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Measures and Affine Semigroups.

Michael Friedberg
Louisiana State University and Agricultural & Mechanical College

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ABSTRACT

The object of this work is the study of measure semigroups as well as general affine semigroups. We divide this study into three parts: finite-dimensional affine semigroups, semigroups of measures, and compact affine topological semigroups.

Finite-dimensional affine semigroups have been investigated by W. E. Clark, and H. Cohen and H. S. Collins. Such semigroups can be realized as matrix semigroups, as well as subalgebras of algebras over the reals. It will be shown in Chapter III that this class of affine semigroups can be used to approximate compact group-extremal affine semigroups.

Measure semigroups have been the object of much investigation; the principal investigators include H. S. Collins, Collins and Koch, I. Glicksberg, B. M. Kloss, J. S. Pym, M. Rosenblatt, S. Schwarz, and J. G. Wendel. Glicksberg and Wendel give conditions under which certain semigroups of measures are the full probability measure semigroup on some compact semigroup. In Chapter II, we add another theorem in the same general area. Collins and Wendel show that under certain conditions, a compact affine semigroup is the continuous, affine homomorphic image of its probability

measure semigroup. We remove these conditions and obtain this theorem for arbitrary compact affine semigroups.

A theory of representations of compact semigroups is lacking due, in part, to the absence of an invariant carrying measure. We show, however, in Chapter III, that a group-extremal semigroup allows both a theory of representations and a theory of characters; we use this theory to show that certain properties of the group are carried over to the semigroup.

In the first chapter, we show that an abelian finite-dimensional affine semigroup can be imbedded in a finite product of finite-dimensional abelian algebras of the form T(C), where C is the complex numbers, T is a finite-dimensional abelian nilpotent algebra over the complexes and $T(C) = C \oplus T$ where multiplication is defined by: $(z,a)o(w,b) = (zw, zb + wa + ab)$. To accomplish this, we use several theorems, due to W. E. Clark, and discovered independently by the author. We conclude this section by identifying the minimal ideal of an abelian, finite-dimensional affine semigroup as the finite direct product of additive reals.

In Chapter II, we extend a result first proved by Wendel, and under less restrictive conditions by Collins, which shows that the resultant map is a continuous affine homomorphism between $\widetilde{S}$ and S when S is a compact affine semigroup. We use this result to reprove the fact that a group-extremal semigroup has a zero. We show, further, that the resultant of an idempotent measure is in the kernel of the closed convex hull of its carrier. Also, if $\mu^2 = \mu$, $\mu \varepsilon \tilde{S}$, where S is compact and abelian, we show $\mu \tilde{S}$ is the measure semigroup on some compact semigroup. Finally, we show that if S is compact and convex, and $\mu \varepsilon \tilde{S}$, then the closed convex hull of the carrier of μ supports a measure. Consequently, a group-extremal semigroup supports a measure.

In Chapter III, we show that a compact group-extremal semigroup admits a sufficient system of representations by finite-dimensional affine semigroups. As a consequence, several properties of the group are extended to the semigroup. Namely, metrizability is extended and, if the group is abelian, we obtain a sufficient system of affine semicharacters. It follows immediately that an abelian, metrizable, group-extremal semigroup is imbeddable in the countable product of discs. Finally, we show that a group-extremal semigroup is the inverse limit of finite-dimensional group-extremal semigroups.

INTRODUCTION

When affine semigroups first appeared in the literature, with studies done by J. G. Wendel [48], and J. E. L. Peck [30], as well as others in measure semigroups, it was assumed that the semigroup was imbedded in a larger space in which there was a multiplication compatible with the semigroup multiplication. Such semigroups, for instance, as the probability semigroup over a compact semigroup, and semigroups of operators on a Banach space are, indeed, imbedded in spaces in which multiplication can be performed outside of the semigroup. The definition we shall use here is due to Cohen and Collins [6]; this definition does not assume a multiplication outside of the semigroup. In Chapter II, we shall show that, under suitable conditions, one can assume such a multiplication does exist outside the semigroup.

Semigroups of measures are of comparatively recent origin; the earliest work seems to be the paper by Kawada and Itô [20] written in 1940. Then, in 1954, Wendel's paper [48] created much interest in the field since he deduced the existence of Haar measure on a compact group by using the structure of the measure semigroup. Since Wendel's paper, there have been several contributors to the theory; among them are Collins [7], [8], [9], [10], [12], Collins and Koch [13], Glicksberg [17], Kloss [21], [22], [23], Pym [32], Rosen [33], Rosenblatt [34], Rosenblatt and Heble [35], Schwarz [42], [43], [44], and Stromberg [45].

In his 1954 paper, Wendel proved that in the probability semigroup over a compact group the only probability measures with inverse are the point measures (i.e., the extreme points). Cohen and Collins then showed in [6] that this was true in any compact affine semigroup with unit; that is, the only elements with inverse are extreme points.

Glicksberg showed in [17] that if μ is an element of $\tilde{S}$ for some compact semigroup S, then $\frac{1}{N}\sum_{i=1}^{N}\mu^i$ converges in the weak-star topology to an element $\lambda \in \tilde{S}$ satisfying $\lambda^2 = \lambda$ and $\lambda\mu = \mu\lambda = \lambda$. Further, in an invited address (unpublished), Wendel showed that if S is a group-extremal affine semigroup, then the resultant is a continuous, affine homomorphism from $\tilde{S}$ onto S. Collins showed in [12] that 'group-extremal' may be replaced by the condition that the extreme points form a compact semigroup. In Chapter II, we remove all these assumptions and show that the resultant is a continuous, affine homomorphism onto S if S is a compact affine topological semigroup. With this fact, together with the result of Glicksberg, it follows that for an arbitrary element x of a compact affine semigroup S, that $\frac{1}{N}\sum_{i=1}^{N}x^i$ converges to an element $e^2 = e \in S$ which satisfies xe = ex = e. We also use the resultant map to prove the fact noted by Peck [30] and Cohen and Collins [6] that a group-extremal semigroup has a zero.

Wendel also showed in [48] that if G is a compact group, $S = \widetilde{G}$, and $\mu^2 = \mu \in S$, then μS is the full probability semigroup over some compact group. Subsequently, Glicksberg showed in [17] that if S is either a compact abelian semigroup or a compact group, and Γ is a subgroup of $\widetilde{S}$, then $\langle \Gamma \rangle$, the closed convex hull of Γ, is the full probability semigroup over some compact group. To complete this sequence of theorems, we show that if S is a compact abelian semigroup, and $\mu^2 = \mu \in \widetilde{S}$, then $\mu \widetilde{S}$ is the full probability semigroup over some compact abelian semigroup.

In attempting to determine the structure of general affine semigroups, Cohen and Collins [6] considered the case where the semigroup is a convex subset of some finite-dimensional space. They showed that the multiplication on the semigroup S may be extended uniquely to V(S), the manifold generated by S, so that V(S) becomes an affine semigroup. In case S has a left or right zero, they showed that S may be realized as a semigroup of matrices. They then characterized completely all one and two-dimensional affine semigroups. Clark then showed in [33] that an affine semigroup can be imbedded in a finite-dimensional algebra over the reals. He also showed that a finite-dimensional affine semigroup has a completely simple kernel (i.e., minimal ideal). In Chapter I, we shall show that an abelian, finite-dimensional affine semigroup may be imbedded in an abelian algebra over the complexes which is the direct sum of finite-dimensional abelian algebras over the complexes of the form T(C), where C is the complex numbers, T is an abelian, finite-dimensional nilpotent algebra over the complexes, and where $T(C) = C \oplus T$, with multiplication defined by:

$$(z,a)o(w,b) = (zw, zb + wa + ab).$$

We conclude Chapter I by showing that the kernel of an abelian, finite-dimensional affine semigroup S satisfying V(S) = S is degenerate or is isomorphic to a finite product of additive reals. Chapter III will demonstrate that finite-dimensional affine semigroups may be used to approximate compact, group-extremal semigroups.

Compact group-extremal affine semigroups are of much interest, as the model is the probability semigroup over a compact group. The works of Glicksberg [17], Cohen and Collins [6], Peck [30], and Wendel [48] all include theorems about such semigroups in some form. We show in Chapter II and III that many properties of the group of extreme points may be carried over to the entire semigroup. In Chapter II, we show that if a probability measure is concentrated on a compact subset, A, of a compact convex set in a locally convex linear space, then there is another probability measure concentrated on the closed convex hull of $\acute{A}$. It will follow immediately that a group-extremal semigroup supports a probability measure, since the group supports Haar measure.

Several authors have contributed to the theory of representations and the theory of characters on semigroups. Among these contributors are Clifford [4], who showed that a completely simple semigroup has a faithful representation by (infinite) matrices, and Preston [31] who proved a similar statement concerning regular semigroups. Further, Hewitt and Zuckerman in [18] and [19] investigated semicharacters on finite and infinite abelian semigroups, as did Schwarz ([37] - [41]). However, in all these studies the underlying semigroup was assumed discrete and, naturally, no continuity conditions are obtained. Schwarz in [41] investigated characters on a compact abelian semigroup from the standpoint of determining the structure of the semigroup of all such characters. He stated explicitly all the semicharacters of the disc, and we shall use this in the sequel to counter a possible conjecture.

The difficulty in obtaining continuous characters in an arbitrary abelian compact semigroup seems to be due in part to the absence of an invariant carrying measure that exists for compact groups. In Chapter III, we show that in spite of the absence of an invariant carrying measure, a group-extremal affine semigroup has a sufficient system of affine representations. If the group is abelian, these representations may be taken to be one-dimensional, so that we obtain a sufficient system of affine semicharacters. As a consequence, a group-extremal affine semigroup is the inverse limit of finite-dimensional group-extremal affine semigroups. Further, if the group is metrizable, then the entire semigroup is as well. In the abelian case, if the group is metrizable, the semigroup can be imbedded in the countable product of discs under coordinate-wise multiplication.

PRELIMINARIES

Definition: A semigroup is a set S together with a function m: $S \times S \longrightarrow S$ satisfying $m(a,m(b,c)) = m(m(a,b),c)$. If S is a Hausdorff topological space and m is jointly continuous on SxS to S, then S is called a topological semigroup. As usual, m is suppressed and m(a,b) is written ab.

Definition: A topological linear space is a vector space V over the reals (or complexes) which possesses a Hausdorff topology in which vector addition and scalar multiplication are continuous in both variables simultaneously. If, in addition, the origin of V possesses a basis in this topology consisting of open convex sets U which satisfy: $x \varepsilon U$, $|\lambda| = 1 \longrightarrow \lambda x \varepsilon U$; then V is called a locally convex linear space. Henceforth, all linear spaces will be locally convex.

A proof of the following well-known theorem may be found in [53;117]:

Theorem A: Let A and B be disjoint compact convex sets in a locally convex linear space V. Then there is a continuous, real-valued linear functional on V satisfying:

$$\max_{z \in A} \{f(z)\} < \min_{z \in B} \{f(z)\}.$$

Remark: Included in Theorem A is the fact that the continuous, real valued functionals on V separate points.

Definition: If A is a subset of a linear space V, the closed convex hull of A (denoted by <A>) is the smallest closed convex subset of V containing A. <A> consists of all those elements of V which may be approximated by elements of the form: $\sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, and $x_i \in A$ for i = 1, 2, ..., n. If A is any set in V, an extreme point of A is an element of A which is interior to no line segment between two points of A.

The following theorem is due to Krein and Milman [24], and proofs may also be found in several standard sources (c.f. Naimark [56;62], Kelly and Namioka [53;130]).

Theorem B: If A is a compact convex set in a linear space V, then A is the closed convex hull of its extreme points.

A proof of the following can be found in Dunford and Schwartz [51;440].

Theorem C: If A is a compact subset of a compact convex set S in a linear space, then the extreme points of <A> are again in A.

Definition: An affine semigroup S is a convex subset of a linear space V which is a semigroup with respect to some multiplication that satisfies:

(a) $[\lambda x + (1 - \lambda)y]z = \lambda(xz) + (1 - \lambda)(yz)$

(b) $z[\lambda x + (1 - \lambda)y] = \lambda(zx) + (1 - \lambda)(zy)$

for $x,y,z \in S$ and $0 \leq \lambda \leq 1$.

If, in addition, S is a topological semigroup with the topology inherited from V, then S is called an affine topological semigroup.

Definition: Two affine semigroups S and T are said to be equivalent if there is a one-to-one affine homomorphism from S onto T. If S and T are affine topological semigroups, we require the homomorphism to be bicontinuous as well.

Since measure semigroups provide much motivation for the study of affine semigroups, we include here a development of the measure semigroups over compact semigroups.

Let S be a compact Hausdorff space, C(S) the Banach space of complex-valued continuous functions on S. Let M(S) denote the space of all complex-valued, regular Borel measures on S. If $\mu \in M(S)$, and we define a function $|\mu|$ on the Borel sets of S by:

$$|\mu|(E) = \sup_{P_E} \sum_{E_i \in P_E} |\mu(E_i)|$$

where P_E is a partition of E by disjoint Borel sets, and the supremum is taken over all partitions of E, then $|\mu|$ is again an element of M(S). Further M(S) is a Banach space under the norm given by $\|\mu\| = |\mu|(S)$. M(S) may also be given the so-called 'weak-star' topology, in which a net of measures $\{\mu_\alpha\}_{\alpha \in D}$ converges to $\nu \in M(S)$ if and only if $\int f d\mu_\alpha \longrightarrow \int f d\nu$ for all $f \in C(S)$.

By the Riesz-Kakutani Theorem M(S) with the above norm is the adjoint of C(S), where the correspondence between a continuous linear functional T on C(S) and the associated measure is given by: $T(f) = \int f d\mu$ for all $f \in C(S)$. In view of this correspondence, we do not distinguish between the measure and the linear functional it defines and write simply:

$$\mu(f) = \int f d\mu$$

The sets $B(S) = \{\mu \in M(S): \|\mu\| \leq 1\}$ and $\widetilde{S} = \{\mu \in M(S): \mu \geq 0, \mu(S) = 1\}$ are compact in the weak-star topology.

