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Further constructions of control-Lyapunov functions and stabilizing feedbacks for systems satisfying the Jurdjevic-Quinn conditions

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Further Constructions of Control-Lyapunov Functions and Stabilizing Feedbacks for Systems Satisfying the Jurdjevic-Quinn Conditions

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Abstract

For a broad class of nonlinear systems, we construct smooth control-Lyapunov functions whose derivatives along the trajectories of the systems can be made negative definite by smooth control laws that are arbitrarily small in norm. We assume our systems satisfy appropriate generalizations of the Jurdjevic-Quinn conditions. We also design state feedbacks of arbitrarily small norm that render our systems integral-input-to-state stable to actuator errors.

Key Words: Control-Lyapunov functions, global asymptotic and integral-input-to-state stabilization

I. Introduction

Lyapunov stability is of paramount importance in nonlinear control theory. In many important applications, it is very beneficial to have a continuously differentiable Lyapunov function whose derivative along the trajectories of the system can be made negative definite by an appropriate choice of feedback. Observe in particular that:

- Recent advances in the stabilization of nonlinear delay systems (e.g., [7], [13], [21]) are based on knowledge of continuously differentiable Lyapunov functions.
- Lyapunov functions are very efficient tools for robustness analysis. For example, many proofs of nonlinear disturbance-to-state L^p stability properties rely on Lyapunov functions; see [6, Chapter 13] and [3], [10], [15]. Moreover, control-Lyapunov function (CLF) based control designs guarantee robustness to different types of deterministic [5] and stochastic disturbances, and to unmodeled dynamics [16], [17].

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• When a CLF satisfying the *small control property* (as defined below) is available, the universal formula in [19] provides an explicit expression for a stabilizing feedback that is also an optimal control for a suitable optimization problem whose value function is the CLF; see [19].

• Backstepping and forwarding require Lyapunov functions of class C^1 for the subsystems [17].

The converse Lyapunov theorem (see [9]) ensures that, for any system that is globally asymptotically stabilizable by C^1 feedback, a CLF exists. Unfortunately, for nonlinear control systems, determining *explicit expressions* for CLFs is in general difficult. Fortunately, for large classes of systems, one can determine functions whose derivatives along the trajectories can be rendered negative *semi*-definite. If the systems satisfy the so-called weak Jurdjevic-Quinn conditions (defined below), which generalize those given in [8], then globally asymptotically stabilizing feedbacks can be constructed. However, in this case, explicit formulas for CLFs are not generally available. This motivates the following fundamental question: *When the Jurdjevic-Quinn method applies, is it possible to design explicit CLFs?*

In [4], where this issue was addressed for the first time, a method was presented for designing explicit CLFs for affine homogeneous systems that satisfy the Jurdjevic-Quinn conditions. Our objective in the present note is to extend the main result of [4] by constructing CLFs for systems satisfying appropriate generalizations of the Jurdjevic-Quinn conditions, but not necessarily having the homogeneity property, including cases where the system may not be control-affine. Our work also complements [14] where strong Lyapunov functions are constructed for a large family of systems satisfying either an appropriate Lie algebraic condition or which can be shown to be stable using the LaSalle invariance principle. The main difference between the present work and [14] is that in [14], only systems without input are considered whereas here we consider systems with input.

We end this introduction by recalling some basic facts on the Jurdjevic-Quinn method. We say (see for example [4] for the relevant definitions) that a nonlinear control-affine system

$$\dot{x} \;=\; f(x) + g(x)u \;,\; g(x) = (g_1(x), \ldots, g_m(x)) \tag{1}$$

satisfies the *(weak) Jurdjevic-Quinn conditions* provided there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the following three properties: V is positive definite and radially unbounded; for all $x \in \mathbb{R}^n$, $L_f V(x) \leq 0$; and there exists an integer l such that the set

$$W(V) = \Big\{ x \in \mathbb{R}^n : \forall k \in \{1, \ldots, m\} \text{ and } i \in \{0, \ldots, l\}, L_f V(x) = L_{ad^i_f(g_k)} V(x) = 0 \Big\}$$

equals $\{0\}$. Here and in the sequel, we assume all functions are sufficiently smooth. If (1) satisfies the weak Jurdjevic-Quinn conditions, then it is globally asymptotically stabilized by any feedback $u = -\xi(x) L_g V(x)^\top$ where ξ is any positive function of class C^1. The proof of this result relies on the LaSalle Invariance Principle.

The remainder of this paper is organized as follows. In Section II, we present our main result. Section III is devoted to a discussion of our main result, Section IV to its proof, and Section V to an illustrating example. Section VI constructs feedbacks for our systems that have arbitrarily small norm and that in addition achieve integral-input-to-state stability relative to actuator errors. Concluding remarks in Section VII end our work.

II. Main result

Recall (cf. [2]) that a C^1 positive definite function $V(\cdot)$ on $\mathbb{R}^n$ is called a *control-Lyapunov function (CLF)* for a system $\dot{\chi} = \varphi_1(\chi) + \varphi_2(\chi)u$ with input u provided it is radially unbounded and satisfies $L_{\varphi_1} V(\chi) \geq 0 \;\; \Rightarrow \;\; [\chi = 0 \;\; \text{or} \;\; L_{\varphi_2} V(\chi) \neq 0]$. We use $\dot{V}(x,u)$ to denote the derivative $\dot{V}(x,u) = L_{\varphi_1} V(x) + L_{\varphi_2} V(x) u$ of V along trajectories of the system. We often suppress the arguments of $\dot{V}$ to simplify the notation. We say that a CLF $V(\cdot)$ for the system $\dot{\chi} = \varphi_1(\chi) + \varphi_2(\chi)u$ satisfies the *small control property* [19] provided for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $0 < |\chi| < \delta(\varepsilon)$, then there exists u (possibly depending on χ) such that $|u| < \varepsilon$ and $L_{\varphi_1} V(\chi) + L_{\varphi_2} V(\chi) u < 0$.

