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The Strict Topology and Compactness in the Space of Measures.

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THE STRICT TOPOLOGY AND COMPACTNESS IN THE SPACE OF MEASURES

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TABLE OF CONTENTS

ABSTRACT

The strict topology β on C(S), the bounded continuous complex valued functions on the locally compact Hausdorff space S, was first introduced by R. C. Buck. The primary concern of this dissertation is the relationship between $C(S)_\beta$ and its adjoint M(S), the bounded Radon measures. In particular, when is $C(S)_\beta$ a Mackey space? From our answer to this question we are able to prove several theorems on various types of compactness and convergence in M(S).

The first chapter contains preliminary material with virtually no proofs. Chapter II contains the basic properties of the strict topology. Some of these results are known and some of the old theorems are presented with new proofs. In particular, we prove Buck's result that $C(S)_\beta{}^* = M(S)$ and we calculate a basis for the β-equicontinuous sets in M(S).

The third and fourth chapters contain the heart of this work. Chapter III begins with necessary and sufficient conditions for β-equicontinuity in M(S) and a proof that (ℓ^∞, β) is a strong Mackey space. Using these two results we prove the principal theorem of this chapter. This result is that if S is paracompact then every β-weak * countably compact subset of M(S) is β-equicontinuous;

consequently, $C(S)_\beta$ is a strong Mackey space. Also we show that if S is the space of ordinal numbers less than the first uncountable then $C(S)_\beta$ is not a Mackey space. Chapter III concludes with a characterization of the closed subspaces of (ℓ^∞, β) which are Mackey spaces, and a proof that (H^∞, β) is not a Mackey space.

After a few preliminary lemmas the fourth chapter begins with some results concerning β -equicontinuity. For example, every β -weak * compact subset of positive measures is β -equicontinuous. If S is metrizable then β-weak * sequential, countable, and conditional compactness are all equivalent to β -equicontinuity. Then we show how the concept of β -equicontinuity and the main theorem of Chapter III can be combined to generalize, improve, and give new proofs of some theorems of J. Dieudonné on various types of sequential convergence in M(S). Finally we use all these facts to prove a weak compactness theorem of A. Grothendieck.

Chapter V contains generalizations of the preceding results to vector valued measures and functions. Also we characterize the weakly compact operators from a Banach space E into M(S). Using this, we show that a weakly compact operator from a subspace of E into M(S) can be extended to a weakly compact operator of E into M(S).

INTRODUCTION

The strict topology β on C(S), the bounded continuous complex valued functions on the locally compact space S, was first introduced by R. C. Buck [10,11,12]. It has also been studied by Glicksberg [18] and Wells [36]. This topology has been used in the study of various problems in spectral synthesis (Herz [22]), spaces of bounded analytic functions (Shields and Rubel [31,32]), and multipliers of Banach algebras (Wells [35] and Wang [34]). In spite of these successful applications, there has as yet been no detailed investigation of the relationship between $C(S)_\beta$ and its adjoint space M(S). In particular, it is not known whether or not $C(S)_\beta$ is a Mackey space (a question asked by Buck [12]). It is one of the purposes of this dissertation to begin such an investigation.

The existence and description of the Mackey topology, the strongest topology yielding a given adjoint space, is a natural object for consideration. This topology can be described for general locally convex spaces, and there are several conditions (e.g. metric) which imply that a topology is a Mackey topology. Nevertheless, for a particular locally convex space E which is not <u>a priori</u> a Mackey space, the question of whether or not E is a Mackey space may be extremely difficult. Moreover, the general

description of the Mackey topology may be totally unsuitable for a concrete space. Indeed, the author knows of no examples in which a space with an intrinsically defined topology is shown to be a Mackey space (unless it has some other formally stronger property like metric; etc.). However, if S is a paracompact non-compact space then $C(S)_\beta$ is a Mackey space which is not metric, barrelled, or bornological (Theorem 3.7).

Also we give necessary and sufficient conditions for a closed subspace of (ℓ^∞, β) to be a Mackey space, and we show that (H^∞, β) is not a Mackey space. In the process we show that (ℓ^∞, β) has closed subspaces which are not Mackey spaces.

The second purpose of this paper is to present the proof of several compactness and sequential convergence criteria for M(S). There is a wealth of literature on this matter. In fact, in addition to some results of our own on these subjects, we succeed in applying our Theorem 3.7 to an investigation of the results of J. Dieudonné [13]. Our work here consists of generalizations to locally compact spaces, improvements, and the elimination of many of Dieudonné's arguments through the use of Theorem 3.7.

Every effort has been made to make the reader's job painless. In addition to the inclusion of detail, which to some may seem tedious, we have also added an index of symbols at the end of Chapter I. All theorems, corollaries,

and lemmas in a given chapter have been numbered consecutively. Also Theorem x.y means theorem number y in chapter x.

CHAPTER I

PRELIMINARIES

In this chapter we have endeavored to present a variety of results and facts which we hope will facilitate the reading of this dissertation. Some of the terms we will use can be found in the literature with a meaning different from ours. For this reason we advise the reader to being with at least a cursory reading of this section.

Topology and continuous functions.

Unless otherwise stated the topological notions used will be that of Kelley [24]. However, there are some notable exceptions. To avoid cumbersome phraseology we shall adapt the following terminology. If X is a topological space and A a subset of X then A is countably compact if and only if every sequence in A has a cluster point in X (not necessarily in A). Also A is sequentially compact if and only if every sequence in A has a subsequence which converges to some point of X. Finally A is conditionally compact if and only if A^- (the closure of A) is compact.

Throughout this work S will always denote a locally compact Hausdorff space, and int A the interior of the set A.

THEOREM 1.1. ([6,p.107]) The space S is paracompact

if and only if S is the union of a pairwise disjoint collection of open and closed σ-compact subsets of S.

In particular, the above theorem says that if S is σ-compact or a topological group then S is paracompact.

Let Ω_o be the space of ordinal numbers less than the first uncountable ordinal Ω, with the order topology.

THEOREM 1.2. (Tong [33]) The space Ω_o is not σ-compact and the closure of any σ-compact set in Ω_o is compact. Every continuous function f on Ω_o is eventually constant (i.e., there is an $x \in \Omega_o$ such that for $y \geq x$ $f(y) = f(x)$). The Stone-Cech compactification of Ω_o is the same as the one point compactification.

We shall let C(S) be the space of bounded continuous complex valued functions on S, $C_o(S)$ those which vanish at infinity (i.e., $\phi \in C_o(S)$ if and only if for every $\epsilon > 0$ $\{s \in S : |\phi(s)| \geq \epsilon\}$ is compact), and $C_c(S)$ the functions in C(S) which vanish off some compact set (a possibly different set for each function). If $\phi \in C(S)$ then $N(\phi) = \{s : \phi(s) \neq 0\}$ and spt $(\phi) = \overline{N(\phi)}$.

The uniform topology on C(S) is the metric topology defined by the supremum norm $\|f\|_\infty$ = the least upper bound of $\{|f(s)| : s \in S\}$. It is easy to see that both C(S) and $C_o(S)$ are complete with respect to this norm. Also $C_c(S)$ is uniformly dense in $C_o(S)$; i.e., if $f \in C_o(S)$ and $\epsilon > 0$ there is a $\phi \in C_c(S)$ such that $\|f-\phi\|_\infty \leq \epsilon$.

A set $A \subset S$ is said to be regularly σ-compact if and only if $A = \bigcup_{n=1}^{\infty} K_n$ where K_n is compact and $K_n \subset$

int K_{n+1}. It is easy to see that every σ-compact set is contained in a regularly σ-compact set. Also, every regularly σ-compact set is open; conversely each open σ-compact set is a regularly σ-compact set.

In all that follows wide use will be made of Urysohn's lemma. Moreover, we will need the following results on the existence of special types of continuous functions.

THEOREM 1.3. (Buck [12]) If $\phi \in C_o(S)$ then $N(\phi)$ is a regularly σ-compact set. Conversely, if A is a regularly σ-compact set there is a function $\phi \in C_o(S)$ with $N(\phi) = A$.

THEOREM 1.4. (Buck [12]) If F is a continuous function on S such that $F\phi \in C_o(S)$ for all $\phi \in C_o(S)$ then F is bounded ($F\phi$ denotes the pointwise product of F and ϕ).

We shall say that a sequence in S is discrete if and only if it is a discrete space when furnished with the relative topology. Hence $\{s_k\}_{k=1}^{\infty}$ is discrete if and only if for every $n \geq 1$ there is an open set U_n such that $\{s_n\} = U_n \cap \{s_k\}_{k=1}^{\infty}$. If a sequence has no limit points then it is easily seen to be discrete. The converse, however, is not true as may be seen by letting $S = [0,1]$ and $s_n = 1/n$.

THEOREM 1.5. (Buck [12]) If $\{s_n\}_{n=1}^{\infty}$ is a discrete sequence in S and $\{c_n\}_{n=1}^{\infty}$ is a sequence of complex numbers converging to zero, then there is a function $\phi \in C_o(S)$ such that $\phi(s_n) = c_n$ for $n \geq 1$.

THEOREM 1.6. (Bourbaki [8,p.49]) If K is a compact subset of S and $\{U_i\}_{i=1}^{n}$ is an open cover of K then there

exist functions $\phi_1, \cdots, \phi_n$ in $C_c(S)$ such that : (a) $0 \leq \phi_i \leq 1$ for $1 \leq i \leq n$; (b) $\sum_{i=1}^{n} \phi_i(s) \leq 1$ for $s \in S$ and $\sum_{i=1}^{n} \phi_i(s) = 1$ for $s \in K$; and (c) $\phi_i(s) = 0$ if $s \notin U_i$, $1 \leq i \leq n$.

Recall that a real valued function f on S is <u>lower semi-continuous</u> (l.s.c.) at $s_o \in S$ if and only if for every real number $a < f(s_o)$ there is an open neighborhood U of s_o such that $f(U) > a$; f is l.s.c. if and only if f is l.s.c. at each point of S. Hence f is l.s.c. if and only if for every real number a $\{s : f(s) > a\}$ is open; or, equivalently, $\{s : f(s) \leq a\}$ is closed. We shall say that a complex valued function is l.s.c. at $s_o \in S$ if and only if both its real and imaginary parts are. For general information on l.s.c. functions we refer the reader to Bourbaki [7,pp.109-116]. However, we shall present some of the notions essential for our development.

Let f be a real valued function on S, $s \in S$, and $\{U_s\}$ the neighborhood system of s. Then $\liminf_{t \to s} f(t) = \sup_{U_s} \inf\{f(t) : t \in U_s\}$, [7,p.100].

<u>THEOREM 1.7</u>. ([7,p.114]) If f is a real valued function on S then f is l.s.c. at $s \in S$ if and only if $f(s) = \liminf_{t \to s} f(t)$.

<u>THEOREM 1.8</u>. ([7,p.114]) If f is a real valued function on S and $g(s) = \liminf_{t \to s} f(t)$ for all $s \in S$, then

g is lower semi-continuous.

In a similar manner we define upper semi-continuity (u.s.c.). A real valued function f on S is u.s.c. if and only if for every real number a, $\{s : f(s) \geq a\}$ is closed; or, equivalently, $\{s : f(s) < a\}$ is open.

Let f be a function (complex valued) on S. Then the oscillation of f at s is defined by $\operatorname{osc}(f,s) = \inf_{U_s} \sup\{|f(t) - f(s)| : t \in U_s\}$, where $\{U_s\}$ is the neighborhood system of s. It is not hard to show that $g(s) = \operatorname{osc}(f,s)$ is an u.s.c. function. Also f is continuous at s if and only if $\operatorname{osc}(f,s) = 0$.

Measure theory.

In general, a knowledge of Halmos [21] or Bourbaki [8] is assumed. However, we shall present some of the essentials here for easy reference.

The Borel sets of S, Borel (S), are usually defined as the elements of the σ-ring generated by the compact sets of S. However, since we will always restrict our attention to bounded Borel measures, we will define Borel (S) to be the σ-algebra generated by the closed sets. If μ is a Borel measure then we define the variation $|\mu|$ of μ by $|\mu|(A) = \sup \sum_{i=1}^{n} |\mu(A_i)|$, where the supremum is taken over all finite Borel partitions $\{A_i\}_{i=1}^{n}$ of A; the total variation $\|\mu\|$ of μ equals $|\mu|(S)$. A measure μ is regular if and only if for every $A \in$ Borel (S) and $\epsilon > 0$ there is a compact set $K \subset A$ and an

open set $U \supset A$ such that $|\mu|(U \setminus K) \leq \epsilon$. When we say measure we will mean a complex valued, countably additive, regular Borel measure μ (hence μ is of bounded variation). If μ is a measure then $|\mu|$ is a measure and $|\mu|$ is positive (i.e., $|\mu|(A) \geq 0$ for all $A \in$ Borel (S)). The totality of all measures is denoted by $M(S)$, and $M(S)$ with total variation norm is a Banach space.

By combining the regularity of a measure with a well known result [14,p.97] we have

THEOREM 1.9. If μ is a measure and $A \in$ Borel (S) then $\sup\{|\mu(K)| : K \text{ is a compact subset of } A\} \leq |\mu|(A) \leq 4 \sup\{|\mu(K)| : K \text{ is a compact subset of } A\}$.

Let S_1 be a subset of S such that with the relative topology S_1 is locally compact (hence S_1 equals the intersection of an open and a closed subset of S). If $\mu \in M(S)$ then by the restriction of μ to S_1 we will mean the measure $\mu_{S_1} \in M(S_1)$ defined by $\mu_{S_1}(A) = \mu(A)$ for all $A \in$ Borel (S_1). If $\mu \in M(S_1)$ then by the extension of μ to S we will mean the measure $\nu \in M(S)$ defined by $\nu(B) = \mu(B \cap S_1)$ for all $B \in$ Borel (S). Usually we will make no distinction between μ and its extension and merely consider μ as an element of $M(S)$. If $A \in$ Borel (S) and $\mu \in M(S)$ then to say that μ vanishes off A means that $\mu(B) = 0$ for all $B \in$ Borel (S) such that $B \cap A = \square$; equivalently, $|\mu|(S \setminus A) = 0$.

Finally if f is a Borel function on S such that f is integrable with respect to μ (i.e., $\int |f| \, d|\mu| < \infty$), then

$f\mu$ will denote the measure $\nu \in M(S)$ such that $\nu(A) = \int_A f d\mu$ for all A $\in$ Borel (S).

Functional Analysis.

Our terminology will be that of Kelley, Namioka, et al [25] and Dunford and Schwartz [14]. All topological vector spaces E will be locally convex Hausdorff spaces; i.e., their topology is defined by a family of semi-norms $\{p\}$ such that $x \in E$ with $p(x) = 0$ for all p implies $x = 0$ (a semi-norm is a function p from E into the non-negative reals such that $p(0) = 0$, $p(ax) = a\,p(x)$ for $a \in \mathbb{C}$ and $x \in E$, and $p(x+y) \leq p(x) + p(y)$ for all $x,y \in E$). All sets of the form $\{x : p(x) \leq \epsilon\}$ where $\epsilon > 0$ form a subbasis for the neighborhood system at the origin. We will denote by E* the space of all continuous linear functionals on E; E* is called the adjoint space of E. If $x \in E$ and $x^* \in E^*$ then $\langle x,x^* \rangle$ is the value of x^* at x.

The weak * topology on E*, denoted by $\sigma(E^*,E)$, is the weakest topology on E* such that the function $x^* \longmapsto \langle x,x^* \rangle$ is continuous for every $x \in E$. The topology $\sigma(E^*,E)$ is a locally convex Hausdorff topology on E* and the defining semi-norms are the functions $x^* \longmapsto |\langle x,x^* \rangle|$. The adjoint space of E* with the weak * topology is (algebraically) E. Finally, if $\{x_i^*\}$ is a net in E* then $x_i^* \longrightarrow x^* \in E^*$ (weak *) if and only if $\langle x,x_i^* \rangle \longrightarrow \langle x,x^* \rangle$ for every $x \in E$.

Similarly, we define the weak topology $\sigma(E,E^*)$ on E.

A net $\{x_i\}$ in E converges weakly to $x \in E$ if and only if $\langle x_i, x^* \rangle \longrightarrow \langle x, x^* \rangle$ for all $x^* \in E^*$. Also the adjoint of E with the weak topology is E^*.

We shall assume a familiarity with the Hahn-Banach theorem, the open mapping theorem, and the principle of uniform boundedness as presented in the two above cited references. Also we assume a knowledge of ''polar sets.'' If $A \subset E$ then A°, the polar of A, is $\{x^* \in E^* : |\langle x, x^* \rangle| \leq 1 \text{ for all } x \in A\}$. If $B \subset E^*$ then $B^\circ = \{x \in E : |\langle x, x^* \rangle| \leq 1 \text{ for all } x^* \in B\}$.

If E is a Banach space then $x^* \in E^*$ if and only if x^* is linear and $\|x^*\| = \sup \{|\langle x, x^* \rangle| : \|x\| \leq 1\} < \infty$. Also, this defines a norm on E^* which makes E^* into a Banach space. If E is a topological vector space then we may define the strong topology on E^* which is analogous to this norm topology (in fact if E is a Banach space then the strong topology on E^* and the norm topology are the same). The strong topology on E^* is the topology of uniform convergence on bounded subsets of E. Hence a net $\{x_i^*\}$ in E^* converges strongly to 0 if and only if for every bounded set $B \subset E$, $\sup\{|\langle x, x_i^* \rangle| : x \in B\} \longrightarrow 0$. Even though E^* with the strong topology is not necessarily complete, we do have the following theorem due to Grothendieck.

THEOREM 1.10. ([25,p.145]) The space E^* is strongly complete if and only if every linear functional on E whose restriction to every bounded, weakly closed, convex and

circled subset of E is continuous is in E*.

If F is a closed subspace of E and $F^{\perp} = \{x^* \in E^*: \langle x,x^* \rangle = 0 \text{ for all } x \in F\}$ then $F^* = E^*/F^{\perp}$, the quotient space E* mod $F^{\perp}$. That is, every continuous linear functional on F is obtained by restricting an element of E* to F. Not only does $F^* = E^*/F^{\perp}$ algebraically, but also topologically if F* has its weak * topology and $E^*/F^{\perp}$ the quotient topology derived from the weak * topology on E*. This topological correspondence does not hold in general for the strong topology (though it does for Banach spaces).

The second adjoint E** of E is the adjoint of E* with the strong topology. There is a canonical embedding of E into E**, where for each $x \in E$ we define $x^{**} \in E^{**}$ by $\langle x^*,x^{**} \rangle = \langle x,x^* \rangle$ for all $x^* \in E^*$. This embedding is in general neither onto nor continuous when E has its initial topology and E** its strong topology, although it is open onto its image. If it is onto then E is called semi-reflexive; if it is also continuous E is reflexive. A space E is semi-reflexive if and only if every bounded weakly closed subset of E is weakly compact [25,p.190].

THEOREM 1.11. ([25,p.190]) If E is semi-reflexive and F is a closed subspace then F is semi-reflexive; and F*, with the strong topology, is topologically isomorphic to $E^*/F^{\perp}$ with the quotient topology derived from the strong topology on E*.

A set $H \subset E^*$ is equicontinuous if and only if for every $\epsilon > 0$ there is a neighborhood W of 0 in E such that if $x \in W$, $|\langle x,x^* \rangle| \leq \epsilon$ for all $x^* \in H$. Hence H is equicontinuous if and only if $H^\circ \subset E$ is a neighborhood of zero. Every equicontinuous set has weak * compact closure, but the converse is false in general.

We define another topology on E called the Mackey topology; this is the topology of uniform convergence on weak * compact convex circled subsets of E^*. Since the initial topology on E is the topology of uniform convergence on equicontinuous subsets of E^*, the Mackey topology is stronger. In fact, the Mackey topology is the strongest topology on E which has E^* as its adjoint [25,p.173]. We will say that E is a Mackey space if and only if its topology is the Mackey topology; hence, if and only if every weak * compact convex circled subset of E^* is equicontinuous. Also, we will call E a strong Mackey space if and only if every weak * compact (not necessarily convex and circled) subset of E^* is equicontinuous.

THEOREM 1.12. (Riesz Representation Theorem [21]) L is a bounded linear functional on $C_o(S)$ if and only if there exists a unique measure $\mu \in M(S)$ such that $L(\phi) = \int \phi d\mu$ for all $\phi \in C_o(S)$. Also, if L and μ correspond in this way then $\|L\| = \|\mu\| = \sup\{ |\int \phi d\mu| : \phi \in C_c(S), \|\phi\|_\infty \leq 1\}$.

COROLLARY 1.13. If U is open in S and $\mu \in M(S)$ then $|\mu|(U) = \sup\{ |\int \phi d\mu| : \phi \in C_c(S), \|\phi\|_\infty = 1, \text{ and } spt(\phi)$

$\subset U\}$.

We will use the following theorems on the weak * and weak topologies for Banach spaces.

THEOREM 1.14. ([14,p.429]) If E is a Banach space then a subspace of E* is weak * closed if and only if its intersection with the unit ball is weak * closed.

THEOREM 1.15. ([14,p.430]) Let A be a subset of the Banach space E. Then the following are equivalent:

(a) A is weakly sequentially compact;

(b) A is weakly countably compact;

(c) A is weakly conditionally compact.

Note: Theorem 1.15 is called the Eberlein-Smulian theorem.

THEOREM 1.16. ([14,p.434]) The closed convex hull of a weakly compact subset of a Banach space is weakly compact.

We will conclude this chapter with a theorem on extreme points. If K is a convex subset of E then $x \in K$ is an extreme point of K if and only if $a,b \in K$ such that $x = \frac{1}{2}(a+b)$ implies $x = a = b$; or, equivalently, if and only if $x = \sum_{i=1}^{n} a_i x_i$, where $x_i \in K$, $a_i > 0$ for $1 \leq i \leq n$, and $\sum_{i=1}^{n} a_i = 1$, implies $x = x_1 = \cdots = x_n$. If K_1, K_2 are convex subsets of E_1, E_2 then a function $\mathcal{U}: K_1 \longrightarrow K_2$ is called affine if and only if $\mathcal{U}(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i \mathcal{U}(x_i)$ whenever $x_i \in K$, $a_i \geq 0$, and $\sum_{i=1}^{n} a_i = 1$.

THEOREM 1.17. Let E_1, E_2 be locally convex Hausdorff topological vector spaces, K_i a compact convex subset of E_i for $i = 1,2$, and $\mathcal{U} : K_1 \longrightarrow K_2$ a continuous affine map of K_1 onto K_2. If x_2 is an extreme point of K_2 then there is an extreme point x_1 of K_1 such that $\mathcal{U}(x_1) = x_2$.

COROLLARY 1.18. If $x \in E$ and $B \subset E^*$ is a weak * compact convex and circled set, then there is an extreme point x^* of B such that $\langle x,x^* \rangle = \sup\{|\langle x,y^* \rangle| : y^* \in B\}$.

Proof. Consider $A = \{\langle x,y^* \rangle : y^* \in B\}$. Then x is weak * continuous and affine from B onto A. But A is a compact convex and circled subset of the plane, and hence must be a closed disk about zero. If r = the radius of A then $r \in A$. Furthermore, r is an extreme point of A. The conclusion is now apparent from the theorem.

COROLLARY 1.19. If E is a Banach space and $x \in E$ then there is an extreme point x^* of ball E^* such that $\|x\| = \langle x,x^* \rangle$.

Proof. If B = ball E^* then this is an immediate consequence of the preceding corollary.

INDEX OF SYMBOLS

1. $x_i \longrightarrow x$: the net $\{x_i\}$ converges to x.
2. $x_i \xrightarrow{cl} x$: the net $\{x_i\}$ clusters to x.
3. c.s.n. : continuous semi-norm.
4. A^- = closure of A.
5. int A = interior of A.
6. C(S) = bounded continuous complex valued functions.
7. $C_o(S)$ = elements of C(S) which vanish at ∞.
8. $C_c(S)$ = elements of C(S) which vanish off some compact set.
9. $N(\phi) = \{s : \phi(s) \neq 0\}$.
10. $spt(\phi) = N(\phi)^-$
11. $osc(f,s) = \inf_{U_s} \sup\{|f(t)-f(s)| : t \in U_s\}$
12. μ_{S_1} = element of $M(S_1)$ such that $\mu(A) = \mu_{S_1}(A)$ for all $A \in$ Borel (S_1).
13. $f\mu$ = measure $\nu \in M(S)$ such that $\nu(A) = \int_A f d\mu$.
14. $\langle x,x^* \rangle$ = value of x^* at x.
15. $\sigma(E^*,E)$ = weak * topology on E^*.
16. $\sigma(E,E^*)$ = weak topology on E.
17. A° = polar of A.
18. $F^\perp = \{x^* \in E^* : \langle x,x^* \rangle = 0$ for all $x \in F\}$.
19. $\mathbb{C}$ = complex numbers.

CHAPTER II

GENERAL PROPERTIES OF THE STRICT TOPOLOGY

In this chapter we have brought together the basic topological properties of the strict topology. Many of the results given here are not new and can be found in the literature referred to in the bibliography. We present these known results (some of which here have new proofs) not only for the sake of completeness, but also to give evidence that the strict topology has many desirable properties and, in many ways, is more manageable than the norm topology when S is not compact.

The strict topology β on C(S) is that locally convex Hausdorff topology defined by semi-norms $\{\|\cdot\|_{\phi}:\phi\in C_o(S)\}$, where $\|f\|_{\phi} = \|\phi f\|_{\infty}$ for all f in C(S). Notice that if S is compact then the strict and norm topologies are one and the same. Also, it is easy to see that we need only consider those semi-norms defined by the functions $\phi \in C_o(S)$ such that $\phi \geq 0$. Hence, the collection of sets $V_{\phi} \equiv \{f \in C(S): \|f\phi\|_{\infty} \leq 1\}$ form a subbase for the neighborhood system at zero. But more than this is true: the sets V_{ϕ} actually form a neighborhood <u>basis</u> at the origin. In fact, if $\phi_1,\cdots,\phi_n \in C_o(S)$ are all non-negative and $\phi(s) = \max\{\phi_i(s):1 \leq i \leq n\}$, then $\phi \in C_o(S)$ and $\|\phi f\|_{\infty} \leq 1$ implies $\|\phi_i f\|_{\infty} \leq 1$ for $1 \leq i \leq n$. That is, $V_{\phi} \subset \bigcap_{i=1}^{n} V_{\phi_i}$.

