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## On moment conditions for the Girsanov Theorem

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ON MOMENT CONDITIONS FOR THE GIRSANOV THEOREM

A Dissertation

Submitted to the Graduate Faculty of the  
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Agricultural and Mechanical College  
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in

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# Abstract

In this dissertation, the well-known Girsanov Theorem will be proved under a set of moment conditions on exponential processes. Our conditions are motivated by the desire to avoid using the local martingale theory in the proof of the Girsanov Theorem. Namely, we will only use the martingale theory to prove the Girsanov Theorem. Many sufficient conditions for the validity of the Girsanov Theorem have been found since the publication of the result by Girsanov [7] in 1960. We will compare our conditions with some of these sufficient conditions. As an application of the Girsanov Theorem, we will show the nonexistence of an arbitrage in a market and will also explain a simplified version of Black-Scholes model.

# Chapter 1

## Introduction

The main result in this dissertation is to show the validity of the Girsanov Theorem under a new condition in terms of moments. Under this new condition, we do not need to use local martingale theory. In many applications, e.g., the Black-Scholes model, this new condition is enough.

Let  $B(t)$  be a Brownian motion in a probability space  $(\Omega, \mathcal{F}, P)$  and let  $\{\mathcal{F}_t; a \leq t \leq b\}$  be a filtration such that  $B(t)$  is  $\mathcal{F}_t$ -measurable for each  $t$  and for any  $s \leq t$ , the random variable  $B(t) - B(s)$  is independent of the  $\sigma$ -field  $\mathcal{F}_s$ . We denote by  $\mathcal{L}_{ad}(\Omega, L^2[a, b])$  the space of all stochastic processes  $h(t, \omega)$ ,  $a \leq t \leq b$ ,  $\omega \in \Omega$  such that  $h(t)$  is  $\mathcal{F}_t$ -adapted and  $\int_a^b |h(t)|^2 dt < \infty$  almost surely. Also we denote by  $L_{ad}^2([a, b] \times \Omega)$  the space of all stochastic processes  $h(t, \omega)$ ,  $a \leq t \leq b$ ,  $\omega \in \Omega$  such that  $h(t)$  is  $\mathcal{F}_t$ -adapted and  $\int_a^b E|h(t)|^2 dt < \infty$ . It is a fact that  $L_{ad}^2([a, b] \times \Omega) \subset \mathcal{L}_{ad}(\Omega, L^2[a, b])$ . An exponential process  $\mathcal{E}_h(t)$ ,  $a \leq t \leq b$ , given by  $h(t) \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$  is a stochastic process defined by

$$\mathcal{E}_h(t) = e^{\int_a^t h(s) dB(s) - \frac{1}{2} \int_a^t h(s)^2 ds}, \quad a \leq t \leq b.$$

Then the Girsanov Theorem states that if the exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$  given by  $h(t) \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  is a martingale, then the process given by

$$W(t) = B(t) - \int_0^t h(s) ds, \quad 0 \leq t \leq T,$$

is a Brownian motion with respect to the probability measure  $Q$  given by  $dQ = \mathcal{E}_h(T) dP$ . An exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$  given by  $h(t) \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  is a martingale if and only if  $E[\mathcal{E}_h(T)] = 1$ . So the Girsanov Theorem is true if  $E[\mathcal{E}_h(T)] = 1$ .

As can be seen in the statement, the Girsanov Theorem is true for stochastic processes  $h(t) \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  satisfying a certain condition. For our result, we restrict the discussion to processes  $h(t)$  in  $L^2_{ad}([a, b] \times \Omega)$ . It is a fact that the stochastic integral  $\int_a^t h(s) dB(s)$  for  $h \in L^2_{ad}([a, b] \times \Omega)$  is a martingale. With the assumption of the new moment conditions, the proof of the Girsanov Theorem is now elementary. The idea behind these new conditions is to make some of the stochastic integrals that appear in the proof to be martingales.

Since Girsanov [7] published his result in 1960, many results in finding a sufficient condition for the validity of the Girsanov Theorem have been found. We will compare some of these sufficient conditions for  $h \in L^2_{ad}([a, b] \times \Omega)$  with our condition in Chapter 4.

In the theory of finance, an arbitrage in a market is regarded as a portfolio that can generate a profit without any risk of losing money. This situation contradicts the real life situation. One of the applications of the Girsanov Theorem is in showing the nonexistence of an arbitrage in a market. In Chapter 5, by using the “new” Girsanov Theorem, we will show the nonexistence of an arbitrage in a market. We will also explain a simplified version of Black-Scholes model, a model that determines the formula for pricing option calls.

# Chapter 2

## Background from Probability Theory

In this chapter, we review some basic ideas from probability theory which will be needed in this dissertation.

### 2.1 Stochastic Processes and Brownian Motion

**Definition 2.1.** A *stochastic process* is a collection  $X = \{X(t, \omega); t \in T, \omega \in \Omega\}$  of random variables defined on a probability space  $(\Omega, \mathcal{F}, P)$  with index set  $T$ .

*Remark 2.2.* A stochastic process can also be regarded as a measurable function  $X(t, \omega)$  defined on the product space  $[0, \infty) \times \Omega$ . In particular,

1. for fixed  $t$ ,  $X(t, \cdot)$  is a random variable;
2. for fixed  $\omega$ ,  $X(\cdot, \omega)$  is a function of  $t$ .

If there is no confusion, we denote  $X(t, \omega)$  by  $X(t)$  or  $X_t$ .

*Remark 2.3.* Usually the set  $T$  represents “time”. In the continuous case, it is an interval of  $\mathbb{R}$ , while in the discrete case, it is a subset of  $\mathbb{N}$ . However the set  $T$  does not necessarily denote the time.

*Example 2.4.* Let  $X_1, X_2, \dots, X_n, \dots$  be independent and identically distributed random variables and let  $S_n = X_1 + X_2 + \dots + X_n$ . Then the sequence  $\{S_n\}$  is a *discrete time* stochastic process.

*Example 2.5.* Let  $T = [t_0, \infty)$ , where  $t_0$  is a real number. For every partition

$$t_0 < t_1 < \dots < t_n, \quad t_i \in T, \quad i = 1, 2, \dots, n,$$



if  $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$  are independent random variables for all possible choices of partitions described above, then  $\{X_t, t \in T\}$  is a stochastic process with *independent increments*.

*Example 2.6.* Let  $X_t, t = 0, 1, 2, \dots$  denote the rock component (e.g., lignite, shale, sandstone, siltstone) of the  $t$ th layer of a rock. This is a discrete stochastic process and here,  $t$  is a space variable.

*Example 2.7.* In the fluctuation problem of electron-photon cascade, let  $X_t^e$  denote the number of particles with the energy value less than  $e$  at an arbitrary thickness  $t$  of the absorber. This is a continuous stochastic process. In this case,  $t$  does not represent time.

Now let's look at the concept of "sameness" between two processes under a probability measure  $P$ .

**Definition 2.8.** Two stochastic processes  $X(t)$  and  $Y(t)$  are *equivalent* if  $P_X = P_Y$ , where  $P_X$  and  $P_Y$  are the distributions for  $X$  and  $Y$ , respectively.

*Example 2.9.* Consider the set  $\Omega_1 = \{a, b, c, d\}$  with uniform probability. Define the random variables  $X$  and  $Y$  such that  $X(a) = X(b) = 1, X(c) = X(d) = -1$  and  $Y(a) = Y(c) = 1, Y(b) = Y(d) = -1$ . Then  $X$  and  $Y$  are two different random variables on the same probability space with the same distribution. Thus they are equivalent.

*Example 2.10.* Consider the sets  $\Omega_1 = \{a, b, c, d\}$  and  $\Omega_2 = \{e, f\}$ , both with uniform probabilities. Define the random variables  $X$  and  $Z$  such that  $X(a) = X(b) = 1, X(c) = X(d) = -1$  and  $Z(e) = 1, Z(f) = -1$ . Then  $X$  and  $Z$  have the same distribution (thus are equivalent), but they arise from different probability spaces.

**Definition 2.11.** A stochastic process  $Y(t)$  is a *version* of a stochastic process  $X(t)$  if  $P\{X(t) = Y(t)\} = 1$  for all  $t$ .

*Remark 2.12.* Two equivalent processes  $X(t)$  and  $Y(t)$  may have different probability spaces, whereas two versions of a process must be defined on the same probability space.

*Remark 2.13.* Two processes  $X(t)$  and  $Y(t)$  which are versions of each other are equivalent, but the converse is not true (Example 2.10).

**Definition 2.14.** A stochastic process  $Y(t)$  is a *realization* of  $X(t)$  if there is a probability space  $\Omega_0$  with  $P(\Omega_0) = 1$  such that for all  $\omega \in \Omega_0$ ,  $X(t, \omega) = Y(t, \omega)$  for all  $t$ , that is

$$P\{\omega; X(\omega, t) = Y(\omega, t), \text{ for all } t \geq 0\} = 1.$$

*Remark 2.15.* A realization is a version, but not conversely. However, a continuous version is a realization.

*Example 2.16.* Define the random variable  $X(t) \equiv 0$  for any  $(t, \omega) \in [0, 1] \times [0, 1]$ .

For fixed  $t \in [0, 1]$ , define  $Y(t) = \begin{cases} 0 & \text{if } t \neq \omega; \\ 1 & \text{if } t = \omega. \end{cases}$

Then  $Y(t)$  is a version of  $X(t)$  because for any  $t \in [0, 1]$ ,

$$P\{Y(t) = X(t)\} = P\{t \neq \omega\} = 1 - P\{t = \omega\} = 1.$$

On the other hand,  $P\{Y(t) = X(t) ; \text{ for any } t \in [0, 1]\} = 0$ . So  $Y(t)$  is not a realization of  $X(t)$ .

A famous example of a stochastic process is Brownian motion.

**Definition 2.17.** A stochastic process  $B(t, \omega)$  is called a *Brownian motion* if it satisfies the following conditions:

1.  $P\{\omega; B(0, \omega) = 0\} = 1$ .
2. For any  $0 \leq s < t$ , the random variable  $B(t) - B(s)$  is normally distributed with mean zero and variance  $t - s$ , i.e., for any  $a < b$ ,

$$P\{a \leq B(t) - B(s) \leq b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{x^2}{2(t-s)}} dx.$$

3.  $B(t, \omega)$  has independent increments, i.e., for any  $0 \leq t_1 < t_2 < \dots < t_n$ , the random variables  $B(t_1), B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1})$  are independent.
4. Almost all sample paths of  $B(t, \omega)$  are continuous functions, i.e.,

$$P\{\omega; B(\cdot, \omega) \text{ is a continuous function of } t\} = 1.$$

One way of constructing a Brownian motion is based on the following theorem by Kolmogorov. Let  $\mathbb{R}^{[0, \infty)}$  denote the space of all real valued functions  $f$  defined on the interval  $[0, \infty)$ . Let  $\mathcal{F}$  be the  $\sigma$ -field generated by sets of the form

$$\{f \in \mathbb{R}^{[0, \infty)}; (f(t_1), \dots, f(t_n)) \in A\},$$

where  $0 \leq t_1 < t_2 < \dots < t_n$  and  $A \in \mathcal{B}(\mathbb{R}_n)$ . These sets are called *cylinder sets*.

**Theorem 2.18.** (Kolmogorov's Extension Theorem) *Suppose that associated with each  $0 \leq t_1 < t_2 < \dots < t_n$ ,  $n \geq 1$ , is a probability measure  $\mu_{t_1, \dots, t_n}$  on  $\mathbb{R}^n$ . Assume that the family*

$$\{\mu_{t_1, \dots, t_n}; 0 \leq t_1 < \dots < t_{n-1} < t_n, n = 1, 2, 3, \dots\}$$

*satisfies the consistency condition*

$$\mu_{t_1, \dots, t_{i-1}, \hat{t}_i, t_{i+1}, \dots, t_n}(A_1 \times A_2) = \mu_{t_1, \dots, t_n}(A_1 \times \mathbb{R} \times A_2),$$

where  $1 \leq i \leq n$ ,  $A_1 \in \mathcal{B}(\mathbb{R}^{i-1})$ ,  $A_2 \in \mathcal{B}(\mathbb{R}^{n-i})$  and  $\widehat{t}_i$  means that  $t_i$  is deleted.

Then there exists a probability measure  $P$  on  $(\mathbb{R}^{[0,\infty)}, \mathcal{F})$  such that

$$P\{f \in \mathbb{R}^{[0,\infty)}; (f(t_1), \dots, f(t_n)) \in A\} = \mu_{t_1, \dots, t_n}(A)$$

for any  $0 \leq t_1 < t_2 < \dots < t_n$ ,  $A \in \mathcal{B}(\mathbb{R}^n)$  and  $n \geq 1$ .

With this theorem, the stochastic process  $X(t)$  defined by

$$X(t, \omega) = \omega(t), \quad \omega \in \mathbb{R}^{[0,\infty)},$$

can be shown to be a Brownian motion. For more on the construction of a Brownian motion, see [14].

*Example 2.19.* Let  $X_{\delta, h}(t)$  be a random walk with jumps  $h$  and  $-h$  equally likely at times  $\delta, 2\delta, 3\delta, \dots$ . Assume that  $h^2 = \delta$ , then for each  $t \geq 0$ , the limit

$$B(t) = \lim_{\delta \rightarrow 0} X_{\delta, \sqrt{\delta}}(t),$$

is a Brownian motion.

*Example 2.20.* Let  $C[0, 1]$  be the Banach space of real-valued continuous functions  $x$  on  $[0, 1]$  with  $x(0) = 0$  and the norm given by  $\|x\|_\infty = \sup_{0 \leq t \leq 1} |x(t)|$ . Consider the mapping  $\mu$  given by

$$\mu(A) = \int_U \prod_{i=1}^n \left( \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp \left[ -\frac{(u_i - u_{i-1})^2}{2(t_i - t_{i-1})} \right] \right) du_1 \dots du_n,$$

where  $U \in \mathcal{B}(\mathbb{R}^n)$ , the Borel  $\sigma$ -field of  $\mathbb{R}^n$  and  $A$  is a set (called a *cylinder set*) of the form

$$A = \{x \in C[0, 1]; (x(t_1), x(t_2), \dots, x(t_n)) \in U\},$$

where  $0 < t_1 < t_2 < \dots < t_n \leq 1$ . Then  $(C[0, 1], \mathcal{B}(C[0, 1]), \mu)$  is a probability space and the stochastic process defined by

$$B(t, x) = x(t), \quad 0 \leq t \leq 1, \quad x \in C[0, 1],$$

is a Brownian motion, a construction due to Nobert Wiener.