We next provide our main CLF and stabilizing feedback constructions for the fully nonlinear system

$$\dot{x} = F(x,u) \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ is the control, $F(0,0) = 0$, and the function F is assumed to be C^1. We further assume that $u \mapsto F(x,u)$ is C^2 (i.e., the second order partial derivatives, with respect to the components of u, of each component of F are continuous), so the functions

$$f(x) := F(x,0), \quad g(x) := \frac{\partial F}{\partial u}(x,0) \tag{3}$$

are at least C^1. Finally, we assume:

Assumption 1: A smooth function $V(x)$ that is radially unbounded and positive definite and such that

$$L_f V(x) \;\leq\; 0 \quad \forall x \in \mathbb{R}^n \tag{4}$$

is known.

Assumption 2: A vector field $G(x)$ such that if $L_gV(x) = 0$ and $x \neq 0$, then we either have $L_f L_G V(x) < 0$ or $L_f V(x) < 0$, is known.

We are ready to state our main result.

Theorem 1: Assume the data (3) satisfy Assumptions 1-2. Then one can determine a positive definite smooth function $\delta : [0, \infty) \to [0, \infty)$ and a function $\Omega : [0, \infty) \to [0, \infty)$ such that

$$V^\sharp(x) \;=\; V(x) + \int_0^{V(x)} \Omega(s)\mathrm{d}s + \delta(V(x))L_G V(x) \tag{5}$$

is a CLF for (2) that satisfies the small control property. In fact, for each real-valued C^1 positive function $\bar{\xi}(\cdot)$, one can determine a function $\delta(\cdot)$, and a C^1 function $\xi : [0, \infty) \to (0, \infty)$ satisfying $\xi(s) \leq \bar{\xi}(s)$ for all $s \geq 0$, such that (5) is a CLF for (2) satisfying the small control property whose derivative along the trajectories of (2) in closed-loop with the feedback

$$u \;=\; -\xi(V(x))L_g V(x)^\top \tag{6}$$

is negative definite.

III. Discussion of Theorem 1

1. Assumptions 1 and 2 are similar to the assumptions of the main result of [4]. In particular, for the special case where F is control-affine, [4] provides an explicit expression for a vector field $G(x)$ such that Assumption 2 holds whenever the so-called “weak Jurdjevic-Quinn conditions” (see the introduction) are satisfied. This vector field is not continuous at the origin but it turns out that there exists an integer $N \geq 1$ such that the vector field $G_N(x) = V(x)^N G(x)$ is of class C^∞ for V satisfying our assumptions. The equality $L_f L_{G_N} V(x) = NV(x)^{N-1} L_f V(x) L_G V(x) + V(x)^N L_f L_G V(x)$ then implies that if $G(x)$ satisfies Assumption 2, and if Assumption 1 also holds, then $G_N(x)$ satisfies Assumption 2 as well. Consequently, one can take advantage of the formula in [4] to determine a C^∞ vector field for which Assumption 2 is satisfied.
2. No restriction on the size of the function $\xi(\cdot)$ in (6) is imposed. Therefore, the family of feedbacks (6) contains elements that are arbitrarily small in (sup) norm. In fact, for any continuous positive function $\epsilon : [0, \infty) \to (0, \infty)$, we can design our stabilizing feedback u so that it satisfies $|u(x)| \leq \epsilon(|x|)$ for all $x \in \mathbb{R}^n$.

3. An important class of dynamics covered by Theorem 1 is described by the so-called *Euler-Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau \tag{7}$$

for the motion of mechanical systems, in which q represents the generalized configuration coordinates, $L = K - P$ is the difference between the kinetic energy K and potential energy P, and τ is the control [22]. In standard cases, $K(q,\dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q}$ where the inertia matrix $M(q)$ is C^1 and everywhere symmetric and positive definite. Then the generalized momenta $\frac{\partial L}{\partial \dot{q}}$ are given by $p = M(q)\dot{q}$, so in terms of the state $x = (q,p)$, the equations (7) become [22]

$$\dot{q} = \frac{\partial H}{\partial p}(q,p)^\top = M^{-1}(q)p, \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p)^\top + \tau, \tag{8}$$