If $\{f_i\}$ is a net in C(S) then $\{f_i\}$ converges strictly to 0 if and only if $\{\phi f_i\}$ converges to zero uniformly for all $\phi \in C_o(S)$.

We will also have need of the compact open topology (denoted by c-op) on C(S). The c-op topology is defined by the family of semi-norms $\{\|\cdot\|_\phi : \phi \in C_c(S)\}$. Clearly the β topology is stronger than the c-op topology since it has a larger class of defining semi-norms. As in the case of the β -topology, it quickly follows that a basis for the c-op neighborhoods at the origin is given by the sets V_ϕ where $\phi \in C_c(S)$ and $\phi \geq 0$. By a routine application of Urysohn's lemma we see that all sets of the form $\{f \in C(S) : |f(s)| \leq \epsilon \text{ for all } s \in K\}$, where $K \subset S$ is compact and $\epsilon > 0$, also form a c-op neighborhood basis at the origin. Hence, a net $\{f_i\}$ in C(S) converges to zero in the c-op topology if and only if it converges to zero uniformly on all compact sets in S. It might be asked if a similar interpretation of the strict topology may be given; i.e., is the strict topology a topology of uniform convergence on a certain class of subsets of S? The following theorem gives an answer which is as close as possible to an affirmative one. Here, as in what follows, $C(S)_\beta$ denotes C(S) with the β -topology.

THEOREM 2.1. (Herz[22]) An equivalent system of neighborhoods for the origin in $C(S)_\beta$ is given as follows: for each sequence $\{K_n, \epsilon_n\}_{n=1}^{\infty}$, where $K_n \subset S$ is compact

with $K_n \subset \text{int } K_{n+1}$ and the ϵ_n form an increasing sequence of positive numbers approaching infinity, we define the neighborhood $U = \{f : \sup_{s \in K_n} |f(s)| \leq \epsilon_n \text{ for } n = 1,2,\ldots\}$.

Proof. Let $\mathfrak{B}$ be the topology defined by the neighborhood basis $\{U\}$ given above. Let $\phi \in C_o(S)$ be such that $\phi \geq 0$. By considering $\phi / \|\phi\|_\infty$ we may suppose $0 \leq \phi \leq 1$. If we put $K_n = \{s : \phi(s) \geq \frac{1}{n+1}\}$ for $n \geq 1$, then K_n is compact and $K_n \subset \text{int } K_{n+1}$. Let U be the $\mathfrak{B}$ - neighborhood of zero determined by the sequence $\{K_n, n\}_{n=1}^{\infty}$. If $f \in U$ and $s \in K_n \setminus K_{n-1}$, $n \geq 2$, then $\frac{1}{n+1} \leq \phi(s) < \frac{1}{n}$ and thus $|f(s)\,\phi(s)| < \frac{n}{n} = 1$. If $s \in K_1$ then $|f(s)\phi(s)| \leq |f(s)| \leq 1$. Therefore $f \in V_\phi$ and we have $\beta \leq \mathfrak{B}$.

Now fix $\{K_n, \epsilon_n\}_{n=1}^{\infty}$. For the sake of convenient notation we will suppose that $\epsilon_1 \leq 2$. If this is not so then a number smaller than $\frac{1}{2}$ would have to be used below. Let n_1 be such that $\frac{1}{\epsilon_{(n_1-1)}} \geq \frac{1}{2} > \frac{1}{\epsilon_{n_1}}$ ($n_1 > 1$ since $\epsilon_1 \leq 2$); let n_k be such that $\frac{1}{\epsilon_{(n_k-1)}} \geq (\frac{1}{2})^k > \frac{1}{\epsilon_{n_k}}$ (these n_k can be chosen since $\epsilon_n \longrightarrow \infty$ monotonically). Thus $(\frac{1}{2})^k \geq \frac{1}{\epsilon_j}$ for $j \geq n_k$ and $k \geq 1$.

Let $C_k = K_k$ for $1 \leq k \leq n_1 - 1$ and $C_k = \bigcup_{j=n_k}^{n_{k+1}-1} K_j$ for $k \geq n_1$. Then $C_k \subset \text{int } C_{k+1}$ for $k \geq 1$. Also for each k there is, by Theorem 1.6, a function $\phi_k \in C_c(S)$ such that $0 \leq \phi_k \leq 1$, $\phi_k(C_k) = 1$, and $\phi_k(s) = 0$ for $s \notin C_{k+1}$.

Finally set $\phi = \sum_{j=1}^{n_1-1} \frac{1}{\epsilon_j}\phi_j + \sum_{j=n_1}^{\infty}(\frac{1}{2})^j\phi_j$. Then $\phi \in C_o(S)$ and $\phi \geq 0$. If $f \in V_\phi$ (i.e., $\|f\phi\|_\infty \leq 1$) then for $s \in K_j$, $1 \leq j \leq n_1 - 1$, $\phi(s) \geq \frac{1}{\epsilon_j}\phi_j(s) = \frac{1}{\epsilon_j}$, and so $|f(s)| \leq 1/\phi(s) \leq \epsilon_j$. If $k \geq 1$, $n_k \leq j < n_{k+1}$, and $s \in K_j$ then $s \in C_k$ and $\phi(s) \geq (\frac{1}{2})^k\phi_k(s) = (\frac{1}{2})^k \geq \frac{1}{\epsilon_j}$. Thus $|f(s)| \leq \epsilon_j$ and $V_\phi \subset U$. Therefore β and $\mathcal{T}$ are the same topology, completing the proof.

If S is compact then the β and c-op topologies agree. As illustrated by the following result, this is also true when $S = \Omega_o$, the space of ordinal numbers less than the first uncountable. First let us prove a lemma.

LEMMA 2.2. If $\phi_1, \phi_2 \in C(S)$ and $\|f\phi_1\|_\infty \leq \|f\phi_2\|_\infty$ for all $f \in C(S)$ then $|\phi_1(s)| \leq |\phi_2(s)|$ for all s in S.

Proof. Clearly we may assume ϕ_1 and ϕ_2 to be non-negative. If $s \in S$ such that $\phi_2(s) < \phi_1(s)$ then by continuity there is an open set U such that $s \in U$ and for $t \in U$, $\phi_2(t) < \phi_1(t)$. Let $f \in C(S)$ such that $0 \leq f \leq 1$, $f(s) = 1$, and $f(t) = 0$ for $t \notin U$. Then $\|f\phi_2\|_\infty < \|f\phi_1\|_\infty$, contradicting our hypothesis.

THEOREM 2.3. (Wang[34]) The following are equivalent:

(a) the β and c-op topologies agree;

(b) the closure of any σ-compact set is compact;

(c) $C_o(S) = C_c(S)$.

Proof. (a) implies (b). If A is any σ-compact set

then by Theorem 1.3 there is a non-negative function $\phi \in C_o(S)$ such that $A \subset N(\phi)$. But by (a) there is a function $\psi \in C_c(S)$ such that $\|f\phi\|_\infty \leq \|f\psi\|_\infty$ for all $f \in C(S)$. By the preceding lemma this implies that $\phi(s) \leq |\psi(s)|$ for all $s \in S$. Hence $A \subset N(\phi) \subset N(\psi) \subset spt(\psi)$, and A^- is compact since $spt(\psi)$ is.

(b) implies (c). If $\phi \in C_o(S)$ then $N(\phi)$ is σ-compact and hence, by (b), $spt(\phi) = N(\phi)^-$ is compact. Therefore $\phi \in C_c(S)$.

That (c) implies (a) is clear since this would say that β and c-op have the same families of defining semi-norms. Therefore, the proof is complete.

Before proceeding with our study of the basic properties of β we will need the following result.

LEMMA 2.4. The Banach algebra $C_o(S)$ has an approximate identity. In particular, there is a net $\{\phi_i\}$ in $C_c(S)$ such that $0 \leq \phi_i \leq 1$ for all i and $\lim_i \phi\phi_i = \phi$ uniformly for each ϕ in $C_o(S)$.

Proof. Let $\{K_i : i \in I\}$ be the collection of all compact subsets of S. We direct I by declaring $i \leq j$ if and only if $K_i \subset K_j$. For each i let $\phi_i \in C_c(S)$ such that $0 \leq \phi_i \leq 1$ and $\phi(K_i) = 1$. Hence $\{\phi_i\}$ is a net in $C_c(S)$. If $\phi \in C_o(S)$ and $\epsilon > 0$ then $\{s \in S : |\phi(s)| \geq \epsilon/2\} = K_{i_o}$ for some $i_o \in I$. If $i \geq i_o$ then $\phi_i(s) = 1$ for $s \in K_{i_o}$; thus, $\sup\{|\phi(s)\phi_i(s) - \phi(s)| : s \in S\} = \sup\{|\phi(s)\phi_i(s) - \phi(s)|:$

$s \notin K_{i_o}\} \leq 2 \cdot \frac{\epsilon}{2} = \epsilon$. Hence $\phi\phi_i \longrightarrow \phi$ uniformly.

THEOREM 2.5. (Buck [12]) The following statements are true:

(a) The β-topology and norm topology agree if and only if S is compact.

(b) $C(S)_\beta$ is complete.

(c) The β-topology is metrizable if and only if S is compact.

(d) A set is β-bounded if and only if it is norm bounded.

(e) On bounded subsets of C(S) the β and c-op topologies agree.

(f) $C_c(S)$ is β-dense in C(S).

Proof. (a) The β-topology is always weaker than the norm topology; if the converse holds then there is a non-negative function $\phi \in C_o(S)$ such that $\|f\|_\infty \leq \|f\phi\|_\infty$ for all $f \in C(S)$. By Lemma 2.2, $\phi(s) \geq 1$ for all $s \in S$. Hence $S = \{s : \phi(s) \geq 1\}$ must be compact.

(b) Let $\{f_i\}$ be a β-Cauchy net in C(S). Then $\{f_i\}$ is also a c-op Cauchy net. But it is well known that the completion of C(S) with the c-op topology is the space of all continuous functions on S (not just the bounded ones). Hence there is a continuous function f on S such that $f_i \longrightarrow f$ uniformly on compact subsets of S. If $\phi \in C_o(S)$ then $\{\phi f_i\}$ is a norm Cauchy net in $C_o(S)$ so there is a $g \in C_o(S)$ such that $\phi f_i \longrightarrow g$ uniformly. But since multiplication is continuous in the c-op topology and $f_i \longrightarrow f$(c-op)

we have that $\phi f_i \longrightarrow \phi f$ (c-op). Therefore $\phi f = g$, and $\phi f_i \longrightarrow \phi f$ uniformly. Since $\phi f \in C_o(S)$ for all $\phi \in C_o(S)$ we have that f is bounded (see Theorem 1.4). That is, $f \in C(S)$ and $f_i \longrightarrow f(\beta)$; hence $C(S)_\beta$ is complete.

(c) The identity map $i : C(S) \longrightarrow C(S)_\beta$, where the domain has the norm topology, is always continuous. By (b) $C(S)_\beta$ is complete and if β is metrizable then the open mapping theorem [14,p.55] implies that i is open. Therefore the β and norm topologies agree and so S is compact by (a).

(d) Since the norm topology is stronger than β, every norm bounded set is β bounded. For the converse, suppose $A \subset C(S)$ is β bounded but not norm bounded. Then for each integer n there is a function $f_n \in A$ with $\|f_n\|_\infty \geq 2n$. Thus there is a point $s_n \in S$ such that $|f_n(s_n)| \geq n$. If $\{s_n\}_{n=1}^{\infty}$ has a cluster point $s \in S$ and V is a compact neighborhood of s then let $\phi \in C_c(S)$ be such that $\phi(V) = 1$. Since $s_n \in V$ for infinitely many n, we have that $\|\phi f_n\|_\infty \geq n$ for infinitely many n; contradicting the fact that A is β bounded. If $\{s_n\}_{n=1}^{\infty}$ has no cluster points then it is a discrete sequence. By Theorem 1.5 there is a function ϕ in $C_o(S)$ such that $\phi(s_n) = 1/\sqrt{n}$ for all n. Therefore $\|\phi f_n\|_\infty \geq |\phi(s_n) f_n(s_n)| \geq \sqrt{n}$ implying again that A is not β bounded.

(e) Let $A \subset C(S)$ be such that $\|f\|_\infty \leq 1$ for all $f \in A$, and let $\{f_i\}$ be a net in A such that $f_i \longrightarrow f \in A$ in the

c-op topology. If $\phi \in C_o(S)$ and $\epsilon > 0$ let $K = \{s : |\phi(s)| \geq \epsilon/4\}$. Then K is compact and $\|\phi f_i - \phi f\|_\infty \leq \sup\{|\phi(s)|\,|f_i(s) - f(s)| : s \in K\} + \sup\{|\phi(s)|\,|f_i(s) - f(s)| : s \notin K\} \leq \|\phi\|_\infty \sup\{|f_i(s) - f(s)| : s \in K\} + \frac{\epsilon}{4}\|f_i - f\|_\infty$. But $f_i \longrightarrow f$ (c-op) implies there is an i_o such that for $i \geq i_o$ $|f_i(s) - f(s)| \leq \epsilon/2\,\|\phi\|_\infty$ for all $s \in K$. Hence $\|f_i - f\|_\infty \leq \|f_i\|_\infty + \|f\|_\infty \leq 2$ implies $\|\phi f_i - \phi f\|_\infty \leq \epsilon$ for all $i \geq i_o$.

(f) Let $f \in C(S)$. By Lemma 2.4 there is an approximate identity $\{\phi_i\}$ for $C_o(S)$ where $\phi_i \in C_c(S)$ for all i. If $f_i = f\phi_i$ then $f_i \in C_c(S)$. Also, if $\phi \in C_o(S)$ then $\phi f_i = \phi_i(\phi f) \longrightarrow \phi f$ uniformly since $\phi f \in C_o(S)$; i.e., $f_i \longrightarrow f(\beta)$ and $C_c(S)$ is β dense in C(S). This concludes the proof of the theorem.

We now turn our attention to another ''description'' of the strict topology and represent $C(S)_\beta$ as the projective (or inverse) limit of certain special Banach spaces. To do this we could use, in our context, a general theorem due to E. Michael [26,p.17]; but a direct approach is available and hence would seem more desirable.

LEMMA 2.6. If $\phi \in C_o(S)$, $S_1 = N(\phi)$, and $T_\phi : C(S) \longrightarrow C_o(S_1)$ is defined by $T_\phi(f) = \phi f|_{S_1}$ (the restriction of ϕf to S_1), then T_ϕ is a continuous linear map onto a norm dense subspace of $C_o(S_1)$.

Proof. It is clear that T_ϕ is linear and continuous.

To see that the image is dense we will show that for each $x \in C_c(S_1)$ there is a function $f \in C(S)$ such that $\phi f(s) = x(s)$ for all $s \in S_1$. In fact if $x \in C_c(S_1)$ let $\bar{x}$ be the extension of x to S such that $\bar{x}(s) = 0$ for $s \notin S_1$. Then $\bar{x}(s) = x(s)$ for $s \in S_1$, an open set, and $\bar{x}(s) = 0$ for $s \in S \setminus \text{spt}(x)$ which is also open. Hence $\bar{x} \in C_c(S)$. Let $\rho = \min\{|\phi(s)| : s \in \text{spt}(x)\}$; then $\rho > 0$. Define f on S by $f(s) = x(s)/\phi(s)$ if $s \in \text{spt}(x)$ and $f(s) = 0$ if $\bar{x}(s) = 0$. Since $|\phi(s)| \geq \rho$ on $\text{spt}(x)$ f is continuous on $\text{spt}(x)$. Hence, $\text{spt}(x)$ and $\{s : \bar{x}(s) = 0\}$ are closed sets imply f is continuous. Also $|x(s)/\phi(s)| \leq \frac{1}{\rho}\|x\|_\infty < \infty$ for $s \in \text{spt}(x)$ implies $f \in C(S)$. Clearly $T_\phi(f) = x$, completing the proof.

Let $I = \{\phi_i : i \in I\}$ be a net of non-negative elements of $C_o(S)$ such that $i \leq j$ if and only if $\phi_i \leq \phi_j$, and $\phi \in C_o(S)$ implies there is some $\phi_i \geq |\phi|$. For each $i \in I$ let $S_i = N(\phi_i)$. If $i,j \in I$ and $i \leq j$ then we will define a map $\pi_{ij} : C_o(S_j) \longrightarrow C_o(S_i)$. If $f \in C(S)$ and $\phi_j(s)f(s) = 0$ for all $s \in S_j$ then $f(s) = 0$ for all $s \in S_j$. But $\phi_i \leq \phi_j$ implies $S_i \subset S_j$ and so $\phi_i(s)f(s) = 0$ for all $s \in S_i$. Hence, if we let $\pi_{ij}(\phi_j f|_{S_j}) = \phi_i f|_{S_i}$ for all $f \in C(S)$ then, by the above remarks and Lemma 2.6, π_{ij} is well defined on a norm dense subspace of $C_o(S_j)$. Also $\|\phi_i f\|_\infty \leq \|\phi_j f\|_\infty$ for all $f \in C(S)$ since $\phi_i \leq \phi_j$. Therefore, considered as a linear map on a norm dense subspace of $C_o(S_j)$, π_{ij} has norm ≤ 1. Thus π_{ij} can be extended to a linear

map of $C_o(S_j)$ into $C_o(S_i)$ because it is uniformly continuous.

THEOREM 2.7. The space $C(S)\beta$ is topologically isomorphic to the projective (inverse) limit $\mathcal{LP}\{C_o(S_i)\}$ of the Banach spaces $C_o(S_i)$, where the bonding maps are the maps π_{ij} described above.

Proof. Let $f \in C(S)$ and put $f_i = \phi_i f\big|_{S_i} \in C_o(S_i)$. Then $\{f_i\}_{i\in I} \in \prod\{C_o(S_i) : i \in I\}$. If $i \leq j$ then $\pi_{ij}(f_j) = f_i$ and hence $\{f_i\}_{i\in I} \in \mathcal{LP}\{C_o(S_i)\}$. Define $T : C(S) \longrightarrow \mathcal{LP}\{C_o(S_i)\}$ by $T(f) = \{f_i\}$ and let π_j be the projection of $\prod\{C_o(S_i)\}$ onto $C_o(S_j)$. To see that T is continuous, observe that $\pi_i \circ T = T_{\phi_i}$, the map described in Lemma 2.6, which was shown to be continuous; thus it follows that T is continuous. Also if $\phi \in C_o(S)$ then there is a $j \in I$ such that $|\phi| \leq \phi_j$. Therefore if $T(f)$ is in $\pi_j^{-1}(\{g \in C_o(S_j) : \|g\|_\infty \leq 1)$ then f_j is in the unit ball of $C_o(S_j)$. That is, $\|\phi_j f\|_\infty \leq 1$ and so $\|\phi f\|_\infty \leq 1$. Therefore, $T(V_\phi) \supset \pi_j^{-1}(\{g \in C_o(S_j) : \|g\|_\infty \leq 1\}) \cap T(C(S))$ and T is open onto its image.

Clearly T is one-one, since $\phi_i f = 0$ for all i implies $\phi f = 0$ for all $\phi \in C_o(S)$; from this it is clear that $f = 0$. This gives that T is a homeomorphism onto its image. Since T is linear and $C(S)\beta$ is complete, $T(C(S))$ is a closed subspace of $\mathcal{LP}\{C_o(S_i)\}$. On the other hand, $T(C(S))$ is dense in $\mathcal{LP}\{C_o(S_i)\}$. In fact the sets of the form $\pi_j^{-1}(\{g \in C_o(S_j) : \|g\|_\infty \leq r\}) \cap \mathcal{LP}$ where $r > 0$

and $j \in I$ form a neighborhood basis at zero in $\mathcal{LP}$. If $\{f_i\} \in \mathcal{LP}$, $r > 0$, and $j \in I$, then, by Lemma 2.6, there is a function $h \in C(S)$ such that $|f_j(s) - h(s)\phi_j(s)| < r$ for $s \in S_j$; that is, $T(h) - \{f_i\} \in \pi_j^{-1}(\{g \in C_o(S_j) : \|g\|_\infty \leq r\}) \cap \mathcal{LP}\{C_o(S_i)\}$. Thus $T(C(S))$ is dense in $\mathcal{LP}\{C_o(S_i)\}$, and this completes the proof.

Let us now turn our attention to the calculation of the adjoint space of $C(S)_\beta$. To do this we will need the following fundamental result. Not only is this lemma needed in the characterization of $C(S)_\beta^*$, but it will be a basic tool for the next two chapters. Recall that a measure μ vanishes off a set A if and only if $\mu(B) = 0$ for all Borel sets disjoint from A; or, $|\mu|(S \setminus A) = 0$.

LEMMA 2.8. If H is a subset of M(S) then the following two statements are equivalent:

(a) H is uniformly bounded and for every $\epsilon > 0$ there is a compact set $K \subset S$ such that $|\mu|(S \setminus K) \leq \epsilon$ for all $\mu \in H$.

(b) There is a non-negative function $\phi \in C_o(S)$ such that $H \subset \{\mu \in M(S) : \mu$ vanishes off $N(\phi)$ and $\|\frac{1}{\phi}\mu\| \leq 1\}$.

Proof. If (b) holds then H is clearly uniformly bounded since for each $\mu \in H$, $\|\mu\| = \|\phi \cdot \frac{1}{\phi}\mu\| \leq \|\phi\|_\infty \|\frac{1}{\phi}\mu\| \leq \|\phi\|_\infty$. If $\epsilon > 0$ then $K = \{s : \phi(s) \geq \epsilon\}$ is compact and for each $\mu \in H$, $|\mu|(S \setminus K) = \int_{S \setminus K} d|\mu| = \int_{S \setminus K} \phi \cdot \frac{1}{\phi}\, d|\mu| \leq \|\frac{1}{\phi}\mu\| \sup\{\phi(s) : s \notin K\} \leq \epsilon$.

For the converse let (a) hold. We may assume that $\|\mu\| \leq 1$ for all $\mu \in H$. From (a) we may obtain inductively a sequence $\{K_n\}_{n=1}^{\infty}$ of compact sets in S such that $K_n \subset \text{int } K_{n+1}$ and $|\mu|(S \setminus K_n) \leq (\frac{1}{4})^n$ for all μ in H. For each integer n let $\phi_n \in C_c(S)$ be such that $0 \leq \phi_n \leq 1$, $\phi_n(K_n) = 1$, and $\phi(S \setminus K_{n+1}) = 0$. Put $\phi = 2\sum_{n=1}^{\infty} (\frac{1}{2})^n \phi_n$; then $\phi \in C_o(S)$ and $N(\phi) = \bigcup_{n=1}^{\infty} K_n$. If $A \in$ Borel (S) is such that $A \cap N(\phi) = \square$ then $A \cap K_n = \square$ for all $n \geq 1$. Hence if $\mu \in H$, $|\mu|(A) \leq (\frac{1}{4})^n$ for all $n \geq 1$ and so $\mu(A) = 0$. Thus, every measure in H vanishes off $N(\phi)$.

If $s \in K_n \setminus K_{n-1}$, $n \geq 2$, then $\phi_1(s) = \cdots = \phi_{n-2}(s) = 0$ and $\phi_k(s) = 1$ for $k \geq n$. Therefore $\phi(s) = \sum_{k=n-1}^{\infty} (\frac{1}{2})^k \phi_k(s) \geq 2\sum_{k=n}^{\infty} (\frac{1}{2})^k = 2 \cdot (\frac{1}{2})^{n-1}$. If $\mu \in H$ then $\int_{K_n \setminus K_{n-1}} \frac{1}{\phi} \, d|\mu| \leq \frac{1}{2} \cdot 2^{n-1}(\frac{1}{4})^{n-1} = \frac{1}{2} \cdot (\frac{1}{2})^{n-1}$. Hence $\int \frac{1}{\phi} \, d|\mu| = \int_{K_1} \frac{1}{\phi} \, d|\mu| + \sum_{n=2}^{\infty} \int_{K_n \setminus K_{n-1}} \frac{1}{\phi} \, d|\mu| \leq \frac{1}{2} \|\mu\| + \frac{1}{2} \sum_{n=2}^{\infty} (\frac{1}{2})^{n-1} \leq \frac{1}{2} + \frac{1}{2} = 1$.

Since μ was chosen arbitrarily, the proof is complete.

REMARKS. The first proof of a result of this type known to the author is found in Buck [12,p.100] where H consists of a single element. However, it is essential for all that follows that this lemma be proved in the generality in which we have proved it, because this furnishes the criterion for β-equicontinuity in M(S) (see Theorem 3.2 below).

THEOREM 2.9. (Buck [12]) A linear functional L on $C(S)_\beta$ is continuous if and only if there is a unique measure $\mu \in M(S)$ such that

$$L(f) = \int f d\mu \qquad \text{for } f \in C(S).$$

Proof. If $\mu \in M(S)$ and H consists of the single measure μ, then, by the inner regularity of μ, H satisfies condition (a) of the preceding lemma. Hence there is a function $\phi \in C_o(S)$, $\phi \geq 0$, such that μ vanishes off $N(\phi)$ and $\|\frac{1}{\phi}\mu\| \leq 1$. If $\{f_i\}$ is a net in C(S) such that $f_i \longrightarrow 0$ (β) then $\phi f_i \longrightarrow 0$ uniformly in $C_o(S)$. From this we have that $\int f_i d\mu = \int \phi f_i \frac{1}{\phi} d\mu \longrightarrow 0$. Therefore if $L(f) = \int f d\mu$ for all $f \in C(S)$ then $L \in C(S)_\beta^*$.

Conversely, suppose $L \in C(S)_\beta^*$. Since β is weaker than the norm topology, the restriction of L to $C_o(S)$ is norm continuous. Thus, by the Riesz representation theorem, there is a unique μ in M(S) such that $L(\phi) = \int \phi d\mu$ for all $\phi \in C_o(S)$. But μ represents a β-continuous linear functional, and L and μ agree on $C_o(S)$ which is β-dense in C(S) (see Theorem 2.5 (f)). Hence $L(f) = \int f d\mu$ for all $f \in C(S)$. This completes the proof.

Since $M(S) = C(S)_\beta^*$, we can now give answers to certain questions concerning the weak topology on $C(S)_\beta$. It should be pointed out that there is a strong analogy between $C(S)_\beta$ and C(S) with the norm topology for S compact. Besides the preceding theorem, this analogy will be further emphasized by the following results. Since the phrasing of criteria for topological properties on C(S)

with the norm topology necessitates passage to the Stone-Cech compactification of S, it might be argued that β is more ''natural'' than the norm topology on C(S).

