Further results on input-to-state stability for nonlinear systems with delayed feedbacks

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Further Results on Input-to-State Stability for Nonlinear Systems with Delayed Feedbacks [⋆]

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Abstract

We consider a class of nonlinear control systems for which stabilizing feedbacks and corresponding Lyapunov functions for the closed loop systems are available. In the presence of feedback delays and actuator errors, we explicitly construct input-to-state stability (ISS) Lyapunov-Krasovskii functionals for the resulting feedback delayed dynamics, in terms of the available Lyapunov functions for the original undelayed dynamics, which establishes that the closed loop systems are input-to-state stable (ISS) with respect to actuator errors. We illustrate our results using a generalized system from identification theory and other examples.

Key words: Delayed systems, Input-to-state stability, Lyapunov function constructions

1 Introduction

Input-to-state stability (ISS) plays an important role in modern stability analysis and controller design (see Sontag (1989, 2006) and Teel (1998)). The ISS paradigm was introduced in Sontag (1989) and extended to delay systems in the seminal paper Teel (1998), which gave sufficient conditions for ISS using control ISS Lyapunov-Razumikhin functions (ISS-LRFs). By contrast, Pepe & Jiang (2006) gave sufficient conditions for ISS of delay systems via Lyapunov-Krasovskii functionals (see Section 2.1 for the relevant definitions and Section 3 for details on the known results). However, none of these works provide general methods for explicitly constructing Lyapunov functionals for feedback delayed dynamics.

In applications, it is often essential to *explicitly construct* Lyapunov functions. For example, Lyapunov functions for delayless systems $\dot{x} = f(x)$ can be used to construct stabilizing feedbacks $K(x)$ for which $\dot{x}(t) = f(x(t)) + g(x(t))[K(x(t)) + d(t)]$ is ISS with respect to the actuator error $d(t)$, under standard assumptions (see Sontag (1989)). In Karafyllis (2006), Lyapunov functionals were used to characterize robust global asymptotic stability of nonlinear time-varying retarded systems. For certain classes of delayed systems, LRFs can be used to construct stabilizing feedbacks as well. For example, see Jankovic (2001), which also constructs the needed LRFs in some cases.

With these observations in mind, we pursue a different objective. We assume that we are given a time-varying control affine system for which a stabilizing feedback $u = u_s(x,t)$ and a Lyapunov function for the corresponding closed loop system

$$\dot{x} = f(x,t) + g(x,t)u_s(x,t) \tag{1}$$

are known. We do not require (1) to be exponentially stable. We introduce a vector of constant delays $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ with $\tau_i \geq 0$, in the feedback signal. We then explicitly construct an ISS Lyapunov-Krasovskii functional (ISS-LKF) for

$$\dot{x}(t) = f(x(t),t) + g(x(t),t)[u_s(\xi_\tau(t),t) + d(t)] \tag{2}$$

with $\xi_\tau(t) = (x_1(t-\tau_1), x_2(t-\tau_2), \cdots, x_n(t-\tau_n))$ and $0 \leq \tau_i \leq \bar{\tau}$ for some bound $\bar{\tau} \geq 0$ (see, e.g., Definition 2 and Nesic & Teel (2004) for applications where general delays of this type such as different delays in the different components of x are used). Our ISS-LKF construction implies that (2) is ISS with respect to the actuator error $d(t)$. The ISS-LKF we construct is an explicit formula involving the known Lyapunov function for (1). Therefore, while the known results largely concern *constructing feedbacks* that achieve stability and with building LRFs (or sufficient conditions for stability), here we address the complementary problems of (a) quantifying

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the effects of introducing feedback delays and actuator errors on the stability performance of *given feedbacks* and (b) explicitly building corresponding ISS-LKFs (but see Section 3 for a comparison of our work with Fan & Arcak (2006), Pepe & Jiang (2006), and Teel (1998)).

In Section 2, we provide the relevant definitions and notation and state our main result. We provide a detailed comparison of our work with the known stability results in Section 3. We prove our main theorem in Section 4. We extend our results to cascades in Section 5. In Section 6, we apply our results to dynamics from identification theory and to other examples. We close in Section 7 with a brief concluding remark. We include some of the technical details and extensions in the appendices.

2 Preliminaries and statement of main result

2.1 Definitions and assumptions

Let $\mathcal{K}_\infty$ denote the set of all continuous, strictly increasing and unbounded functions $\rho : [0,\infty) \to [0,\infty)$ with $\rho(0) = 0$. Let $\mathcal{KL}$ denote the set of all continuous functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying (1) $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \geq 0$, (2) $\beta(s, \cdot)$ is non-increasing for each $s \geq 0$, and (3) $\beta(s,t) \to 0$ as $t \to +\infty$ for each $s \geq 0$. Given $\phi : \mathcal{I} \to \mathbb{R}^p$, defined on an interval $\mathcal{I}$, let $|\phi|_{\mathcal{I}}$ denote its (essential) supremum. Let $|\cdot|$ denote the usual Euclidean norm (or the induced matrix norm).

A function $V : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called *(uniformly) proper and positive definite* provided there are $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(|x|) \leq V(x,t) \leq \bar{\alpha}(|x|)$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. When we say V is C^1, we understand its partial derivatives at $t = 0$ to be one-sided derivatives. Let $\mathcal{C}_n(\mathcal{I})$ denote the set of all continuous $\mathbb{R}^n$-valued functions on any interval $\mathcal{I}$, with the locally uniform topology.

Consider a general delayed control system (2) with constant $\tau_i \in (0, \bar{\tau}]$, $f(0,t) \equiv 0$, $u(0,t) \equiv 0$, and $d \in \mathcal{L}_m^\infty([0,\infty))$(=all measurable essentially bounded functions $[0,\infty) \to \mathbb{R}^m$) being an *actuator error*. We refer to the constant vector $\tau \in (0,\bar{\tau}]^n$ as a vector *feedback (time) delay*. We stipulate $\bar{\tau}$ later, although in many cases of interest, the bound $\bar{\tau}$ can be raised to an arbitrary large positive constant (see Remarks 4 and A.4 below). Given constants $t_o \geq 0$, $\tau \in (0,\bar{\tau}]^n$, $x_o \in \mathcal{C}_n([t_o - \bar{\tau}, t_o])$, and $d \in \mathcal{L}_m^\infty([0,\infty))$, consider the initial value problem

$$\begin{aligned} &\dot{x}(t) = f(x(t),t) + g(x(t),t)[u(\xi_\tau(t),t) + d(t)] \\ &\forall t \geq t_o \text{ a.e. and } x(r) = x_o(r)\ \forall r \in [t_o - \bar{\tau}, t_o]\,. \end{aligned} \tag{IP}$$

The assumptions we make below imply that (2) has *globally well defined solutions*, i.e., for all choices of $t_o \geq 0$, τ, $x_o \in \mathcal{C}_n([t_o - \bar{\tau}, t_o])$, and $d \in \mathcal{L}_m^\infty([0,\infty))$, (IP) has a unique solution $t \mapsto x(t; t_o, x_o, d, \tau)$ defined on $[t_o - \bar{\tau}, +\infty)$ (see Assumption H and Appendix A.1 below). We then call x_o the *initial function* for $t \mapsto x(t; t_o, x_o, d, \tau)$. When (2) has globally well defined solutions, we denote $x(t; t_o, x_o, d, \tau)$ simply by $x(t)$ when t_o, x_o, d, and τ are clear. We extend the functions $x(t)$ to $\mathbb{R}$ by setting $x_o(t) = x_o(t_o - \bar{\tau})$ for $t \leq t_o - \bar{\tau}$.

The following generalize the ISS notions from Sontag (1989) and Sontag & Wang (1995) to delayed systems (see also, e.g., Pepe & Jiang (2006)). The first definition is unchanged if $|d|_{[t_o,t]}$ is replaced by $|d|_\infty$.