where $H(q,p) = \frac{1}{2}p^\top M^{-1}(q)p + P(q)$ is the total energy of the system. We make the following additional assumptions: (a) $P(q)$ is positive definite and radially unbounded and (b) $\nabla P(q) \neq 0$ whenever $q \neq 0$. (These two assumptions are not too restrictive since one can often modify H and τ to get a new system that satisfies these assumptions. Condition (a) can be weakened by assuming there is a constant c such that $q \mapsto P(q) + c$ is radially unbounded and positive definite in which case we simply add c to the function V in what follows.) Then H is positive definite and radially unbounded, so $V = H$ satisfies Assumption 1. The radial unboundedness follows from the continuity of the (positive) eigenvalues of the positive definite matrix $M^{-1}(q)$ as functions of q [20, Appendix A4], which implies that each compact set $\mathcal{S}$ of q values admits a constant $c_{\mathcal{S}} > 0$ such that $p^\top M^{-1}(q)p \geq c_{\mathcal{S}}|p|^2$ for all $q \in \mathcal{S}$ and all p. In our general notation with $x = (q,p)$, we get $L_f V(x) \equiv 0$ and $L_g V(x) = H_p(x) = p^\top M^{-1}(q)$. Choosing $G(x) = [\ 0 \quad \nabla P(q)^\top\ ]^\top$ gives $L_G V(x) = H_p(x)\nabla P(q)^\top$. Therefore, if $L_g V(x) = p^\top M^{-1}(q) = 0$ and $x \neq 0$, then $p = 0$ and therefore also $L_f L_G V(x) = -\nabla P(q) M^{-1}(q) \nabla P(q)^\top$ and $q \neq 0$. Since M^{-1} is everywhere positive definite, Assumption 2 therefore reduces to our assumption (b) and therefore is satisfied as well. We study a special case of (8) in Section V below, where we explicitly compute the corresponding CLF and stabilizing feedback.

IV. Proof of Theorem 1

A. Control Affine Case

We fix a positive function $\bar{\xi} : [0,\infty) \to (0,\infty)$, and functions V and G satisfying Assumptions 1-2. We begin by proving Theorem 1 for the case where (2) is control affine, i.e., of the form

(1). In this control affine case, the conclusions of our theorem will hold with $\Omega \equiv 0$ and $\xi \equiv \bar{\xi}$. In Section IV-B, we will modify our constructions to handle the fully nonlinear system (2).

First step. We exhibit a family of functions $\delta(\cdot)$ for which the function

$$U(x) \;:=\; V(x) + \delta(V(x))L_G V(x) \tag{9}$$

is positive definite and radially unbounded. One can determine $\alpha_i(\cdot)$ of class $\mathcal{K}_\infty$ such that $\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$ and $|L_G V(x)| \le \alpha_3(|x|)$ for all $x \in \mathbb{R}^n$. It follows that

$$U(x) \;\ge\; \alpha_1(|x|) - \delta(V(x))\alpha_3(|x|) \;\ge\; \alpha_1(\alpha_2^{-1}(V(x))) \;-\; \delta(V(x))\alpha_3(\alpha_1^{-1}(V(x))) \tag{10}$$

for all $x \in \mathbb{R}^n$. We can use standard results to find a C^1 function $\delta : [0, \infty) \to [0, \infty)$ such that

$$\delta(v) \le \frac{\alpha_1(\alpha_2^{-1}(v))}{1 + 2\alpha_3(\alpha_1^{-1}(v))} \quad \forall v \ge 0 \ . \tag{11}$$

With such a function $\delta(\cdot)$, the inequality $U(x) \ge \frac{1}{2}\alpha_1(\alpha_2^{-1}(V(x)))$ for all $x \in \mathbb{R}^n$ follows from (10). Since $V(x)$ is positive definite and radially unbounded and $\frac{1}{2}\alpha_1(\alpha_2^{-1}(\cdot))$ is of class $\mathcal{K}_\infty$, this implies that $U(x)$ is positive definite and radially unbounded as well. In the next steps, we impose further restrictions on δ.

Second step. Along the trajectories $x(t)$ of our system (1) in closed-loop with the feedback $u = -\bar{\xi}(V(x))L_g V(x)^\top$, the derivative $\dot{U}$ of $U(x)$ from (9) reads

$$\begin{aligned} \dot{U} \;\;&=\;\; \left[L_f V(x) - \bar{\xi}(V(x))|L_g V(x)|^2\right]\left[1 + \delta'(V(x))L_G V(x)\right] \\ &\quad + \delta(V(x))L_f L_G V(x) \;-\; \bar{\xi}(V(x))\delta(V(x))L_g L_G V(x) L_g V(x)^\top . \end{aligned} \tag{12}$$

We restrict our attention to functions δ such that

$$\delta'(V(x))L_G V(x) \ge -\frac{1}{4} \;\; \forall x \in \mathbb{R}^n. \tag{13}$$

Recalling (4) and (12) therefore gives the inequality

$$\begin{aligned} \dot{U} \;\;&\le\;\; \tfrac{3}{4}\left[L_f V(x) - \bar{\xi}(V(x))|L_g V(x)|^2\right] \;+\; \delta(V(x))L_f L_G V(x) \\ &\quad + \bar{\xi}(V(x))\delta(V(x))|L_g L_G V(x)||L_g V(x)|. \end{aligned} \tag{14}$$

¿From (4), we deduce that

$$\begin{aligned} \dot{U} \;\;&\le\;\; \frac{1}{2}\left[L_f V(x) - \bar{\xi}(V(x))|L_g V(x)|^2\right] + \delta(V(x))L_f L_G V(x) \\ &\quad + \bar{\xi}(V(x))\delta^2(V(x))|L_g L_G V(x)|^2 \ . \end{aligned} \tag{15}$$

Third step. The remaining part of the proof relies extensively on the following:

Lemma 2: Assume that the system (1) satisfies Assumptions 1-2. Then, there exist continuous positive definite functions Γ and N satisfying the following: If $|L_gV(x)| \leq \Gamma(|x|)$, then either $L_fV(x) \leq -N(|x|)$ or $L_fL_GV(x) \leq -N(|x|)$.

