}| \geq 3$ and no cycle in $G - H$ collates $\mathcal{R}$. We shall inductively define an ascending sequence $A_0, A_1, \ldots$ of finite connected subgraphs of $G$ as follows. Since each of $H$ and $\mathcal{R}$ is
finite, there is a finite connected subgraph $A_0$ of $G$ that contains $H$ and meets all rays in $\mathcal{R}$. Suppose $i$ is a positive integer and $A_{i-1}$ has been defined so that it is connected. Clearly, $A_{i-1}$ is finite and connected. Let $A_i$ be the subgraph of $G$ induced by the vertices whose distance to $A_{i-1}$ is less than two. Then $A_i$ is also finite and connected.

Let $B_0 = A_0$. For a positive integer $i$, let $B_i = A_i \setminus A_{i-1}$, and let $V_i$ denote the vertices common to $B_{i-1}$ and $B_i$. From the construction, we conclude the following:

1. If $i$ and $j$ are non-negative integers such that $|j - i| \geq 2$, then $B_i$ and $B_j$ are disjoint.
2. If $i$ and $j$ are non-negative integers such that $|j - i| = 1$, then $B_i \cap B_j$ is a non-null edgeless graph with vertex set $V_i$.
3. $B_0$ has exactly one bridge $\bigcup_{j > 0} B_j$, and the set of the vertices of attachment of this bridge is $V_1$.
4. For every positive integer $i$, the graph $B_i$ has two bridges: the finite $A_{i-1}$, and the infinite $\bigcup_{j > i} B_j$. The sets of vertices of attachment of these bridges are $V_i$ and $V_{i+1}$, respectively.

It follows from (3) that all vertices in $V_1$ lie in the same face $\varphi$ of $B_0$. Hence, by adding edges to $B_0$ if necessary, we may form a plane graph $B'_0$ that has a path $P_1$ in the boundary of the infinite face of $B_0$ such that $V(P_1) \supseteq V_1$.

Let now $i$ be a positive integer. From (4) we conclude that there is a face $\varphi$ of $B_i$ whose boundary contains all elements of $V_i$. Likewise, there is a face $\varphi'$ of $B_i$ whose boundary contains all elements of $V_{i+1}$. We shall show that

5. $\varphi = \varphi'$.

Suppose not. Then $B_i$ has a cycle $C$ that meets every path joining a vertex in $V_i$ to a vertex in $V_{i+1}$. Observe that, for each ray $\rho$ in $\mathcal{R}$, the intersection $\rho \cap B_i$ is a path joining a vertex of $V_i$ to a vertex in $V_{i+1}$, and hence meeting $C$. Let $D$ be a subpath of $C$ that has its endvertices in different rays of $\mathcal{R}$ and is internally disjoint from $\bigcup_{\rho \in \mathcal{R}} \rho$. Then the path $P = C \setminus D$ is contained in $G \setminus H$ and meets all rays in $\mathcal{R}$. By Lemma 2.1, there is a path $Q$ that is contained in $P \cup \bigcup_{\rho \in \mathcal{R}} P_\rho'$ and collates $\mathcal{R}$. Let $\rho$ and $\sigma$ denote the elements of $\mathcal{R}$ that contain the endvertices of $Q$. It follows that the graph $Q \cup (C \setminus D) \cup \rho \cup \sigma$ contains a cycle that avoids $H$ and collates $\mathcal{R}$; a contradiction. Hence (5) follows.

Let $v_1, v_2, \ldots, v_n$ be the list of vertices from $V_i \cup V_{i+1}$ (with possible repetitions) in the cyclic order in which they appear on the boundary of $\varphi$. By (3), $B_i$ has two bridges $A_{i-1}$ and $\bigcup_{j > i} B_j$, whose sets of vertices of attachments are $V_i$ and $V_{i+1}$. Thus, as $G$ is a plane graph and each of its bridges is connected, it follows that there are integers $a$ and $b$ such that $V_i = \{v_a, v_{a+1}, \ldots, v_{a+b}\}$ where the arithmetic is carried out modulo $n$. Hence, by adding edges to $B_i$ if necessary, we form a plane graph $B'_i$ that has two disjoint paths $P_i$ and $P_{i+1}$ in the boundary of its infinite face such that $V(P_i) \supseteq V_i$ and $V(P_{i+1}) \supseteq V_{i+1}$.