Definition 1 *Assume that (2) has globally well defined solutions. Given a vector delay $\tau \in (0,\bar{\tau}]^n$, we call (2)* (uniformly) input-to-state stable (ISS) *provided that there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that for all $d \in \mathcal{L}_m^\infty([0,\infty))$,*

$$|x(t; t_o, x_o, d, \tau)| \leq \beta(|x_o|_{[t_o - \bar{\tau}, t_o]}, t - t_o) + \gamma(|d|_{[t_o,t]})$$

for all $t_o \geq 0$, $x_o \in \mathcal{C}_n([t_o - \bar{\tau}, t_o])$, and $t \geq t_o$.

Let $\bar{\tau}$ be a positive real number and κ be a positive integer. For a given $t \geq 0$, $x_t(\cdot)$ denotes the restriction of $x(\cdot)$ to the interval $[t - \kappa\bar{\tau}, t]$ translated to $[-\kappa\bar{\tau}, 0]$, i.e., $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-\kappa\bar{\tau}, 0]$.

Definition 2 *A continuous functional $U : \mathcal{C}_n(\mathbb{R}) \times [0,\infty) \to [0,\infty)$ is called an* ISS Lyapunov-Krasovskii functional (ISS-LKF) *for (2) provided for all $\tau \in (0,\bar{\tau}]^n$ and all trajectories $x(t) := x(t; t_o, x_o, d, \tau)$ of (2), the function $t \mapsto U(x_t, t)$ is locally absolutely continuous and there exist functions $\alpha_i \in \mathcal{K}_\infty$ for $i = 1,2,3,4$ and $\kappa \in \mathbb{N}$ such that for all $\phi \in \mathcal{C}_n([-\kappa\bar{\tau}, 0])$, all trajectories $x(t)$ of (2), and all $t \geq t_o + \kappa\bar{\tau}$, we have (i) $\alpha_1(|\phi(0)|) \leq U(\phi, t) \leq \alpha_2(|\phi|_{[-\kappa\bar{\tau},0]})$ and (ii) the time derivative $D_t U(x_t, t)$ of $U(x_t, t)$ satisfies $D_t U(x_t, t) \leq -\alpha_3(U(x_t, t)) + \alpha_4(|d|_{[t_o,t]})$ a.e..*

Our assumptions will guarantee that if (2) admits an ISS-LKF, then it is ISS (see Appendix A.1 below).

2.2 Statement of main result

Set $F(x,t,u_s) = f(x,t) + g(x,t)u_s(x,t)$. Consider (2) with $d \in \mathcal{L}_m^\infty([0,\infty))$ an unknown disturbance, $\tau \in (0,\bar{\tau}]^n$, and $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$, $g : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^{n\times m}$ and $u_s : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ satisfying:

- H The function u_s is C^1 and $u_s(0,t) \equiv 0$, and f and g are locally Lipschitz. Also, there exist a $\sigma \in \mathcal{K}_\infty$ for which $\sigma(r) \leq r$ for all $r \geq 0$, a C^1 uniformly proper and positive definite $V : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$; positive constants $\bar{L}$ and K_1, and $K_i \geq 0$ $(i = 2,3,4)$ such that for all $x \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, and $l \geq 0$, we have:

H1 $V_t(x,l) + V_x(x,l)F(x,l,u_s) \leq -\sigma^2(\sqrt{n}|x|)$.

H2 $|V_x(x,l)g(x,l)| \leq K_1\sigma(|x|)$, $\left|\frac{\partial u_s}{\partial x}(x,l)\right| \leq \bar{L}$.

H3 $|f(x,l)|^2 \leq K_2\sigma^2(|x|)$, $|g(x,l)|^2 \leq K_3(\sigma(|x|) + 1)$.

H4 $[|g(x,l)||u_s(q,l)|]^2 \leq K_4[\sigma^2(|x|) + \sigma^2(|q|)]$.

Assumption H allows many cases where (1) is a stable linear system with bounded g (using a quadratic Lyapunov function), as well as cases where the closed loop system is not exponentially stable or g is unbounded (see Section 6.2 below). In what follows,

$$\bar{\tau} = \frac{0.49}{\bar{L}K_1\sqrt{2K_2 + 8K_4 + 0.25}}, \tag{3}$$

but see Remarks 4 and A.4 for larger delay bounds.

Theorem 3 *Under Assumption H and with $\bar{\tau}$ in (3), the feedback delayed system (2) with any $\tau \in (0, \bar{\tau}]^n$, in closed loop with the feedback u_s, admits the ISS-LKF*

$$U(x_t, t) = V(x(t), t) + \frac{1}{4\bar{\tau}} \int_{t-2\bar{\tau}}^{t} \left(\int_r^t \sigma^2(\sqrt{n}|x(l)|)\mathrm{d}l \right) \mathrm{d}r \tag{4}$$

and therefore is ISS.

Remark 4 *Often, $\bar{\tau}$ can be raised to an arbitrarily large positive constant. For example, if (2) satisfies Assumption H with $V_t \equiv 0$ and $f \equiv 0$, then for any constant $\eta \in (0,1)$, Assumption H also holds if we replace u_s, $\bar{L}$, σ, K_1, and K_3 with ηu_s, $\eta \bar{L}$, $\eta^{1/2}\sigma$, $K_1/\eta^{1/2}$, and $K_3/\eta^{1/2}$, respectively. Plugging into (3) and taking η arbitrarily small makes (3) arbitrarily large. Our control affineness assumption can be relaxed as well (see Remark 6 below).*

Remark 5 *Growth restrictions on f and g are required. Indeed, the following system, evolving on $\mathbb{R}^2$,*

$$\dot{x}_1(t) = -x_1(t) + x_1^4(t)x_2(t), \quad \dot{x}_2(t) = u(x(t-\tau)),$$

with τ being a nonnegative scalar delay, is globally asymptotically stabilized by the feedback $u(x) = -x_2 - x_1^5$ when $\tau = 0$, and $V(x) = |x|^2$ is a Lyapunov function for the corresponding closed-loop system. However, this system is not globally asymptotically stabilizable by any continuous $u(x(t-\tau))$ for any constant $\tau > 0$ (see (Mazenc & Bliman, 2006, Appendix 1)).

3 Comparison with the literature

Much of the delay systems literature uses (control) Lyapunov-Razumikhin functions (LRFs) (see Jankovic (2001)). ISS-LRFs were used by Teel (1998) to give sufficient conditions for nonlinear delayed systems to be ISS. Using LRFs, one can build stabilizing feedbacks for some control affine systems, and the stability of the closed loop dynamics enjoys some robustness relative to unmodeled dynamics (see Jankovic (2001)). LKF methods can be viewed as generalizations of Razumikhin methods and have been used to give sufficient conditions for ISS as well (see Pepe & Jiang (2006)). Michiels *et al.* (2002) uses Lyapunov techniques to study delayed partially linear systems. For delayed *linear* systems, stabilizing feedbacks can often be built using linear matrix inequalities (see, e.g., Fattouh *et al.* (2000), Niculescu (1998), and Tarbouriech & da Silva, Jr. (2000)).

Our approach differs from these earlier works in our explicit construction of LKFs and our use of ISS to quantify the effect of introducing feedback delays and actuator errors into a priori stable closed loop nonlinear dynamics. Moreover, we allow undelayed closed loop dynamics that are not necessarily exponentially stable, and our results lead to new stabilizing feedbacks that guarantee ISS of cascades (see Section 5 below). The results Fan & Arcak (2006) and Teel (1998) lead to an ISS small gain approach for proving robustness to feedback delays. However, Fan & Arcak (2006) and Teel (1998) do not include our results. For example, they do not explicitly construct ISS-LKFs. See also Section 6.2 for an example covered by our results which is beyond the scope of Fan & Arcak (2006).

The work Pepe & Jiang (2006) assumes the availability of a Lyapunov-like functional for the delayed dynamics, which we do not require here. The ISS-LKFs we construct are explicit expressions involving the available Lyapunov functions for the original undelayed dynamics. Our hypotheses correspond to cases where a feedback is known to stabilize an undelayed system but the state that is observed by the feedback in the implementation has a small unknown delay. It is desirable to find conditions under which the feedback continues to stabilize the system when feedback delays or actuator errors are introduced, which our work provides.