Proof: We first show that the continuous function

$$S(x) \;=\; \min\{0, L_fL_GV(x)\} + \min\{0, L_fV(x)\} - |L_gV(x)| \tag{16}$$

is negative definite. Observe first that $S(0) = 0$ and $S(x) \leq 0$ for all x. Assume $S(x) = 0$. Each term of $S(x)$ is nonpositive, so $\min\{0, L_fL_GV(x)\} = \min\{0, L_fV(x)\} = |L_gV(x)| = 0$. By Assumption 2, $x = 0$, which gives the negative definiteness. Therefore $-S(x)$ is positive definite, so we can determine a continuous positive definite real-valued function ρ such that $\rho(|x|) \leq -S(x)$ (e.g., $\rho(s) = \min\{-S(r) : |r| = s\}$). We prove that $|L_gV(x)| \leq \frac{1}{2}\rho(|x|)$ implies that either $L_fL_GV(x) \leq -\frac{1}{4}\rho(|x|)$ or $L_fV(x) \leq -\frac{1}{4}\rho(|x|)$. To this end, consider x such that $|L_gV(x)| \leq \frac{1}{2}\rho(|x|)$. Then $\rho(|x|) \leq -\min\{0, L_fL_GV(x)\} - \min\{0, L_fV(x)\} + \frac{1}{2}\rho(|x|)$, by our choices of ρ and S. We deduce that $\min\{0, L_fL_GV(x)\} + \min\{0, L_fV(x)\} \;\leq\; -\frac{1}{2}\rho(|x|)$. It follows that either $\min\{0, L_fL_GV(x)\} \leq -\frac{1}{4}\rho(|x|)$ or $\min\{0, L_fV(x)\} \leq -\frac{1}{4}\rho(|x|)$. Therefore, $|L_gV(x)| \leq \frac{1}{2}\rho(|x|)$ implies $L_fL_GV(x) \leq -\frac{1}{4}\rho(|x|)$ or $L_fV(x) \leq -\frac{1}{4}\rho(|x|)$, so we can take $\Gamma(s) = \frac{1}{2}\rho(s)$ and $N(s) = \frac{1}{4}\rho(s)$. ■

Fourth step. We prove that the right hand side of (15) is negative definite when the smooth positive definite function $\delta(\cdot)$ is suitably chosen. By the preceding lemma, there are three cases:

First Case. $|L_gV(x)| \leq \Gamma(|x|)$ and $L_fV(x) \leq -N(|x|)$. Then the inequality (15) implies that

$$\dot{U} \;\leq\; -\tfrac{1}{2}N(|x|) + \delta(V(x))L_fL_GV(x) \;+\; \bar{\xi}(V(x))\delta^2(V(x))|L_gL_GV(x)|^2\;. \tag{17}$$

Choosing $\delta(\cdot)$ such that

$$\delta(V(x))L_fL_GV(x) \leq \frac{1}{8}N(|x|), \quad \bar{\xi}(V(x))\delta^2(V(x))|L_gL_GV(x)|^2 \leq \frac{1}{8}N(|x|) \tag{18}$$

for all $x \in \mathbb{R}^n$. Therefore, (17) gives $\dot{U} \leq -\frac{1}{4}N(|x|) < 0$ for all $x \neq 0$.

Second Case. $|L_gV(x)| \leq \Gamma(|x|)$ and $L_fL_GV(x) \leq -N(|x|)$. Then the inequalities (4) and (15) imply $\dot{U} \leq -\delta(V(x))N(|x|) + \bar{\xi}(V(x))\delta^2(V(x))|L_gL_GV(x)|^2$. Choosing $\delta(\cdot)$ such that

$$\delta(V(x))\bar{\xi}(V(x))|L_gL_GV(x)|^2 \;\leq\; \frac{1}{2}N(|x|) \tag{19}$$

we obtain $\dot{U} \leq -\frac{1}{2}\delta(V(x))N(|x|) < 0$ for all $x \neq 0$.

<u>Third Case.</u> $|L_gV(x)| \geq \Gamma(|x|)$. Then the inequality (15) implies that

$$\dot{U} \;\leq\; -\tfrac{1}{2}\bar{\xi}(V(x))\Gamma^2(|x|) \;+\; \delta(V(x))L_fL_GV(x) \;+\; \bar{\xi}(V(x))\delta^2(V(x))|L_gL_GV(x)|^2 \;.$$

Arguing as above provides $\delta(\cdot)$ such that

$$\delta(V(x))L_fL_GV(x) \leq \frac{1}{8}\Gamma^2(|x|)\bar{\xi}(V(x)), \quad \delta^2(V(x))|L_gL_GV(x)|^2 \leq \frac{1}{8}\Gamma^2(|x|) \;, \tag{20}$$

so we obtain $\dot{U} \leq -\frac{1}{4}\bar{\xi}(V(x))\Gamma^2(|x|) < 0$ for all $x \neq 0$.

Fifth step. To conclude the proof for the control affine case, one has to prove that one can determine a C^1 and positive definite function $\delta(\cdot)$ simultaneously satisfying the requirements (11), (13), (18), (19), (20). This can be done as follows. We can first find a C^1 positive definite function δ satisfying the requirements (11), (18), (19), (20) that is increasing on $[0,1]$, non-increasing on $[1,\infty)$ and bounded by 1. We denote this initial choice of δ by δ_a. Next, we minorize $1/(1+4|L_GV(x)|)$ by a positive function of the form $x \mapsto \mathcal{P}(V(x))$ (using, e.g., $\mathcal{P}(s) = \inf\{1/(1+4|L_GV(x)|) : x \in \mathbb{R}^n, V(x) = s\}$). One can easily determine an everywhere positive, non-increasing C^1 function $\omega(\cdot)$ such that $\omega(s) \leq \frac{1}{2}\min\{\mathcal{P}(s), \mathcal{P}(2s), 1\}$ for all $s \geq 0$. Now consider the function

$$\delta(s) = \int_{\frac{1}{2}s}^{s} \frac{\delta_a(l)\delta_a(2l)\omega(l)}{1+4l^2}dl \;. \tag{21}$$