We shall use the graphs $B'_i$ to inductively define a sequence of flat segments as follows. Let $h_0$ denote the trivial isomorphism between null graphs. Then $(B'_0, (\emptyset, h_0), P_1)$
is a flat segment. Hence, by Theorem 3.2, there is an integer \( n_0 \) and an isomorphism \( f_0 \) from \( B_0 \) to a minor of \( \Gamma_{n_0} \) that agrees with \((B'_0, (0, h_0), P)\). Inductively, suppose that \( i \) is a positive integer, \( n_0, n_1, \ldots, n_{i-1} \) is an increasing sequence of integers, \((B'_{i-1}, (P_{i-1}, h_{i-1}), P)\) is a flat segment, and \( f_{i-1} \) is an isomorphism from \( B'_{i-1} \) to a minor of \( \Gamma_{n_{i-1}} \) that agrees with \((B'_{i-1}, (P_{i-1}, h_{i-1}), P)\).

Suppose \( v \) is a vertex of \( P_i \). Then the set \( V(f_{i-1}(v)) \) contains an element \((1, \beta)\) for some \( \beta \in [2^{n-1}] \). Define the isomorphism \( h_i \) from \( P_i \) to a minor of \( \Gamma_{n_{i-1}} \) by letting \( h_i(v) = (0, \beta) \). Then \((B'_i, (P_i, h_i), P_{i+1})\) is a flat segment, and by Theorem 3.2, there is an integer \( n_i \) exceeding \( n_{i-1} \) and an isomorphism from \( B'_i \) to a minor of \( \Gamma_{n_i} \) that agrees with \((B'_i, (P_i, h_i), P_{i+1})\).

Finally, we shall combine all isomorphisms \( f_i \) into an isomorphism \( f \) from \( G \) to a minor of \( G'_{\mathbb{Z} \times \mathbb{N}} \). For a vertex \( v \) of \( G \), we define \( f(v) \) as follows. Suppose first that \( v \notin \bigcup_{i \geq 1} V_i \). Then it follows from (1)–(3) that there is exactly one integer \( i \) such that \( v \in V(B_i) \). We let \( f(v) = f_i(v) \). Suppose now that \( v \in V_i \) for some positive integer \( i \). Then \( v \in V(P_{i-1}) \), and thus \( V(f_{i-1}(v)) \) contains an element of the form \((1, \alpha)\) for some \( \alpha \in [2^{n-1}] \). Then \( h_i(v) = (0, \alpha) \), and, as \( f_i \) agrees with \((B'_i, (P_i, h_i), P_{i+1})\), the set \( V(f_i(v)) \) contains a vertex of the form \((0, \beta)\) for some binary sequence \( \beta \in [2^n] \) that has \( \alpha \) as the initial segment. We let \( f(v) = f_{i-1}(v) \cup P_{1/2} \cup f_i(v) \).

To define \( f \) on the edge set of \( G \), observe from (1)–(3) that every edge \( e \) of \( G \) is contained in exactly one \( B'_i \). We let \( f(e) = f_i(e) \). It is easy to verify that \( f \) satisfies the conclusion of Lemma 4.3.

5. Round graphs

In this section, we prove analogs of the results from Section 4 for round graphs. The main result of this section is the following:

**Theorem 5.1.** Every round graph is minor-equivalent to \( G_{\mathbb{Z} \times \mathbb{Z}} \).

Just as in Section 4, we shall introduce a new graph that is the most convenient representative of the minor-equivalence class of round graphs. Let \( G'_{\mathbb{Z} \times \mathbb{Z}} \) be the graph obtained by taking the disjoint union of the graphs \( \Theta_n \) over all integers \( n \geq 2 \), and then, for each \( n \geq 2 \) and each \( \beta \in [2^n] \), adding two new edges that join the vertex \((1, \beta)\) to \((0, n+1, \beta + 0)\) and to \((0, n+1, \beta + 1)\). As noted in Proposition 1.10, \( G_{\mathbb{Z} \times \mathbb{Z}} \) is round. Thus, to prove Theorem 5.1, it suffices to show that all round graphs are in the same minor-equivalence class. This will be accomplished by proving the following two lemmas.
Lemma 5.2. If $G$ is round, then $G'_{\mathbb{Z} \times \mathbb{Z}}$ is isomorphic to a minor of $G$.

Lemma 5.3. If $G$ is round, then $G$ is isomorphic to a minor of $G'_{\mathbb{Z} \times \mathbb{Z}}$.

The remainder of this section is devoted to proving Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. Observe that, roughly speaking, the graph $G'_{\mathbb{Z} \times \mathbb{Z}}$ is the union of an infinite set of pairwise-disjoint cycles and a subdivision of the infinite binary tree. This observation captures the main idea of the proof.