COROLLARY 2.10. If $\{f_n\}_{n=1}^{\infty}$, f are in C(S) then $\{f_n\}$ converges to f weakly if and only if $f(s) = \lim f_n(s)$ for all $s \in S$ and $\{f_n\}$ is uniformly bounded.

Proof. If $\{f_n\}$ converges to f weakly then $\{f_n\}$ is weakly bounded and hence β bounded. But β boundedness and norm boundedness are equivalent by Theorem 2.5. Therefore $\{f_n\}$ is uniformly bounded. Since $\delta(s)$, the unit point mass at s, is in M(S), we readily get that $f_n(s) = \int f_n d\delta(s) \rightarrow \int f d\delta(s) = f(s)$.

The converse follows immediately from the Lebesgue bounded convergence theorem.

THEOREM 2.11. (Glicksberg [17]) If $F \subset C(S)_\beta$ then F is weakly conditionally compact if and only if F is uniformly bounded and conditionally compact for the topology of pointwise convergence on S.

Proof. Since we will not use this theorem and the proof we would present is exactly that of Glicksberg, we shall content ourselves with the above cited reference to his paper.

THEOREM 2.12. If $F \subset C(S)$ then the following are equivalent:

(a) F is β conditionally compact;

(b) F is uniformly bounded and c-op conditionally compact;

(c) F is uniformly bounded and for every compact set $K \subset S$, $F|_K = \{f|_K : f \in F\}$ is norm conditionally compact in C(K);

(d) F is uniformly bounded and an equicontinuous family.

Proof. Clearly (a) implies (b). To see that (b) implies (a) note that F uniformly bounded implies that we may assume $F \subset$ ball C(S). But ball C(S) is c-op and β-closed and on ball C(S) the β and c-op topologies agree (Theorem 2.5 (e)). Hence, (b) easily implies (a) and the two are equivalent.

(b) implies (c). Let K be compact and $\{f_n\}_{n=1}^{\infty}$ a sequence in $F|_K \subset C(K)$. Then for each integer $n \geq 1$ there is a $g_n \in F$ such that $g_n|_K = f_n$. But F is c-op conditionally compact and hence there is a function $g \in C(S)$ such that $g_n \xrightarrow{cl} g$(c-op). If $f = g|_K \in C(K)$ then $f_n \xrightarrow{cl} f$ uniformly on K, and so $F|_K$ is conditionally compact in C(K).

(c) implies (d). Let $s \in S$ and let K be a compact neighborhood of s. By (c) $F|_K$ is norm conditionally compact in C(K) and hence equicontinuous [14,p.266]. Therefore for every $\epsilon > 0$ there exists a neighborhood U of s in K such that $|f(s) - f(t)| < \epsilon$ for all $t \in U$ and $f \in F$. But since $s \in$ int K we may choose U sufficiently small so that U is open in S. Hence F is equicontinuous.

(d) implies (c). This follows easily from [14,p.266].

(c) implies (b). Suppose $F \subset$ ball $C(S)$; then F^- = the c-op closure of F is contained in ball C(S). Also $F^-|_K \subset (F|_K)^- \subset C(K)$ for every compact set $K \subset S$. Hence for K compact $F^-|_K$ is totally bounded in C(K). Therefore if $\epsilon > 0$ there are $f_1, \cdots, f_n \in F^-$ such that $F^-|_K \subset \bigcup_{i=1}^{n} \{g \in C(K) : \|g - f_i|_K\|_\infty < \epsilon\}$. That is, $F^- \subset \bigcup_{i=1}^{n} \{f \in C(S) : \sup_{s \in K} |f(s) - f_i(s)| < \epsilon\}$ and F^- is c-op totally bounded. Also $F^- \subset$ ball C(S) and β = c-op on ball C(S) implies F^- is c-op complete since ball C(S) is β complete. Therefore F^- is c-op compact and the proof of the theorem is complete.

We will conclude this chapter with a study of the polars of the sets $V_\phi = \{f \in C(S) : \|f\phi\|_\infty \leq 1\}$ and their extreme points. Before doing this, let us say a word here about notation. We will denote by '' β -weak *'' the weak star topology on M(S) which it has as the adjoint of $C(S)_\beta$; i.e., the $\sigma(M(S), C(S))$ topology. This is in order to distinguish it from the weak * topology which M(S) has as the adjoint of the Banach space $C_o(S)$; i.e., the $\sigma(M(S), C_o(S))$ topology. For an example of their difference, note that ball M(S) is weak * compact but it is far from being β-weak * compact if S is not compact.

The next theorem is due to Glicksberg [18]. His proof is, however, quite complicated and we will furnish a comparatively simple one.

THEOREM 2.13. If $\phi \in C_o(S)$ then $V_\phi^\circ \equiv \{\mu : \left|\int f d\mu\right| \leq 1$ for all $f \in V_\phi\} = \{\mu : \mu$ vanishes off $N(\phi)$ and $\|\frac{1}{\phi}\mu\| \leq 1\}$.

Proof. Define the map $T_\phi : C(S)_\beta \longrightarrow C_o(S)$ by $T_\phi(f) = \phi f$ for all $f \in C(S)$. Let B = ball $C_o(S)$ and B^* = ball $M(S)$. Then $T_\phi^{-1}(B) = V_\phi$ and T_ϕ is continuous. Therefore its adjoint $T_\phi^* : M(S) \longrightarrow M(S)$ is well defined and continuous when both domain and range have their weak * and β - weak * topologies respectively. But $\int f dT_\phi^*(\mu) = \int f\phi d\mu$ for all f in $C(S)$; therefore $T_\phi^*(\mu) = \phi\mu$. If $\mu \in B^*$ and $f \in V_\phi$ then $\left|\int f\phi d\mu\right| \leq \|f\phi\|_\infty \|\mu\| \leq 1$. Hence $T_\phi^*(B^*) \subset V_\phi^\circ$. Let $f \in [T_\phi^*(B^*)]^\circ$; i.e., $\left|\int f\phi d\mu\right| \leq 1$ for all $\mu \in B^*$. Thus $\|f\phi\|_\infty = \sup\{\left|\int \phi f d\mu\right| : \mu \in B^*\} \leq 1$, or $f \in V_\phi$. That is, $[T_\phi^*(B^*)]^\circ \subset V_\phi$ and so $V_\phi^\circ \subset [T_\phi^*(B^*)]^{\circ\circ}$ = the β-weak * closure of $T_\phi^*(B^*)$. But B^* is weak * compact in $M(S)$ and T_ϕ^* is continuous implies $T_\phi^*(B^*)$ is β-weak * compact and $V_\phi^\circ = T_\phi^*(B^*) = \{\phi\nu : \|\nu\| \leq 1\}$.

If $\mu = \phi\nu$ and $\|\nu\| \leq 1$ then μ vanishes off $N(\phi)$ and $(\frac{1}{\phi}\mu)(A) = (\frac{1}{\phi} \cdot \phi\nu)(A) = \nu(A \cap N(\phi))$. Hence $\|\frac{1}{\phi}\mu\| = |\nu|(N(\phi)) \leq \|\nu\| \leq 1$. If, conversely, μ vanishes off $N(\phi)$ and $\|\frac{1}{\phi}\mu\| \leq 1$ then $\mu = \phi\nu$ where $\nu = \frac{1}{\phi}\mu$; hence $\mu \in V_\phi^\circ$, concluding the proof.

We will now calculate the extreme points of V_ϕ° where $\phi \in C_o(S)$. In order to do this we need an extension of a result of Arens and Kelley [3] for which we give a new proof.

THEOREM 2.14. A measure μ is an extreme point of the unit ball of M(S) if and only if $\mu = \lambda \delta_{(s)}$ where λ is a unimodular complex number and $\delta_{(s)}$ is the unit point mass at some $s \in S$.

Proof. If $\lambda \delta_{(s)} = \frac{1}{2}(\sigma + \nu)$ where $\|\sigma\|, \|\nu\| \leq 1$ and $|\lambda| = 1$ then $\delta_{(s)} = \frac{1}{2}(\lambda^{-1}\sigma + \lambda^{-1}\nu)$. Hence we may assume $\lambda = 1$. If $A \in$ Borel (S) and $s \in A$ then $1 = \frac{1}{2}[\sigma(A) + \nu(A)]$ and $|\sigma(A)|, |\nu(A)| \leq 1$. Thus $\sigma(A) = \nu(A) = 1$ since 1 is an extreme point of the unit disk in the plane. If $s \notin A$ then $1 = \sigma(A \cup \{s\}) = \sigma(A) + 1$ and hence $\sigma(A) = 0$; similarly, $\nu(A) = 0$. Therefore $\sigma = \nu = \delta_{(s)}$ and so $\delta_{(s)}$ is an extreme point.

Suppose now that μ is an extreme point of ball M(S) and let $\mathcal{K} = \{K : K$ is a compact subset of S and $|\mu|(K) > 0\}$. Since $\mu \neq 0$ $\mathcal{K} \neq \emptyset$. We will show that $\mathcal{K}$ has the finite intersection property. Suppose $K_1, K_2 \in \mathcal{K}$ and $K_1 \cap K_2 = \emptyset$; let $K_3 = S \setminus (K_1 \cup K_2)$. If $|\mu|(K_3) \neq 0$ then $\mu = \sum_{i=1}^{3} |\mu|(K_i)\mu_i$ where $\mu_i(A) = |\mu|(K_i)^{-1}\mu(A \cap K_i)$ for all $A \in$ Borel (S). But $\|\mu_i\| = 1$ for $i = 1,2,3$ and $\sum_{i=1}^{3} |\mu|(K_i) = \|\mu\| = 1$. Since μ is an extreme point of ball M(S) $\mu = \mu_1$. If $|\mu|(K_3) = 0$ we reach the same conclusion by just having two terms in the above sums.

Hence $|\mu|(K_2) = |\mu_1|(K_2) = |\mu|(K_1)^{-1}|\mu|(K_1 \cap K_2) = 0$ since $K_1 \cap K_2 = \emptyset$, a contradiction. Therefore $K_1 \cap K_2 \neq \emptyset$. If F is a compact subset of $K_1 \setminus K_2$ then $F \cap K_2 =$

$\square$; thus $F \notin \mathcal{K}$ and $|\mu|(F) = 0$. By inner regularity $|\mu|(K_1 \setminus K_2) = 0$ and so $|\mu|(K_1 \cap K_2) = |\mu|(K_1) - |\mu|(K_1 \setminus K_2) = |\mu|(K_1) \neq 0$. This shows that $K_1, K_2 \in \mathcal{K}$ implies $K_1 \cap K_2 \in \mathcal{K}$. Now by a simple induction argument we have that $\mathcal{K}$ is closed under finite intersections and so there exists $s \in \bigcap\{K : K \in \mathcal{K}\}$. If U is open and $s \notin U$ then for every compact set $K \subset U$ $|\mu|(K) = 0$ since $s \notin K$. Therefore $|\mu|(U) = 0$ and so μ is concentrated on $\{s\}$. Thus $\mu = \lambda \delta_{(s)}$ where $\lambda = \mu(S)$.

THEOREM 2.15. If $\phi \in C_0(S)$ then μ is an extreme point of V_ϕ° if and only if $\mu = \lambda \phi(s) \delta_{(s)}$ where $|\lambda| = 1$ and $s \in N(\phi)$.

Proof. Let μ be an extreme point of V_ϕ° and let $T_\phi : C(S)_\beta \longrightarrow C_0(S)$ be defined by $T_\phi(f) = \phi f$. Therefore its adjoint map $T_\phi^* : M(S) \longrightarrow M(S)$ is given by $T_\phi^*(\mu) = \phi\mu$. Now $T_\phi^* :$ ball $M(S) \longrightarrow V_\phi^\circ$ is a continuous onto affine map when domain and range have their relative weak * and β-weak * topologies (see the proof of Theorem 2.13). By Theorems 1.17 and 2.14 there exists a point $s \in S$ and a unimodular scalar such that $\mu = T_\phi^*(\lambda \delta_{(s)}) = \lambda\phi(s)\delta_{(s)}$. Since $\mu \neq 0$, $s \in N(\phi)$.

For the converse suppose $\mu = \phi(s)\delta_{(s)}$ where $\phi(s) \neq 0$. Then $\mu \in V_\phi^\circ$, and if $\mu = \frac{1}{2}(\nu + \sigma)$ where $\nu, \sigma \in V_\phi^\circ$ then $\delta_{(s)} = \frac{1}{\phi}\mu = \frac{1}{2}(\frac{1}{\phi}\nu + \frac{1}{\phi}\sigma)$ and $\|\frac{1}{\phi}\nu\|, \|\frac{1}{\phi}\sigma\| \leq 1$. By Theorem 2.14 $\delta_{(s)} = \frac{1}{\phi}\nu = \frac{1}{\phi}\sigma$. Since ν and σ

vanish off $N(\emptyset)$ we have that $\mu = \emptyset(s)\delta_{(s)} = \nu = \sigma$ and μ is an extreme point. This completes the proof.

In Chapter V we will give an interesting extension of this result to vector-valued measures.

CHAPTER III

THE MACKEY TOPOLOGY ON $C(S)_\beta$

This chapter is divided into two sections. The first section investigates a question asked by Buck [12,p.100]; is $C(S)_\beta$ a Mackey space? In Theorem 3.7 we show that the answer is yes if S is paracompact. Actually, as the theorem states, we show much more than this; it is this stronger result which we use to develope Chapter IV. We conclude section one with an example to show that $C(S)_\beta$ is not always a Mackey space.

Section two treats the question of whether or not a subspace of $C(S)_\beta$ is a Mackey space provided $C(S)_\beta$ is a Mackey space. In particular, we give necessary and sufficient conditions for a β-closed subspace of ℓ^∞ to be a Mackey space; and we show that H^∞, the space of bounded analytic functions on the open unit disk, is not a Mackey space if it has the strict topology.

Section 1. $C(S)_\beta$ is a Mackey space for S paracompact.

As in the preceding chapter "β-weak *" will be used to denote the weak star topology on M(S) which it has as the adjoint of $C(S)_\beta$. When referring to properties of subsets of $M(S) = C(S)_\beta^*$ we will invariably prefix the symbol β. This we hope will alleviate some of the confusion that may arise from the fact that M(S) is also the

adjoint of $C_o(S)$. Thus, we shall refer to β -equicontinuous subsets H of M(S). A subset H of M(S) is β -equicontinuous if and only if $H^o = \{f \in C(S) : \left|\int f d\mu\right| \leq 1$ for all μ in H$\}$ is a β -neighborhood of the origin in C(S).

If E is a topological vector space then E is a Mackey space if and only if every weak * compact convex circled subset of E* is equicontinuous; E is a strong Mackey space if and only if every weak * compact subset of E* is equicontinuous (see Chapter I).

The following is a classical result due to J. Schur [30]. Proofs can also be found in [4,p.137] and [14,p.296]. We will reproduce the proof here not only for the sake of convenience, but also because this result is the cornerstone of this paper. Of course, ℓ^1 is the Banach space of absolutely summable sequences of complex numbers. If $x = \{z_i\}_{i=1}^{\infty} \in \ell^1$ then the norm of x is $\|x\| = \sum_{i=1}^{\infty} |z_i|$. The Banach space adjoint of ℓ^1 is ℓ^∞, the space of bounded sequences of complex numbers. Also, c_o is the space of complex sequences converging to zero. Note that if S is the space of positive integers with the discrete topology then $c_o = C_o(S)$, $\ell^\infty = C(S)$, and $\ell^1 = M(S)$.

THEOREM 3.1. In the space ℓ^1 a sequence converges weakly if and only if it converges in norm.

Proof. Clearly we need only consider the case where a sequence $\{x_n\}_{n=1}^{\infty}$ of elements $x_n \in \ell^1$ converges weakly

to zero. That is, if $x_n = \{z_i^{(n)}\}_{i=1}^{\infty}$ for $n \geq 1$ and for every bounded sequence $y = \{y_i\}_{i=1}^{\infty}$, $\langle x_n, y\rangle = \sum_{i=1}^{\infty} y_i z_i^{(n)}$ converges to zero as $n \to \infty$, then we must show that $\lim_{n \to \infty} \sum_{i=1}^{\infty} |z_i^{(n)}| = 0$. If $y_i = 1$ for $i = j$ and 0 for $i \neq j$ then we have $\lim_{n \to \infty} z_j^{(n)} = \lim_{n \to \infty} \langle x_n, y\rangle = 0$ for $j = 1,2,\cdots$.

Suppose that $\{x_n\}$ does not converge to zero in norm; hence $\limsup_{n \to \infty} \sum_{i=1}^{\infty} |z_i^{(n)}| > \epsilon$ for some $\epsilon > 0$. From this we may define, by induction, two monotonically increasing sequences of integers $\{n_k\}$ and $\{r_k\}$ such that:

(a) n_1 is the first integer such that $\sum_{i=1}^{\infty} |z_i^{(n_1)}| > \epsilon$;

(b) r_1 is the first integer such that $\sum_{i=1}^{r_1} |z_1^{(n_1)}| > \epsilon/2$ and $\sum_{i=r_1+1}^{\infty} |z_i^{(n_1)}| < \epsilon/5$;

(c) n_k is the first integer greater than n_{k-1} such that $\sum_{i=1}^{\infty} |z_i^{(n_k)}| > \epsilon$ and $\sum_{i=1}^{r_{k-1}} |z_i^{(n_k)}| < \epsilon/5$;

(d) r_k is the first integer greater than r_{k-1} such that $\sum_{i=r_{k-1}+1}^{r_k} |z_i^{(n_k)}| > \epsilon/2$ and $\sum_{i=r_k+1}^{\infty} |z_i^{(n_k)}| < \epsilon/5$.

Now define a bounded sequence $y = \{y_i\}_{i=1}^{\infty}$ by $y_i = |z_i^{(n_1)}|^{-1} \overline{z_i^{(n_1)}}$ for $1 \leq i \leq r_1$ and $y_i = |z_i^{(n_{k+1})}|^{-1} \overline{z_i^{(n_{k+1})}}$ for $r_k < i \leq r_{k+1}$ and $k = 1,2,\cdots$.

Then $\sum_{i=r_{k-1}+1}^{r_k} |z_i^{(n_k)}| = \sum_{i=r_{k-1}+1}^{r_k} y_i z_i^{(n_k)} = \sum_{i=1}^{\infty} y_i z_i^{(n_k)} - \sum_{i=1}^{r_{k-1}} y_i z_i^{(n_k)} - \sum_{i=r_k+1}^{\infty} y_i z_i^{(n_k)}$. Hence $\sum_{i=r_{k-1}+1}^{r_k} |z_i^{(n_k)}| \leq |\sum_{i=1}^{\infty} y_i z_i^{(n_k)}| + \sum_{i=1}^{r_{k-1}} |z_i^{(n_k)}| + \sum_{i=r_k+1}^{\infty} |z_i^{(n_k)}|$. From this it follows that $|\sum_{i=1}^{\infty} y_i z_i^{(n_k)}| \geq \sum_{i=r_{k-1}+1}^{r_k} |z_i^{(n_k)}| - \sum_{i=1}^{r_{k-1}} |z_i^{(n_k)}| - \sum_{i=r_k+1}^{\infty} |z_i^{(n_k)}| \geq \frac{\epsilon}{2} - \frac{\epsilon}{5} - \frac{\epsilon}{5} = \frac{\epsilon}{10}$ for $k = 1,2,\cdots$. But $\lim_{n\to\infty} |\sum_{i=1}^{\infty} y_i z_i^{(n)}| = 0$ and thus we have a contradiction. This completes the proof.

Now let us give our basic theorem characterizing β-equicontinuous sets.

THEOREM 3.2. If $H \subset M(S)$ then the following are equivalent:

(a) H is β-equicontinuous;

(b) H is uniformly bounded and for every net $\{\phi_i\}$ in $C_c(S)$ such that $\|\phi_i\|_\infty \leq 1$ for all i and $\phi_i \to 0$ (c-op), we have that $\phi_i \to 0$ uniformly on H;

(c) H is uniformly bounded and for every $\epsilon > 0$ there is a compact set $K \subset S$ such that $|\mu|(S \setminus K) \leq \epsilon$ for all μ in H;

(d) there is a function $\phi \in C_0(S)$ such that $\phi \geq 0$ and $H \subset \{\mu : \mu$ vanishes off $N(\phi)$ and $\|\frac{1}{\phi}\mu\| \leq 1\}$.

Proof. (a) implies (b). If $\{\phi_i\}$ is such a net then

$\|\phi_i\|_\infty \leq 1$ for all i implies $\phi_i \to 0(\beta)$ by Theorem 2.5 (e). Since H is β-equicontinuous, H° is a β-neighborhood of zero in C(S). Therefore if $\epsilon > 0$ there is an i_o such that for $i \geq i_o$ $\phi_i \in (\epsilon \cdot H^\circ)$. Hence, if $i \geq i_o$ and $\mu \in H$ then $\left|\int \frac{1}{\epsilon} \phi_i d\mu\right| \leq 1$ or $\left|\int \phi_i d\mu\right| \leq \epsilon$; that is, $\phi_i \to 0$ uniformly on H.

(b) implies (c). Suppose that (c) does not hold. If $\{K_i : i \in I\}$ is the collection of all compact sets in S then there exists an $\epsilon > 0$ such that for every $i \in I$, $|\mu_i|(S \setminus K_i) > \epsilon$ for some $\mu_i \in H$. Thus, by Corollary 1.13 there is a function ϕ_i in $C_c(S)$ with $\|\phi_i\|_\infty = 1$, $spt(\phi_i) \subset S \setminus K_i$, and $\left|\int \phi_i d\mu_i\right| > \epsilon$. Declare that $i \leq j$ if and only if $K_i \subset K_j$. Therefore $\{\phi_i\}$ is a net in $C_c(S)$ and $\|\phi_i\|_\infty \leq 1$ for all i. If $i_o \in I$ and $i \geq i_o$ then $\sup\{|\phi_i(s)| : s \in K_{i_o}\} = 0$ since $spt(\phi_i) \cap K_{i_o} = \square$. Hence, $\phi_i \to 0$ (c-op) and $\{\phi_i\}$ satisfies the conditions of (b). By (b), $\phi_i \to 0$ uniformly on H. Therefore there is an i_o such that if $i \geq i_o$ then $\left|\int \phi_i d\mu\right| < \epsilon$ for all $\mu \in H$. In particular, $\left|\int \phi_i d\mu_i\right| < \epsilon$ for $i \geq i_o$, contradicting the choice of the μ_i.

(c) implies (d). This is the substance of our Lemma 2.8.

(d) implies (a). By Glicksberg's theorem (2.13), (d) says that $H \subset V_\phi^\circ$ and hence it must be β-equicontinuous.

LEMMA 3.3. The strong topology on $M(S) = C(S)_\beta^*$ is

exactly the norm topology. Hence $C_o(S)$ and $C(S)_\beta$ have the same second adjoint. Also $C(S)_\beta$ is semi-reflexive if and only if S is discrete.

Proof. The strong topology on M(S) is, by definition, the topology of uniform convergence on β -bounded sets in C(S). By Theorem 2.5 (d) these are exactly the norm bounded sets. Hence the strong topology is exactly the topology of uniform convergence on the unit ball of C(S). But this is the norm topology on M(S).

For the last part of the theorem let S be discrete. Then $M(S) = \ell^1(S)$; i.e., $\mu \in M(S)$ if and only if $\mu = \sum_{n=1}^{\infty} a_n \delta_{(s_n)}$ where $\{s_n\}_{n=1}^{\infty}$ is some sequence in S and $\sum_{n=1}^{\infty} |a_n| < \infty$ (in fact $\|\mu\| = \sum_{n=1}^{\infty} |a_n|$). Also, C(S) consists of all bounded functions on S. If $L \in M(S)^*$ then define $f(s) = L(\delta_{(s)})$ for all $s \in S$. Since $\|\delta_{(s)}\| = 1$ for all $s \in S$, $\|f\|_\infty \leq \|L\| < \infty$ and hence $f \in C(S)$. If $\mu = \sum_{n=1}^{\infty} a_n \delta_{(s_n)}$ then $L(\mu) = \sum_{n=1}^{\infty} a_n L(\delta_{(s_n)}) = \sum_{n=1}^{\infty} a_n f(s_n) = \int f d\mu$. Therefore $C(S)_\beta{}^{**} = C(S)$.

Conversely, suppose $C(S)_\beta{}^{**} = C(S)$ and let $s \in S$. Then $L(\mu) = \mu(\{s\})$ defines a bounded linear functional L on M(S). Thus there is a function $f \in C(S)$ such that $\int f d\mu = \mu(\{s\})$ for all $\mu \in M(S)$. In particular, $f(s) = \int f d\delta_{(s)} = 1$ and for $t \neq s$ $f(t) = \int f d\delta_{(t)} = \delta_{(t)}(\{s\}) = 0$. Therefore f is the characteristic function of singleton

{s} and, since f is continuous, {s} is open and closed. This implies S is discrete and concludes the proof.

THEOREM 3.4. If S is the space of positive integers with the discrete topology and $H \subset \ell^1 = M(S)$, then the following are equivalent: (a) H is weakly conditionally compact; (b) H is β-weak * conditionally compact; (c) H is norm conditionally compact; (d) H is β-equicontinuous.

Proof. It is clear that (a) and (b) are equivalent since $(\ell^\infty, \beta) = C(S)_\beta$ is semi-reflexive. Also, it is trivial that (d) implies (b) since this is true for an arbitrary space S. To see that (b) implies (c) let $\{x_n\}_{n=1}^\infty$ be a sequence in H. By Theorem 1.15 we can get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $x \in \ell^1$ such that $x_{n_k} \longrightarrow x$ weakly. But according to the theorem of Schur this implies $x_{n_k} \longrightarrow x$ in norm. Thus H is norm conditionally compact.

If (c) holds and we wish to prove (d) then let $\epsilon > 0$ and choose $x_1, \cdots, x_n \in H$ such that $H \subset \bigcup_{k=1}^n \{x \in \ell^1 : \|x - x_n\| < \epsilon/2\}$. If for each $k = 1, \cdots, n$ we let $x_k = \{z_i^{(k)}\}_{i=1}^\infty$ then there is an integer N such that $\sum_{i=N+1}^\infty |z_i^{(k)}| \leq \epsilon/2$ for $1 \leq k \leq n$. Hence, if $x = \{z_i\}_{i=1}^\infty \in H$ then there is an integer k, $1 \leq k \leq n$, such that $\|x - x_k\| < \epsilon/2$. Therefore $\sum_{i=N+1}^\infty |z_i| \leq \sum_{i=N+1}^\infty |z_i - z_i^{(k)}| +$

$\sum_{i=N+1}^{\infty} |z_i^{(k)}| \leq \epsilon$. But for ℓ^1 this is exactly the formulation of β-equicontinuity given in Theorem 3.2 (c).