If S is also a compact semigroup, then for $\mu, \nu \in M(S)$ there is a unique third measure in M(S) called the convolution of μ and ν (written $\mu * \nu$) which satisfies:

$$\int f d(\mu * \nu) = (\mu * \nu)(f) = \int\int f(xy) d\mu(x) d\nu(y).$$

This measure is obtained via the Riesz-Kakutani Theorem, and under this multiplication and the norm in M(S), M(S) is a Banach algebra.

Further, on B(S) and $\widetilde{S}$ the operation is binary and jointly continuous in the weak-* topology. Hence, B(S) and $\widetilde{S}$ are compact, affine topological semigroups. B(S) is called the 'ball' semigroup of S and $\widetilde{S}$ is called the 'probability' semigroup of S.

If S is compact Hausdorff, and $\mu \varepsilon \widetilde{S}$, the carrier of μ, written $C(\mu)$, is the complement of the largest open set having μ-measure zero. Consequently, $C(\mu)$ is compact and for any open subset V of S, $\mu(V) > 0$ iff $V \cap C(\mu) \neq \emptyset$.

A compact semigroup possesses a minimal ideal K which may be written as the disjoint union of minimal left (right) ideals and also as the disjoint union of maximal groups [5]. A semigroup is simple if it does not contain any proper ideals.

Theorem D [17]: If μ and ν are elements of $\widetilde{S}$, where S is a compact semigroup, then $C(\mu * \nu) = C(\mu)C(\nu)$.

Theorem E [23];[8]: If $\mu^2 = \mu \; \varepsilon \; \widetilde{S}$, where S is a compact semigroup, then $C(\mu)$ is a compact simple semigroup, and for $f \; \varepsilon \; C(S)$ the mapping $x \longrightarrow \int f(yx)d\mu(y)$ is constant on each minimal left ideal of $C(\mu)$ and $x \longrightarrow \int f(xy)d\mu(y)$ is constant on each minimal right ideal of $C(\mu)$.

Theorem F [17]: If S is compact, and either an abelian semigroup or a group, and Γ is a group in $\widetilde{S}$, then $\langle \Gamma \rangle$ is the full probability semigroup over some compact group.

Theorem G [48]: If G is a compact group and $\mu^2 = \mu \; \varepsilon \; \widetilde{G}$, then $\mu\widetilde{G} = \widetilde{G}\mu$ and is the full probability semigroup over some compact group.

Theorem H [1]: The weak-star closed convex hull of the collection of all point measures of M(S) is $\widetilde{S}$.

Theorem I Wendel (unpublished) and [6]: If S is a compact affine topological semigroup with identity, then every element of S with inverse is an extreme point of S.

We shall need the following theorem in Chapter I, but an independent proof will be given in Chapter II.

Theorem J [30]; [6]: If S is a compact affine topological semigroup with identity, and if each extreme point of S has an inverse, then S has a zero.

Theorem K [6]: If S is a compact affine topological semigroup, then:

(a) Each minimal left (right) ideal os S is convex.

(b) $x \varepsilon K$ (the minimal ideal of S) iff $xSx = \{x\}$; in particular, each element of K is idempotent.

CHAPTER I

In this chapter, we discuss affine semigroups where the containing linear space is of finite dimension over the reals. Such semigroups are referred to as finite-dimensional.

A linear manifold in a linear space X is a translate of a linear subspace. If A is a non-void subset of X, the manifold generated by A, written V(A), is the smallest linear manifold containing A.

If S is a finite-dimensional affine semigroup, Cohen and Collins [6] show that the multiplication on S may be uniquely extended to V(S), and relative to this multiplication, V(S) is a finite-dimensional affine semigroup.

Remark: In this extension, multiplication is expressed in terms of coordinates relative to a fixed affine basis for V(S), the coordinates of a product being polynomials in the coordinates of the elements multiplied. It follows easily that a finite-dimensional affine semigroup is a topological semigroup (*i.e.*, multiplication is jointly continuous).

Clark shows in [3] that a finite-dimensional affine semigroup which is a manifold is equivalent to a subsemigroup of an algebra of finite dimension over the reals. If the original semigroup is abelian, Clark's construction yields an abelian algebra. Combining these results, we have the following:

<u>Theorem 1.1</u> A finite-dimensional affine semigroup S is equivalent to a subsemigroup of a finite-dimensional algebra over the reals. If S is abelian, then the algebra is as well.

The following theorem is well-known, but was rediscovered independently by the author:

<u>Theorem 1.2</u> Let A be a finite-dimensional abelian algebra over a field $\mathfrak{F}$ where A contains a non-zero idempotent. Then there exists $e_1, \ldots, e_n \in A$ satisfying:

1) $e_i^2 = e_i$, $e_i e_j = 0$ for $i \neq j$.

2) $e_1, \ldots, e_n$ are linearly independent.

3) If $e^2 = e \in A$ then e can be expressed: $e = \sum_{i=1}^{n} \lambda_i e_i$ where λ_i is 0 or 1.

<u>Proof</u>: If $B \subset A$, S(B) will denote the subspace generated by B.

Suppose we have constructed a set $e_1, \ldots, e_r$ satisfying 1) and 2). Let $e^2 = e \in S(\{e_1, \ldots, e_r\})$, so that $e = \sum_{i=1}^{r} \lambda_i e_i$, where $\lambda_i \in \mathfrak{F}$. Then we have $ee_j = \sum_{i=1}^{r} \lambda_i e_i e_j = \lambda_j e_j$. Hence, $e = ee = \sum_{i=1}^{r} \lambda_i (e_i e) = \sum_{i=1}^{r} \lambda_i (\lambda_i e_i) =$

$\sum_{i=1}^{r} \lambda_i^2 e_i$, and since $e_1, \ldots, e_r$ are independent, it follows that $\lambda_i^2 = \lambda_i$, $\lambda_i = 0$ or $\lambda_i = 1$. Thus, if every idempotent of A is in $S(\{e_1, \ldots, e_r\})$, then 3) is satisfied and we are finished.

Now suppose $g^2 = g \in A$, $g \notin S(\{e_1, \ldots, e_r\})$; then we have two cases:

<u>Case I</u> Suppose $ge_i \in S(\{e_1, \ldots, e_r\})$ for all $i = 1, 2, \ldots, r$. In this case, let $e_{r+1} = -ge_1 - ge_2 - \ldots -ge_r + g$, so that $e_{r+1} \notin S(\{e_1, \ldots, e_r\})$. Further, $ge_{r+1} = e_{r+1}$ and $e_{r+1}e_j = -g(e_1e_j) - \ldots -g(e_je_j) - \ldots -g(e_re_j) + ge_j = -ge_j + ge_j = 0$. Then we have $e_{r+1}e_{r+1} = -(ge_1e_{r+1}) - \ldots -(ge_re_{r+1}) + ge_{r+1} = ge_{r+1} = e_{r+1}$, and, since $e_{r+1} \notin S(\{e_1, \ldots, e_r\})$, $e_1, \ldots, e_{r+1}$ are independent and also satisfy condition 1).

<u>Case II</u> Suppose $ge_k \notin S(\{e_1, \ldots, e_r\})$ for some k, $1 \leq k \leq r$. Let $e_k' = e_k - ge_k$, then $ge_k' = 0$, and $e_k'e_k = e_k'$ so that $e_k'e_k' = e_ke_k' - (ge_k)e_k' = e_k'$. We show that $e_1, \ldots, e_{k-1}, e_k', e_{k+1}, \ldots, e_r$ are independent. Suppose $\lambda_1 e_1 + \ldots + \lambda_k e_k' + \ldots + \lambda_r e_r = 0$. Then $0 = (\lambda_1 e_1 + \ldots + \lambda_k e_k' + \ldots + \lambda_r e_r)e_k = \lambda_1(e_1e_k) + \ldots + \lambda_k e_k'e_k + \ldots + \lambda_r e_r e_k = \lambda_k e_k' e_k = \lambda_k e_k'$ by condition 1). Since $e_k' \neq 0$, $\lambda_k = 0$, and since $e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_r$ are independent, it follows that $\lambda_i = 0$ for $i = 1, 2, \ldots, r$.

Now, $ge_k, e_1, e_2, \ldots, e_k', \ldots, e_r$ are independent by similar reasoning; hence, we let $e_{r+1} = ge_k$. Then $e_{r+1}e_j = (ge_k)e_j = g(e_ke_j) = 0$ for $j \neq k$, and $e_{r+1}e_k' = (ge_k)e_k' = ge_k' = 0$. Clearly, ${e_{r+1}}^2 = e_{r+1}$, so that $e_1, \ldots, e_{k-1}, e_k', \ldots, e_r, e_{r+1}$ satisfy 1) and 2).

If m is the largest integer for which there is a set $\{e_1, \ldots, e_m\}$ satisfying 1) and 2), then by the preceding discussion, 3) is also satisfied. This completes the proof of the theorem.

<u>Theorem 1.3</u> An abelian, finite-dimensional algebra A over a field $\mathfrak{F}$ can be imbedded in the direct sum of abelian, finite dimensional algebras $T_1, \ldots, T_n$ where each T_i has an identity e_i and such that if $e^2 = e \in T_i$ then $e = e_i$ or $e = 0$.

<u>Proof</u>: Suppose first that A has an identity u. By Theorem 1.2, there exist linearly independent idempotents $e_1, \ldots, e_n$ such that $e_ie_j = 0$ for $i \neq j$ and if $e^2 = e \in A$, then $e = \sum_{i=1}^{n} \lambda_i e_i$ where $\lambda_i = 1$ or $\lambda_i = 0$.

Let $T_i = Ae_i$; then T_i is an abelian, finite-dimensional algebra over $\mathfrak{F}$ and has identity e_i. If $x \in Ae_i \cap Ae_1 + \ldots + Ae_{i-1}$, then $xe_i = x$, and $x = \sum_{j=1}^{i-1} a_je_j$, where $a_j \in A$. But then $x = xe_i = \sum_{j=1}^{i-1} a_j(e_je_i) = 0$, and it follows that $Ae_1 + \ldots + Ae_n$ is a direct sum.

Further, since $u^2 = u$, $u = \sum_{i=1}^{n} \lambda_i e_i$, where $\lambda_i = 1$ or

$\lambda_i = 0$; however, $e_j = ue_j = \sum_{i=1}^{n} \lambda_i(e_i e_j) = \lambda_j e_j$, so that $\lambda_j = 1$ and $u = \sum_{i=1}^{n} e_i$. Then if $x \in A$, $x = xu = \sum_{i=1}^{n} xe_i \in \sum_{i=1}^{n} \oplus Ae_i$.

Finally, if $e^2 = e \in Ae_i$, then $ee_i = e$, and $ee_j = (ee_i)e_j = e(e_i e_j) = 0$ for $i \neq j$. Now, since $e = \sum_{j=1}^{n} \lambda_j e_j$, then $0 = ee_j = \sum_{k=1}^{n} (\lambda_k e_k)e_j = \sum_{k=1}^{n} \lambda_k(e_k e_j) = \lambda_j e_j$ so that for $i \neq j$, $\lambda_j = 0$. Moreover, since $ee_i = e$, and $e = \lambda_i e_i$, so that $e = ee = \lambda_i(e_i e) = \lambda_i(e_i e) = \lambda_i e = \lambda_i(\lambda_i e_i) = \lambda_i^2 e_i$, it follows that $\lambda_i^2 = \lambda_i$ and $e = e_i$ or $e = 0$. Hence, the only idempotents in Ae_i are 0 and e_i, so that the conclusion follows if A has an identity.

If A does not have an identity, we form $\mathfrak{F} \oplus A$ and define multiplication by:

$$(f,a)o(g,b) = (fg, fb + ga + ab)$$

Then $\mathfrak{F} \oplus A$ with this multiplication is an abelian finite-dimensional algebra over $\mathfrak{F}$. Further, A is isomorphic to the subset of $\mathfrak{F} \oplus A$ consisting of those elements of the form (o,a), where $a \in A$. Finally, if we let $u = (1,0)$, then for $(f,a) \in \mathfrak{F} \oplus A$, $(1,0)o(f,a) = (f, a + f\cdot 0 + 0\cdot a) = (f,a)$. Thus, u is an identity for $\mathfrak{F} \oplus A$, and the theorem follows from the preceding argument.

If V is a linear manifold in the finite-dimensional space X, then $V - a = \{v - a : v \in V\}$ is a subspace of X whenever $a \in V$. Further, if $a,b \in V$, then $V - a = V - b$, so that associated to V is a unique linear subspace. The dimension of this subspace is the dimension of V.

If the dimension of V is n, and $x_1, \ldots, x_s \in V$, then if we let m be the largest integer satisfying $x_m \notin V(\{x_1, \ldots, x_{m-1}\})$ then $m \leq n + 1$. We also note that if $A \subset V$, then $V(A) \subset V$ and consists of all elements of the form $\sum_{i=1}^{r} \lambda_i a_i$ where $a_i \in A$, and $\sum_{i=1}^{r} \lambda_i = 1$.

<u>Theorem 1.4</u> [6] If S is a one-dimensional affine semigroup, then V(S) is equivalent to the real line under one of the following multiplications:

(a) usual (b) $xy = 0$ all x,y (c) $xy = x + y$

(d) $xy = x$ all x,y (e) $xy = y$ all x,y.

Using Theorem 1.4, we give a new proof of the following theorem due to Clark [3]:

<u>Theorem 1.5</u> Let S be a finite-dimensional affine semigroup satisfying $S = V(S)$; then some power of each element lies in a subgroup of S.

<u>Proof</u>: We argue by induction on the dimension of S.

If dim S = 1, then by inspection of Theorem 1.4, the conclusion follows. Hence, we assume the statement true for dimension less than n, and let dim S = n. Let $x \in S$, then there is an integer $m \leq n + 1$ such that $x^{m+1} \in V(\{x, x^2, \ldots, x^m\})$, but $x^{k+1} \notin V(\{x, \ldots, x^k\})$ for $1 \leq k <$

m. Then $x^{m+1} = \sum_{i=1}^{m} \lambda_i x^i$ where $\sum_{i=1}^{m} \lambda_i = 1$.