It follows from Theorem 4.6 and Lemma 4.7 that $G$ contains a subgraph $T$ that is isomorphic to a subdivision of the infinite binary tree. A ray of $T$ is proper if its vertices are linearly ordered by $<_T$. A fork in $T$ is the graph consisting of two proper rays of $T$ that meet in exactly one vertex, the root of the fork. For a binary sequence $\alpha$, let $t_{\alpha}$ denote the vertex of $T$ that corresponds to the vertex $\alpha$ of $T'$, and let $\rho_{\alpha}$ denote the ray of $T$ containing all vertices $t_{\gamma}$ for which $\gamma$ consists of $\alpha$ followed by nothing but zeros.

For each binary sequence $\alpha$ of length exceeding one, let $G_{\alpha}$ denote the subgraph of $G'_{\mathbb{Z} \times \mathbb{Z}}$ induced by the vertices $(\tau, \beta)$ for which $\tau < \beta$. We shall proceed by induction on the set $\{1, 0, 1\}$ ordered by the relation $\preceq$ to define a sequence of isomorphisms $f_0, f_1, f_{10}, \ldots$ such that, for each finite binary sequence $\alpha$ of length exceeding one, the following hold.

1. $f_\alpha$ is an isomorphism from $G_\alpha$ to a minor of $G$.
2. $f_\alpha$ a restriction of $f_\alpha$.
3. For every vertex of $G'_{\mathbb{Z} \times \mathbb{Z}}$ that has the form $(\alpha, \beta)$, there is a fork $F$ in $T$ such that the set $V(F) \cap V(f_\alpha(G_{\alpha}))$ consists of the root $r$ of $F$, and $r \in V(f_\alpha((\alpha, \beta)))$.
4. Let $n$ denote the length of $\alpha$. Then there is a set $R_\alpha$ of $n$ pairwise-disjoint proper rays of $T$ such that, for all $\beta \in [2^n]$, the image of the vertex $(\alpha, \beta)$ under $f_\alpha$ meets exactly one of the rays $\rho$ of $R_\alpha$, and the intersection $f_\alpha((\alpha, \beta)) \cap \rho$ is a path.

Let $A_0 = \{r_{00}, r_{01}, r_{10}, r_{11}\}$. Clearly, all rays in $A_0$ are pairwise disjoint. Since $G$ is round, it has a cycle $C_0$ that collates $A_0$. Let $\beta \in \{00, 01, 10, 11\}$. As $C_0$ is finite and $G$ is locally finite, there is a fork $F_{\beta}$ in $T$ that is disjoint from $C_0$, and whose root $r_{\beta}$ is such that $C_0 < r_{\beta}$.

Let $f_0(00, \beta)$ be the minimal path of $p_\beta$ that contains $C_0 \cap p_{\beta}$ and $r_{\beta}$. It is clear that (1)–(4) hold for $\alpha = 00$.

Inductively, suppose that $f_\gamma$ has been defined so that (1)–(4) hold for $\gamma = \gamma$. We shall consider two cases depending on the number of entries of $\gamma$. Suppose first that the length of $\gamma'$ equals the length of $\gamma$ and denote this common length by $n$. Since $G$ is round, there is a cycle $C_{\gamma'}$ in $G$ that collates $R_\gamma$. Let $\beta \in [2^n]$ and let $\rho_{\beta}$ be the ray from $R_\gamma$ that meets $f_\gamma((\gamma, \beta))$. Since $C_{\gamma'}$ is finite and $G$ is locally finite, there is a fork $F_{\beta}$ in $T$ that is disjoint from $C_{\gamma'}$ and whose root $r_{\beta}$ is such that $C_{\gamma'} < r_{\beta}$. For all $x \in V(G_{\gamma}) \cup E(G_{\gamma})$, let $f_{\gamma'}(x) = f_\gamma(x)$, and, for all $\beta \in [2^n]$, let $f_{\gamma'}((\gamma, \beta))$ be the minimal path of $\rho_{\beta}$ that contains $C_{\gamma'} \cap \rho_{\beta}$ and $r_{\beta}$. Let $B_{\gamma'}$ be the subgraph of $G'_{\mathbb{Z} \times \mathbb{Z}}$ that is induced by the vertices of the form $(\gamma, \beta)$ for all $\beta \in [2^n]$. Observe that, by the inductive application of (4), the image of $B_{\gamma'}$ under $f_{\gamma'}$ is a cycle collating $R_\gamma$. Now, $C_{\gamma'}$ collates $R_\gamma$ as well, and, by Lemma 2.4,
tr_{A_y}(C_+^+) = tr_{A_y}(f_y(B_+^+)). Thus, it is easy to define \( f_y^+ \) on the edges of \( G_y^+ \setminus G_y \) so that (1)–(4) hold with \( x = y^+ \).

