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Pruefer Rings.

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ABSTRACT

In this paper R will denote a commutative ring with identity. A Prüfer ring is a ring R in which the ideals are linearly ordered in the quotient ring R_P for each proper prime ideal P of R. R is an A-ring if and only if the distributive law $A \cap (B + C) = A \cap B + A \cap C$ holds for all ideals A, B, and C of R. A semi-multiplication ring is a ring in which $A \subset B$ and B finitely generated imply there exists an ideal C such that $A = BC$.

In Chapter II the following ten conditions are considered for a ring R:

I. R is an A-ring.

II. R is a Prüfer ring.

III. R is a semi-multiplication ring.

IV. If A, B, and C are ideals of R with C finitely generated, then $(A + B):C = A:C + B:C$.

V. If A, B, and C are ideals of R with A and B finitely generated, then $C:(A \cap B) = C:A + C:B$.

VI. $A = \cap A^{ec}$ where the intersection is taken over all quotient rings of R which have linearly ordered ideal systems and e and c denote ideal extension and contraction, respectively.

VII. $A(B \cap C) = AB \cap AC$ where A, B, and C are ideals with one of B or C regular.

VIII. $(A + B)(A \cap B) = AB$ where one of A or B is regular.

IX. Finitely generated regular ideals are invertible.

X. A, B, and C ideals of R, A finitely generated and regular and $AB = AC$ imply $B = C$.

Proofs are given showing that I through VI are equivalent and VII through X are equivalent. Also, a Prüfer ring satisfies VII through X. Several properties of Prüfer rings are given, including the property that any ring between a Prüfer ring and its total quotient ring is also a Prüfer ring.

The third chapter deals with the primary ideal structure of a Prufer ring. Also, α-rings, which are rings satisfying the ascending chain condition for prime ideals and in which primary ideals are prime power ideals, are studied. It is shown if P is a prime ideal of a Prüfer ring or of an α-ring and the kernel of the natural map from R to R_P, denoted by K_P, is contained in P^n then P^n is P-primary. If P is a non-idempotent prime ideal of a Prüfer ring, Q is a P-primary ideal, and K_P is properly contained in Q then Q is a prime power. An example is given verifying that an α-ring is not necessarily a Prüfer ring, but an α-ring will always satisfy the equivalent properties VII through X.

CHAPTER I

INTRODUCTION AND NOTATIONS

The rings considered in this work will be commutative with a multiplicative identity $1 \neq 0$. The notation will, in general, be that of [15] with $A \subset B$ meaning A is contained in B and $A < B$ meaning A is properly contained in B. Whenever M is a multiplicative system of R, R_M will denote the quotient ring of R with respect to M as defined in [15]. If P is a prime ideal, then we will follow the convention of using R_P to represent R_M, where $M = R \setminus P$. An ideal A is regular if it contains a regular element. We say the ideals of R are linearly ordered through A if the following two conditions are satisfied:

1) If B is an ideal then either $A \subset B$ or $B \subset A$.
2) If B and C are ideals and $B \not\subset A$ then either $B \subset C$ or $C \subset B$.

If the ideals of R are linearly ordered through (0) we simply say the ideals of R are linearly ordered. In case R is an integral domain this is the same as saying R is a valuation ring.

A Prüfer domain is an integral domain R such that R_P is a valuation ring for each proper prime ideal P of R. Prüfer domains have been investigated in detail. In particular it is known that for an integral domain R with quotient field K the following statements are equivalent:

a) Every finitely generated, non-zero ideal of R is invertible.

b) R is a Prüfer domain.

c) If $A \neq (0)$, B and C are ideals of R such that A is finitely generated and $AB = AC$, then $B = C$.

d) If A, B, and C are ideals of R then $A \cap (B + C) = (A \cap B) + (A \cap C)$ or, equivalently, $A + (B \cap C) = (A + B) \cap (A + C)$.

e) Every ideal A of R is complete, that is to say, $A = \cap AR_v$ where the intersection is taken over all valuation rings R_v such that $R \subset R_v < K$.

f) Every ideal of R is an intersection of valuation ideals [16].

g) The Chinese Remainder theorem is valid in R.

Also, other conditions have been given by Jensen [8]. Verification of the equivalence of the above can be found in [10], [13], [15], and [6].

Rings (not necessarily domains) which satisfy property d) are considered in [1] and [5], and are called "Arithmetischer Ringe", or, simply, A-rings. It is shown in [15] that d) and g) are equivalent in arbitrary rings. Also, Prüfer domains have appeared in works in Homological Algebra [3] and [7]. The first part of this paper will be devoted to the study of these conditions (or their analogues in the ring case) in arbitrary rings.

Definition 1.1. R is a Prüfer ring if and only if the ideals are linearly ordered in R_P for each proper prime ideal P.

Definition 1.2. A semi-multiplication ring is a ring R having the following property: if A and B are ideals of R with B finitely generated and $A \subset B$ then there exists an ideal C in R such that $A = BC$.

If we omit the condition "B finitely generated" in the Definition 1.2 we have the ordinary multiplication ring of Krull which is studied in [10] and [11].

Although not all of the conditions a) through g) (or their analogues in the ring case) are equivalent, we will show the equivalence of some. In particular, in Chapter 2 we will prove the concepts of a Prüfer ring, an A-ring, and a semi-multiplication ring coincide. Also, other equivalent conditions will be given.

The remainder of this paper is devoted to the study of the ideal structure in Prüfer rings. Primary ideal structure and dimension problems are considered. In Chapter 3, α-rings are defined and their relation to Prüfer rings is studied.

CHAPTER II

PRÜFER RINGS AND EQUIVALENT CONDITIONS

Our first results in this chapter concern Prüfer rings. Specifically, we will prove the equivalence of several properties, some corresponding to those given in Chapter 1 for integral domains. Recall that the Chinese Remainder Theorem is valid in R if and only if R is an A-ring, so in this chapter we refer only to the latter condition.

The collection of all quotient rings of R of the type R_M, in which the ideals are linearly ordered will be denoted by $\mathcal{L}$. If A is an ideal of R, let $\mathcal{L}(A) = \cap A^{ec}$ where the intersection is taken over all rings $S \in \mathcal{L}$, and e and c denote ideal extension and contraction, respectively [15; 218]. With this notation we can now state the result we wish to prove.

<u>Proposition 2.1</u>. In a ring R the following are equivalent:

I. R is an A-ring.

II. R is a Prüfer ring.

III. R is a semi-multiplication ring.

IV. If A, B, and C are ideals of R with C finitely generated, then $(A + B):C = A:C + B:C$.

V. If A, B, and C are ideals of R with A and B finitely generated, then $C:(A \cap B) = C:A + C:B$.

VI. $A = \quad (A)$ for each ideal A in R.

The proof follows and is in several parts. First we need three lemmas.

<u>Lemma 2.2</u>. If extension is with respect to a quotient ring R_M in which $A^e \subset B^e$ or $B^e \subset A^e$, then $(A \cap B)^e = A^e \cap B^e$.

Proof: Suppose $A^e \subset B^e$. Let $x \in A^e \cap B^e = A^e$. If f is the natural map from R to R_M, then there exist $a \in A$ and $m \in M$ such that $x = f(a)[f(m)]^{-1}$. Since $A^e \subset B^e$ there exists $k \in M$ such that $ak = b \in B$. Therefore, $x = f(ak)[f(mk)]^{-1} \in (A \cap B)^e$. The other containment is always true, giving us $(A \cap B)^e = A^e \cap B^e$.

<u>Lemma 2.3</u>. If R is a ring in which the principal ideals are linearly ordered, then all of the ideals are linearly ordered.

Proof: This is clear.

<u>Lemma 2.4</u>. If $B \subset (b_1,\ldots,b_n)$ are ideals of R and $B^e = (b_1\ldots,b_n)^e = (b_n)^e$ in R_M, then $(A:B)^e = A^e:B^e$.

Proof: We always have $(A:B)^e \subset A^e:B^e$ so we need only prove the other containment. Let $x \in R$ with $f(x) \in A^e:B^e$ (f is the map from R to R_M). Now $B^e(x)^e = (b_n)^e(x)^e = (b_nx)^e \subset A^e$. Therefore, there is a $y_n \in M$ such that $b_nxy_n \in A$. For each i, $1 \leq i \leq n - 1$, let $y_i \in M$ be such that $b_iy_i \in (b_n)$. Set $y = y_1y_2 \ldots y_n$. For $b \in B$ we have $by_1 \ldots y_{n-1} \in (b_n)$, hence $byx \in A$. Therefore $yx \in A:B$, but $y \in M$ implies $f(x) \in (A:B)^e$. As a result $(A:B)^e = A^e:B^e$.

I implies II

Proof: Suppose a and b are in R, an A-ring, and P is a proper prime ideal of R. Then $(a - b) + [(a) \cap (b)] = (a,b)$, hence there exist $x \in R$ and $y \in (a) \cap (b)$ such that $a = x(a - b) + y$. If $x \in P$ then $1 - x \not\in P$. Since $(1 - x)a = y - xb \in (b)$ we have $(a)^e \subset (b)^e$. On the other hand, if $x \not\in P$ then $bx = ax - a + y \in (a)$, hence $(b)^e \subset (a)^e$. Therefore, the principal ideals of R_P are linearly ordered, and by Lemma 2.3 the ideals of R_P are linearly ordered.

