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# Further Remarks on Strict Input-to-State Stable Lyapunov Functions for Time-Varying Systems <sup>★</sup>

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## Abstract

We study the stability properties of a class of time-varying nonlinear systems. We assume that non-strict input-to-state stable (ISS) Lyapunov functions for our systems are given and posit a mild persistency of excitation condition on our given Lyapunov functions which guarantee the existence of strict ISS Lyapunov functions for our systems. Next, we provide simple direct constructions of explicit strict ISS Lyapunov functions for our systems by applying an integral smoothing method. We illustrate our constructions using a tracking problem for a rotating rigid body.

*Key words:* Lyapunov functions, input-to-state stabilization, nonautonomous systems.

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## 1 Introduction

The theory of input-to-state stable (ISS) systems plays a central role in modern non-linear control analysis and controller design (see (Malisoff *et al.*, 2004; Malisoff & Sontag, 2004; Sontag, 1998, 2001; Sontag & Wang, 1995)). The ISS property was introduced by Sontag in (Sontag, 1989) and an ISS Lyapunov characterization was obtained by Sontag and Wang in (Sontag & Wang, 1995). The ISS Lyapunov characterization provides necessary and sufficient conditions for time-invariant systems to be ISS, in terms of the existence of so-called strict ISS Lyapunov functions; see Section 2 below for the relevant definitions and (Edwards *et al.*, 2000) for an extension to time-varying

systems. Strict Lyapunov functions have been used to design stabilizing feedback laws that render asymptotically controllable systems ISS to actuator errors and small observation noise; see (Malisoff & Sontag, 2004; Sontag, 2001). Such control laws are expressed in terms of gradients of Lyapunov functions and therefore require explicit strict Lyapunov functions in order to be implemented. This has motivated a great deal of research devoted to constructing explicit strict Lyapunov functions.

One obstacle to these constructions is that the known strict Lyapunov functions from the existence theory are optimal control value functions, involving a supremum of a cost criterion over infinitely many possible solution paths (see (Bacciotti & Rosier, 2001; Edwards *et al.*, 2000; Sontag & Wang, 1995; Teel & Praly, 2000)), and therefore are not explicit. Although value functions can often be expressed as unique solutions of Hamilton-Jacobi (HJ) equations subject to appropriate side conditions, the usual techniques for computing value functions in terms of HJ equation solutions can be difficult to implement. For certain special kinds of systems, strict ISS Lyapunov functions can be explicitly constructed by ad hoc means. On the other hand, there are numerous

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important cases where it is relatively straightforward to use backstepping or other known methods to construct explicit *non*-strict ISS Lyapunov functions (see our definitions of ISS and non-strict ISS Lyapunov functions in Section 2 and Section 4 for an explicit example). For instance, applying the methods of (Jiang & Nijmeijer, 1997) to tracking problems for nonholonomic systems in chained form results in non-strict Lyapunov functions. The constructions in (Mazenc & Praly, 2000) also frequently give rise to non-strict Lyapunov functions.

This motivates the search for techniques for constructing strict ISS Lyapunov functions for time-varying systems, in terms of known non-strict ISS Lyapunov functions. This search is the focus of this note. For time-varying systems with no controls, the paper (Mazenc, 2003) constructed strict globally asymptotically stable (GAS) Lyapunov functions in terms of given non-strict GAS Lyapunov functions. Here we further develop the approach in (Mazenc, 2003). We provide the necessary background on ISS systems and Lyapunov functions in Section 2. We then introduce a non-strict generalization of ISS in which the dissipation rate depends on a non-negative time-dependent decay parameter. The parameter can be zero along intervals of positive length. However, when the parameter is identically one, our non-strict ISS property agrees with the usual ISS condition. Under a mild non-degeneracy assumption on this parameter, which is of persistency of excitation type (see for instance (Loria *et al.*, 2002) and (Loria & Panteley, 2002) for definitions and discussions of the concept of persistency of excitation), we show that our non-strict ISS property is equivalent to the existence of a strict ISS Lyapunov function and is therefore also equivalent to the standard ISS condition. We prove these equivalences in Section 3. They are proved by explicitly constructing strict ISS Lyapunov functions. In Section 4, we illustrate our constructions using a tracking example. Concluding remarks in Section 5 end the paper.