Now, if $\lambda_1 \neq 0$, then we define: $p = (-\frac{\lambda_2}{\lambda_1})x + (-\frac{\lambda_3}{\lambda_1})x^2 + \ldots + (-\frac{\lambda_m}{\lambda_1})x^{m-1} + \frac{1}{\lambda_1}x^m$. Note that $(-\frac{\lambda_2}{\lambda_1}) + \ldots + (-\frac{\lambda_m}{\lambda_1}) + \frac{1}{\lambda_1} = 1$, so that $p \varepsilon V$. Further $px = (-\frac{\lambda_2}{\lambda_1})x^2 + \ldots + (-\frac{\lambda_m}{\lambda_1})x^m + \frac{1}{\lambda_1}x^{m+1} = (-\frac{\lambda_2}{\lambda_1})x^2 + \ldots + (-\frac{\lambda_m}{\lambda_1})x^m + \frac{1}{\lambda_1}(\lambda_1 x + \ldots + \lambda_m x^m) = (-\frac{\lambda_2}{\lambda_1})x^2 + \ldots + (-\frac{\lambda_m}{\lambda_1})x^m + x + \frac{\lambda_2}{\lambda_1}x^2 + \ldots + \frac{\lambda_m}{\lambda_1}x^m = x$. Similarly, $xp = x$; hence $x^n p = px^n = x^n$ for all n. Then $p^2 = p[(-\frac{\lambda_2}{\lambda_1})x + \ldots + (-\frac{\lambda_m}{\lambda_1})x^{m-1} + \frac{1}{\lambda_1}x^m] = (-\frac{\lambda_2}{\lambda_1})xp + \ldots + (-\frac{\lambda_m}{\lambda_1})x^{m-1}p + \frac{1}{\lambda_1}x^m p = (-\frac{\lambda_2}{\lambda_1})x + \ldots + (-\frac{\lambda_m}{\lambda_1})x^{m-1} + \frac{1}{\lambda_1}x^m = p$. Thus, $p^2 = p$, and $xp = px = x$. Note also that $p = (-\frac{\lambda_2}{\lambda_1})x + \ldots + (-\frac{\lambda_m}{\lambda_1})x^{m-1} + \frac{1}{\lambda_1}x^m = x[(-\frac{\lambda_2}{\lambda_1})p + \ldots + \frac{1}{\lambda_1}x^{m-1})] = [(-\frac{\lambda_2}{\lambda_1})p + \ldots + \frac{1}{\lambda_1}x^{m-1}]x$. Setting $y = (-\frac{\lambda_2}{\lambda_1})p + \ldots + \frac{1}{\lambda_1}x^{m-1}$ then $yp = py = y$ and $xy = yx = p$. Hence, x is in the subgroup of S determined by $p^2 = p$.

If, on the other hand, $\lambda_1 = 0$, then $x^{m+1} = \lambda_2 x^2 + \ldots + \lambda_m x^m$ so that $x^{m+1} \varepsilon V(\{x^2, \ldots, x^m\})$. Further, since $x^{m+1} \varepsilon V(\{x^2, \ldots, x^m\})$ it follows that $V(\{x^2, \ldots, x^m\})$ is a subsemigroup of S, and has dimension less than n since $m \leq n + 1$. By the induction assumption, some power of x^2 is in a subgroup of $V(\{x^2, \ldots, x^m\})$ and hence a subgroup of S. Therefore, some power of x is in a subgroup of S, and the proof is complete.

The following theorem appears in more general form in [3]; we shall give a proof of the version we require. First, we give the following:

<u>Definition</u>: An element x of a semigroup S with zero 0 is called nilpotent if $x^n = 0$ for some integer $n \geq 1$. N(S) is the set of nilpotents in S.

<u>Theorem 1.6</u> An abelian, finite-dimensional affine semigroup S with zero θ, satisfying $S = V(S)$ and $N(S) = \{\theta\}$ is equivalent to a finite direct sum of reals and complexes.

<u>Proof</u>: Let $T = S - \theta$; then T is an algebra over the reals and is equivalent to S [6]. By assumption, $N(S) = \{\theta\}$, so that $N(T) = \{0\}$.

By Theorem 1.2, there exist linearly independent idempotents $e_1, \ldots, e_r$ satisfying:

(a) $e_i e_j = 0$ for $i \neq j$ and

(b) $e^2 = e \in T$ then $e = \sum_{i=1}^{r} \lambda_i e_i$ where $\lambda_i = 0$ or $\lambda_i = 1$.

Let $A_i = Te_i$, then A_i is an abelian algebra of finite dimension over the reals with identity e_i. As in the proof of Theorem 1.3, A_i has no other idempotents besides e_i and 0. Further, since $A_i \subset T$, $N(A_i) = \{0\}$.

By Theorem 1.5, some power of each element of A_i is in a group in A_i. Let $x \neq 0$; since $x \notin N(A_i)$ it follows that

x^r is in some group in A_i determined by a non-zero idempotent, hence by e_i. It follows that each $x \neq 0$ is invertible with respect to the identity. Thus A_i is a field of finite dimension over the reals; by the Frobenius Theorem, A_i is either the reals or complexes.

Let $I = \sum_{i=1}^{r} A_i = \sum_{i=1}^{r} Te_i$; then I is an ideal in T. We show $I = T$. Let $p \in T \setminus I$, and let $z = -pe_1 - \ldots -pe_r + p$; then since $p \notin I$ we have $z \notin I$. Note also that $ze_i = 0$ for $i = 1, 2, \ldots, r$. Hence by (b), $ze = 0$ for every $e^2 = e \in T$. By Theorem 1.5, z^r, for some r, is in a subgroup of T, and, since $N(T) = \{0\}$, the idempotent, e, of this subgroup is different from 0. Thus, $z^r e = ez^r = z^r$; but since $ze = 0$, $z^r e = 0$ so that $z^r = 0$, and, $z \in N(T)$. Thus, $z = 0$ and $p \in I$. Therefore, $T = I = \sum_{i=1}^{r} A_i = \sum_{i=1}^{r} Te_i$. Now, $A_i A_j = (0)$ and $A_j \cap \sum_{i=1}^{j-1} A_i = \{0\}$ so that $T = \sum_{i=1}^{r} \oplus A_i$; clearly, $u = \sum_{i=1}^{r} e_i$ is an identity for T. This completes the proof.

<u>Definition</u>: If A is an algebra over a field $\mathfrak{F}$ let $A(\mathfrak{F}) = \mathfrak{F} \oplus A$ with multiplication defined by

$$(f,a)o(g,b) = (fg, fb + ga + ab)$$

<u>Remark</u>: If A is abelian and finite-dimensional over $\mathfrak{F}$, then $A(\mathfrak{F})$ is also. The element $u = (1,0)$, where 1 is the identity of $\mathfrak{F}$, is an identity for $A(\mathfrak{F})$; A is imbedded in $A(\mathfrak{F})$.

<u>Theorem 1.7</u> Let A be an abelian, finite-dimensional algebra over the reals. Further, suppose A has an identity $u \neq 0$ and no other non-zero idempotents. Then there exists an abelian, finite-dimensional nilpotent algebra T over the complexes such that A is imbedded in $T(\mathbb{C})$.

<u>Proof</u>:

<u>Case I</u> Suppose $x^2 + u = 0$ has a solution $x_0 \in A$. Suppose a,b are real and $ax_0 + bu = 0$; if $a \neq 0$, then $x_0 = -\frac{b}{a}u$. Hence $x_0^2 = \frac{b^2}{a^2}u$ and hence $\frac{b^2}{a^2}u = -u$. Thus, $\frac{b^2}{a^2} + 1 = 0$ or $(\frac{b}{a})^2 + 1 = 0$; but, a and b are real. Hence $a = b = 0$, and x_0 and u are independent over the reals.

If $x \in A$, $y \in N(A)$, then clearly $xy \in N(A)$. Further, if $x,y \in N(A)$ then $ax + by \in N(A)$ for all a,b real. Thus N(A) is an ideal in A; we show A/N(A) is isomorphic to the complex numbers. Since A has an identity, A/N(A) also has an identity. Further, A/N(A) has no other non-zero idempotents since the same is true of A. Now if $x \in A$ and $x^r \equiv 0 \pmod{N(A)}$ for some integer r, then $x^r \in N(A)$; hence $(x^r)^s = 0$ for some integer s. Thus, $x \in N(A)$ and $x \equiv 0 \pmod{N(A)}$. Thus, $N(A/N(A)) = \{0\}$; as in the proof of Theorem 1.6, A/N(A) is either the reals or complexes.

Suppose $x_0 \equiv \lambda u \pmod{N(A)}$ for some λ real, $\lambda \neq 0$. Hence, $x_0 - \lambda u = c_0 \in N(A)$; since x_0, u are independent over the reals, $c_0 \neq 0$. Now, there is an integer $n \geq 1$ for which $c_0^n \neq 0$ but $c_0^{n+1} = 0$. We then have $(x_0 - \lambda u)c_0^n = c_0^{n+1} =$

0 and $x_o c_o^n - \lambda c_o^n = 0$. Consequently, $x_o c_o^n = \lambda c_o^n$ and $x_o^2 c_o^n = \lambda x_o c_o^n = \lambda^2 c_o^n$; but $x_o^2 c_o^n = (-u) c_o^n = -c_o^n$, so that $\lambda^2 c_o^n = -c_o^n$. Since $c_o^n \neq 0$, we have $\lambda^2 = -1$ and λ is real. Of course, this is impossible and $\lambda = 0$, so we have x_o and u are independent over the reals, modulo N(A). Thus A/N(A) is two-dimensional over the reals, and the classes containing x_o and u are independent. Thus, if $x \in A$ there are unique real numbers μ_o, λ_o such that $x = \mu_o u + \lambda_o x_o + c$ for some $c \in N(A)$. Clearly, this c is unique.

If we let C be the subspace of A spanned by u and x_o then C is clearly isomorphic to the complexes. By the above remarks, $A = C \oplus N(A)$ and since N(A) is an ideal we have for $z, w \in C$ $x, y \in N(A)$, $(z + x)(w + y) = zw + zy + wx + xy$. Hence, $A = C \oplus N(A) = N(A)(C)$, and N(A) is an algebra over C.

<u>Case II</u> Suppose $x^2 + u = 0$ has no solution in A. Let $T = A \oplus A$, where $(a,b)(x,y) = (ax - by, ay + bx)$; then T is a finite-dimensional abelian algebra over the reals. Further, A is isomorphic to the subset of T defined by $\{(a,0) : a \in A\}$.

The element $(u,0)$ is an identity for T, and the element $(0,u)$ is a solution of $x^2 + (u,0) = 0$. We show $(u,0)$ is the only non-zero idempotent of T.

Suppose $(a,b)^2 = (a,b)$, so that $(a^2 - b^2, 2ab) = (a,b)$.

Hence, $a^2 - b^2 = a$ and $2ab = b$. If b is not nilpotent, then by an argument used in Theorem 1.6, b is invertible. It follows that $2a = u$, or $a = \frac{u}{2}$. Then $\frac{u}{2} = a = a^2 - b^2 = \frac{u}{4} - b^2$, and $b^2 = -\frac{u}{4}$. Thus $(2b)^2 + u = 0$ which contradicts the assumption of this case.

Thus, b is nilpotent; we show $b = 0$. Suppose $b \neq 0$; then there exists an integer n such that $b^n \neq 0$ and $b^{n+1} = 0$. Now $a^2 - b^2 = a$ so that $a^2b^n - b^{n+2} = ab^n$; since $b^{n+2} = 0$ we have $a^2b^n = ab^n$. However, $2ab = b$, so that $2ab^n = b^n$ as well, and, hence, $2a^2b^n = ab^n$. Thus, $2a^2b^n = a^2b^n$, from which it follows that $a^2b^n = 0$. Consequently, $ab^n = 0$. But $b^n = 2ab^n = 0$, while $b^n \neq 0$. This contradiction shows that for no integer n is $b^n \neq 0$; hence, $b = 0$. In view of $a^2 - b^2 = a$ we have $a^2 = a$, and $a = u$ or $a = 0$. Thus, $(a,b) = (0,0)$ or $(a,b) = (u,0)$ are the only idempotents in T.

By Case I, $T = N(T)(C)$, and the proof is complete.

<u>Corollary 1.7</u> Let S be a finite dimensional abelian affine semigroup. Then there exist $T_1, \ldots, T_n$, where each T_i is an abelian nilpotent finite-dimensional algebra over the complexes, and S is equivalent to a subsemigroup of $\sum_{i=1}^{r} \oplus T_i(C)$.

<u>Proof</u>: A direct consequence of Theorems 1.1, 1.3, and 1.7.

By Theorem J, a compact, abelian affine topological semigroup has a zero. Without compactness, this need not be true.

<u>Example</u>: Let $S = R \oplus R$ (R is the real numbers) where $(x,y)o(a,b) = (0,y + b)$. Then S is an abelian affine semigroup and $K(S) = \{(0,a): a \in R\}$. Note that K(S) is isomorphic to the additive reals.

Clark [3] shows that a finite-dimensional affine semigroup has a completely simple minimal ideal. The next theorem shows that the kernel of the above example is typical for abelian, finite-dimensional affine semigroups.

<u>Theorem 1.8</u> Let S be an abelian, finite-dimensional affine semigroup satisfying $S = V(S)$. The kernel of S consists of a zero or is isomorphic to a finite product of additive reals.

<u>Proof</u>: Let K be the minimal ideal of S; assume K is not degenerate. Let $x \in K$, then $Kx \subset K$ and is an ideal; hence $Kx = K$. It follows easily that K is a group. If $e^2 = e$ is the identity of K, then $Se \subset K$ and, hence, $Se = K$.

By the remark preceding Theorem 1.1, S is a topological semigroup. Further, Se is locally compact since it is a linear manifold. Thus, Se is a locally compact topological semigroup which is algebraically a group. By a theorem of Ellis [15], Se is a topological group. Consequently, Se is a locally compact abelian topological group. By the Principle Structure Theorem [59;40], Se contains an open subgroup G which is isomorphic to the direct product of finitely many additive reals together with a compact group H. Now Se is connected, and, since G is both open and closed, Se = G.