II implies III

Proof: Suppose R is a Prüfer ring with ideals A and C, A is finitely generated, and $C \subset A$. Let $B = \{x \in R \mid xA \subset C\}$. B is an ideal. By showing $(AB)^e = C^e$ where extension is done to R_P, for each proper prime P, we have $AB = C$. Moreover, it is clear that $(AB)^e \subset C^e$ so we need only verify the other containment. Let $c \in C$. Set $c = a_k$ and $A = (a_1,\ldots,a_k,\ldots,a_n)$, where the a_i are indexed so that $(a_1)^e \subset \ldots \subset (a_k)^e \subset \ldots \subset (a_n)^e$. For $1 \leq i \leq n - 1$ there exist x_i, y_i in R, with $x_i \not\in P$, such that $a_ix_i = a_{i+1}y_i$. Let $y = (\prod_{i=1}^{k-1} x_i)(\prod_{i=k}^{n-1} y_i)$. It is easily seen that $y \in B$ and hence $a_ny \in AB$. Moreover, $a_ny = c(\prod_{i=1}^{n-1} x_i)$ and $(\prod_{i=1}^{n-1} x_i) \not\in P$ imply

$(c)^e \subset (AB)^e$. Therefore, $AB = C$ and R is a semi-multiplication ring.

III implies II

Proof: Suppose R is a semi-multiplication ring. Let a and b be in the proper prime ideal P of R. $(b) \subset (a,b)$ implies there exists an ideal C such that $(a,b)C = (b)$. Let $x,y \in C$ such that $ax + by = b$. If $y \notin P$ since $ay \in (b)$ we have $(a)^e \subset (b)^e$. On the other hand, if $y \in P$ then $1 - y \notin P$. Moreover, $b(1 - y) = ax \in (a)$ and $1 - y \notin P$ imply $(b)^e \subset (a)^e$. It follows that R is a Prüfer ring.

II implies I

Proof: This implication is obtained using Lemma 2.2 and the fact that the distributive laws required for an A-ring are valid in a ring with linearly ordered ideal system.

II implies IV and II implies V

Proof: It is easily seen that IV and V are valid in a ring with linearly ordered ideal system. Since this is true for each R_P it is necessary only to show that the operations of addition, intersection, and quotient ideal formation are preserved in going from R to R_P, with the desired ideals. Intersection is preserved by Lemma 2.2. Also, for the ideals considered in IV and V we have the hypothesis of Lemma 2.4 satisfied, giving us the preservation of quotient ideal formation. Addition is always preserved.

IV implies II

Proof: Suppose $(A + B):C = A:C + B:C$, whenever C is finitely generated. Let $a,b \in R$ and P be a proper prime ideal of R. For $A = (a)$, $B = (b)$, and $C = (a,b)$ the above formula becomes $R = (a):(b) + (b):(a)$. Therefore there exist $x \in (a):(b)$ and $y \in (b):(a)$ such that $1 = x + y$. Now x, an element of $(a):(b)$, implies $bx = za$ for some $z \in R$. Moreover, $b = bx + by = za + by$. $y \in (b):(a)$ implies $ay \in (b)$. If $y \notin P$ then $(a)^e \subset (b)^e$. If $y \in P$ then $1 - y \notin P$. Since $(1 - y)b = za \in (a)$ we have $(b)^e \subset (a)^e$. Therefore, by Lemma 2.3, the ideals of R_P are linearly ordered, hence R is a Prüfer ring.

V implies II

Proof: Suppose $C:(A \cap B) = C:A + C:B$ whenever A and B are finitely generated. Let $a,b \in P$, a proper prime ideal of R. Set $C = (a) \cap (b)$, $A = (a)$, and $B = (b)$. The above formula then gives us $R = [(a) \cap (b)]:(a) + [(a) \cap (b)]:(b)$. Therefore, there exist $x \in [(a) \cap (b)]:(a)$ and $y \in [(a) \cap (b)]:(b)$ such that $1 = x + y$. Since $ax \in (a) \cap (b) \subset (b)$ there exists a z in R such that $ax = bz$. Thus $a = ax + ay = bz + ay$. If $y \in P$ then $1 - y \notin P$ and since $(1 - y)a = bz$ we get $(a)^e \subset (b)^e$. If $y \notin P$ then $y \in [(a) \cap (b)]:(b)$ and we have $yb \in (a)$. Therefore, $(b)^e \subset (a)^e$. Thus the ideals of R_P are linearly ordered and R is a Prüfer ring.

II implies VI

Proof: If $a \in \mathcal{L}(A) \setminus A$ then $A:(a) = \{x \in R \mid xa \in A\} < R$. Let P be a proper prime ideal such that $A:(a) \subset P$. Thus $(A:(a))^e < R_P$. But, $R_P \in \mathcal{L}$ and $a \in \mathcal{L}(A)$ imply $a \in A^{ec}$. Hence, if $f: R \longrightarrow R_P$ is the natural map, $f(a) \in A^e$. Therefore, there exists $x \notin P$ such that $ax \in A$, which says $x \in A:(a)$. However, $f(x)$ is a unit in R_P so $(A:(a))^e = R_P$, a contradiction. Thus $\mathcal{L}(A) \subset A$. The other containment always holds so they are equal.

VI implies I

Proof: Let A, B, and C be ideals of R. If R_M is in $\mathcal{L}$ then R_M is an A-ring. When extension is done to $R_M \in \mathcal{L}$ we get, using Lemma 2.2, $[A \cap (B + C)]^e = A^e \cap (B^e + C^e) = (A^e \cap B^e) + (A^e \cap C^e) = [(A \cap B) + (A \cap C)]^e$. Therefore, $\mathcal{L}[A \cap (B + C)] = \mathcal{L}[(A \cap B) + (A \cap C)]$ and R must be an A-ring. This completes the proof of Proposition 2.1.

Several properties of Prüfer rings are immediate corollaries to Proposition 2.1. Corollaries 2.5 and 1.6, which follow, correspond to the properties a) and c) given in Chapter I. They are equivalent to I through VI if R is an integral domain, but, in a ring with proper zero divisors this need not be true.

Corollary 2.5. Finitely generated regular ideals of a Prüfer ring are invertible.

Proof: Let a be a regular element of the finitely generated ideal A. By property III there exists an ideal C such that AC = (a). Since a is regular this implies A is invertible.

Corollary 2.6. Suppose A, B, and C are ideals in a Prüfer ring with A finitely generated and regular. If AB = AC then B = C.

Proof: Apply Corollary 2.5.

Corollary 2.7. A Prüfer ring is integrally closed in its total quotient ring.

Proof: It is easily shown using the same method as in [13], that any ring satisfying the cancellation law given in Corollary 2.6 is integrally closed in its total quotient ring.

Corollary 2.8. If $P \not\subset Q$ and $Q \not\subset P$ are prime ideals of a Prufer ring R, then P + Q = R.

Proof: Let $p \in P \setminus Q$ and $q \in Q \setminus P$. Using property I, $(p - q) + (p) \cap (q) = (p,q)$. Therefore, there exist $x \in R$ and $y \in (p) \cap (q)$ such that $q = x(p - q) + y$. Moreover, $x \in Q$ and $1 + x \in P$ imply $1 \in P + Q$.

One of the properties valid for Prüfer domains is that any ring between a Prüfer domain and its quotient field is a Prüfer domain. The next theorem shows that this property is valid for Prüfer rings.

<u>Lemma 2.9</u>. Suppose R and S are rings such that $R \subset S \subset T$, where T is the total quotient ring of R. If the ideals of R are linearly ordered, then so are those of S.

Proof: Let $a = \frac{x}{y}$ and $b = \frac{w}{z}$ be in S, with $x, y, w, z \in R$ and y and z regular in R. The lemma is verified if we show either $a \in (b)S$ or $b \in (a)S$. This is clearly true if both a and b are in R. However, if $a \notin R$ then $y \in (x)R$ which implies that x is regular in R and $\frac{y}{x} \in S$. Hence $b \in (a)S = S$. This completes the proof of the lemma.

<u>Theorem 2.10</u>. If R is a Prüfer ring with total quotient ring T and if S is a ring such that $R \subset S \subset T$, then S ia a Prüfer ring.