Two important properties of a Brownian motion are listed below.

**Theorem 2.21.** *Let  $B(t)$  be a Brownian motion. Then for any  $s, t \geq 0$ , we have  $E[B(s)B(t)] = \min\{s, t\}$ .*

Using this theorem and the definition of Brownian motion, we see that a stochastic process  $X(t)$ ,  $t \geq 0$  which is normally distributed with mean zero and variance  $t$  and satisfying  $E[X(s)X(t)] = \min\{s, t\}$ , is a Brownian motion.

**Theorem 2.22.** *The path of a Brownian motion is nowhere differentiable almost surely.*

## 2.2 Absolute Continuity and Equivalence of Probability Measures

**Definition 2.23.** Let  $(\Omega, \mathcal{F})$  be a measurable space. A probability measure  $Q$  is *absolutely continuous* with respect to a probability measure  $P$  if  $P(A) = 0$  implies  $Q(A) = 0$ , for any  $A \in \mathcal{F}$ . We denote this by  $Q \ll P$ .

*Example 2.24.* Let  $X$  be a nonnegative random variable on  $(\Omega, \mathcal{F}, P)$  such that  $\int_{\Omega} X dP = 1$ . Define  $Q : \Omega \rightarrow [0, \infty)$  by

$$Q(A) = \int_A X dP, \quad A \in \mathcal{F}.$$

Then  $Q(\Omega) = 1$  and  $Q(A) \geq 0$  for all  $A \in \mathcal{F}$ . If  $A_1, A_2, \dots$  are disjoint sets in  $\mathcal{F}$ , then

$$\begin{aligned} Q\left(\bigcup_{n=1}^{\infty} A_n\right) &= \lim_{n \rightarrow \infty} Q\left(\bigcup_{i=1}^n A_i\right) = \lim_{n \rightarrow \infty} \int_{\Omega} \sum_{i=1}^n 1_{A_i} X dP \\ &= \sum_{n=1}^{\infty} \int_{A_n} X dP \\ &= \sum_{n=1}^{\infty} Q(A_n). \end{aligned}$$

Thus  $Q$  is a probability measure on  $(\Omega, \mathcal{F})$ . If  $P(A) = 0$ , then  $Q(A) = \int_A X dP = 0$ . In fact, since  $1_A X = 0$  on  $A^c$  (the complement of  $A$ ), it follows that  $1_A X = 0$   $P$ -almost surely. It is a fact in analysis that if a measurable function  $f = 0$   $\mu$ -almost everywhere, then  $\int_{\Omega} f d\mu = 0$ . Thus we get  $\int_A X dP = \int_{\Omega} 1_A X dP = 0$ . So  $Q$  is absolutely continuous to  $P$ .

**Definition 2.25.** Two measures  $P$  and  $Q$  are *equivalent* if  $P$  and  $Q$  are absolutely continuous with respect to each other, namely  $P \ll Q$  and  $Q \ll P$ . We denote this by  $P \sim Q$  (or  $Q \sim P$ ).

## 2.3 Conditional Expectation

**Definition 2.26.** Let  $X$  be an integrable random variable in a probability space  $(\Omega, \mathcal{F}, P)$  and let  $\mathcal{G} \subset \mathcal{F}$  be a sub- $\sigma$ -field of  $\mathcal{F}$ . The *conditional expectation of  $X$  given  $\mathcal{G}$*  is the unique random variable  $Y$  such that

1.  $Y$  is  $\mathcal{G}$ -measurable.
2.  $\int_G Y dP = \int_G X dP$  for all  $G \in \mathcal{G}$ .

We usually write  $Y = E[X|\mathcal{G}]$ .

*Remark 2.27.* The existence and uniqueness of the conditional expectation is guaranteed by the *Radon-Nikodym Theorem*.

**Theorem 2.28.** (Radon-Nikodym Theorem) *Suppose  $(\Omega, \mathcal{F}, P)$  is a probability space. Let  $\mu$  be a signed measure (namely  $\mu : \Omega \rightarrow [-\infty, \infty]$  is a  $\sigma$ -additive function on  $(\Omega, \mathcal{F})$  such that  $\mu(\phi) = 0$  for null set  $\phi$ ) such that  $\mu$  is absolutely continuous with respect to  $P$ . Then there exists a unique integrable function  $f$  such that*

$$\mu(A) = \int_A f dP, \quad A \in \mathcal{F}.$$

*Remark 2.29.* The function  $f$  is called the *density* or the *Radon-Nikodym derivative* of  $\mu$  with respect to  $P$ . We write  $f = \frac{d\mu}{dP}$ .

The following are some simple properties of conditional expectation.

**Theorem 2.30.** *Let  $(\Omega, \mathcal{F}, P)$  be a probability space,  $\mathcal{G}$  is a sub- $\sigma$ -field of  $\mathcal{F}$  and  $X \in L^1(\Omega, \mathcal{F})$ . Then each of the following hold almost surely:*

- (a)  $E(E[X|\mathcal{G}]) = EX$ .
- (b) If  $X$  is  $\mathcal{G}$ -measurable, then  $E[X|\mathcal{G}] = X$ .
- (c) If  $X$  and  $\mathcal{G}$  are independent, then  $E[X|\mathcal{G}] = EX$ .
- (d) If  $Y$  is  $\mathcal{G}$ -measurable and  $E|XY| < \infty$ , then  $E[XY|\mathcal{G}] = YE[X|\mathcal{G}]$ .
- (e) If  $\mathcal{H}$  is a sub- $\sigma$ -field of  $\mathcal{G}$ , then  $E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$ .
- (f) Let  $\varphi$  be a convex function on  $\mathbb{R}$  and suppose that  $\varphi(X)$  is integrable with respect to  $P$ . Then  $\varphi(E[X|\mathcal{G}]) \leq E[\varphi(X)|\mathcal{G}]$ .
- (g) Let  $X_n \geq 0$ ,  $X_n \in L^1(\Omega)$ ,  $n = 1, 2, \dots$ , and assume that  $\liminf_{n \rightarrow \infty} X_n \in L^1(\Omega)$ . Then  $E\left[\liminf_{n \rightarrow \infty} X_n \middle| \mathcal{G}\right] \leq \liminf_{n \rightarrow \infty} E[X_n|\mathcal{G}]$ .

## 2.4 Martingales

One of the important properties of a Brownian motion is the martingale property.

In this section, we define the concepts of martingales and local martingales.

**Definition 2.31.** Let  $T$  be either  $\mathbb{Z}_+$  (the set of positive integers) or an interval in  $\mathbb{R}$ . A *filtration* on  $T$  is an increasing family  $\{\mathcal{F}_t : t \in T\}$  of  $\sigma$ -fields. A stochastic process  $\{X_t; t \in T\}$  is said to be *adapted* to the filtration  $\{\mathcal{F}_t : t \in T\}$  if for each  $t$ , the random variable  $X_t$  is  $\mathcal{F}_t$ -measurable.

*Remark 2.32.* We always assume that all  $\sigma$ -fields  $\mathcal{F}_t$  are *complete*, namely if  $A \in \mathcal{F}_t$  and  $P(A) = 0$ , then  $B \in \mathcal{F}_t$  for any subset  $B$  of  $A$ .

**Definition 2.33.** For a filtration  $\{\mathcal{F}_t : t \in T\}$  on a probability space  $(\Omega, \mathcal{F}, P)$ , we define  $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$  for any  $t \in T$ . We say that the filtration  $\{\mathcal{F}_t : t \in T\}$  is *right continuous* if  $\mathcal{F}_{t+} = \mathcal{F}_t$  for every  $t \in T$ . In particular, if  $t \in [a, b]$ , a filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$  is said to be *right continuous* if  $\mathcal{F}_t = \bigcap_{n=1}^{\infty} \mathcal{F}_{t+\frac{1}{n}}$  for all  $t \in [a, b)$ , where by convention  $\mathcal{F}_t = \mathcal{F}_b$  when  $t > b$ .

**Definition 2.34.** Let  $X_t$  be a stochastic process adapted to a filtration  $\{\mathcal{F}_t : t \in T\}$  and  $E|X_t| < \infty$  for all  $t \in T$ . Then  $X_t$  is called a *martingale* with respect to  $\{\mathcal{F}_t\}$  if for any  $s \leq t$  in  $T$ ,

$$E\{X_t | \mathcal{F}_s\} = X_s, \quad \text{almost surely.} \quad (2.1)$$

*Remark 2.35.* If the filtration is not explicitly specified, then the filtration  $\{\mathcal{F}_t\}$  is understood to be the one given by  $\mathcal{F}_t = \sigma\{X_s; s \leq t\}$ .

*Remark 2.36.* If the equality in Equation 2.1 is replaced by  $\geq$  (or  $\leq$ ), then  $X_t$  is called a *submartingale* (or *supermartingale*) with respect to  $\{\mathcal{F}_t\}$ .

*Example 2.37.* A Brownian motion  $B(t)$  is a martingale. In fact, for  $s < t$ ,

$$\begin{aligned} E[B(t) | \mathcal{F}_s] &= E[(B(t) - B(s)) + B(s) | \mathcal{F}_s] \\ &= E[B(t) - B(s) | \mathcal{F}_s] + E[B(s) | \mathcal{F}_s] \\ &= E[B(t) - B(s)] + B(s) \\ &= B(s), \end{aligned}$$

where we had used properties 2 and 3 of Definition 2.17 to get the last two equalities.



*Example 2.38.* For a Brownian motion  $B(t)$ , the process  $B(t)^2$  is a submartingale.

In fact, for  $s < t$ ,

$$\begin{aligned}
E[B(t)^2 | \mathcal{F}_s] &= E[\{(B(t) - B(s)) + B(s)\}^2 | \mathcal{F}_s] \\
&= E[(B(t) - B(s))^2 | \mathcal{F}_s] + 2B(s)E[B(t) - B(s) | \mathcal{F}_s] \\
&\quad + E[B(s)^2 | \mathcal{F}_s] \\
&= E[(B(t) - B(s))^2] + 2B(s)E[B(t) - B(s)] + B(s)^2 \\
&= (t - s) + 0 + B(s)^2 \\
&> B(s)^2.
\end{aligned} \tag{2.2}$$

From Equation 2.2 we can see that the process  $B(t)^2 - t$  is a martingale.

**Definition 2.39.** A random variable  $\tau : \Omega \rightarrow [a, b]$  is a *stopping time* with respect to the filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$  if  $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$  for all  $t \in [a, b]$ , i.e., the set  $\{\tau \leq t\}$  is  $\mathcal{F}_t$ -measurable.

*Remark 2.40.* The  $b$  in the above definition is allowed to be  $\infty$ .

*Remark 2.41.* In the case of discrete  $t$ , the requirement in Definition 2.39 is equivalent to  $\{\tau = t\} \in \mathcal{F}_t$  because  $\{\tau = t\} = \{\tau \leq t\} - \{\tau \leq t - 1\}$  and  $\{\tau \leq t\} = \bigcup_{k=a}^t \{\tau = k\}$ .

Given a right continuous filtration, we have the following characterization of a stopping time.

**Theorem 2.42.** Let  $\{\mathcal{F}_t; a \leq t \leq b\}$  be a right continuous filtration. The random variable  $\tau : \Omega \rightarrow [a, b]$  is a stopping time with respect to  $\{\mathcal{F}_t\}$  if and only if  $\{\omega; \tau(\omega) < t\} \in \mathcal{F}_t$  for all  $t \in [a, b]$ .

*Remark 2.43.* A random variable  $\tau$  is a stopping time if and only if  $\{\omega; \tau(\omega) > t\} \in \mathcal{F}_t$  for all  $t \in [a, b]$ .

*Example 2.44.* If  $\tau \equiv c \in [a, b]$ , then  $\tau$  is a stopping time because  $\{\tau = n\}$  is either an empty set  $\phi$  or  $\Omega$  for any  $n$ ,  $1 \leq n < \infty$ .

*Example 2.45.* Let  $\{X_t\}$  be a sequence of  $\mathcal{F}_t$ -adapted random variables defined on a probability space  $(\Omega, \mathcal{F}, P)$  with filtration  $\{\mathcal{F}_t\}$ . For  $A \in \mathcal{B}(\mathbb{R})$ , define

$$\tau(\omega) = \begin{cases} \inf\{t; X_t(\omega) \in A\}, & \text{for } t \geq 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Then  $\tau$  is a stopping time since for any finite  $t \geq 1$ ,

$$\{\tau = t\} = \{X_1 \notin A, X_2 \notin A, \dots, X_{t-1} \notin A, X_t \in A\} \in \mathcal{F}_t,$$

and for  $t = \infty$ ,

$$\{\tau = \infty\} = \Omega - \{\tau < \infty\} \in \mathcal{F}_t.$$

**Definition 2.46.** An  $\mathcal{F}_t$ -adapted stochastic process  $X_t$ ,  $a \leq t \leq b$  is called a *local martingale* with respect to  $\{\mathcal{F}_t\}$  if there exists a sequence of stopping times  $\tau_n, n = 1, 2, \dots$ , such that

1.  $\tau_n$  increases monotonically to  $b$  almost surely as  $n \rightarrow \infty$ ;
2. For each  $n$ ,  $X_{t \wedge \tau_n}$  is a martingale with respect to  $\{\mathcal{F}_t; a \leq t \leq b\}$ .

*Remark 2.47.* A martingale is a local martingale (let  $\tau_n = b$  for all  $n$ ). However the converse is not true. For an example of a local martingale which is not a martingale, refer to [22](page 37), [11](page 168) or Example 3.18 below.

*Example 2.48.* Since a Brownian motion  $B(t)$  is a martingale, by the above remark, it is a local martingale.

A cornerstone result in martingale theory is the *Doob-Meyer decomposition Theorem*. This theorem states that under certain conditions, a submartingale  $X(t)$

with respect to a right continuous filtration  $\{\mathcal{F}_t\}$  can be decomposed as a sum of a martingale  $M(t)$  and an increasing process  $A(t)$ , i.e.,

$$X(t) = M(t) + A(t). \quad (2.3)$$

For details, see [11].