## 2 Preliminaries

Let  $\mathcal{K}_\infty$  denote the set of all continuous functions  $\rho : [0, \infty) \rightarrow [0, \infty)$  for which (i)  $\rho(0) = 0$  and (ii)  $\rho$  is increasing and unbounded. Let  $\mathcal{KL}$  denote the set of all continuous functions  $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  for which (1) for each  $t \geq 0$ ,  $\beta(\cdot, t)$  is strictly increasing and  $\beta(0, t) = 0$  (2)  $\beta(s, \cdot)$  is non-increasing for each  $s \geq 0$ , and (3)  $\beta(s, t) \rightarrow 0$  as  $t \rightarrow +\infty$  for each  $s \geq 0$ .

We study the stability properties of the fully nonlinear nonautonomous system

$$\dot{x} = f(t, x, u), \quad t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m \quad (1)$$

where we always assume  $f$  is locally Lipschitz in  $(t, x, u)$ . Following (Mazenc, 2003), we also assume  $f$  is periodic in  $t$ , which means there exists a constant  $T > 0$  such

that  $f(t + T, x, u) = f(t, x, u)$  for all  $t \geq 0$ ,  $x \in \mathbb{R}^n$ , and  $u \in \mathbb{R}^m$ . However, most of our arguments remain valid if this periodicity assumption is weakened to requiring  $f$  to be uniformly locally bounded in  $t$ , meaning,

$$\sup\{|f(t, x, u)| : (x, u) \in K, t \geq 0\} < +\infty \quad (2)$$

where  $|\cdot|$  is the usual Euclidean norm. The control functions for our system (1) comprise the set of all measurable locally essentially bounded functions  $\alpha : [0, \infty) \rightarrow \mathbb{R}^m$ ; we denote this set by  $\mathcal{U}$ . We let  $|\alpha|_I$  denote the essential supremum of any control  $\alpha \in \mathcal{U}$  restricted to any interval  $I \subseteq [0, \infty)$ . For each  $t_o \geq 0$ ,  $x_o \in \mathbb{R}^n$ , and  $\alpha \in \mathcal{U}$ , we let  $I \ni t \mapsto \phi(t; x_o, t_o, \alpha)$  denote the unique trajectory of (1) for the input  $\alpha$  satisfying  $x(t_o) = x_o$  and defined on its maximal interval  $I \subseteq [t_o, \infty)$ . This trajectory will be denoted by  $\phi$  when this would not lead to confusion. We say that  $f$  is *forward complete* provided each such trajectory  $\phi$  is defined on all of  $[t_o, \infty)$ .

A  $C^1$  function  $V : [0, \infty) \times \mathbb{R}^n \rightarrow [0, \infty)$  is said to be of class UPPD (written  $V \in \text{UPPD}$ ) provided it is uniformly proper and positive definite, which means there exist  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$  such that, for all  $t \geq 0$ ,  $x \in \mathbb{R}^n$ ,

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|), \quad |\nabla V(t, x)| \leq \alpha_3(|x|). \quad (3)$$

We say that  $V$  has period  $\tau$  in  $t$  provided there exists a constant  $\tau > 0$  such that  $V(t + \tau, x) = V(t, x)$  for all  $t \geq 0$  and  $x \in \mathbb{R}^n$ ; in this case, the bound on  $\nabla V$  in (3) is redundant. We assume  $\alpha_1$  and  $\alpha_2$  in (3) are  $C^1$ , e.g., by taking  $\alpha_2(s) = \int_0^s \alpha_3(r) dr$  and minorizing  $\alpha_1$  by a  $C^1$  function of class  $\mathcal{K}_\infty$ . Given  $V \in \text{UPPD}$ , we set

$$\dot{V}(t, x, u) := \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)f(t, x, u).$$

Notice that  $s \mapsto \sup\{|\dot{V}(t, x, u)| : t \geq 0, |x| \leq \chi(s), |u| \leq s\} + s$  is of class  $\mathcal{K}_\infty$  for each  $\chi \in \mathcal{K}_\infty$  (by (2)-(3)). We let  $\mathcal{P}$  denote the set of all continuous functions  $p : \mathbb{R} \rightarrow [0, \infty)$  that admit constants  $\tau, \varepsilon, \bar{p} > 0$  for which

$$\int_{t-\tau}^t p(s) ds \geq \varepsilon \quad \text{and} \quad p(t) \leq \bar{p}, \quad \forall t \geq 0. \quad (4)$$