Proof: Let M be a proper prime ideal in S and set $P = M \cap R$. Consider the following diagram, where f and g are the natural maps:

$$\begin{array}{ccc} M \subset S & \xrightarrow{f} & S_M \\ \cup & & \uparrow h \\ P \subset R & \xrightarrow{g} & R_P \end{array}$$

To prove the theorem we will define the homomorphism h so that $h(R_P) \subset S_M \subset K$, where K is the total quotient ring of $h(R_P)$. Since the ideals are linearly ordered in R_P, hence in $h(R_P)$, we get by Lemma 2.9, the same is true in S_M. Therefore S is a Prüfer ring. Define $h: R_P \longrightarrow S_M$ by $h(\frac{g(x)}{g(y)}) = \frac{f(x)}{f(y)}$ where $x, y \in R$. It is easy to verify that h is a well defined homomorphism using the characterization of the kernels of the maps f and g given in [15]. To show $S_M \subset K$ let $a \in S_M$. There exist b and c in S such that $a = \frac{f(b)}{f(c)}$ where $c \not\in M$. Let $b = \frac{x}{y}$ and $c = \frac{w}{z}$, where x, y, w, z $\in$ R with y and z regular in R. We must then have $f(yb) = f(x)$ and $f(zc) = f(w)$. Both y and z regular in R implies f(y) and f(z) are regular in $h(R_P)$. Therefore $f(b) = \frac{f(x)}{f(y)}$ and $f(c) = \frac{f(w)}{f(z)}$ are elements of K. But f(c) is regular in S_M and thus f(w) is also regular in $h(R_P)$. Therefore, $a = \frac{f(b)}{f(c)}$ is an element of K. This completes the proof of the theorem.

The next theorem deals with the relationship between the dimension of a Prufer ring and that of its polynomial ring. Seidenberg [14] proves if R is an n dimensional Prüfer domain, then R[x] is n + 1 dimensional. The same holds in case R is a Prüfer ring.

<u>Theorem 2.11</u>. If R is an n dimensional Prüfer ring and x is an indeterminate over R, then R[x] has dimension n + 1.

Proof: Let $(0) < Q_1 < \dots < Q_r < R[x]$ be a chain of prime ideals in R[x]. $P_1 = Q_1 \cap R$ is a prime ideal in R and $P_1[x] = \{\sum_{i=0}^{k} a_i x^i \mid a_i \in P$ and k a positive integer$\}$ is a prime ideal of R[x]. Moreover $(0) \subset P_1[x] \subset Q_1$. If $P_1 = (0)$ then R is an integral domain and we have the result by [14]. Therefore, we assume $(0) < P_1$ and hence, $(0) < P_1[x]$. Now R/P_1 is a Prüfer domain and $\dim (R/P_1) \leq n - 1$.

By Seidenberg's theorem we have dim $(R/P_1[x]) \leq n$. But $R/P_1[x] \simeq R[x]/P_1[x]$, and $P_1[x] \subset Q_1$ imply $r - 1 \leq n$. Hence $r \leq n + 1$ which verifies that $R[x]$ has finite dimension and, if $s = \dim (R[x])$ then $s \leq n + 1$. Now, let $(0) < P_1 < \ldots < P_n < R$ be a chain of prime ideals in R. Again, R/P_1 is a Prüfer domain of dimension $n - 1$ so $R/P_1[x]$ has dimension n. But $R/P_1[x] \simeq R[x]/P_1[x]$ and $(0) < P_1[x]$ imply $n \leq s - 1$. Therefore $n + 1 = s$ which completes the proof of the theorem.

We now return to the properties given in Corollaries 2.5 and 2.6. First, we will state the main proposition and then, after proving three lemmas, we will complete the proof.

<u>Proposition 2.12</u>. In a ring R the following are equivalent:

VII. $A(B \cap C) = AB \cap AC$ where A, B, and C are ideals with one of B or C regular.

VIII. $(A + B)(A \cap B) = AB$ where one of A or B is regular.

IX. Finitely generated regular ideals are invertible.

X. A, B, and C ideals of R, A finitely generated and regular, and $AB = AC$ imply $B = C$.

<u>Lemma 2.13</u>. Let a, b, and c be in R, a regular, P a proper prime ideal of R, and $(a)^e \subset (b)^e$ in R_P. If property VIII is valid in R then either $(b)^e \subset (c)^e$ or $(c)^e \subset (b)^e$.

Proof: Suppose $x \in R$ and $y \in R \setminus P$ are such that $ay = bx$.

By VIII, $(a,b,c)[(a,b) \cap (c)] = (a,b)(c)$. Therefore $bc = x_1a + x_2b + x_3c$ where $x_i \in (a,b) \cap (c)$, for $i = 1, 2$, and 3. If $x_3 = au + bv$ then $x_3y = bxu + byv$. Now $bc = x_1a + x_2b + acu + bcv$ so $bcy = x_1bx + x_2yb + bxcu + bcvy$. Therefore, $bc[y - (xu + yv)] = (x_1x + x_2y)b$. If $z = xu + yv \notin P$ then $(b)^e \subset (c)^e$ because $x_3y = zb \in (c)$. On the other hand, if $z \in P$ since $y \notin P$ we have $y - z \notin P$. Also x_1 and x_2 are in (a,b) so we get $(c)^e(b)^e \subset (a,b)^e(b)^e = (b)^e(b)^e$. Moreover, $(b)^e$ is regular in R_P which implies $(c)^e \subset (b)^e$. This proves the lemma.

<u>Lemma 2.14</u>. Suppose R is a ring in which property VIII is valid. If A is finitely generated, P is a proper prime ideal, and c is a regular element of A then there exist $a_1,\ldots,a_n \in A$ indexed so that $A = (a_1,\ldots,a_n)$, $c = a_k$ for some k, and in $R_P(a_1,\ldots,a_{k-1})^e \subset (a_k)^e \subset (a_{k+1})^e \subset \ldots \subset (a_n)^e$.

Proof: For each generator b of A we have, by letting $a = c^2$ in Lemma 2.13, either $(c)^e \subset (b)^e$ or $(b)^e \subset (c)^e$. Let $a_1,\ldots,a_{k-1}$ be all the generators of A for which $(a_i)^e \subset (c)^e$. Let $c = a_k$. If b_1,b_2 are other generators of A, again by Lemma 2.13, with $a = c$, $b_1 = b$, and $b_2 = c$, either $(b_1)^e \subset (b_2)^e$ or $(b_2)^e \subset (b_1)^e$. Therefore we can index the remaining generators in the desired manner.

<u>Lemma 2.15</u>. Suppose $a,b \in R$, a ring in which property X holds, and P is a proper prime ideal of R. If c is regular

in R, and $(c)^e \subset (a^2)^e$ in R_P, then either $(a)^e \subset (b)^e$ or $(b)^e \subset (a)^e$.

Proof: Since X is valid in R and $(ab)(a,b,c) \subset (a,b,c)(a^2,b^2,c)$ we have $\overline{ab} \in (a^2,b^2,c)$. Moreover, $(c)^e \subset (a^2)^e$ implies $(ab)^e \subset (a^2,b^2,c)^e = (a^2,b^2)^e$. Therefore, there exists $y \not\in P$ such that $aby = xa^2 + zb^2$. Also $(zb)(a,b,c) \subset (a,c)(a,b,c)$ implies $zb \in (a,c)$. Since $(c)^e \subset (a^2)^e \subset (a)^e$ we have $(zb)^e \subset (a,c)^e = (a)^e$. Therefore, there exists $v \not\in P$ such that $zbv = au$. But we had $aby = xa^2 + zb^2$ so $abyv = xva^2 + zvb^2$ or $abyv = xva^2 + abu$ which implies $(a)(b)(yv - u) \subset (a)(a)$. If $u \not\in P$ then $(a)^e \subset (b)^e$ because $zbv = au$. On the other hand, $u \in P$ implies $yv - u \not\in P$. Therefore $(a)^e(yv - u)^e(b)^e = (a)^e(b)^e \subset (a)^e(a)^e$. But c regular in R and $(c)^e \subset (a)^e$ imply $(a)^e$ is regular in R_P. Hence $(b)^e \subset (a)^e$ completing the proof of the lemma.

<u>VII implies VIII</u>

Proof: If VII is valid in R then $(A + B)(A \cap B) = [(A + B)A] \cap [(A + B)B] = [A^2 + AB] \cap [AB + B^2] \supset AB$. The other containment is always true giving us $(A + B)(A \cap B) = AB$.

<u>VIII implies IX</u>

Proof: Let A be finitely generated and c be a regular and c be a regular element of A. Set $B = (c):A$. To show A is invertible it is sufficient to prove $AB = (c)$. Since

$AB \subset (c)$ we reduce the problem to showing $(c)^e \subset (AB)^e$ for each proper prime ideal P in R. By Lemma 2.14, let $a_1, \ldots, a_n \in A$ be indexed such that $A = (a_1, \ldots, a_n), c = a_k$ for some k, and in R_P $(a_1, \ldots, a_{k-1})^e \subset (a_k)^e \subset \ldots \subset (a_n)^e$. For each i, $1 \leq i \leq n - 1$ let $x_i, y_i \in R$ with $y_i \not\in P$, be such that $a_i y_i = a_k x_i$ if $1 \leq i \leq k - 1$, and $a_i y_i = a_{i+1} x_i$ if $k \leq i \leq n - 1$. If $b = y_1 \ldots y_{k-1} x_k \ldots x_{n-1}$ then $b \in B$, hence $a_n b \in AB$. But $a_n b = y_1 \ldots y_{n-1} a_k$ and $y_1 \ldots y_{n-1} \not\in P$ implies $(a_k)^e = (c)^e \subset (AB)^e$. This completes the proof of VIII implies IX.

IX implies X

Proof: This is clear.