**Definition 2.49.** The process  $A(t)$  in Equation 2.3 is called the *compensator* of  $X(t)$ .

*Example 2.50.* The compensator of  $B(t)^2$  for a Brownian motion  $B(t)$  is  $t$  since

$$B(t)^2 = (B(t)^2 - t) + t,$$

and  $B(t)^2$  and  $B(t)^2 - t$  are submartingale and martingale respectively, by Example 2.38.

## 2.5 Some Inequalities

We end this chapter with a discussion of some inequalities that may be needed in this dissertation.

**Theorem 2.51.** (Hölder's inequality)

(a) (Analysis version) *Let  $(X, \mathcal{B}, \mu)$  be a measure space and let  $f$  and  $g$  be two measurable functions on  $X$  such that  $|f|, |g| < \infty$  almost everywhere on  $X$ .*

*Then for any  $p, q \in (1, \infty)$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , we have*

$$\int_X |fg| d\mu \leq \left[ \int_X |f|^p d\mu \right]^{\frac{1}{p}} \left[ \int_X |g|^q d\mu \right]^{\frac{1}{q}}.$$

(b) (Probability version) *Let  $X$  and  $Y$  be two random variables in a probability space  $(\Omega, \mathcal{F}, P)$  such that  $E|X|^p < \infty$  and  $E|Y|^q < \infty$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ .*

*Then*

$$E|XY| \leq (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}.$$

When  $p = q = 2$  in the Hölder's inequality, we have the following celebrated inequality.

**Theorem 2.52.** (Schwarz's inequality)

(a) (Analysis version or Integral form) *Let  $(X, \mathcal{B}, \mu)$  be a measure space with  $f$  and  $g$  two measurable functions on  $X$  such that  $|f|, |g| < \infty$  almost everywhere on  $X$ . Then*

$$\int_X |fg| d\mu \leq \left[ \int_X |f|^2 d\mu \right]^{\frac{1}{2}} \left[ \int_X |g|^2 d\mu \right]^{\frac{1}{2}}.$$

(b) (Probability version or Expectation form) *Suppose  $X$  and  $Y$  are random variables with finite variances in a probability space  $(\Omega, \mathcal{F}, P)$ . Then*

$$E|XY| \leq \sqrt{E(X^2)E(Y^2)}.$$

**Theorem 2.53.** (Jensen's inequality)

(a) (Analysis version) *Let  $(X, \mathcal{B}, \mu)$  be a measure space. Let  $g$  be a real valued  $\mathcal{B}$ -measurable and  $\mu$ -integrable function on a set  $A \in \mathcal{B}$  with  $\mu(A) \in (0, \infty)$ . If  $f$  is a convex function on an open interval  $I$  in  $\mathbb{R}$  and if  $g(A) \subset I$ , then*

$$f\left(\frac{1}{\mu(A)} \int_A g d\mu\right) \leq \frac{1}{\mu(A)} \int_A (f \circ g) d\mu,$$

*where  $f \circ g$  denotes the composition of  $f$  and  $g$ .*

(b) (Probability version) *Let  $X$  be a random variable on a probability space  $(\Omega, \mathcal{F}, P)$ . Let  $f$  be a convex function on  $\mathbb{R}$ , and suppose that  $X$  and  $f(X)$  are integrable. Then*

$$f(EX) \leq E(f(X)).$$

# Chapter 3

## Stochastic Integrals

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and let  $B(t)$  be a Brownian motion with respect to  $P$ . In this chapter, we discuss integrals of the form

$$\int_a^b f(t) dB(t)$$

where  $f$  is in certain classes of functions. For each class of functions, some properties of the integral will be given. For a more detail discussion on these stochastic integrals, the reader can refer to [14].

### 3.1 Wiener Integral

Let  $f$  be a real-valued square integrable function on  $[a, b]$ , i.e.,  $f \in L^2[a, b]$ . Then the integral

$$\int_a^b f(t) dB(t, \omega), \quad f \in L^2[a, b],$$

is called a *Wiener integral*.

The integrals  $\int_0^1 e^t dB(t)$ ,  $\int_0^1 t \sin(\frac{1}{t}) dB(t)$  and  $\int_0^1 t dB(t)$  are examples of Wiener integrals.

*Remark 3.1.* Let  $C[0, 1]$  be the set of real-valued continuous functions  $x(t)$  on the interval  $[0, 1]$  with  $x(0) = 0$ . The integral on  $C[0, 1]$  with respect to the Wiener measure  $w$  in  $C[0, 1]$  is called a Wiener integral. The Wiener measure  $w$  is defined by

$$w(I) = \frac{1}{\sqrt{(2\pi)^n t_1(t_2 - t_1) \cdots (t_n - t_{n-1})}} \int_E \exp \left[ -\frac{1}{2} \left( \frac{u_1^2}{t_1} + \frac{(u_2 - u_1)^2}{t_2 - t_1} + \cdots + \frac{(u_n - u_{n-1})^2}{t_n - t_{n-1}} \right) \right] du_1 du_2 \cdots du_n,$$

where  $E$  is a Borel subset of  $\mathbb{R}^n$  and  $I$  is the cylinder set  $I = \{x \in C[0, 1] : (x(t_1), \dots, x(t_n)) \in E\}$  for  $0 < t_1 < t_2 < \dots < t_n \leq 1$  (see [25]).

Let  $L^2(\Omega)$  denote the Hilbert space of square integrable real-valued random variables on  $\Omega$  with inner product  $\langle X, Y \rangle = E(XY)$ . We outline the construction of the Wiener integral  $\int_a^b f(t) dB(t, \omega)$ .

**Step 1.** *f is a step function*

Suppose  $f$  is a step function given by

$$f = \sum_{i=1}^n a_i 1_{[t_{i-1}, t_i)},$$

where  $a = t_0 < t_i < \dots < t_n = b$ . Define

$$I_{step}(f) = \sum_{i=1}^n a_i (B(t_i) - B(t_{i-1})).$$

Then  $I_{step}$  is linear and the random variable  $I_{step}(f)$  is Gaussian with mean zero and variance  $E[\{I_{step}(f)\}^2] = \int_a^b f(t)^2 dt$ .

**Step 2.** *f ∈ L<sup>2</sup>[a, b]*

Choose a sequence  $\{f_n\}_{n=1}^\infty$  of step functions such that  $f_n$  approaches  $f$  in  $L^2[a, b]$ . The sequence  $\{I_{step}(f_n)\}_{n=1}^\infty$  is Cauchy in  $L^2(\Omega)$ , hence it is convergent in  $L^2(\Omega)$ . We set

$$I(f) = \lim_{n \rightarrow \infty} I_{step}(f_n) \quad \text{in } L^2(\Omega),$$

and write

$$I(f)(\omega) = \left( \int_a^b f(t) dB(t) \right) (\omega), \quad \omega \in \Omega, \quad \text{almost surely.}$$

This  $I(f)$  is the Wiener integral. We also denote it by  $\int_a^b f(t) dB(t, \omega)$  or just  $\int_a^b f(t) dB(t)$ .

**Theorem 3.2.** *For each  $f \in L^2[a, b]$ , the Wiener integral  $\int_a^b f(t) dB(t)$  is a Gaussian random variable with mean zero and variance  $\|f\|_{L^2[a, b]}^2 = \int_a^b |f(t)|^2 dt$ .*

*Example 3.3.* The (Wiener) integral  $\int_0^1 t^2 dB(t)$  is a Gaussian random variable with mean zero and variance  $\int_0^1 t^4 dt = \frac{1}{5}$ .

It is easy to check that  $I : L^2[a, b] \rightarrow L^2(\Omega)$  is a linear transformation, whence we have the following:

**Corollary 3.4.** *If  $f, g \in L^2[a, b]$ , then*

$$E[I(f)I(g)] = \int_a^b f(t)g(t) dt.$$

## 3.2 Itô Integral

Suppose  $B(t)$  is a Brownian motion, and let  $\{\mathcal{F}_t; a \leq t \leq b\}$  be a filtration such that

- (a) for each  $t$ ,  $B(t)$  is  $\mathcal{F}_t$ -measurable,
- (b) for any  $s \leq t$ , the random variable  $B(t) - B(s)$  is independent of the  $\sigma$ -field  $\mathcal{F}_s$ .

Let  $L_{ad}^2([a, b] \times \Omega)$  denote the space of all stochastic processes  $f(t, \omega)$ ,  $a \leq t \leq b$ ,  $\omega \in \Omega$ , satisfying

- (i)  $f(t)$  is adapted to the filtration  $\{\mathcal{F}_t\}$ ;
- (ii)  $\int_a^b E|f(t)|^2 dt < \infty$ .

*Remark 3.5.* Note that by *Fubini's theorem* (see [2]), condition (ii) can also be written as  $E \int_a^b |f(t)|^2 dt < \infty$ .

The stochastic integral

$$\int_a^b f(t, \omega) dB(t, \omega), \quad f \in L_{ad}^2([a, b] \times \Omega)$$

is called an *Itô integral*. For convenience, we suppress the  $\omega$  and we just write  $\int_a^b f(t) dB(t)$ . Before presenting some examples, let us consider the construction of the Itô integral.

**Step 1.** *f is a step stochastic process in  $L_{ad}^2([a, b] \times \Omega)$*

Suppose  $f$  is a step stochastic process given by

$$f(t, \omega) = \sum_{i=1}^n \xi_{i-1}(\omega) 1_{[t_{i-1}, t_i)}(t),$$

where  $\xi_{i-1}$  is  $\mathcal{F}_{i-1}$ -measurable and  $E[(\xi_{i-1})^2] < \infty$ . Define

$$I_{step}(f) = \sum_{i=1}^n \xi_{i-1} (B(t_i) - B(t_{i-1})).$$

Then  $I_{step}$  is linear,  $E[I_{step}(f)] = 0$  and

$$E[|I_{step}(f)|^2] = \int_a^b E|f(t)|^2 dt.$$

**Step 2.** *Approximation of  $f \in L_{ad}^2([a, b] \times \Omega)$  by step processes*

Suppose  $f \in L_{ad}^2([a, b] \times \Omega)$ . Then there exists a sequence  $\{f_n(t); n \geq 1\}$  of step stochastic processes in  $L_{ad}^2([a, b] \times \Omega)$  such that

$$\lim_{n \rightarrow \infty} \int_a^b E[|f(t) - f_n(t)|^2] dt = 0,$$

i.e.,  $f_n \rightarrow f$  in  $L_{ad}^2([a, b] \times \Omega)$ .

**Step 3.**  *$f \in L_{ad}^2([a, b] \times \Omega)$*

By Steps 1 and 2, there exists a sequence  $\{f_n(t, \omega); n \geq 1\}$  of adapted step stochastic processes such that

$$\lim_{n, m \rightarrow \infty} E(|I_{step}(f_n) - I_{step}(f_m)|^2) = 0.$$



Hence the sequence  $\{I_{step}(f_n)\}$  is Cauchy in  $L^2(\Omega)$ . For  $f \in L^2_{ad}([a, b] \times \Omega)$ , define

$$I(f) = \lim_{n \rightarrow \infty} I_{step}(f_n), \quad \text{in } L^2(\Omega).$$

Then denote  $I(f, \omega) = \int_a^b f(t, \omega) dB(t, \omega)$  for  $f \in L^2_{ad}([a, b] \times \Omega)$ .

*Remark 3.6.* For a deterministic function  $f(t)$ , the Itô integral  $\int_a^b f(t) dB(t, \omega)$  agrees with the Wiener integral defined in section 3.1.

*Example 3.7.* Let  $f(t, \omega) = B(t, \omega)$ . Since  $B(t)$  is adapted to the filtration  $\{\mathcal{F}_t\}$ , it follows that  $f(t)$  is  $\mathcal{F}_t$ -adapted. Also

$$\int_a^b E|B(t)|^2 dt = \int_a^b t dt = \frac{1}{2}(b^2 - a^2) < \infty.$$

So  $\int_a^b B(t) dB(t)$  is an Itô integral. In fact it can be shown (see Example 3.23) that

$$\int_a^b B(t) dB(t) = \frac{1}{2} (B(b)^2 - B(a)^2 - (b - a)). \quad (3.1)$$

*Example 3.8.* The integral  $\int_a^b e^{B(t)} dB(t)$  is an Itô integral because  $e^{B(t)}$  is  $\mathcal{F}_t$ -adapted and

$$\begin{aligned} E|e^{2B(t)}| &= \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\ &= \int_{-\infty}^{\infty} e^{2t} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-2t)^2}{2t}} dx \\ &= e^{2t}, \end{aligned}$$

thus  $\int_a^b E|e^{2B(t)}|^2 dt = \int_a^b e^{2t} dt = \frac{1}{2}(e^{2b} - e^{2a}) < \infty$ .

**Theorem 3.9.** *Suppose that  $f \in L^2_{ad}([a, b] \times \Omega)$ . Then the Itô integral  $I(f) = \int_a^b f(t) dB(t)$  is a random variable with mean  $E[I(f)] = 0$  and variance*

$$E(|I(f)|^2) = \int_a^b E|f(t)|^2 dt.$$

*Example 3.10.* Consider  $f(t) = \text{sgn}(B(t))$ . Since

$$\int_a^b E|\text{sgn}(B(t))|^2 dt = \int_a^b E(1) dt = b - a < \infty,$$

it follows that  $f(t) = \text{sgn}(B(t)) \in L_{ad}^2([a, b] \times \Omega)$ . By Theorem 3.9, the random variable  $\int_a^b \text{sgn}(B(t)) dB(t)$  has mean 0 and variance  $\int_a^b E|\text{sgn}(B(t))|^2 dt = b - a$ .

Suppose that  $f \in L_{ad}^2([a, b] \times \Omega)$ . Then for any  $t \in [a, b]$ ,  $\int_a^t E|f(t)|^2 dt \leq \int_a^b E|f(t)|^2 dt < \infty$ . So  $f \in L_{ad}^2([a, t] \times \Omega)$  and the integral  $\int_a^t f(s) dB(s)$  is well-defined. Consider a stochastic process given by

$$X_t = \int_a^t f(s) dB(s), \quad a \leq t \leq b.$$

Note that by Theorem 3.9, we have

$$E(|X_t|^2) = E \left| \int_a^t f(s) dB(s) \right|^2 \leq \int_a^b E|f(s)|^2 ds < \infty.$$

So by Theorem 2.52,  $E|X_t| \leq [E(|X_t|^2)]^{1/2} < \infty$ . Hence for each  $t$ , the random variable  $X_t$  is integrable.