COROLLARY 3.5. The space (ℓ^∞, β) is a strong Mackey space.

Before proceeding we must prove a well known fact concerning uniform boundedness in M(S) (actually what we show can be shown for barrelled spaces [25]). This lemma will be used not only for the next theorem but also throughout Chapter IV.

LEMMA 3.6. If $H \subset M(S)$ is weak * bounded then H is uniformly bounded.

Proof. Let $B = \{\phi \in C_o(S) : \left|\int \phi d\mu\right| \leq 1$ for all μ in H$\}$. Then B is a weakly closed convex circled subset of $C_o(S)$. Since H is weak * bounded B is absorbing; i.e., $\phi \in C_o(S)$ implies there is a scalar $a > 0$ such that $a\phi \in B$. Hence there is a constant $r > 0$ such that $\|\phi\|_\infty \leq r$ implies $\phi \in B$. Therefore if $\phi \in C_o(S)$ such that $\|\phi\|_\infty \leq 1$ then $r\phi \in B$ and so $\left|\int r\phi d\mu\right| \leq 1$ for all $\mu \in H$. This implies that $\|\mu\| \leq 1/r$ for all $\mu \in H$ and the proof is complete.

We are now in a position to prove one of our main theorems (almost all of Chapter IV will follow from this theorem).

THEOREM 3.7. Let S be paracompact; if $H \subset M(S)$ is β-weak * countably compact then H is β-equicontinuous. Consequently $C(S)_\beta$ is a strong Mackey space.

Proof. Since H is β -weak * countably compact it is weak * bounded and hence uniformly bounded by the preceding lemma. For the remainder of the proof we will suppose that S is σ -compact, and afterwards we will indicate the proof for the general case. Let $S = \bigcup_{n=1}^{\infty} D_n$ where D_n is compact and $D_n \subset$ int D_{n+1}, and suppose H is not β -equicontinuous. Since H is uniformly bounded there exists, by Theorem 3.2, an $\epsilon > 0$ such that for each compact set $K \subset S$, $|\mu|(S \setminus K) > \epsilon$ for some μ in H. We claim that there is a sequence $\{(\mu_n, \phi_n, K_n, U_n)\}_{n=1}^{\infty}$ having the following properties:

(a) $\mu_n \in H$, $\phi_n \in C_c(S)$, K_n is compact, U_n is open in S with U_n^- compact and $U_n^- \cap K_n = \emptyset$;

(b) $D_n \cup K_n \cup U_n^- \subset$ int K_{n+1};

(c) $|\mu_n(U_n)| > \epsilon/4$;

(d) $\|\phi_n\|_\infty = 1$, $\text{spt}(\phi_n) \subset U_n$, and $|\mu_n|(U_n) < \left|\int \phi_n d\mu_n\right| + \epsilon/8$.

To see this let $K_1 = D_1$; then there is a $\mu_1 \in H$ such that $|\mu_1|(S \setminus K_1) > \epsilon$. But $|\mu_1|(S \setminus K_1) \leq 4 \sup\{|\mu_1(C)| : C \text{ is a compact subset of } S \setminus K_1\}$ (Theorem 1.9). Hence there exists a compact set $C \subset S \setminus K_1$ with $|\mu_1(C)| > \epsilon/4$. Choose an open set U_1 such that U_1^- is compact, $C \subset U_1 \subset U_1^- \subset S \setminus K_1$, and $|\mu_1|(U_1 \setminus C) < \frac{1}{2}[|\mu_1(C)| - \frac{\epsilon}{4}]$. From this we have $|\mu_1(U_1)| > \epsilon/4$. By Corollary 1.13 there is a function $\phi_1 \in C_c(S)$ with $\text{spt}(\phi_1)$

$\subset U_1$, $\|\phi_1\|_\infty = 1$, and $|\mu_1|(U_1) < \left|\int \phi_1 d\mu_1\right| + \epsilon/8$. Since $K_1 \cup \bar{U}_1$ is compact we can find a compact set K_2 such that $K_1 \cup \bar{U}_1 \subset \text{int } K_2$. This completes the first step in the argument necessary to obtain the sequence. The rest of the induction is similar.

Since $S = \bigcup_{n=1}^{\infty} D_n$ and $D_n \subset \text{int } K_{n+1}$ we have that $S = \bigcup_{n=1}^{\infty} \text{int } K_n$.

Claim 1: $F = \bigcup_{n=1}^{\infty} \text{spt}(\phi_n)$ is closed. In fact if $s \in \bar{F}$ then $s \in \text{int } K_n$ for some $n \geq 1$. Hence, for every open neighborhood W of s such that $W \subset \text{int } K_n$ we have $\emptyset \neq W \cap F = W \cap \bigcup_{i=1}^{n-1} \text{spt}(\phi_i)$. Thus $s \in \bigcup_{i=1}^{n-1} \text{spt}(\phi_i) \subset F$ and F is closed.

Claim 2: If $x = \{x^{(n)}\}_{n=1}^{\infty} \in \ell^\infty$ then $f_x(s) = \sum_{n=1}^{\infty} x^{(n)} \phi_n(s)$ is a well defined bounded continuous function on S and $\|f_x\|_\infty = \|x\|_\infty$. In fact it is clearly well defined since at most one term in the sum is not zero. Also, for this same reason, $|f_x(s)| = \sum_{n=1}^{\infty} |x^{(n)}||\phi_n(s)|$ for all $s \in S$, and, since each $|\phi_n|$ achieves its maximum, we have $\|f_x\|_\infty = \|x\|_\infty$. To see that f_x is continuous let $s \in S$ and let $\{s_i\}$ be a net in S such that $s_i \to s$. If $s \notin F$ then F is closed implies that there is an i_o such that for $i \geq i_o$, $s_i \notin F$. Hence for $i \geq i_o$ $f_x(s) = f_x(s_i) = 0$ and thus $f_x(s_i) \to f_x(s)$. If $s \in F$ then $s \in \text{spt}(\phi_n)$

$\subset U_n$ for a unique integer n. Therefore there is an i_o such that for $i \geq i_o$, $s_i \in U_n$. But then, if $i \geq i_o$, $f_x(s_i) = x^{(n)}\phi_n(s_i) \longrightarrow x^{(n)}\phi_n(s) = f_x(s)$ and so $f_x \in C(S)$.

Now define the map $T: \ell^\infty \longrightarrow C(S)$ by $T(x) = f_x$. Then T is an isometry and

Claim 3: $T : (\ell^\infty, \beta) \longrightarrow C(S)_\beta$ is continuous. Let $\{x_i\}$ be a net in ℓ^∞ such that $x_i \longrightarrow 0$ (β). If $\phi \in C_o(S)$ and $\epsilon > 0$ then there is an integer N such that for $x \notin K_N$, $|\phi(s)| < \epsilon$. Thus, for $n \geq N$ $spt(\phi_n) \cap K_N = \square$ and hence $\|\phi\phi_n\|_\infty \leq \epsilon$; i.e., the sequence $x = \{\|\phi\phi_n\|_\infty\}_{n=1}^\infty$ is an element of c_o. Since $x_i \longrightarrow 0$ (β) there is for every $\epsilon > 0$ an i_o such that for $i \geq i_o$, $\|xx_i\|_\infty \leq \epsilon$. If $x_i = \{x_i^{(n)}\}_{n=1}^\infty$ for all i then this says that for $i \geq i_o$ $\epsilon > \sup\{|x_i^{(n)}||\phi(s)\phi_n(s)| : s \in S \text{ and } n \geq 1\} = \sup\{\sum_{n=1}^\infty |x_i^{(n)}||\phi(s)\phi_n(s)| : s \in S\} = \|\phi T(x_i)\|_\infty$. Hence $T(x_i) \longrightarrow 0$ (β) in C(S) and so T is continuous.

Therefore T has a well defined adjoint $T^* : M(S) \longrightarrow \ell^1$ which is continuous when both range and domain have their β-weak * topologies. Thus, $T^*(H)$ is β-weak * countably compact in ℓ^1 and, by Theorems 3.4 and 1.15, $T^*(H)$ is β-equicontinuous in ℓ^1. Now if $\mu \in M(S)$ and $x \in \ell^\infty$ then $\langle x, T^*(\mu)\rangle = \int T(x)d\mu = \int f_x d\mu = \sum_{n=1}^\infty x^{(n)} \int \phi_n d\mu$, so that $T^*(\mu) = \{\int \phi_n d\mu\}_{n=1}^\infty$. Interpreting our β-equicontinuity condition for $T^*(H) \subset \ell^1$ we have that

there is an integer N such that $\sum_{n=N+1}^{\infty} \left| \int \phi_n d\mu \right| < \epsilon/8$ for all $\mu \in H$. In particular, if $n > N$ $\left| \int \phi_n d\mu_n \right| < \epsilon/8$ and so $|\mu_n(U_n)| \leq |\mu_n|(U_n) < \left| \int \phi_n d\mu_n \right| + \epsilon/8 \leq \epsilon/4$ by condition (d). But this contradicts condition (c) on our sequence and so the proof is complete.

Note that the reason for condition (b) on the sequence was to ensure that $\bigcup_{n=1}^{\infty} K_n$ was both open and closed; from this it followed that F was closed and f_x was continuous. This same method yields a proof if S is a topological group, and condition (b) is replaced by the requirement that each K_n be a symmetric neighborhood of the identity and $K_n^2 \cup U_n^- \subset \text{int } K_{n+1}$. Then $\bigcup_{n=1}^{\infty} K_n$ is an open and closed subgroup. Both of these proofs may be subsumed in the proof of the case when S is paracompact. By Theorem 1.1 $S = \bigcup \{S_a : a \in A\}$ where the S_a are pairwise disjoint open and closed σ-compact subsets of S. Let $S_a = \bigcup_{n=1}^{\infty} C(n,a)$ where each $C(n,a)$ is compact and $C(n,a) \subset \text{int } C(n+1,a)$ for $n \geq 1$ and $a \in A$. By an induction process similar to that used in the above proof, we obtain a sequence of integers $\{k_n\}_{n=1}^{\infty}$ such that $k_{n+1} > k_n$, and a sequence $\{a_n\}_{n=1}^{\infty}$ in A as well as the sequence $\{(\mu_n, \phi_n, U_n, K_n)\}_{n=1}^{\infty}$. This sequence of quadruples has all the properties it had in the proof of the theorem except

that condition (b) is replaced by $K_n = \bigcup_{i=1}^{k_n} C(k_n, a_i)$, and $U_n^- \subset$ int K_{n+1}. We now proceed as above and $\bigcup_{n=1}^{\infty} K_n = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{k_n} C(k_n, a_i) = \bigcup_{n=1}^{\infty} S_{a_n}$ since $k_{n+1} > k_n$. Therefore $\bigcup_{n=1}^{\infty} K_n$ is both open and closed.

REMARKS. There is a class of spaces for which the preceding method of proof cannot work - namely the pseudo-compact, non-compact spaces. For in such a space no matter how the K_n are chosen the set F will never be closed (see [16]).

In addition, there are spaces for which Theorem 3.7 is not true, as the following example (of a pseudocompact space) illustrates.

THEOREM 3.8. If Ω_o is the space of ordinal numbers less than the first uncountable with the order topology, then $C(\Omega_o)_\beta$ is not a Mackey space (and a fortiori not a strong Mackey space).

Proof. Let H be the β-weak * closed convex circled hull of the set of all measures of the form $\frac{1}{2}[\delta_{(s)} - \delta_{(s+1)}]$, where s is a non-limit ordinal and s + 1 is its immediate successor. Besides the properties of Ω_o in Theorem 1.2 we will need the fact that if $s \in \Omega_o$ then the characteristic function of [1,s] is continuous.

Let Ω be the first uncountable ordinal and Ω_1 the Stone-Cech compactification of Ω_o. Hence $\Omega_1 = \Omega_o \cup \{\Omega\}$ and $M(\Omega_1) = M(\Omega_o) \oplus C\{\delta_{(\Omega)}\}$, where

$\mathbb{C}\{\delta_{(\Omega)}\} = \{c\delta_{(\Omega)} : c \in \mathbb{C}\}$. We claim that if we consider H as a subset of $M(\Omega_1)$ then H is weak * closed. If this is so then, since H is clearly contained in the unit ball of $M(\Omega_1)$, we would have that H is weak * compact[1]; but the weak * topology of $M(\Omega_1)$ relativized to $M(\Omega_0)$ is exactly the β-weak * topology, and so H is β-weak * compact.

To prove the claim, suppose that μ is in the weak * closure of H in $M(\Omega_1)$. Then there is a unique measure $\nu \in M(\Omega_0)$ and a scalar $c \in \mathbb{C}$ such that $\mu = \nu + c\delta_{(\Omega)}$. We must show that $c = 0$. If $\sigma = \frac{1}{2}[\delta_{(s)} - \delta_{(s+1)}]$ then $\sigma(\Omega_1) = \int d\sigma = 0$. Therefore $\sigma(\Omega_1) = 0$ for all $\sigma \in H$ and so $\mu(\Omega_1) = 0$. Thus $c = -\nu(\Omega_0)$. Since Ω_0 is not σ-compact and ν vanishes off a σ-compact subset of Ω_0, there exists a limit ordinal x such that ν vanishes off [1,x]. Let f = the characteristic function of [1,x]. If s is any non-limit ordinal then either $s < x$ or $x < s$. If $s < x$ then $s + 1 < x$ and $f(s) = f(s + 1) = 1$. If $x < s$ then $f(s) = f(s+1) = 0$. Hence $\int f d\sigma = 0$ for all $\sigma \in H$ and therefore $0 = \int f d\mu = \int f d\nu + cf(\Omega) = \int f d\nu = \nu([1,x]) = \nu(\Omega_0)$. Thus $c = 0$ and $\mu = \nu \in M(\Omega_0)$. But the weak * topology on

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[1]This is because the unit ball of the adjoint of a Banach space is always weak * compact. This theorem is well known and will be used often without specific reference. See [14,p.424].

$M(\Omega_1)$ relativized to $M(\Omega_o)$ is the β-weak * topology. Since μ is in the weak * closure of H in $M(\Omega_1)$ and $\mu \in M(\Omega_o)$ we have that μ is in the β-weak * closure of H. But H is β-weak * closed in $M(\Omega_o)$ and so $\mu \in H$. Thus H is weak * closed in $M(\Omega_1)$ and so is β-weak * compact in $M(\Omega_o)$.

However H is not β-equicontinuous. In fact, if it were there would be a σ-compact set of the type $N(\emptyset)$ such that each measure in H vanishes off $N(\emptyset)$. From the definition of H it is clear that this cannot be.

Section 2. Subspaces of $C(S)_\beta$ and the spaces (ℓ^∞, β) and (H^∞, β)

A natural question to ask is whether or not subspaces of $C(S)_\beta$ are Mackey spaces if they have the relative topology and if $C(S)_\beta$ is a Mackey space. Along these lines it is known that the completion of a Mackey space is a Mackey space; the converse, however, is false. In fact $C(S)_\beta$ is the completion of $C_o(S)_\beta$; but $C_o(S)_\beta$ is never a Mackey space (unless S is compact), since the norm topology on $C_o(S)$ is stronger than the strict topology and yields the same adjoint M(S).

The difficulties encountered in attacking our problem may be visualized as follows. Let E be a subspace of C(S) and $i : E_\beta \to C(S)_\beta$ the injection map, with $i^* : M(S) \to E_\beta^*$ its adjoint. In order to show that a subset $H \subset E_\beta^*$ is β-equicontinuous it is necessary and sufficient to

show that there is a β-equicontinuous subset $H_1 \subset M(S)$ such that $i^*H_1 = H$. Therefore if $C(S)_\beta$ is a Mackey space and $H \subset E_\beta{}^*$ is a β-weak * compact convex circled set, then to show that H is β-equicontinuous we need to find a β-weak * compact convex circled set $H_1 \subset M(S)$ such that $i^*H_1 = H$. Since $E_\beta{}^*$ with its β-weak * topology is topologically isomorphic to a quotient space of M(S), it would seem that what is needed is a version of a theorem of Bartle and Graves (see [5] or [27,p.375]) where both the domain and range have their β-weak * topologies. Unfortunately, no such theorem is available in general, although in the special case of ℓ^∞ one can use this theorem to great advantage (see Theorem 3.10 below).

Let E be a β-closed subspace of ℓ^∞ and recall that we proved that (ℓ^∞, β) is a strong Mackey space by using Schur's theorem. It is not difficult to prove Schur's theorem if we assume that (ℓ^∞, β) is a strong Mackey space; hence the two theorems are equivalent. A statement similar to Schur's theorem turns out to be exactly what is needed to characterize those closed subspaces of (ℓ^∞, β) which are Mackey spaces.

LEMMA 3.9. Let E be a β-closed subspace of ℓ^∞. Then E_β is semi-reflexive and $E_\beta{}^*$ is a Banach space when furnished with its strong topology. Consequently, the β-weak * topology on $E_\beta{}^*$ is its weak topology which it has as a Banach space.

Proof. Since (ℓ^∞, β) is semi-reflexive (Lemma 3.3) and E is a closed subspace of (ℓ^∞, β) we have that E is semi-reflexive by Theorem 1.11. By the same theorem we have that $E_\beta{}^*$ with its strong topology is topologically isomorphic to ℓ^1/N with its quotient norm, where $N = E^\perp \subset \ell^1$. Hence $E_\beta{}^*$ is a Banach space.

THEOREM 3.10. If E is closed in (ℓ^∞, β) then a subset of $E_\beta{}^*$ is β-equicontinuous if and only if it is norm conditionally compact.

Proof. Let $H \subset E_\beta{}^*$ be β-equicontinuous and let $i : E_\beta \longrightarrow (\ell^\infty, \beta)$ be the injection map. Then there is a $\phi \in c_o$ such that $i^* V_\phi^\circ \supset H$, where $i^* : \ell^1 \longrightarrow E_\beta{}^* = \ell^1/N$ is the adjoint map of i. But V_ϕ° is β-equicontinuous and norm closed in ℓ^1. Therefore V_ϕ° is norm compact in ℓ^1 (Theorem 3.4), and, since i^* is norm continuous, H has norm compact closure.

Assume that H is norm compact. Then $E_\beta{}^*$ is a Banach space and i^* is a map of ℓ^1 onto $E_\beta{}^*$. By the Bartle-Graves selection theorem [27,p.375] there is a continuous function $f : E_\beta{}^* \longrightarrow \ell^1$ such that $f(I) \in i^{*-1}(I)$ for all $I \in E_\beta{}^*$. Hence f(H) is norm compact in ℓ^1 and thus β-equicontinuous. Therefore $i^*(f(H)) = H$ implies H is β-equicontinuous.

THEOREM 3.11. If E is a β-closed subspace of ℓ^∞ then the following are equivalent:

(a) E_β is a Mackey space;

(b) E_β is a strong Mackey space;

(c) every β-weak * compact subset of E_β^* is norm compact;

(d) every β-weak * convergent sequence in E_β^* is norm convergent.

Proof. (a) is equivalent to (b). This is immediate from Theorem 1.16 since the β-weak * and the weak topologies on E_β^* are the same.

(b) implies (c). If $H \subset E_\beta^*$ is β-weak * compact then H is β-equicontinuous by (b). Since it is norm closed the preceding theorem implies H is norm compact.

(c) implies (d). A β-weak * convergent sequence with its limit point is β-weak * compact and hence norm compact by (c). From this it is easy to see that the sequence converges in norm.

(d) implies (b). If $H \subset E_\beta^*$ is β-weak * compact then it is β-weak * sequentially compact by Theorem 1.15. Thus (d) implies it is norm compact, and hence β-equicontinuous in virtue of the preceding theorem. This concludes the proof of the theorem.

Now let us turn our attention to H^∞, the space of bounded analytic functions on the open unit disk D. For general information on H^∞ see [23]. Theorems 3.12 and 3.13 below were obtained quite recently by Shields and Rubel ([31] and [32]). We have decided to present them here because we obtain them by different methods, and

because they form a direct path to our result that H^∞ with its strict topology is not a Mackey space (it should be pointed out that in the process we show there exists a closed subspace of (ℓ^∞, β) which is not a Mackey space).

We follow a method of Brown, Shields, and Zeller [9] and get a sequence $\{a_n\}_{n=1}^\infty$ in D which has no limit points in D and such that for all $f \in H^\infty$, $\|f\|_\infty = \sup\{|f(a_n)| : n \geq 1\}$. Hence if E is the subspace of ℓ^∞ consisting of all sequences $\{f(a_n)\}_{n=1}^\infty$ where $f \in H^\infty$ then the map $T : H^\infty \longrightarrow E$, defined by $T(f) = \{f(a_n)\}$, is a linear isometry. Moreover, since $\{a_n\}$ has no limit points in D it is a discrete sequence. Thus, if $\{f_i\}$ is a net in H^∞ such that $f_i \longrightarrow 0$ (β) and $x = \{x_n\}_{n=1}^\infty \in c_o$, then there is a function $\phi \in C_o(D)$ such that $\phi(a_n) = x_n$ for $n \geq 1$ (Theorem 1.5). If $\epsilon > 0$ then there is an i_o such that for $i \geq i_o$ $\|\phi f_i\|_\infty \leq \epsilon$. But for $i \geq i_o$ $\sup\{|x_n f_i(a_n)| : n \geq 1\} = \sup\{|\phi(a_n) f_i(a_n)| : n \geq 1\} \leq \|\phi f_i\|_\infty \leq \epsilon$ and so $T(f_i) \longrightarrow 0$ (β) in E. Therefore T is a continuous map from (H^∞, β) onto E_β (a fact which is crucial in our development).

THEOREM 3.12. A subset of (H^∞, β) is β-compact if and only if it is β-closed and bounded.

Proof. Let $A \subset H^\infty$ be β-closed and bounded. Since the β and c-op topologies agree on bounded sets in C(D), and since the c-op topology is metric on H^∞ [12, p.98], we need only show that every sequence in A has a c-op convergent subsequence. But $\{f_n\}_{n=1}^\infty \subset A$ and A uniformly

bounded implies $\{f_n\}$ is an equicontinuous family [1,p.171]. Therefore there is a subsequence of $\{f_n\}$ which converges to $f \in A$ in the c-op topology.

THEOREM 3.13. If I is a linear functional on H^∞ which is β-continuous on the unit ball then I is β-continuous on H^∞. Hence $(H^\infty, \beta)^*$ with its strong topology is a Banach space and H^∞ is its adjoint.

Proof. Consider the map $T : (H^\infty, \beta) \rightarrow E_\beta \subset \ell^\infty$ defined prior to Theorem 3.12. Since T is continuous and ball H^∞ is β-compact, T (ball H^∞) is β-compact in ℓ^∞ and hence $\sigma(\ell^\infty, \ell^1)$ compact. Therefore $E \cap$ ball ℓ^∞ = T (ball H^∞) is $\sigma(\ell^\infty, \ell^1)$ closed and thus, by Theorem 1.14, E is $\sigma(\ell^\infty, \ell^1)$ closed. But β is stronger than the $\sigma(\ell^\infty, \ell^1)$ topology and hence E is a β-closed subspace of ℓ^∞. Also ball H^∞ is β-compact implies that the restriction of T to ball H^∞ is a homeomorphism. Hence $I \circ T^{-1}$ is a linear functional on E which is β-continuous on ball E. By Lemma 3.9 $E_\beta{}^*$ is strongly complete and so it follows from Grothendieck's completeness theorem (Theorem 1.10) that $I \circ T^{-1} \in E_\beta{}^*$. Therefore $I = T^*(I \circ T^{-1}) \in (H^\infty, \beta)^*$ and $(H^\infty, \beta)^*$ is strongly complete (also by Theorem 1.10). Since it is clearly a normed space we have that $(H^\infty, \beta)^*$ with its strong topology is a Banach space. Finally, ball H^∞ β-compact implies that every bounded set in (H^∞, β) is weakly conditionally compact, and hence (H^∞, β) is semi-reflexive. Therefore $H^\infty = (H^\infty, \beta)^{**}$.

COROLLARY 3.14. A linear functional I on H^∞ is β-continuous if and only if there is a Lebesgue integrable function g on $[-\pi,\pi]$ such that $I(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})g(\theta)d\theta$ for all $f \in H^\infty$.

Proof. Let $g \in L^1(-\pi,\pi)$ and B = ball H^∞. If $\{f_n\}$ is a sequence in B and $f_n \longrightarrow f$ (β) then identify each f_n and f with its boundary values on the unit circle [23]. Therefore each f_n and f determines an element of $L^\infty(-\pi,\pi)$ and $L^\infty(-\pi,\pi) = L^\infty$ is the adjoint of the separable Banach space $L^1(-\pi,\pi) = L^1$. Hence the $\sigma(L^\infty,L^1)$ topology on ball L^∞ is metrizable, and ball L^∞ is compact implies there is an $h \in L^\infty$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \longrightarrow h$ $\sigma(L^\infty,L^1)$. Therefore $\hat{h}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)e^{-in\theta}d\theta = \lim \hat{f}_{n_k}(n)$, for $n = 0,\pm1,\ldots$.[2] Hence $\hat{h}(n) = 0$ for $n < 0$, and for $n \geq 0$ $\hat{f}_{n_k}(n) = \frac{1}{n!} f_{n_k}^{(n)}(0)$. But $f_{n_k} \longrightarrow f$ (β) implies $f_{n_k} \longrightarrow f$ (c-op) and so $\hat{f}(n) = \frac{1}{n!} f^{(n)}(0) = \lim \frac{1}{n!} f_{n_k}^{(n)}(0)$. Therefore $\hat{h}(n) = \hat{f}(n)$ for all n, and so $f(e^{i\theta}) = h(\theta)$ for almost all θ (see [29,p.17]). What we have shown is that every $\sigma(L^\infty,L^1)$ convergent subsequence of $\{f_n\}$ converges to f. Since ball L^∞ is $\sigma(L^\infty,L^1)$ compact we have that $f_n \longrightarrow f$ $\sigma(L^\infty,L^1)$. Therefore

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[2] Of course $\hat{h}$ denotes the Fourier-Stieltjes transform of h [29].