It is positive definite, of class C^2, and (since δ_a is bounded by 1) satisfies, for all $s \geq 0$,

$$|\delta'(s)| \;=\; \left|\tfrac{\delta_a(s)\delta_a(2s)\omega(s)}{1+4s^2} - \tfrac{1}{2}\tfrac{\delta_a(\frac{1}{2}s)\delta_a(s)\omega(\frac{1}{2}s)}{1+s^2}\right| \leq \omega(s) + \tfrac{1}{2}\omega(\tfrac{1}{2}s) \leq \mathcal{P}(s) \;. \tag{22}$$

¿From this inequality, one can deduce that δ defined in (21) satisfies (13). On the other hand, since ω is smaller than 1, the inequality $\delta(s) \leq \int_{s/2}^{s}\delta_a(l)\delta_a(2l)/(1+4l^2)dl$ is satisfied for all $s \geq 0$. Now, we distinguish between two cases. *First case:* If $s \in [0,1]$, then, since δ_a is a nonnegative function smaller than 1 and increasing on $[0,1]$, we get $\delta(s) \leq \int_{s/2}^{s}\delta_a(l)dl \leq \delta_a(s)$. *Second case:* $s \geq 1$, then, since δ_a is a nonnegative function smaller than 1 and nonincreasing on $[1,+\infty)$, we get

$$\delta(s) \leq \int_{\frac{1}{2}s}^{s}\frac{\delta_a(2l)}{1+s^2}dl \leq \frac{s}{2(1+s^2)}\delta_a(s) \leq \delta_a(s) \;. \tag{23}$$

Hence, the function δ defined in (21) satisfies the requirements (11), (13), (18), (19), (20).

Remark 3: The proof of Lemma 2 provides explicit formulae for the functions Γ and N required for our constructions. On the other hand, the function δ in (5) can be obtained by simply verifying the requirements in the fifth step of our proof.

B. Fully Nonlinear Case

We now extend the construction to our original fully nonlinear system (2). We can write

$$F(x,u) = f(x) + g(x)u + h(x,u)u, \quad \text{where} \quad h(x,u) = \int_0^1 \left[\frac{\partial F}{\partial u}(x,\lambda u) - \frac{\partial F}{\partial u}(x,0)\right] \mathrm{d}\lambda. \quad (24)$$

Along the trajectories of (2), it follows that $\dot{V} = L_fV(x) + L_gV(x)u + \nabla V(x)h(x,u)u$. Since F is C^2 in u, we can find a continuous function $H : [0,\infty)\times[0,\infty) \to (0,\infty)$ that is nondecreasing in both variables such that $|h(x,u)u| \leq H(V(x),|u|)|u|^2$ for all x and u. One can find $\alpha_4 \in \mathcal{K}_\infty$ such that $|\nabla V(x)| \leq \alpha_4(|x|)$ for all x. Taking u to be a feedback of the form (6) gives

$$\dot{V} \;\leq\; L_fV(x) - \xi(V(x))|L_gV(x)|^2 + H_*(V(x), |\xi(V(x))L_gV(x)^\top|)\xi^2(V(x))|L_gV(x)|^2$$

with $H_*(r,s) = \alpha_4(\alpha_1^{-1}(r))H(r,s)$. We now restrict our attention to the set $\mathcal{F}[\bar{\xi}]$ of all feedbacks (6) such that $\xi(s) \leq \overline{\xi}(s)$ for all $s \geq 0$, where we assume the positive function $\overline{\xi}$ is such that

$$H_*(V(x), \overline{\xi}(V(x))|L_gV(x)|)\overline{\xi}(V(x)) \;\leq\; \frac{1}{2} \quad \forall x \in \mathbb{R}^n. \quad (25)$$

Condition (25) can be satisfied by minorizing $\bar{\xi}$ as necessary without relabelling. (The proof that $\bar{\xi}$ can be chosen to satisfy (25) is similar to the construction of the function δ in the first part of the proof.) Fixing a feedback from this family $\mathcal{F}[\bar{\xi}]$, we get

$$\dot{V} \leq L_fV(x) - \frac{1}{2}\xi(V(x))|L_gV(x)|^2 \quad (26)$$

along the closed loop trajectories of (2). Applying the construction from the first part of the proof to the control affine system (1) with $\bar{\xi} = \xi$ provides a function δ and a CLF U of the form (9) such that $W(x) := -\left\{L_fU(x) - L_gU(x)\xi(V(x))L_gV(x)^\top\right\}$ is positive definite. Therefore, $\dot{U}$ along the trajectories of (2) in closed-loop with the feedback (6), with ξ satisfying $\xi(s) \leq \bar{\xi}(s)$ for all $s \geq 0$, reads $\dot{U} = -W(x) - \nabla U(x)h(x, -\xi(V(x))L_gV(x)^\top)\xi(V(x))L_gV(x)^\top$. Therefore, since H is non-decreasing in its second argument, it follows from our choices of ξ and H that $\dot{U} \leq -W(x) + \left\{|\nabla U(x)|H(V(x), \overline{\xi}(V(x))|L_gV(x)|)\overline{\xi}(V(x))\right\}\xi(V(x))\,|L_gV(x)|^2$ for all x. One can construct a positive nondecreasing function Δ such that, along the closed loop trajectories,

$$\dot{U} \;\;\leq\;\; -W(x) + \Delta(V(x))\xi(V(x))\,|L_gV(x)|^2\,. \quad (27)$$