The next two theorems discuss the martingale and continuity properties of the Itô integral.

**Theorem 3.11.** *Suppose  $f \in L_{ad}^2([a, b] \times \Omega)$ . Then the stochastic process*

$$X_t = \int_a^t f(s) dB(s), \quad a \leq t \leq b$$

*is a martingale with respect to the filtration  $\{\mathcal{F}_t : a \leq t \leq b\}$ .*

*Example 3.12.* The stochastic processes  $\int_a^t B(s) dB(s)$  and  $\int_a^t e^{B(s)} dB(s)$  are martingales.

**Theorem 3.13.** *Suppose  $f \in L_{ad}^2([a, b] \times \Omega)$ . Then the stochastic process*

$$X_t = \int_a^t f(s) dB(s), \quad a \leq t \leq b$$

*is continuous, i.e., almost all its sample paths are continuous functions on  $[a, b]$ .*

*Example 3.14.* Consider  $f(t) = \text{sgn}(B(t))$ . In Example 3.10 we showed that  $f(t) = \text{sgn}(B(t)) \in L^2_{ad}([a, b] \times \Omega)$ . Therefore  $X_t = \int_0^t \text{sgn}(B(s)) dB(s)$ ,  $a \leq t \leq b$ , is a continuous martingale by Theorems 3.11 and 3.13.

In Theorem 3.11, we showed that if  $f \in L^2_{ad}([a, b] \times \Omega)$ , then the stochastic process  $X_t = \int_a^t f(s) dB(s)$ ,  $a \leq t \leq b$ , is a martingale with respect to the filtration  $\{\mathcal{F}_t\}$ . The converse is also true, i.e., any  $\mathcal{F}_t$ -martingale can be represented as an Itô integral. In particular we have the following result due to Itô (see [20]).

**Theorem 3.15.** *Let  $F \in L^2(\mathcal{F}_T, P)$ , then there exists a stochastic process  $f \in L^2_{ad}([0, T] \times \Omega)$  such that*

$$F = E[F] + \int_0^T f(t) dB(t).$$

### 3.3 An Extension of the Itô Integral

As in previous section, we fix a Brownian motion  $B(t)$  and a filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$  such that

- (a) for each  $t$ ,  $B(t)$  is  $\mathcal{F}_t$ -measurable,
- (b) for any  $s \leq t$ , the random variable  $B(t) - B(s)$  is independent of the  $\sigma$ -field  $\mathcal{F}_s$ .

In this section, we define the stochastic integral  $\int_a^b f(t) dB(t)$  for the stochastic process  $f(t, \omega)$  satisfying

- (a)  $f(t)$  is adapted to the filtration  $\{\mathcal{F}_t\}$ ;
- (b)  $\int_a^b |f(t)|^2 dt < \infty$  almost surely.

Condition (b) tells us that almost all sample paths are functions in the Hilbert space  $L^2[a, b]$ . Hence the map  $\omega \mapsto f(\cdot, \omega)$  is a measurable function from  $\Omega$  to  $L^2[a, b]$ .

We will use the notation  $\mathcal{L}_{ad}(\Omega, L^2[a, b])$  to denote the space of all stochastic processes  $f(t, \omega)$  satisfying conditions (a) and (b) above. Now we briefly outline the definition of the stochastic integral  $\int_a^b f(t) dB(t)$ ,  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ .

**Step 1.** Approximation of  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$  by processes in  $L_{ad}^2([a, b] \times \Omega)$

Let  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ . Then there exists a sequence  $\{f_n\}$  in  $L_{ad}^2([a, b] \times \Omega)$  such that

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(t) - f(t)|^2 dt = 0$$

almost surely, and hence in probability.

**Step 2.** Approximation of  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$  by step processes in  $L_{ad}^2([a, b] \times \Omega)$

Let  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ . Then there exists a sequence  $\{f_n\}$  of step processes in  $L_{ad}^2([a, b] \times \Omega)$  such that

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(t) - f(t)|^2 dt = 0$$

in probability.

**Step 3.** General case

With the sequence  $\{f_n\}$  of step stochastic processes in  $L_{ad}^2([a, b] \times \Omega)$  from Step 2, define for each  $n$

$$I_{step}(f_n) = \int_a^b f_n(t) dB(t).$$

It can be shown that the sequence  $\{I_{step}(f_n)\}$  converges in probability. Then let

$$\int_a^b f(t) dB(t) = \lim_{n \rightarrow \infty} I_{step}(f_n), \quad \text{in probability}$$

for  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ .

In the previous section, we saw that for  $f \in L_{ad}^2([a, b] \times \Omega)$ ,  $E \int_a^b |f(t)|^2 dt = \int_a^b E(|f(t)|^2) dt < \infty$ . It follows that  $\int_a^b |f(t)|^2 dt < \infty$  almost surely since if  $\int_a^b |f(t)|^2 dt = \infty$ , then  $E \int_a^b |f(t)|^2 dt = \infty$ , which is absurd. This shows that

we have a larger class of integrands  $f(t, \omega)$  for the stochastic integral  $\int_a^b f(t) dB(t)$ , namely  $L_{ad}^2([a, b] \times \Omega) \subset \mathcal{L}_{ad}(\Omega, L^2[a, b])$ . The difference between them is the possible lack of integrability for  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ .

*Example 3.16.* Consider the stochastic process  $f(t) = e^{B(t)^2}$ . Note that

$$\begin{aligned} E(|f(t)|^2) &= E\left[e^{2B(t)^2}\right] \\ &= \int_{-\infty}^{\infty} e^{2x^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\ &= \frac{1}{\sqrt{1-4t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\left(\frac{t}{1-4t}\right)}} e^{-\frac{x^2}{2\left(\frac{t}{1-4t}\right)}} dx \\ &= \begin{cases} \frac{1}{\sqrt{1-4t}}, & \text{if } 0 < t < \frac{1}{4}; \\ \infty, & \text{if } t \geq \frac{1}{4}. \end{cases} \end{aligned}$$

So  $\int_0^1 E|f(t)|^2 dt = \infty$ , which implies that  $f \notin L_{ad}^2([0, 1] \times \Omega)$ . However,  $f \in L_{ad}^2([0, c] \times \Omega)$ , where  $0 \leq c < \frac{1}{4}$ . On the other hand, since  $f(t)$  is a continuous function of  $t$ , we have that  $\int_0^1 |f(t)|^2 dt = \int_0^1 e^{2B(t)^2} dt < \infty$ . So  $f \in \mathcal{L}_{ad}(\Omega, L^2[0, 1])$ .

As stated above, the stochastic process  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$  may lack the integrability property. So the stochastic integral  $\int_a^b f(t) dB(t)$  is just a random variable and may have infinite expectation as seen in Example 3.16. Thus the stochastic process  $X_t = \int_a^t f(s) dB(s)$  may not be a martingale for  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ . However, we have the following:

**Theorem 3.17.** *Let  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ . Then the stochastic process*

$$X_t = \int_a^t f(s) dB(s) \quad a \leq t \leq b$$

*is a local martingale with respect to the filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$ .*

*Example 3.18.* By Theorem 3.17 and Example 3.16, the stochastic process  $X_t = \int_0^1 e^{B(t)^2} dB(t)$  is a local martingale but not a martingale.

In Theorem 3.13, we saw that for  $f \in L_{ad}^2([a, b] \times \Omega)$ , the stochastic process  $X_t = \int_0^t f(s) dB(s)$  is a continuous function of  $t$ . For  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ , we have the following theorem.

**Theorem 3.19.** *Let  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ . Then the stochastic process*

$$X_t = \int_a^t f(s) dB(s) \quad a \leq t \leq b$$

*has a continuous realization.*

Now consider the stochastic integral  $\int_0^t B(1) dB(s)$ , where  $0 \leq t < 1$ . Note that this integral is not defined as an integral we have seen in this chapter because  $B(1)$  is non-adapted. However this integral can be defined by extending the stochastic integral  $\int_a^b f(t) dB(t)$  to non-adapted integrand  $f(t)$ , as one may see in [9].

### 3.4 Itô's Formula

In ordinary calculus, we deal with deterministic functions. One of the most important rules in differentiation is the Chain Rule, which states that for any differentiable functions  $f$  and  $g$ , the composite function  $f \circ g$  is also differentiable and

$$\frac{d}{dt}(f \circ g)(t) = \frac{d}{dt}f(g(t)) = f'(g(t))g'(t).$$

In terms of the Fundamental Theorem of Calculus, we have

$$f(g(t)) - f(g(a)) = \int_a^t f'(g(s))g'(s) ds.$$

In Itô calculus, we deal with random functions, i.e., stochastic processes and we have the counterpart of the above Chain Rule. One must note that there is no differentiation theory in Itô calculus since almost all sample paths of a Brownian motion  $B(t)$  are nowhere differentiable (Theorem 2.22). Nevertheless we have the integral version which we call the *Itô formula* or the *change of variables formula*.

In this section, we will see several versions of Itô's formula. For the proofs, the reader can refer to [14].

Let  $B(t)$  be a Brownian motion. We start with the simplest form of the Itô formula.

**Theorem 3.20.** *Let  $f$  be a  $C^2$ -function, i.e.,  $f$  is twice differentiable and  $f''$  is continuous. Then*

$$f(B(t)) - f(B(a)) = \int_a^t f'(B(s)) dB(s) + \frac{1}{2} \int_a^t f''(B(s)) ds. \quad (3.2)$$

*Remark 3.21.* The first integral on the right is an Itô integral as defined in Section 3.2 and the second integral is a Riemann integral for each sample path of  $B(s)$ .

*Remark 3.22.* The extra term  $\frac{1}{2} \int_a^t f''(B(s)) ds$  is a consequence of the nonzero quadratic variation of the Brownian motion  $B(t)$ . This extra term distinguishes Itô calculus from ordinary calculus.

*Example 3.23.* Let  $f(x) = x^2$ . Then by Equation 3.2, we get

$$B(t)^2 - B(a)^2 = 2 \int_a^t B(s) dB(s) + (t - a)$$

namely

$$\int_a^t B(s) dB(s) = \frac{1}{2} [B(t)^2 - B(a)^2 - (t - a)].$$

This is equivalent to Equation 3.1 in Example 3.7 with  $b = t$ .

*Example 3.24.* Let  $f(x) = x^3$ . Then by Equation 3.2,

$$B(t)^3 = 3 \int_0^t B(s)^2 dB(s) + 3 \int_0^t B(s) ds.$$

So,

$$\int_0^t B(s)^2 dB(s) = \frac{1}{3} B(t)^3 - \int_0^t B(s) ds.$$

*Example 3.25.* Let  $f(x) = e^{x^2}$ . Then by Equation 3.2,

$$e^{B(t)^2} - 1 = 2 \int_0^t B(s) e^{B(s)^2} dB(s) - \int_0^t \left( e^{B(s)^2} + 2B(s)^2 e^{B(s)^2} \right) ds.$$

Now consider a function  $f(t, x)$  of  $x$  and  $t$ . Set  $x = B(t, \omega)$  to get a stochastic process  $f(t, B(t))$ . Notice that now  $t$  appears in two places: as a variable of  $f$  and in the Brownian motion  $B(t)$ . For the first  $t$ , we can apply ordinary calculus. For the second  $t$  in  $B(t)$ , we need to use Itô calculus. This leads to the second version of Itô's formula:

**Theorem 3.26.** *Let  $f(t, x)$  be a continuous function and have continuous partial derivatives  $\frac{\partial f}{\partial t}$ ,  $\frac{\partial f}{\partial x}$  and  $\frac{\partial^2 f}{\partial x^2}$ . Then*

$$\begin{aligned} f(t, B(t)) &= f(a, B(a)) + \int_a^t \frac{\partial f}{\partial x}(s, B(s)) dB(s) + \int_a^t \frac{\partial f}{\partial s}(s, B(s)) ds \\ &\quad + \frac{1}{2} \int_a^t \frac{\partial^2 f}{\partial x^2}(s, B(s)) ds. \end{aligned} \tag{3.3}$$

*Example 3.27.* Let  $f(t, x) = x^2 - t$ . Then by Equation 3.3,

$$\begin{aligned} B(t)^2 - t &= (B(a)^2 - a) + \int_a^t 2B(s) dB(s) + \int_a^t (-1) ds + \frac{1}{2} \int_a^t 2 ds \\ &= B(a)^2 - a + 2 \int_0^t B(s) dB(s) - (t - a) + (t - a) \end{aligned}$$

which gives

$$\int_a^t B(s) dB(s) = \frac{1}{2} [B(t)^2 - B(a)^2 - (t - a)],$$

which is the same as in Example 3.23.

*Example 3.28.* Let  $f(t, x) = e^{x - \frac{1}{2}t}$ . Then by Equation 3.3,

$$\begin{aligned} e^{B(t) - \frac{1}{2}t} &= 1 + \int_0^t e^{B(s) - \frac{1}{2}s} dB(s) - \frac{1}{2} \int_0^t e^{B(s) - \frac{1}{2}s} ds + \frac{1}{2} \int_0^t e^{B(s) - \frac{1}{2}s} ds \\ &= 1 + \int_0^t e^{B(s) - \frac{1}{2}s} dB(s). \end{aligned}$$

Note that by Theorem 3.11,  $e^{B(t) - \frac{1}{2}t}$  is a martingale.



Now let  $\{\mathcal{F}_t; a \leq t \leq b\}$  be a filtration as specified for Itô integrals in Sections 3.2 and 3.3, namely

- (a) for each  $t$ ,  $B(t)$  is  $\mathcal{F}_t$ -measurable,
- (b) for any  $s < t$ , the random variable  $B(t) - B(s)$  is independent of the  $\sigma$ -field  $\mathcal{F}_s$ .

Recall that  $\mathcal{L}_{ad}(\Omega, L^2[a, b])$  is the class consists of all  $\mathcal{F}_t$ -adapted stochastic processes  $f(t)$  such that  $\int_a^b |f(t)|^2 dt < \infty$  almost surely. Now we introduce the class  $\mathcal{L}_{ad}(\Omega, L^1[a, b])$ , that is the class of all  $\mathcal{F}_t$ -adapted stochastic processes  $f(t)$  such that  $\int_a^b |f(t)| dt < \infty$  almost surely.

**Definition 3.29.** An *Itô process* is a stochastic process of the form

$$X_t = X_a + \int_a^t f(s) dB(s) + \int_a^t g(s) ds, \quad a \leq t \leq b,$$

where  $X_a$  is  $\mathcal{F}_a$ -measurable,  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$  and  $g \in \mathcal{L}_{ad}(\Omega, L^1[a, b])$ .