X implies VII

Proof: Let A, B and C be ideals of R with B regular, and let P be a proper prime ideal. If $C^e \not\subset B^e$ then there exists $c \in C$ such that $(c)^e \not\subset B^e$. If b is regular in B then b^2 is regular, and $(b^2)^e \subset (b)^e$ implies by Lemma 2.5 either $(b^e) \subset (c^e)$ or $(c)^e \subset (b^e)$. Since $(c)^e \not\subset (b)^e$ we have $(b)^e \subset (c)^e$. If x is any element of B, since $(b^2)^e \subset (c)^e$ we have by Lemma 2.5, $(x)^e \subset (c)^e$. Therefore $B^e \subset C^e$. We have shown if B and C are ideals of R, with one of B or C regular, then either $B^e \subset C^e$ or $C^e \subset B^e$. Therefore $(AB)^e \subset (AC)^e$ or $(AC)^e \subset (AB)^e$ and by Lemma 2.2 $[A(B \cap C)]^e = A^e(B \cap C)^e = A^e(B^e \cap C^e) = A^eB^e \cap A^eC^e = (AB)^e \cap (AC)^e = [AB \cap AC]^e$. This is true for each proper prime ideal P of R, so $A(B \cap C) = AB \cap AC$. This completes the proof of Proposition 2.12.

In case R has a unique maximal ideal one other condition can be added to the list in Proposition 2.12.

Corollary 2.16. Suppose R is a ring with a unique maximal ideal P. The conditions given in Proposition 2.12 are equivalent to the following:

XI. The regular ideals of R are linearly ordered and the zero divisors, Z, form a prime ideal which is contained in every regular ideal.

Proof: If XI holds in R and one of A or B is regular, then either $A \subset B$ or $B \subset A$. It follows that $(A + B)(A \cap B) = AB$ and hence property VIII is valid. Conversely, suppose property VIII is valid in K. Since $R = R_P$ we have by Lemma 2.13, if $a,b \in R$ with a regular, then $(a) \subset (b)$ or $(b) \subset (a)$. This gives us the regular ideals are linearly ordered and for each regular ideal A, $Z \subset A$. If $a,b \in Z$ and $a + b \not\subset Z$ then $a + b$ is regular. Therefore there exist $x,y \in R$ such that $a = x(a + b)$ and $b = y(a + b)$. Moreover, $x,y \in Z \subset P$. But $(a + b) = (x + y)(a + b)$ implies $1 = x + y \in P$ which is a contradiction. As a result, Z is closed under addition. The other properties required for Z to be a prime ideal are clearly satisfied.

Remark. If R is a ring for which R_P satisfies property XI for each proper prime ideal P of R, then R satisfies VIII and its equivalent conditions. This is verified by noticing if A or B is regular, then A^e or B^e is regular. As a result, if R_P satisfies property XI, either $A^e \subset B^e$ or $B^e \subset A^e$. Therefore, $[(A + B)(A \cap B)]^e = [AB]^e$ for each proper prime ideal P which implies $(A + B)(A \cap B) = AB$. The converse of this statement is not true. We have the following example.

<u>Example 2.17</u>. Let Q be the rational numbers and x,y, and z be indeterminates over Q. Set $R = Q[x,y,z]_{(x,y,z)}$, with $P_1 = (x,y,z)^e$, $P_2 = (x,y)^e$, and $A = (x^2)^e$ ideals of R. $S = R/P_1A$ has one maximal ideal, P_1/P_1A, and every non-unit is a zero divisor. Therefore S vacuously satisfies the properties about regular ideals. $P' = P_2/P_1A$ is a prime ideal. $S_{P'} = \dfrac{[R/P_1A]_{M+P_1A}}{P_1A} \simeq R_{P_2}/(P_1A)^e$ where extension is to R_{P_2}. This is by the permutability of residue and quotient ring formation. But, $P_2 < P_1$ and hence $(P_1A)^e = P_1{}^eA^e = A^e$. Therefore $S_{P_1} \simeq \dfrac{Q[x,y,z]_{(x,y)}}{(x^2)^e}$. Using bars to denote residues from $Q[x,y,z]_{(x,y)}$ to $S_{P'}$ we have $\bar{y}$ and $\bar{x} + \bar{y}$ are regular elements. In $S_{P'}$, neither $\bar{y} \in (\bar{x} + \bar{y})$ nor $\bar{x} + \bar{y} \in (\bar{y})$, hence the regular ideals are not linearly ordered. This gives us the desired example.

<u>Remark</u>. As we noticed in Corollary 2.5 and Corollary 2.6, any ring satisfying properties I through VI will also satisfy VII through X. There are simple examples showing

the converse is not true, since any ring in which every non-unit is a zero divisor will satisfy VII through X. In this type of ring, properties VII through X place very little restriction on the ideal theory, which is one reason the stronger conditions I through VI were used in defining a Prüfer ring.

CHAPTER III

α-RINGS AND PRIMARY IDEALS OF PRÜFER RINGS

This chapter is concerned primarily with the primary ideal structure of a Prüfer ring. Ohm [12] considers this in the integral domain case. Also we consider α-rings, which are studied in [2], and their relationship with Prüfer rings.

Definition 3.1. R satisfies the ascending chain condition for prime ideals (a.c.c. for primes) if and only if every strictly ascending chain of prime ideals is finite.

Definition 3.2. R is an α-ring if and only if the a.c.c. for prime ideals is valid and every primary ideal is a prime power.

The natural questions which arise are: Under what conditions is an α-ring a Prüfer ring? Conversely, when is a Prüfer ring an α-ring? In either an α-ring or a Prüfer ring, when are prime power ideals primary? Answers will be given to these questions in this chapter. Note, that if R is an α-ring then R_P is an α-ring for each proper prime ideal P. As a result, the question of when an α-ring is a Prüfer ring can be reduced to: "When does an α-ring, with a unique maximal ideal, have a linearly ordered ideal system?". It is not always true that such a ring has a linearly ordered ideal system as will be seen later by example. The next few theorems deal with this problem. We will use the terminology of [16] and call a ring with a unique maximal ideal a quasi-local ring.

<u>Theorem 3.3</u>. Suppose R is an α-ring with unique maximal ideal P. If P_1 is prime and $P_1 < P$, then $P_1 \subset P^n$ for every positive integer n and $P^2 < P$.

Proof: We can assume P_1 is maximal in P since the a.c.c. for prime ideals holds in R. Therefore, if $x \in P \setminus P_1$ then $x^i \not\in P_1$ and $\sqrt{P_1 + (x)^i} = P$ for each positive integer i. P maximal implies that $P_1 + (x)^i$ is P-primary, and hence there exists a positive integer e_i such that $P_1 + (x)^i = P^{e_i}$. Moreover, $i < j$ implies that $e_i \neq e_j$, for otherwise $P_1 + (x^i) = P_1 + (x^j)$ and in the integral domain R/P_1 $x + P_1$ would be a unit. Therefore $e_1 < e_2 < \ldots$ and also $P > P^2 > \ldots$, which is the second assertion. For each positive integer n there exists $e_i > n$ which implies that $P_1 \subset P_1 + (x^i) = P^{e_i} \subset P^n$.

<u>Corollary 3.4</u>. Suppose R, P_1, and P are as in Theorem 3.3. If P_1 is maximal in P then $P_1 = \cap P^n$, the intersection taken over all positive integers n.

Proof: It is sufficient to show $\cap P^n$ is prime. Suppose $ab \in \cap P^n$ but $a,b \not\in \cap P^n$. There exists an integer m such that $P^m \subset [P_1 + (a)][P_1 + (b)] \subset P_1 + (ab)$. Therefore $P^m = P^{m+1} = \ldots$, which is a contradiction.

Corollary 3.5. Let P and P_1 be prime ideals in an α-ring R. If $P < P_1$ then $P \subset P_1^n$ for each positive integer n and $P_1 \neq P_1^2$.

Proof: Let P_2 be a maximal prime for which $P \subset P_2 < P_1$. In R_{P_1}, we have by Corollary 3.4, $P_2^e = \cap(P_1^n)^e$. Therefore, $P_2 = P_2^{ec} = \cap(P_1^n)^{ec}$. Since $(P_1^n)^{ec}$ is a P_1-primary ideal there exists a sequence of positive integers $n_1 \leq n_2 \leq \ldots$ such that $P_2 = \prod_{i=1}^{\infty} P_1^{n_i}$. Moreover, P_2 prime implies that the sequence $\{n_i\}$ can be taken to be strictly increasing. Therefore $P \subset P_2 \subset P_1^n$ for each positive integer n.

Remark. We observe (under the conditions of Corollary 3.5) that $\cap P_1^n = P'$ is the unique prime ideal such that $P' < P_1$ and there are no prime ideals properly between P_1 and P'.

Theorem 3.6. If R is a quasi-local α-ring then the prime ideals of R are linearly ordered.