4 Proof of Theorem 3

In what follows, all inequalities and equalities should be understood to hold globally unless otherwise indicated. Consider the dynamics (2). From Assumption H1,

$$\begin{aligned}\dot{V} &= V_t(x(t),t) + V_x F(x(t),t,u_s) + V_x g(x(t),t)d(t) \\ &\quad + V_x g(x(t),t)[u_s(\xi_\tau(t),t) - u_s(x(t),t)] \\ &\le -\sigma^2(\sqrt{n}|x(t)|) + |V_x g(x(t),t)|\,|d(t)| \\ &\quad + |V_x g(x(t),t)|\,|u_s(\xi_\tau(t),t) - u_s(x(t),t)|\end{aligned}$$

along any trajectory $x(t) := x(t; t_o, x_o, d, \tau)$ of (2), where we omitted the argument $(x(t),t)$ of V_x. From Assumption H2 and the relation $wz \le w^2/(2\delta) + \delta z^2/2$ with $w = \sigma(\sqrt{n}|x|)$, and with $\delta = 8$ and then $\delta = 2$, we get

$$\begin{aligned}\dot{V} &\le -\sigma^2(\sqrt{n}|x(t)|) + K_1\sigma(|x(t)|)|d(t)| \\ &\quad + K_1\bar{L}\sigma(|x(t)|)|\xi_\tau(t) - x(t)| \\ &\le -\tfrac{5}{8}\sigma^2(\sqrt{n}|x(t)|) + 4K_1^2|d(t)|^2 \\ &\quad + K_1^2\bar{L}^2|\xi_\tau(t) - x(t)|^2.\end{aligned} \tag{5}$$

Jensen's Inequality and the fact that $\tau_i \le \bar{\tau}$ for all i give

$$\begin{aligned}|\xi_\tau(t) - x(t)|^2 &= \sum_{i=1}^n \left(\int_{t-\tau_i}^t \dot{x}_i(r)\mathrm{d}r \right)^2 \\ &\le \sum_{i=1}^n \tau_i \int_{t-\tau_i}^t \dot{x}_i^2(r)\mathrm{d}r \le \bar{\tau} \int_{t-\bar{\tau}}^t |\dot{x}(r)|^2\mathrm{d}r\end{aligned} \tag{6}$$

for each $t \ge t_o + 2\bar{\tau}$. From Assumptions H3-H4,

$$\begin{aligned}&|\dot{x}(r)|^2 \le 2\,|f|^2 + 2\,|g[u_s(\xi_\tau(r),r) + d(r)]|^2 \\ &\le 2K_2\sigma^2(|x(r)|) + 4K_4(\sigma^2(|x(r)|) + \sigma^2(|\xi_\tau(r)|)) \\ &\quad + 4K_3(\sigma(|x(r)|) + 1)|d(r)|^2\end{aligned} \tag{7}$$

where we omitted the arguments $(x(r),r)$ from f and g. By the relation $wz \le w^2/4 + z^2$ with $w = \sigma(|x(r)|)$ and $z = 4K_3|d(r)|^2$, we conclude from (7) that

$$\begin{aligned}&|\dot{x}(r)|^2 \le (2K_2 + 4K_4 + 0.25)\sigma^2(|x(r)|) \\ &+ 4K_4\sigma^2(|\xi_\tau(r)|) + 16K_3^2|d(r)|^4 + 4K_3|d(r)|^2 \\ &\le (2K_2 + 4K_4 + 0.25)\sigma^2(|x(r)|) + 16K_3^2|d(r)|^4 \\ &+ 4K_4\sigma^2(\sqrt{n}\max_i|x_i(r-\tau_i)|) + 4K_3|d(r)|^2.\end{aligned}$$

Set $\Theta(r) = 4K_3(4K_3+1)(r^4+r^2)$. Since $\tau_i \le \bar\tau$ for all i, it follows that for all $t \ge t_o + 2\bar\tau$, we get

$$\begin{aligned}&\int_{t-\bar\tau}^t |\dot x(r)|^2 \mathrm{d}r \le \int_{t-\bar\tau}^t (2K_2+4K_4+0.25)\sigma^2(|x(r)|)\mathrm{d}r\\ &\quad + \int_{t-\bar\tau}^t 4K_4\sigma^2(\sqrt{n}\max_i |x_i(r-\tau_i)|)\mathrm{d}r + \bar\tau\Theta(d_{[t_o,t]})\\ &\le \int_{t-\bar\tau}^t (2K_2+4K_4+0.25)\sigma^2(|x(r)|)\mathrm{d}r\\ &\quad +4K_4\int_{t-2\bar\tau}^t \sigma^2\left(\sqrt{n}|x(r)|\right)\mathrm{d}r + \bar\tau\Theta(d_{[t_o,t]})\\ &\le \int_{t-2\bar\tau}^t (2K_2+8K_4+0.25)\sigma^2(\sqrt{n}|x(r)|)\mathrm{d}r + \bar\tau\Theta(d_{[t_o,t]}).\end{aligned}$$

Combining this inequality, (5), and (6) yields

$$\begin{aligned}\dot V \le& -\tfrac{5}{8}\sigma^2(\sqrt{n}|x(t)|) + 4K_1^2|d(t)|^2 + \bar L^2K_1^2\bar\tau^2\Theta(d_{[t_o,t]})\\ &+\bar L^2K_1^2\bar\tau\int_{t-2\bar\tau}^t (2K_2+8K_4+0.25)\sigma^2(\sqrt{n}|x(r)|)\mathrm{d}r.\end{aligned}$$

Set $\mathcal{R}(\bar\tau) = \bar L^2K_1^2\bar\tau(2K_2+8K_4+0.25) - 1/(4\bar\tau)$ and $\alpha_4(r) = \bar L^2K_1^2\bar\tau^2\Theta(r) + 4K_1^2r^2$. Our bound (3) on $\bar\tau$ gives $\mathcal{R}(\bar\tau) \le -1/(128\bar\tau)$. By our choice of α_4, our bound on $\mathcal{R}(\bar\tau)$, and the fact (cf. Mazenc *et al.* (2006)) that

$$\begin{aligned}&\tfrac{d}{dt}\int_{t-2\bar\tau}^t\int_r^t \sigma^2(\sqrt{n}|x(l)|)\,\mathrm{d}l\,\mathrm{d}r\\ &\quad = 2\bar\tau\sigma^2(\sqrt{n}|x(t)|) - \int_{t-2\bar\tau}^t \sigma^2(\sqrt{n}|x(l)|)\mathrm{d}l,\end{aligned} \tag{8}$$

we deduce that when $t \ge t_o + 2\bar\tau$, U from (4) gives

$$\begin{aligned}\dot U \le& -\tfrac{5}{8}\sigma^2(\sqrt{n}|x|) + 4K_1^2|d|^2 + \tfrac{1}{2}\sigma^2(\sqrt{n}|x|)\\ &+\bar L^2K_1^2\bar\tau\int_{t-2\bar\tau}^t(2K_2+8K_4+\tfrac{1}{4})\sigma^2(\sqrt{n}|x(r)|)\mathrm{d}r\\ &+\bar L^2K_1^2\bar\tau^2\Theta(d_{[t_o,t]}) - \tfrac{1}{4\bar\tau}\int_{t-2\bar\tau}^t\sigma^2(\sqrt{n}|x(l)|)\mathrm{d}l\\ \le& -\tfrac{1}{8}\sigma^2(\sqrt{n}|x|) - \tfrac{1}{128\bar\tau}\int_{t-2\bar\tau}^t \sigma(\sqrt{n}|x(r)|)^2\mathrm{d}r\\ &+\alpha_4(|d|_{[t_o,t]}) \quad \forall t\ge t_o+2\bar\tau,\end{aligned} \tag{9}$$

where we omitted the argument t from x and d.