It is common to write the equation above in the “stochastic differential” form:

$$dX_t = f(t) dB(t) + g(t) dt.$$

Again, note that this “stochastic differential” form has no meaning because Brownian motion paths are nowhere differentiable.

*Example 3.30.* Let  $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ . Then

$$X_t = X_a + \int_a^t f(s) dB(s) + \int_a^t f(s)^2 ds, \quad a \leq t \leq b,$$

is an Itô process. For example, let  $f(t) = B(t)$  or  $f(t) = e^{B(t)}$  or  $f(t) = e^{B(t)^2}$ .

Next is the third (more general) version of the Itô formula.

**Theorem 3.31.** Let  $X_t$  be an Itô process given by

$$X_t = X_a + \int_a^t f(s) dB(s) + \int_a^t g(s) ds, \quad a \leq t \leq b.$$

Suppose  $\theta(t, x)$  is a continuous function with continuous partial derivatives  $\frac{\partial \theta}{\partial t}$ ,  $\frac{\partial \theta}{\partial x}$  and  $\frac{\partial^2 \theta}{\partial x^2}$ . Then  $\theta(t, X_t)$  is also an Itô process and

$$\begin{aligned} \theta(t, X_t) &= \theta(a, X_a) + \int_a^t \frac{\partial \theta}{\partial x}(s, X_s) f(s) dB(s) \\ &\quad + \int_a^t \left[ \frac{\partial \theta}{\partial t}(s, X_s) + \frac{\partial \theta}{\partial x}(s, X_s) g(s) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(s, X_s) f(s)^2 \right] ds. \end{aligned} \quad (3.4)$$

In using Equation 3.4, the following table called the *Itô table* is very useful:

**Table 1 : Itô table 1**

$\times$	$dB(t)$	$dt$
$dB(t)$	$dt$	0
$dt$	0	0

For example, if  $dX_t = f(t) dB(t) + g(t) dt$ , then

$$(dX_t)^2 = f(t)^2 (dB(t))^2 + 2f(t)g(t)dB(t)dt + g(t)^2 (dt)^2 = f(t)^2 dt$$

*Example 3.32.* Let  $f \in \mathcal{L}_{ad}(\Omega, L^2[0, 1])$ . Consider the Itô process

$$X_t = \int_0^t f(s) dB(s) - \frac{1}{2} \int_0^t f(s)^2 ds, \quad 0 \leq t \leq 1,$$

and the function  $\theta(x) = e^x$ . Then  $dX_t = f(t)dB(t) - \frac{1}{2}f(t)^2 dt$ . Apply the Taylor expansion and use Itô table 1 to get

$$\begin{aligned} d\theta(X_t) &= e^{X_t} dX_t + \frac{1}{2} e^{X_t} (dX_t)^2 \\ &= e^{X_t} \left( f(t) dB(t) - \frac{1}{2} f(t)^2 dt \right) + \frac{1}{2} e^{X_t} f(t)^2 dt \\ &= f(t) e^{X_t} dB(t). \end{aligned}$$

Therefore, we have

$$e^{\int_0^t f(s) dB(s) - \frac{1}{2} \int_0^t f(s)^2 ds} = 1 + \int_0^t f(s) e^{\int_0^s f(u) dB(u) - \frac{1}{2} \int_0^s f(u)^2 du} dB(s).$$

By Theorem 3.17, the stochastic process  $Y_t = e^{\int_0^t f(s) dB(s) - \frac{1}{2} \int_0^t f(s)^2 ds}$  is a local martingale.

We can extend the general form of Itô's formula in Theorem 3.31 to the multidimensional case. Let  $B_1(t), B_2(t), \dots, B_m(t)$  be  $m$  independent Brownian motions. Consider  $n$  Itô processes  $X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n)}$  given by

$$X_t^{(i)} = X_a^{(i)} + \sum_{j=1}^m \int_a^t f_{ij}(s) dB_j(s) + \int_a^t g_i(s) ds, \quad 1 \leq i \leq n, \quad a \leq t \leq b, \quad (3.5)$$

where  $f_{ij} \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$  and  $g_i \in \mathcal{L}_{ad}(\Omega, L^1[a, b])$ . Equation 3.5 can be written as a matrix equation

$$X_t = X_a + \int_a^t f(s) dB(s) + \int_a^t g(s) ds, \quad a \leq t \leq b, \quad (3.6)$$

where

$$B(t) = \begin{bmatrix} B_1(t) \\ \vdots \\ B_m(t) \end{bmatrix}, \quad X_t = \begin{bmatrix} X_t^{(1)} \\ \vdots \\ X_t^{(n)} \end{bmatrix},$$

$$f(t) = \begin{bmatrix} f_{11}(t) & \cdots & f_{1m}(t) \\ \vdots & \ddots & \vdots \\ f_{n1}(t) & \cdots & f_{nm}(t) \end{bmatrix}, \quad g(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}. \quad (3.7)$$

With this notation, we have the Itô formula in the multi-dimensional case.

**Theorem 3.33.** *Let  $X_t$  be an  $n$ -dimensional Itô process given by*

$$X_t = X_a + \int_a^t f(s) dB(s) + \int_a^t g(s) ds, \quad a \leq t \leq b,$$

with  $X_t$ ,  $f(s)$ ,  $g(s)$  and  $B(s)$  as in Equation 3.7. Suppose  $\theta(t_1, x_1, \dots, x_n)$  is a continuous function on  $[a, b] \times \mathbb{R}^n$  with continuous partial derivatives  $\frac{\partial \theta}{\partial t}$ ,  $\frac{\partial \theta}{\partial x_i}$  and  $\frac{\partial^2 \theta}{\partial x_i \partial x_j}$  for  $1 \leq i, j \leq n$ . Then the stochastic differential of  $\theta(t, X_t^{(1)}, \dots, X_t^{(n)})$  is given by

$$\begin{aligned} \partial \theta(t, X_t^{(1)}, \dots, X_t^{(n)}) &= \frac{\partial \theta}{\partial t}(t, X_t^{(1)}, \dots, X_t^{(n)}) dt + \sum_{i=1}^n \frac{\partial \theta}{\partial x_i}(t, X_t^{(1)}, \dots, X_t^{(n)}) dX_t^{(i)} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \theta}{\partial x_i \partial x_j}(t, X_t^{(1)}, \dots, X_t^{(n)}) dX_t^{(i)} dX_t^{(j)}. \end{aligned}$$

The product  $dX_t^{(i)} dX_t^{(j)}$  can be computed by using the following table

**Table 2 : Itô table 2**

$\times$	$dB_j(t)$	$dt$
$dB_i(t)$	$\delta_{ij} dt$	0
$dt$	0	0

The product  $dB_i(t)dB_j(t) = 0$  for  $i \neq j$  is the symbolic expression of the following fact:

*Fact 3.34.* Let  $B_1(t)$  and  $B_2(t)$  be two independent Brownian motions and let  $\Delta_n = \{t_0, t_1, \dots, t_{n-1}, t_n\}$  be a partition of  $[a, b]$ . So

$$\sum_{i=1}^n (B_1(t_i) - B_1(t_{i-1}))(B_2(t_i) - B_2(t_{i-1})) \longrightarrow 0$$

in  $L^2(\Omega)$  as  $\|\Delta_n\| = \max_{1 \leq i \leq n} (t_i - t_{i-1})$  tends to 0.

*Example 3.35.* Let  $\theta(x, y) = xy$ . Then we have  $\frac{\partial \theta}{\partial x} = y$ ,  $\frac{\partial \theta}{\partial y} = x$ ,  $\frac{\partial^2 \theta}{\partial x \partial y} = \frac{\partial^2 \theta}{\partial y \partial x} = 1$  and  $\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial y^2} = 0$ . Hence by Theorem 3.33 for two processes  $X_t$  and  $Y_t$ , we have

$$\begin{aligned} d(X_t Y_t) &= Y_t dX_t + X_t dY_t + \frac{1}{2} dX_t dY_t + \frac{1}{2} dX_t dY_t \\ &= Y_t dX_t + X_t dY_t + dX_t dY_t \end{aligned} \tag{3.8}$$

Therefore,

$$X_t Y_t = X_a Y_a + \int_a^t Y_s dX_s + \int_a^t X_s dY_s + \int_a^t dX_s dY_s. \quad (3.9)$$

Equations 3.8 and 3.9 are called *the product formulas* for Itô processes.

### 3.5 Applications of Itô's Formula

The Itô formula plays an important role in Itô calculus. It has many useful applications in stochastic analysis. In this section, we see some of its applications. The first is to find the Doob-Meyer decomposition for submartingales that are functions of a Brownian motion  $B(t)$ .

Let  $f \in L_{ad}^2([a, b] \times \Omega)$  and consider a stochastic process  $M(t)$  defined by

$$M(t) = \int_a^t f(s) dB(s), \quad a \leq t \leq b.$$

By Theorem 3.11, we know that  $M(t)$  is a martingale. Let  $\varphi$  be a  $C^2$ -function. Then by Itô's formula (Equation 3.2),

$$\varphi(M(t)) = \varphi(0) + \int_a^t \varphi'(M(s)) f(s) dB(s) + \frac{1}{2} \int_a^t \varphi''(M(s)) f(s)^2 ds. \quad (3.10)$$

Furthermore, suppose that  $\varphi$  is convex and  $E \int_a^b |\varphi'(M(t)) f(t)|^2 dt < \infty$ . Then  $\varphi(M(t))$  is a submartingale by the conditional Jensen's inequality (see Theorem 2.30(f)). Hence Equation 3.10 gives the Doob-Meyer decomposition of the submartingale  $\varphi(M(t))$ .

*Example 3.36.* Let  $\varphi(x) = x^2$ ,  $M(t) = B(t)$  and  $f \equiv 1$ . Then by Equation 3.10,

$$B(t)^2 = 2 \int_0^t B(s) dB(s) + t.$$

The compensator of  $B(t)^2$  for Brownian motion  $B(t)$  is given by  $\langle B \rangle_t = t$ . More generally, for  $f \in L_{ad}^2([a, b] \times \Omega)$ , the compensator  $\langle M \rangle_t$  of  $M(t)^2$  is given by

$$\langle M \rangle_t = \int_a^t f(s)^2 ds.$$

The next application of Itô's formula is in the proof of the Lévy Characterization Theorem [14]. This theorem gives condition for a stochastic process to be a Brownian motion under a certain probability measure. In the next chapter, we use this theorem in the proof of our main result.

**Theorem 3.37.** (Lévy Characterization Theorem) *A stochastic process  $M(t)$ ,  $a \leq t \leq b$ , is a Brownian motion if and only if there exist a probability measure  $Q$  and a filtration  $\{\mathcal{F}_t\}$  such that  $M(t)$  is a continuous martingale with respect to  $\{\mathcal{F}_t\}$  under  $Q$ ,  $Q\{M(0) = 0\} = 1$  and  $\langle M \rangle_t = t$  almost surely with respect to  $Q$  for each  $t$ .*

*Example 3.38.* Let  $B(t)$ ,  $0 \leq t \leq 1$  be a Brownian motion with respect to the probability measure  $P$  in a probability space  $(\Omega, \mathcal{F}, P)$ . Note that the process  $W(t) = B(t) - t$ ,  $0 \leq t \leq 1$  is not a Brownian motion with respect to  $P$  because  $E_P[B(t) - t] = -t$ , which is not constant.

Define  $Q : \mathcal{F} \rightarrow [0, \infty)$  by

$$Q(A) = \int_A e^{B(1) - \frac{1}{2}} dP, \quad A \in \mathcal{F}. \quad (3.11)$$

Observe that

$$Q(\Omega) = \int_{\Omega} e^{B(1) - \frac{1}{2}} dP = e^{-\frac{1}{2}} \int_{\mathbb{R}} e^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2} dx = 1.$$

So  $Q$  is a probability measure. We will show that  $W(t)$  is a Brownian motion with respect to  $Q$  using Theorem 3.37.

Let  $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$ . Note that the probability measures  $P$  and  $Q$  are equivalent since from Equation 3.11 we can write  $P(A) = \int_A e^{\frac{1}{2} - B(1)} dQ$  for  $A \in \mathcal{F}$ .

Thus

$$Q\{\omega; W(0, \omega) = 0\} = P\{\omega; B(0, \omega) = 0\} = 1$$

and

$$Q\{\omega; W(t, \omega) \text{ is continuous in } t\} = P\{\omega; B(t, \omega) \text{ is continuous in } t\} = 1.$$

To show that  $W(t)$  is a martingale with respect to  $Q$ , first note that  $e^{B(1)-\frac{1}{2}}$  is a martingale as seen in Example 3.28 with  $t = 1$ . For any  $A \in \mathcal{F}_t$ ,

$$\begin{aligned} \int_A W(t) dQ &= \int_A W(t) e^{B(1)-\frac{1}{2}} dP \\ &= E_P \left[ 1_A W(t) e^{B(1)-\frac{1}{2}} \right] \\ &= E_P \left\{ E_P \left[ 1_A W(t) e^{B(1)-\frac{1}{2}} \middle| \mathcal{F}_t \right] \right\} \\ &= E_P \left\{ 1_A W(t) E_P \left[ e^{B(1)-\frac{1}{2}} \middle| \mathcal{F}_t \right] \right\} \\ &= E_P \left[ 1_A W(t) e^{B(t)-\frac{1}{2}t} \right] \\ &= \int_A W(t) e^{B(t)-\frac{1}{2}t} dP \end{aligned}$$

With this equality, we can show that  $W(t)$  is a martingale with respect to  $Q$  if and only if  $W(t)e^{B(t)-\frac{1}{2}t}$  is a martingale with respect to  $P$ .

For  $0 < s \leq t$ , suppose  $W(t)e^{B(t)-\frac{1}{2}t}$  is a  $P$ -martingale. Then for any  $A \in \mathcal{F}_s$ ,

$$\begin{aligned} \int_A E_Q [W(t) | \mathcal{F}_s] dQ &= \int_A W(t) dQ \\ &= \int_A W(t) e^{B(t)-\frac{1}{2}t} dP \\ &= \int_A E_P \left[ W(t) e^{B(t)-\frac{1}{2}t} \middle| \mathcal{F}_s \right] dP \\ &= \int_A W(s) e^{B(s)-\frac{1}{2}s} dP \\ &= \int_A W(s) dQ. \end{aligned}$$

So  $E_Q [W(t) | \mathcal{F}_s] = W(s)$ , i.e.,  $W(t)$  is a  $Q$ -martingale.