Now we may assume that the length of \( y^+ \) exceeds the length of \( y \). For each vertex of \( G'_{y \times z} \) that has the form \((y', \beta')\), let \( F_{\beta'} \) be the fork as described in (4), and let \( r_{\beta'} \) be the root of \( F_{\beta'} \). Observe that the graph \( F_{\beta'} \setminus \{r_{\beta'}\} \) consists of two disjoint rays. Denote these rays \( \rho_{\beta'+0} \) and \( \rho_{\beta'+1} \). Let \( \mathcal{A}_{n+1} = \bigcup_{\beta \in [2^n]} \{\rho_{\beta'+0}, \rho_{\beta'+1}\} \). It is clear that \( \mathcal{A}_{n+1} \) is a set of \( 2^{n+1} \) rays each two of which are disjoint. By (C3), there is a cycle \( C_{y^+} \) in \( G - f_y^+(G_y^+) \) that collates \( \mathcal{A}_{n+1} \). It is now easy to use the ideas of the previous paragraph to extend \( f_y \) to a function \( f_y^+ \) that is an isomorphism from \( G_y^+ \) to a minor of \( G'_{y \times z} \) such that (1)–(4) hold with \( x = y^+ \). This completes the proof of Lemma 5.2.

**Proof of Lemma 5.3.** This proof is similar to the proof of Lemma 4.3. Recall from that proof that, for a finite subgraph \( A \) of \( G \), the union of \( A \) with all its finite bridges in \( G \) is denoted by \( \overline{A} \). Let \( A_0 \) be an arbitrarily chosen finite subgraph of \( G \) on at least three vertices. Follow the inductive step of the definition of the \( A_i \)'s from the proof of Lemma 4.3 by letting \( A_i \) be the subgraph of \( G \) induced by the vertices whose distance from \( A_{i-1} \) is less than two. Let \( B_0 = A_0 \), and, for a positive integer \( i \), let \( B_i = A_i \setminus \overline{A_{i-1}} \). Let \( V_i = V(B_{i-1}) \cap V(B_i) \). Then all of (1)–(4) from the proof of Lemma 4.3 hold.

It follows from (3) that all vertices in \( V_i \) lie in the boundary of the same face \( \varphi \) of \( B_0 \). Hence, by adding edges to \( B_0 \) if necessary, we form a plane graph \( B'_i \) that has a cycle \( C_1 \) that bounds a face and contains all elements of \( V_i \).

Now let \( i \) be a positive integer. From (4) we conclude that there is a face \( \varphi \) of \( B_i \) whose boundary contains all elements of \( V_i \). Similarly, there is a face \( \varphi' \) of \( B_i \) whose boundary contains all elements of \( V_{i+1} \). Again, by adding edges if necessary, we form a plane graph \( B'_i \) that has two disjoint cycles \( C_i \) and \( C_{i+1} \) such that each bounds a face of \( B' \) and \( V(C_i) \supseteq V_i \) and \( V(C_{i+1}) \supseteq V_{i+1} \).

Let \( h_0 \) denote the trivial isomorphism between null graphs. Then \( (B'_0, (\emptyset, h_0), C_1) \) is a round segment. Hence, by Theorem 3.3, there is an integer \( n_0 \) and an isomorphism \( f_0 \) from \( B'_0 \) to a minor of \( \Theta_{n_0} \) that agrees with \( (B'_0, (\emptyset, h_0), C_1) \).

Inductively, suppose that \( i \) is a positive integer, \( n_0, n_1, \ldots, n_{i-1} \) is an increasing sequence of integers, \( (B'_{i-1}, (C_{i-1}, h_{i-1}), C_i) \) is a round segment, and \( f_{i-1} \) is an isomorphism from \( B'_{i-1} \) to a minor of \( \Theta_{n_{i-1}} \) that agrees with \( (B'_{i-1}, (C_{i-1}, h_{i-1}), C_i) \). Suppose \( v \) is a vertex of \( P_i \). Then the set \( V(f_{i-1}(v)) \) contains a vertex \( (1, \beta_v) \) for some \( \beta_v \in [2^{n_{i-1}}] \). Define the isomorphism \( h_i \) from \( C_i \) to a minor of \( \Theta_{n_{i-1}} \) by letting \( h_i(v) = (0, \beta_v) \). Then \( (B'_i, (C_i, h_i), P_{i+1}) \) is a round segment, and by Theorem 3.3, there is an integer \( n_i \) exceeding \( n_{i-1} \) and an isomorphism from \( B'_i \) to a minor of \( \Theta_{n_i} \) that agrees with \( (B'_i, (C_i, h_i), P_{i+1}) \).