Since $\sigma \in \mathcal{K}_\infty$ and V is proper and positive definite, we can easily find $\gamma \in \mathcal{K}_\infty$ such that $\gamma(s) \le s$ for all $s \ge 0$ and, $\frac{1}{8}\sigma^2(\sqrt{n}|x|) \ge \gamma(V(x,t))$ for all x and t. Since $\gamma(s) \le s$ for all $s \ge 0$, the relation $\gamma(a+b) \le \gamma(2a) + 2b$ (for $a, b \ge 0$) with $a = V(x(t),t)/2$ and (9) give

$$\begin{aligned}\dot U \;\le&\; -\gamma\left(\tfrac{1}{2}\left\{V(x(t),t) + \tfrac{1}{128\bar\tau}\int_{t-2\bar\tau}^t\sigma(|x(l)|)^2\mathrm{d}l\right\}\right)\\ &+\alpha_4(|d|_{[t_o,t]})\\ \le&\; -\alpha_3(U(x_t,t)) + \alpha_4(|d|_{[t_o,t]}),\ \ \forall t\ge t_o+2\bar\tau,\end{aligned}$$

where $\alpha_3(s) := \gamma(s/\{128\bar\tau+2\})$. Hence (ii) from the ISS-LKF definition holds with $\kappa = 2$. To check (i) from the ISS-LKF definition with $\kappa = 2$, we first choose $\underline{\alpha}, \bar\alpha \in \mathcal{K}_\infty$ so that $\underline{\alpha}(|x|) \le V(x,t) \le \bar\alpha(|x|)$ everywhere. Given $\phi \in \mathcal{C}_n([-2\bar\tau,0])$ and $t \ge 0$, the function $s \mapsto x(s) := \phi(s-t)$ satisfies $x_t = \phi$. Hence,

$$\begin{aligned}&\alpha_1(|\phi(0)|) \;\le\; V(x(t),t) \;\le\; U(\phi,t)\\ &\le\; V(x(t),t) + \int_{t-2\bar\tau}^t\sigma^2(\sqrt{n}|x(l)|)\mathrm{d}l \;\le\; \alpha_2(|\phi|_{[-2\bar\tau,0]})\end{aligned}$$

where $\alpha_1 = \underline{\alpha}$ and $\alpha_2(s) = \bar\alpha(s) + 2\bar\tau\sigma^2(\sqrt{n}s)$. Thus, U is an ISS-LKF for (2). Hence, (2) is ISS by Remark A.2.

Remark 6 *We can extend Theorem 3 to cover systems that are not control affine, e.g.,* $\frac{d}{dt}x(t) = f(x(t),t) + g(x(t),t)\Sigma(u_s(\xi_\tau(t),t) + d(t))$ *where* $\Sigma : \mathbb{R}^m \to \mathbb{R}^m$ *is globally Lipschitz (e.g., saturation) if we change Assumption H1 to* $V_t(x,l) + V_x(x,l)[f(x,l) + g(x,l)\Sigma(u_s(x,l))] \le -\sigma(\sqrt{n}|x|)^2$*. The proof is as before except we add and subtract* $V_xg(x(t),t)\Sigma(u_s(x(t),t))$ *at the start to get* $D_tV \le -\sigma^2(\sqrt{n}|x(t)|) + \bar B|V_x(x,t)g(x(t),t)|\{|d(t)| + |u_s(\xi_\tau(t),t) - u_s(x(t),t)|\}$*, where* $\bar B$ *is a Lipschitz constant for* Σ*.*

5 Extension to cascades

We extend Theorem 3 to cascades of the form

$$\begin{aligned}&\dot x(t) = f(x(t)) + g(x(t))z(t),\\ &\dot z(t) = u(x(t-2\tau), x(t-\tau), z(t-\tau)) + d(t),\end{aligned} \tag{10}$$

evolving on $\mathbb{R}^n\times\mathbb{R}$ with $d \in \mathcal{L}_1^\infty([0,\infty)$ and constant delay $\tau > 0$. We assume that the x-dynamics (with fictitious input z) satisfies Assumption H for the case where the vector fields, feedback, and Lyapunov function are time independent. Let $u_s : \mathbb{R}^n \to \mathbb{R}^m$ and the constants $\bar L, K_1, K_2, \ldots, K_4$ be as in Assumption H. Mazenc & Bliman (2006) designed stabilizing controls for a family of delayed systems that includes (10), but those control laws do not yield ISS. Here we show that a suitable u renders (10) ISS, i.e., there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $t \ge t_o \ge 0$, all sufficiently small delays $\tau > 0$, all $d \in \mathcal{L}_1^\infty([0,\infty))$, and all trajectories $(x,z)(t) := (x(t;t_o,(x_o,z_o),d,\tau), z(t;t_o,(x_o,z_o),d,\tau))$ of (10),

$$|(x,z)(t)| \le \beta(|(x,z)|_{[t_o-\tau,t_o]}, t-t_o) + \gamma(|d|_{[t_o,t]}) \tag{11}$$

with the convention that the initial functions $(x_o, z_o) \in \mathcal{C}_{n+1}([t_o-2\tau, t_o])$ are constant on $[t_o-2\tau, t_o-\tau]$.

Corollary 7 *Let Assumption H (with time invariant* f*,* g*, and* u_s*) hold for* $\dot x = f(x) + g(x)u_s(x)$*. Set*

$$\bar\tau_c := \min\{1/\sqrt{8}, \bar\tau\},\ \ Z(t) := z(t) - u_s(x(t-\tau)), \tag{12}$$

where $\bar\tau$ *is defined in (3). Then for each constant delay* $\tau \in (0, \bar\tau_c]$*, the dynamics (10) is ISS when we choose*

$$\begin{aligned}&u(x(t-2\tau), x(t-\tau), z(t-\tau)) = -Z(t-\tau)+\\ &\tfrac{\partial u_s}{\partial x}(x(t-\tau))\left[f(x(t-\tau)) + g(x(t-\tau))z(t-\tau)\right].\end{aligned} \tag{13}$$

Let us sketch the proof of the preceding result. Set $q = (x, Z)$. Note that (13) transforms (10) into

$$\dot q(t) = F(q(t)) + G(q(t))[U_s(q(t-\tau)) + D(t)] \tag{14}$$

where $U_s(q) = (u_s(x), Z)^T$, $D(t) = (0, -d(t))^T$,

$$F(q) = \begin{pmatrix} f(x) + g(x)Z \\ 0 \end{pmatrix}, \ \text{ and }\ G(q) = \begin{pmatrix} g(x) & 0 \\ 0 & -1\end{pmatrix}.$$

The proof of Theorem 3 provides an ISS-LKF $U(x_t)$ and $\alpha_3 \in \mathcal{K}_\infty$ such that along the trajectories (14),

$$\begin{aligned}\dot U \le& -\alpha_3(U(x_t)) + 4K_1^2Z^2(t)\\ &+16\bar L^2K_1^2K_3\bar\tau\int_{t-\tau}^t\left[K_3Z^4(l) + Z^2(l)\right]\mathrm{d}l\end{aligned} \tag{15}$$

for all $t \ge t_0 + 2\tau$ and all $\tau \in (0, \bar\tau]$. The relation $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ and Jensen's Inequality give

$$\begin{aligned}&Z(t)[Z(t) - Z(t-\tau)] \;=\\ &-Z(t)\int_{t-\tau}^t Z(r-\tau)\mathrm{d}r + Z(t)\int_{t-\tau}^t d(r)\mathrm{d}r\\ &\le \tfrac{1}{2}Z^2(t) + \tfrac{\tau}{2}\int_{t-2\tau}^t Z^2(r)\mathrm{d}r + Z(t)\int_{t-\tau}^t d(r)\mathrm{d}r,\end{aligned} \tag{16}$$

since $\dot{Z}(t) = -Z(t-\tau) + d(t)$. Consider the functionals

$M(Z_t) = \frac{1}{2}Z(t)^2 + \tau \int_{t-2\tau}^{t} \int_{r}^{t} Z(l)^2 \mathrm{d}l\mathrm{d}r,$
$\Gamma(d,t) = 4(1+\tau)|d|^2_{[t_o,t]}, \quad Q_1(Z_t) = 4M(Z_t) + 4M^2(Z_t)$
and $Q_2(Z_t) = Q_1(Z_t) + \frac{1}{16}\int_{t-2\tau}^{t}\int_{r}^{t}(Z(l)^2 + Z(l)^4)\mathrm{d}l\mathrm{d}r.$