By Theorem J, $\langle H \rangle$ has a zero element. But $\langle H \rangle \subset Se$, and Se is a group so that $\langle H \rangle = \{e\}$. Thus Se = G is isomorphic to the direct product of finitely many additive reals.

CHAPTER II

In this chapter we study the relationship between a compact, affine, topological semigroup S and its associated probability semigroup $\tilde{S}$. We show that the resultant map is a continuous affine homomorphism of $\tilde{S}$ onto S. Several properties of S are deduced from this fact. We add to the sequence of theorems of the same category as Theorem F and Theorem G by showing that if S is a compact abelian semigroup and $\mu^2 = \mu \in \tilde{S}$, then $\mu\tilde{S}$ is the full probability semigroup of some compact semigroup.

Finally, we show that a group-extremal affine semigroup supports a probability measure. This theorem is a consequence of a general theorem to be proved about measures on compact convex sets in linear spaces.

If S is a compact convex set in a locally convex linear space, by A(S) we mean the collection of all complex-valued continuous affine functions on S. With the norm defined by: $\|f\|_\infty = \sup_{x \in S} |f(x)|$, A(S) is a closed subspace of C(S). By the remark following Theorem A of the Preliminaries, A(S) separates points of S.

If X is a locally convex topological linear space, by X* we mean the collection of all continuous, complex-valued linear functionals on S. Let $F \varepsilon X^*$, $x_1, \ldots, x_n \varepsilon X$, and $\varepsilon > 0$; then define:

$U(F, x_1, \ldots, x_n, \varepsilon) = \{G \varepsilon X^* : |F(x_i) - G(x_i)| < \varepsilon$ for $i = 1, 2, \ldots, n\}$. The collection of all possible sets of this form is a basis for a locally convex topology, called the 'weak-star' topology, on X*. If X is a Banach space, X* is a Banach space with norm defined by:

$F \varepsilon X^*$, then $\|F\| = \sup_{\|x\| = 1} |F(x)|$, where $x \varepsilon X$. The unit ball in X* is compact in the weak-star topology.

I. <u>The Resultant map</u>: In case S is a compact convex set in a linear space, Choquet [2] shows that there is a continuous affine homomorphism from $\widehat{S}$ to S. Loomis makes use of this fact in [26]. We give two proofs of the existence of such a map in two settings; the general case, where S is a compact convex set, and another assuming S to be a compact affine topological semigroup with identity. In both cases, we show that if S is also an affine semigroup, then this map is a homomorphism.

<u>Theorem 2.1</u> If S is a compact convex set in a locally convex linear space, then there exists $R:\widetilde{S} \longrightarrow S$ satisfying:

1) R is continuous, affine, onto S.
2) $f(R(\mu)) = \int f(y) d\mu(y)$ for $\mu \varepsilon \widetilde{S}$ and $f \varepsilon A(S)$.
3) R is a homomorphism if S is an affine semigroup.

Proof 1: Let S be a compact, convex set in the locally convex linear space X. Imbed S in A(S)* by defining:
(1) $\hat{x}(f) = f(x)$ where $x \varepsilon S$, $f \varepsilon A(S)$. The mapping $x \longrightarrow \hat{x}$ is affine and continuous in the weak-star topology on A(S)*. Further, $\|\hat{x}\| = 1$ for all $x \varepsilon S$, so that $\hat{S} = \{\hat{x}: x \varepsilon S\}$ is a weak-star compact subset of the unit ball of A(S)*.

Fix $\mu \varepsilon \tilde{S}$, and define:
(2) $T_\mu f = \int f d\mu$ for $f \varepsilon A(S)$. It is clear that $T_\mu \varepsilon A(S)^*$, in fact $\|T_\mu\| = 1$. We show that T_μ is in the weak-star closure of $\hat{S}$, and, hence, in $\hat{S}$.

Let $U(T_\mu, f_1, \ldots, f_n, \varepsilon)$ be a weak-star basis neighborhood of T_μ. By definition (2), $T_\mu f_i = \int f_i d\mu$ for each i; hence, there exist partitions $P_1, \ldots, P_n$ of S into disjoint Borel sets such that if $Q = \{E_j\}_{j=1}^r$ is a refinement of P_i, and $z_j \varepsilon E_j$ then
(3) $\left| T_\mu(f_i) - \sum_{j=1}^r f_i(z_j)\mu(E_j) \right| < \varepsilon$. Taking Q to be a common refinement of $P_1, \ldots, P_n$, where $Q = \{E_j\}_{j=1}^r$, and $z_j \varepsilon E_j$, we have:
(4) $\left| T_\mu(f_i) - \sum_{j=1}^r f_i(z_j)\mu(E_j) \right| < \varepsilon$ for $i = 1, 2, \ldots, n$. Setting $x_o = \sum_{j=1}^r z_j \mu(E_j)$, then $x_o \varepsilon S$ since $\sum_{j=1}^r \mu(E_j) = 1$ and $z_j \varepsilon E_j$. Further, $\left| T_\mu(f_i) - f_i(x_o) \right| = \left| T_\mu(f_i) - f_i\left(\sum_{j=1}^r z_j \mu(E_j)\right) \right| = \left| T_\mu(f_i) - \sum_{j=1}^r f_i(z_j)\mu(E_j) \right| < \varepsilon$ for $i = 1, 2, \ldots, n$. Thus, since $\hat{x}_o(f_i) = f_i(x_o)$, we have

that $\hat{x}_0 \in U(T_\mu, f_1, \ldots, f_n, \varepsilon)$. Hence, T_μ is in the weak-star closure of $\hat{S}$, and, consequently, in $\hat{S}$. By definition of $\hat{S}$, there exists an $x \in S$ such that $T_\mu = \hat{x}$. Thus, $\int f d\mu = T_\mu f = \hat{x}(f) = f(x)$ for all $f \in A(S)$.

We have shown that for $\mu \in \tilde{S}$, there is an $x \in S$ satisfying (5) $f(x) = \int f d\mu$ for all $f \in A(S)$. Since $A(S)$ separates points of S, this element is unique with respect to (5). We set $x = R(\mu)$ and show that $\mu \longrightarrow R(\mu)$ is the desired function.

Let $\mu, \nu \in \tilde{S}$, $0 \leq \lambda \leq 1$, and $f \in A(S)$. Then $f(\lambda R(\mu) + (1 - \lambda)R(\nu)) = \lambda f(R(\mu)) + (1-\lambda)f(R(\nu)) = \lambda\int f d\mu + (1-\lambda)\int f d\nu = \int f d[\lambda\mu + (1-\lambda)\nu] = f(R(\lambda\mu + (1-\lambda)\nu))$. Again, since $A(S)$ separates points, $R(\lambda\mu + (1-\lambda)\nu) = \lambda R(\mu) + (1-\lambda)R(\nu)$ and $\mu \longrightarrow R(\mu)$ is affine.

If $\mu_\alpha \longrightarrow \mu$, $\alpha \in \Lambda$, in the weak-star topology on S, then by definition, $\mu_\alpha(f) \longrightarrow \mu(f)$ for each $f \in C(S)$. The net $\{R(\mu_\alpha)\}_{\alpha \in \Lambda} \subset S$ must have a cluster point p. Suppose $p \neq R(\mu)$; then there is an $f \in A(S)$ satisfying $f(p) \neq f(R(\mu))$. Since $\{R(\mu_\alpha)\}_{\alpha \in \Lambda}$ clusters to p, by continuity of f, $\{f(R(\mu_\alpha))\}_{\alpha \in \Lambda}$ clusters at $f(p)$. However, $f(R(\mu_\alpha)) = \int f d\mu_\alpha = \mu_\alpha(f)$, and $\mu_\alpha(f)$ converges to $\mu(f) = \int f d\mu = f(R(\mu))$. Thus, $f(p) = f(R(\mu))$ and, hence, $p = R(\mu)$. It follows that the only cluster point of $\{R(\mu_\alpha)\}_{\alpha \in \Lambda}$ is $R(\mu)$, so that $R(\mu_\alpha) \longrightarrow R(\mu)$. Consequently, $\mu \longrightarrow R(\mu)$ is continuous.

Now if $\check{x}$ is the point mass concentrated at $x \varepsilon S$, and $f \varepsilon A(S)$, then $f(R(\check{x})) = \int f d\check{x} = f(x)$. It follows that $R(\check{x}) = x$ so that $\mu \longrightarrow R(\mu)$ takes $\tilde{S}$ onto S.

Finally, suppose S is a compact, affine topological semigroup, and let $f \varepsilon A(S)$. Define

(6) $f^a(x) = f(xa)$, $f_a(x) = f(ax)$; then f^a, $f_a \varepsilon A(S)$. Further, if $\mu \varepsilon \tilde{S}$, $a \varepsilon S$ then $f(R(\mu)a) = f^a(R(\mu)) = \int f^a(y)d\mu(y) = \int f(ya)d\mu(y)$. If $\mu, \nu \varepsilon \tilde{S}$, and $f \varepsilon A(S)$, then $f(R(\mu\nu)) = \int fd\mu*\nu = \int(\int f(xy)d\mu(x))d\nu(y) = \int f(R(\mu)y)d\nu(y) = \int f_{R(\mu)}(y)d\nu(y) = f_{R(\mu)}(R(\nu)) = f(R(\mu)R(\nu))$. This shows that $f(R(\mu\nu)) = f(R(\mu)R(\nu))$, and again, since A(S) separates points, $R(\mu\nu) = R(\mu)R(\nu)$, and $\mu \longrightarrow R(\mu)$ is a homomorphism.

Proof 2: Here we assume S is a compact affine topological semigroup with identity u.

Let B be the linear space of bounded linear operators on A(S). We describe two topologies in which B is a locally convex linear space, by describing a neighborhood basis at each point.

(P)-Topology: $F \varepsilon B$, $f_1, \ldots, f_n \varepsilon A(S)$, $s_1, \ldots, s_m \varepsilon S$, $\varepsilon > 0$; $U_p(F, f_1, \ldots, f_n, s_1, \ldots, s_m, \varepsilon) = \{G \varepsilon B: |Gf_i(s_j) - Ff_i(s_j)| < \varepsilon, 1 \leq i \leq n, 1 \leq j \leq m\}$.

(SOT) [Strong Operator Topology]: $F \varepsilon B$, $f_1, \ldots, f_n \varepsilon A(S)$, $\varepsilon > 0$; $U_s(F, f_1, \ldots, f_n, \varepsilon) = \{G \varepsilon B: \|Gf_i - Ff_i\| <$

ε, $1 \leq i \leq n\}$.

For $s \in S$, define $R_s \in B$ by: $(R_s f)(x) = f(xs) = f^s(x)$. DeLeeuw and Glicksberg [25] show that $s \longrightarrow R_s$ is a one-to-one homomorphism of S into B which is SOT continuous. They do so by showing that, for fixed $f \in C(S)$, the map $s \longrightarrow f^s$ is norm continuous. It is clear that $s \longrightarrow R_s$ is affine.

Setting $\hat{S} = \{R_s : s \in S\}$, then $\hat{S}$ is a compact, affine topological semigroup with the strong operator topology, and the operation of composition.

For $\mu \in \tilde{S}$, we define $T_\mu \in B$ by:

(7) $(T_\mu f)(s) = \int f(sx)d\mu(x)$ where $s \in S$ and $f \in A(S)$.

To show $T_\mu : A(S) \longrightarrow A(S)$, let $f \in A(S)$, $s, t \in S$, $0 \leq \lambda \leq 1$; then $(T_\mu f)(\lambda s + (1 - \lambda)t) =$
$\int f([\lambda s + (1 - \lambda)t]x)d\mu(x) = \int f(\lambda sx + (1 - \lambda)tx)d\mu(x)$
$= \int [\lambda f(sx) + (1 - \lambda)f(tx)]d\mu(x) = \lambda \int f(sx)d\mu(x) +$
$(1 - \lambda)\int f(tx)d\mu(x) = \lambda T_\mu f(s) + (1 - \lambda)T_\mu f(t)$, so that $T_\mu f$ is affine. That $T_\mu f$ is continuous follows immediately from the fact that $s \longrightarrow f_s$ is norm continuous.

Let $f_1, \ldots, f_n \in A(S)$, $s_1, \ldots, s_m \in S$, and $\varepsilon > 0$. Fix i,j where $1 \leq i \leq n$, $1 \leq j \leq m$; then there is a partition $P_{i,j}$ such that if $Q = \{E_k\}_{k=1}^r$ is a refinement of $P_{i,j}$ and $z_k \in E_k$, then $\left| T_\mu f_i(s_j) - \sum_{k=1}^{r} f_i(s_j z_k)\mu(E_k) \right| < \varepsilon$.

Let Q be a common refinement of $P_{i,j}$ where $1 \leq i \leq n$ and

$1 \leq j \leq m$, and $Q = \{E_k\}_{k=1}^{r}$; then if $z_k \in E_k$,

$\left| T_\mu f_i(s_j) - \sum_{k=1}^{r} f_i(s_j z_k)\mu(E_k) \right| < \varepsilon$ for all i,j. Setting $T_\mu x_o = \sum_{k=1}^{r} z_k \mu(E_k)$, then $x_o \in S$ and $(R_{x_o} f_i)(s_j) = f_i^{x_o}(s_j) =$ $f_i(s_j x_o) = f_i(s_j[\sum_{k=1}^{r} z_k \mu(E_k)]) = f_i(\sum_{k=1}^{r} s_j z_k \mu(E_k)) =$ $f_i(s_j x_o) = f_i(s_j[\sum_{k=1}^{r} z_k \mu(E_k)]) = f_i(\sum_{k=1}^{r} s_j z_k \mu(E_k)) = \sum_{k=1}^{r} f_i(s_j z_k)\mu(E_k)$. Hence, $\left| T_\mu f_i(s_j) - R_{x_o} f_i(s_j) \right| < \varepsilon$ for all i,j, and $R_{x_o} \in U_p(T_\mu, f_1, \ldots, f_n, s_1, \ldots, s_m, \varepsilon)$.