Now consider the function (5) with the above choice of δ and $\Omega(s) = 4\Delta(s)$, which is positive definite and radially unbounded. Then, according to our Assumption 1, (26), and (27), we get

$$\dot{V}^\sharp \;\;\leq\;\; -W(x) - \Delta(V(x))\xi(V(x))|L_gV(x)|^2 \;\; \forall x \in \mathbb{R}^n$$

The right-hand-side of this inequality is negative definite, so we can satisfy the requirements of the theorem using $\Omega(s) = 4\Delta(s)$ and the CLF $V^\sharp$. This concludes our proof.

V. Example

We illustrate Theorem 1 by applying it to the two-link manipulator (see [1]). This system is a fully actuated system described by the Euler-Lagrange equations

$$\left(mr^2 + M\tfrac{L^2}{3}\right)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} \;=\; \tau\;, \quad m\ddot{r} - mr\dot{\theta}^2 \;=\; F\;, \tag{28}$$

where M is the mass of the arm; L is its length; m is the mass of the gripper; r and θ denote the angle of the link and the position of the gripper, respectively; and τ and F are forces acting on the system. It is well-known that (28) can be stabilized by bounded control laws. On the other hand, this system is globally feedback linearizable so a quadratic CLF can be determined. The novelty is that we determine a CLF whose derivative along the trajectory is made negative definite by an appropriate choice of *bounded* feedback. Without loss of generality, we take $m = M = 1$ and $L = \sqrt{3}$. With $x_1 := \theta, x_2 := \dot{\theta}, x_3 := r, x_4 := \dot{r}$, the system (28) becomes

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\tfrac{2x_3x_2x_4}{x_3^2+1} + \tfrac{\tau}{x_3^2+1}, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = x_3x_2^2 + F. \tag{29}$$

We construct a globally stabilizing feedback, bounded in norm by 1, and an associated CLF for (29). We set $\langle p\rangle = 1/(2\sqrt{1+p^2}\,)$ for all $p \in \mathbb{R}$ throughout the sequel.

Consider the function

$$V(x) = \frac{1}{2}\left[(x_3^2+1)x_2^2 + x_4^2 + \sqrt{1+x_1^2} + \sqrt{1+x_3^2} - 2\right]\;. \tag{30}$$

This function is composed of the kinetic energy of the system with additional terms. It is positive definite and radially unbounded and its derivative along trajectories of (29) satisfies $\dot{V}(x) = x_2\tau + x_4F + x_1\langle x_1\rangle x_2 + x_3\langle x_3\rangle x_4$. Therefore the change of feedback

$$\tau = -x_1\langle x_1\rangle + \tau_b\;,\;\; F = -x_3\langle x_3\rangle + F_b \tag{31}$$

yields $\dot{V}(x) = x_2\tau_b + x_4F_b$. On the other hand, after the change of feedback (31), the equations of the system take the control affine form $\dot{x} = f(x) + g(x)u$ with

$$f(x) = \begin{bmatrix} x_2 \\ \frac{-2x_3x_2x_4 - x_1\langle x_1\rangle}{x_3^2+1} \\ x_4 \\ x_2^2x_3 - x_3\langle x_3\rangle \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 & 0 \\ \frac{1}{x_3^2+1} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} \tau_b \\ F_b \end{bmatrix}$$

Next consider the vector field $G(x) = (0, x_1, 0, x_3)^\top$. Simple calculations yield

$$L_G V(x) = \frac{\partial V}{\partial x_2}(x) x_1 + \frac{\partial V}{\partial x_4}(x) x_3 = (x_3^2 + 1) x_2 x_1 + x_4 x_3. \tag{32}$$

Since $\nabla(L_G V(x)) = (x_2(x_3^2+1), x_1(x_3^2+1), x_4 + 2x_1x_2x_3, x_3)$, we get

$$L_f L_G V(x) \;=\; x_2^2(2x_3^2+1) + x_4^2 - x_1^2\langle x_1\rangle - x_3^2\langle x_3\rangle. \tag{33}$$

We now check that Assumptions 1 and 2 are satisfied. Since $L_f V(x) = 0$ and $L_g V(x) = [x_2 \;\; x_4]$ everywhere, Assumption 1 is satisfied. If $L_g V(x) = 0$, then $x_2 = x_4 = 0$, in which case we get $L_f L_G V(x) = -x_1^2\langle x_1\rangle - x_3^2\langle x_3\rangle$. It follows that if $x \neq 0$ and $L_g V(x) = 0$, then $L_f L_G V(x) < 0$. Therefore Assumption 2 is satisfied. Hence, Theorem 1 applies. Consider the function

$$V^\sharp(x) \;=\; 40[2 + 2V(x)]^6 + L_G V(x) - 40(2^6)\ . \tag{34}$$

Simple multiplications show $80[2+2V(x)]^6 \geq V^\sharp(x) \geq 3\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)$ for all x, so $V^\sharp$ is positive definite and radially unbounded. Moreover, we see that along the trajectories of (29) after the change of feedback (31),

$$\begin{aligned} \dot V^\sharp(x) \;&=\; 480[2+2V(x)]^5(x_2\tau_b + x_4F_b) + x_2^2(2x_3^2+1) + x_4^2 - x_1^2\langle x_1\rangle - x_3^2\langle x_3\rangle \\ &\quad + x_1\tau_b + x_3F_b\ , \end{aligned} \tag{35}$$

since $\dot V(x) = x_2\tau_b + x_4F_b$. Therefore, from the triangle inequality, we deduce that

$$\begin{aligned} \dot V^\sharp(x) \;&\leq\; \sqrt{1+x_1^2}\tau_b^2 + 480[2+2V(x)]^5 x_2\tau_b + x_2^2(2x_3^2+1) \\ &\quad + \sqrt{1+x_3^2}F_b^2 + 480[2+2V(x)]^5 x_4F_b + x_4^2 - \tfrac{1}{2}x_1^2\langle x_1\rangle - \tfrac{1}{2}x_3^2\langle x_3\rangle\ . \end{aligned} \tag{36}$$