For two binary sequences \( \alpha \) and \( \beta \) such that \( \alpha \) is an initial segment of \( \beta \), let \( P_{\alpha \beta} \) be the path of \( G'_{y \times N} \) defined as in the proof of Lemma 4.3. As \( G'_{y \times N} \) is a subgraph of \( G'_{z \times N} \), each path \( P_{\alpha \beta} \) is also a subgraph of \( G'_{y \times z} \).

Upon following the last two paragraphs of the proof of Lemma 4.3 with \( P_{\alpha \beta} \) replaced by \( C_{\alpha \beta} \), we obtain an isomorphism \( f \) from \( G \) to a minor of \( G'_{y \times z} \) that satisfies the conclusion of Lemma 5.3. □
6. Subgraphs of the full grid

In this section we shall investigate subgraphs of $G_{\mathbb{Z} \times \mathbb{Z}}$. Two edges $e$ and $f$ of $G_{\mathbb{Z} \times \mathbb{Z}}$ are **linked** if there is a face of $G_{\mathbb{Z} \times \mathbb{Z}}$ that has both $e$ and $f$ in its boundary. A subgraph $H$ of $G_{\mathbb{Z} \times \mathbb{Z}}$ is **linked** if, for every two edges $e$ and $f$ of $H$, there is a sequence $e = e_0, e_1, \ldots, e_n = f$ of edges of $G_{\mathbb{Z} \times \mathbb{Z}}$ such that $e_i$ is linked to $e_{i-1}$ for all $i \in \{1, 2, \ldots, n\}$. Observe that there is a close relationship between linked subgraphs of $G_{\mathbb{Z} \times \mathbb{Z}}$ and connected duals of subgraphs of $G_{\mathbb{Z} \times \mathbb{Z}}$. We shall not pursue this relationship, however, as it will not be needed here. A **link-component** of a subgraph $H$ of $G_{\mathbb{Z} \times \mathbb{Z}}$ is a maximal subgraph of $H$ that is linked. The remainder of this section is devoted to proving the following two lemmas.

**Lemma 6.1.** If $H$ is an infinite linked subgraph of $G_{\mathbb{Z} \times \mathbb{Z}}$ such that $G_{\mathbb{Z} \times \mathbb{Z}} \setminus H \leq_m G_{\mathbb{Z} \times \mathbb{N} \setminus \mathbb{N}}$, then $G_{\mathbb{Z} \times \mathbb{Z}} \setminus H \leq_m G_{\mathbb{Z} \times \mathbb{Z} \setminus \mathbb{N}}$.

**Lemma 6.2.** If $H$ is a subgraph of $G_{\mathbb{Z} \times \mathbb{Z}}$ such that all link-components of $H$ are finite and $G_{\mathbb{Z} \times \mathbb{Z}} \setminus H$ contains infinitely many pairwise disjoint rays, then $G_{\mathbb{Z} \times \mathbb{Z}} \leq_m G_{\mathbb{Z} \times \mathbb{Z} \setminus H}$.

As preparation for proving Lemmas 6.1 and 6.2, we state and prove two easy lemmas about link graphs. For a cycle $C$ of $G_{\mathbb{Z} \times \mathbb{Z}}$, the number of faces of $G_{\mathbb{Z} \times \mathbb{Z}}$ contained in the finite region of the plane cut off by $C$ will be denoted by $a(C)$. Suppose $H$ is a finite linked subgraph of $G_{\mathbb{Z} \times \mathbb{Z}}$. Let $\mathcal{C}(H)$ be the set of cycles $C$ of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus H$ such that all vertices of $H$ lie on $C$ or in the finite region of the plane cut off by $C$. A cycle $C_0$ in $\mathcal{C}(H)$ **surrounds** $H$ if $a(C_0) < a(C)$ for every cycle $C$ in $\mathcal{C}(H)$.

**Lemma 6.3.** Let $L$ be a subgraph of $G_{\mathbb{Z} \times \mathbb{Z}}$ that is a cycle or a two-way-infinite path. Then $L$ induces a decomposition of $G_{\mathbb{Z} \times \mathbb{Z}}$ into pairwise-edge-disjoint union of three graphs: $G'$, $G''$, and $L$ such that $G'$ and $G''$ are on the opposite sides of $L$. Suppose $K$ is a subgraph of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus L$ and $K'$ in a link-component of $K$. Then $K'$ meets at most one of $G' - L$ and $G'' - L$.

