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Siu Hung Ng
Iowa State University

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Siu-Hung Ng [a,b]

[a] *Mathematics Department, Iowa State University, Ames, IA 50011, USA*
[b] *Mathematics Department, Towson University, Baltimore, MD 21252, USA*

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Abstract

Let H be a non-semisimple Hopf algebra with antipode S of dimension pq over an algebraically closed field of characteristic 0 where $p \leqslant q$ are odd primes. We prove that $\mathrm{Tr}(S^{2p}) = p^2 d$ where $d \equiv pq \pmod 4$. As a consequence, if p, q are twin primes, then any Hopf algebra of dimension pq is semisimple.

1. Introduction

Let p be an odd prime and k an algebraically closed field of characteristic 0. If H is a semisimple Hopf algebra of dimension p^2 over k, then H is isomorphic to $k[\mathbb{Z}_{p^2}]$ or $k[\mathbb{Z}_p \times \mathbb{Z}_p]$ by [8]. A more general result for semisimple Hopf algebras of dimension pq, where p, q are primes, was obtained by [4]. In [10], the author proved that non-semisimple Hopf algebras of dimension p^2 over k are Taft algebras and hence completed the classification of Hopf algebras of dimension p^2. However, if p, q are distinct primes, there is still no example of non-semisimple Hopf algebras of dimension pq. In fact, it was shown in [1] and [3] that there is no non-semisimple Hopf algebras over k of dimension 14, 15, 21, 35, 55, 77, 65, 91 or 143.

By [10], if $p \leqslant q$ are odd primes and H is a non-semisimple Hopf algebra with antipode S of dimension pq, then $S^{4p} = \mathrm{id}_H$ and $\mathrm{Tr}(S^{2p}) = p^2 d$ for some odd integer d. In this paper, we prove that $d \equiv pq \pmod 4$. As a consequence, we prove that if p, q are twin primes, any Hopf algebra of dimension pq over k is semisimple. Recently, Etingof

E-mail addresses: rng@math.iastate.edu, rng@towson.edu.

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and Gelaki also announce an even more general result [5] which covers the cases when $p < q \leqslant 2p + 1$.

2. Notation and preliminaries

Throughout this paper, $p \leqslant q$ are odd primes, k denotes an algebraically closed field of characteristic 0, and H denotes a finite-dimensional Hopf algebra over k with antipode S. Its comultiplication and counit are, respectively, denoted by Δ and ε. We will use Sweedler's notation [16]:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$

A non-zero element $a \in H$ is called group-like if $\Delta(a) = a \otimes a$. The set of all group-like elements $G(H)$ of H is a linearly independent set, and it forms a group under the multiplication of H. For the details of elementary aspects for finite-dimensional Hopf algebras, readers are referred to [9,16].

Let $\lambda \in H^*$ be a non-zero right integral of H^* and $\Lambda \in H$ a non-zero left integral of H. There exists $\alpha \in \mathrm{Alg}(H, k) = G(H^*)$, independent of the choice of Λ, such that $\Lambda a = \alpha(a)\Lambda$ for $a \in H$. Likewise, there is a group-like element $g \in H$, independent of the choice of λ, such that $\beta\lambda = \beta(g)\lambda$ for $\beta \in H^*$. We call g the distinguished group-like element of H and α the distinguished group-like element of H^*. Then we have a formula for S^4 in terms of α and g [11]:

$$S^4(a) = g\big(\alpha \rightharpoonup a \leftharpoonup \alpha^{-1}\big)g^{-1} \quad \text{for } a \in H, \tag{2.1}$$

where $\rightharpoonup$ and $\leftharpoonup$ denote the natural actions of the Hopf algebra H^* on H described by

$$\beta \rightharpoonup a = \sum a_{(1)}\beta(a_{(2)}) \quad \text{and} \quad a \leftharpoonup \beta = \sum \beta(a_{(1)})a_{(2)}$$

for $\beta \in H^*$ and $a \in H$. There are some useful formulae for the trace of a linear endomorphism on H in terms of λ and Λ.