We write  $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$  to indicate that (i)  $p \in \mathcal{P}$  and (ii)  $\tau, \varepsilon, \bar{p} > 0$  are constants such that (4) holds. In particular, any continuous periodic function  $p : \mathbb{R} \rightarrow [0, \infty)$  that is not identically zero admits constants  $\tau, \varepsilon, \bar{p} > 0$  satisfying (4). On the other hand, (4) also allows non-periodic  $p$  with arbitrarily large null sets, e.g., for fixed  $r > 0$ , set  $p_r(t) = (1 + e^{-t}) \max\{0, \sin^3(\frac{t}{r})\}$ . The elements of  $\mathcal{P}$  serve as the decay rates for our non-strict Lyapunov functions as follows:

**Definition 1** *Let  $p \in \mathcal{P}$ . A function  $V \in \text{UPPD}$  is called an ISS(p) Lyapunov function for (1), provided there exist  $\chi \in \mathcal{K}_\infty$  and  $\mu \in \mathcal{K}_\infty \cap C^1$  such that*

$$|x| \geq \chi(|u|) \Rightarrow \dot{V}(t, x, u) \leq -p(t)\mu(|x|) \quad \forall t \geq 0. \quad (5)$$

An ISS( $p$ ) Lyapunov function for (1) and  $p(t) \equiv 1$  is also called a strict ISS Lyapunov function.

Notice that (5) allows  $\dot{V}(t, x, u) = 0$  for those  $t$  where  $p(t) = 0$ . This corresponds to allowing  $V$  to non-strictly decrease along the solutions  $\phi$  of (1).

**Definition 2** Let  $p \in \mathcal{P}$ . We say that (1) is ISS( $p$ ), or that it is input-to-state stable (ISS) with decay rate  $p$ , provided there exist  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}_\infty$  such that for all  $t_o \geq 0$ ,  $x_o \in \mathbb{R}^n$ ,  $u_o \in \mathcal{U}$  and  $h \geq 0$ ,

$$|\phi(t_o + h; x_o, t_o, u_o)| \leq \beta \left( |x_o|, \int_{t_o}^{t_o+h} p(s) ds \right) + \gamma(|u_o|_{[t_o, t_o+h]}). \quad (6)$$

If (1) is ISS( $p$ ) with  $p \equiv 1$ , then we say that (1) is ISS.

Notice that ISS( $p$ ) systems are automatically forward complete. We also study dissipation-type decay conditions as follows:

**Definition 3** Let  $p \in \mathcal{P}$ . A function  $V \in \text{UPPD}$  is called a non-strict dissipative Lyapunov function for (1) and  $p$ , or a DIS( $p$ ) Lyapunov function, provided there exist  $\Omega \in \mathcal{K}_\infty$  and  $\mu \in \mathcal{K}_\infty \cap C^1$  such that, for all  $t \geq 0$ ,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$

$$\dot{V}(t, x, u) \leq -p(t)\mu(|x|) + \Omega(|u|). \quad (7)$$

A DIS( $p$ ) Lyapunov function for (1) and  $p(t) \equiv 1$  is also called a strict DIS Lyapunov function.

**Remark 4** Definition 3 is a nonlinear version of the property used in (Loria & Panteley, 2002) to ensure the global uniform exponential stability of time-varying linear systems belonging to a specific family of systems. Thus, the explicit construction of a strict DIS Lyapunov function in terms of a given DIS( $p$ ) Lyapunov function we present in the next section, extends (Loria & Panteley, 2002) where only linear systems are studied and no strict Lyapunov function is constructed.

We use the following elementary observations:

**Lemma 5** Let  $\tau, \varepsilon, \bar{p} > 0$  be constants and  $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$  be given. Then:

- (i)  $0 \leq \int_{t-\tau}^t \left( \int_s^t p(r) dr \right) ds \leq \frac{\tau^2 \bar{p}}{2}$  for all  $t \geq 0$  and
- (ii)  $[0, \infty) \ni h \mapsto \underline{p}(h) = \inf \left\{ \int_t^{t+h} p(r) dr : t \geq 0 \right\}$  is continuous, non-decreasing, and unbounded.