**Proof.** Suppose that $K'$ meets both $G' - L$ and $G'' - L$. Then there is a sequence $e_0, e_1, \ldots, e_n$ of edges of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus L$ in which every two consecutive elements are linked, and such that $e_0$ is incident with a vertex of $G' - L$ and $e_n$ is incident with a vertex of $G'' - L$. Let $k$ be the largest index such that $e_k$ is incident with a vertex $v'$ of $G' - L$. Clearly, $k < n$. Since $e_{k+1}$ is not an edge of $L$, the choice of $k$ implies that at least one of its vertices $v''$ lies in $G'' - L$. Thus $v'$ and $v''$ lie in the boundary of the same face of $G_{\mathbb{Z} \times \mathbb{Z}}$, and, at the same time, they lie on opposite sides of $L$; a contradiction. □

**Lemma 6.4.** Let $H$ be a finite linked subgraph of $G_{\mathbb{Z} \times \mathbb{Z}}$ and let $C$ be a cycle of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus H$ that surrounds $H$. Then, for every edge $e$ of $C$, there is an edge $f$ of $H$ and a face $\phi$ of $G_{\mathbb{Z} \times \mathbb{Z}}$ such that $e$ and $f$ are in the boundary of $\phi$. 
Proof. Suppose the lemma fails. Then there is an edge $e$ of $C$ that is not linked with any edges of $H$. Let $\varphi$ denote the face of $G_{\mathbb{Z} \times \mathbb{Z}}$ that has $e$ in its boundary and lies inside the finite region of the plane cut off by $C$. Let $D$ denote the boundary of $\varphi$. Observe that, as $H$ is linked and no edges of $D$ are in $H$, it is impossible for $C \cap D$ to consist of two non-adjacent edges. Hence $C \cap D$ is a path. Construct a cycle $C'$ from $C$ by replacing $C \cap D$ by $D \setminus C$. Then $C'$ is in $\mathcal{C}_H$ and $a(C') < a(c)$; a contradiction. \[\square\]

Now we are ready to prove Lemmas 6.1 and 6.2.

Proof of Lemma 6.1. Observe that each edge of $H$ is linked to finitely many and possibly no other edges of $H$. Hence, as $H$ is linked and infinite, there is a sequence of edges $e_0, e_1, e_2, \ldots$ of $H$ such that

(1) $e_i$ and $e_j$ are linked if and only if $|i - j| \leq 1$.

Let $K$ be the subgraph of $G_{\mathbb{Z} \times \mathbb{Z}}$ induced by the edges in $\{e_0, e_1, \ldots\}$. Clearly, $K$ is a subgraph of $H$, and hence, to prove Lemma 6.1, it suffices to show that

(2) $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K \cong_{\text{m}} G_{\mathbb{Z} \times \mathbb{N}}$.

It is clear that $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$ is planar and locally finite. We shall show that

(3) $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$ has at most one end.

Suppose $\rho$ and $\sigma$ are two rays in distinct ends of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$. Let $P$ be a path in $G_{\mathbb{Z} \times \mathbb{Z}}$ that joins $\rho$ to $\sigma$. Then $\rho \cup \sigma \cup P$ has a subgraph $L$ that is a two-way-infinite path. Now $G_{\mathbb{Z} \times \mathbb{Z}}$ can be represented as the union of three edge-disjoint graphs, $G_1$, $G_2$, and $L$, such that $G_1$ and $G_2$ are on opposite sides of $L$. Clearly, each of $G_1$ and $G_2$ contains infinitely many pairwise-disjoint paths each of which joins $\rho$ to $\sigma$. As $\rho$ and $\sigma$ are in different rays of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$, the graph $K$ must meet all but finitely many of such $\rho\sigma$-paths. Hence, for infinitely many indices $n$, the edges $e_n$ and $e_{n+1}$ are on opposite sides of $L$. Since $P$ is finite, there is an index $n_0$ such that none of the edges $e_{n_0}$ and $e_{n_0+1}$ is incident with vertices of $P$, yet $e_{n_0}$ and $e_{n_0+1}$ are on opposite sides of $L$. This is clearly impossible, and hence (3) follows.

Next we show that

(4) $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$ is connected.