Conversely, suppose that  $W(t)$  is a  $Q$ -martingale. Then we can show in a similar manner that  $W(t)e^{B(t)-\frac{1}{2}t}$  is a  $P$ -martingale.

With this fact, let  $f(t, x) = (x - t)e^{x - \frac{1}{2}t}$ . Then  $\frac{\partial f}{\partial t} = -e^{x - \frac{1}{2}t} - \frac{1}{2}(x - t)e^{x - \frac{1}{2}t}$ ,  $\frac{\partial f}{\partial x} = e^{x - \frac{1}{2}t} + (x - t)e^{x - \frac{1}{2}t}$  and  $\frac{\partial^2 f}{\partial x^2} = 2e^{x - \frac{1}{2}t} + (x - t)e^{x - \frac{1}{2}t}$ . So by Itô's formula,

$$W(t)e^{B(t) - \frac{1}{2}t} = \int_0^t (1 + B(s) - s) e^{B(s) - \frac{1}{2}s} dB(s),$$

which implies that  $W(t)e^{B(t) - \frac{1}{2}t}$  is a martingale with respect to  $P$ . Thus  $W(t)$  is a martingale with respect to  $Q$ .

Also since  $d\langle W \rangle_t = (dW(t))^2 = (dB(t) - dt)^2 = dt$ , it follows that  $\langle W \rangle_t = t$ . Therefore by Theorem 3.37,  $W(t)$  is a Brownian motion with respect to  $Q$ .

*Example 3.39.* Let  $B(t)$  be a Brownian motion with respect to a probability measure  $P$  and let  $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$  be a filtration. Consider the random variable  $X_t = \int_0^t \text{sgn}(B(s)) dB(s)$ . Then obviously  $P\{X_0 = 0\} = 1$  and also  $X_t$  is a continuous martingale with respect to  $P$  and  $\mathcal{F}_t$  by Example 3.14. The compensator of  $X_t^2$  is given by

$$\langle X \rangle_t = \int_0^t |\text{sgn}(B(s))|^2 ds = \int_0^t 1 ds = t.$$

Hence by Theorem 3.37, the stochastic process  $X_t$  is a Brownian motion with respect to the probability measure  $P$ . From Example 3.10, we have that  $X_t - X_s$  has mean zero and variance  $t - s$ . So  $X_b - X_a$  is a Gaussian random variable with mean 0 and variance  $b - a$ . This example shows that the stochastic integral  $\int_a^b f(t) dB(t)$  can be Gaussian even when the integrand  $f(t)$  is not deterministic.



# Chapter 4

## Girsanov Theorem

In this chapter, we prove our main result, the Girsanov Theorem. The result of this theorem is well known for a condition on exponential process given by  $h$  in  $\mathcal{L}_{ad}(\Omega, L^2[a, b])$  (see [14]). Here we show the result for the exponential process given by  $h$  in  $L^2_{ad}([a, b] \times \Omega)$  which satisfy some new moment conditions. We begin by introducing the exponential process.

### 4.1 Exponential Processes

**Definition 4.1.** The *exponential process* given by  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  is defined to be the stochastic process

$$\mathcal{E}_h(t) = e^{\int_0^t h(s) dB(s) - \frac{1}{2} \int_0^t h(s)^2 ds}, \quad 0 \leq t \leq T.$$

*Example 4.2.* Let  $h(t) = 1$ . Then

$$\mathcal{E}_1(t) = e^{\int_0^t 1 dB(s) - \frac{1}{2} \int_0^t 1^2 ds} = e^{B(t) - \frac{1}{2}t}, \quad 0 \leq t \leq T$$

is an exponential process.

*Example 4.3.* Let  $h(t) = \text{sgn}(B(t))$ , namely  $h(0) = 0$  and  $h(t) = \frac{B(t)}{|B(t)|}$  for  $t \neq 0$ .

Then

$$\mathcal{E}_h(t) = e^{\int_0^t \text{sgn}(B(s)) dB(s) - \frac{1}{2}t}, \quad 0 \leq t \leq T,$$

is an exponential process.

*Example 4.4.* Let  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$ . Then

$$\mathcal{E}_h(t) = e^{\int_0^t h(s) dB(s) - \frac{1}{2} \int_0^t h(s)^2 ds}, \quad 0 \leq t \leq T.$$

Let  $X_t = \int_0^t h(s) dB(s) - \frac{1}{2} \int_0^t h(s)^2 ds$ . By applying the Itô formula (Equation 3.4) with  $\theta(x) = e^x$ , we get

$$\begin{aligned} d\mathcal{E}_h(t) &= \mathcal{E}_h(t) dX_t + \frac{1}{2} \mathcal{E}_h(t) (dX_t)^2 \\ &= \mathcal{E}_h(t) \left[ h(t) dB(t) - \frac{1}{2} h(t)^2 dt \right] + \frac{1}{2} \mathcal{E}_h(t) (h(t)^2 dt) \\ &= \mathcal{E}_h(t) h(t) dB(t). \end{aligned}$$

So,  $\mathcal{E}_h(t) = 1 + \int_0^t \mathcal{E}_h(s) h(s) dB(s)$ . By Theorem 3.17,  $\mathcal{E}_h(t)$  is a local martingale.

In general, we have the following:

**Theorem 4.5.** *The exponential process  $\mathcal{E}_h(t)$  given by  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  is a local martingale and a supermartingale.*

*Proof.* The process  $\mathcal{E}_h(t)$  is a local martingale is shown in Example 4.4.

Since  $\mathcal{E}_h(t)$  is a local martingale, there exists a sequence of stopping times  $\tau_n$  increasing to  $T$  almost surely such that  $\mathcal{E}_h(t \wedge \tau_n)$  is a martingale, namely for  $s < t$ ,  $E[\mathcal{E}_h(t \wedge \tau_n) | \mathcal{F}_s] = \mathcal{E}_h(s \wedge \tau_n)$ . Since  $\mathcal{E}_h(s \wedge \tau_n) \rightarrow \mathcal{E}_h(s)$  almost surely as  $n \rightarrow \infty$  for any  $s \in [0, T]$ , we have by *Conditional Fatou's lemma*

$$\begin{aligned} E[\mathcal{E}_h(t) | \mathcal{F}_s] &= E[\liminf_{n \rightarrow \infty} \mathcal{E}_h(t \wedge \tau_n) | \mathcal{F}_s] \\ &\leq \liminf_{n \rightarrow \infty} E[\mathcal{E}_h(t \wedge \tau_n) | \mathcal{F}_s] \\ &= \liminf_{n \rightarrow \infty} \mathcal{E}_h(s \wedge \tau_n) \\ &= \mathcal{E}_h(s). \end{aligned}$$

By Remark 2.35, the process  $\mathcal{E}_h(t)$  is a supermartingale. □

We know that in general a local martingale is not necessarily a martingale. The following theorem gives a condition for which an exponential process given by  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  is a martingale.

**Theorem 4.6.** *Let  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$ . Then the exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$ , is a martingale if and only if  $E[\mathcal{E}_h(t)] = 1$ , for each  $t \in [0, T]$ .*

*Proof.* Refer [11], [14]. □

*Example 4.7.* Consider the exponential process in Example 4.2. Note that

$$\begin{aligned}
 E[\mathcal{E}_1(t)] &= E \left[ e^{B(t) - \frac{1}{2}t} \right] \\
 &= e^{-\frac{1}{2}t} E \left[ e^{B(t)} \right] \\
 &= e^{-\frac{1}{2}t} \int_{\mathbb{R}} e^x \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2} \frac{x^2}{t}} dx \\
 &= e^{-\frac{1}{2}t} \int_{\mathbb{R}} e^{\frac{1}{2}t} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2} \frac{(x-t)^2}{t}} dx \\
 &= 1.
 \end{aligned}$$

So by Theorem 4.6,  $\mathcal{E}_1(t) = e^{B(t) - \frac{1}{2}t}$ ,  $0 \leq t \leq T$  is a martingale. Note that in Example 3.28, we also showed that  $\mathcal{E}_1(t) = e^{B(t) - \frac{1}{2}t}$  is a martingale.

We have seen in Chapter 3 that  $L_{ad}^2([0, T] \times \Omega) \subset \mathcal{L}_{ad}(\Omega, L^2[0, T])$ . Thus for  $h \in L_{ad}^2([0, T] \times \Omega)$ , the exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$  is a martingale if  $E[\mathcal{E}_h(t)] = 1$  by Theorem 4.6. The next theorem gives another sufficient condition for the exponential process  $\mathcal{E}_h(t)$  given by  $h \in L_{ad}^2([0, T] \times \Omega)$  to be a martingale.

**Theorem 4.8.** *Let  $h \in L_{ad}^2([0, T] \times \Omega)$ . Then the exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$  is a martingale if*

$$E \int_0^T \mathcal{E}_h(t)^2 h(t)^2 dt < \infty.$$

*Proof.* As in Example 4.4, use the Itô formula to get

$$\mathcal{E}_h(t) = 1 + \int_0^t \mathcal{E}_h(s) h(s) dB(s), \quad 0 \leq t \leq T. \quad (4.1)$$

So if  $E \int_0^t \mathcal{E}_h(s)^2 h(s)^2 ds < \infty$ ,  $0 \leq t \leq T$ , then  $\mathcal{E}_h(t)h(t) \in L_{ad}^2([0, T] \times \Omega)$ . Thus  $\mathcal{E}_h(t)$  is a martingale by Theorem 3.11. □

*Example 4.9.* For  $h(t) = 1$ ,  $0 \leq t \leq T$ ,

$$E \int_0^t \mathcal{E}_h(s)^2 h(s)^2 ds = E \int_0^t \mathcal{E}_1(s)^2 ds = \int_0^t E [\mathcal{E}_1(s)^2] ds.$$

Note that

$$\begin{aligned} E [\mathcal{E}_1(s)^2] &= E [e^{2B(s)-s}] \\ &= e^{-s} E [e^{2B(s)}] \\ &= e^{-s} \int_{\mathbb{R}} e^{2x} \frac{1}{\sqrt{2\pi s}} e^{-\frac{1}{2} \frac{x^2}{s}} dx \\ &= e^{-s} \int_{\mathbb{R}} e^{2s} \frac{1}{\sqrt{2\pi s}} e^{-\frac{1}{2} \frac{(x-2s)^2}{s}} dx \\ &= e^s. \end{aligned}$$

Thus  $E \int_0^t \mathcal{E}_h(s)^2 h(s)^2 ds = \int_0^t e^s ds = e^t - 1 < \infty$ . So by Theorem 4.8,  $\mathcal{E}_1(t) = e^{B(t) - \frac{1}{2}t}$  is a martingale, further confirming what we demonstrated in Example 4.7.

*Example 4.10.* Let  $h(t)$  be a deterministic function in  $L^2[0, T]$ . Since

$$\mathcal{E}_h(t)^2 = e^{2 \int_0^t h(s) dB(s) - \int_0^t h(s)^2 ds} = e^{-\int_0^t h(s)^2 ds} e^{2 \int_0^t h(s) dB(s)},$$

it follows that

$$E [\mathcal{E}_h(t)^2] = e^{-\int_0^t h(s)^2 ds} E \left[ e^{2 \int_0^t h(s) dB(s)} \right].$$

Furthermore  $\int_0^t h(s) dB(s)$  is a Wiener integral with mean 0 and variance  $\sigma^2 = \int_0^t h(s)^2 ds$  (Theorem 3.2), so we have

$$\begin{aligned} E \left[ e^{2 \int_0^t h(s) dB(s)} \right] &= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{2x} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= e^{\sigma^2} \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{(x-2\sigma^2)^2}{2\sigma^2}} dx \\ &= e^{\int_0^t h(s)^2 ds}. \end{aligned}$$

Thus

$$E [\mathcal{E}_h(t)^2] = e^{-\int_0^t h(s)^2 ds} e^{\int_0^t h(s)^2 ds} = 1,$$

and hence

$$E \int_0^T \mathcal{E}_h(t)^2 h(t)^2 dt = \int_0^T h(t)^2 E [\mathcal{E}_h(t)^2] dt = \int_0^T h(t)^2 dt < \infty.$$

Therefore the condition in Theorem 4.8 is satisfied for deterministic functions  $h(t)$  in  $L^2[0, T]$ .

**Theorem 4.11.** *If  $h \in L_{ad}^2([0, T] \times \Omega)$  satisfies the condition that*

$$E \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt < \infty, \quad (4.2)$$

then

$$E \int_0^T h(t)^2 \mathcal{E}_h(t)^2 dt < \infty \quad (4.3)$$

and

$$E [\mathcal{E}_h(t)^2] = 1 + E \int_0^t h(s)^2 \mathcal{E}_h(s)^2 ds, \quad 0 \leq t \leq T. \quad (4.4)$$

*Remark 4.12.* Equation 4.2 gives another sufficient condition for the exponential process given by  $h \in L_{ad}^2([0, T] \times \Omega)$  to be a martingale.

*Proof.* Let us write  $h(t)^2 \mathcal{E}_h(t)^2 = (h(t))(h(t)\mathcal{E}_h(t)^2)$ . Then by using Theorem 2.52(a), we have

$$\int_0^T h(t)^2 \mathcal{E}_h(t)^2 dt \leq \left( \int_0^T h(t)^2 dt \right)^{\frac{1}{2}} \left( \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}}.$$

Now by using Theorem 2.52(b), we get

$$\begin{aligned} E \int_0^T h(t)^2 \mathcal{E}_h(t)^2 dt &\leq E \left[ \left( \int_0^T h(t)^2 dt \right)^{\frac{1}{2}} \left( \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}} \right] \\ &\leq \left( E \int_0^T h(t)^2 dt \right)^{\frac{1}{2}} \left( E \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Hence Equation 4.2 implies Equation 4.3 since for  $h \in L_{ad}^2([0, T] \times \Omega)$ , we have  $E \int_0^T h(t)^2 dt < \infty$ .