We leave the proof of this lemma to the reader as a simple exercise.

### 3 Equivalent Characterizations of Non-Strict ISS

We next relate the Lyapunov functions and stability notions we introduced in the last section. We show that ISS( $p$ ) is equivalent to the existence of an ISS( $p$ ) Lyapunov function and the existence of a strict ISS Lyapunov function. Our proof explicitly constructs a strict ISS Lyapunov function for (1) in terms of a given DIS( $p$ ) Lyapunov function. Moreover, if  $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$  and our given DIS( $p$ ) Lyapunov function both have period  $\tau$ , then the strict ISS Lyapunov function we construct also has period  $\tau$ . We next prove:

**Theorem 6** Let  $p \in \mathcal{P}$  and  $f$  be as above. The following are equivalent:

- (C<sub>1</sub>)  $f$  admits an ISS( $p$ ) Lyapunov function.
- (C<sub>2</sub>)  $f$  admits a strict ISS Lyapunov function.
- (C<sub>3</sub>)  $f$  admits a DIS( $p$ ) Lyapunov function.
- (C<sub>4</sub>)  $f$  admits a strict DIS Lyapunov function.
- (C<sub>5</sub>)  $f$  is ISS( $p$ ).
- (C<sub>6</sub>)  $f$  is ISS.

We prove the following implications: (C<sub>1</sub>)  $\Rightarrow$  (C<sub>2</sub>)  $\Rightarrow$  (C<sub>4</sub>)  $\Rightarrow$  (C<sub>1</sub>), (C<sub>3</sub>)  $\Leftrightarrow$  (C<sub>4</sub>), (C<sub>2</sub>)  $\Leftrightarrow$  (C<sub>6</sub>), and (C<sub>5</sub>)  $\Leftrightarrow$  (C<sub>6</sub>). We fix  $\tau, \varepsilon, \bar{p} > 0$  such that  $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$ .

**Step 1:** (C<sub>1</sub>)  $\Rightarrow$  (C<sub>2</sub>). If (C<sub>1</sub>) holds, then we can find an ISS( $p$ ) Lyapunov function  $V$  for  $f$ , and therefore  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty \cap C^1$  satisfying (3) and  $\chi \in \mathcal{K}_\infty$  and  $\mu \in \mathcal{K}_\infty \cap C^1$  satisfying (5). Set

$$\begin{aligned} \tilde{\alpha}_2(s) &:= \max \left\{ \frac{\tau \bar{p}}{2}, 1 \right\} (\alpha_2(s) + \mu(s) + s), \\ w(s) &:= \frac{1}{4\tau} \mu(\tilde{\alpha}_2^{-1}(s)). \end{aligned} \quad (8)$$

Then  $\tilde{\alpha}_2, \tilde{\alpha}_2^{-1} \in \mathcal{K}_\infty \cap C^1$ . Since  $V(t, x) \leq \tilde{\alpha}_2(|x|)$  for all  $t \geq 0$  and  $x \in \mathbb{R}^n$ , the following holds for all  $t \geq 0$ :

$$|x| \geq \chi(|u|) \Rightarrow \dot{V}(t, x, u) \leq -p(t)\mu(\tilde{\alpha}_2^{-1}(V(t, x))). \quad (9)$$

Note too that  $w \in \mathcal{K}_\infty \cap C^1$ . We later use the fact that

$$\begin{aligned} 0 \leq w'(s) &\leq \frac{\mu'(\tilde{\alpha}_2^{-1}(s))}{4\tau \max \left\{ \frac{\tau \bar{p}}{2}, 1 \right\} (\mu'(\tilde{\alpha}_2^{-1}(s)) + 1)} \\ &\leq \frac{1}{2\tau^2 \bar{p}} \end{aligned} \quad (10)$$

for all  $s \geq 0$ . Consider the UPPD function

$$V^\sharp(t, x) = V(t, x) + \xi(t)w(V(t, x)) \quad (11)$$

with  $\xi(t) = \int_{t-\tau}^t \left( \int_s^t p(r) dr \right) ds$ . Then

$$\begin{aligned} \dot{V}^\sharp(t, x, u) &= [1 + \xi(t)w'(V(t, x))] \dot{V}(t, x, u) \\ &\quad + \left[ \tau p(t) - \int_{t-\tau}^t p(r) dr \right] w(V(t, x)) \end{aligned}$$