Suppose not. Then, from (3), it follows that $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$ has a finite component $G$. Let $E$ be the set of edges of $K$ that are incident with vertices of $G$. The plane embedding of $G_{\mathbb{Z} \times \mathbb{Z}}$ induces a cyclic order on the elements of $E$ with two successive elements in this order lying in the boundary of the same face. We obtain a contradiction to (1) as, clearly, $E$ has more than two elements.

The next statement is the last we need in order to conclude that $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$ is flat.

(5) One of the following holds:

(i) $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$ fails to have three pairwise-disjoint rays; or

(ii) there is a finite subgraph $L$ of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$ and a set $\mathcal{R}$ of three pairwise-disjoint rays of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$ such that no cycle of $(G_{\mathbb{Z} \times \mathbb{Z}} \setminus K) - L$ collates $\mathcal{R}$.

Suppose neither (i) nor (ii) holds. Then there is a set $\mathcal{R}$ of three pairwise-disjoint rays of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$. Let $L$ be a connected finite subgraph of $G_{\mathbb{Z} \times \mathbb{Z}} \setminus K$ that meets all rays
of $\mathcal{R}$ and both endvertices of the edge $e_0$ of $K$. Then $(G_{x\times z}\setminus K) - L$ contains a cycle $C$ that collates $\mathcal{R}$.

Observe now that, from the definition of $K$, it follows that there is a sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ of faces of $G_{x\times z}$ such that

(6) for all positive integers $i$, the only edges of $K$ in the boundary of $\varphi_i$ are $e_{i-1}$ and $e_i$; and

(7) if an edge $e$ of $G_{x\times z}$ is in the boundaries of $\varphi_i$ and $\varphi_{i+1}$ for some integer $i$, then $e$ is an edge of $K$.

Construct a new graph $G'_{x\times z}$ from $G_{x\times z}$ by subdividing each edge $e_i$ of $K$ with a new vertex $v_i$, and then, for all positive integers $i$, joining the vertices $v_{i-1}$ and $v_i$ with a new edge $f_i$ across the face $\varphi_i$. Let $\rho'$ be the ray induced by the edges $f_1, f_2, f_3, \ldots$. Then $\mathcal{R} \cup \{\rho'\}$ is a set of four pairwise-disjoint rays, all of which are in the same end of $G'_{x\times z}$. Let $L'$ be the subgraph of $G'_{x\times z}$ that is induced by $L$ together with the two new edges obtained by subdividing $e_0$. Then $L'$ is finite and connected. Note that $G'_{x\times z}$ is planar and locally finite, and $C$ is a subgraph of $G'_{x\times z}$ that meets three elements of $\mathcal{R} \cup \{\rho'\}$. Thus, by Lemma 2.5, $C$ also meets $\rho'$; a contradiction. Hence (5) holds.

In (3)–(5), we established that $G_{x\times z}\setminus K$ is flat. Hence, by Lemma 4.2, $G_{x\times z}\setminus K \leq m G_{x\times z}$. As $K$ is a subgraph of $H$, the theorem holds.

Proof of Lemma 6.2. We shall show that

(1) $G_{x\times z}\setminus H$ satisfies $(C_3)$.

Let $\mathcal{R}$ be a set of pairwise-disjoint rays of $G_{x\times z}\setminus H$ such that $3 \leq |\mathcal{R}| < \infty$. Let $K$ be a finite subgraph of $G_{x\times z}\setminus H$. Let $K'$ be a connected subgraph of $G_{x\times z}$ that contains $K$ and meets all rays in $\mathcal{R}$. Let $\mathcal{C}$ be the set of cycles of $G_{x\times z} - K'$ that collate $\mathcal{R}$. Since $G_{x\times z}$ is round, $\mathcal{C}$ is non-empty. For each cycle $C$ in $\mathcal{C}$, let $r(C)$ denote the number of link-components of $H$ whose edges are in $C$. Let $C_0$ be a cycle from $\mathcal{C}$ such that $r(C_0) \leq r(C)$ for all $C$ in $\mathcal{C}$. We shall prove that

(2) $C_0$ is contained in $(G_{x\times z}\setminus H) - K$.