Next note that we have from Equation 4.1 that  $d\mathcal{E}_h(t) = \mathcal{E}_h(t)h(t) dB(t)$ . By using the Itô product formula (Equation 3.8), we get

$$\begin{aligned} d(\mathcal{E}_h(t)^2) &= 2\mathcal{E}_h(t)[d\mathcal{E}_h(t)] + [d\mathcal{E}_h(t)]^2 \\ &= 2\mathcal{E}_h(t)[\mathcal{E}_h(t)h(t)dB(t)] + [\mathcal{E}_h(t)h(t)dB(t)]^2 \\ &= 2\mathcal{E}_h(t)^2h(t)dB(t) + \mathcal{E}_h(t)^2h(t)^2dt. \end{aligned}$$

Thus,

$$\mathcal{E}_h(t)^2 = 1 + 2 \int_0^t \mathcal{E}_h(s)^2 h(s) dB(s) + \int_0^t \mathcal{E}_h(s)^2 h(s)^2 ds.$$

Taking the expectation on both sides and since  $\int_0^t \mathcal{E}_h(s)^2 h(s) dB(s)$  is a martingale with mean zero (section 3.2), we get Equation 4.4.  $\square$

*Example 4.13.* Consider  $h(t) = 1$ ,  $0 \leq t \leq T$  and suppose that  $E \int_0^T \mathcal{E}_1(t)^4 dt < \infty$ .

Then by Theorem 2.52 (a) and (b),

$$\begin{aligned} E \int_0^T h(t)^2 \mathcal{E}_1(t)^2 dt &= E \int_0^T (1) \mathcal{E}_1(t)^2 dt \\ &\leq E \left[ T^{\frac{1}{2}} \left( \int_0^T \mathcal{E}_1(t)^4 dt \right)^{\frac{1}{2}} \right] \\ &\leq T^{\frac{1}{2}} \left( E \int_0^T \mathcal{E}_1(t)^4 dt \right)^{\frac{1}{2}} \\ &< \infty. \end{aligned}$$

This verifies Equation 4.3 in Theorem 4.11. In Example 4.9, we saw that  $E[\mathcal{E}_1(t)^2] = e^t$ . On the other hand, we have

$$E \int_0^t (1) \mathcal{E}_1(s)^2 ds = e^t - 1.$$

So Equation 4.4 is satisfied for  $h \equiv 1$ .

## 4.2 Transformation of Probability Measures

In probability theory, Girsanov theorem tells how stochastic processes change under changes in (probability) measure. In this section we discuss briefly the notion of transformation underlying probability measures.

In basic probability theory, when considering a certain probability measure  $P$ , it is common to bear in mind a *shape* and a *location* for the density of the random variable. The former is determined by the variance while the latter is determined by the mean of the random variable. With this, a probability distribution is subjected to two types of transformation:

1. Keep the shape of the distribution but move the density to a different location. This is equivalent to saying that the mean is changed without changing the variance.
2. Change the shape of the distribution but keep the density at the same location.

We are more interested in the first type of transformation, namely changing the mean without changing the variance. There are two methods for changing the mean of a random variable: *operation on the possible values assumed by the random variable* or *operation on the probabilities associated with the random variable*.

*Example 4.14.* A fair die is rolled and the values of the random variable  $X$  are defined as follows:

$$X = \begin{cases} -1, & \text{roll of 1 or 4;} \\ 1, & \text{roll of 2 or 5;} \\ 3, & \text{roll of 3 or 6.} \end{cases}$$

Then the mean of  $X$  is

$$E[X] = \frac{1}{3}(-1) + \frac{1}{3}(1) + \frac{1}{3}(3) = 1,$$

and the variance is

$$\begin{aligned} \text{Var}(X) &= E[X - E[X]]^2 \\ &= \frac{1}{3}(-1 - 1)^2 + \frac{1}{3}(1 - 1)^2 + \frac{1}{3}(3 - 1)^2 \\ &= \frac{8}{3}. \end{aligned}$$

Now define  $\tilde{X} = X - 1$ . Then

$$E[\tilde{X}] = \frac{1}{3}(-1 - 1) + \frac{1}{3}(1 - 1) + \frac{1}{3}(3 - 1) = 0,$$

and

$$\text{Var}(\tilde{X}) = E[\tilde{X}]^2 = \frac{1}{3}(-2)^2 + \frac{1}{3}(0)^2 + \frac{1}{3}(2)^2 = \frac{8}{3}.$$

So we have changed the mean of  $X$  to zero without changing its variance by defining the new random variable  $\tilde{X} = X - 1$  (*operation on the possible values*).

*Example 4.15.* Consider the random variable as in Example 4.14. Again we want to change the mean of  $X$  from 1 to 0 and keep the variance unchanged. Define a new probability measure  $Q$  as follows:

$$\begin{aligned} P(\text{getting 1 or 4}) &= \frac{1}{3} \longrightarrow Q(\text{getting 1 or 4}) = \frac{17}{24}, \\ P(\text{getting 2 or 5}) &= \frac{1}{3} \longrightarrow Q(\text{getting 2 or 5}) = \frac{1}{12}, \\ P(\text{getting 3 or 6}) &= \frac{1}{3} \longrightarrow Q(\text{getting 3 or 6}) = \frac{5}{24}. \end{aligned}$$

Then

$$E_Q[X] = \frac{17}{24}(-1) + \frac{1}{12}(1) + \frac{5}{24}(3) = 0,$$

and

$$\text{Var}_Q(X) = \frac{17}{24}(-1)^2 + \frac{1}{12}(1) + \frac{5}{24}(3) = \frac{8}{3}.$$

Note that the method applied here *operated on the probability measure*.



Consider a normally distributed random variable  $Z \sim N(m, 1)$ . Let  $f(z)$  be the density function and denote the implied probability measure by  $P$  with

$$dP = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-m)^2} dz. \quad (4.5)$$

Now define the function

$$\xi(z) = e^{\frac{1}{2}m^2 - zm}. \quad (4.6)$$

Multiply  $\xi(z)$  by  $dP$ , we get a new probability measure

$$\xi(z) dP = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz.$$

By denoting the expression  $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$  by  $d\tilde{P}$ , we have a new probability measure  $\tilde{P}$  defined by

$$d\tilde{P} = \xi(z) dP = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz. \quad (4.7)$$

Note that by Equations 4.5 and 4.7, the random variable  $Z$  has mean  $m$  and 0, respectively under the probability measures  $P$  and  $\tilde{P}$ , while the variance is equal to 1 under both  $P$  and  $\tilde{P}$ . So the transformation from the probability  $P$  to probability  $\tilde{P}$  changes the mean of  $Z$ .

*Remark 4.16.* If we define the function  $\xi(z)$  above to be  $e^{zm - \frac{1}{2}m^2}$  and let  $\hat{P}$  be the corresponding probability measure, then transformation from  $P$  to  $\hat{P}$  will change the mean of  $Z$  from  $m$  to  $2m$ .

Now from Equation 4.7, by dividing  $d\tilde{P}$  by  $dP$ , we get

$$\frac{d\tilde{P}}{dP} = \xi(z).$$

Thus  $\xi(z)$  is actually the Radon-Nikodym derivative of  $\tilde{P}$  with respect to  $P$ . By the Radon-Nikodym theorem (Theorem 2.28), we know that the function  $\xi(z)$  exists when the probability measures  $P$  and  $\tilde{P}$  are equivalent.

Let  $B(t)$  be a Brownian motion with respect to the probability  $P$  in a probability space  $(\Omega, \mathcal{F}, P)$ . Consider the process

$$\mathcal{E}_h(t) = e^{\int_0^t h(s) dB(s) - \frac{1}{2} \int_0^t h(s)^2 ds}, \quad 0 \leq t \leq T, \quad h \in \mathcal{L}_{ad}(\Omega, L^2[0, T]), \quad (4.8)$$

namely the exponential process discussed in Section 4.1. Suppose  $h(t) = m$ , then Equation 4.8 becomes

$$\mathcal{E}_m(t) = e^{mB(t) - \frac{1}{2}m^2t},$$

which is similar to the  $\xi(z)$  discussed above.

Define the function

$$dQ = \mathcal{E}_h(T) dP = e^{\int_0^T h(t) dB(t) - \frac{1}{2} \int_0^T h(t)^2 dt} dP. \quad (4.9)$$

Suppose that  $E[\mathcal{E}_h(t)] = 1$  for  $0 \leq t \leq T$ . Then by Example 2.24,  $Q$  is a probability measure on  $(\Omega, \mathcal{F})$  and  $Q$  is absolutely continuous with respect to  $P$ . If we rewrite Equation 4.9 as

$$dP = (\mathcal{E}_h(T))^{-1} dQ = e^{\int_0^T h(t) dB(t) - \frac{1}{2} \int_0^T h(t)^2 dt} dQ,$$

we get that  $P$  is absolutely continuous with respect to  $Q$ . Therefore  $P$  and  $Q$  are equivalent probability measures.

Now we look at an example which shows how transformation of probability measures is useful.

*Example 4.17.* Consider the probability measure  $dQ = e^{B(1) - \frac{1}{2}} dP$ , where we take  $h \equiv 1$  and  $T = 1$  in Equation 4.9. We can use this  $Q$  to compute the expectation of  $B(t)^2 e^{B(1) - \frac{1}{2}}$ ,  $0 < t \leq 1$ , i.e.,  $E \left[ B(t)^2 e^{B(1) - \frac{1}{2}} \right]$ .

Note that for

$$E \left[ B(t)^2 e^{B(1) - \frac{1}{2}} \right] = \int_{\Omega} B(t)^2 e^{B(1) - \frac{1}{2}} dP = \int_{\Omega} B(t)^2 dQ = E_Q \left[ B(t)^2 \right],$$

where  $E_Q$  is the expectation with respect to  $Q$ . In Example 3.38, we showed that  $W(t) = B(t) - t$ ,  $0 \leq t \leq 1$  is a Brownian motion with respect to  $Q$ . Therefore

$$\begin{aligned} E \left[ B(t)^2 e^{B(1) - \frac{1}{2}} \right] &= E_Q [B(t)^2] = E_Q [(W(t) + t)^2] \\ &= E_Q [W(t)^2 + 2tW(t) + t^2] \\ &= t + t^2. \end{aligned}$$

In fact Example 4.17 is just a special case of the following theorem.

**Theorem 4.18.** *Let  $B(t)$ ,  $0 \leq t \leq 1$ , be a Brownian motion with respect to a probability measure  $P$ . Let  $Q$  be the probability measure defined by  $dQ = e^{B(1) - \frac{1}{2}} dP$ . Then for any function  $f$  such that  $E_P |f(B(t))| < \infty$ , we have*

$$\int_{\Omega} f(B(t) - t) dQ = \int_{\Omega} f(B(t)) dP, \quad (4.10)$$

which can also be expressed as  $E_Q[f(B(t) - t)] = E_P[f(B(t))]$ .

*Proof.* Refer [14], page 140. □

*Example 4.19.* Let  $f(x) = e^{i\lambda x}$ , for some  $\lambda \in \mathbb{R}$ . By Equation 4.10 we see

$$\int_{\Omega} e^{i\lambda(B(t) - t)} dQ = \int_{\Omega} e^{i\lambda B(t)} dP = e^{-\frac{1}{2}\lambda^2 t}, \quad \forall \lambda \in \mathbb{R},$$

which is equivalent to writing  $E_Q[e^{i\lambda(B(t) - t)}] = E_P[e^{i\lambda B(t)}] = e^{-\frac{1}{2}\lambda^2 t}$ . So the characteristic function of  $B(t) - t$  under  $Q$  is  $e^{-\frac{1}{2}\lambda^2 t}$ , which implies that  $B(t) - t$  is normally distributed with mean 0 and variance  $t$ .

### 4.3 Girsanov Theorem

In this section we present the main result in this dissertation, namely the *Girsanov theorem*. This result is well-known for exponential process given by  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  satisfying a certain condition, which we will state in Theorem

4.22. Then we present our result in which the exponential process given by  $h \in L_{ad}^2([0, T] \times \Omega)$  satisfies certain moment conditions.

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $B(t)$  be a Brownian motion with respect to the probability  $P$ . Consider a stochastic process  $\varphi(t)$ . Is the process  $B(t) - \varphi(t)$  a Brownian motion? Let us look at some examples.

*Example 4.20.* Let  $\varphi(t) = c$ . Then  $B(t) - \varphi(t) = B(t) - c$  is just a translation of the Brownian motion  $B(t)$ . So  $B(t) - c$  is still a Brownian motion with respect to  $P$ , but starts from  $-c$ .

*Example 4.21.* Let  $\varphi(t) = t$ ,  $0 \leq t \leq 1$ . Then by Example 3.38, we know that  $B(t) - t$  is not a martingale with respect to  $P$ , but it is a martingale with respect to  $Q$  given by  $dQ = e^{B(1) - \frac{1}{2}} dP$ .

So it is natural to ask whether the process  $B(t) - \varphi(t)$  is a Brownian motion with respect to some probability measure. The Girsanov Theorem answers this question for a certain kind of stochastic processes.

**Theorem 4.22.** (Girsanov Theorem) *Let  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  and assume that  $E_P[\mathcal{E}_h(t)] = 1$  for all  $t \in [0, T]$ . Then the stochastic process*

$$W(t) = B(t) - \int_0^t h(s) ds, \quad 0 \leq t \leq T$$

*is a Brownian motion with respect to the probability measure  $Q$  defined by  $dQ = \mathcal{E}_h(T) dP$ , namely  $Q(A) = \int_A \mathcal{E}_h(T) dP$  for  $A \in \mathcal{F}$ .*

*Proof.* Refer [14] page 143. □

Before continue to show our result, we look at some lemmas.

**Lemma 4.23.** *Let  $\theta \in L^1(P)$  be nonnegative such that  $d\mu = \theta dP$  defines a probability measure. Then for any  $\sigma$ -field  $\mathcal{G} \subset \mathcal{F}$  and  $X \in L^1(\mu)$ , we have*

$$E_\mu[X|\mathcal{G}] = \frac{E_P[X\theta|\mathcal{G}]}{E_P[\theta|\mathcal{G}]}, \quad \mu - \text{almost surely.}$$

*Proof.* First note that  $E_P|X\theta| = \int_\Omega |X|\theta dP = \int_\Omega |X| d\mu < \infty$ . So the conditional expectation  $E_P[X\theta|\mathcal{G}]$  is defined.