follows from a simple calculation. When  $|x| \geq \chi(|u|)$ , condition (9) gives  $\dot{v}(t, x, u) \leq 0$  and therefore also

$$\begin{aligned} \dot{V}^\#(t, x, u) &\leq -p(t)\mu(\tilde{\alpha}_2^{-1}(V(t, x))) \\ &\quad + \left[ \tau p(t) - \int_{t-\tau}^t p(r) dr \right] \frac{1}{4\tau}\mu(\tilde{\alpha}_2^{-1}(V(t, x))) \\ &\leq -\frac{3}{4}p(t)\mu(\tilde{\alpha}_2^{-1}(V(t, x))) \\ &\quad - \left( \int_{t-\tau}^t p(r) dr \right) \frac{1}{4\tau}\mu(\tilde{\alpha}_2^{-1}(V(t, x))) \\ &\leq -\frac{\varepsilon}{4\tau}\mu(\tilde{\alpha}_2^{-1}(\alpha_1(|x|))) \quad \forall t \geq 0. \end{aligned}$$

Since  $\mu \circ \tilde{\alpha}_2^{-1} \circ \alpha_1 \in C^1 \cap \mathcal{K}_\infty$ , it follows that  $V^\#$  is a strict ISS Lyapunov function for (1).

**Step 2:**  $(C_2) \Rightarrow (C_4)$ . Assume  $(C_2)$ , so  $f$  admits a strict ISS Lyapunov function  $V$ . Let  $\mu$  and  $\chi$  satisfy condition (5) with  $p \equiv 1$ . Then the strict dissipative condition (7) with  $p \equiv 1$  follows by choosing any  $\Omega \in \mathcal{K}_\infty$  satisfying

$$\Omega(s) \geq \max_{\{t \geq 0, |x| \leq \chi(s), |u| \leq s\}} \{\dot{V}(t, x, u) + \mu(|x|)\} \quad \forall s \geq 0.$$

Such an  $\Omega$  exists by our assumptions (2)-(3). Therefore,  $V$  is itself a strict DIS Lyapunov function for  $f$ .

**Step 3:**  $(C_4) \Rightarrow (C_1)$ . Assume  $(C_4)$ , so  $f$  admits a strict DIS Lyapunov function  $V$ . Let  $\mu, \Omega \in \mathcal{K}_\infty$  satisfy (7) with  $p \equiv 1$ ; then if  $|x| \geq \chi(|u|) := \mu^{-1}(2\Omega(|u|))$ , then

$$\dot{V}(t, x, u) \leq -\frac{1}{2}\mu(|x|), \quad \text{so} \quad \dot{V}(t, x, u) \leq -\frac{p(t)}{2\bar{p}}\mu(|x|)$$

for all  $t \geq 0$ . Therefore,  $V$  is also an ISS(p) Lyapunov function for  $f$ , so  $(C_1)$  is satisfied.

**Step 4:**  $(C_3) \Leftrightarrow (C_4)$ . Since  $p \in \mathcal{P}$  is bounded, we easily conclude that  $(C_4)$  implies  $(C_3)$ . Conversely, assume  $V \in \text{UPPD}$  is a DIS(p) Lyapunov function for  $f$  and  $\alpha_1, \alpha_2, \mu, \Omega \in \mathcal{K}_\infty$  satisfy (3) and the DIS(p) requirements. Define  $\tilde{\alpha}_2, w \in \mathcal{K}_\infty \cap C^1$  and  $V^\#$  by (8) and (11). As before, when  $\tilde{\mu} = \mu \circ \tilde{\alpha}_2^{-1}$ , we have  $\dot{v}(t, x, u) \leq -p(t)\tilde{\mu}(V(t, x)) + \Omega(|u|)$  for all  $t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m$ . It follows from Lemma 5(i) and (10) that

$$1 + \xi(t)w'(V(t, x)) \in \left[ 1, \frac{5}{4} \right], \quad \forall t \geq 0, x \in \mathbb{R}^n. \quad (12)$$