Suppose (2) fails. Then there is a link-component $H_0$ of $H$ some of whose edges are in $C_0$. Recall that $K'$ is connected and meets all of the rays in $\mathcal{R}$. Thus it follows from Lemma 6.3 that $H_0$ meets at most two rays $\rho_1$ and $\rho_2$ in $\mathcal{R}$, and that $\rho_1$ and $\rho_2$ are consecutive in the cyclic order induced on $\mathcal{R}$ by the plane embedding of $G_{x\times z}$. Consequently, $H_0$ does not contain $C_0$. Let $P = C_0 \setminus H_0$. Then $P$ is a disjoint union of paths each of which has both endvertices in the cycle $D$ that surrounds $H_0$. Since $C_0$ avoids $K'$ and collates $\mathcal{R}$, there is a connected component of $P$ that collates $\mathcal{R} - \{\rho_1, \rho_2\}$. Thus the graph $D_H \cup P \cup \rho_1 \cup \rho_2$ contains a cycle that collates $\mathcal{R}$. From among all such cycles, let $C_1$ be such that $a(C_1)$ is as large as possible. It is clear from the definition of $K'$ and $C_1$ that $C_1$ avoids $K$. Suppose that an edge $e$ of $C_1$ is in a link component $H_1$ of $H$. Then, clearly, $e$ is an edge of $D$. Thus $H_1 \neq H_0$ and, by Lemma 6.4, there is an edge $f$ of $H_0$ and a face $\varphi$ of $G_{x\times z}$ such that $e$ and $f$ are in the boundary of $\varphi$; a contradiction. Hence $C_1$ avoids $H$ and thus it is in $\mathcal{C}$. But then $r(C_1) < r(C_0)$, which contradicts the choice of $C_0$ and thus proves (2). The lemma follows.
7. Proof of Theorem 1.4

In this section, we present the formal proof of Theorem 1.4. In the proof, we shall employ the concept of plane duals. For a plane graph $G$, its plane dual will be denoted by $G^*$. The following two lemmas are well-known results on graphs.

**Lemma 7.1.** Let $G$ be a plane graph and let $E$ be a subset of the edge set of $G$ such that $G\setminus E$ is isomorphic to a minor of a 3-connected graph $K$. Then $G^*/E$ is isomorphic to a minor of $K^*$.

**Lemma 7.2.** Let $G$ be a plane graph and let $E$ be a subset of the edge set of $G$ such that $G\setminus E$ has a minor isomorphic to a 3-connected graph $K$. Then $G^*/E$ has a minor isomorphic to $K^*$.

**Proof of Theorem 1.4.** Observe that the full grid is self-dual, and while the half-grid is not, it is minor-equivalent to its dual. It follows from Propositions 1.9, 1.10 and Theorems 4.1 and 5.1 that

$$G_{Z\times N} \preceq_m G_{Z\times Z}.$$  

Let $K$ be a graph such that $G_{Z\times N} \preceq_m K \preceq_m G_{Z\times Z}$. To complete the proof of Theorem 1.4, we need to show that

(1) $G_{Z\times N} \preceq_m G_{Z\times Z}$.

(2) either $K \preceq_m G_{Z\times N}$ or $G_{Z\times Z} \preceq_m K$.

Let $E$ and $F$ denote sets of edges of $G_{Z\times Z}$ such that $K$ is isomorphic to $G_{Z\times Z}\setminus E/F$. If some link-component of $E$ is infinite, then, by Lemma 6.1, $G_{Z\times Z}\setminus E \preceq_m G_{Z\times N}$, and consequently $K \preceq_m G_{Z\times N}$. Suppose now that $F$ has an infinite link-component. Then $(G_{Z\times Z}\setminus F) \preceq_m G_{Z\times N}$, and, as $G_{Z\times Z}$ is 3-connected and minor-equivalent to its plane dual, we conclude from Lemma 7.1 that $G_{Z\times Z}^*/F \preceq_m G_{Z\times N}$. Since $G_{Z\times N}^*$ is isomorphic to $G_{Z\times Z}$, we have $K \preceq_m (G_{Z\times Z}/F) \preceq_m G_{Z\times N}$.

Now we may assume that all link-components of $E$ and all link-components of $F$ are finite. Let $L = G_{Z\times Z}\setminus E$. Then, by Lemma 6.2, $G_{Z\times Z} \preceq_m L$, and clearly

(3) $(G_{Z\times Z}/F) \preceq_m (L/F) = K$.

We apply Lemma 6.2 to conclude that $G_{Z\times Z} \preceq_m (G_{Z\times Z}/F)$. Upon applying the fact that $G_{Z\times Z}$ is 3-connected and isomorphic to its plane dual, we conclude from Lemma 7.2 that $G_{Z\times Z} \preceq_m (G_{Z\times Z}/F)$. This together with (3) implies that $G_{Z\times Z} \preceq_m K$, as required. 

**Acknowledgements**

This research was partially supported by a grant from the Louisiana Education Quality Support Fund through the Board of Regents.

**References**