One can easily prove that the derivative of M along the trajectories of (14) for all $t \geq t_0 + 2\tau$ satisfies

$$\begin{aligned} \dot{M} &= Z(t)[Z(t) - Z(t-\tau)] - [1 - 2\tau^2]Z^2(t) \\ &\quad -\tau \int_{t-2\tau}^{t} Z^2(l)\mathrm{d}l + Z(t)d(t) \\ &\leq -\tfrac{1}{4}Z(t)^2 - \tfrac{\tau}{2}\int_{t-2\tau}^{t} Z(r)^2\mathrm{d}r + Z(t)d(t) \\ &\quad + Z(t)\int_{t-\tau}^{t} d(r)\mathrm{d}r \quad \leq \quad -M/4 + \Gamma(d,t), \end{aligned} \tag{17}$$

where we applied $ab \leq a^2/16 + 4b^2$ with $a = Z(t)$ twice, (16), $\tau \leq 1/\sqrt{8}$, and Jensen's Inequality.

From this and $ab \leq a^2/4 + b^2$ with $b = M$, one easily gets

$$\dot{Q}_1 \leq -\tfrac{1}{2}Z(t)^2 - \tfrac{1}{4}Z(t)^4 + 4\Gamma(d,t) + 16\Gamma^2(d,t) \tag{18}$$

for all $t \geq t_0 + 2\tau$. The estimate (18) easily gives

$$\begin{aligned} \dot{Q}_2 &\leq -\tfrac{1}{8}Z(t)^2 - \tfrac{1}{8}Z(t)^4 - \tfrac{1}{16}\int_{t-2\tau}^{t}(Z(l)^2 + Z(l)^4)\mathrm{d}l \\ &\quad + \alpha_u(|d|_{[t_0,t]}),\ \forall t \geq t_0 + 2\tau \text{ and } \tau \leq \bar{\tau}_c, \end{aligned}$$

where $\alpha_u(r) = 16(1+\tau^2)^2(r^2 + r^4)$. From (15), we can then determine a constant $C > 0$ and $\alpha_a \in \mathcal{K}_\infty$ such that the time derivative of $U_f(x_t, Z_t) = U(x_t) + CQ_2(Z_t)$ satisfies

$$\dot{U}_f \leq -\alpha_a(U_f(x_t, Z_t)) + C\alpha_u(|d|_{[t_0,t]}) \quad \forall t \geq t_0 + 2\tau.$$

By viewing Z as a disturbance in the x-dynamics, one can find $\Omega \in \mathcal{K}_\infty$ such that $|q(t)| \leq \Omega(|q|_{[t_0-\tau,t_0]}) + \Omega(|d|_{[t_0,t_0+2\tau]})$ for all $t \in [t_0 - \tau, t_0 + 2\tau]$ and trajectories $q(t)$ of (14); see Appendix A.1 for similar arguments. This gives (cf. Appendix A.1) an ISS estimate of the form

$$|q(t)|^2 \leq \tilde{\beta}\left(|q|^2_{[t_o-\tau,t_o]}, t - t_o\right) + \tilde{\gamma}(|d|_{[t_o,t]}) \tag{19}$$

for all $t \geq t_o$. Moreover, $|z(t)| \leq |Z(t)| + \bar{L}|x(t-\tau)|$, by our choice (12) of Z. Therefore, for all $t \geq t_o$, (19) gives

$$\begin{aligned} |(x,z)(t)|^2 &\leq 2\tilde{\beta}\left(|q|^2_{[t_o-\tau,t_o]}, t - t_o\right) \\ &\quad + 2\bar{L}^2\tilde{\beta}^c\left(|q|^2_{[t_o-\tau,t_o]}, \max\{0, t-\tau-t_o\}\right) \\ &\quad + 2\tilde{\gamma}(|d|_{[t_o,t]}) + 2\bar{L}^2\tilde{\gamma}(|d|_{[t_o,t]}), \quad \tilde{\beta}^c(s,r) := \tilde{\beta}(s,r) + se^{-r}. \end{aligned}$$

Substituting $|q|^2_{[t_o-\tau,t_o]} \leq 2(1+\bar{L}^2)|(x,z)|^2_{[t_o-\tau,t_o]}$ into the preceding estimate gives (11) with

$$\begin{aligned} \gamma(s) &:= \{4(1+\bar{L}^2)\tilde{\gamma}(s)\}^{1/2} \\ \beta(s,r) &:= \{4\tilde{\beta}(2\{1+\bar{L}^2\}s^2, r) \\ &\quad + 4\bar{L}^2\tilde{\beta}^c(2\{1+\bar{L}^2\}s^2, \max\{0, r-\tau\})\}^{1/2}. \end{aligned}$$

6 Illustrations

6.1 An example from identification theory

Consider a dynamics (cf. Mazenc *et al.* (2006) and several references contained therein)

$$\dot{x} = -m(t)m^T(t)u \tag{20}$$

in which $m : \mathbb{R} \to \mathbb{R}^n$ is continuous and satisfies $|m(t)| = 1$ for all $t \in \mathbb{R}$ and admits constants $\alpha' \in (0,1)$ and $\beta', \tilde{c} > 0$ such that

$$\alpha' I \leq \int_{t}^{t+\tilde{c}} m(\tau)m^T(\tau)\mathrm{d}\tau \leq \beta' I, \quad \forall t \in \mathbb{R}. \tag{21}$$

A Lyapunov function whose derivative along the trajectories of (20) is negative definite is not obvious; no time independent Lyapunov function has this property, because $m^T(t)x$ may be zero when $x \neq 0$. Fortunately, (Mazenc *et al.*, 2006, Lemma 12) gives:

Lemma 8 *Let the function $m(t)$ satisfy the preceding conditions. Then, with $\kappa = 1 + \frac{\tilde{c}}{2} + \frac{1}{4\alpha'}\tilde{c}^4$ and*

$$P(t) = \kappa I + \int_{t-\tilde{c}}^{t}\int_{s}^{t} m(l)m^T(l)\,\mathrm{d}l\,\mathrm{d}s, \tag{22}$$

the function $V(x,t) := x^T P(t)x$ satisfies $\dot{V} \leq -\alpha'|x|^2/2$ along all trajectories of $\dot{x}(t) = -m(t)m^T(t)x(t)$. Moreover, $|P(t)| \leq \kappa + \tilde{c}^2$ everywhere.

Using Lemma 8, one checks that Assumption H holds for $\dot{x}(t) = -m(t)m^T(t)[u(\xi_\tau(t)) + d(t)]$ with $u(x) = x$, $\sigma(r) = r(\alpha'/\{2n\})^{1/2}$, $K_1 = 2(2n/\alpha')^{1/2}(\kappa + \tilde{c}^2)$, $K_4 = 2n/\alpha'$, $\bar{L} = 1$, $K_3 = 1$, and $K_2 = 0$, so Theorem 3 gives:

Corollary 9 *Let the preceding hypotheses hold. Choose P as in (22) and $\bar{\tau}$ from (3). Then for all constants $\tau_i \in (0, \bar{\tau}]$, $\dot{x}(t) = -m(t)m^T(t)[\xi_\tau(t) + d(t)]$ has the ISS-LKF*

$$U(x_t, t) = x^T(t)P(t)x(t) + \tfrac{\alpha'}{8\bar{\tau}}\int_{t-2\bar{\tau}}^{t}\left(\int_{r}^{t}|x(l)|^2\mathrm{d}l\right)\mathrm{d}r$$

and therefore is ISS.