Since  $w = \frac{1}{4\tau}\tilde{\mu}$ , we deduce that

$$\begin{aligned} \dot{V}^\# &\leq -p(t)\tilde{\mu}(V(t, x)) + \frac{5}{4}\Omega(|u|) \\ &\quad + \tau p(t)w(V(t, x)) - \left( \int_{t-\tau}^t p(r) dr \right) w(V(t, x)) \\ &\leq -\varepsilon w(\alpha_1(|x|)) + \frac{5}{4}\Omega(|u|). \end{aligned}$$

Since  $w \circ \alpha_1 \in C^1 \cap \mathcal{K}_\infty$ , it follows that  $V^\#$  is the desired strict DIS Lyapunov function.

**Step 5:**  $(C_2) \Leftrightarrow (C_6)$ . The implication  $(C_2) \Rightarrow (C_6)$  follows from (Khalil, 2002, Theorem 4.19, p.176). (In (Khalil, 2002), the controls are bounded piecewise

continuous functions  $\alpha : [0, \infty) \rightarrow \mathbb{R}^m$ , but the result from (Khalil, 2002) can be extended to our general control set  $\mathcal{U}$  using a standard denseness argument (see e.g. Remark C.1.2 and the proof of Theorem 1 in (Sontag, 1998)).) The converse was announced in (Edwards *et al.*, 2000, Theorem 1) and can be deduced from (Bacciotti & Rosier, 2001) as follows. If  $f$  is ISS, then (Sontag & Wang, 1995) provides  $\chi \in \mathcal{K}_\infty$  such that the constrained input system  $\dot{x} = f_\chi(t, x, d) := f(t, x, d\chi^{-1}(|x|))$ ,  $|d| \leq 1$  is uniformly globally asymptotically stable (UGAS); i.e., there exists  $\beta \in \mathcal{KL}$  such that for each  $t_o \geq 0$  and  $x_o \in \mathbb{R}^n$  and each trajectory  $y$  of  $f_\chi$  satisfying  $y(t_o) = x_o$ , we have  $|y(t_o + h)| \leq \beta(|x_o|, h)$  for all  $h \geq 0$ . By minorizing  $\chi^{-1}$ , we can assume it is  $C^1$ . This means the locally Lipschitz set-valued dynamics  $F(t, x) = \{f(t, x, u) : \chi(|u|) \leq |x|\}$  is UGAS, as is its convexification  $\overline{\text{co}}(F)$ , namely  $(t, x) \mapsto \overline{\text{co}}\{F(t, x)\}$  where  $\overline{\text{co}}$  denotes the closed convex hull (cf. (Bacciotti & Rosier, 2001, Proposition 4.2)). Since  $\overline{\text{co}}(F)$  is continuous and compact and convex valued, and since we are assuming  $f$  is periodic in  $t$ , (Bacciotti & Rosier, 2001, Theorem 4.5) provides a time-periodic  $V \in \text{UPPD}$  such that, for all  $x \in \mathbb{R}^n, t \geq 0, w \in F(t, x)$ ,

$$\frac{d}{dt}V(t, x) + \frac{d}{dx}V(t, x)w \leq -V(t, x).$$

Recalling the definition of  $F$  and assuming (without loss of generality) that  $V$  satisfies (3) with  $\alpha_1 \in \mathcal{K}_\infty \cap C^1$ ,

$$\begin{aligned} |x| \geq \chi(|u|) &\Rightarrow f(t, x, u) \in F(t, x) \\ &\Rightarrow \dot{V}(t, x, u) \leq -V(t, x) \leq -\alpha_1(|x|) \end{aligned}$$

for all  $t \geq 0$ , so  $V$  is the desired strict ISS Lyapunov function for  $f$ . This establishes  $(C_6) \Rightarrow (C_2)$ .