We demonstrate now that $V^\sharp$ is a CLF for (29) by showing that the right hand side of (36) is negative definite for the feedbacks

$$\tau_b = -x_2\langle x_2\rangle\ ,\ F_b = -x_4\langle x_4\rangle. \tag{37}$$

To this end, notice that we have

$$\dot V^\sharp(x) \;\leq\; T_1(x)x_2^2\langle x_2\rangle + T_2(x)x_4^2\langle x_4\rangle - \tfrac{1}{2}\left[x_1^2\langle x_1\rangle + x_2^2\langle x_2\rangle + x_3^2\langle x_3\rangle + x_4^2\langle x_4\rangle\right] \tag{38}$$

where we define the T_i's by $T_1(x) = \sqrt{1+x_1^2} - 480(2+2V(x))^5 + 2\sqrt{1+x_2^2}(2x_3^2+1) + \frac{1}{2}$ and $T_2(x) = \sqrt{1+x_3^2} - 480(2+2V(x))^5 + 2\sqrt{1+x_4^2} + \frac{1}{2}$. From the expression of $V(x)$, we deduce that T_1 and T_2 are nonpositive and therefore

$$\dot V^\sharp(x) \leq -\frac{1}{2}\left[x_1^2\langle x_1\rangle + x_2^2\langle x_2\rangle + x_3^2\langle x_3\rangle + x_4^2\langle x_4\rangle\right]\ . \tag{39}$$

The right hand side of this inequality is negative definite and the feedbacks resulting from (31) and (37) are bounded in norm by 1. This concludes the proof.

VI. Robustness to Actuator Errors

Theorem 1 provided a stabilizing feedback $u = K_1(x)$ such that $\dot{x} = f(x) + g(x)K_1(x)$ is globally asymptotically stable (GAS) to $x = 0$. Moreover, for each $\varepsilon > 0$, we can choose K_1 to satisfy $|K_1(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^n$.

One natural and widely used generalization of the GAS condition is the so-called input-to-state stable (ISS) property [18]. For a general nonlinear system $\dot{x} = F(x, d)$ evolving on $\mathbb{R}^n \times \mathbb{R}^m$ (where d represents the disturbance), the ISS property is the requirement that there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the following holds for all measurable essentially bounded functions $\mathbf{d} : [0, \infty) \to \mathbb{R}^m$ and corresponding trajectories $x(t)$ for $\dot{x}(t) = F(x(t), \mathbf{d}(t))$:

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(|\mathbf{d}|_\infty) \ \ \forall t \geq 0. \tag{ISS}$$

Here $|\cdot|_\infty$ is the essential supremum norm. The ISS property reduces to GAS to 0 for systems with no controls, in which case the overshoot term $\gamma(|\mathbf{d}|_\infty)$ in the ISS decay condition is 0; see also [11], [12] for the relationship between the ISS property and asymptotic controllability. It is therefore natural to look for a feedback $K(x)$ for (1) (which could in principle differ from K_1) for which

$$\dot{x} = F(x, d) := f(x) + g(x)[K(x) + d] \tag{40}$$

is ISS, and for which $|K(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^n$, where ε is any prescribed positive constant. In other words, we would want an arbitrarily small feedback K that renders (1) GAS to $x = 0$ and that has the additional property that (40) is also ISS with respect to actuator errors d.

However, it is clear that this objective cannot be met, since there is no *bounded* feedback $K(x)$ such that the one-dimensional system $\dot{x} = K(x) + d$ is ISS. On the other hand, if we add

Assumption 3: A positive nondecreasing smooth function $\mathcal{D}$ such that (i) $\int_0^{+\infty} \frac{1}{\mathcal{D}(s)}\, \mathrm{d}s = +\infty$ and (ii) $|L_g V(x)| \leq \mathcal{D}(V(x))$ for all $x \in \mathbb{R}^n$ is known.

where V satisfies our continuing Assumptions 1-2, then any feedback $K := -\xi(V(x))L_g V(x)^\top$, obtained from Theorem 1 for the control affine system $\dot{x} = f(x) + g(x)u$ and chosen such that $|\xi(V(x))L_g V(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^n$, also renders (40) *integral-input-to-state stable (iISS)*. For a general nonlinear system $\dot{x} = F(x, d)$ evolving on $\mathbb{R}^n \times \mathbb{R}^m$, the iISS condition is the following: There exist $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for all measurable locally essentially bounded functions $\mathbf{d} : [0,\infty) \to \mathbb{R}^m$ and corresponding trajectories $x(t)$ for $\dot{x}(t) = F(x(t), \mathbf{d}(t))$,

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \gamma(|\mathbf{d}(s)|)\mathrm{d}s \quad \forall t \geq 0. \tag{iISS}$$

The iISS condition reflects the qualitative property of having small overshoots when the disturbances have finite energy. It provides a nonlinear analog of "finite H^2 norm" for linear systems, and thus has obvious physical relevance and significance [2], [3]. Assumptions 1-3 hold for our example in the previous section, since in that case, $|L_g V(x)| \leq 2(V(x)+2)$ for all $x \in \mathbb{R}^n$, so we can take $\mathcal{D}(s) = 2(s+2)$. In fact, our assumptions hold for a broader class of Hamiltonian systems as well; see Remark 5 below.