This argument shows that T_μ is in the (P) closure of $\hat{S}$. However, the (P) closure of $\hat{S}$ is clearly contained in the SOT closure of $\hat{S}$ and, hence, $T_\mu \in \hat{S}$. There exists, therefore, an element $x_o \in S$ such that $T_\mu = R_{x_o}$; this means that for $f \in A(S)$, $s \in S$, $\int f(sy)d\mu(y) = T_\mu f(s) = R_{x_o} f(s) = f^{x_o}(s) = f(sx_o)$. Taking $s = u$, we have $f(x_o) = \int f(y)d\mu(y)$; setting $x_o = R(\mu)$ the map $\mu \longrightarrow R(\mu)$ is again a continuous affine homomorphism of $\tilde{S}$ onto S, the proof being the same as in Proof 1. This completes the proof of Theorem 2.1.

<u>Definition</u>: A compact affine topological semigroup S with identity u is called <u>group-extremal</u> if the extreme points of S have inverses. In this case, in view of Theorem I of the Preliminaries, the extreme points form the maximal group of the idempotent u which is well-known to be compact.

<u>Corollary 2.1.1</u> [6] A compact, group-extremal semigroup has a zero.

<u>Proof</u>: Let $\mu^2 = \mu \in \widetilde{S}$ be Haar measure on the extreme points G. Then for $x \in G$, $\underset{\sim}{x}\mu = \mu\underset{\sim}{x} = \mu$ by the invariance of Haar measure. By Theorem 2.1, $R(\mu) = R(\mu\underset{\sim}{x}) = R(\mu)R(\underset{\sim}{x}) = R(\mu)x$, and, similarly, $R(\mu) = xR(\mu)$. If $x = \sum_{i=1}^{n} \lambda_i x_i$, where $x_i \in G$, $\sum_{i=1}^{n}\lambda_i = 1$, $\lambda_i \geq 0$, then $R(\mu)x = \sum_{i=1}^{n} \lambda_i R(\mu)x_i = \sum_{i=1}^{n}\lambda_i R(\mu) = R(\mu)$. Similarly $xR(\mu) = R(\mu)$; by the Krein-Milman Theorem, the elements of the above form are dense in S. Thus, $xR(\mu) = R(\mu)x = R(\mu)$ for all $x \in S$, and $R(\mu)$ is a zero for S. Note, then, that if S is group-extremal with zero θ, then $\theta = R(\mu)$, where μ is Haar measure on the extreme points. Thus, $f(\theta) = \int f d\mu$ for all $f \in A(S)$.

In [17], Glicksberg shows that if S is a compact semigroup, and $\mu \in \widetilde{S}$, then the sequence $\frac{1}{N}\sum_{i=1}^{N}\mu^i$ converges weak-star to an element $\lambda^2 = \lambda \in \widetilde{S}$ satisfying $\mu\lambda = \lambda\mu = \lambda$.

<u>Corollary 2.1.2</u> If S is a compact affine topological semigroup, and $x \in S$, then $\frac{1}{N}\sum_{i=1}^{N}x^i$ converges to an element $e^2 = e \in S$ satisfying $ex = xe = e$.

<u>Proof</u>: If $x \in S$, then $\frac{1}{N}\sum_{i=1}^{N}\underset{\sim}{x}^i$ converges to an element $\lambda^2 = \lambda \in \widetilde{S}$ satisfying $\lambda\underset{\sim}{x} = \underset{\sim}{x}\lambda = \lambda$. Then $\frac{1}{N}\sum_{i=1}^{N}x^i = \frac{1}{N}\sum_{i=1}^{N}R(\underset{\sim}{x}^i) = R(\frac{1}{N}\sum_{i=1}^{N}\underset{\sim}{x}^i)$, and, hence, converges to $R(\lambda)$. Further, $R(\lambda) = R(\lambda\lambda) = R^2(\lambda)$ and $R(\lambda) = R(\lambda\underset{\sim}{x}) = R(\lambda)x$ and $R(\lambda) = xR(\lambda)$.

Remark: In the definition of an affine semigroup S, it is not assumed that a multiplication exists outside of S in the linear space in which S is imbedded. However, we can make this assumption if S is compact and has an identity, since the second proof of Theorem 2.1 shows that S can be imbedded in the algebra of bounded linear operators on A(S).

If S is a compact semigroup and $\mu^2 = \mu \in \widetilde{S}$ then, by Theorem E, $C(\mu)$ is a compact simple semigroup. Hence $\langle C(\mu)\rangle$ is a compact semigroup.

Theorem 2.2 If S is a compact affine topological semigroup, and $\mu^2 = \mu \in \widetilde{S}$, then $R(\mu)$ is in the kernel of $\langle C(\mu)\rangle$.

Proof: Let $e^2 = e = R(\mu)$ and $\mu_o = \delta_e \mu \delta_e$. Then $\mu_o \in \widetilde{S}$, $\delta_e \mu_o = \mu_o \delta_e = \mu_o$ and $R(\mu_o) = e$.

Now $\frac{1}{N}\sum_{i=1}^{N} \mu_o^{\,i}$ converges to $\lambda^2 = \lambda \in \widetilde{S}$ which satisfies $\mu_o \lambda = \lambda \mu_o = \lambda$. Since $R(\mu_o) = e$, by continuity, $R(\lambda) = e$. Also, $\lambda = \mu_o \lambda = \delta_e \mu_o \lambda = \delta_e \lambda$ and similarly, $\lambda = \lambda \delta_e$. Thus, by Theorem D, $C(\lambda) = C(\lambda \delta_e) = C(\delta_e \lambda \delta_e) = C(\delta_e)C(\lambda)C(\delta_e) = eC(\lambda)e \subset eSe$. Further, since $\lambda^2 = \lambda$, $C(\lambda)$ is a compact simple semigroup (Theorem E).

We show, next, that $e \in \langle C(\lambda)\rangle$. If not there exists, by Theorem A, an $f \in A(S)$ such that $f(e) < c_o < \min_{x \in \langle C(\lambda)\rangle}\{f(x)\}$. However, $e = R(\lambda)$ so that $f(e) = \int_{C(\lambda)} f(y)\,d\lambda(y)$ and, thus, $\int_{C(\lambda)} f(e)d\lambda(y) = \int_{C(\lambda)} f(y)d\lambda(y)$. Then $\int_{C(\lambda)}[f(y) - f(e)]d\lambda(y) = 0$, and $f(y) > f(e)$, imply that $f(y) = f(e)$ for all $y \in C(\lambda)$, which contradicts the choice of f.

Consequently, $e \in \langle C(\lambda)\rangle$ and, since $\langle C(\lambda)\rangle \subset eSe$, it follows that e is an identity for $\langle C(\lambda)\rangle$. By Theorem I, e is an extreme point of $\langle C(\lambda)\rangle$, and by Theorem C, $e \in C(\lambda)$. Since $C(\lambda)$ is simple and has an identity, it must be a group [50;12]. Thus, $C(\lambda)$ is a group and λ is Haar measure on $C(\lambda)$.

Now for $x \in C(\lambda)$, $f \in A(S)$, $f(x) = f(ex) = f^x(e) = \int f^x(y)d\lambda(y) = \int f(xy)d\lambda(y) = \int f(y)d\lambda(y) = f(e)$. Since f is arbitrary, it follows that $C(\lambda) = \{e\}$, and $\lambda = \varepsilon$.

Since $\mu_o\lambda = \lambda\mu_o = \lambda$, and $\lambda = \varepsilon$, we have $\varepsilon\mu_o = \mu_o\varepsilon = \varepsilon$. However, $\mu_o\varepsilon = \varepsilon\mu_o = \mu_o$, so that $\mu_o = \varepsilon$; that is $\varepsilon\mu\varepsilon = \varepsilon$. Again by Theorem D, $eC(\mu)e = C(\varepsilon)C(\mu)C(\varepsilon) = C(\varepsilon\mu\varepsilon) = C(\varepsilon) = \{e\}$. It follows that $e\langle C(\mu)\rangle e = \{e\}$, and by repetition of a previous argument $e \in \langle C(\mu)\rangle$. The conclusion now follows by Theorem K(b).

Theorem 2.2 seems to be the closest statement one can make in analogy to Corollary 2.1.1. One might conjecture that if the extreme points of a compact affine semigroup S consist of a finite union of groups, then S has a zero.

To see that this need not be true consider the following:

Example: Let $S = D \times I$ where D is the complex unit disc under ordinary multiplication, and where I is the interval $[0,1]$ with multiplication defined by $xy = x$ for all $x,y \in I$. Then S is a compact affine topological semigroup. The extreme points of S are $S^1 \times \{1\} \cup S^1 \times \{0\}$, while the kernel of S is $\{0\} \times I$.

II. Subsemigroups of $\tilde{S}$. We now prove the theorem promised in the Introduction which completes the series of theorems given by Theorem F and Theorem G.

Theorem 2.3 Let S be a compact, abelian topological semigroup, and $\mu^2 = \mu \in \tilde{S}$. Then $\mu \tilde{S}$ is equivalent to $\tilde{T}$ for some compact, abelian semigroup T.

Proof: In view of Theorem E, and the fact that S is abelian, $C(\mu)$ is an abelian group.

Define $R = \{(x,y) \in S \times S : C(\mu)x = C(\mu)y\}$; then R is a closed congruence on S, and S/R is a compact abelian topological semigroup. Let $\varphi : S \longrightarrow S/R$ be the natural homomorphism of S onto S/R.

We show that $(x,y) \in R$ iff $\int f(xz)d\mu(z) = \int f(yz)d\mu(z)$ for all $f \in C(S)$. If $(x,y) \in R$, then $C(\mu)x = C(\mu)y$; hence there exist $p,q \in C(\mu)$ for which $px = qy$. Let $f \in C(S)$; then $\int f(xz)d\mu(z) = \int f(xpz)d\mu(z) = \int f(yqz)d\mu(z) = \int f(yz)d\mu(z)$. On the other hand, suppose

$C(\mu)x \not\subset C(\mu)y$. Then there exists $p \in C(\mu)$ for which $px \not\in C(\mu)y$. There exists $f \in C(S)$, $0 \leq f \leq 1$, and $f(px) = 1$ while $f(z) = 0$ for $z \in C(\mu)y$. There exists an open set U containing p for which $f(tx) > \frac{1}{2}$ for $t \in U$. Since $p \in C(\mu)$, $\mu(U) > 0$; therefore, $\int f(xz)d\mu(z) \geq \int_U f(xz)d\mu(z) \geq \frac{1}{2}\mu(U) > 0$, and $\int f(yz)d\mu(z) = 0$, since $f(yz) = 0$ for $z \in C(\mu)$. Thus, $\int f(xz)d\mu(z) \neq \int f(yz)d\mu(z)$ and the assertion is proved.

Let e be the identity of $C(\mu)$. Then $C(\mu)e = C(\mu)$, so that for $x \in S$, $C(\mu)x = C(\mu)ex$ and, therefore, $\varphi(x) = \varphi(ex) = \varphi(e)\varphi(x)$. Clearly, then, $\varphi(e)$ is an identity for T, so that $\varphi_o(e)$ is an identity for $\widetilde{T}$.

Let $f \in C(S)$ and define $f'(\varphi(x)) = \int f(yx)d\mu(x)$. Then f' is well-defined and $f' \in C(T)$. Let $\nu \in \widetilde{T}$ and define $(P\nu)(f) = \int f'd\nu$ for $f \in C(S)$. Then, by the argument used in [17] to prove Theorem F, P is a continuous, affine homomorphism of $\widetilde{T}$ into $\widetilde{S}$.

On the other hand, define $\varphi^*:\widetilde{S} \longrightarrow \widetilde{T}$ by $[\varphi^*(\nu)](f) = \int f(\varphi(x))d\nu(x)$ for all $f \in C(T)$, where $\nu \in \widetilde{S}$. It is well-known (and easy to show) that φ^* is a continuous, affine homomorphism. Further, $\varphi^*(\underset{\delta}{x}) = \varphi_o(x)$, so that φ^* takes $\widetilde{S}$ onto $\widetilde{T}$, as a consequence of Theorem H.

If $x \in C(\mu)$, then $C(\mu)x = C(\mu)e$ so that $\varphi(x) = \varphi(e)$ for all $x \in C(\mu)$. Then for $f \in C(T)$, $(\varphi^*\mu)(f) =$

$\int f(\varphi(x))d\mu(x) = \int_{C(\mu)} f(\varphi(x))d\mu(x) = \int_{C(\mu)} f(\varphi(e))d\mu(x)$ $= f(\varphi(e)) = \int f(\varphi(x))d\delta(x) = \varphi^*(\delta)(f) = \varphi_0(e)(f)$. We have shown, then, that $\varphi^*\mu = \varphi_0(e)$ and is thereby an identity for $\hat{T}$.