$I(f_n) \longrightarrow I(f)$ and I is β-continuous on ball H^∞. By the previous theorem, $I \in (H^\infty, \beta)^*$.

For the converse let $u : H^\infty \longrightarrow L^\infty$ be the canonical imbedding. What we have shown is that u is continuous if H^∞ has its weak topology from $(H^\infty, \beta)^*$ and L^∞ has its $\sigma(L^\infty, L^1)$ topology. If $E = u(H^\infty)$ and $I \in (H^\infty, \beta)^*$ then, by an argument similar to that used in the theorem, we have that $I \circ u^{-1}$ is a $\sigma(L^\infty, L^1)$ continuous linear functional on E. By the Hahn-Banach theorem we may extend $I \circ u^{-1}$ to a $\sigma(L^\infty, L^1)$ continuous linear functional on L^∞. Thus there is a $g \in L^1(-\pi, \pi)$ such that $I \circ u^{-1}(h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta)g(\theta)d\theta$ for all $h \in E$. Therefore if $f \in H^\infty$,

$$I(f) = I \circ u^{-1}(u(f)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})g(\theta)d\theta.$$

THEOREM 3.15. A subset of $(H^\infty, \beta)^*$ is β-equicontinuous if and only if it is norm conditionally compact.

Proof. If A is a norm compact subset of $(H^\infty, \beta)^*$ and $T : (H^\infty, \beta) \longrightarrow E_\beta$ is the map described prior to Theorem 3.12 then $T^* : E_\beta{}^* \longrightarrow (H^\infty, \beta)^*$ is an isometry onto $(H^\infty, \beta)^*$. Therefore $T^{*-1}(A)$ is norm compact in $E_\beta{}^*$ and hence, by Theorem 3.10, $T^{*-1}(A)$ is β-equicontinuous. This implies that there exists a $\phi \in c_0$ such that $T^{*-1}(A) \subset V_\phi{}^\circ$. But T is continuous and so $T^{-1}(V_\phi)$ is a β-neighborhood of zero in H^∞. It is

routine to show that $A \subset [T^{-1}(V_\emptyset)]^\circ$ and hence A is β-equicontinuous.

Now suppose that A is β-equicontinuous and $\{I_n\}_{n=1}^{\infty}$ is a sequence in A. Hence A is β-weak * conditionally compact and, since the β-weak * and weak topologies on $(H^\infty, \beta)^*$ are the same, A is weakly sequentially compact by Theorem 1.15. Therefore some subsequence of $\{I_n\}$ converges weakly (= β-weak *) to an element I of $(H^\infty, \beta)^*$. Assume that $I_n \longrightarrow I$ β-weak *. Since ball H^∞ is β-compact, there is for each integer n a function $f_n \in$ ball H^∞ such that $\|I - I_n\| = (I - I_n)(f_n)$. Then there is an $f \in$ ball H^∞ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \longrightarrow f$ (β). Since $\{I - I_n\}_{n=1}^{\infty}$ is β-equicontinuous, $f_{n_k} \longrightarrow f$ uniformly on $\{I - I_n\}$. Therefore if $\epsilon > 0$ there is an integer N_1 such that for $n_k \geq N_1$, $|(I - I_n)(f - f_{n_k})| \leq \frac{\epsilon}{2}$ for all $n \geq 1$. Also there is an integer $N_2 \geq N_1$ such that for $n \geq N_2$ $|I(f) - I_n(f)| < \frac{\epsilon}{2}$. Hence for $n_k \geq N_2$, $\|I - I_{n_k}\| = (I - I_{n_k})(f_{n_k}) \leq |(I - I_{n_k})(f - f_{n_k})| + |(I - I_{n_k})(f)| \leq \epsilon$. That is $I_{n_k} \longrightarrow I$ in norm and A is norm conditionally compact.

COROLLARY 3.16. If $\{I_n\}$, I are in $(H^\infty, \beta)^*$ then $I_n \longrightarrow I$ in norm if and only if (a) $I_n \longrightarrow I$ β-weak * and (b) $\{I_n\}_{n=1}^{\infty}$ is β-equicontinuous.

THEOREM 3.17. (H^∞, β) is not a Mackey space.

Proof. Since $(H^\infty, \beta)^*$ is a Banach space with adjoint space H^∞, it is sufficient to show that (H^∞, β) is not a strong Mackey space (see Theorem 1.16). For each integer $n \geq 1$ define $I_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta = \hat{f}(n)$, for all $f \in H^\infty$. Then $I_n \in (H^\infty, \beta)^*$ by Corollary 3.14 and $I_n \longrightarrow 0$ (β-weak $*$)[3]. Clearly $\|I_n\| \leq 1$ and since for $f(z) = z^n \in H^\infty$ $I_n(f) = 1$, we have that $\|I_n\| = 1$ for all $n \geq 1$. Hence $\{I_n\}$ cannot approach zero in norm. By the preceding corollary $\{I_n\}$ is not β-equicontinuous. This concludes the proof.

REMARK. Theorem 3.17 answers a question posed by Shields and Rubel [32].

Let us close this chapter with some remarks on the strict topology. There are no examples known to the author of topological vector spaces which are Mackey spaces, except by virtue of some other formally stronger property (e.g. barrelled, metric, etc.). However this is not true of $C(S)_\beta$. We have already seen that $C(S)_\beta$ is metric if and only if S is compact. This same statement can be made with respect to ''barrelled'' and ''bornological.'' In fact, the unit ball is β-closed and absorbs bounded sets, but it is a β-neighborhood of zero if and only if

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[3] Again, $\hat{f}$ is the Fourier-Stieltjes transform of f. It is a well known fact that $\lim \hat{f}(n) = 0$ [29].

S is compact. Another interesting point is that the strict topology gives an example of a semi-reflexive Mackey space (ℓ^∞, β) which has a closed subspace which is not a Mackey space - the image E of H^∞ under the mapping discussed before Theorem 3.12.

CHAPTER IV

COMPACTNESS AND SEQUENTIAL CONVERGENCE IN M(S)

This chapter will be devoted to an exploitation of our Theorem 3.7, the principal result of Chapter III. After a number of preliminary lemmas we present some results relating β-weak * compactness and sequential compactness with β-equicontinuity. We then begin a discussion of a paper by J. Dieudonné [13]. In this paper he gives necessary and sufficient conditions for several modes of sequential convergence in M(S) for S compact. Our treatment of these results will fall into three categories. First, by making use of the concept of β-equicontinuity, we will generalize the results of Dieudonné to locally compact spaces (note that β-equicontinuity may be intuitively regarded as saying that the underlying space S is ''approximately compact'' relative to the measures involved). For the most part these generalizations are connected with the sufficiency of the conditions of his theorems. It can be seen that the necessity arguments used all have a noticeable similarity with one another. The second facet of our development will be to replace these arguments by a judicious application of Theorem 3.7. Finally we will strengthen some of these results, and show how a theorem of A. Grothendieck on weak compactness in

$M(S)$ can be obtained.

Before proceeding we will need the following preliminary lemmas.

LEMMA 4.1. If K_1 is a compact subset of S and A is a countable subset of $M(S)$ then there exists a compact G_δ set $K_2 \supset K_1$ such that $|\mu|(K_2 \setminus K_1) = 0$ for all $\mu \in A$. Furthermore, if S is compact then $S \setminus K_2$ is a regularly σ-compact set.

Proof. Let $A = \{\mu_n\}_{n=1}^{\infty}$ and $\nu = \sum_{n=1}^{\infty} (\frac{1}{2})^n \frac{1}{\|\mu_n\|} |\mu_n|$. Then $\nu \in M(S)$ and each $|\mu_n|$ is absolutely continuous with respect to ν. We will produce a sequence $\{U_n\}$ of open sets such that $K_1 \subset U_{n+1} \subset \bar{U}_{n+1} \subset U_n$ for $n \geq 1$, $\nu(U_n \setminus K_1) \leq \frac{1}{n}$, and $\bar{U}_n$ is compact. By regularity, there is an open set U_1 with compact closure such that $K_1 \subset U_1$ and $\nu(U_1 \setminus K_1) \leq 1$. Hence, suppose that the sets $U_1, \ldots, U_n$ are constructed having the desired properties. Again, by outer regularity there is an open set $O \supset K_1$ such that $\nu(O \setminus K_1) \leq \frac{1}{n+1}$. But K_1 is compact and disjoint from the closed set $S \setminus (O \cap U_n)$. Therefore there is an open set $U_{n+1} \supset K_1$ such that $\bar{U}_{n+1} \subset O \cap U_n \subset U_n$. Thus $\bar{U}_{n+1}$ is compact and $\nu(U_{n+1} \setminus K_1) \leq \nu(O \setminus K_1) \leq \frac{1}{n+1}$.

Let $K_2 = \bigcap_{n=1}^{\infty} U_n$; then $K_1 \subset K_2$ and $K_2 \subset \bar{U}_1$ implies that $\bar{K}_2$ is compact. But $S \setminus K_2 = \bigcup_{n=1}^{\infty} (S \setminus U_n)$ and $\bar{U}_{n+1} \subset U_n$ implies that $S \setminus U_n \subset S \setminus \bar{U}_{n+1} \subset S \setminus U_{n+1}$. Hence $S \setminus U_n \subset$

int $(S \setminus U_{n+1})$ and so $S \setminus K_2$ is open. Thus, $K_2 = K_2^-$ is compact. Finally, $\nu(K_2 \setminus K_1) \leq \nu(U_n \setminus K_1) \leq \frac{1}{n}$ for all $n \geq 1$ implies $\nu(K_2 \setminus K_1) = 0$. Since each $|\mu_n|$ is absolutely continuous with respect to ν we have $|\mu_n|(K_2 \setminus K_1) = 0$ for all $n \geq 1$.

If S is compact then $S \setminus U_n$ is compact and so $S \setminus K_2$ is (by construction) a regularly σ-compact set. This completes the proof.

If $B \in$ Borel (S) and $\{\mu_n\}$ is a sequence in M(S) then we will say that B is a $\{\mu_n\}$-quarrable set if and only if $|\mu_n|(B^- \setminus \text{int } B) = 0$ for all $n \geq 1$. We will also say that B is quarrable for $\{\mu_n\}$.

LEMMA 4.2. ([13,p.277]) If $\{\mu_n\} \subset M(S)$ then every point $s \in S$ has a fundamental neighborhood system consisting of open $\{\mu_n\}$-quarrable sets.

Proof. As in the above proof we need only consider the lemma for a single positive measure μ. If V is an open neighborhood of t then let $f \in C(S)$ be such that $0 \leq f \leq 1$, $f(t) = 1$, and $f(S \setminus V) = 0$. We will show that there are at most countably many real numbers r, $0 \leq r \leq 1$, with $\mu(\{s : f(s) = r\}) \neq 0$. In fact, if this is not the case and there is an uncountable collection of such r, then for some integer $n > 1$ there are uncountably many such $r \geq \frac{1}{n}$. Applying the same type of reasoning, we obtain an integer $m > 1$ and a sequence $\{r_k\}$ of real numbers such that $r_k \geq \frac{1}{n}$ and $\mu(\{s : f(s) = r_k\}) \geq \frac{1}{m}$ for all $k \geq 1$.

Therefore $\int f d\mu \geq \sum_{k=1}^{\infty} \int_{f^{-1}(r_k)} f d\mu \geq \sum_{k=1}^{\infty} \frac{1}{m} \cdot r_k = \infty$, which is a contradiction. Hence, we may find an r, $0 < r < 1$, such that $(\{s : f(s) = r\}) = 0$. Let $U = \{s : f(s) > r\}$; then $t \in U \subset V$ and $\mu(U^- \setminus U) \leq \mu(\{s : f(s) = r\}) = 0$.

Lemma 4.3. If K is a compact subset of S and $A \subset M(S)$ is countable then there is an open set $W \supset K$ which is A-quarrable and such that W^- is compact.

Proof. Using Lemma 4.2 and the compactness of K, we may find a finite collection $\{U_1, \cdots, U_m\}$ of open A-quarrable sets with compact closures such that $K \subset \bigcup_{k=1}^{m} U_k$. Let $W = \bigcup_{k=1}^{m} U_k$; then W has the properties stated in the lemma.

LEMMA 4.4.[4] ([13,p.279]) If $\{\mu_n\}, \mu$ are uniformly bounded in $M(S)$ and $\mu_n(U) \to \mu(U)$ for every open $\{\mu_n\}$-quarrable set U then $\mu_n \to \mu$ (β-weak *).

Proof. We may assume $\|\mu_n\| \leq 1$ for $n \geq 1$. Clearly it is only necessary to show that $\int f d\mu_n \to \int f d\mu$ for all real valued functions in $C(S)$; so let $\epsilon > 0$ and $f \in C(S)$ be real valued. Choose a finite sequence $r_1, \cdots, r_m$ of real numbers such that $r_1 \leq -\|f\|_\infty$, $\|f\|_\infty \leq r_m$, and $0 < r_{k+1} - r_k < \frac{\epsilon}{4}$ for $1 \leq k \leq m-1$. If $\nu = |\mu| + \sum_{n=1}^{\infty} (\frac{1}{2})^n |\mu_n|$ then $\nu \in M(S)$ and by an argument like that used in Lemma 4.2 there are at most a countable number of r's such that $\nu(\{s : f(s) = r\}) \neq 0$. Hence (possibly

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[4] See also Alexandroff [2,p.182].

after a small adjustment of the original r_k) we may choose the r_k such that $\nu(\{s : f(x) = r_k\}) = 0$, for $1 \leq k \leq m$. Let $U_k = \{s : r_k < f(s) < r_{k+1}\}$ for $1 \leq k \leq m-1$, $A_1 = \{s : r_1 \leq f(s) \leq r_2\}$, and $A_k = \{s : r_k < f(s) \leq r_{k+1}\}$ for $2 \leq k \leq m-1$.

If $g = \sum_{k=1}^{m-1} r_k \chi_{U_k}$, where χ_{U_k} is the characteristic function of U_k, then $\left|\int f d\mu - \int f d\mu_n\right| \leq \left|\int (f-g) d(\mu - \mu_n)\right| + \left|\int g d(\mu - \mu_n)\right|$. But $\left|\int (f-g) d(\mu - \mu_n)\right| \leq \sum_{k=1}^{m-1} \left|\int_{A_k} (f-g) d(\mu - \mu_n)\right| \leq \sum_{k=1}^{m-1} \left|\int_{U_k} (f-r_k) d(\mu - \mu_n)\right|$, this last inequality holding since $|\mu - \mu_n|(A_k \setminus U_k) \leq |\mu|(A_k \setminus U_k) + |\mu_n|(A_k \setminus U_k) = 0$ for all $n \geq 1$ and $1 \leq k \leq m-1$. But for $s \in U_k$ $|f(s) - r_k| < \epsilon/4$ so that $\left|\int (f-g) d(\mu - \mu_n)\right| \leq \sum_{k=1}^{m-1} \frac{\epsilon}{4} |\mu - \mu_n|(U_k) \leq \frac{\epsilon}{4} \|\mu - \mu_n\| \leq \epsilon/2$ for all $n \geq 1$. Also U_k is $\{\mu_n\}$-quarrable for $1 \leq k \leq m-1$ implies that there is an integer N such that for $n \geq N$ $\left|\int g d\mu_n - \int g d\mu\right| \leq \epsilon/2$. Hence for $n \geq N$, $\left|\int f d\mu - \int f d\mu_n\right| \leq \epsilon/2 + \epsilon/2 = \epsilon$, completing the proof.

Many of the results that follow will make use of β-equicontinuity as well as uniform boundedness. The following theorem gives a sufficient condition for β-equicontinuity; for a result on uniform boundedness see Lemma 3.6.

THEOREM 4.5. If H is a β-weak * compact subset of positive measures in M(S) then H is β-equicontinuous.

Proof. Let $\{K_i\}$ be the collection of all compact subsets of S. For each i let $\phi_i \in C_c(S)$ be such that $0 \leq \phi_i \leq 1$ and $\phi_i(K_i) = 1$. If $f_i = 1 - \phi_i$ then $f_i \in C(S)$, $0 \leq f_i \leq 1$, and $f_i(K_i) = 0$. Let $\epsilon > 0$; if $\mu \in M(S)$ then by inner regularity there is a K_i such that $|\mu|(S \setminus K_i) < \epsilon$. Therefore $\left|\int f_i d\mu\right| = \left|\int_{S \setminus K_i} f_i d\mu\right| \leq |\mu|(S \setminus K_i) < \epsilon$ and so the sets $\{\mu : \left|\int f_i d\mu\right| < \epsilon\}$ form a β-weak * open cover of H. Since H is β-weak * compact we can find $K_1, \cdots, K_m$ such that $H \subset \bigcup_{i=1}^{m} \{\mu : \left|\int f_i d\mu\right| < \epsilon\}$. Let $C_i = \text{spt}(\phi_i)$ and $K = \bigcup_{i=1}^{m} C_i$. Then $\mu \in H$ implies that $\left|\int f_i d\mu\right| < \epsilon$ for some i, $1 \leq i \leq m$. Hence $|\mu|(S \setminus K) = \mu(S \setminus K) \leq \mu(S) - \mu(C_i)$; but $\phi_i \leq \chi_{C_i}$ and so $\mu(C_i) \geq \int \phi_i d\mu$. Combining these facts we have that $|\mu|(S \setminus K) \leq \mu(S) - \int \phi_i d\mu = \int (1-\phi_i) d\mu = \int f_i d\mu < \epsilon$ and H is β-equicontinuous. This concludes the proof.

It is always true that β-equicontinuity implies conditional β-weak * compactness. However it is often the case that sequential compactness is more useful than compactness. For this reason we prove the following interesting result.

THEOREM 4.6. If S is locally metrizable and $H \subset M(S)$ is β-equicontinuous then H is β-weak * sequentially compact.

Proof. Recall that S is locally metrizable if and

only if every point has a metrizable neighborhood. Since S is locally compact this is equivalent to the requirement that every compact set be metrizable. Now H is β - equicontinuous implies we may find a sequence $\{K_n\}$ of compact sets such that $K_n \subset K_{n+1}$ and $|\mu|(S \setminus K_n) \leq (\frac{1}{2})^n$ for all $\mu \in H$. Let $\{\mu_j\}_{j=1}^{\infty}$ be a sequence in H; we may suppose $\|\mu_j\| \leq 1$ for all $j \geq 1$. Since S is locally metrizable K_n is a compact metric space and $C(K_n)$ is separable for all $n \geq 1$ ([14,p.501]). Therefore the weak * topology on the unit ball of $M(K_n)$ is metrizable ([14, p.426]). Consider $\{\mu_j^{(1)}\}_{j=1}^{\infty}$, where $\mu_j^{(1)}$ is the element of $M(K_1)$ which is the restriction of μ_j to K_1 ($\mu_j^{(1)}(A) = \mu_j(A)$ for all $A \in$ Borel (K_1); see Chapter I). Then this sequence is in ball $M(K_1)$ so there is a measure $\nu_1 \in M(K_1)$, $\|\nu_1\| \leq 1$, and a subsequence $\{\mu_{1j}\}_{j=1}^{\infty}$ of $\{\mu_j\}$ such that $\mu_{1j}^{(1)} \longrightarrow \nu_1$ (weak *) in $M(K_1)$. By an induction argument we obtain for each integer $i \geq 2$ a subsequence $\{\mu_{ij}\}_{j=1}^{\infty}$ of $\{\mu_{i-1\,j}\}_{j=1}^{\infty}$ and a measure $\nu_i \in M(K_i)$ with $\|\nu_i\| \leq 1$, such that $\mu_{ij}^{(i)} \longrightarrow \nu_i$ (weak *) in $M(K_i)$.

Let $i \geq 1$ and $f \in C(K_{i+1})$ be such that $\|f\|_\infty \leq 1$. Let $\overline{\nu_i}$ be the extension of ν_i to K_{i+1} (i.e., $\overline{\nu_i}(A) = \nu_i(A \cap K_i)$ for all $A \in$ Borel (K_{i+1})). Then

$$\left|\int_{K_{i+1}} fd(\nu_{i+1} - \overline{\nu_i})\right| = \left|\lim_{j\to\infty} \int_{K_{i+1}} fd\mu_{i+1\,j}^{(i+1)} - \lim_{j\to\infty} \int_{K_i} fd\mu_{i\,j}^{(i)}\right| = \lim_{j\to\infty} \left|\int_{K_{i+1}} fd\mu_{i+1\,j}^{(i+1)} - \int_{K_i} fd\mu_{i+1\,j}^{(i)}\right|.$$

But $\left|\int_{K_{i+1}} fd\mu_{i+1\,j}^{(i+1)} - \int_{K_i} fd\mu_{i+1\,j}^{(i)}\right| \leq$

$|\mu_{i+1j}|(K_{i+1} \setminus K_i) \leq (\frac{1}{2})^i$ for all $j \geq 1$ and $i \geq 1$. Therefore $\|\nu_{i+1} - \overline{\nu_i}\| \leq (\frac{1}{2})^i$ for all $i \geq 1$. If we regard each ν_i as a measure in M(S) by considering its extension to S then $\|\nu_{i+1} - \nu_i\| \leq (\frac{1}{2})^i$ for all $i \geq 1$. Thus $\mu = \nu_1 + \sum_{i=1}^{\infty}(\nu_{i+1} - \nu_i)$ is a measure in M(S). We will show that the diagonal sequence $\{\mu_{jj}\}_{j=1}^{\infty}$ converges to μ β-weak *. Let $f \in C(S)$ be such that $\|f\|_\infty \leq 1$. If $\epsilon > 0$ choose $n \geq 1$ with $(\frac{1}{2})^n < \epsilon/3$; then $\left|\int fd\mu_{jj} - \int fd\mu\right| \leq$

$$\left|\int fd\mu_{jj} - \int fd\nu_1 - \sum_{i=1}^{n}\int fd(\nu_{i+1} - \nu_i)\right| + \sum_{i=n+1}^{\infty}\left|\int fd(\nu_{i+1} - \nu_i)\right| \leq \left|\int_{K_{n+1}} fd\mu_{jj} - \int_{K_{n+1}} fd\nu_{n+1}\right|$$

$$+ \left|\int_{S\setminus K_{n+1}} fd\mu_{jj}\right| + \sum_{i=n+1}^{\infty}(\tfrac{1}{2})^i \leq \left|\int_{K_{n+1}} fd(\mu_{jj} - \nu_{n+1})\right|$$

$$+ (\tfrac{1}{2})^{n+1} + (\tfrac{1}{2})^n \leq \left|\int_{K_{n+1}} fd(\mu_{jj} - \nu_{n+1})\right| + \epsilon/2.$$

But $\{\mu_{jj}^{(n+1)}\}_{j=1}^{\infty}$ is a subsequence of $\{\mu_{n+1\,j}^{(n+1)}\}_{j=1}^{\infty}$ except possibly for the first n terms. Hence there is a j_o such that for $j \geq j_o$, $\left|\int_{K_{n+1}} fd(\mu_{jj} - \nu_{n+1})\right| < \frac{\epsilon}{2}$.

Therefore if $j \geq j_o$ we have $\left|\int f d\mu_{jj} - \int f d\mu\right| < \epsilon$ which completes the proof.

COROLLARY 4.7. If S is a metric space and $H \subset M(S)$ then the following are equivalent:

(a) H is β -equicontinuous;

(b) H is β -weak * sequentially compact;

(c) H is β -weak * countably compact;

(d) H is β -weak * conditionally compact.

Proof. Notice that we always have that (a) implies (d) implies (c) and (b) implies (c). Since S is metrizable it is paracompact and hence, by Theorem 3.7, (c) implies (a). By the preceding theorem we have that (a) implies (b).

COROLLARY 4.8. If S is a metric space and $\{f_n\}$, f are in C(S), then $f_n \longrightarrow f(\beta)$ if and only if $f_n \longrightarrow f$ uniformly on β -weak * convergent sequences in M(S).

Proof. Suppose $f_n \longrightarrow f$ (β) and let $\{\mu_n\}_{n=1}^{\infty}$ be a β-weak * convergent sequence in M(S). By the preceding corollary we have that $\{\mu_n\}$ is a β -equicontinuous set and hence $f_n \longrightarrow f$ uniformly on $\{\mu_n\}$.

For the converse let $\mathcal{T}$ be the topology on C(S) of uniform convergence on β -weak * convergent sequences in M(S). Suppose that $f_n \longrightarrow f(\mathcal{T})$. If there is a β -equicontinuous set $H \subset M(S)$ such that $\{f_n\}$ does not converge to f uniformly on H, then there is an $\epsilon > 0$ and an increasing sequence of integers $\{n_k\}$ with

$\sup\{\left|\int f_{n_k} d\mu - \int f d\mu\right| : \mu \in H\} > \epsilon$ for all $k \geq 1$. Hence for each $k \geq 1$ there is a measure $\mu_k \in H$ with $\left|\int (f_{n_k} - f) d\mu_k\right| > \epsilon$. But S is metrizable implies there is a subsequence $\{\mu_{k_i}\}_{i=1}^{\infty}$ of $\{\mu_k\}$ and a $\mu \in M(S)$ such that $\mu_{k_i} \longrightarrow \mu$ β-weak *. But $f_n \longrightarrow f$ ($\mathfrak{T}$) implies there is an integer N such that for $n \geq N$ $\left|\int f_n d\mu_{k_i} - \int f d\mu_{k_i}\right|$ $< \epsilon$ for all $i \geq 1$. Clearly this gives us a contradiction. Hence $f_n \longrightarrow f$ uniformly on every β -equicontinuous set; that is, $f_n \longrightarrow f$ (β). This concludes the proof.

We will now turn our attention to the results of Dieudonné [13] and Grothendieck [19]. If $\{\mu_n\}_{n=1}^{\infty}$ is a sequence in M(S) and f is a complex valued function on S then f is $\{\mu_n\}$-continuous if and only if f is bounded and and the set of discontinuities of f has $|\mu_n|$-measure zero for all $n \geq 1$. We will say that a sequence $\{\mu_n\}$ in M(S) converges R-weak * (''R'' stands for Riemann) if and only if $\int f d\mu_n \longrightarrow \int f d\mu$ for every $\{\mu_n\}$-continuous function f. Even though there is no R-weak * topology, we give the obvious meaning to the terms R-weak * countably compact and R-weak * sequentially compact.[5]

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[5] In what follows there may arise some confusion as to the originality of the results. We will endeavor to settle such questions with bibliographical footnotes. If no footnote or reference is given then, to the best of our knowledge, the result is our own.