Proof: Suppose the theorem is not true. Let $F = \{(P_i,P_j) \mid P_i \not\subset P_j \text{ and } P_j \not\subset P_i \text{ with } P_i, P_j \text{ primes}\}$. Partially order F as follows: $(P_i,P_j) \leq (P_m,P_n)$ if and only if $P_i \subset P_m$ and $P_j \subset P_n$. Let C be a maximal chain in F and $P_1 = \cup P_i$, $P_2 = \cup P_j$ where i and j are such that (P_i,P_j) is in C. Since the a.c.c. for prime ideals holds in R, there exists a j such that (P_1,P_j) is in C. This says $P_j \not\subset P_1$, hence $P_2 \not\subset P_1$. A similar argument gives us $P_1 \not\subset P_2$. Therefore, (P_1,P_2) is in C. Moreover, if $P_1 < P$, a prime ideal, then $P_2 < P$, and conversely. If P and Q are prime ideals such that $P_1 < P$ and $P_1 < Q$ then $P \subset Q$ or $Q \subset P$. If $P = \cap Q_i$ where the Q_i are all of the prime ideals such that $P_1 < Q_i$, then $P_2 \subset P$ and hence $P_1 < P$ and $P_2 < P$. Since there is a one-to-one, order preserving correspondence between the prime ideals in R_P and those in R contained in P, we assume P is the maximal ideal of R. But Corollary 3.4 gives us $P_1 = \cap P^n = P_2$ which is a contradiction. Therefore $F = \emptyset$ and the prime ideals of R are linearly ordered.

The next result is an immediate consequence of the preceding theorem.

Corollary 3.7. If R is a quasi-local α-ring then the set of prime and primary ideals of R is linearly ordered.

Notation: In an α-ring R we denote by N(P) the largest prime ideal properly contained in P. If P is a minimal prime of (0), let N(P) = (0).

Theorem 3.8. If R is an α-ring with a unique maximal ideal, then the set of zero divisors of R is a prime ideal.

Proof: For the remainder of this chapter we will let Z denote the set of zero divisors of R. If $Z > (0)$, there exists a prime ideal P such that $[P \setminus N(P)] \cap Z \neq \emptyset$. Let P_0 be the largest prime ideal with this property. Clearly,

$Z \subset P_0$. Also, by the choice of P_0, there exists $x \in [P_0 \setminus N(P_0)] \cap Z$. In R_{P_0}, $(x)^e$ is primary for $P_0{}^e$ so $(x)^e = (P_0{}^e)^n$ for some positive integer n. Therefore if $p \in P_0$ there exist $v \notin P_0$ and $u \in R$ such that $p^n v = ux$. But $p^n v \in Z$ and $v \notin Z$ imply $p^n \in Z$, hence $p \in Z$. Thus $P_0 = Z$. The theorem is obvious in case $Z = (0)$.

Lemma 3.9. Let R be a quasi-local α-ring. Suppose A is an ideal such that

1) Either A is prime and $N(Z) \subset A$, or $N(Z) = A$.
2) P prime and $A < P$ imply P has the following property: if $x,y \in R$ with $x \notin P$ then either $x \in (y)$ or $y \in (x)$.

Then A itself satisfies the property given in 2).

Proof: Let $x,y \in R$ with $x \notin A$. If P is the minimal prime ideal of (x), then $A < P$. Moreover, since the prime ideals of R are linearly ordered, $(x)^e$ is P^e-primary in R_P and therefore $(x)^e = (P^n)^e$ for some n. Now $y \notin P$ implies $(x)^e \subset (y)^e$, $y \in N(P)$ implies $(y)^e \subset (x)^e = (P^n)^e$, and $y \in P \setminus N(P)$ implies that there exists an m such that $(y)^e = (P^m)^e$. In all cases we get either $(x)^e \subset (y)^e$ or $(y)^e \subset (x)^e$.

Case 1: $(x)^e \subset (y)^e$. Therefore, there exist $u \in R$ and $v \in R \setminus P$ such that $xv = yu$. But $v \notin P$ implies either $v \in (u)$ or $u \in (v)$. If $v \in (u)$ then $xv = xwu = yu$. Also, $v \notin P$ implies that v is regular and hence u is regular. Thus $y = wx \in (x)$. If $u \in (v)$ then $xv = yu = ywv$, hence $x = yw \in (y)$.

Case 2: $(y)^e \subset (x)^e$. This is treated in the same manner as Case 1. This proves the lemma.

Theorem 3.10. If R is a quasi-local α-ring then the ideals are linearly ordered through N(Z).

Proof: By using the a.c.c. for primes and Lemma 3.9 we can conclude if $x \notin N(Z)$ and $y \in R$ then either $x \in (y)$ or $y \in (x)$. This gives us if A is any ideal then either $A \subset N(Z)$ or $N(Z) \subset A$. Moreover, if A and B are any two ideals such that $N(Z) \subset A$ then either $A \subset B$ or $B \subset A$.

The preceding theorem in conjunction with Theorem 3.8 gives us: if R is an α-ring and P is a proper prime ideal of R, then in R_P the regular ideals are linearly ordered, and the zero divisors form a prime ideal contained in every regular ideal. Together with the remark after Corollary 2.16 in Chapter II this gives us

Corollary 3.11. An α-ring satisfies the equivalent properties VII through X given in Proposition 2.12.

Of course we also have, in case R has a unique maximal ideal, that the ideals of R are linearly ordered whenever N(Z) = (0). This happens if Z is a minimal prime of (0). If (0) is primary then (0) is Z-primary and again we have N(Z) = (0). Other conditions will be given later.

Lemma 3.12. Suppose R is a quasi-local α-ring and P is a proper prime ideal of R. If $y \in P \setminus N(P)$ then for some integer k, $(y)^e = (P^k)^e$ in R_P. Moreover, $P^{k+1} \subset (y)$.

Proof: Let F be the set of primes which do not satisfy the above property. If P is the maximal element of F then $P < M$, the maximal ideal of R, for clearly M is not in F. Now, $y \in P \setminus N(P)$ implies that $(y)^e$ is P^e-primary in R_P and hence $(y)^e = (P^k)^e$ for some integer k. If $x \in P^k$ then there exist $v \notin P$ and $w \in R$ such that $xv = yw$. Since P is maximal in F, there is a prime Q such that $P < Q$ and $Q^n \subset (v)$ for some integer n. By Corollary 3.5, $P \subset Q^n \subset (v)$. If $p \in P$ then $p = vz$ for some z, hence $xvz = ywz = xp \in (y)$. Therefore $P^{k+1} \subset (y)$, which implies $P \in F$, a contradiction. Thus every prime ideal of R satisfies the desired property.

If R is a ring and P is a proper prime ideal of R, K_P will denote the kernel of the natural mapping of R into R_P. Using the characterization of this kernel given in [15], $K_P = \{x \mid \text{there exists } y \notin P \text{ such that } xy = 0\}$. If P is a minimal prime of (0), then K_P is a P-primary ideal.

Corollary 3.13. Suppose R is a quasi-local α-ring and P is the minimal prime ideal of (0). If $K_P = P^n$ then $P^{n+1} = (0)$.

Proof: By the remarks preceding this corollary, K_P is P-primary, hence for some n, $K_P = P^n$. Let $x \in P^n$ and $p \in P$. There exists a $y \notin P$ such that $xy = 0$. If $y \in Q \setminus N(Q)$ where Q is prime, then by Lemma 3.12, $(p) \subset P \subset Q^k \subset (y)$, for some integer k. Therefore $px = 0$ which implies $P^{n+1} =$

(0).

<u>Corollary 3.14</u>. If R is a quasi-local α-ring and if prime power ideals are primary, then the ideals of R are linearly ordered.

Proof: Corollary 3.13 gives us with our added hypothesis that (0) is a primary ideal. The corollary follows from the remarks after Corollary 3.11.

<u>Remark</u>. It will be shown later that in an α-ring all the powers of a prime ideal are primary through K_{P_0} where P_0 is a minimal prime of zero. Of course, if P is not minimal this includes all powers. In the quasi-local ring case this says all the prime powers are primary except possibly those which are zero.

<u>Theorem 3.15</u>. Let R be a quasi-local α-ring and let P_0 be the minimal prime ideal of (0). The ideals of R are linearly ordered through K_{P_0}.

Proof: We assume $P_0 < Z$, for otherwise we have all the ideals are linearly ordered by Theorem 3.10. By Corollary 3.13 we have for some positive integer n, $P_0^n = (0) < P_0^{n-1}$ where $K_{P_0} = P_0^{n-1}$. Suppose x,y ε R with $x \not\in P_0^{n-1}$. The proof will be complete if we can show either $x \varepsilon (y)$ or $y \varepsilon (x)$. We say a prime ideal P has property * if and only if x,y ε R with $x \varepsilon P \setminus [N(p) \cup K_{P_0}]$ implies $x \varepsilon (y)$ or $y \varepsilon (x)$. Using the a.c.c. for primes it is sufficient to show: if every prime ideal Q properly containing P has property *, then P has property *. Suppose P is such a prime. Clearly (in view of Lemma 3.12), the only case we are required to consider is when both x and y are in $P \setminus [N(P) \cup K_{P_0}]$. Therefore in R_P, $(x)^e$ and $(y)^e$ are P^e-primary so either $(x)^e \subset (y)^e$ or $(y)^e \subset (x)^e$. The two cases are treated similarly so we consider the case $(x)^e \subset (y)^e$. There exist $v \notin P$ and $w \in R$ such that $xv = wy$. Also, $v \notin P$ and every prime Q properly containing P has property * imply that $v \in (w)$ or $w \in (v)$.