6.2 Further illustrations

Example 10 *One easily checks that $\dot{x} = ux^2/(1+x^2)$ with $x \in \mathbb{R}$ is not locally exponentially stabilizable by any C^1 feedback $u(x)$ (using e.g. (Khalil, 2002, Theorem 4.14, p.162)). However, it satisfies Assumption H with*

$$V(x) = \tfrac{1}{2}x^2, \quad \sigma(r) = r^2/\sqrt{1+r^2}, \quad u_s(x) = -x,$$

$\bar{L} = K_1 = 1$, $K_2 = 0$ and $K_4 = 2$. Hence, by Theorem 3,

$$\dot{x}(t) = -\tfrac{x^2(t)}{1+x^2(t)}[x(t-\tau) - d(t)] \tag{23}$$

is ISS when $0 < \tau \leq \bar{\tau} = 0.98/\sqrt{65}$. Moreover, Theorem 3 provides an explicit ISS-LKF. In fact, by scaling u_s, we can allow an arbitrarily large bound $\bar{\tau}$ (see Remark 4).

Example 11 *The system $\dot{x} = u(1+x^2)^{1/2}$ on $\mathbb{R}$ has an unbounded function g and satisfies Assumption H′ from Appendix A.2 below with $K_1 = K_4 = 1$,*

$$u_s(x) = -\int_0^x \tfrac{1}{\sqrt{1+l^2}}\mathrm{d}l, \ \sigma(r) = -u_s(r), \ V(x) = \tfrac{1}{2}u_s^2(x)$$

and $K_2 = 0$ and so is tractable by Lemma A.3:

$$\dot{x}(t) = \sqrt{1+x^2(t)}\left[-\int_0^{x(t-\tau)}\tfrac{1}{\sqrt{1+l^2}}\mathrm{d}l + d(t)\right]$$

is ISS when $0 < \tau \leq 1/8$. Moreover, we can again allow an arbitrarily large delay bound (see Remark A.4 below for details). Since g is unbounded, this example is not covered by Fan & Arcak (2006).

7 Conclusions

We gave new explicit constructions of ISS Lyapunov-Krasovskii functionals for delayed dynamics, in terms of given Lyapunov functions for the corresponding undelayed dynamics. This led to general conditions under which a time-varying delayless system with a stabilizing feedback remains ISS with respect to actuator errors when time delays are introduced into the feedback.

APPENDICES

A.1 Explicit ISS estimate

Throughout the appendices, we assume the following which automatically holds under our Assumption H:

A The functions f, g, and u are locally Lipschitz and there exists a constant $\bar{L} > 0$ such that for all $x \in \mathbb{R}^n$ and $t \geq 0$, we have (A1) $|f(x,t)| \leq \bar{L}|x|$, (A2) $|g(x,t)| \leq \bar{L}(|x|+1)$, and (A3) $|\partial u_s/\partial x(x,t)| \leq \bar{L}$.

Lemma A.1 *Let $\bar{\tau} > 0$ be a given constant. Then, for each $\kappa \in \mathbb{N}$ and $\tau \in (0,\bar{\tau}]^n$, there exists a $\bar{\gamma}_{\kappa,\tau} \in \mathcal{K}_\infty$ (depending on κ and τ) such that for all $t_o \geq 0$, $x_o \in \mathcal{C}_n([t_o - \bar{\tau}, t_o])$, and $d \in \mathcal{L}^\infty_m([0,\infty))$, the corresponding solution $t \mapsto x(t; t_o, x_o, d, \tau)$ of (IP) above satisfies*

$$|x(t; t_o, x_o, d, \tau)| \leq \bar{\gamma}_{\kappa,\tau}(|x_o|_{[t_o-\bar{\tau},t_o]}) + \bar{\gamma}_{\kappa,\tau}(|d|_{[t_o,t]})$$

for all $t \in [t_o, t_o + \kappa\bar{\tau}]$. In particular, we have globally well defined solutions.

To prove Lemma A.1, let $t_o \geq 0$, $\tau \in (0,\bar{\tau}]^n$, $x_o \in \mathcal{C}_n([t_o - \bar{\tau}, t_o])$, and $d \in \mathcal{L}^\infty_m([0,\infty))$ be given. Set $\hat{\tau} = \min_i \tau_i$. The existence of a unique maximal solution $x(t)$ of (IP) on some interval $[t_o - \bar{\tau}, t_o + \varepsilon)$ for some $\varepsilon > 0$ follows from the classical theory applied with $x_o(t)$ and $d(t)$ viewed as the inputs. Moreover, Assumption A gives $|\dot{x}(t)| \leq |x(t)|\{\bar{L} + \bar{L}^2|\xi_\tau(t)| + \bar{L}|d(t)|\} + \bar{L}^2|\xi_\tau(t)| + \bar{L}|d(t)|$ for all $t \geq t_o$ for which $x(t)$ is defined. Integrating up to any such $t \in [t_o, t_o + \hat{\tau}]$, applying Gronwall's Inequality, and setting $\mathcal{D} := \bar{L}^2|x_o|_{[t_o-\bar{\tau},t_o]} + \bar{L}|d|_{[t_o,t]}$ gives

$$|x(t)| \leq \{|x(t_o)| + \bar{t}\mathcal{D}\}\, e^{\bar{t}(\bar{L}+\bar{L}^2|x_o|_{[t_o-\bar{\tau},t_o]}+\bar{L}|d|_{[t_o,t]})}$$

on all intervals $[t_o, t_o + \bar{t}] \subseteq [t_o, t_o + \hat{\tau}]$ for which the solution is defined. A standard maximality and local existence argument now easily allows us to conclude that $x(t)$ is uniquely defined at least on $[t_o, t_o + \hat{\tau}]$. Taking $\bar{t} = \hat{\tau}$ in the above estimate and using $e^{a+b} \leq e^{2a} + e^{2b} - 1$ and $ab \leq a^2 + b^2$ for $a, b \geq 0$ provides a $\gamma_1 \in \mathcal{K}_\infty$ such that

$$\begin{aligned}
|x(t)| \leq\ & \left[\tilde{L}|x_o|_{[t_o-\bar{\tau},t_o]} + \hat{\tau}\bar{L}|d|_{[t_o,t]}\right] e^{\bar{L}\hat{\tau}} \\
& \times \left\{e^{2\bar{L}^2\hat{\tau}|x_o|_{[t_o-\bar{\tau},t_o]}} + e^{2\bar{L}\hat{\tau}|d|_{[t_o,t]}} - 1\right\} \\
\leq\ & e^{\bar{L}\bar{\tau}}\left\{\tilde{L}|x_o|_{[t_o-\bar{\tau},t_o]} e^{2\bar{L}^2\bar{\tau}|x_o|_{[t_o-\bar{\tau},t_o]}}\right. \\
& + (\tilde{L}|x_o|_{[t_o-\bar{\tau},t_o]})^2 + (e^{2\bar{L}^2\bar{\tau}|x_o|_{[t_o-\bar{\tau},t_o]}} - 1)^2 \\
& + \left(e^{2\bar{L}\tau|d|_{[t_o,t]}} - 1\right)^2 + (\bar{\tau}\bar{L}|d|_{[t_o,t]})^2 \\
& \left.+ \bar{\tau}\bar{L}|d|_{[t_o,t]} e^{2\bar{L}\bar{\tau}|d|_{[t_o,t]}}\right\} \\
\leq\ & \gamma_1(|x_o|_{[t_o-\bar{\tau},t_o]}) + \gamma_1(|d|_{[t_o,t]})
\end{aligned} \tag{A.1}$$

for all $t \in [t_o, t_o + \hat{\tau}]$, where $\tilde{L} = 1 + \bar{\tau}\bar{L}^2$. Repeating this argument on $[t_o + \hat{\tau}, t_o + 2\hat{\tau}]$ gives a $\gamma_2 \in \mathcal{K}_\infty$ such that

$$|x(t)| \ \leq\ \gamma_2(|x|_{[t_o,t_o+\hat{\tau}]}) + \gamma_2(|d|_{[t_o,t]}) \tag{A.2}$$

for all $t \in [t_o + \hat{\tau}, t_o + 2\hat{\tau}]$. Taking the supremum on the left side of (A.1) over $[t_o, t_o + \hat{\tau}]$, substituting into (A.2), and using the relation $\gamma_2(a+b) \leq \gamma_2(2a) + \gamma_2(2b)$ for $a, b \geq 0$ gives $|x(t)| \leq \gamma_3(|x_o|_{[t_o-\bar{\tau},t_o]}) + \gamma_3(|d|_{[t_o,t]})$ for all $t \in [t_o, t_o + 2\hat{\tau}]$, where $\gamma_3(s) = \gamma_2(s) + \gamma_2(2\gamma_1(s))$. Lemma A.1 now follows by the obvious inductive argument.