THEOREM 4.9. If $H \subset M(S)$ is R-weak * countably compact then H is β -equicontinuous.

Proof. Let H be a R-weak * countably compact subset of M(S). By Lemma 3.6 H is uniformly bounded. Therefore if H is not β -equicontinuous there is an $\epsilon > 0$ such that for every compact set $K \subset S$, $|\mu|(S \setminus K) > \epsilon$ for some $\mu \in H$. We will show by induction that there is a sequence $\{(\mu_n, K_n)\}_{n=1}^{\infty}$ where $\mu_n \in H$ and K_n is a compact subset of S contained in int K_{n+1} such that: (a) $|\mu_n|(S \setminus K_n) > \epsilon$ for all $n \geq 1$, and (b) $|\mu_k|(S \setminus K_{n+1}) \leq \frac{1}{n}$ for $1 \leq k \leq n$ and $n \geq 1$. To accomplish this let K_1 be an arbitrary compact set and $\mu_1 \in H$ with $|\mu_1|(S \setminus K_1) > \epsilon$. If (μ_1, K_1), $\cdots, (\mu_n, K_n)$ are chosen then by inner regularity there is a compact set K_{n+1} such that $K_n \subset$ int K_{n+1} and $|\mu_k|(S \setminus K_{n+1}) \leq \frac{1}{n}$ for $1 \leq k \leq n$. Since H is not β - equicontinuous there is a $\mu_{n+1} \in H$ with $|\mu_{n+1}|(S \setminus K_{n+1}) > \epsilon$. This completes the induction.

Let $S_1 = \bigcup_{n=1}^{\infty} K_n$; then S_1 is an open σ-compact set. If $A \in$ Borel (S) with $A \cap S_1 = \emptyset$ then $A \cap K_n = \emptyset$ for all $n \geq 1$. Thus, if $k \geq 1$ then $|\mu_k|(A) \leq \frac{1}{n}$ for all $n \geq k$ which implies that $|\mu_k|(A) = 0$ for all $k \geq 1$. Therefore each μ_k vanishes off S_1. If $\nu \in M(S)$ is an R-weak * cluster point of some subsequence $\{\nu_n\}$ of $\{\mu_n\}$, and K is a compact set such that $K \cap S_1 = \emptyset$ then $f = \chi_K$ is $\{\nu_n\}$-continuous. Thus $\int f d\nu_n \xrightarrow{cl} \int f d\nu$. But

$\int f d\nu_n = \nu_n(K) = 0$ for all $n \geq 1$ implies that $\nu(K) = 0$, and so ν is also concentrated on S_1.

Let $H_1 = \{\mu_n\}$, considered as a subset of $M(S_1)$. If $\{\nu_n\}$ is a sequence in H_1 then there is a $\nu \in M(S)$ such that $\nu_n \xrightarrow{cl} \nu$ R-weak *. From the preceding paragraph we have that ν is concentrated on S_1, and so ν may be considered as an element of $M(S_1)$. If $f \in C(S_1)$ then let $\bar{f}$ be the extension of f to S such that $\bar{f}(s) = 0$ for $s \notin S_1$. Then $\bar{f}$ is $\{\nu_n\}$-continuous and so $\int_{S_1} f d\nu_n = \int \bar{f} d\nu_n \xrightarrow{cl} \int \bar{f} d\nu = \int f d\nu$. Hence H_1 is a β-weak * countably compact subset of $M(S_1)$. Since S_1 is σ-compact we have that H_1 is β-equicontinuous by Theorem 3.7. Therefore there exists a compact set $K \subset S_1$ such that $|\mu_n|(S_1 \setminus K) \leq \epsilon$ for all $n \geq 1$. But $S_1 = \bigcup_{n=1}^{\infty} \text{int } K_n$ implies there is an integer n such that $K \subset K_n$. From here it follows that $|\mu_n|(S \setminus K_n) = |\mu_n|(S_1 \setminus K_n) \leq |\mu_n|(S_1 \setminus K) \leq \epsilon$, and we have a contradiction to the choice of the μ_n and K_n. This concludes the proof.

REMARKS. Because of the fact that there is no R-weak * topology there are several pathologies which can occur. For example, if $\{\mu_n\}$ is an R-weak * convergent sequence then it is not *a priori* true that every subsequence of $\{\mu_n\}$ is R-weak * convergent. However, by a proof similar to the preceding one it can be shown that every R-weak * convergent sequence in $M(S)$ is β-

equicontinuous. This fact allows us to prove the following result.

THEOREM 4.10.[6] If $\{\mu_n\}$, μ are elements of M(S), then the following are euqivalent:

(a) $\mu_n \longrightarrow \mu$ R-weak *;

(b) i) $\{\mu_n\}$ is β-equicontinuous and ii) $\mu_n(U) \longrightarrow \mu(U)$ for every $\{\mu_n\}$-quarrable open set U;

(c) i) $\{\mu_n\}$ is β-equicontinuous, ii) $\mu_n \longrightarrow \mu$ β-weak *, and iii) for every $\epsilon > 0$ and every compact set $K \subset S$ such that $|\mu_n|(K) = 0$ for all $n \geq 1$ there is an open set $V \supset K$ such that $|\mu_n|(V) \leq \epsilon$ for all $n \geq 1$.

Proof. (a) implies (b). If $\mu_n \longrightarrow \mu$ R-weak * then $\{\mu_n\}$ is β-equicontinuous by the remarks preceding this theorem. If U is an open $\{\mu_n\}$-quarrable set and f the characteristic function of U then f is $\{\mu_n\}$-continuous. Hence $\mu_n(U) \longrightarrow \mu(U)$.

(b) implies (c). By Lemma 4.4 we have (ii). To see (iii) let $\epsilon > 0$ and K be such a compact set. By Lemma 4.1 we may suppose that K is a compact G_δ set. Since $\{\mu_n\}$ is β-equicontinuous there is a compact set $D \subset S$ such that $|\mu_n|(S \setminus D) \leq \frac{\epsilon}{2}$ for all n. Applying Lemma 4.3 to the set $D \cup K$ with $A = \{\mu, \mu_n : n \geq 1\}$ we obtain an open set $W \supset D \cup K$ such that W^- is compact, $|\mu_n|(S \setminus W^-) \leq \epsilon/2$ for all $n \geq 1$, and $|\mu|(W^- \setminus W) = |\mu_n|(W^- \setminus W) = 0$ for $n \geq 1$.

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[6] The equivalence of (a) and (c) for S compact is due to Dieudonné [13,p.29]; the equivalence of (a) and (b) is our own and strengthens a result of Dieudonné [13,p.279].

From Lemma 4.1 we have that $S_1 = W^- \setminus K$ is a regularly σ-compact subset of W^-. Let ν_n and ν be the elements of $M(S_1)$ which are the restrictions of μ_n and μ, respectively to S_1. If U is an open subset of S_1 which is $\{\nu_n\}$-quarrable then $U \cap W$ is open in S. If C = the closure of U in S_1 and U^- is the closure of U in S then $U^- \setminus U = (C \setminus U) \cup (K \cap U^-)$; in fact this follows quite readily since $U^- \subset W^- = S_1 \cup K$. Thus $|\mu_n|(U^- \setminus U) = |\mu_n|(C \setminus U) + |\mu_n|(K \cap U^-) = 0$, since U is $\{\nu_n\}$-quarrable in S_1. Also since $K \subset W$, $S_1 = (W^- \setminus W) \cup (W \setminus K)$ and $|\mu_n|(U \cap W) \leq |\mu_n|((U \cap W)^-) \leq |\mu_n|(U^- \cap W^-) = |\mu_n|((U^- \setminus U) \cap W^-) + |\mu_n|(U \cap (W^- \setminus W)) + |\mu_n|(U \cap W) = |\mu_n|(U \cap W)$. Hence $U \cap W$ is an open $\{\mu_n\}$-quarrable subset of S. Also, if $\lambda = \mu_n$ or μ for some $n \geq 1$ then $|\lambda|(U) = |\lambda|(U \cap W) + |\lambda|(U \cap (W^- \setminus W)) = |\lambda|(U \cap W)$ and so $\lambda(U) = \lambda(U \cap W)$. Therefore $\nu_n(U) = \mu_n(U) = \mu_n(U \cap W) \longrightarrow \mu(U \cap W) = \mu(U) = \nu(U)$, so that $\nu_n(U) \longrightarrow \nu(U)$ for every open $\{\nu_n\}$-quarrable subset of S_1. From Lemma 4.4 we have that $\nu_n \longrightarrow \nu$ (β-weak $*$) in $M(S_1)$. Since S_1 is σ-compact our main theorem (3.7) implies that $\{\nu_n\}$ is β-equicontinuous in $M(S_1)$. There is, then, a compact set $C \subset S_1$ such that $|\nu_n|(S_1 \setminus C) \leq \epsilon/2$ for all n. Let $V = S \setminus C$; then V is open and $K \subset V$. Also $V = S_1 \setminus C \cup S \setminus W^- \cup K$ and thus $|\mu_n|(V) = |\nu_n|(S_1 \setminus C) + |\mu_n|(S \setminus W^-) + |\mu_n|(K) \leq \epsilon/2 + \epsilon/2 + 0 = \epsilon$.

(c) implies (a). Assume $\|\mu_n\| \leq 1$ and let f be a $\{\mu_n\}$-continuous function such that $\|f\|_\infty \leq 1$. If $\epsilon > 0$

then, since $\{\mu_n\}$ is β-equicontinuous, there is a compact set $K_1 \subset S$ such that $|\mu - \mu_n|(S \setminus K_1) \leq \epsilon/8$ for all $n \geq 1$. From Chapter I we know that $K = \{s : \text{osc}(f,s) \geq \epsilon/8\} \cap K_1$ is compact; but K is a subset of the discontinuities of f and so $|\mu_n|(K) = 0$ for all $n \geq 1$. By (iii) we may find an open set $V \supset K$ such that $|\mu_n|(V) \leq \epsilon/16$ for all $n \geq 1$. Let $\phi \in C_c(S)$ such that $\text{spt}(\phi) \subset V$ and $\|\phi\|_\infty \leq 1$. Then $\int \phi d\mu_n \longrightarrow \int \phi d\mu$ by (ii) and $\left|\int \phi d\mu_n\right| = \left|\int_V \phi d\mu_n\right| \leq |\mu_n|(V) \leq \epsilon/16$ for all $n \geq 1$. Thus $\left|\int \phi d\mu\right| \leq \epsilon/16$ and so, by Corollary 1.13, $|\mu|(V) \leq \epsilon/16$. Therefore $|\mu - \mu_n|(V) \leq |\mu|(V) + |\mu_n|(V) \leq \epsilon/8$.

For every point $s \in K_1 \setminus V$, $\text{osc}(f,s) < \epsilon/8$. Since $K_1 \setminus V$ is compact there are points $s_1, \cdots, s_m \in K_1 \setminus V$ and open sets $U_1, \cdots, U_m$ such that $s_k \in U_k$, $\left|f(s) - f(s_k)\right| < \frac{\epsilon}{8}$ for $s \in U_k$, and $K_1 \setminus V \subset \bigcup_{k=1}^{m} U_k$. By Theorem 1.6 there exist functions $\phi_1, \cdots, \phi_m \in C_c(S)$ such that $0 \leq \phi_k \leq 1$, $\sum_{k=1}^{m} \phi_k(s) \leq 1$ for all $s \in S$ with $\sum_{k=1}^{m} \phi_k(s) = 1$ for $s \in K_1 \setminus V$, and $\phi_k(s) = 0$ for $s \notin U_k$, $1 \leq k \leq m$. Let $g(s) = \sum_{k=1}^{m} f(s_k)\phi_k(s)$; then g is in C(S). If $s \in K_1 \setminus V$ then $\left|g(s) - f(s)\right| = \left|\sum_{k=1}^{m} \phi_k(s)[f(s_k) - f(s)]\right|$ and this is a convex combination of complex numbers in the disk of radius $\epsilon/8$. Hence $\left|g(s) - f(s)\right| \leq \epsilon/8$ for $s \in K_1 \setminus V$. If $s \in S$ then $\left|g(s) - f(s)\right| \leq 2$.

By (ii) $\int g d\mu_n \longrightarrow \int g d\mu$ and so there is an integer N such that $\left|\int g d\mu_n - \int g d\mu\right| < \epsilon/4$ for $n \geq N$. Therefore

if $n \geq N$, $\left|\int f d\mu - \int f d\mu_n\right| \leq \left|\int g d(\mu - \mu_n)\right| +$
$\left|\int (f-g)d(\mu-\mu_n)\right| \leq \frac{\epsilon}{4} + \left|\int_{K_1 \setminus V} (f-g)d(\mu-\mu_n)\right| +$
$\left|\int_{K_1 \cap V} (f-g)d(\mu-\mu_n)\right| + \left|\int_{S \setminus K_1} (f-g)d(\mu-\mu_n)\right| \leq$
$\frac{\epsilon}{4} + \frac{\epsilon}{8}\|\mu - \mu_n\| + 2|\mu - \mu_n|(V) + 2|\mu - \mu_n|(S \setminus K_1)$
$\leq 4 \cdot \frac{\epsilon}{4} = \epsilon$. Therefore $\mu_n \longrightarrow \mu$ R-weak * completing the proof.

The following is a well known result (see [14,p.308]) which we present here for the sake of completeness.

LEMMA 4.11. Let $\{\mu_n\}$, μ be elements of M(S). Then $\mu_n \longrightarrow \mu$ weakly if and only if $\{\mu_n\}$ is uniformly bounded and $\mu_n(A) \longrightarrow \mu(A)$ for every Borel set A.

Proof. Clearly the conditions are necessary; to prove sufficiency we can assume $\|\mu_n\| \leq 1$ for all $n \geq 1$, $\mu = 0$, and $\mu_n(A) \longrightarrow 0$ for all A ϵ Borel (S). Let $\nu = \sum_{n=1}^{\infty} (\frac{1}{2})^n |\mu_n|$; then $\nu \in M(S)$ and $|\mu_n|$ is absolutely continuous with respect to ν for $n = 1,2,\cdots$. By the Radon Nikodym theorem there is, for each $n \geq 1$, a function $f_n \in L^1(\nu)$ such that $\mu_n(A) = \int_A f_n d\nu$ for all A ϵ Borel (S). Now $L^1(\nu)$ can be isometrically and isomorphically identified with a closed subspace of M(S), and so $\mu_n \longrightarrow 0$ weakly if and only if $f_n \longrightarrow 0$ weakly in $L^1(\nu)$.

Now I is in $L^1(\nu)^*$ if and only if there is a bounded Borel function g on S such that $I(f) = \int fg d\nu$ for all $f \in L^1(\nu)$. It is sufficient to suppose that g is real

valued. Thus, if $\epsilon > 0$ let $a_1, \cdots, a_m$ be real numbers such that $a_1 = -\|g\|_\infty$, $a_m = \|g\|_\infty$, and $0 < a_{k+1} - a_k < \frac{\epsilon}{2}$ for $1 \leq k \leq m-1$. Let $A_1 = \{s : a_1 \leq g(s) \leq a_2\}$ and $A_k = \{s : a_k < g(s) \leq a_{k+1}\}$ for $2 \leq k \leq m-1$. Put $h(s) = \sum_{k=1}^{m-1} a_k \chi_{A_k}(s)$. Clearly $S = \bigcup_{k=1}^{m-1} A_k$ and so if $s \in S$ there is a unique k, $1 \leq k \leq m-1$, such that $s \in A_k$. Therefore $|g(s) - h(s)| = |g(s) - a_k| < \epsilon/2$ and so $\|g - h\|_\infty < \epsilon/2$. But $A_k \in$ Borel (S) for $1 \leq k \leq m-1$ and so by hypothesis $\int h f_n d\nu = \int h d\mu_n \longrightarrow 0$. Choose an integer N such that for $n \geq N$ $\left|\int h f_n d\nu\right| \leq \epsilon/2$; therefore if $n \geq N$ $\left|\int f_n g d\nu\right| \leq \left|\int f_n (g-h) d\nu\right| + \left|\int f_n h d\nu\right| \leq \|g-h\|_\infty \|\nu\| + \epsilon/2 \leq \epsilon$.

THEOREM 4.12.[7] If $\{\mu_n\}$, μ are in $M(S)$ then the following are equivalent:

(a) $\mu_n \longrightarrow \mu$ weakly in $M(S)$;

(b) $\{\mu_n\}$ is uniformly bounded and $\mu_n(U) \longrightarrow \mu(U)$ for every open set U;

(c) $\int f d\mu_n \longrightarrow \int f d\mu$ for every bounded l.s.c. function f;

(d) (i) $\{\mu_n\}$ is β-equicontinuous, (ii) $\mu_n \longrightarrow \mu$ β-weak $*$, and (iii) for every $\epsilon > 0$ and every compact set $K \subset S$ there is an open set $V \supset K$ such that $|\mu_n|(V \setminus K) \leq \epsilon$ for all $n \geq 1$.

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[7] It is known that (b) implies (c). Dieudonné [13,p.32] showed that (c) and (d) are equivalent for S compact, and that (a) and (b) are equivalent if S is a compact metric space [13,p.35]. Later Grothendieck [19,p.150] showed that (a) and (b) are equivalent for an arbitrary space S.

Proof. Clearly (a) implies (b). The pattern of the remainder of the proof will be:

(b) implies (c) implies (d) implies (b) implies (a).

(b) implies (c). If F is closed in S and $U = S \setminus F$ then $\mu_n(F) = \mu_n(S) - \mu_n(U) \longrightarrow \mu(S) - \mu(U) = \mu(F)$. If F is any closed set and U is any open set then $\mu_n(F \cup U) = \mu_n(F \setminus U) + \mu_m(U) \longrightarrow \mu(F \setminus U) + \mu(U) = \mu(F \cup U)$. Similarly $F \cap U = (F \cup U) \setminus [(F \setminus U) \cup (U \setminus F)]$ implies that $\mu_n(F \cap U) \longrightarrow \mu(F \cap U)$.

Let f be a real valued bounded l.s.c. function. Then $\{s : f(s) > a\}$ is open and $\{s : f(s) \leq a\}$ is closed for all real numbers a. If $a < b$ and $B = \{s : a < f(s) \leq b\} = \{s : a < f(x)\} \cap \{s : f(s) \leq b\}$ then $\mu_n(B) \longrightarrow \mu(B)$. If $\epsilon > 0$ then choose real numbers $a_1, \cdots, a_m$ such that $a_1 < -\|f\|_\infty$, $\|f\|_\infty < a_m$, and $0 < a_{k+1} - a_k < \epsilon/4$ for $1 \leq k \leq m-1$. Put $A_k = \{s : a_k < f(s) \leq a_{k+1}\}$ for $1 \leq k \leq m-1$ and $g(s) = \sum_{k=1}^{m-1} a_k \chi_{A_k}(s)$. Therefore $\int g d\mu_n \longrightarrow \int g d\mu$ and $\|g - f\|_\infty \leq \epsilon/4$ (see the proof of Lemma 4.11 for a similar argument). Assume $\|\mu_n\| \leq 1$ for all $n \geq 1$, and let N be an integer such that $\left|\int g d\mu_n - \int g d\mu\right| < \epsilon/2$ for $n \geq N$. Then $\left|\int f d\mu_n - \int f d\mu\right| \leq \left|\int (f-g) d(\mu - \mu_n)\right| + \left|\int g d\mu_n - \int g d\mu\right| \leq \frac{\epsilon}{4} \|\mu - \mu_n\| + \frac{\epsilon}{2} \leq \epsilon$ for all $n \geq N$; that is, $\int f d\mu_n \longrightarrow \int f d\mu$. Since every l.s.c. function is of the form $f_1 + i f_2$ where f_1, f_2 are real valued l.s.c. functions we have proved that (b) implies (c).

(c) implies (d). Clearly (c) implies $\mu_n(U) \longrightarrow \mu(U)$ for every open set since the characteristic function of an open set is l.s.c. In particular, this holds if U is an open $\{\mu_n\}$-quarrable set. Hence $\mu_n \longrightarrow \mu$ R-weak * (Theorem 4.10) and so (i) and (ii) hold. Assume that $\|\mu_n\| \leq 1$. Also, the proof of the general case will quickly follow from the case where $\mu = 0$ if we make use of the outer regularity of μ. Hence, let us also assume that $\mu = 0$.

To prove (iii) let $K \subset S$ be compact and $\epsilon > 0$. By Lemma 4.1 we may assume that K is a G_δ set. Combining the β-equicontinuity of $\{\mu_n\}$ with Lemma 4.3, we obtain an open $\{\mu_n\}$-quarrable set $W \subset S$ such that W^- is compact, $|\mu_n|(S \setminus W^-) \leq \epsilon/2$ for all $n \geq 1$, and $K \subset W$. By Lemma 4.1 $S_1 = W^- \setminus K$ is an open σ-compact subset of W^-. Let ν_n be the restriction of μ_n to S_1 (i.e., $\nu_n \in M(S_1)$), and let $f \in C(S_1)$ be a real valued function. If $\overline{f}$ is the extension of f to S such that $\overline{f}(s) = 0$ for $s \notin S_1$ then let $g(s) = \liminf_{t \to s} \overline{f}(t)$ for all $s \in S$. By Theorem 1.8 g is l.s.c., and since $W \setminus K$ is open and $\overline{f}$ is continuous here, $g(s) = f(s)$ for $s \in W \setminus K$. Also $S \setminus W^-$ is open implies $g(s) = 0$ for $s \notin W^-$. From (c) and the fact that $|\mu_n|(W^- \setminus W) = 0$ for $n \geq 1$ we have $\int_{S_1} f d\nu_n = \int_{W \setminus K} f d\mu_n = \int g d\mu_n \longrightarrow 0$. That is, $\nu_n \longrightarrow 0$ (β-weak *) in $M(S_1)$. Since S_1 is a σ-compact space our main theorem implies the existence of a compact set $C \subset S_1$ such that $|\nu_n|(S_1 \setminus C) \leq \epsilon/2$ for

$n \geq 1$. Let $V = S \setminus C$; then V is open, $K \subset V$, and $|\mu_n|(V \setminus K) = |\mu_n|(S \setminus W^-) + |\mu_n|(S_1 \setminus C) \leq \epsilon$.

(d) implies (b). Again, by the outer regularity of μ we may assume $\mu = 0$; also we can suppose that $\|\mu_n\| \leq 1$ for all $n \geq 1$. We will show that $\mu_n(F) \longrightarrow 0$ for all closed sets F. From this it follows that if U is open then $\mu_n(U) = \mu_n(S) - \mu_n(S \setminus U) \longrightarrow 0$.

Fix the closed set $F \subset S$ and $\epsilon > 0$. Since $\{\mu_n\}$ is β-equicontinuous there is a compact set $K \subset S$ such that $|\mu_n|(S \setminus K) \leq \epsilon/2$ for all $n \geq 1$. Therefore $F \cap K$ is compact and by (iii) there is an open set $V \supset F \cap K$ with $|\mu_n|(V \setminus (F \cap K)) \leq \epsilon/4$ for $n \geq 1$. Let $f \in C(S)$ be such that $0 \leq f \leq 1$, $f(F \cap K) = 1$, and $f(s) = 0$ for $s \notin V$. Then $|\mu_n(F)| \leq |\mu_n(F \cap K)| + |\mu_n(F \cap (S \setminus K))| \leq \left|\int \chi_{F \cap K} d\mu_n\right| + |\mu_n|(S \setminus K) \leq \left|\int (\chi_{F \cap K} - f) d\mu_n\right| + \left|\int f d\mu_n\right| + \epsilon/2$. But $f(s) = \chi_{F \cap K}(s) = 0$ for $s \notin V$ and $f(s) = \chi_{F \cap K}(s) = 1$ for $s \in F \cap K$. Thus $\left|\int (\chi_{F \cap K} - f) d\mu_n\right| = \left|\int_{V \setminus F \cap K} f d\mu_n\right| \leq \|f\|_\infty \cdot |\mu_n|(V \setminus (F \cap K)) \leq \epsilon/4$ for all $n \geq 1$. Also $\int f d\mu_n \longrightarrow 0$ by (ii) and so there is an integer N such that for $n \geq N$ $\left|\int f d\mu_n\right| \leq \epsilon/4$. Combining these facts we have that $|\mu_n(F)| \leq \epsilon$ for $n \geq N$.

(b) implies (a). Clearly we may again assume $\mu = 0$ and $\|\mu_n\| \leq 1$ for all n. Fix a Borel set A and an $\epsilon > 0$. There exists a sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of S such that $K_n \subset K_{n+1} \subset A$ and $|\mu_k|(A \setminus K_n) \leq \frac{1}{n}$ for $1 \leq k$

$\leq n$ and $n \geq 1$. In fact K_1 exists by inner regularity. If $K_1, \cdots, K_n$ exist then by regularity there is a compact set $C \subset A$ such that $|\mu_k|(A \setminus C) \leq \frac{1}{n+1}$ for $1 \leq k \leq n+1$. Let $K_{n+1} = K_n \cup \bigcup_{k=1}^{n+1} C_k$. This easily completes the induction and establishes the existence of the sequence $\{K_n\}$.

If $F = \bigcup_{n=1}^{\infty} K_n$ then $F \subset A$ and $|\mu_n|(A) = |\mu_n|(F)$ for all $n \geq 1$. We have already seen that (b) implies (d); therefore for each integer $n \geq 1$ there is an open set $V_n \supset K_n$ such that $|\mu_k|(V_n \setminus K_n) \leq \epsilon\,(\frac{1}{2})^{n+1}$ for $k = 1,2,\cdots$. Let $U = \bigcup_{n=1}^{\infty} V_n$; then U is open and $F \subset U$. Also $|\mu_k|(V_n \setminus F) = |\mu_k|(\bigcap_{i=1}^{\infty}(V_n \setminus K_i)) \leq |\mu_k|(V_n \setminus K_n) \leq \epsilon \cdot (\frac{1}{2})^{n+1}$ for $k \geq 1$. Thus $|\mu_k|(U \setminus F) = |\mu_k|(\bigcup_{n=1}^{\infty}(U_n \setminus F)) \leq \sum_{n=1}^{\infty} |\mu_k|(V_n \setminus F) \leq \epsilon \sum_{n=1}^{\infty} (\frac{1}{2})^{n+1} = \epsilon/2$ for all $k \geq 1$. But $\mu_n(U) \to 0$ and so there is an integer N such that for $n \geq N$ $|\mu_n(U)| \leq \epsilon/2$. Therefore if $n \geq N$, $|\mu_n(A)| = |\mu_n(F)| \leq |\mu_n(F) - \mu_n(U)| + |\mu_n(U)| \leq \epsilon$ and $\mu_n \to 0$ weakly by Lemma 4.11.