Case 1: $v \in uw$. This says that $w \notin P$ and $u \notin P$. Moreover, $xv = xuw = yw$ so $(xu - y)w = 0$. Therefore $xu - y \in K_{P_0}$. But $(y)^e = (P^s)^e$ and $y \notin K_{P_0} \cup N(P)$ imply that $K_{P_0} \subset P^{s+1} \subset (y)$ (see Lemma 3.12). Thus there exists $z \in R$ such that $xu - y = zy$. But $y \notin K_{P_0}$ and $xu - y \in K_{P_0}$ imply that $z \in P_0$ since K_{P_0} is primary. Therefore $1 + z$ is a unit in R and $y = (1 + z)^{-1}ux \in (x)$.

Case 2: $w = uv$. Then $xv = yuv$ which implies $(x - yu)v = 0$. Therefore, $x - yu \in K_{P_0}$. By Lemma 3.12 it follows that $x - yu \in (x)$, hence $x - yu = zx$ for some $z \in R$. Again, $x - yu \in K_{P_0}$ and $x \notin K_{P_0}$ imply $z \in P_0$. Therefore, $1 - z$ is a unit in R and $x = (1 - z)^{-1}uy \in (y)$. Thus, $x \in (y)$ or $y \in (x)$ and P has property *. As a result, every prime ideal has property * which proves the theorem.

We have shown in a quasi-local α-ring the ideals are linearly ordered through $K_{P_0} = P_0^{\,n}$, for some positive integer n. We also have $P_0^{\,n+1} = (0)$. Suppose $P_0^{\,n} < P_0$

(i.e., $n > 1$ and $P_0 > (0)$). Let $z \varepsilon P_0 \setminus P_0^n$. If x and y are non-zero elements of P_0^n, then there exist w,u such that $x = wz$ and $y = uz$. Moreover, w and u are not elements of $P_0^n = K_{P_0}$. Therefore, either $w \varepsilon (u)$ or $u \varepsilon (w)$. But $w \varepsilon (u)$ implies $x \varepsilon (y)$, while $u \varepsilon (w)$ implies $y \varepsilon (x)$. As a result we have the ideals of R are linearly ordered.

Summarizing, we have if either $P_0^n = K_{P_0} = (0)$ or if $n > 1$, the ideals of R are linearly ordered. The question that arises is whether or not there exists a quasi-local α-ring, with maximal ideal P, zero-divisors Z, minimal prime P_0, and $(0) = P_0^2 < P_0 = K_{P_0} < Z \subset P < R$, in which the ideals are not linearly ordered. There is one as will be shown by an example at the end of this chapter.

First we consider the converse of some of the preceding results. We have shown a quasi-local α-ring has no proper idempotent primes. Our next goal is to show that a quasi-local ring with no idempotent primes, and ideals linearly ordered through K_P where P is a minimal prime ideal of (0), is an α-ring. At the same time we will develop some theorems on the primary ideal structure of a Prüfer ring.

<u>Theorem 3.16</u>. If the ideals of R are linearly ordered through P, a prime ideal, and if $K_P \subset P^n$, then P^n is P-primary.

Proof: In R_P, P^e is maximal, hence $(P^n)^e = (P^e)^n$ is P^e-primary. Therefore, it is sufficient to show $(P^n)^{ec} \subset P^n$

as the other containment is always true. Let $x \in (P^n)^{ec}$. There exist $y \notin P$ and $b \in P^n$ such that $xy = b = \sum_{i=1}^{m} b_i p_i$, where $b_i \in P^{n-1}$ and $p_i \in P$ (assuming that $n > 1$, for if $n = 1$ the result is obvious). For each i, since $y \notin P$ and $p_i \in P$ there exists $p_i' \in P$ such that $p_i = p_i' y$. Hence $xy = \sum b_i p_i' y = (\sum b_i p_i')y$. Moreover $y \notin P$ implies $x - \sum b_i p_i' \in K_P \subset P^n$. Therefore, there exists $u \in P^n$ such that $x = \sum b_i p_i' + u \in P^n$. This completes the proof of the theorem.

The following lemma is a generalization of a lemma in a paper by Ohm [12].

Lemma 3.17. If $\sqrt{Q} = P$, a prime ideal, and for each maximal ideal M of R, Q^e is P^e-primary in R_M, then Q is P-primary.

Proof: Let $Q^* = Q^{ec}$ where extension and contraction is with respect to R_P. Since $\sqrt{Q} = P$ then Q^* is a P-primary ideal. If M is a maximal ideal, then $Q^e \subset (Q^*)^e$ in R_M. Let $x \in (Q^*)^e$ and f be the natural map from R to R_M, then $x = f(q)[f(m)]^{-1}$ where $q \in Q^*$ and $m \notin M$. Now $q \in Q^*$ implies there exists $y \notin P$ such that $qy = q' \in Q$. Therefore, $xf(y) = f(qy)[f(m)]^{-1} = f(q')[f(m)]^{-1}$ which is in Q^e. But Q^e is P^e-primary (by assumption) and $f(y) \notin P^e$, so $x \in Q^e$. Thus, $Q^e = (Q^*)^e$ for each maximal ideal M of R which implies $Q = Q^*$ and hence Q is P-primary.

Lemma 3.18. Suppose P, M, and A are ideals of R with P and

M prime and $P \subset M$. Let K_P be the kernel of the map from R to R_P and K_{P^e} be the kernel of the map from R_M to $(R_M)_{P^e}$. If $K_P \subset A$ then $K_{P^e} \subset A^e$ where extension is to R_M.

Proof: x is in K_{P^e} if and only if there exists a $y \not\in P^e$ such that $xy = 0$. Let $f:R \longrightarrow R_M$. For some a, b, m, n ε R with $m,n \not\in M$ and $b \not\in P$ we have $x = f(a)[f(m)]^{-1}$ and $y = f(b)[f(n)]^{-1}$. Now $xy = 0$ implies that $f(a)f(b) = f(ab) = 0$ and therefore, there exists $z \not\in M$ (hence $z \not\in P$) such that $abz = 0$. But $bz \not\in P$ implies $a \,\varepsilon\, K_P \subset A$. Therefore, $x \,\varepsilon\, A^e$ completing the proof.

<u>Theorem 3.19</u>. Suppose R is either a Prüfer ring or an α-ring. If P is a prime ideal of R and $K_P \subseteq P^n$ then P^n is P-primary.

Proof: Let M be a maximal ideal of R. If R is a Prüfer ring the ideals of R_M are linearly ordered. By Theorem 3.15, if R is an α-ring the ideals are at least linearly ordered through P^e. Using Lemma 3.17, if $P \subset M$ we have $K_{P^e} \subset (P^e)^n$ in R_M which implies by Theorem 3.16 $(P^e)^n$ is P^e-primary. This is also true if $P \not\subset M$. Since this argument is valid for each maximal ideal M we have (by Lemma 3.17) P^n is P-primary.

<u>Remark</u>. Suppose R is a ring, P a prime ideal of R and P^n is P-primary. If $x \,\varepsilon\, K_P$ then there exists $y \not\in P$ such that $xy = 0$. Therefore, $xy \,\varepsilon\, P^n$ which implies $x \,\varepsilon\, P^n$. We get then a partial converse of Theorem 3.19: if P^n is P-primary

then $K_P \subset P^n$. We also note that Theorem 3.19 verifies the remark made after Corollary 3.14.

Lemma 3.20. Suppose $P \neq P^2$ is a prime ideal and A is a P-primary ideal. If the ideals of R are linearly ordered through A then $A = P^n$ for some integer n.

Proof: We first show there exists an integer k such that $P^k \subset A$. Since the ideals are linearly ordered through A either $P^n \subset A$ or $A \subset P^n$ for each positive integer n. Suppose A is always contained in P^n. Since A is P-primary it follows that $P^* = \cap P^n$ is not prime. Thus, there exist $a, b \not\in P^*$ such that $ab \in P^*$. Since $A \subset P^*$ implies $a, b \not\in A$, we may assume $(a) \subset (b)$. For some positive integer m we have $A \subset P^* \subset P^m \subset (a) \subset (b)$, hence $(a^2) \subset (ab) \subset P^* \subset P^{2m} \subset (a^2)$. Therefore, we have for a positive integer k, $(a^2) = P^* = P^k = P^{k+1} = \ldots$. Now $a^2 \in (a^4) = P^{2k}$ so there exists $u \in R$ such that $a^2 = ua^4$, hence $a^2(1 - ua^2) = 0 \in A$. But $1 - ua^2 \not\in P$ and A is P-primary so $a^2 \in A$. Therefore $P^k = (a^2) = A$. Let n be the least positive integer such that $P^n \subset A < P^{n-1}$ (assuming that $A \neq P$). If $x \in A$ and $y \in P^{n-1} \setminus A$ then $x \in A < (y) \subset P^{n-1}$. Therefore, there exists $z \in R$ such that $x = zy \in A$. But $y \not\in A$ implies that $z \in P$, hence $x = zy \in P^n$. Thus $A = P^n$ which completes the proof.