Theorem 2.1 [13, Theorem 1]. *Let H be a finite-dimensional Hopf algebra with antipode S over the field k. Suppose that λ is a right integral of H^*, and that Λ is a left integral of H such that $\lambda(\Lambda) = 1$. Then for any $f \in \mathrm{End}_k(H)$,*

$$\mathrm{Tr}(f) = \sum \lambda\big(S(\Lambda_{(2)})f(\Lambda_{(1)})\big) = \sum \lambda\big((S \circ f)(\Lambda_{(2)})\Lambda_{(1)}\big) = \sum \lambda\big((f \circ S)(\Lambda_{(2)})\Lambda_{(1)}\big).$$

Following [10, Section 2], the index of H is the least positive integer n such that

$$S^{4n} = \mathrm{id}_H \quad \text{and} \quad g^n = 1.$$

Suppose that H is a finite-dimensional non-semisimple Hopf algebra of odd index n, and that $\omega \in k$ is a primitive nth root of unity. Since $g^n = 1$ and α is an algebra map, $\alpha(g)$ is a nth root of unity. There exists a unique element $x(\omega, H) \in \mathbb{Z}_n$ such that

$$\alpha(g) = \omega^{x(\omega,H)}.$$

Following the notation in [10], we let

$$H^{\omega}_{a,i,j} = \{u \in H \mid S^2(u) = (-1)^a \omega^i u \text{ and } ug = \omega^j u\}$$

for any $(a, i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n$. Since $r(g) \in \mathrm{End}_k(H)$, defined by $r(g)(a) = ag$ for $a \in H$, commutes with S^2, we have

$$H = \bigoplus_{\mathbf{a} \in \mathcal{K}_n} H^{\omega}_{\mathbf{a}}, \tag{2.2}$$

where $\mathcal{K}_n$ denotes the group $\mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n$.

Using the eigenspace decomposition of H in (2.2), the diagonalization of a left integral Λ of H admits the following form (cf. [10]),

$$\Delta(\Lambda) = \sum_{\mathbf{a} \in \mathcal{K}_n} \Big(\sum u_{\mathbf{a}} \otimes v_{-\mathbf{a}+\mathbf{x}}\Big), \tag{2.3}$$

where $\sum u_{\mathbf{a}} \otimes v_{-\mathbf{a}+\mathbf{x}} \in H^{\omega}_{\mathbf{a}} \otimes H^{\omega}_{-\mathbf{a}+\mathbf{x}}$ and $\mathbf{x} = (0, -x(\omega, H), x(\omega, H))$ in $\mathcal{K}_n$.

In the sequel, we will call the expression in Eq. (2.3) the *normal form* of $\Delta(\Lambda)$ associated with ω. We will simply write $u_{\mathbf{a}} \otimes v_{-\mathbf{a}+\mathbf{x}}$ for the sum $\sum u_{\mathbf{a}} \otimes v_{-\mathbf{a}+\mathbf{x}}$ in the normal form of $\Delta(\Lambda)$.

Let $E^{\omega}_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{K}_n$, be the set of orthogonal projections associated with the decomposition (2.2). Then

$$\dim(H^{\omega}_{\mathbf{a}}) = \mathrm{Tr}(E^{\omega}_{\mathbf{a}})$$

and we have the following lemma.

Lemma 2.2. *Let H be a finite-dimensional non-semisimple Hopf, algebra with antipode S of odd index n over k, and $\omega \in k$ a primitive nth root of unity. Then we have*

$$\dim(H^{\omega}_{\mathbf{a}}) = \dim(H^{\omega}_{\mathbf{x}-\mathbf{a}})$$

for all $\mathbf{a} \in \mathcal{K}_n$ where $\mathbf{x} = (0, -x(\omega, H), x(\omega, H))$.

Proof. Let Λ be a left integral of H and let λ be a right integral of H^* such that $\lambda(\Lambda) = 1$. Using the normal form of $\Delta(\Lambda)$ associated with ω in (2.3) and Theorem 2.1, we have

$$\mathrm{Tr}(E^{\omega}_{\mathbf{a}}) = \sum_{\mathbf{b} \in \mathcal{K}_n} \lambda\big(S(v_{-\mathbf{b}+\mathbf{x}}) E^{\omega}_{\mathbf{a}}(u_{\mathbf{b}})\big) = \lambda\big(S(v_{-\mathbf{a}+\mathbf{x}}) u_{\mathbf{a}}\big).$$

By Theorem 2.1 again, we also have

$$\mathrm{Tr}\big(E^{\omega}_{-\mathbf{a}+\mathbf{x}}\big) = \sum_{\mathbf{b}\in\mathcal{K}_n} \lambda\big(S\big(E^{\omega}_{-\mathbf{a}+\mathbf{x}}(v_{-\mathbf{b}+\mathbf{x}})\big)u_{\mathbf{b}}\big) = \lambda\big(S(v_{-\mathbf{a}+\mathbf{x}})u_{\mathbf{a}}\big).$$