Corollary A.2 *If (2) satisfies Assumption A and admits an ISS-LKF, then it is ISS.*

To prove this corollary, let U and the α_is be as in Definition 2. Take any trajectory $x(t)$ of (2). Then $s \mapsto \chi(s) := \alpha_3^{-1}(2\alpha_4(s))$ satisfies

$$U(x_t, t) \geq \chi(|d|_\infty) \ \Rightarrow\ D_t U(x_t, t) \leq -\alpha_3(U(x_t,t))/2$$

for all $t \geq t_o + \kappa\bar{\tau}$. Also, $\mathcal{S} := \{t \geq t_o + \kappa\bar{\tau} : U(x_t,t) \geq \chi(|d|_\infty)\}$ is a (possibly empty) interval which we denote by $\mathcal{I}$. Hence, either (a) the left endpoint of $\mathcal{I}$ is $t_o + \kappa\bar{\tau}$ or (b) $U(x_t,t) \leq \chi(|d|_\infty)$ for all $t \geq t_o + \kappa\bar{\tau}$ (by the argument from Sontag (1989)).

If case (a) occurs, then a standard comparison argument applied to $\mathcal{I} \ni t \mapsto U(x_t,t)$ and (i) in the ISS-LKF definition (with $\phi(r) = x(t+r)$ and then $\phi(r) = x(t_o + \kappa\bar{\tau} + r)$ to get an upper bound) provide a $\beta^a \in \mathcal{KL}$ so that $U(x_t,t) \geq \chi(|d|_\infty) \Rightarrow |x(t)| \leq \beta^a(|x|_{[t_o,t_o+\kappa\bar{\tau}]}, t - t_o)$ for all $t \geq t_o + \kappa\bar{\tau}$, which gives $|x(t)| \leq \beta^a(|x|_{[t_o,t_o+\kappa\bar{\tau}]}, t - t_o) + \gamma^c(|d|_{[t_o,t]})$ for all $t \geq t_o + \kappa\bar{\tau}$, where $\gamma^c := \alpha_1^{-1} \circ \chi$. Hence, for all trajectories $x(t)$, $|x(t)| \leq \beta^c(|x|_{[t_o,t_o+\kappa\bar{\tau}]}, t - t_o) + \gamma^c(|d|_{[t_o,t]})$ for all $t \geq t_o$, where $\beta^c(s,r) = \beta^a(s,r) + se^{\kappa\bar{\tau}-r}$. Lemma A.1 provides a $\gamma_{\kappa,\tau} \in \mathcal{K}_\infty$ so that $|x(t)| \leq \gamma_{\kappa,\tau}(|x_o|_{[t_o-\bar{\tau},t_o]}) + \gamma_{\kappa,\tau}(|d|_{[t_o,t]})$ for all $t \in [t_o, t_o + \kappa\bar{\tau}]$. Since $\beta^c(a+b, r) \leq \beta^c(2a, r) + \beta^c(2b, 0)$ for all $a, b, r \geq 0$, we can satisfy the ISS requirements with $\beta(s,r) := \beta^c(2\gamma_{\kappa,\tau}(s), r)$ and $\gamma(r) := \gamma^c(r) + \beta^c(2\gamma_{\kappa,\tau}(r), 0)$.

A.2 The case of equal delays

When $\tau_1 = \tau_2 = \ldots = \tau_n$, we can relax Assumption H to the following, which we call Assumption H′: *The function $u_s \in C^1$. Also, there exist a $\sigma \in \mathcal{K}_\infty$ for which $\sigma(r) \leq r$ for all $r \geq 0$, a C^1 uniformly proper and positive definite $V : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$, and constants $K_1 > 0$ and $K_i \geq 0$ ($i = 2,3,4$) such that for all $x \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, $l \geq 0$, and $t \geq 0$: (H1′) $V_t(x,t) + V_x(x,t)[f(x,t) + g(x,t)u_s(x,t)] \leq -\sigma^2(|x|)$; (H2′) $|V_x(x,t)g(x,t)| \leq K_1\sigma(|x|)$; (H3′) $|\partial u_s/\partial x(x,t)f(x,l)|^2 \leq K_2\sigma^2(|x|)$ and $|\partial u_s/\partial x(x,t)g(x,l)|^2 \leq K_3(\sigma(|x|)+1)$; and (H4′) $[|\partial u_s/\partial x(x,t)g(x,l)|\,|u_s(q,l)|]^2 \leq K_4[\sigma^2(|x|) + \sigma^2(|q|)]$.* Notice the use of both l and t in Assumptions (H3′)-(H4′). Example 11 above satisfies H′ but violates Assumption H4. We now use the delay bound

$$\bar{\tau} = \frac{1}{4K_1\sqrt{3K_2 + 3K_4 + 1}}, \tag{A.3}$$

but see Remark A.4 for results with larger bounds.

Lemma A.3 *Under Assumptions A and H′ with* $\tau := \tau_1 = \tau_2 = \ldots = \tau_n \in (0, \bar{\tau}]$ *constant, the delayed system (2) in closed loop with the feedback* $u = u_s$ *admits the ISS-LKF*

$$U(x_t, t) = V(x(t), t) + \tfrac{1}{8\bar{\tau}} \int_{t-2\tau}^{t} \left(\int_r^t \sigma^2(|x(l)|)\mathrm{d}l \right) \mathrm{d}r$$

and therefore is ISS.

Let us prove Lemma A.3. Our dynamics have globally well defined solutions, by Lemma A.1. By adding and subtracting $V_x g(x(t), t) u_s(x(t), t)$, (H1′) gives

$$\begin{aligned} \dot{V} \ \le\ & -\sigma(|x(t)|)^2 + |V_x g(x(t), t)|\,|d(t)| \\ & + |V_x g(x(t), t)|\,|u_s(x(t-\tau), t) - u_s(x(t), t)| \end{aligned}$$

along any trajectory $x(t) := x(t; t_o, x_o, d, \tau)$ of the dynamics, where we omitted the argument $(x(t), t)$ of V_x. From (H2′) and $wz \le \frac{1}{4}w^2 + z^2$ with $w = \sigma(|x|)$,

$$\begin{aligned} \dot{V} \ \le\ & -\sigma(|x(t)|)^2 + K_1\sigma(|x(t)|)|d(t)| \\ & + K_1\sigma(|x(t)|)|u_s(x(t-\tau), t) - u_s(x(t), t)| \\ \le\ & -\tfrac{1}{2}\sigma^2(|x(t)|) + K_1^2|d(t)|^2 \\ & + K_1^2|u_s(x(t-\tau), t) - u_s(x(t), t)|^2. \end{aligned} \tag{A.4}$$

By the Fundamental Theorem of Calculus applied to $[t-\tau, t] \ni l \mapsto u_s(x(l), t)$ and Jensen's inequality,

$$\begin{aligned} & |u_s(x(t-\tau), t) - u_s(x(t), t)|^2 \le \\ & \tau \int_{t-\tau}^{t} T_t(x(l), x(l-\tau), d(l), l)\mathrm{d}l, \ \text{where} \end{aligned} \tag{A.5}$$

$$T_t(a, b, d, l) = \left[\tfrac{\partial u_s}{\partial x}(a, t)\left\{f(a, l) + g(a, l)(u_s(b, l) + d)\right\}\right]^2.$$

By the Cauchy Inequality, we get

$$\begin{aligned} & T_t(a, b, d, l) \le 3 \left[\left|\tfrac{\partial u_s}{\partial x}(a, t) f(a, l)\right|^2 \right. \\ & \left. + \left|\tfrac{\partial u_s}{\partial x}(a, t) g(a, l)\right|^2 |u_s(b, l)|^2 + \left|\tfrac{\partial u_s}{\partial x}(a, t) g(a, l)\right|^2 |d|^2 \right]. \end{aligned}$$