**Step 6:**  $(C_5) \Leftrightarrow (C_6)$ . Assuming  $(C_6)$ , there are  $\beta \in \mathcal{KL}$  such that for all  $t_o \geq 0, x_o \in \mathbb{R}^n, u_o \in \mathcal{U}$ , and  $h \geq 0$ ,

$$\begin{aligned} |\phi(t_o + h; x_o, t_o, u_o)| &\leq \beta(|x_o|, \bar{p}h) + \gamma(|u_o|_{[t_o, t_o+h]}) \\ &\leq \beta(|x_o|, \int_{t_o}^{t_o+h} p(s) ds) \\ &\quad + \gamma(|u_o|_{[t_o, t_o+h]}), \end{aligned}$$

where  $\phi$  is the trajectory of  $f$  we defined in Section 2. Therefore,  $f$  is ISS(p) so  $(C_6) \Rightarrow (C_5)$ . Conversely, if  $f$  is ISS(p), then we can find  $\beta \in \mathcal{KL}$  such that for all  $t_o \geq 0, x_o \in \mathbb{R}^n, u_o \in \mathcal{U}$ , and  $h \geq 0$ ,

$$\begin{aligned} |\phi(t_o + h; x_o, t_o, u_o)| &\leq \beta(|x_o|, \int_{t_o}^{t_o+h} p(s) ds) \\ &\quad + \gamma(|u_o|_{[t_o, t_o+h]}) \\ &\leq \beta(|x_o|, \underline{p}(h)) + \gamma(|u_o|_{[t_o, t_o+h]}). \end{aligned}$$

By Lemma 5(ii),  $\hat{\beta}(s, t) := \beta(s, \underline{p}(t)) \in \mathcal{KL}$ , so  $(C_5) \Rightarrow (C_6)$ , as desired. This proves Theorem 6.

**Remark 7** Observe that if the functions  $V$ ,  $\alpha_2$ ,  $\mu$ ,  $p$  are of class  $C^k$ , where  $k$  is a positive integer or  $\infty$ , then the particular function  $\tilde{\alpha}_2$  in (8) we have chosen implies that the function  $V^\#(t, x)$  is of class  $C^k$ .

**Remark 8** Our proof of Theorem 6 shows that if  $V$  is a strict ISS Lyapunov function for  $f$ , then  $V$  is also a strict DIS Lyapunov function for  $f$ . The preceding implication is no longer true if our boundedness requirement (2) on  $f$  is dropped, as illustrated by the following example from (Edwards et al., 2000): Take the one-dimensional single input system  $\dot{x} = f(t, x, u) := -x + (1+t)q(u - |x|)$ , where  $q: \mathbb{R} \rightarrow \mathbb{R}$  is any  $C^1$  function for which  $q(r) \equiv 0$  for  $r \leq 0$  and  $q(r) > 0$  otherwise. Then  $V(x) = x^2$  is a strict ISS Lyapunov function for the system since  $|x| \geq |u| \Rightarrow \dot{V} \leq -x^2$  but  $V$  does not satisfy the strict DIS condition (7) for any choices of  $\mu$  and  $\Omega$ . This does not contradict our results because (2) is not satisfied. This contrasts with the time-invariant case where strict ISS Lyapunov functions are automatically strict DIS Lyapunov functions.

## 4 Illustration

We next use our results to construct a strict ISS Lyapunov function for a tracking problem for a rotating rigid body (see (Crouch, 1984; Morin et al., 1995; Morin & Samson, 1997) for the background and motivation for this problem). Following Lefeber (Lefeber, 2000, p.31), we only consider the dynamics of the velocities, which, after a change of feedback, are

$$\dot{\omega}_1 = \delta_1 + u_1, \quad \dot{\omega}_2 = \delta_2 + u_2, \quad \dot{\omega}_3 = \omega_1 \omega_2. \quad (13)$$

where  $\delta_1$  and  $\delta_2$  are the inputs and  $u_1$  and  $u_2$  are the disturbances. We consider the reference state trajectory

$$\omega_{1r}(t) = \sin(t), \quad \omega_{2r}(t) = \omega_{3r}(t) = 0 \quad (14)$$

but our method applies to more general reference trajectories as well; see Remark 9 below. The substitution  $\tilde{\omega}_i(t) = \omega_i(t) - \omega_{ir}(t)$  transforms (13) into the error equations

$$\begin{aligned} \dot{\tilde{\omega}}_1 &= \delta_1 + u_1 - \cos(t), \\ \dot{\tilde{\omega}}_2 &= \delta_2 + u_2, \\ \dot{\tilde{\omega}}_3 &= (\tilde{\omega}_1 + \sin(t))\tilde{\omega}_2. \end{aligned} \quad (15)$$