Therefore, $\mathrm{Tr}(E^{\omega}_{\mathbf{a}}) = \mathrm{Tr}(E^{\omega}_{\mathbf{x}-\mathbf{a}})$. Since $\dim(H^{\omega}_{\mathbf{a}}) = \mathrm{Tr}(E^{\omega}_{\mathbf{a}})$ for any $\mathbf{a} \in \mathcal{K}_n$, the result follows. □

Theorem 2.3 [10]. *Let H be a non-semisimple Hopf algebra of dimension pq over k with antipode S, where $p \leqslant q$ are odd primes. Then the index of H and the order of S^4 are equal to p, and $\mathrm{Tr}(S^{2p}) = p^2 d$ for some odd integer d.*

Lemma 2.4. *Suppose that H is a non-semisimple Hopf algebra of dimension pq over k where $p \leqslant q$ are odd primes, and that $\omega \in k$ is a primitive pth root of unity. Let g and α be the distinguished group-like elements of H and H^*, respectively. If g is non-trivial, then the integer d in Theorem 2.3 is given by*

$$\dim\big(H^{\omega}_{0,i,j}\big) - \dim\big(H^{\omega}_{1,i,j}\big) = d$$

for all $i, j \in \mathbb{Z}_p$. Moreover, if both g and α are not trivial, then

$$\dim\big(H^{\omega}_{a,i,j'}\big) = \dim\big(H^{\omega}_{a,i,j}\big)$$

for any $a \in \mathbb{Z}_2$ and $i, j, j' \in \mathbb{Z}_p$.

Proof. If α is trivial and $g \neq 1$, then by [10, Lemma 4.3],

$$\dim\big(H^{\omega}_{0,i,j}\big) - \dim\big(H^{\omega}_{1,i,j}\big) = d.$$

If both g and α are non-trivial, then by the proof of [10, Proposition 5.3], H is isomorphic to the biproduct

$$R \times B \tag{2.4}$$

as Hopf algebras (cf. [12]) where $B = k[g]$ and R is a left B-comodule subalgebra of H. It was shown in [2, Section 4] that R is invariant under S^2. Moreover, in the identification $H \cong R \otimes B$ given by multiplication, one has

$$S^2 = T \otimes \mathrm{id}_B, \tag{2.5}$$

where T is the restriction of S^2 on R. Let

$$R_{a,i} = \big\{x \in R \mid S^2(x) = (-1)^a \omega^i x\big\}$$

for any $(a, i) \in \mathbb{Z}_2 \times \mathbb{Z}_p$. It follows from the proof of [10, Proposition 5.3] that

$$\dim(R_{0,i}) - \dim(R_{1,i}) = d.$$

By (2.4),

$$H^{\omega}_{a,i,j} = R_{a,i} \otimes e_j$$

for all $(a,i,j) \in \mathcal{K}_p$ where e_j is the central idempotent of B such that $e_j g = \omega^j e_j$. Thus,

$$\dim\left(H^{\omega}_{a,i,j}\right) = R_{a,i}$$

for all $(a,i,j) \in \mathcal{K}_p$ and hence

$$\dim\left(H^{\omega}_{0,i,j}\right) - \dim\left(H^{\omega}_{1,i,j}\right) = d. \qquad \square$$

3. Proofs of main results

Lemma 3.1. *Let H be a finite-dimensional non-semisimple Hopf algebra with antipode S of odd index n over k, and $\omega \in k$ a primitive nth root of unity. Let $\ell \in \mathbb{Z}_n$ such that $2\ell = x(\omega, H)$. Then*

$$\dim\left(H^{\omega}_{1,-\ell,\ell}\right)$$

is even.

Proof. Let V be the space of all $f \in H^*$ such that $f(H^{\omega}_{a,i,j}) = \{0\}$ whenever $(a,i,j) \neq (1,-\ell,\ell)$. Obviously, V is isomorphic to $(H^{\omega}_{1,-\ell,\ell})^*$ and so $\dim(V) = \dim(H^{\omega}_{1,-\ell,\ell})$. Let Λ be a non-zero left integral of H and

$$\Delta(\Lambda) = \sum_{\mathbf{a}\in\mathcal{K}_n} u_{\mathbf{a}} \otimes v_{-\mathbf{a}+\mathbf{x}}$$

the normal form of $\Delta(\Lambda)$ associated with ω where $\mathbf{x} = (0, -2\ell, 2\ell)$. Then

$$(f,h) = (f \otimes h)\Delta(\Lambda)$$

defines a non-degenerate bilinear form on H^*. Let $f \in V$ such that $(f,h) = 0$ for all $h \in V$. For any $h' \in H^*$, there exists $h \in V$ such that $h'(u) = h(u)$ for all $u \in H^{\omega}_{1,-\ell,\ell}$. Thus

$$\left(f,h'\right) = \sum_{\mathbf{a}\in\mathcal{K}_n} f(u_{\mathbf{a}})h'(v_{-\mathbf{a}+\mathbf{x}}) = f(u_{1,-\ell,\ell})h'(v_{1,-\ell,\ell}) = (f,h) = 0.$$