From (H3′)-(H4′) and the relation $wz \le w^2 + z^2/4$, we get $T_t(x(l), x(l-\tau), d(l), l) \le 3K_3^2|d(l)|^4 + 3K_3|d(l)|^2 + (3K_2 + 3K_4 + 1)\sigma(|x(l)|)^2 + 3K_4\sigma(|x(l-\tau)|)^2$, where we took $z = 3K_3|d(l)|^2$. Hence, when $t \ge t_o + 2\tau$,

$$\begin{aligned} & |u_s(x(t-\tau), t) - u_s(x(t), t)|^2 \\ & \le \tau \int_{t-\tau}^{t} \left[(3K_2 + 3K_4 + 1)\sigma(|x(l)|)^2 \right. \\ & \quad \left. + 3K_4\sigma(|x(l-\tau)|)^2\right] \mathrm{d}l \\ & \quad + \tau \int_{t-\tau}^{t} \left[3K_3^2|d(l)|^4 + 3K_3|d(l)|^2\right] \mathrm{d}l \\ & \le \tau(3K_2 + 3K_4 + 1) \int_{t-2\tau}^{t} \sigma(|x(l)|)^2 \mathrm{d}l \\ & \quad + 3\tau K_3 \int_{t-\tau}^{t} \left[K_3|d(l)|^4 + |d(l)|^2\right] \mathrm{d}l. \end{aligned} \tag{A.6}$$

Substituting (A.6) into (A.4), and arguing analogously to the proof of (8), we deduce that when $t \ge t_o + 2\tau$,

$$\begin{aligned} \dot{U} \ \le\ & \tfrac{1}{4\bar{\tau}}\tau\sigma^2(|x(t)|) - \tfrac{1}{8\bar{\tau}} \int_{t-2\tau}^{t} \sigma^2(|x(l)|)\mathrm{d}l - \tfrac{1}{2}\sigma(|x|)^2 \\ & + \tau K_1^2(3K_2 + 3K_4 + 1) \int_{t-2\tau}^{t} \sigma(|x(l)|)^2\mathrm{d}l + \alpha_4(|d|_{[t_o, t]}) \\ \le\ & -\tfrac{1}{4}\sigma(|x(t)|)^2 + \left(-\tfrac{1}{8\bar{\tau}} + \bar{\tau}K_1^2(3K_2 + 3K_4 + 1)\right) \\ & \times \int_{t-2\tau}^{t} \sigma(|x(l)|)^2\mathrm{d}l + \alpha_4(|d|_{[t_o, t]}) \end{aligned}$$

along all of the trajectories of (2), where $\alpha_4(r) = 3\bar{\tau}^2K_1^2K_3(K_3 + 1)(r^4 + r^2) + K_1^2r^2$. The rest of the proof is essentially the same as in the proof of Theorem 3.

Remark A.4 *Our bound (A.3) can sometimes be raised to an arbitrarily large positive constant. To show why, first note that if Assumption H′ holds with* $V_t \equiv 0$ *and* $f \equiv 0$*, and if* $\eta \in (0, 1)$ *is any fixed constant, then Assumption H′ also holds with* u_s, σ, K_1, *and* K_4 *replaced by* u_s^η, $\eta^{1/2}\sigma$, $K_1/\eta^{1/2}$, *and* η^3K_4 *respectively, and* $K_2 = 0$*, with the feedback* $u_s^\eta(x, t) := \eta u_s(x, t)$*. Next, note that applying* $wz \le K_4w^2 + z^2/(4K_4)$ *with* $z = 3K_3|d(l)|^2$ *and arguing in almost the same way as in the proof of Lemma A.3 gives* $T_t(x(l), x(l-\tau), d(l), l) \le (3K_2 + 4K_4)\sigma^2(|x(l)|) + 3K_4\sigma^2(|x(t-\tau)|) + 3K_3^2|d(l)|^4/K_4 + 3K_3|d(l)|^2$*. A slight variant of the rest of the proof of Lemma A.3 then gives ISS of (2) for all* $0 < \tau \le \bar{\tau}_c := 1/\{4K_1\sqrt{3K_2 + 4K_4}\}$.

Combining the preceding observations (with $K_2 = 0$*), we conclude that if (2) satisfies H′ with* $V_t \equiv 0$, $f \equiv 0$*, and equal delays, then* $u = u_s^\eta := \eta u_s$ *renders the system* $\dot{x}(t) = g(x(t), t)[u(x(t-\tau), t) + d(t)]$ *ISS as long as*

$$0 \ <\ \tau \ \le\ \frac{1}{4\frac{K_1}{\eta^{1/2}}\sqrt{4\eta^3K_4}} \ =\ \frac{1}{8\eta K_1\sqrt{K_4}} \ =:\ \bar{\tau}_\eta.$$

Since $\bar{\tau}_\eta \to +\infty$ *as* $\eta \to 0$*, we get ISS with arbitrarily large delay bounds* $\bar{\tau} = \bar{\tau}_\eta$*, if* u_s *is properly selected. Moreover, we can find explicit ISS-LKFs for all* $\tau > 0$.

References

Fan, X., & Arcak, M. (2006). Delay robustness of a class of nonlinear systems and applications to communication networks. *IEEE Transactions on Automatic Control*, *51*(1), 139-144.

Fattouh, A., Sename, O., & Dion, J.-M. (2000). A LMI approach to robust observer design for linear time-delay systems. In *Proceedings of the 39th IEEE Conference on Decision and Control* (pp. 1495-1500), Sydney, Australia.

Jankovic, M. (2001). Control-Lyapunov-Razumikhin functions and robust stabilization of time delay systems. *IEEE Transactions on Automatic Control*, *46*(7), 1048-1060.

Karafyllis, I. (2006). Lyapunov theorems for systems described by retarded functional differential equations. *Nonlinear Analysis: Theory Methods and Applications*, *64*(3), 590-617.

Khalil, H. (2002). *Nonlinear Systems* (3rd ed.). Upper Saddle River, NJ: Prentice Hall.

Mazenc, F., & Bliman, P. (2006). Backstepping design for time-delay nonlinear systems. *IEEE Transactions on Automatic Control*, *51*(1), 149-154.

Mazenc, F., Malisoff, M., & de Queiroz, M. (2006). Further results on strict Lyapunov functions for rapidly time-varying nonlinear systems. *Automatica*, *42*(10), 1663-1671.

Michiels, W., Sepulchre, R., Roose, D., & Moreau, L. (2002). A perturbation approach to the stabilization of nonlinear cascade systems with time-delay. In *Proceedings of the 41st IEEE Conference on Decision and Control* (pp. 5782-5787), Las Vegas, NV.

Nesic, D., & Teel, A. (2004). Input-to-state stability of networked control systems. *Automatica*, *40*, 2121-2128.

Niculescu, S.-I. (1998). $\mathcal{H}_\infty$ memoryless control and an α-stability constraint for time-delay systems: An LMI approach. *IEEE Transactions on Automatic Control*, *43*(5), 739-743.

Pepe, P., & Jiang, Z.-P. (2006). A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems and Control Letters*, *55*(12), 1006–1014.

Sontag, E.D. (1989). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, *34*(4), 435-443.

Sontag, E.D. (2006). Input-to-State Stability: Basic concepts and results. *Nonlinear and Optimal Control Theory* (pp. 163-220). Berlin: Springer.

Sontag, E.D., & Wang, Y. (1995). On characterizations of the Input-to-State Stability property. *Systems and Control Letters*, *24*(5), 351-359.

Tarbouriech, S., & da Silva, Jr., J. (2000). Synthesis of controllers for continuous time delay systems with saturating controls via LMI's. *IEEE Transactions on Automatic Control*, *45*(1), 105-111.

Teel, A. (1998). Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Transactions on Automatic Control*, *43*(7), 960-964.