To verify that the Theorem 1 feedback also renders (40) iISS, we begin by fixing $\varepsilon > 0$ and V satisfying our Assumptions 1-3, and applying our theorem to $\dot{x} = f(x) + g(x)u$. This provides a CLF U for (1) and a corresponding positive function ξ that satisfies $|\xi(V(x))L_g V(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^n$. The CLF U has the form (9). By reducing δ and δ' from Section IV-A, and replacing $\mathcal{D}(p)$ with $p \mapsto \mathcal{D}(2p) + 1$ in Assumption 3 without relabelling, we can assume

$$|L_g U(x)| \leq \mathcal{D}(U(x)) \ \ \forall x \in \mathbb{R}^n. \tag{41}$$

Then

$$\tilde{U}(x) = \int_0^{U(x)} \frac{\mathrm{d}p}{\mathcal{D}(p)}, \ \ \text{where} \ \ U(x) = V(x) + \delta(V(x))L_G V(x) \tag{42}$$

is again a CLF for our dynamic (1), since our choice of $\mathcal{D}$ gives $\tilde{U}(x) \to +\infty$ as $|x| \to \infty$ because U is radially unbounded, and because $\nabla\tilde{U}(x) \equiv \nabla U(x)/\mathcal{D}(U(x))$ (which gives the CLF decay condition). The smoothness of $\tilde{U}$ follows because U and $\mathcal{D}$ are both smooth. Finally, (41) gives

$$|L_g \tilde{U}(x)| = |L_g U(x)/\mathcal{D}(U(x))| \leq 1 \ \ \forall x \in \mathbb{R}^n. \tag{43}$$

We next choose the smooth feedback $K_1(x) = -\xi(V(x))L_g V(x)^\top$, where ξ is a smooth positive function satisfying the above requirements, so K_1 renders (1) GAS to $x = 0$, by Theorem 1. To check that $K(x) := K_1(x)$ also renders (40) iISS, notice that our choice of K_1 and (43) give

$$\begin{aligned} \nabla\tilde{U}(x)F(x,d) &= \nabla\tilde{U}(x)[f(x) + g(x)K_1(x)] + L_g\tilde{U}(x)d \\ &\leq -\alpha_5(|x|) + |L_g\tilde{U}(x)|\,|d| \ \ \leq \ \ -\alpha_5(|x|) + |d| \end{aligned} \tag{44}$$

for all x and d and some continuous positive definite function α_5. Inequality (44) says (see [3]) that the positive definite radially unbounded smooth function $\tilde{U}$ is an iISS-CLF for (40). The fact that (40) is iISS now follows from the iISS Lyapunov characterization [3, Theorem 1]. We conclude as follows:

Corollary 4: Let the data (3) satisfy Assumptions 1-3 for some vector field $G : \mathbb{R}^n \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$, and let $\varepsilon > 0$ be given. Then there exist smooth functions $\delta, \xi : [0, \infty) \to [0, \infty)$ such that (i) the system (40) with the feedback $K(x) := -\xi(V(x))L_gV(x)^\top$ is iISS and has a smooth iISS-CLF of the form (42) and (ii) $|K(x)| \le \varepsilon$ for all $x \in \mathbb{R}^n$.

Remark 5: Assume the Hamiltonian system (8) satisfies the conditions (a)-(b) we introduced in Section III as well as the following additional condition: (c) There exist $\underline{\lambda}, \bar{\lambda} > 0$ such that $\mathrm{spectrum}\{M^{-1}(q)\} \subseteq [\underline{\lambda}, \bar{\lambda}]$ for all q. (Assumption (c) means there are *positive* constants $\underline{c}$ and $\bar{c}$ such that $\underline{c}|p|^2 \le p^\top M(q)p \le \bar{c}|p|^2$ for all q and p.) Then (8) satisfies our Assumptions 1-3 and so is covered by the preceding corollary. In fact, we saw on p. 5 that (a)-(b) imply that Assumptions 1-2 hold with $V = H$, and then Assumption 3 follows from (c) because $|L_gV(x)|^2 = |p^\top M^{-1}(q)|^2 \le \bar{\lambda}^2|p|^2 \le (\bar{\lambda}^2/\underline{\lambda})p^\top M^{-1}(q)p \le 2(\bar{\lambda}^2/\underline{\lambda})V(x)$ for all $x = (q, p)$. We can choose $\mathcal{D}(s) := \sqrt{2(\bar{\lambda}^2/\underline{\lambda})(s + 1)}$.

VII. Conclusion

We showed how to construct control-Lyapunov functions for fully nonlinear systems satisfying appropriate generalizations of the Jurdjevic-Quinn conditions. We also constructed feedbacks of arbitrarily small norm that render our systems integral-input-to-state stable to actuator errors. Our constructions apply to important families of nonlinear systems, and in particular to systems described by Euler-Lagrange equations. Redesign and further robustness analysis for our systems via our construction of control-Lyapunov functions will be subjects of future work.

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