By applying the backstepping approach as it is applied in (Jiang & Nijmeijer, 1997), or through direct calculations, one shows that the derivative of the class UPPD function

$$V(t, \tilde{\omega}) = \frac{1}{2} [\tilde{\omega}_1^2 + (\tilde{\omega}_2 + \sin(t)\tilde{\omega}_3)^2 + \tilde{\omega}_3^2] \quad (16)$$

with  $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)^\top$  along the trajectories of (15) in closed-loop with the control laws

$$\begin{aligned} \delta_1(t, \tilde{\omega}) &= -\tilde{\omega}_1 - \tilde{\omega}_2 \tilde{\omega}_3 + \cos(t) \\ \delta_2(t, \tilde{\omega}) &= -[1 + \sin(t)\tilde{\omega}_1 + \sin^2(t)]\tilde{\omega}_2 \\ &\quad - (2\sin(t) + \cos(t))\tilde{\omega}_3 \end{aligned} \quad (17)$$

satisfies

$$\begin{aligned} \dot{V} &= -\tilde{\omega}_1^2 - (\tilde{\omega}_2 + \sin(t)\tilde{\omega}_3)^2 - \sin^2(t)\tilde{\omega}_3^2 \\ &\quad + \tilde{\omega}_1 u_1 + (\tilde{\omega}_2 + \sin(t)\tilde{\omega}_3)u_2 \\ &\leq -\frac{1}{2}\tilde{\omega}_1^2 - \frac{1}{2}(\tilde{\omega}_2 + \sin(t)\tilde{\omega}_3)^2 - \sin^2(t)\tilde{\omega}_3^2 \\ &\quad + \frac{1}{2}(u_1^2 + u_2^2) \\ &\leq -p(t)\tilde{\mu}(V(\tilde{\omega})) + \Omega(|u|) \end{aligned} \quad (18)$$

with  $u = (u_1, u_2)^\top \in \mathbb{R}^2$ ,  $p(t) = \sin^2(t)$ ,  $\tilde{\mu}(s) = s$  and  $\Omega(s) = \frac{1}{2}s^2$ . Therefore  $V$  is a DIS(p) Lyapunov function for (15) in closed-loop with the control laws (17). Observe that, in this case,  $p \in \mathcal{P}(\pi, \pi/2, 1)$ . Setting  $\tau = \pi$  and  $w(s) = \frac{1}{8\tau}\tilde{\mu}(s) = \frac{s}{8\pi}$ , it follows that (12) also holds. Therefore, Steps 3-4 from our proof of Theorem 6 show

$$\begin{aligned} V^\#(t, \tilde{\omega}) &= V(t, \tilde{\omega}) + \left[ \int_{t-\tau}^t \left( \int_s^t p(r) dr \right) ds \right] w(V(t, \tilde{\omega})) \\ &= \left[ 1 + \frac{\pi}{32} - \frac{1}{32} \sin(2t) \right] V(t, \tilde{\omega}) \end{aligned}$$

is a strict DIS Lyapunov function and also a strict ISS Lyapunov function for the system (15) in closed-loop with the control laws (17).

**Remark 9** We chose to work with the reference trajectory (14) because it leads to the simple error equations (15). However, one can easily check that a strict ISS Lyapunov function can be constructed for any reference state trajectory  $(\omega_{1r}(t), \omega_{2r}(t), \omega_{3r}(t))$  such that

$$\begin{aligned} \sup_t \left| \int_0^t \omega_{1r}(s)\omega_{2r}(s) ds \right| &< \infty \text{ and} \\ \int_{t-\tau}^t [\omega_{1r}^2(s) + \omega_{2r}^2(s)] ds &\geq \varepsilon, \quad \forall t \geq \tau \end{aligned}$$

for some constants  $\tau, \varepsilon > 0$ .

## 5 Conclusion

For ISS time-varying systems, we provided explicit strict Lyapunov function *constructions* that can easily be performed in practice. The knowledge of these Lyapunov functions allows us to extend the well-known and useful theory of ISS systems to a broad class of time-varying nonlinear dynamics. We conjecture that a discrete-time version of our main result can be proved.

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