Now, if $f \in C(S)$, $s \in C(\mu)$, then $(f^s)'(\varphi(x)) = \int f^s(yx)d\mu(y) = \int f(yx)d\mu(y) = f'(\varphi(x))$. Then $(f^s)' = f'$, and $(P\nu * \mu)(f) = \int\int f(xy)dP\nu(x)d\mu(y) = \int_{C(\mu)} (P\nu)(f^y)d\mu(y) = \int_{C(\mu)}\int_T (f^y)'(z)d\nu(z)d\mu(y) = \int_{C(\mu)}\int f'(z)d\nu(z)d\mu(y) = \int f'(z)d\nu(z) = (P\nu)(f)$. This shows that $(P\nu)*\mu = P\nu$, or $P\nu \in \mu\tilde{S}$. Also, if $\nu \in \tilde{S}$, then $[(P\varphi^*)(\nu)(f)] = [P(\varphi^*(\nu))](f) = \int f'(z)d\varphi^*\nu(z) = \int f'(\varphi(x))d\nu(x) = \int\int f(yx)d\mu(y)d\nu(x) = (\mu*\nu)(f)$; hence, $(P\varphi^*)(\nu) = \mu*\nu$. If $\mu*\nu = \nu$, (<u>i.e.</u>, if $\nu \in \mu\tilde{S}$), then $(P\varphi^*)(\nu) = \nu$. Thus, if $\mu*\nu = \nu$, then $(P\varphi^*)(\nu) = \nu$, and P takes $\tilde{T}$ onto $\mu\tilde{S}$. Suppose $P\nu_1 = P\nu_2$, where $\nu_1, \nu_2 \in \tilde{T}$. Then there exist $\xi_1, \xi_2 \in \tilde{S}$ for which $\varphi^*(\xi_1) = \nu_1$, $\varphi^*(\xi_2) = \nu_2$. Let $\tau_1 = \mu\xi_1$, $\tau_2 = \mu\xi_2$, then $\varphi^*(\tau_1) = \varphi^*(\mu\xi_1) = \varphi^*(\mu)\varphi^*(\xi_1) = \varphi^*(\xi_1)$; similarly, $\varphi^*(\tau_2) = \xi_2$. Now $\tau_1, \tau_2 \in \mu\tilde{S}$ so that $(P\varphi^*)(\tau_1) = \tau_1$, $(P\varphi^*)(\tau_2) = \tau_2$. However, $\tau_1 = P\varphi^*(\tau_1) = P\varphi^*(\xi_1) = P\nu_1 = P\nu_2 = P\varphi^*(\xi_2) = P\varphi^*(\tau_2) = \tau_2$, so that $\nu_1 = \varphi^*(\tau_1)$, $\nu_2 = \varphi^*(\tau_2)$ imply that $\nu_1 = \nu_2$. Hence, P is one-to-one. $\mu\tilde{S}$ and $\tilde{T}$ are now equivalent, which was to be shown.

III. <u>Probability measures on compact, convex sets</u>. We show here that given a measure $\mu \in \tilde{S}$, where S is a compact convex set in a linear space X, then there is a measure $\nu \in \tilde{S}$ satisfying $C(\nu) = \langle C(\mu) \rangle$.

<u>Lemma 2.4.1</u> Let S,K be compact Hausdorff spaces, and $f: K \longrightarrow S$, a continuous function. Then f induces $f^*: \tilde{K} \longrightarrow \tilde{S}$ which is continuous and satisfies $C(f^*(\mu)) = f(C(\mu))$.

<u>Proof</u>: Define $f^*: \tilde{K} \longrightarrow \tilde{S}$ by:
$(f^*\mu)(g) = \int g(f(x))d\mu(x)$ where $\mu \in \tilde{K}$, $g \in C(S)$.
Clearly, $f^*: \tilde{K} \longrightarrow \tilde{S}$ and is weak-star continuous.

Let $\mu \in \tilde{K}$, we show $C(f^*(\mu)) = f(C(\mu))$:
(1) $C(f^*(\mu)) \subset f(C(\mu))$. Suppose $x_o \in C(f^*(\mu))$ and $x_o \notin f(C(\mu))$. Then there exists $g \in C(S)$, $0 \leq g \leq 1$, $g(x_o) = 1$ and $g \equiv 0$ on $f(C(\mu))$. There is an open set V containing x_o on which $g(y) > \frac{1}{2}$ for $y \in V$. Then $(f^*\mu)(g) = \int g(y)d(f^*\mu)(y) \geq \int_V g(y)df^*(\mu)(y) \geq \frac{1}{2}(f^*\mu)(V) > 0$; $(f^*\mu)(V) > 0$, since $V \cap C(f^*(\mu)) \neq \emptyset$. However, $(f^*\mu)(g) = \int g(f(y))d\mu(y) = \int_{C(\mu)} g(f(y))d\mu(y) = 0$. This contradiction establishes (1).

(2) $f(C(\mu)) \subset C(f^*(\mu))$. Let $x_o = f(y_o)$, where $y_o \in C(\mu)$ and $x_o \notin C(f^*(\mu))$. There is a $g \in C(S)$, $0 \leq g \leq 1$, and an open set V containing x_o such that $g(y) > \frac{1}{2}$ on V and $g \equiv 0$ on $C(f^*(\mu))$. There is an open set U containing

y_o such that $f(U) \subset V$. Thus, for all $y \in U$, $g(f(y)) > \frac{1}{2}$. Then we have $(f^*\mu)(g) = \int_{C(f^*\mu)} g(y)df^*\mu(y) = 0$ since $g \equiv 0$ on $C(f^*\mu)$. On the other hand, $(f^*\mu)(g) = \int g(f(y))d\mu(y) \geq \int_U g(f(y))d\mu(y) \geq \frac{1}{2}\mu(U) > 0$; $\mu(U) > 0$ since $y_o \in U \cap C(\mu)$. Hence (2) is established, and, therefore, the lemma.

Lemma 2.4.2 Let S be compact, Hausdorff, and $\{\mu_i\}_{i=1}^{\infty} \subset \tilde{S}$, then $\sum_{n=1}^{\infty} \frac{1}{2}n\mu_n$ converges weak-star to $\mu_o \in \tilde{S}$, where $C(\mu_o) = \overline{\bigcup_1^{\infty} C(\mu_n)}$.

Proof: Since $\left\|\sum_{n=1}^{m} \frac{1}{2}n\mu_n - \sum_{n=1}^{k} \frac{1}{2}n\mu_n\right\| \leq \sum_{n=k}^{m} \frac{1}{2}n$, it follows that $\{\sum_{n=1}^{m} \frac{1}{2}n\mu_n\}_{m=1}^{\infty}$ converges in norm and, hence, weak-star to an element $\mu_o \in M(S)$. Each $\mu_i \in \tilde{S}$, so that $\mu_o \in \tilde{S}$.

(1) $C(\mu_o) \subset \overline{\bigcup_1^{\infty} C(\mu_n)}$. If not, there exists a $g \in C(S)$, $0 \leq g \leq 1$, such that $\int g d\mu_o > 0$, but $\int g d\mu_n = 0$ for all n. But $\int g d\mu_o = \lim_m \sum_{n=1}^{m} (\frac{1}{2}n \int g d\mu_n) = 0$; this contradiction establishes (1).

(2) $\overline{\bigcup_{n=1}^{\infty} C(\mu_n)} \subset C(\mu_o)$. Note that $\frac{1}{2}n\mu_n(f) \leq \mu_o(f)$ if $f \in C(S)$, $f \geq 0$. If $C(\mu_n) \not\subset C(\mu_o)$ for some n, then there is a $g \in C(S)$, $0 \leq g \leq 1$, for which $\int g d\mu_n > 0$, but $\int g d\mu_o = 0$. But $\frac{1}{2}n \int g d\mu_n \leq \int g d\mu_o$; so that $\int g d\mu_n = 0$. This contradiction establishes (2), and the lemma is proved.

<u>Theorem 2.4</u> If S is compact and convex in the linear space X, and $\mu \in \tilde{S}$, then there exists $\nu \in \tilde{S}$ for which $C(\nu) = \langle C(\mu)\rangle$.

<u>Proof</u>: Fix $n \geq 1$, and let

(1) $A_n = \{(\lambda_1, \ldots, \lambda_n) \in E^n : 0 \leq \lambda_i \leq 1, \sum_{i=1}^{n} \lambda_i \leq 1\}$.

Let $m_n \in \tilde{A}_n$ satisfy $C(m_n) = A_n$ (note that Legesque measure suitably restricted and normalized will do).

(2) Set $K_n = A_n \times \underbrace{C(\mu) \times \ldots \times C(\mu)}_{n+1}$, and

(3) $\nu_n = m_n \times \underbrace{\mu \times \ldots \times \mu}_{n+1}$ where ν_n is the product measure on K_n. Note that $\nu_n \in \tilde{K}_n$ and $C(\nu_n) = K_n$, since the measure of any product set is the product of the measures.

Define $h_n : K_n \longrightarrow S$ by:

(4) $h_n(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n} \lambda_i x_i + (1 - \sum_{i=1}^{n} \lambda_i) x_{n+1}$ where $(\lambda_1, \ldots, \lambda_n) \in A_n$, $x_i \in C(\mu)$ for $1 \leq i \leq n+1$. Clearly, h_n is continuous and

$h_n(K_n) = \{\sum_{i=1}^{n+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, x_i \in C(\mu), 1 \leq i \leq n+1\}$.

By Lemma 2.4.1, h_n induces $h_n^* : \tilde{K}_n \longrightarrow \tilde{S}$ and $C(h_n^*(\xi)) = h_n(C(\xi))$ for any $\xi \in \tilde{K}_n$. Let $\xi_n = h_n^*(\nu_n)$; then $C(\xi_n) = C(h_n^*(\nu_n)) = h_n(C(\nu_n)) = h_n(K_n) =$

$\{\sum_{i=1}^{n+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, x_i \in C(\mu), 1 \leq i \leq n+1\}$.

Lemma 2.4.2 then gives a measure $\mu_o \in \tilde{S}$ satisfying:
$C(\mu_o) = \overline{\bigcup_{n=1} C(\varepsilon_n)} = \overline{\bigcup_{n=1} \{\sum_{i=1}^{n+1} \lambda_i x_i : \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0, x_i \in C(\mu)\}} = \langle C(\mu) \rangle$. The last equality is justified by Theorem B of the Preliminaries. This measure μ_o is the desired extension of μ.

<u>Corollary 2.4.1</u> Let S be a group-extremal affine semigroup, then S supports a measure.

<u>Proof</u>: By assumption, the extreme points of S form a compact group G. A compact group supports a Haar measure μ, and we may assume, by suitable extension of μ to S, that $\mu \in \tilde{S}$. Thus $C(\mu) = G$, and Theorem 2.4 now gives a supporting measure for S.

CHAPTER III

In this chapter, we give a representation theory for compact, group-extremal affine semigroups. In the abelian case, we produce a sufficient system of affine semicharacters.

Definition: A representation of an affine topological semigroup S is a continuous affine homomorphism from S into the set of nxn complex matrices for some n.

Definition: If H is a Hilbert space, a completely-continuous symmetric operator is a bounded linear operator T from H into H which satisfies:

(1) T takes a uniformly bounded set in H to a relatively compact set.

(2) $(Tx,y) = (x,Ty)$ for all $x, y \in H$.

The following theorem is well-known, and an excellent proof may be found in [58;232].

Theorem 3.1 Let H be a Hilbert space, and T a completely continuous, symmetric operator from H to H. Then there exists a sequence $\{\varphi_i\}_{i=1}^{\infty} \subset H$ satisfying:

(1) $T\varphi_i = \lambda_i \varphi_i$ for some real number $\lambda_i \neq 0$.

(2) $(\varphi_i, \varphi_j) = \delta_{ij}$ (δ_{ij} is the Kronecker delta function).

(3) For $x \in H$, $Tx = \sum_{n=1}^{\infty} (Tx, \varphi_n)\varphi_n$.

(4) For a fixed $\lambda \neq 0$, $M_\lambda = \{x \in H : Tx = \lambda x\}$ is a finite-dimensional subspace of H.

If G is a compact group, $\mathcal{L}^2(G)$ is a complex Hilbert space with inner product:

(1) $(f,g) = \int f(x)\overline{g(x)}dx$ for $f,g \in \mathcal{L}^2(G)$. The norm in $\mathcal{L}^2(G)$ is denoted by $\|\cdot\|_2$. Let $k \in C(G)$, where k is real, and $k(y) = k(y^{-1})$ for all $y \in G$. Define

$T: \mathcal{L}^2(G) \longrightarrow \mathcal{L}^2(G)$ by

(2) $Tf(x) = \int k(xy^{-1})f(y)dy$, where $f \in \mathcal{L}^2(G)$, $x \in G$; then T is a completely continuous, symmetric operator in H (c.f. [49;204], [57;221], [55;49]).

Let S be a compact, group-extremal affine topological semigroup, where the extreme points are the compact group G. By $A_G(S)$, we shall mean the collection of functions in C(G) which are restrictions to G of elements of A(S).

Lemma 3.2.1 $A_G(S)$ is a norm closed subspace of C(G).

Proof: Let $\{f_n\}_{n=1}^{\infty} \subset A_G(S)$ and suppose $f_n \longrightarrow g \in C(G)$. There exists $\{f_n^*\}_{n=1}^{\infty} \subset A(S)$, where f_n^* restricted to G is f_n. We show $\{f_n^*\}_{n=1}^{\infty}$ is Cauchy in A(S).

Let $\varepsilon > 0$; there exists $N \geq 1$ such that for $m \geq N$, and $x \in G$, $|f_m(x) - g(x)| < \varepsilon/2$. Thus, for $n,m \geq N$, and $x \in G$, $|f_m(x) - f_n(x)| < \varepsilon$. If $x = \sum_{i=1}^{r} \lambda_i x_i$, $\sum_{i=1}^{r} \lambda_i = 1$, $\lambda_i \geq 0$, and $x_i \in G$, then $|f_m^*(x) - f_n^*(x)| = |f_m^*(\sum_{i=1}^{r}\lambda_i x_i) -$

$f_n^*(\sum_{i=1}^{r}\lambda_i x_i)| = |\sum_{i=1}^{r}\lambda_i[f_n^*(x_i) - f_m^*(x_i)]| =$
$|\sum_{i=1}^{r}\lambda_i[f_n(x_i) - f_m(x_i)]| \leq \sum_{i=1}^{r}\lambda_i |f_n(x_i) - f_m(x_i)| <$
$\sum_{i=1}^{r}\lambda_i\varepsilon = \varepsilon$. Since the elements of this form are dense in S by Theorem B, $|f_n^*(x) - f_m^*(x)| \leq \varepsilon$ for all $x \in S$. This shows that $\|f_n^* - f_m^*\|_\infty \leq \varepsilon$ for $n,m \geq N$, so that $\{f_n^*\}_{n=1}^{\infty}$ is Cauchy in A(S). Thus, $f_n^* \longrightarrow f \in A(S)$ in the uniform norm and, since $f_n \longrightarrow g$ on G, we must have $g = f$ on G. Thus, $g \in A_G(S)$.

Remark: Included in the proof of Lemma 3.1.1 are the following facts:

(a) If a sequence of elements of A(S) converge uniformly on G, they converge uniformly on S.