By the non-degeneracy of $(\cdot,\cdot)$, $f = 0$. Therefore, $(\cdot,\cdot)$ induces a non-degenerate bilinear form on V. Using [14, Theorem 3(d)], we have

$$\Delta^{\mathrm{op}}(\Lambda) = \sum_{(a,i,j)\in\mathcal{K}_n} (-1)^a \omega^{-i-j} \left(\sum u_{a,i,j} \otimes v_{a,-2\ell-i,2\ell-j}\right).$$

Therefore, for any $f, h \in V$,

$$(h, f) = (f \otimes h)\Delta^{\mathrm{op}}(\Lambda) = -f(u_{1,-\ell,\ell})h(v_{1,-\ell,\ell}) = -(f, h).$$

Hence, V admits a non-degenerate alternating form and so $\dim(V)$ is even. □

If H is a finite-dimensional Hopf algebra of index n, we define

$$H_- := \{u \in H \mid S^{2n}(u) = -u\},$$
$$H_+ := \{u \in H \mid S^{2n}(u) = u\}.$$

Corollary 3.2. *Suppose H is a finite-dimensional non-semisimple Hopf algebra with antipode S of odd index n over k. Then, the subspace H_- is of even dimension.*

Proof. Let $\omega \in k$ be a nth root of unity and $\ell \in \mathbb{Z}_n$ such that $2\ell = x(\omega, H)$. We then have

$$H_- = \bigoplus_{i,j \in \mathbb{Z}_n} H^{\omega}_{1,i,j} = H^{\omega}_{1,-\ell,\ell} \oplus \left(\bigoplus_{\substack{\text{some } i,j \in \mathbb{Z}_n \\ (i,j) \neq (-\ell,\ell)}} H^{\omega}_{1,i,j} \oplus H^{\omega}_{1,-2\ell-i,2\ell-j} \right).$$

It follows from Lemmas 2.2 and 3.1 that $\dim(H_-)$ is even. □

Theorem 3.3. *Let H be a non-semisimple Hopf algebra with antipode S of dimension pq where $p \leqslant q$ are odd primes. Then*

$$\mathrm{Tr}(S^{2p}) = p^2 d \quad \textit{and} \quad d \equiv pq \pmod 4.$$

Proof. By Theorem 2.3, H is of index p and $\mathrm{Tr}(S^{2p}) = p^2 d$ for some odd integer d. Since

$$\dim(H_+) + \dim(H_-) = pq$$

and

$$\mathrm{Tr}(S^{2p}) = \dim(H_+) - \dim(H_-) = p^2 d,$$

we have

$$\dim(H_-) = p(q - pd)/2.$$

By Corollary 3.2, $p(q - pd) \equiv 0 \pmod 4$ or $d \equiv pq \pmod 4$. □

Theorem 3.4. *For any pair of twin primes $p < q$, if H is a Hopf algebra of dimension pq, then H is semisimple.*

Proof. Suppose there is a non-semisimple Hopf algebra H of dimension pq. By [6], H^* is also non-semisimple. Since $\dim(H)$ is odd, by [7, Theorem 2.1], H and H^* can not be both unimodular. By duality, we may simply assume that H^* is not unimodular. It follows from Theorem 2.3 that $|G(H)| = p$ and so

$$\dim(C) \geqslant p,$$

where C is the coradical of H. If $\dim(C) = p$, then H is pointed and hence, by [15, Corollary 4], H is semisimple. Therefore, $\dim(C) > p$ and so we have

$$\operatorname{Tr}\bigl(S^{2p}|_{H/C}\bigr) \geqslant -\bigl(pq - \dim(C)\bigr) > -pq + p = -p^2 - p.$$

It follows from [6, Lemma 3.2] that

$$\operatorname{Tr}\bigl(S^{2p}|_{C}\bigr) \geqslant p.$$

Thus, we have

$$\operatorname{Tr}\bigl(S^{2p}\bigr) = \operatorname{Tr}\bigl(S^{2p}|_{C}\bigr) + \operatorname{Tr}\bigl(S^{2p}|_{H/C}\bigr) > -p^2. \tag{3.1}$$

Since $pq \equiv -1 \pmod 4$, by Theorem 3.3,

$$\operatorname{Tr}\bigl(S^{2p}\bigr) = -p^2$$

but this contradicts (3.1). □

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