This completes the proof.

REMARKS. Suppose $\{\mu_n\}_{n=1}^{\infty}$, μ are elements of $M(S)$ such that (a) $\{\mu_n\}$ is β-equicontinuous, (b) $\{\mu_n\}$ clusters to μ β-weak *, and (c) for every compact set K and $\epsilon > 0$ there is an open set $V \supset K$ such that $|\mu_n|(V \setminus K) \leq \epsilon$ for all $n \geq 1$. Then, as in the proof above that (d) implies (b), we get that $\mu_n(U) \xrightarrow{cl} (U)$ for every open

set U. Also, as in the proof that (b) implies (a), we get that $\{\mu_n\}$ clusters to μ weakly.

Using these comments we obtain the following known (but rephrased in our terminology) result on weak compactness.

THEOREM 4.13. (Grothendieck [19,p.146]) If $H \subset M(S)$ then H is weakly conditionally compact if and only if (a) H is β-equicontinuous, and (b) for every compact set $K \subset S$ and $\epsilon > 0$ there is an open set $V \supset K$ such that $|\mu|(V \setminus K) \leq \epsilon$ for all $\mu \in H$.

Proof. Suppose H is weakly compact. Then H is R-weak * countably compact and hence β-equicontinuous by Theorem 4.9. If (b) does not hold then there is an $\epsilon > 0$ and a compact set K such that for every open set $V \supset K$ there is a $\mu \in H$ with $|\mu|(V \setminus K) > \epsilon$. From this we obtain a sequence $\{(\mu_n, V_n)\}_{n=1}^{\infty}$ where (i) $\mu_n \in H$, V_n is an open set containing K with $\bar{V}_n$ compact and contained in V_n for $n \geq 1$; (ii) $|\mu_k|(V_{n+1} \setminus K) \leq \frac{1}{n+1}$ for $1 \leq k \leq n$; and (iii) $|\mu_n|(V_n \setminus K) > \epsilon$ for all $n \geq 1$. The existence of this sequence is established by induction in a manner similar to that used in several previous proofs. Since H is weakly compact the Eberlein-Smulian theorem (1.15) implies that $\{\mu_n\}$ has a subsequence $\{\mu_{n_k}\}$ such that $\mu_{n_k} \longrightarrow \mu$ weakly for some $\mu \in M(S)$. Now if $K_1 = \bigcap_{n=1}^{\infty} V_n$ then K_1 is compact by (i) and $K \subset K_1$. By the preceding theorem there

is an open set $W \supset K_1$ such that $|\mu_{n_k}|(W \setminus K_1) \leq \epsilon$ for all n_k. But by (ii) $|\mu_n|(K_1 \setminus K) = 0$ for all $n \geq 1$ and so $|\mu_{n_k}|(W \setminus K) \leq \epsilon$ for all n_k. Also $K_1 = \bigcap_{n=1}^{\infty} V_n^- \subset W$ implies there is an integer N such that $V_N^- \subset W$. Hence if $n_k \geq N$, $K \subset V_{n_k} \subset W$ and $|\mu_{n_k}|(V_{n_k} \setminus K) \leq |\mu_{n_k}|(W \setminus K) \leq \epsilon$, contradicting (iii).

Suppose that H satisfies conditions (a) and (b); by Theorem 1.15 we need only show that H is weakly countably compact. Therefore let $\{\mu_n\}$ be a sequence in H. By (a) H is β-weak * countably compact and so there is a measure $\mu \in M(S)$ such that $\mu_n \xrightarrow{cl} \mu$ β-weak *. But from the remarks following Theorem 4.12 we have that $\mu_n \xrightarrow{cl} \mu$ weakly, and the proof is complete.

A few comments on the hypothesis of the theorems of this chapter may be in order. In the statements of many of our conditions we assume that a sequence $\{\mu_n\}$ converged β-weak * and also that $\{\mu_n\}$ was β-equicontinuous. If S is paracompact then the β-equicontinuity of $\{\mu_n\}$ is superfluous in virtue of Theorem 3.7. Furthermore it is an open question as to whether or not a β-weak * convergent sequence is β-equicontinuous (note that in $M(\Omega_o)$ every β-weak * convergent sequence is β-equicontinuous even though Theorem 3.7 does not hold for Ω_o).

CHAPTER V

VECTOR VALUED FUNCTIONS AND MEASURES

This chapter may be viewed as an appendix, in so far as it is concerned with generalizations of some of the preceding results. In particular we show how the strict topology can be introduced on the space $C(S,E)$ of bounded continuous vector valued functions (see [12] and [36]), and we will indicate that many of the properties of $C(S)_\beta$ hold also for $C(S,E)_\beta$. The principal endeavor of this chapter will be to show that the adjoint space of $C(S,E)_\beta$ is a certain space of vector valued measures. This result has already been obtained by Wells [36], but we will give a presentation based on the theory of topological tensor products as originated by Grothendieck [20]. Our approach will be to represent $C(S,E)_\beta$ as a certain topological tensor product, and apply a general theorem on adjoints of tensor products. After this we will state some results on extreme points similar to Theorems 2.14 and 2.15. Finally, we will conclude the chapter with some results of ours on weakly compact operators.

If E is a locally convex topological vector space, we will denote by $C(S,E)$ all those continuous functions f from S into E such that $f(S)$ is a bounded subset of E; i.e., for every continuous semi-norm (c.s.n.) p on E,

$\sup\{p(f(s)) : s \in S\} < \infty$. A function f in C(S,E) is in $C_o(S,E)$ if and only if f vanishes at infinity; i.e., if and only if for every $\epsilon > 0$ and each c.s.n. p on E $\{s : p(f(s)) \geq \epsilon\}$ is compact.

The <u>uniform topology</u> on C(S,E) is defined by the semi-norms $\|\cdot\|_p$, where p ranges over the continuous semi-norms on E and, $\|f\|_p = \sup\{p(f(s)) : s \in S\}$ for all $f \in C(S,E)$. Note that if $E = \mathbb{C}$ = the complexes then $C(S,E) = C(S)$ and the uniform topology on C(S,E) is the uniform topology on C(S). It is easy to see that C(S,E) is uniformly complete if and only if E is complete. Also, if $\hat{E}$ is the completion of E then the uniform completion of C(S,E) is $C(S,\hat{E})$. Hence we will always assume that E is complete. If p is a c.s.n. on E and $\phi \in C_o(S)$ then let $V_{p\phi} = \{f \in C(S,E) : \|f\phi\|_p \leq 1\}$. The <u>strict topology</u> on C(S,E) is the topology which has as a neighborhood subbasis for the origin all the sets of the form $V_{p\phi}$, where ϕ is in $C_o(S)$ and p is a c.s.n. on E. Hence a net $\{f_i\}$ in C(S,E) converges to zero strictly if and only if $\phi f_i \longrightarrow 0$ uniformly for all $\phi \in C_o(S)$.

The space $C(S,E)_\beta$ has many of the properties enjoyed by $C(S)_\beta$. In particular $C(S,E)_\beta$ is complete and $C_o(S,E)$ is β-dense in C(S,E). The proofs of these and other properties can be found in Buck [12], or they may be proved by rephrasing the analogous proofs in Chapter II.

Before proceeding we will present the fundamentals from the theory of tensor products. If E and F are vector

spaces and E', F' their algebraic duals then let B(E',F') be the space of bilinear functionals on E' $\times$ F'. If $x \in E$ and $y \in F$ we define the element $x \otimes y$ of B(E',F') by $x \otimes y(x',y') = \langle x,x'\rangle\langle y,y'\rangle$ for all $(x',y') \in E' \times F'$. The tensor product $E \otimes F$ of E and F is the linear span of $\{x \otimes y : x \in E, y \in F\}$ in B(E',F'). Hence, if $(x_i,y_i) \in E \times F$ for $1 \leq i \leq n$ then $b = \sum_{i=1}^{n} x_i \otimes y_i$ means $b(x',y') = \sum_{i=1}^{n} \langle x_i,x'\rangle\langle y_i,y'\rangle$ for all $(x',y') \in E' \times F'$.

LEMMA 5.1. If $b \in E \otimes F$ and $b \neq 0$ then there exist $(x_i,y_i) \in E \times F$, $1 \leq i \leq n$, such that $b = \sum_{i=1}^{n} x_i \otimes y_i$ and $\{x_1,\cdots,x_n\}$ and $\{y_1,\cdots,y_n\}$ are linearly independent in E and F respectively.

Proof. Suppose that $b = \sum_{i=1}^{n} x_i \otimes y_i$ and that the x_i are not linearly independent. Then $x_n = \sum_{i=1}^{n-1} a_i x_i$ for some scalars $a_1,\cdots,a_{n-1}$. Hence $b = \sum_{i=1}^{n-1} x_i \otimes y_i + (\sum_{i=1}^{n-1} a_i x_i) \otimes y_n = \sum_{i=1}^{n-1} x_i \otimes y_i + \sum_{i=1}^{n-1} x_i \otimes (a_i y_n) = \sum_{i=1}^{n-1} x_i \otimes (y_i + a_i y_n)$. Continuing this reduction process we arrive at a set of x_i which is linearly independent. Hence, assume that $\{x_1,\cdots,x_n\}$ is linearly independent. If $y_n = \sum_{i=1}^{n-1} a_i y_i$ then by a similar argument $b = \sum_{i=1}^{n-1} (x_i + a_i x_n) \otimes y_i$ and $\{x_1 + a_1 x_n,\cdots,x_{n-1} + a_{n-1} x_n\}$ is linearly independent since $\{x_1,\cdots,x_n\}$ is. Continuing in this manner we arrive

at a set of y_i which is linearly independent. This concludes the proof.

Now let E and F be locally convex Hausdorff spaces. If $b \in E \otimes F$ is such that $b(x^*,y^*) = 0$ for all $x^* \in E^*$, $y^* \in F^*$ then $b = 0$. In fact, if $b \neq 0$ then let $b = \sum_{i=1}^{n} x_i \otimes y_i$ where the x_i and y_i form linearly independent sets in E and F respectively. By the Hahn-Banach theorem there exists $x_1^*,\cdots,x_n^* \in E^*$ such that for each j $\langle x_i,x_j^*\rangle = 0$ for $i \neq j$ and 1 for $i = j$. Hence if $1 \leq j \leq n$ then $0 = b(x_j^*,y^*) = \langle y_j,y^*\rangle$ for $y^* \in F^*$. Thus $y_j = 0$ for $1 \leq j \leq n$ and so $b = 0$. Thus we need only consider $E \otimes F$ as a subspace of the space of bilinear forms on $E^* \otimes F^*$. Also note that if $b \in E \otimes F$ and both E^* and F^* have their weak * topologies then $b : E^* \otimes F^* \longrightarrow \mathbb{C}$ is separately continuous.

The <u>biprojective</u> or <u>biequicontinuous topology</u> γ on $E \otimes F$ is defined by the semi-norms $p(b) = \sup \{|b(x^*,y^*)| : x^* \in P,\ y^* \in Q\}$ where P and Q are arbitrary weak * closed equicontinuous subsets of E^* and F^* respectively. We will let $(E \otimes F)_\gamma$ denote $E \otimes F$ with the γ-topology and $E \hat{\otimes} F$ be the completion of $(E \otimes F)_\gamma$.

<u>THEOREM 5.2.</u> (Grothendieck [20,p.124]) A linear functional L on $E \otimes F$ is γ-continuous if and only if there exist weak * closed equicontinuous sets P and Q contained in E^* and F^* respectively, and a measure $\mu \in M(P \times Q)$ such that $L(b) = \int_{P \times Q} b(x^*,y^*)d\mu(x^*,y^*)$ for

all $b \in E \otimes F$.

Proof. If L is given by such a measure then $|L(b)| \leq \|\mu\| \sup\{|b(x^*,y^*)| : x^* \in P, y^* \in Q\}$ and so L is γ-continuous.

Let $L \in (E \otimes F)_\gamma^*$; then there are sets P and Q of the prescribed type and a constant $a > 0$ such that $|L(b)| \leq a \sup\{|b(x^*,y^*)| : x^* \in P, y^* \in Q\} = a\, p(b)$. Let $x \in E$ and $y \in F$ and consider the function $f : P \times Q \longrightarrow \mathbb{C}$ defined by $f(x^*,y^*) = x \otimes y(x^*,y^*) = \langle x,x^*\rangle\langle y,y^*\rangle$ for all $(x^*,y^*) \in P \times Q$. Clearly f is separately continuous if both P and Q have their relative weak * topologies. Moreover $f \in C(P \times Q)$; in fact, if $(x_i^*,y_i^*) \longrightarrow (x^*,y^*)$ in $P \times Q$ then $|f(x^*,y^*) - f(x_i^*,y_i^*)| \leq |f(x^* - x_i^*, y^*)| + |f(x_i^*, y_i^* - y^*)| \leq |\langle x, x^* - x_i^*\rangle|\,|\langle y,y^*\rangle| + \sup\{|\langle x,x^*\rangle| : x^* \in P\}\,|\langle y, y_i^* - y^*\rangle|$. Hence, if $c_1 = |\langle y,y^*\rangle|$ and $c_2 = \sup\{|\langle x,x^*\rangle| : x^* \in P\}$ then there is an i_o such that for $i \geq i_o$ $|\langle x, x_i^* - x^*\rangle| \leq \epsilon/2c_1$ and $|\langle y, y_i^* - y^*\rangle| \leq \epsilon/2c_2$. Therefore if $i \geq i_o$ then $|f(x_i^*,y_i^*) - f(x^*,y^*)| \leq \epsilon$ and so f is in $C(P \times Q)$. Clearly we have that each $b \in E \otimes F$ defines an element of $C(P \times Q)$ in the same manner. Thus we can define a linear map $T : E \otimes F \longrightarrow C(P \times Q)$ by $T(b)(x^*,y^*) = b(x^*,y^*)$ for all $(x^*,y^*) \in P \times Q$. If $T(b) = 0$ then $|L(b)| \leq a\, p(b) = 0$; so if we set $L'(T(b)) = L(b)$, L' is a well defined bounded linear functional on a subspace of $C(P \times Q)$. If we extend L' to all of $C(P \times Q)$ and apply the Riesz representation theorem, we obtain a measure $\mu \in M(P \times Q)$

such that $L(b) = L'(T(b)) = \int_{P\times Q} bd\mu$. This completes the proof.

We will now prove the following known analogue of the Stone-Weierstrass theorem.

THEOREM 5.3. Let A be a uniformly closed subspace of $C_o(S,E)$ such that $\phi A \subset A$ for all ϕ in $C_c(S)$. Then $A = C_o(S,E)$ if and only if for every $s \in S$ $A(s) \equiv \{f(s) : f \in A\} = E$.

Proof. Clearly if $A = C_o(S,E)$ then the condition holds. Hence suppose that $A(s) = E$ for all $s \in S$ and let $f \in C_o(S,E)$. If p is a c.s.n. on E then there is a compact set $K \subset S$ such that $p(f(s)) \leq \frac{1}{3}$ for $s \notin K$. If $s_o \in K$ then, by hypothesis, there is a function $g \in A$ (g depends on s_o) such that $g(s_o) = f(s_o)$. By the continuity of f and g there is an open neighborhood U of s_o such that $p(f(t) - g(t)) \leq \frac{1}{3}$ for all $t \in U$. From this and the compactness of K we obtain a finite open cover $\{U_1,\cdots,U_n\}$ of K and functions $g_1,\cdots,g_n \in A$ such that $p(f(s) - g_i(s)) \leq \frac{1}{3}$ for $s \in U_i$ and $1 \leq i \leq n$. By Theorem 1.6 we obtain functions $\phi_1,\cdots,\phi_n \in C_c(S)$ such that $0 \leq \phi_i \leq 1$, $\phi_i(s) = 0$ for $s \notin U_2$, $\sum_{i=1}^{n}\phi_i(s) \leq 1$ and $\sum_{i=1}^{n}\phi_i(s) = 1$ for $s \in K$. Let $g(s) = \sum_{i=1}^{n}\phi_i(s)g_i(s)$; by hypothesis $\phi_i g_i \in A$ for $1 \leq i \leq n$ and so $g \in A$.

If $s \in K$ then $|g(s) - f(s)| = \left|\sum_{i=1}^{n}\phi_i(s)[g_i(s) - f(s)]\right|$. But if $s \in U_i$ $p(g_i(s) - f(s)) \leq \frac{1}{3}$, and if $s \notin U_i$ then

$\phi_i(s) = 0$. Thus the above sum is a convex combination of elements of $\{x : p(x) \leq \frac{1}{3}\}$ which is a convex set. This implies that $p(g_i(s) - f(s)) \leq \frac{1}{3} < 1$. If $s \notin K$ then $g(s) - f(s) = \sum_{i=1}^{n} \phi_i(s)[g_i(s) - f(s)] + (\sum_{i=1}^{n} \phi_i(s))f(s) - f(s)$. But $p(-f(s)) \leq \frac{1}{3}$ and $a = \sum_{i=1}^{n} \phi_i(s) \leq 1$ implies $p(a\, f(s)) \leq \frac{1}{3}$. If $a = 0$ then $\phi_i(s) = 0$ for $1 \leq i \leq n$ and so $p(g(s) - f(s)) = p(-f(s)) \leq \frac{1}{3} < 1$. If $a > 0$ then $\sum_{i=1}^{n} \phi_i(s)[g_i(s)-f(s)]$ $= a \sum_{i=1}^{n} \frac{1}{a} \phi_i(s)\,[g_i(s) - f(s)]$. But $\sum_{i=1}^{n} \frac{1}{a} \phi_i(s) = 1$ and $a \leq 1$ implies that this last expression is in $\{x \in E : p(x) \leq \frac{1}{3}\}$. Therefore $p(f(s) - g(s)) \leq 3 \cdot \frac{1}{3} \leq 1$ for all $s \in S$. Since p was an arbitrary c,s.n., we have that A is uniformly dense in $C_o(S,E)$ as well as closed. Thus $A = C_o(S,E)$ and the proof is complete.

COROLLARY 5.4. The set of all functions g in $C_o(S,E)$ of the form $g(s) = \sum_{i=1}^{n} \phi_i(s)x_i$, where $\phi_i \in C_c(S)$ and $x_i \in E$ for $1 \leq i \leq n$, is uniformly dense in $C_o(S,E)$.

We define a map $T : C(S) \otimes E \longrightarrow C(S,E)$ by $T(\sum_{i=1}^{n} f_i \otimes x_i)(s) = \sum_{i=1}^{n} f_i(s)x_i$ for $s \in S$. To see that T is well defined let $\sum_{i=1}^{n} f_i \otimes x_i = 0$; if $s \in S$ and $x^* \in E^*$ then $0 = \sum_{i=1}^{n} f_i(s)\langle x_i, x^*\rangle = \langle \sum_{i=1}^{n} f_i(s)x_i, x^*\rangle$. Since x^* was arbitrary we have that $\sum_{i=1}^{n} f_i(s)x_i = 0$ for all $s \in S$ and so T is well defined. We will actually show that T

induces a homeomorphism between $C(S)_\beta \hat{\otimes} E$ and $C(S,E)_\beta$.

LEMMA 5.5. If $Q \subset E^*$ is equicontinuous and p is the gauge of $Q^\circ \subset E$ then $p(x) = \sup\{|\langle x,x^*\rangle| : x^* \in Q\}$ for each x in E.

Proof. By definition $p(x) = \inf\{\frac{1}{a} : a > 0 \text{ and } ax \in U = Q^\circ\}$. Therefore let $a > 0$ with $ax \in U$. If $x^* \in Q$ then $|\langle ax,x^*\rangle| \leq 1$ and so $|\langle x,x^*\rangle| \leq \frac{1}{a}$. Hence $q(x) = \sup\{|\langle x,x^*\rangle| : x^* \in Q\} \leq \frac{1}{a}$ and since a was arbitrary we have that $q(x) \leq p(x)$. If $x \in E$ such that $q(x) \leq 1$ then $x \in U = Q^\circ$; thus $p(x) \leq 1$ and we have, by a property of semi-norms, the inverse inequality.

THEOREM 5.6. The map $T : (C(S)_\beta \hat{\otimes} E)_\gamma \longrightarrow C(S,E)_\beta$ is a topological isomorphism onto a dense subspace of $C(S,E)_\beta$. Consequently $C(S,E)_\beta$ is topologically isomorphic to $C(S)_\beta \hat{\otimes} E$.

Proof. Let $b = \sum f_i \otimes x_i \in C(S)_\beta \otimes E$ and suppose that $T(b) = 0$. If $b \neq 0$ we may suppose that the x_i are all linearly independent. But then $\sum f_i(s)x_i = 0$ for all $s \in S$ implies $f_1(s) = \cdots = f_n(s) = 0$ for all $s \in S$. Therefore $b = 0$ and T is one-one. Let $\phi \in C_o(S)$, $\phi \geq 0$, and let Q be a weak * closed equicontinuous set in E^* with p = the gauge of Q°. If $x^* \in Q$ is fixed then the map $s \longrightarrow \sum f_i(s) \langle x_i,x^*\rangle$ is in $C(S)$ and so

$\sup\{\left|\int(\sum f_i\langle x_i,x^*\rangle)d\mu\right| : \mu \in V_\phi^\circ\} =$

$\sup\{\left|\sum f_i(s)\phi(s)\langle x_i,x^*\rangle\right| : s \in N(\phi)\}$ by Corollary 1.18 and Theorem 2.15. Hence $P_{\phi \times Q}(\sum f_i \otimes x_i) \equiv$

$\sup\{ \left| \sum \left(\int f_i d\mu \right) \langle x_i, x^* \rangle \right| : \mu \in V_\phi^\circ, x^* \in Q\} =$

$\sup\{ |\phi(s)| \left| \sum f_i(s) \langle x_i, x^* \rangle \right| : x^* \in Q, s \in N(\phi)\} =$

$\sup\, p(\phi(s) \sum f_i(s) x_i) : s \in S\}$ by Lemma 5.5; i.e., $P_{\phi \times Q}(\sum f_i \otimes x_i) = \| \phi \cdot \sum f_i x_i \|_p$. It now quickly follows that T is a homeomorphism.

Finally, by Corollary 5.4 the image of T contains a uniformly dense subspace of $C_o(S,E)$. But $C_o(S,E)$ is β-dense in $C(S,E)$ and so the image of T is β-dense in $C(S,E)$. Since T is a linear homeomorphism it extends to the completion of $(C(S)_\beta \otimes E)$; i.e., T can be extended to a topological isomorphism of $C(S)_\beta \hat{\otimes} E$ onto $C(S,E)_\beta$. This completes the proof.

In a similar manner one can prove the following theorem.

THEOREM 5.7. (Grothendieck [20,p.90]) The map $T : (C_o(S) \otimes E)_\gamma \longrightarrow C_o(S,E)$ is a topological isomorphism onto a dense subpsace of $C_o(S,E)$. Consequently, $C_o(S,E)$ with the uniform topology is topologically isomorphic to $C_o(S) \hat{\otimes} E$.

REMARK. If E is a Banach space then the mapping T in Theorem 5.7 becomes an isometry if $(C_o(S) \otimes E)_\gamma$ is given the norm $\|b\| = \sup\{|b(\mu, x^*)| : \mu \in$ ball $M(S)$, $x^* \in$ ball $E^*\}$.

COROLLARY 5.8. If S and R are locally compact Hausdorff spaces then $C_o(S, C_o(R)) = C_o(S) \hat{\otimes} C_o(R) = C_o(S \times R)$.

Proof. By the preceding theorem and the remark following it we have the first equality. If we define a map $T_2 : C_o(S) \times C_o(R) \longrightarrow C_o(S \times R)$ by $T_2(\sum f_i \otimes g_i)(s,r) = \sum f_i(s)g_i(r)$ then it is easy to show (using the Stone-Weierstrass theorem) that T_2 is an isometry onto a dense subspace of $C_o(S \times R)$. Hence we can get an isometry of $C_o(S) \otimes C_o(R)$ onto $C_o(S \times R)$. This completes the proof.

Following Gil de Lamadrid [15], we let $N(S,E^*)$ be the space of measures μ defined on the Borel sets in S, having values in E^*, and such that: (a) for each $x \in E$, $\mu_x(A) = \langle x, \mu(A)\rangle$ for all $A \in$ Borel (S) defines a measure $\mu_x \in M(S)$; (b) there is a c.s.n. p on E and a constant $c > 0$ such that for every Borel partition $\{A_1,\cdots,A_n\}$ of S and each finite sequence $\{a_1,\cdots,a_n\}$ of unimodular scalars, $\sup\{|\langle x, \sum a_i \mu(A_i)\rangle| : p(x) \leq 1\} \leq c$. We let $\|\mu\|_p$ equal the smallest of all such constants c and call it the p semi-variation of μ. If $A \in$ Borel (S) and $\mu \in N(S,E^*)$ then let $|\mu|_p(A) = \sup\{|\sum \langle x_i, \mu(A_i)\rangle| : p(x_i) \leq 1$ and $\{A_i\}$ is a finite Borel partition of A$\}$. Then $|\mu|_p$ is a countably additive measure on Borel (S). We call $|\mu|_p$ the p-variation of μ, and let $M(S,E^*)$ be the collection of all those $\mu \in N(S,E^*)$ such that $|\mu|_p(S) < \infty$ for some c.s.n. p on E. If $\mu \in M(S,E^*)$ and $|\mu|_p(S) < \infty$ then $|\mu|_p \in M(S)$.

If $f \in C(S,E)$ and $\mu \in M(S,E^*)$ then we define $\int d\mu f$ to be the limit (which exists) of sums

$\sum_{i=1}^{n} \langle f(s_i), \mu(A_i) \rangle$, where $\{A_i\}_{i=1}^{n}$ is a Borel partition of S, $s_i \in A_i$, and the limit is with respect to successive refinements of the partitions. If p is a c.s.n. on E such that $|\mu|_p(S) < \infty$ then $\left| \int d\mu f \right| \leq \|f\|_p \, |\mu|_p(S)$.

THEORME 5.9. A linear functional L on C(S,E) is β-continuous if and only if there is a unique measure μ in M(S,E*) such that $L(f) = \int d\mu f$ for all $f \in C(S,E)$.