(b) If two elements of A(S) agree on G, they agree everywhere on S.

Lemma 3.2.2 If T is defined as in (2), then $Tf \in C(G)$ for all $f \in \mathcal{L}^2(G)$ and $T:\mathcal{L}^2(G) \longrightarrow C(G)$ is continuous with the supremum norm on C(G). Further, if $f \in A_G(S)$, then $Tf \in A_G(S)$.

Proof: Let $f \in \mathcal{L}^2(G)$, $x,y \in G$; then $|Tf(x) - Tf(y)|^2 =$
$|\int[k(xz^{-1}) - k(yz^{-1})]f(z)dz|^2 \leq \int|k(xz^{-1}) - k(yz^{-1})|^2dz \cdot$
$\int|f(z)|^2dz \leq \|k_x - k_y\|_\infty^2 \cdot \|f\|_2^2$. Continuity follows from the continuity of $x \longrightarrow k_x$.

Now for $f,g \in \mathcal{L}^2(G)$, $x \in G$, $|Tg(x) - Tf(x)|^2 =$

$\left| \int k(xy^{-1})[g(y) - f(y)]dy \right|^2 \leq \int |k(xy^{-1})|^2 dy \int |g(y) - f(y)|^2 dy$ $= \|k\|_2^2 \|f - g\|_2^2$. Hence, $\|Tg - Tf\|_\infty \leq \|k\|_2 \|f - g\|_2$ and T is continuous.

Finally, if $f \in A_G(S)$, there exists $g \in A(S)$, where $g = f$ on G. Let $f^*(x) = \int k(y^{-1})g(yx)dy$; then $f^* \in A(S)$ and for $x \in G$, $f^*(x) = \int k(y^{-1})g(yx)dy = \int k(y^{-1})f(yx)dy =$ $\int k(xy^{-1})f(y)dy = (Tf)(x)$. Thus on G, $Tf = f^*$, so that $Tf \in A_G(S)$.

<u>Theorem 3.2</u> Let S be a compact, group-extremal affine semigroup with compact group G. If $x,y \in S$ and $x \neq y$, there exists a representation P of S satisfying

(1) $P(x) \neq P(y)$

(2) $P^*(\sigma) \in P(S)$ for all $\sigma \in S$; ($P^*(\sigma)$ denotes the adjoint of the operator $P(\sigma)$).

<u>Proof</u>: Denote the identity of G by u. Then there exists an open subset U of G containing u, and where $\langle U\rangle x \cap \langle U\rangle y = \emptyset$. If not, let $\mathcal{U} = \{U : U \text{ open in } G,\ u \in U\}$; $\mathcal{U}$ is a directed set with the partial order defined by:

(3) $U \leq V$ iff $V \subseteq U$. By assumption, for each $U \in \mathcal{U}$, there are elements $p_U, t_U \in \langle U\rangle$ satisfying $p_U x = t_U y$. For each open subset W of S containing u, there exists an open convex subset V of S, $u \in V$, and for which $\overline{V} \subset W$. Let $V_o = V \cap G$; then $V_o \in \mathcal{U}$ and if $U \in \mathcal{U}$, $V_o \leq U$, $\langle U\rangle \subset \langle V_o\rangle \subset \overline{V} \subset W$. It follows that $p_U, t_U \in W$, and, therefore, $p_U \longrightarrow u$ and $t_U \longrightarrow u$; hence $p_U x \longrightarrow x$, $t_U y \longrightarrow y$

so that $x = y$. This establishes the existence of a $U \varepsilon \mathcal{U}$ where $\langle U\rangle x \cap \langle U\rangle y = \emptyset$. Obviously, we may assume $U = U^{-1}$.

By Theorem A, there exists an $f_o \varepsilon A(S)$ which satisfies $\min_{z\varepsilon\langle U\rangle x} \{f_o(z)\} > r_o > \max_{z\varepsilon\langle U\rangle y} \{f_o(z)\}$, and where f_o is a real-valued function. Further, there exists $h \varepsilon C(G)$ satisfying $h(u) = 1$, $h \equiv 0$ outside of U, and $0 \leq h \leq 1$. Setting $k(z) = \frac{h(z) + h(z^{-1})}{2}$, then $k(u) = 1$, $0 \leq k \leq 1$, $k(z) = k(z^{-1})$, and $k \equiv 0$ outside of U.

Then $\int k(z^{-1})f_o(zx)dz = \int_U k(z^{-1})f_o(zx)dz > r_o\int_U k(z^{-1})dz > \int_U k(z^{-1})f_o(zy)dz = \int k(z^{-1})f_o(zy)dz$. We have shown that $\int k(z^{-1})f_o(zx)dz \neq \int k(z^{-1})f_o(zy)dz$.

Now, let T be the operator defined by (2) which corresponds to the function k. We have that $Tf_o(x) \neq Tf_o(y)$, and $f_o \varepsilon A(S)$. By Lemma 3.2.2, $T:A_G(S) \longrightarrow A_G(S)$; if we let $H = \overline{A_G(S)}^{\mathcal{L}^2}$, then again by Lemma 3.2.2, $T(H) = T(\overline{A_G(S)}^{\mathcal{L}^2}) \subset \overline{A_G(S)}^{\|\cdot\|_\infty} = A_G(S)$. The last equality comes from Lemma 3.2.1. It follows that H is an invariant subspace of $\mathcal{L}^2(G)$; denote the restriction of T to H by T_G. Then:

(4) $T_G f(z) = \int k(zy^{-1})f(y)dy = \int k(y^{-1})f(yz)dy$ for $f \varepsilon H$, and $z \varepsilon G$. Also, $T_G f_o(x) \neq T_G f_o(y)$.

Since T is completely continuous and symmetric, the same is true for T_G. By Theorem 3.1, there exists $\{\varphi_i\}_{i=1}^{\infty} \subset H$, $T_G\varphi_i = \lambda_i\varphi_i$ for some real $\lambda_i \neq 0$, $(\varphi_i, \varphi_j) = \delta_{ij}$,

$T_G f = \sum_{n=1}^{\infty}(T_G f, \varphi_n)\varphi_n$ for all $f \in H$, and for fixed $\lambda \neq 0$, $M_\lambda = \{f \in H : T_G f = \lambda f\}$ is finite dimensional. Since $\varphi_i = T_G(\frac{1}{\lambda_i}\varphi_i)$, it follows that $\varphi_i \in A_G(S)$. Thus, there exists $\{\varphi_i^*\}_{i=1}^{\infty} \subset A(S)$, where $\varphi_i^* = \varphi_i$ on G.

Define $T_S : A(S) \longrightarrow A(S)$ by

(5) $T_S f(z) = \int k(y^{-1})f(yz)dy$ where $f \in A(S)$, $z \in S$. For $f \in A(S)$, let $g = f|_G$, the restriction of f to G. Then $T_S f(z) = \int k(y^{-1})f(yz)dy = \int k(zy^{-1})f(y)dy = \int k(zy^{-1})g(y)dy = T_G g(z)$ whenever $z \in G$. In particular, let $g_o = f_o|_G$, then $T_G g_o = \sum_{i=1}^{\infty}(T_G g_o, \varphi_i)\varphi_i$, where the series converges in $\mathcal{L}^2(G)$. Fix n,m, then we have

$$\sum_{i=n}^{m}(T_G g_o, \varphi_i)\varphi_i = \sum_{i=n}^{m}(g_o, T_G\varphi_i)\varphi_i = \sum_{i=n}^{m}\lambda_i(g_o, \varphi_i)\varphi_i = \sum_{i=n}^{m}(g_o, \varphi_i)T_G\varphi_i = T_G\left(\sum_{i=n}^{m}(g_o, \varphi_i)\varphi_i\right).$$

Now for $z \in G$,

$$\left|T_G\left[\sum_{i=n}^{m}(g_o, \varphi_i)\varphi_i\right](z)\right|^2 =$$

$$\left|\int k(zy^{-1})\left[\sum_{i=n}^{m}(g_o, \varphi_i)\varphi_i\right](y)dy\right|^2 \leq \|k\|_2^2 \cdot \left\|\sum_{i=n}^{m}(g_o, \varphi_i)\varphi_i\right\|_2^2 = \|k\|_2^2 \sum_{i=n}^{m}|(g_o, \varphi_i)|^2. \quad \sum_{i=n}^{m}|(g_o, \varphi_i)|^2$$

goes to zero with n,m by Bessel's inequality. It follows that $\sum_{i=1}^{\infty}(T_G g_o, \varphi_i)\varphi_i$ converges uniformly in $C(G)$, and, hence, converges uniformly to $T_G g_o$. By the remark following Lemma 3.2.1, $\sum_{i=1}^{\infty}(T_G g_o, \varphi_i)\varphi_i^*$ converges uniformly on S. Since $T_S f_o = T_G g_o$ on G, and $\sum_{i=1}^{\infty}(T_G g_o, \varphi_i)\varphi_i^*$ converges uniformly to $T_G g_o$ on G, it follows that $\sum_{i=1}^{\infty}(T_G g_o, \varphi_i)\varphi_i^*$

converges uniformly to $T_S f_o$ on S.

Since $T_S f_o(x) \neq T_S(f_o(y))$, it follows that for some $i \geq 1$, $\varphi_i^*(x) \neq \varphi_i^*(y)$. Now for $z \in G$, $T_S \varphi_i^*(z) = T_G \varphi_i(z) = \lambda_i \varphi_i(z) = \lambda_i \varphi_i^*(z)$. Since $T_S \varphi_i^*$ and $\lambda_i \varphi_i^*$ are elements of A(S), and, since they agree on G, we have:

(6) $T_S \varphi_i^* = \lambda_i \varphi_i^*$ on S.

Since $M_i = \{f \in H : T_G f = \lambda_i f\}$ is finite-dimensional, there exists an orthonormal set $f_1, \ldots, f_n \in M_i$ which span M_i. We define $M_i^* = \{f \in A(S) : T_S f = \lambda_i f\}$; then M_i^* is finite-dimensional. In fact if $f_i = f_i^*|_G$ where $f_i^* \in A(S)$, and $f \in M_i^*$, then $f|_G = g \in M_i$ and $g = \sum_{i=1}^{n} a_i f_i$. It follows that $f = \sum_{i=1}^{n} a_i f_i^*$ by previous arguments.

In view of (6), $\varphi_i^* \in M_i^*$. Denote by B_i the bounded linear operators on M_i^*. For $\sigma \in S$, $f \in M_i^*$, define $P(\sigma)f = f^\sigma$. Then $T_S f^\sigma(z) = \int k(y^{-1}) f^\sigma(yz)dy = \int k(y^{-1}) f(yz\sigma)dy = \lambda_i f(z\sigma) = \lambda_i f^\sigma(z)$ so that $P(\sigma)f \in M_i^*$. Further, if $\sigma, \tau \in S$, then $P(\sigma\tau)f(z) = f^{\sigma\tau}(z) = f(z\sigma\tau) = f^\tau(z\sigma) = P(\tau)f(z\sigma) = P(\sigma)[P(\tau)f](z)$; hence $\sigma \longrightarrow P(\sigma)$ is a homomorphism. Also, if $0 \leq \lambda \leq 1$ $P(\lambda\sigma + (1-\lambda)\tau)(f)(z) = f(z[\lambda\sigma + (1-\lambda)\tau]) = f(\lambda(z\sigma) + (1-\lambda)(z\tau)) = \lambda f(z\sigma) + (1-\lambda)f(z\tau) = [\lambda P(\sigma)f + (1-\lambda)P(\tau)f](z) = [\lambda P(\sigma) + (1-\lambda)P(\tau)](f)(z)$. Thus, $\sigma \longrightarrow P(\sigma)$ is affine. As noted previously, $\sigma \longrightarrow f^\sigma$ is continuous for fixed $f \in C(S)$, so that $\sigma \longrightarrow P(\sigma)$ is SOT continuous. Since M_i^* is finite-dimensional, $\sigma \longrightarrow P(\sigma)$ is continuous in any locally convex topology on B_i. The map $\sigma \longrightarrow P(\sigma)$ is, therefore, a representation of S.

Further $P(x)\varphi_i^*(u) = \varphi_i^*(x) \neq \varphi_i^*(y) = P(y)\varphi_i^*(u)$ so that $P(x)\varphi_i^* \neq P(y)\varphi_i^*$ and, hence, $P(x) \neq P(y)$. If we introduce the following bilinear form on M_i^*:

(7) $(f,g) = \int f(x)\overline{g(x)}dx$, then (f,g) is an inner product on M_i^*. In fact, if $(f,f) = 0$, then $f \equiv 0$ on G and, since $f \in A(S)$, $f \equiv 0$ on S.

For $z \in G$, and $f,g \in M_i^*$, we have $(P(z)f,g) = \int f^z(x)\overline{g(x)}dx = \int f(xz)\overline{g(x)}dx = \int f(x)\overline{g(xz^{-1})}dx = \int f(x)\overline{g^{z^{-1}}(x)}dx = (f,P(z^{-1})g)$. Hence, $P^*(z) = P(z^{-1}) = P^{-1}(z) \in Gl_n$. Further, if $z = \sum_{i=1}^{n} \lambda_i z_i \in S$, where $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \geq 0$, and $z_i \in G$, then $[P(z)]^* = [\sum_{i=1}^{n} \lambda_i P(z_i)]^* = \sum_{i=1}^{n} \lambda_i P^*(z_i) = \sum_{i=1}^{n} \lambda_i P(z_i^{-1}) = P(\sum_{i=1}^{n} \lambda_i z_i^{-1}) \in P(S)$. Then by continuity, $P^*(z) \in P(S)$ for all $z \in S$. This establishes Theorem 3.2.

Corollary 3.2.1 Let S be a compact group-extremal affine semigroup with group G. If G is metrizable, then S is metrizable.