Corollary 3.21. If R is a Prüfer ring and $K_P \subset P^n$ then there does not exist a primary ideal A such that $P^n < A < P^{n-1}$.

Proof: $K_P \subset P^n$ implies by Theorem 3.19 that P^n and P^{n-1} are P-primary ideals. The corollary is an immediate consequence of Lemma 3.20 and the one-to-one correspondence between the primary ideals of R and those of R_P.

<u>Theorem 3.22</u>. If R is a ring with a linearly ordered ideal system and P is a prime ideal for which P^n is not primary for some integer n, then $P^n = (0)$.

Proof: Let n be the least integer such that P^n is not primary. By Theorem 3.16 $K_P \not\subset P^n$, hence $P^n < K_P \subset P^{n-1}$. $P^n < K_P$ implies P is the minimal prime ideal of R, hence K_P is P-primary. We then have $K_P = P^{n-1}$ by Lemma 3.20. Let $x \in P^{n-1}$ and $y \in P$. Now $x \in P^{n-1} = K_P$ implies there exists a $z \notin P$ such that $xz = 0$. Moreover, $z \notin P$ and $y \in P$ imply $y \in (z)$, hence $xy = 0$. Therefore $(0) = P^n = P^{n+1} = \dots$.

<u>Theorem 3.23</u>. If R is a Prüfer ring, P_0 and P prime ideals with $(0) \subset P_0 < P$, then P^n is P-primary for every positive integer n.

Proof: Suppose P^n is not P-primary for some n. Lemma 3.17 gives us that for some maximal ideal M, $(P^n)^e$ is not P^e-primary. Since R is a Prüfer ring we have by Theorem 3.22 $(P^n)^e = (0)$. Therefore, $P^n \subset K_M \subset P_0$ which implies $P \subset P_0$, a contradiction. Thus, P^n is P-primary for every positive integer n.

<u>Lemma 3.24</u>. Suppose P_0 and P are prime ideals of R with $P_0 < P < R$. If the ideals of R are linearly ordered through P_0, then $\cap P^n$ is prime.

Proof: For each positive integer n, $P^n \not\subset P_0$ so $P_0 \subset \cap P^n = P^*$. Therefore $K_P \subset P_0 < P^n$ for each integer n which implies P^n is P-primary (Theorem 3.16). Suppose $ab \in P^*$ where neither a nor b is in P^*. Since $a,b \notin P_0$ there exists a positive integer n such that $P^n \subset (a) \subset (b)$ (or $b \in (a)$). This gives us that $P^{2n} \subset (ab) \subset P^* \subset P^{2n}$, hence $(ab) = P^* = P^{2n} = P^{2n+1} = \ldots = (ab)^2$. Therefore, there exists a u in R such that $ab = u(ab)^2$, or $ab(1 - uab) = 0$. $ab \notin P_0$ (otherwise one of a or b would be in P_0, hence in P^*) implies that $1 - uab \in P_0 \subset P$. Hence $1 \in P$, a contradiction. Thus we have P^* is a prime ideal.

<u>Theorem 3.25</u>. If R is a Prüfer ring and $(0) \subset P_0 < P$ are prime ideals, then $\cap P^n$ is prime.

Proof: By Theorem 3.19, P^n is P-primary for each integer n. In R_P we have by Lemma 3.24, $\cap(P^n)^e = (P^*)^e$ where P^* is a prime ideal of R. Therefore, $P^* = [\cap(P^n)^e]^c = \cap P^n$ is prime.

<u>Theorem 3.26</u>. Suppose R is a ring with P_0 the minimal prime of (0) and the ideals of R are linearly ordered through K_{P_0}. If Q is P-primary and P is not idempotent then $Q = P^n$ for some positive integer n.

Proof: Q a P-primary ideal implies $K_{P_0} \subset Q$. We can then apply Lemma 3.20 to get the desired result.

Remark. Suppose R is a ring with no idempotent prime ideals, in which the ideals are linearly ordered through K_{P_0} where P_0 is the minimal prime ideal os zero. If P_1 and P_2 are prime ideals with $P_1 < P_2$ then $P_1 < P_2{}^2$ also. As a result if $P_1 < P_2 < \ldots$ is an infinite, strictly increasing chain of prime ideals, then $P = \cup P_i = \cup P_i^2 = P^2$. But P is a prime ideal which contradicts our hypothesis. Therefore, in such a ring as described, the a.c.c. for prime ideals is valid. This, together with Theorem 3.26, gives us the "converse" mentioned before Theorem 3.16.

Theorem 3.27. Suppose $P \neq P^2$ is a prime ideal of a Prüfer ring R. If Q is a P-primary ideal, and $K_P < Q$ then Q is a prime power ideal. In case P is not a minimal prime ideal of (0) this is all the P-primaries.

Proof: In the proof we consider two cases.

Case 1: P^n is P-primary for each positive integer n. In R_P the ideals are linearly ordered, and $(P^2)^e < P^e$, hence $Q^e = (P^m)^e$ for some integer m, by Theorem 3.26. Therefore, $Q = P^m$.

Case 2: For some integer n, P^n is not P-primary. By Lemma 3.17 there exists a maximal ideal M for which $(P^n)^e$ is not P^e primary. Let k be the least positive integer for which there exists a maximal ideal M such that $(P^k)^e$ is not P^e-primary in R_M. By Theorem 3.22, $(P^k)^e = (0)$. Moreover, by our assumtion on k, if $t < k$ then P^t is P-primary. Therefore, we have $P^k \subset K_M \subset K_P \subset P^{k-1}$. The P-primary ideals of R are linearly ordered and all P-primary ideals contain K_P. If $P^{k-1} \subset Q$ then Q is a prime power by Corollary 3.21. Suppose $K_P < Q \subset P^{k-1}$. In R_P we have $(K_P)^e < Q^e \subset (P^{k-1})^e$, hence by Lemma 3.20, $Q^e = (P^{k-1})^e$. Therefore, $Q = P^{k-1}$. The second assertion of the theorem is obvious.

This theorem completes our discussion of primary ideals in Prüfer rings. Our final goal of this paper is to construct the example mentioned after Theorem 3.15.

Example 3.28. We will construct a ring D having exactly two proper prime ideals, P_1 and P_2, and such that P_1 will be the only P_1-primary ideal and $(0) = P^2 < P_1 < P_2 < D$. Every proper ideal C which is not contained in P_1 will be a power of P_2. As a result D will be an α-ring, but we will show the ideals are not linearly ordered in D. We first construct the ring D and then verify the statements number 1) through 8) which we assume in the construction of D.

Let $R = Q[x,y,z]_{(x,y,z)}$ where Q is the field of rational numbers, and x,y, and z are indeterminates over Q. We note that R is a unique factorization domain with x a prime element. Let $P = (y,z)R$. Then each non-zero p in P has a unique factorization $p = p_1x^t$ $(t \geq 0)$ where $\gcd(x,p_1) = 1$

and $p_1 \in P$. For notational convenience we will usually disregard unit factors of elements of R, since $d \in R$ implies that $d = fg$ where $f \in Q[x,y,z]$ and g is a unit in R. Moreover, if d is not a unit in R then f has constant term zero.

Set $S = \{[rx^s + p]x^{-s} \mid r \in R,\ p \in P$, and s is a non-negative integer$\}$. It is clear that S is a ring between R and $Q(x,y,z)$. For every element a of S we denote by $\theta(a)$ a set of associates in R, defined as follows: If $a \in R$ let $\theta(a)$ be all the associates of a in R. If $a \in S \setminus R$ we can write a in the form $u[rx^s + p]x^{-s}$ where $s > 0$, $\gcd(x,p) = 1$, u is a unit in R, and $r \in Q[x]$. Let $\theta(a)$ be all of the associates of r in R. To show $\theta(a)$ is well defined we must prove that $u[r_1x^s + p_1]x^{-s} = [r_2x^t + p_2]x^{-t}$ with $s,t > 0$, $\gcd(x,p_1) = 1$, $\gcd(x,p_2) = 1$, and $r_1,r_2 \in Q[x]$ implies r_1 and r_2 are associates in R. Since $x \nmid p_1p_2$ it follows that $s = t$, hence $u[r_1x^s + p_1] = r_2x^s + p_2$. Evaluating these polynomials at $y = z = 0$ we get $u'r_1x^s = r_2x^s$, where u' is u evaluated at $y = z = 0$, which is also a unit in R. Therefore $u'r_1 = r_2$ which proves for each a in S, $\theta(a)$ is unique. We note that in defining $\theta(a)$ we have - if $a \in S \setminus R$ there exists an $r \in \theta(a)$ such that $a = u[rx^s + p]x^{-s}$ where $r \in Q[x]$, $(x,p) = 1$, and $s > 0$. Such a representation of a will be called a normalized form of a.

1) a is a unit in S if and only if a is a unit in R.

2) $a \in (x^n)S \setminus R$ if and only if $x^n \mid r$ where $r \in \theta(a)$. $(x)S$ is a prime ideal of S.