Proof. Let $\mu \in M(S,E^*)$ and let p be a c.s.n. on E such that $|\mu|_p(S) < \infty$. Then $|\mu|_p \in M(S)$ implies, by Lemma 2.8, that there is a $\varphi \in C_o(S)$, $\varphi \geq 0$, such that $|\mu|_p$ vanishes off $N(\varphi)$ and $\frac{1}{\varphi} |\mu|_p \in M(S)$. Therefore, if $c = \int \frac{1}{\varphi} d\,|\mu|_p < \infty$ then $|L(f)| = \left| \int d\mu f \right| = \left| \int d(\frac{1}{\varphi}\mu)(\varphi f) \right| \leq c \, \|\varphi f\|_p$, and so $L \in C(S,E)_\beta^*$.

Conversely, if $L \in C(S,E)_\beta^*$ and $T : (C(S)_\beta \otimes E)_\gamma \longrightarrow C(S,E)_\beta$ is the map described in Theorem 5.6 then $L \circ T$ is a γ-continuous linear functional on $C(S)_\beta \otimes E$. By theorem 5.2 there is a β-weak * closed β-equicontinuous subset of M(S), which we may suppose to be V_φ° for some $\varphi \in C_o(S)$, $\varphi \geq 0$, a weak * closed equicontinuous set $Q \subset E^*$, and a measure $\lambda \in M(V_\varphi^\circ \times Q)$ such that $L(\sum f_i x_i) = L \circ T(\sum f_i \otimes x_i) = \int (\sum f_i \otimes x_i) d\lambda$. We may assume $\|\lambda\| \leq 1$ and $V_\varphi^\circ \subset$ ball M(S). Let p = the gauge of Q°. Now for each $A \in$ Borel (S) and $x \in E$ let $\langle x, \mu(A) \rangle = \int \nu(A) \langle x, x^* \rangle d\lambda(\nu, x^*)$. Obviously $\mu(A)$ is a linear functional on E. Furthermore $\mu(A) \in E^*$, for if $\{x_i\}$ is a net in E such that $x_i \longrightarrow 0$ in E, then $x_i \longrightarrow 0$ uniformly on

Q since Q is equicontinuous. Hence if $\epsilon > 0$ there is an i_o such that for $i \geq i_o$ $|\langle x_i, x^* \rangle| \leq \epsilon$ for all $x^* \in Q$. Thus for $i \geq i_o$ $|\langle x_i, \mu(A) \rangle| = |\int \nu(A) \langle x_i, x^* \rangle d\lambda(\nu, x^*)|$ $\leq \int |\nu|(A) \, |\langle x_i, x^* \rangle| \, d|\lambda|(\nu, x^*) \leq \epsilon$, so that $|\langle x_i, \mu(A) \rangle| \longrightarrow 0$. If $\{A_i\}_{i=1}^n$ is a Borel partition of S and $x_i \in Q^o$ for $1 \leq i \leq n$ then $\left| \sum_{i=1}^n \langle x_i, \mu(A_i) \rangle \right| =$

$$\left| \sum_{i=1}^n \int \nu(A_i) \langle x_i, x^* \rangle d\lambda(\nu, x^*) \right| \leq$$

$$\sum_{i=1}^n \int |\nu|(A_i) p(x_i) \, d|\lambda|(\nu, x^*) \leq \sum_{i=1}^n \int |\nu|(A_i) \, d|\lambda|(\nu, x^*)$$

$= \|\nu\| \, \|\lambda\| \leq \|\lambda\| \leq 1$, so μ is of bounded p-variation. We must still show that $\mu_x \in M(S)$ for all $x \in E$. To see this let $x \in E$ be fixed, f a real valued element of C(S), $\epsilon > 0$. Let $a_1, \cdots, a_m$ be real numbers with $a_1 = -\|f\|_\infty$, $a_m = \|f\|_\infty$, and $0 < a_{k+1} - a_k < \epsilon/2$ for $1 \leq k \leq m-1$. Put $A_1 = \{s : a_1 \leq f(s) \leq a_2\}$ and $A_k = \{s : a_k < f(s) \leq a_{k+1}\}$ for $2 \leq k \leq m-1$; and let $g = \sum_{k=1}^{m-1} a_k \chi_{A_k}$. Then $\{A_i\}_{i=1}^n$ is a Borel partition of S and $\|g - f\|_\infty \leq \epsilon/2$. If $s_k \in A_k$ then $\left| \int \left(\int_S f d\nu \right) \langle x, x^* \rangle d\lambda(\nu, x^*) - \right.$

$$\left. \sum_{k=1}^{m-1} f(s_k) \mu_x(A_k) \right| \leq \left| \int \left[\int_S (f-g) d\nu \right] \langle x, x^* \rangle d\lambda(\nu, x^*) \right| +$$

$$\left| \int \left(\int_S g d\nu \right) \langle x, x^* \rangle d\lambda(\nu, x^*) - \sum_{k=1}^{m-1} f(s_k) \mu_x(A_k) \right| \leq$$

$$\frac{\epsilon}{2} p(x) + \left| \int \left(\sum_{k=1}^{m-1} [a_k - f(s_k)] \nu(A_k) \right) \langle x, x^* \rangle \, d\lambda(\nu, x^*) \right| =$$

$$\frac{\epsilon}{2} p(x) + \left| \sum_{k=1}^{m-1} (a_k - f(s_k) \mu_x(A_k) \right| \leq \frac{\epsilon}{2} p(x) + \frac{\epsilon}{2} \sum_{k=1}^{m-1} |\mu_x(A_k)|.$$

But if b_k is a unimodular scalar such that $b_k\langle x, \mu(A_k)\rangle = |\langle x, \mu(A_k)\rangle|$ then $\sum_{k=1}^{m-1} |\mu_x(A_k)| = \left|\sum_{k=1}^{m-1} \langle b_k x, \mu(A_k)\rangle\right| \leq p(x)$. Therefore

$$\left|\int\left(\int_S f d\nu\right)\langle x, x^*\rangle d\lambda(\nu, x^*) - \sum_{k=1}^{m-1} f(s_k)\mu_x(A_k)\right| \leq \epsilon\, p(x)$$

and so $\int f d\mu_x$ exists and is equal to $\int\left(\int_S f d\nu\right)\langle x, x^*\rangle d\lambda(\nu, x^*)$. But if $\{f_i\}$ is a net in $C(S)$ such that $f_i \to 0$ (β) then $f_i \otimes x \to 0$ in $(C(S)_\beta \otimes E)_\gamma$, and so $L_x(f) = L(f \cdot x)$ defines an element $L_x \in C(S)_\beta^* = M(S)$. But $L_x(f) = L(f \cdot x) = \int f d\mu_x$ and so $\mu_x \in M(S)$. Therefore $\mu \in M(S, E^*)$. Also if $f_i \in C(S)$, $x_i \in E$ for $1 \leq i \leq n$, then $L(\sum f_i x_i) = \sum L(f_i x_i) = \sum f_i d\mu_{x_i} = \int d\mu(\sum f_i x_i)$. But $T(C(S) \otimes E)$ is dense in $C(S,E)_\beta$ and both L and μ are β-continuous. Hence $L(f) = \int d\mu f$ for all $f \in C(S,E)$ and the proof is complete.

This same method of proof could have been used to obtain the following generalization of the Riesz representation theorem (see [15] and [36]).

THEOREM 5.10. A linear functional L on $C_o(S,E)$ is bounded if and only if there is a measure $\mu \in M(S,E^*)$ such that $L(f) = \int d\mu f$ for all $f \in C_o(S,E^*)$. Also if p is a c.s.n. on E such that $|\mu|_p(S) < \infty$ then $|\mu|_p(S) = \sup\{|\int d\mu f| : \|f\|_p \leq 1\}$.

REMARKS. A different approach could have been used to obtain Theorem 5.9. We could have proved Theorem 5.10

first and then used it, as we used the Riesz representation theorem for the proof of the scalar version, to prove Theorem 5.9. This approach was used by Wells [36]. For more information as well as a proof that $C_o(S,E)^* = M(S,E^*)$ see [15].

It might be asked whether or not a space of the type $C(S,E)$ is actually a space of the type $C(Q)$. If $E = C(R)$ and S and R are compact then Corollary 5.8 says that $C(S,E) = C(Q)$ where $Q = S \times R$. Theorem 5.14 gives a partial converse to this result. Before proceeding, we state two theorems on extreme points. Their proofs can be transcribed from the proofs of their scalar valued analogues, Theorems 2.14 and 2.15, and hence will not be repeated.

THEOREM 5.11. If E is a Banach space then a measure μ in $M(S,E^*)$ is an extreme point of the unit ball of $M(S,E^*)$ if and only if $\mu = \delta_{(s)}x^*$, where x^* is an extreme point of ball E^* and $s \in S$. (If $\nu \in M(S)$ and $x^* \in E^*$ then $\mu = \nu x^*$ means $\mu(A) = \nu(A)x^*$ for all $A \in$ Borel (S)).

THEOREM 5.12. If E is a Banach space, $\phi \in C_o(S)$, and $V_\phi = \{f \in C(S,E) : \|\phi(s)f(s)\| \leq 1$ for all $s \in S\}$ then $\mu \in M(S,E^*)$ is an extreme point of $V_\phi^\circ = \{\lambda \in M(S,E^*) : \left|\int d\lambda\, f\right| \leq 1$ for all $f \in V_\phi\}$ if and only if $\mu = \phi(s)\,\delta_{(s)}x^*$, where x^* is an extreme point of ball E^* and $s \in N(\phi)$.

LEMMA 5.13. If S is compact, E is a Banach space, and $\mathcal{E}$ the extreme points of ball E^*, then the set of

extreme points of ball $M(S,E^*)$ with the weak * topology is homeomorphic to $S \times \mathcal{E}$ under a natural identification.

Proof. Let $\Delta(S) = \{\delta_{(s)} : s \in S\}$ and define $h: S \times \mathcal{E} \longrightarrow \Delta(S)\mathcal{E} = \{\delta_{(s)}x^* : s \in S, x^* \in \mathcal{E}\}$ by $h(s,x^*) = \delta_{(s)}x^*$. Then h is clearly one-one and onto. Suppose $\{(s_i,x_i^*)\}$ is a net in $S \times \mathcal{E}$ which converges to $(s,x^*) \in S \times \mathcal{E}$. If $f \in C(S,E)$ then $f(s_i) \longrightarrow f(s)$; i.e., $\|f(s_i) - f(s)\| \longrightarrow 0$. But $|\langle f(s_i),x_i^*\rangle - \langle f(s),x^*\rangle| \leq |\langle f(s_i) - f(s),x_i^*\rangle| + |\langle f(s),x_i^* - x^*\rangle| \leq \|f(s_i) - f(s)\| + |\langle f(s),x_i^* - x^*\rangle|$ and both these terms can be made arbitrarily small. Thus $\delta_{(s_i)}x_i^* \longrightarrow \delta_{(s)}x^*$ weak * in $M(S,E^*)$ and h is continuous.

Consider $h^{-1} : \Delta(S)\mathcal{E} \rightarrow S \times \mathcal{E}$. If $\{\delta_{(s_i)}x_i^*\}$ is a net in $\Delta(S)\mathcal{E}$ which converges weak * to $\delta_{(s)}x^*$, then $1 \otimes x \in C(S,E)$ for all $x \in E$ and so $\langle x,x_i^*\rangle = \langle 1 \otimes x, \delta_{(s_i)}x_i^*\rangle \longrightarrow \langle 1 \otimes x, \delta_{(s)}x^*\rangle = \langle x,x^*\rangle$. Therefore $x_i^* \longrightarrow x^*$ weak * in E^*. If $\phi \in C(S)$ then $|\phi(s_i)| \leq \|\phi\|_\infty$ for all i implies there is a $c \in C$ such that $\phi(s_i) \xrightarrow{cl} c$. Hence, $|\phi(s_i)\langle x,x_i^*\rangle - c\langle x,x^*\rangle| \leq |\langle x,x_i^*\rangle|\,|\phi(s_i) - c| + |c|\,|\langle x,x_i^* - x^*\rangle| \leq \|x\|\,|\phi(s_i) - c| + |c|\,|\langle x,x_i^* - x^*\rangle|$. If $c \neq 0$ and $\epsilon > 0$ then some i_o, $|\langle x,x_i^* - x^*\rangle| < \epsilon/2\,c$ for $i \geq i_o$. Also, $\phi(s_i) \xrightarrow{cl} c$ implies there is an $i \geq i_o$ such that $|\phi(s_i) - c| < \epsilon/2\,\|x\|$. Therefore it follows that $\phi(s_i)\langle x,x_i^*\rangle \xrightarrow{cl} \langle x,x^*\rangle c$. If $c = 0$ then $\phi(s_i) \xrightarrow{cl} 0$ still gives this conclusion. But $\phi(s_i)\langle x,x_i^*\rangle \longrightarrow \phi(s)\langle x,x^*\rangle$ since $\phi \otimes x \in C(S,E)$, and so $\phi(s) = c$. Thus there is

one cluster point of $\{\phi(s_i)\}$ in the closed disk of radius $\|\phi\|_\infty$ in $\mathbb{C}$, namely $\phi(s)$; this implies $\phi(s_i) \longrightarrow \phi(s)$. Since ϕ was arbitrary we have $s_i \longrightarrow s$ and so h^{-1} is also continuous.

THEOREM 5.14. If S is compact and E is a Banach space then C(S,E) is isometrically isomorphic to C(R) for some compact space R if and only if there is a compact space Q such that C(S,E) is isometric to C(S,C(Q)) = C(S × Q); i.e., R is homeomorphic to S × Q.

Proof. If the condition holds then let R = S × Q and we are done. Now suppose that $T : C(S,E) \longrightarrow C(R)$ is an isometry. Let $\mathcal{E}$ = the extreme points of ball E* and let $\Delta(S)$ and $\Delta(R)$ be as in the proof of Lemma 5.13. If T* is the adjoint of T and Γ = the unit circle in $\mathbb{C}$ then T* defines a homeomorphism between $\Gamma\Delta(R)$ and $\mathcal{E}\Delta(S)$. Applying the preceding lemma we have that $S \times \mathcal{E}$ and $R \times \Gamma$ are homeomorphic. Thus R and Γ are compact implies $S \times \mathcal{E}$ is compact and so $\mathcal{E}$ is compact.

Now $(\mathcal{E}, \Gamma)$ is a compact transformation group where the action of Γ on $\mathcal{E}$ is defined by multiplication [28]. If we define $x^* \sim y^*$ $(x^*, y^* \in \mathcal{E})$ to mean that $x^* = e^{i\theta}y^*$ for some θ, then "$\sim$" is an equivalence relation on $\mathcal{E}$ and the equivalence classes are the orbits induced by Γ on $\mathcal{E}$. Hence, if $Q = \mathcal{E}/\sim$ then Q is a compact Hausdorff space. Let $p : \mathcal{E} \longrightarrow Q$ be the natural map.

If $r \in R$ then $T^*(\delta_{(r)}) = \delta_{(\sigma(r))}x(r)^*$ where $\sigma(r) \in S$ and $x(r)^* \in \mathcal{E}$. Define $H : R \longrightarrow S \times Q$ by $H(r) =$

$[\sigma(r), p(x(r)^*)]$. Since $\Delta(S)\mathcal{E} = \Delta(S) \times \mathcal{E}$ and projections are continuous, we have that the map $r \rightarrow \sigma(r)$ of R into S is continuous. Also, $r \rightarrow p(x(r)^*)$ is continuous and so H is continuous. If $H(r_1) = H(r_2)$ then $x(r_1)^* = ax(r_2)^*$ for some $a \in \mathbb{C}$, $|a| = 1$. Hence $T^*(\delta_{(r_1)}) = \delta_{(\sigma(r_1))} x(r_1)^* = \delta_{(\sigma(r_2))} ax(r_2)^* = aT^*(\delta_{(r_2)})$. But T^* is one-one implies $r_1 = r_2$ and $a = 1$. Thus H is one-one; also, since T^* is onto H is onto. But R is compact and so H is a homeomorphism. Therefore $C(S,E) = C(R) = C(\Sigma \times Q)$. This completes the proof.

We will end this chapter with a characterization of the weakly compact operators from a Banach space E into M(S). An operator $T : E \rightarrow M(S)$ is <u>weakly compact</u> if and only if T (ball E) is weakly conditionally compact in M(S) (see [14, p.482]).

If $\mu \in N(S,E^*)$ and we set $T(x) = \mu_x$ for all $x \in E$ then T is a bounded linear transformation from E into M(S), and $\|T\| = \|\mu\|$, the semi-variation of μ. If, conversely, T is given and $T^* : M(S) \rightarrow E^*$ is its adjoint, we may find a measure $\mu \in N(S,E^*)$ such that $T(x) = \mu_x$. This is accomplished by letting $\mu(A) = T^*(\chi_A)$ for all A $\in$ Borel (S) (here $\chi_A \in M(S)^*$ and $\langle \nu, \chi_A \rangle = \nu(A)$ for all $\nu \in M(S)$). Thus the space of bounded linear transformations from E into M(S) is $N(S,E^*)$. The details of the proofs of the above statements may be found in [15] or [14, p.498]. Since $M(S,E^*)$ is a subspace of $N(S,E^*)$ we

might ask if the measures of bounded variation can be characterized by the linear operators which they represent. The answer is yes and is given by the following result.

THEOREM 5.15. If E is a Banach space then a bounded linear transformation of E into M(S) is weakly compact if and only if it can be represented by a measure in M(S,E*).

Proof. Let $T(x) = \mu_x$ and suppose μ is of bounded variation. Hence $|\mu|(A) = \sup\{\sum \|\mu(A_i)\| : \{A_i\}$ is a finite Borel partition of $A\}$ defines an element $|\mu|$ of M(S). We will apply Theorem 4.13 to show that $\{\mu_x : \|x\| \leq 1\}$ is weakly conditionally compact in M(S). If $\epsilon > 0$ then there is a compact set $K \subset S$ such that $|\mu|(S \setminus K) < \epsilon$. It is easy to see that $|\mu_x|(S \setminus K) \leq \|x\| \, |\mu|(S \setminus K) \leq \epsilon \|x\|$. Thus $\{\mu_x : \|x\| \leq 1\}$ is β-equicontinuous. If K is a compact subset of S and $\epsilon > 0$ then there is an open set $V \supset K$ such that $|\mu|(V \setminus K) \leq \epsilon$. Thus if $\|x\| \leq 1$ then $|\mu_x|(V \setminus K) \leq |\mu|(V \setminus K) \leq \epsilon$ and so T is weakly compact.

Conversely suppose T is weakly compact. Then [14, p.306] there is a positive measure λ in M(S) such that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $A \in$ Borel (S) and $\lambda(A) \leq \delta$ then $|\mu_x(A)| \leq \epsilon$ for all $x \in$ ball E. Therefore if $\lambda(A) = 0$ then $\lambda(B) = 0$ for all Borel sets $B \subset A$ and so $\mu_x(B) = 0$ for all $x \in$ ball E. This implies that $|\mu|(A) = 0$; that is, $|\mu|$ is absolutely continuous with respect to λ. Hence there is a constant $c > 0$ such that $\lambda(A) \leq c$ implies $|\mu|(A) \leq 1$. Since λ is bounded there are at most a countable number of points $\{s_i\}_{i=1}^{\infty}$ such

that $\lambda(\{s_i\}) > 0$. Also $\sum_{i=1}^{\infty} \lambda(\{s_i\}) < \infty$ implies there is an integer $n \geq 1$ such that $c \geq \sum_{i=n+1}^{\infty} \lambda(\{s_i\}) = \lambda(\{s_i : i \geq n+1\})$. Therefore $|\mu|(\{s_i : i \geq 1\}) = \sum_{i=1}^{n} |\mu|(\{s_i\}) + |\mu|(\{s_i : i \geq n+1\}) \leq 1 + \sum_{i=1}^{n} |\mu|(\{s_i\}) < \infty$.

If $S_o = S \setminus \{s_i : i \geq 1\}$ then there is a compact set $K \subset S_o$ such that $\lambda(S_o \setminus K) \leq c$; hence $|\mu|(S_o \setminus K) \leq 1$. Also for all $s \in K$ $\lambda(\{s\}) = 0$ and so there is an open neighborhood U_s of s such that $\lambda(U_s) \leq c$. Since K is compact we can find open sets $U_1, \cdots, U_m$ such that $K \subset \bigcup_{i=1}^{m} U_i$ and $\lambda(U_i) \leq c$. Therefore $|\mu|(K) \leq \sum_{i=1}^{m} |\mu|(U_i) \leq m$. Combining these results we get that $|\mu|(S) = |\mu|(\{s_i : i \geq 1\}) + |\mu|(K) + |\mu|(S_o \setminus K) < \infty$ and so $\mu \in M(S, E^*)$.

COROLLARY 5.16. If E is a Banach space then for every subspace E_1 of E there is a constant $a = a(S, E_1)$ such that if T_1 is a weakly compact transformation from E_1 into $M(S)$ then T_1 can be extended to a weakly compact operator T of E into $M(S)$ such that $\|T\| \leq a \|T_1\|$.

Proof. It is easy to see that if $\mu \in M(S, E_1^*)$ then $\|\mu\| \leq |\mu|(S)$. Let $\mathcal{W}_1$ be the space of weakly compact linear transformations from E_1 into $M(S)$ furnished with the operator norm. Thus $\mathcal{W}_1$ is a Banach space and if $U : M(S, E_1^*) \longrightarrow \mathcal{W}_1$ is defined for each $\mu \in M(S, E_1^*)$ by $U(\mu)(x) = \mu_x$ for all $x \in E_1$, then, by the preceding theorem, U is a one-one linear mapping from $M(S, E_1^*)$ onto

$\mathcal{W}_1$. Also $\|U(\mu)\| = \|\mu\| \leq |\mu|(S)$, and so if $M(S,E_1^*)$ has its total variation norm then $M(S,E_1^*)$ is a Banach space and U is norm decreasing. By the open mapping theorem [14,p.57] there is a constant $a > 0$ such that $|\mu|(S) \leq a\|U(\mu)\| = a\|\mu\|$. If $T_1 \in \mathcal{W}_1$ then let $\mu_1 \in M(S,E_1^*)$ be such that $U(\mu_1) = T_1$. But $M(S,E_1^*) = C_o(S,E_1)^*$ by Theorem 5.10 and so $L_1(f) = \int d\mu_1 f$ for all $f \in C_o(S,E_1)$ defines a bounded linear functional L_1 on $C_o(S,E_1)$. But $C_o(S,E_1)$ is clearly a subspace of $C_o(S,E)$ and so by the Hahn-Banach theorem there is a bounded linear functional L on $C_o(S,E)$ which extends L_1 such that $\|L\| = \|L_1\|$. If $\mu \in M(S,E^*)$ such that $L(f) = \int d\mu f$ for all $f \in C_o(S,E)$, and T is the weakly compact operator from E into M(S) represented by μ, then $\|T\| = \|\mu\| \leq |\mu|(S) = \|L\| = \|L_1\| = |\mu_1|(S) \leq a\|T_1\|$. Also if $x \in E_1$ and $\phi \in C_o(S)$ then $\phi \otimes x \in C_o(S,E_1)$ and $\int \phi d\mu_x = \int d\mu(\phi \otimes x) = L(\phi \otimes x) = L_1(\phi \otimes x) = \int \phi d\mu_{1x}$. Since ϕ was arbitrary we have that $\mu_x = \mu_{1x}$ for all $x \in E_1$; i.e., T is an extension of T_1 and the proof is complete.

CHAPTER VI

UNSOLVED PROBLEMS

In this chapter we have listed some unsolved problems and questions. The list is not exhaustive, but is indicative of our current interest in the strict topology.

1. A natural probelm is to try to characterize those spaces S such that $C(S)_\beta$ is a Mackey space (or a strong Mackey space). This seems rather difficult and at present there are no approaches which seem promising to us.

2. It would be useful to know if $C(S)_\beta$ is a Mackey space implies $C(S)_\beta$ is a strong Mackey space. If the answer is yes then we believe that Theorem 3.7 is the strongest possible result along these lines.

3. Show that if S is a pseudocompact non-compact space then $C(S)_\beta$ is not a strong Mackey space.

4. If $C(S)_\beta$ is a Mackey space then characterize those spaces E such that $C(S,E)_\beta$ is a Mackey space.

5. Corollary 4.8 interests us because of its similarity with a theorem on metrizable topological vector spaces (see [25,p.212]). Can we draw the same conclusion for the β-convergence of a net? That is, if S is metrizable and $\{f_i\}$ is a net in C(S) such that $f_i \longrightarrow 0$ uniformly on β-weak * convergent sequences in M(S), then

does $f_i \longrightarrow 0$ (β)?

6. If $\{\mu_n\}$ is a β-weak * convergent sequence in M(S) then is $\{\mu_n\}$ β-equicontinuous? An affirmative answer would simplify the hypotheses of the theorems in Chapter IV. If the answer is negative then characterize those spaces S for which the answer is yes.

7. Prove a version of the Bartle-Graves selection theorem ([5] and [27,p.375]) where both domain and range have weak or weak * topologies.

8. It is easy to see that if N is a norm closed subspace of ℓ^1 (and hence weakly or β-weak * closed) and $E = N^{\perp} \subset \ell^\infty$ then the existence of a bounded projection of ℓ^1 onto N implies that E_β is a Mackey space. Thus Theorem 3.17 and the remarks following it prove the existence of spaces N such that no bounded projection of ℓ^1 onto N exists. The following question presents itself. If E_β is a Mackey space then is it necessary that there is a bounded projection of ℓ^1 onto $E_\beta{}^* = N$?

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BIOGRAPHY

John Bligh Conway was born in New Orleans, Louisiana on September 22, 1939. He attended parochial grammar schools in New Orleans and graduated from Jesuit High School in 1957. That same year he enrolled at Loyola University in New Orleans from which he received a Bachelor of Science in May, 1961. During the next two years he held National Science Foundation Fellowships, first at Indiana University and then at New York University. In the summer of 1963 he came to Louisiana State University where he held a teaching assistantship for the academic year 1963-64. In June, 1964 he began his tenure as a National Science Foundation Cooperative Fellow, and in the following August he married the former Ann Flattery. Mr. Conway is currently being supported by National Science Foundation grant GP 1449 at Louisiana State University, where he is a candidate for the degree of Doctor of Philosophy in the Department of Mathematics.

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Major Field: Mathematics

Title of Thesis: The Strict Topology and Compactness in the Space of Measures.

Approved:

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Max Goodrich
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Date of Examination:

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