Proof: Let $\{U_i\}_{i=1}^{\infty}$ be a neighborhood basis at the identity u, where U_i is open in G, $U_i = U_i^{-1}$. To each U_i, we associate a function $k_i \in C(G)$ as in Theorem 3.2 which satisfies $0 \leq k_i \leq 1$, $k_i(u) = 1$, $k_i(z) = k_i(z^{-1})$, and $k_i \equiv 0$ outside U_i. Each k_i gives rise to a countable number of representations of S. Since for $x \neq y$, it is only necessary to find a neighborhood U satisfying $\langle U \rangle x \cap \langle U \rangle y = \emptyset$, we can take $U = U_i$ for some i and, thus, a representation arising from k_i separates x from y.

We therefore have a countable number of representations by metrizable spaces which separate points of S. Then S is imbedded in a countable number of metric spaces and the conclusion follows.

Definition: Let S be an affine topological semigroup. An affine semicharacter on S is a continuous, affine homomorphism from S to the complex unit disc.

Theorem 3.3 Let S be a compact, abelian, group-extremal semigroup. Then for $x \neq y$, $x,y \in S$, there exists an affine semicharacter p such that $p(x) \neq p(y)$.

Proof: By Theorem 3.2, there is a representation P of S by elements of B(M), where M is a finite-dimensional linear space over the complexes and B(M) is the linear space of bounded linear operators on M, and which satisfies $P(x) \neq P(y)$ and $P^*(\sigma) \in P(S)$ for all $\sigma \in S$.

Let M_1 be a subspace of M minimal with respect to invariance under all $P(\sigma)$ for $\sigma \in S$, and $M_1 \neq \{0\}$ and

$\Delta = \{\alpha \in B(M_1) : \alpha P(\sigma) = P(\sigma)\alpha$ for $\sigma \in S\}$. Δ is clearly an algebra of finite dimension over the complexes. For $\alpha \in \Delta$, let $R(\alpha) = \{x \in M_1 : \alpha(x) = 0\}$; then $R(\alpha)$ is a subspace of M_1 and is invariant under all $P(\sigma)$ for $\sigma \in S$. Since M_1 is minimal, we must have $R(\alpha) = \{0\}$ or $R(\alpha) = M_1$. If $R(\alpha) = \{0\}$, then α is invertible; if $R(\alpha) = M_1$, then $\alpha \equiv 0$. Thus, Δ is a division algebra over the complexes, and, since Δ is finite-dimensional, it is complete. By [16] and [27], Δ is one-dimensional over the complexes; this means that for $\alpha \in \Delta$, there exists a complex number λ where $\alpha(x) = \lambda x$ for $x \in M_1$.

Now for each $\sigma \in S$, $P(\sigma) \in \Delta$ since S is abelian. Thus, there exists a complex number $p_1(\sigma)$ for which $P(\sigma)x = p_1(\sigma)x$ for $x \in M_1$. Let $e_1 \neq 0$ be an element of M_1. Then $\{\lambda e_1\}$, the space spanned by e_1, is invariant under all $P(\sigma)$; it follows that $M_1 = \{\lambda e_1\}$.

Note that $P(\sigma\tau)e_1 = p_1(\sigma\tau)e_1$, and $P(\sigma)(P(\tau)e_1) = P(\sigma)(p_1(\tau)e_1) = p_1(\sigma)p_1(\tau)e_1$. But $P(\sigma\tau) = P(\sigma)P(\tau)$, so that $p_1(\sigma\tau) = p_1(\sigma)p_1(\tau)$. Similarly, p_1 is an affine map, since the same is true for P. Further,

$$|p_1(\sigma) - p_1(\tau)| = |p_1(\sigma) - p_1(\tau)| \frac{\|e_1\|}{\|e_1\|} = \frac{\|[p_1(\sigma) - p_1(\tau)]e_1\|}{\|e_1\|} = \frac{\|[P(\sigma) - P(\tau)]e_1\|}{\|e_1\|}.$$

This shows that p_1 is continuous and, hence, an affine semi-character.

Now, suppose we have constructed an orthonormal set $e_1, \ldots, e_k$ along with affine semicharacters $p_1, \ldots, p_k$ which satisfy $P(\sigma)e_k = p_k(\sigma)e_k$. Let Q be the subspace of M spanned by $e_1, \ldots, e_k$. Suppose $Q \neq M$, then $Q^{\perp}$, the orthogonal complement of Q, is different from 0. If $\sigma \in S$, there exists $\tau \in S$ for which $P^*(\sigma) = P(\tau)$. Then if $x \in Q^{\perp}$, and $y = \sum_{i=1}^{k} a_i e_i$, then $(P(\sigma)x,y) = (x,P^*(\sigma)y) = (x,P(\tau)y) = (x,P(\tau)[\sum_{i=1}^{k} a_i e_i]) = \sum_{i=1}^{k} \bar{a}_i (x,P(\tau)e_i) = \sum_{i=1}^{k} \bar{a}_i (x,p_i(\tau)e_i) = \sum_{i=1}^{k} \bar{a}_i \overline{p_i(\tau)}(x,e_i) = 0$. Thus, $P(\sigma)x \in Q^{\perp}$, and $Q^{\perp}$ is an invariant subspace of the representation. Replacing M by Q in the previous argument, we obtain $e_{k+1} \in Q^{\perp}$, and an affine semicharacter p_{k+1} which satisfies $P(\sigma)e_{k+1} = p_{k+1}(\sigma)e_{k+1}$ for $\sigma \in S$.

Repeating this argument, we finally obtain an orthonormal basis $e_1, e_2, \ldots, e_n$ for M, and affine semicharacters $p_1, \ldots, p_n$, for which $P(\sigma)e_i = p_i(\sigma)e_i$ for $1 \leq i \leq n$ and $\sigma \in S$. Now $P(x) \neq P(y)$; thus for some i, $P(x)e_i \neq P(y)e_i$ and, consequently, $p_i(x) \neq p_i(y)$. This is the desired separating affine semicharacter.

One might approach Theorem 3.3 by attempting to extend each character on the group to an affine semicharacter. We give two examples: the first is an example of an abelian group-extremal semigroup in which every character may be extended to be an affine semicharacter; the second shows that, in general, not every character can be extended.

Example 1: Let G be an arbitrary compact abelian group, and $S = \tilde{G}$. Clearly, S is abelian and group-extremal. Now, for each continuous character γ on G, define $F_\gamma(\mu) = \int \gamma \, d\mu$ where $\mu \in S$. Clearly, F_γ is a continuous, affine function. Further $F_\gamma(\mu * \nu) = \int \gamma \, d\mu * \nu =$ $\iint \gamma(xy) d\mu(x) d\nu(y) = \iint \gamma(x)\gamma(y) d\mu(x) d\nu(y) =$ $F_\gamma(\mu) \cdot F_\gamma(\nu)$. Therefore, F_γ is an affine semicharacter. Further, if $x \in G$, then $F_\gamma(\delta_x) = \int \gamma \, d\delta_x = \gamma(x)$, so that $F_\gamma = \gamma$ on G.

Example 2: Let S be the complex unit disc, S^1 the circle group. Let p be an affine semicharacter $p \not\equiv 1$, $p \not\equiv 0$; then clearly $p(0) = 0$. If $|x| < 1$, then $x^n \longrightarrow 0$, so that $[p(x)]^n = p(x^n) \longrightarrow 0$; thus, $|p(x)| < 1$. It follows that $p^{-1}(1)$ is subset of S^1. However, $p^{-1}(1)$ is convex, so that $p^{-1}(1) = \{1\}$. Now on S^1, p is a character; there exists an integer n for which $p(z) = z^n$ for all $z \in S^1$. But then $p^{-1}(1)$ contains the n-th roots of unit, so that $n = 1$, $p(z) = z$ for $z \in S^1$ and, hence, for $z \in S$.

Therefore, the only affine semicharacters on S are $p \equiv 0$, $p \equiv 1$, and $p(z) = z$. This example can be justified as well by noting that Schwarz [41] has computed all semicharacters of the disc; they are:

(a) $\chi(z) \equiv 0$ (b) $\chi(z) \equiv 1$

(c) $\chi(z) = \begin{cases} 0 \text{ for } z = 0 \\ |z|^{\alpha + i\beta} \cdot z^n \text{ for } z \neq 0 \text{ (} n \text{ an integer, } \beta \text{ real, } n + \alpha > 0) \end{cases}$

By D_∞, we mean the countable product of discs under coordinate-wise multiplication. D_∞ is an abelian group-extremal affine semigroup.

Corollary 3.3.1 An abelian, metrizable, compact, group-extremal affine semigroup S is equivalent to a subsemigroup of D_∞.

Proof: In the proof of Corollary 3.2.1, it was shown that S has a countable number of representations which separate points of S. Each representation gives rise to a finite number of affine semicharacters; consequently, a countable number of affine semicharacters, say p_1, p_2, ..., separate points. If we define $F:S \longrightarrow D_\infty$ by

(1) $[F(x)]_i = p_i(x)$, then F is clearly an equivalence between S and a subsemigroup of D_∞.

Theorem 3.4 A compact, group-extremal affine semigroup S is equivalent to the inverse limit of compact, finite-dimensional group-extremal semigroups.

Proof: Let A be a finite collection of representations of S, say $A = \{P_1, \ldots, P_n\}$, where P_i is a representation of S in the finite-dimensional space M_i. Thus, $P_i(s) \in B(M_i)$ for all $s \in S$, and $P_i(S)$ is a compact, group-extremal affine semigroup.

Define $f_A : S \longrightarrow B(M_1) \oplus \ldots \oplus B(M_n)$ by:

(1) $f_A(\sigma) = (P_1(\sigma), \ldots, P_n(\sigma))$. Clearly, f_A is a continuous, affine homomorphism. We define:

(2) $Q_A = f_A(S)$; then Q_A is a compact, group-extremal, finite-dimensional affine semigroup.

Let λ be the collection of all finite sets of representations of S, and partial order λ by containment. λ is then a directed set in this partial order. If $A, B \in \lambda$, $A \subseteq B$, define $\varphi^B_A : Q_B \longrightarrow Q_A$ as follows:

(3) Let $x_o = f_B(s_o) \in Q_B$; define $\varphi^B_A(x_o) = f_A(s_o)$. φ^B_A merely consists of the function which projects from Q_B to Q_A by deleting the coordinates in $B \setminus A$. In view of this, and the fact that f_B is a continuous affine homomorphism for all $B \in \lambda$, φ^B_A is a continuous, affine homomorphism onto Q_A. Clearly, if $C \geq B \geq A$ then $\varphi^B_A \varphi^C_B = \varphi^C_A$. Thus, $\{Q_A, \varphi^B_A, \lambda\}$ is an inverse system, and, therefore, we set

(4) $Q = \varprojlim \{Q_A, \varphi^B_A, \lambda\}$. We wish to show S is equivalent to Q. To do this, we define a function F on S to $\prod_{A \in \lambda} Q_A$ by

(5) $[F(s)]_A = f_A(s)$ for $A \in \lambda$, and for $s \in S$. Note that $[F(s)]_A \in Q_A$; if $B \geq A$, then $\varphi^B_A([F(s)]_B) = \varphi^B_A(f_B(s)) = f_A(s) = [F(s)]_A$. Thus, $F(s) \in Q$; F is clearly a continuous, affine homomorphism of S into Q. If $x \neq y$, $x, y \in S$, there exists a representation P such that $P(x) \neq P(y)$. Let

$A = \{P\}$; then $f_A(x) = P(x) \neq P(y) = f_A(y)$. Therefore, $[F(x)]_A = f_A(x) \neq f_A(y) = [F(y)]_A$, so that $F(x) \neq F(y)$; this shows that F is one-to-one.

We wish to show F is onto. Let $z \in Q$; for each $A \in \mathcal{A}$, $z_A \in Q_A$, so that there exists $x_A \in S$ such that $z_A = f_A(x_A)$. Define $H(A) = \{x \in S: f_A(x) = f_A(x_A)\}$; $H(A)$ is a compact subset of S for each $A \in \mathcal{A}$. For $A, B \in \mathcal{A}$, let $C = A \cup B$; $C \in \mathcal{A}$, and $C \geq A$ and $C \geq B$. Thus, $\varphi_A^C(z_C) = z_A$ and $\varphi_B^C(z_C) = z_B$, since $z \in Q$. If $x \in H(C)$, then $f_C(x) = f_C(x_C) = z_C$. Then $z_A = \varphi_A^C(z_C) = \varphi_A^C(f_C(x)) = f_A(x)$, and, similarly, $z_B = f_B(x)$. Hence, $f_A(x) = z_A = f_A(x_A)$, and $f_B(x) = z_B = f_B(x_B)$, so that $x \in H(A) \cap H(B)$. This shows that $H(C) \subset H(A) \cap H(B)$, and that $\{H(A)\}_{A \in \mathcal{A}}$ is a directed family of compact subsets of S. There exists an $x \in S$, $x \in \bigcap_{A \in \mathcal{A}} H(A)$, since S is compact. Then $f_A(x) = f_A(x_A) = z_A$ for all $A \in \mathcal{A}$. It follows easily that $F(x) = z$, so that F is onto. Thus, S is equivalent to Q, and the proof is complete.

Remark: In view of the close similarity between compact affine semigroups and measure semigroups, we propose as a conjecture that every compact affine semigroup is equivalent to a semigroup of measures. It would also be interesting to see what uses can be made of Theorem 3.3 in the analysis of compact semigroups.

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BIOGRAPHY

Michael Friedberg was born on August 7, 1939 in New York, New York. He attended public schools in Brooklyn, New York and Miami, Florida, and was graduated from Miami Senior High School in June, 1957. After graduation, he entered the University of Miami in September, 1957, where he received the Bachelor of Science degree in January, 1961.

He attended the Graduate School of the University of Miami from February, 1961 to August, 1961. In September, 1961, he married Lila B. Glossman of Miami, Florida. From September, 1961 to June, 1964 he held a National Defense Education Act fellowship at Louisiana State University. From July, 1964 to the present he has held a Graduate Assistantship at Louisiana State University where he is currently a candidate for the degree of Doctor of Philosophy in the Department of Mathematics.

EXAMINATION AND THESIS REPORT

Candidate: Michael Friedberg

Major Field: Mathematics

Title of Thesis: Measures and Affine Semigroups

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

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