3) $A = \{px^{-s} \mid p \in P\} = (y, yx^{-1}, \ldots, z, zx^{-1}, \ldots)S$ is a prime ideal of S and $A < (x)S$. Moreover $A = \bigcap_{n=1}^{\infty} (x^n)S$.

4) In $T = S_{(x)}$, $B^e < A^e$ where $B = (y,z)S$. In particular neither yx^{-1} nor zx^{-1} is in B^e.

5) If M is an ideal of T such that $M \not\subset \bigcap_{n=1}^{\infty} (x^n)T$, then $M = (x^k)T$ for some integer k.

For our example we let $D = T/B^e$ and (using bars to denote residue formation) $P_2 = \overline{(x)T}$ with $P_1 = \overline{AT}$. By 2) and 3), P_1 and P_2 are prime ideals and by 4), $(0) < P_1 < P_2$. 5) gives us if C is an ideal, $C \subset P_2$ and $C \not\subset P_1$ then $C = P_2{}^k$ for some positive integer k.

6) In S if Q is A-primary and $B \subset Q \subset A$ then $Q = A$. Also $A^2 \subset B$.

This gives us P_1 is the only P_1-primary ideal and $(0) = P_1{}^2 < P_1$.

7) If $a = \overline{(yx^{-1})}$ and $b = \overline{(zx^{-1})}$ in D then $a \notin (b)D$ and $b \notin (a)D$.

Therefore D satisfies all the required properties. We now prove statements 1) through 7).

1) a is a unit in S if and only if a is a unit in R.

Proof: Let a be a unit in S. If $a \notin R$ then a has a normalized form $u[rx^s + p]x^{-s}$. Also, $rx^s + p$ not a unit and $\gcd(x,p) = 1$ imply a^{-1} is not in R, hence for another normalized form we get $[rx^s + p]^{-1}x^s = uv[r_1x^t + p_1]x^{-t}$. However s and t positive integers imply that $x \mid p_1p_2$, a contradiction. Therefore if a is a unit in S then a is in R. Since a^{-1} is also a unit in S we have $a^{-1} \in R$, hence a is a unit in R. The other implication is obvious.

2) $a \in (x^n)S \setminus R$ if and only if $x^n \mid r$ where $r \in \theta(a)$. (x)S is a prime ideal in S.

Proof: Suppose $a \in (x^n)S \setminus R$ and $r \in \theta(a)$ is such that $a = u[rx^s + p]x^{-s}$ is a normalized form of a. Therefore, for some normalized form $v[r_1x^t + p_1]x^{-t}$ we have $u[rx^s + p]x^{-s} = x^nv[r_1x^t + p_1]x^{-t}$. Since $\gcd(x,p) = 1$ we must have $t > n$. Hence $a = v[(r_1x^n)x^{t-n} + p_1]x^{n-t}$ is a normalized form of a. Thus $r_1x^n \in \theta(a)$ which implies that $x^n \mid r$. Any element of S can be expressed in the form $u[r + px^{-s}]$, where u is a unit and $r \in Q[x]$. Moreover, $px^{-s} \in (x)S$. Therefore, in order to show (x)S is prime it is sufficient to prove if r_1 and r_2 are in Q[x] and $r_1r_2 \in (x)S$ then either r_1 or r_2 is in (x)S. Suppose there is a normalized form $a = u[rx^s + p]x^{-s}$ such that $r_1r_2 = xa$. Clearly $s = 1$ and we have $r_1r_2 = urx^s + up$. This cannot happen if r_1 and r_2 are in Q[x]. As a result, if $r_1r_2 \in (x)S$ and $r_1, r_2 \in Q[x]$ then $r_1r_2 \in (x)R$ a prime ideal of R.

Hence, either $r_1 \in (x)R$ or $r_2 \in (x)R$. This completes the proof.

3) $A = \{px^{-s} \mid p \in P\} = (y, yx^{-1}, \ldots, z, zx^{-1} \ldots)S$ is a prime ideal of S and $A < (x)S$. Moreover $A = \prod_{n=1}^{\infty}(x^n)S$.

Proof: Clearly x is not an element of A. Since $px^{-s} = x^n(px^{-s-n}) \in (x^n)S$ we have $A \subset \prod_{n=1}^{\infty}(x^n)S$. $(x)S$ a principal prime ideal of S implies $\prod_{n=1}^{\infty}(x^n)S$ is a prime ideal of S. Therefore we need only show $\prod_{n=1}^{\infty}(x^n)S \subset A$. Let a be a non-zero element of $\prod_{n=1}^{\infty}(x^n)S$. Since $\prod_{n=1}^{\infty}(x^n)R = (0)$ there is an element of $S \setminus R$ having a normalized form $u[rx^s + p]x^{-s}$ and an integer $m > 0$ such that $a = x^m u[rx^s + p]x^{-s}$. But both a and upx^{m-s} are in $(x^n)S$ for each integer n, hence $urx^m \in (x^n)S$ for each integer n. Therefore $r \in \prod_{n=1}^{\infty}(x^n)S$. If $r \neq 0$ we again get for some $e > 0$ that $r = x^e v[r_1 x^t + p_1]x^{-t}$ (where again $v[r_1 x^t + p_1]x^{-t}$ is a normalized form). Evaluating the above at $y = z = 0$, with v' representing v at $y = z = 0$, we have $r = x^{e-t}v'r_1 x^t = v'r_1 x^e$. However, this must happen for e arbitrarily large. Therefore $r = 0$, a contradiction. As a result we have shown if a is a non-zero element of $\prod_{n=1}^{\infty}(x^n)S$ then a is of the form upx^{m-s}. Therefore $\prod_{n=1}^{\infty}(x^n)S = A$.

4) In $T = S_{(x)}$, $B^e < A^e$ where $B = (y,z)S$. In particular neither yx^{-1} nor zx^{-1} is in B^e.

Proof: We first note that $a \in S \setminus (x)S$ if and only if each r in $\theta(a)$ is a unit in R. To show $yx^{-1} \in A^e \setminus B^e$

we must show there does not exist an $a \in S \setminus (x)S$ such that $a(yx^{-1}) \in B$. Suppose $a = u[rx^s + p]x^{-s}$ where r is a unit in R has this property (the case where a itself is a unit is trivial since $yx^{-1} \not\in B$). Therefore,
$[yx^{-1}][u(rx^s + p)x^{-s}] = [r_2x^m + p_2]x^{-m}y + [r_3x^n + p_3]x^{-n}z$. This implies if $k = m + n + s$ that urx^ky is an element of $(x^{k+1}y, x^{k+1}z, y^2, yz, z^2)R$. But ur is a unit in R and yx^k is not an element of $(yx^{k+1}, zx^{k+1}, y^2, yz, z^2)R$. Therefore $yx^{-1} \not\in B^e$. A similar argument gives us $zx^{-1} \not\in B^e$.

5) If M is an ideal of T such that $M \not\subset \bigcap_{n=1}^{\infty}(x^n)T$, then $M = (x^k)T$ for some integer k.

Proof: This follows easily since $(x)S$ is a principal prime ideal of S and $A = \bigcap_{n=1}^{\infty}(x^n)S$.

6) In S if Q is A-primary and $B \subset Q \subset A$ then $Q = A$. Also, $A^2 \subset B$.

Proof: Let $px^{-s} \in A$. Since $x^s \not\in A$ and $x^s[px^{-s}] = p$ is in $B \subset Q$ then $px^{-s} \in Q$. Therefore $A = Q$. Moreover, $[p_1x^{-s}][p_2x^{-t}] = [p_1x^{-s-t}]p_2 \in B$ which proves the second assertion.

7) If $a = \overline{(yx^{-1})}$ and $b = \overline{(zx^{-1})}$ in D then $a \not\in (b)D$ and $b \not\in (a)D$.

Proof: If $b \in (a)D$ then there exist $[rx^s + p]x^{-s} \in S$, and $[ux^t + p_1]x^{-t} \in S \setminus (x)S$ such that $[rx^s + p][yx^{-s-1}] - [ux^t + p_1][zx^{-t-1}] = c$ is in B. We write $[ux^t + p_1]x^{-t}$ so that u is a unit in R (this can be done in general by allowing p_1 to be zero). Since $c \in B$ for some non-negative integers m and n we get $c = [r_2x^m + p_2]yx^{-m} + [r_3x^n + p_3]zx^{-n}$. This gives us if $k = s + t + m + n + 1$ that $uzx^k \in (x^{k+1}, yx^k, y^2, yz, z^2)R$ which is a contradiction. A similar argument gives us $a \notin (b)D$.

This completes the construction of the example.

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BIOGRAPHY

William Walker Smith was born on September 26, 1940, in Duncan, Oklahoma. He attended the public schools of Lawton, Oklahoma City, and Walters, Oklahoma and of Fort Worth, Texas. After graduation from high school he attended Lubbock Christian College in Lubbock, Texas, for one year and then attended Southeastern State College in Durant, Oklahoma, where he received the degree of Bachelor of Science in Education in August, 1961.

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Major Field: Mathematics

Title of Thesis: Prüfer Rings

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EXAMINING COMMITTEE:

Date of Examination:

April 26, 1965