For any  $G \in \mathcal{G}$ , by using the definition of conditional expectation and the definition of  $\mu$ , we have

$$\int_G E_P[X\theta|\mathcal{G}] dP = \int_G X\theta dP = \int_G X d\mu = \int_G E_\mu[X|\mathcal{G}] d\mu. \quad (4.11)$$

Now, by the definition of conditional expectation and by Theorem 2.30 (d),

$$\begin{aligned} \int_G E_\mu[X|\mathcal{G}] d\mu &= \int_G E_\mu[X|\mathcal{G}]\theta dP = \int_G E_P[E_\mu[X|\mathcal{G}]\theta|\mathcal{G}] dP \\ &= \int_G E_\mu[X|\mathcal{G}] E_P[\theta|\mathcal{G}] dP. \end{aligned} \quad (4.12)$$

From Equations 4.11 and 4.12 we get

$$E_P[X\theta|\mathcal{G}] = E_\mu[X|\mathcal{G}] E_P[\theta|\mathcal{G}],$$

which implies the conclusion of the lemma.  $\square$

**Lemma 4.24.** *Suppose for  $h \in L^2_{ad}([0, T] \times \Omega)$ , the exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$  satisfies the condition  $E_P \int_0^T \mathcal{E}_h(t)^4 dt < \infty$ . Then  $E_P \int_0^t B(t)^2 \mathcal{E}_h(t)^2 dt < \infty$ .*

*Proof.* By Theorem 2.52 (a) and (b), we have

$$\begin{aligned} E_P \int_0^T B(t)^2 \mathcal{E}_h(t)^2 dt &\leq E_P \left[ \left( \int_0^T B(t)^4 dt \right)^{\frac{1}{2}} \left( \int_0^T \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}} \right] \\ &\leq \left( E_P \int_0^T B(t)^4 dt \right)^{\frac{1}{2}} \left( E_P \int_0^T \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}} \\ &< \infty \end{aligned}$$

since  $E_P \int_0^T \mathcal{E}_h(t)^4 dt < \infty$  and it is a fact that  $E_P[B(t)^4] = 3t^2$ .  $\square$

Now suppose the exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$  given by  $h \in L_{ad}^2([0, T] \times \Omega)$  satisfies the condition in Equation 4.3, namely  $E \int_0^T h(t)^2 \mathcal{E}_h(t)^2 dt < \infty$ . Then by Theorem 4.8, the exponential process

$$\mathcal{E}_h(t) = e^{\int_0^t h(s) dB(s) - \frac{1}{2} \int_0^t h(s)^2 ds}, \quad 0 \leq t \leq T,$$

is a martingale. Let  $Q$  be the probability measure in  $(\Omega, \mathcal{F})$  defined by  $dQ = \mathcal{E}_h(T) dP$ , i.e.,

$$Q(A) = \int_A \mathcal{E}_h(T) dP, \quad A \in \mathcal{F}.$$

Then  $Q$  and  $P$  are equivalent probability measures as discussed in Section 4.2.

**Theorem 4.25.** *Consider the stochastic process  $W(t) = B(t) - \int_0^t h(s) ds$ ,  $0 \leq t \leq T$ . Suppose for  $h \in L_{ad}^2([0, T] \times \Omega)$  we have the following:*

$$(a) \quad E_P \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt < \infty,$$

$$(b) \quad E_P \int_0^T h(t)^2 B(t)^8 dt < \infty,$$

$$(c) \quad E_P \left( \int_0^T h(t)^2 dt \right)^5 < \infty,$$

$$(d) \quad E_P \int_0^T \mathcal{E}_h(t)^4 dt < \infty.$$

Then  $W(t)$  and  $W(t)^2 - t$ ,  $0 \leq t \leq T$ , are  $Q$ -martingales.

*Remark 4.26.* Conditions (b) and (c) are needed only in proving  $W(t)^2 - t$  is a  $Q$ -martingale. In proving  $W(t)$  is a  $Q$ -martingale, we need  $E_P \int_0^T h(t)^2 B(t)^4 dt < \infty$  and  $E_P \left( \int_0^T h(t)^2 dt \right)^3 < \infty$ . However these conditions can be derived easily from conditions (b) and (c).

*Proof.* (I)  $W(t)$ ,  $0 \leq t \leq T$ , is a  $Q$ -martingale

First we will show that  $W(t)$ ,  $0 \leq t \leq T$ , is a  $Q$ -martingale. Note that under condition (a), we have by Equation 4.4 that  $E_P[\mathcal{E}_h(t)^2] < \infty$  for any  $0 \leq t \leq T$ .

Moreover, by using the fact  $(x + y)^2 \leq 2(x^2 + y^2)$  and Theorem 2.52 (b), we get  $E_P[W(t)^2] < \infty$  for all  $0 \leq t \leq T$ . Indeed,

$$\begin{aligned}
E_P[W(t)^2] &\leq 2 E_P \left[ B(t)^2 + \left( \int_0^t h(s) ds \right)^2 \right] \\
&\leq 2 E_P \left[ B(t)^2 + t \left( \int_0^t h(s) ds \right) \right] \\
&= 2 \left[ t + t E_P \int_0^t h(s) ds \right] \\
&< \infty.
\end{aligned}$$

Hence by Theorem 2.52 (b),

$$E_P(|W(t)\mathcal{E}_h(T)|) \leq (E_P(W(t)^2))^{\frac{1}{2}} (E_P(\mathcal{E}_h(T))^2)^{\frac{1}{2}} < \infty.$$

Thus we can consider the conditional expectation of  $W(t)^2\mathcal{E}_h(T)$  with respect to a  $\sigma$ -field.

Let  $0 \leq s < t \leq T$ . By Theorem 2.30 (e) and (d), we have

$$\begin{aligned}
E_P[W(t)\mathcal{E}_h(T)|\mathcal{F}_s] &= E_P(E_P[W(t)\mathcal{E}_h(T)|\mathcal{F}_t]|\mathcal{F}_s) \\
&= E_P(W(t)E_P[\mathcal{E}_h(T)|\mathcal{F}_t]|\mathcal{F}_s) \\
&= E_P[W(t)\mathcal{E}_h(t)|\mathcal{F}_s], \tag{4.13}
\end{aligned}$$

where the last equality follows from the fact that  $\mathcal{E}_h(t)$  is a martingale (Remark 4.12).

On the other hand, by Lemma 4.23,

$$E_Q[W(t)|\mathcal{F}_s] = \frac{E_P[W(t)\mathcal{E}_h(T)|\mathcal{F}_s]}{E_P[\mathcal{E}_h(T)|\mathcal{F}_s]} = \frac{E_P[W(t)\mathcal{E}_h(T)|\mathcal{F}_s]}{\mathcal{E}_h(s)}. \tag{4.14}$$

It follows from Equations 4.13 and 4.14 that

$$E_Q[W(t)|\mathcal{F}_s] = \frac{E_P[W(t)\mathcal{E}_h(t)|\mathcal{F}_s]}{\mathcal{E}_h(s)}. \tag{4.15}$$

From Equation 4.15, we see that if we can prove  $W(t)\mathcal{E}_h(t)$  is a  $P$ -martingale, then Equation 4.15 becomes

$$E_Q[W(t)|\mathcal{F}_s] = \frac{E_P[W(t)\mathcal{E}_h(t)|\mathcal{F}_s]}{\mathcal{E}_h(s)} = \frac{W(s)\mathcal{E}_h(s)}{\mathcal{E}_h(s)} = W(s)$$

for all  $s \leq t$ , which shows  $W(t)$ ,  $0 \leq t \leq T$ , is a  $Q$ -martingale.

Note that we have  $dW(t) = dB(t) - h(t)dt$  and also by Equation 4.1 that  $d\mathcal{E}_h(t) = h(t)\mathcal{E}_h(t)dB(t)$ . Apply the Itô product formula (Equation 3.8) to obtain

$$\begin{aligned} d[W(t)\mathcal{E}_h(t)] &= [dW(t)]\mathcal{E}_h(t) + W(t)d\mathcal{E}_h(t) + [dW(t)][d\mathcal{E}_h(t)] \\ &= [dB(t) - h(t)dt]\mathcal{E}_h(t) + W(t)h(t)\mathcal{E}_h(t)dB(t) + h(t)\mathcal{E}_h(t)dt \\ &= [1 + h(t)W(t)]\mathcal{E}_h(t)dB(t). \end{aligned}$$

Hence we have for  $0 \leq t \leq T$ ,

$$\begin{aligned} W(t)\mathcal{E}_h(t) &= \int_0^t [1 + h(s)W(s)]\mathcal{E}_h(s)dB(s) \\ &= \int_0^t \mathcal{E}_h(s)dB(s) + \int_0^t h(s)W(s)\mathcal{E}_h(s)dB(s). \end{aligned} \quad (4.16)$$

In order to show that  $W(t)\mathcal{E}_h(t)$  is a  $P$ -martingale, we show that the integrals  $\int_0^t \mathcal{E}_h(s)dB(s)$  and  $\int_0^t h(s)W(s)\mathcal{E}_h(s)dB(s)$  are  $P$ -martingales. Namely we show that  $\mathcal{E}_h(t)$  and  $h(t)W(t)\mathcal{E}_h(t)$  are in  $L^2_{ad}([0, T] \times \Omega)$ .

Recall that we have  $E_P[\mathcal{E}_h(t)^2] < \infty$ . Thus  $\int_0^T E_P[\mathcal{E}_h(t)^2]dt < \infty$ . So  $\mathcal{E}_h(t) \in L^2_{ad}([0, T] \times \Omega)$ . Next write  $h(t)^2W(t)^2\mathcal{E}_h(t)^2$  as  $(h(t)W(t)^2)(h(t)\mathcal{E}_h(t)^2)$  and apply Theorem 2.52 (a) and (b) to get

$$\begin{aligned} &E_P \int_0^T h(t)^2W(t)^2\mathcal{E}_h(t)^2 dt \\ &\leq E_P \left[ \left( \int_0^T h(t)^2W(t)^4 dt \right)^{\frac{1}{2}} \left( \int_0^T h(t)^2\mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}} \right] \\ &\leq \left[ E_P \int_0^T h(t)^2W(t)^4 dt \right]^{\frac{1}{2}} \left[ E_P \int_0^T h(t)^2\mathcal{E}_h(t)^4 dt \right]^{\frac{1}{2}} \end{aligned} \quad (4.17)$$



The second factor on the right hand side is finite by condition (a). For the first factor, we use the inequality  $(x + y)^4 \leq 8(x^4 + y^4)$  and Theorem 2.52 (a) to show that

$$\begin{aligned}
\int_0^T h(t)^2 W(t)^4 dt &= \int_0^T h(t)^2 \left( B(t) - \int_0^t h(s) ds \right)^4 dt \\
&\leq 8 \int_0^T h(t)^2 \left[ B(t)^4 + \left( \int_0^t h(s) ds \right)^4 \right] dt \\
&\leq 8 \int_0^T h(t)^2 \left[ B(t)^4 + \left( \int_0^t 1 ds \right)^2 \left( \int_0^t h(s)^2 ds \right)^2 \right] dt \\
&\leq 8 \int_0^T h(t)^2 \left[ B(t)^4 + T^2 \left( \int_0^T h(s)^2 ds \right)^2 \right] dt \\
&= 8 \left[ \int_0^T h(t)^2 B(t)^4 dt + T^2 \left( \int_0^T h(t)^2 dt \right)^3 \right].
\end{aligned}$$

So,

$$E_P \int_0^T h(t)^2 W(t)^4 dt \leq 8 E_P \int_0^T h(t)^2 B(t)^4 dt + T^2 E_P \left( \int_0^T h(t)^2 dt \right)^3.$$

By condition (c),  $E_P \left( \int_0^T h(t)^2 dt \right)^3 < \infty$ . By writing  $h(t)^2 B(t)^4$  as  $h(t) (h(t) B(t)^4)$  and using Theorem 2.52 (a) and (b), we get

$$\begin{aligned}
E_P \int_0^T h(t)^2 B(t)^4 dt &\leq E_P \left[ \left( \int_0^T h(t)^2 dt \right)^{\frac{1}{2}} \left( \int_0^T h(t)^2 B(t)^8 dt \right)^{\frac{1}{2}} \right] \\
&\leq \left( E_P \int_0^T h(t)^2 dt \right)^{\frac{1}{2}} \left( E_P \int_0^T h(t)^2 B(t)^8 dt \right)^{\frac{1}{2}},
\end{aligned}$$

which is finite by conditions (b) and (c). Hence  $E_P \int_0^T h(t)^2 W(t)^4 dt < \infty$ . By Equation 4.17 we get that

$$E_P \int_0^T h(t)^2 W(t)^2 \mathcal{E}_h(t)^2 dt < \infty. \quad (4.18)$$

This shows that  $h(t)W(t)\mathcal{E}_h(t)$  is in  $L_{ad}^2([0, T] \times \Omega)$ . Therefore we have proved that  $W(t)$ ,  $0 \leq t \leq T$ , is a  $Q$ -martingale.

(II)  $\underline{W(t)^2 - t, 0 \leq t \leq T}$ , is a  $Q$ -martingale

Now we prove that  $W(t)^2 - t$ , for  $0 \leq t \leq T$ , is a  $Q$ -martingale. Similarly as in deriving Equation 4.15, by Theorem 2.30 (e) and Lemma 4.23,

$$\begin{aligned} E_Q [W(t)^2 - t | \mathcal{F}_s] &= \frac{E_P [(W(t)^2 - t) \mathcal{E}_h(T) | \mathcal{F}_s]}{E_P [\mathcal{E}_h(T) | \mathcal{F}_s]} \\ &= \frac{E_P [E_P \{(W(t)^2 - t) \mathcal{E}_h(T) | \mathcal{F}_t\} | \mathcal{F}_s]}{\mathcal{E}_h(s)} \\ &= \frac{1}{\mathcal{E}_h(s)} E_P [(W(t)^2 - t) \mathcal{E}_h(t) | \mathcal{F}_s]. \end{aligned} \quad (4.19)$$

From Equation 4.19, we can see that if  $[W(t)^2 - t] \mathcal{E}_h(t)$ ,  $0 \leq t \leq T$ , is a  $P$ -martingale, then Equation 4.19 will become

$$\begin{aligned} E_Q [W(t)^2 - t | \mathcal{F}_s] &= \frac{E_P [(W(t)^2 - t) \mathcal{E}_h(t) | \mathcal{F}_s]}{\mathcal{E}_h(s)} \\ &= \frac{(W(s)^2 - s) \mathcal{E}_h(s)}{\mathcal{E}_h(s)} \\ &= W(s)^2 - s. \end{aligned}$$

This shows that  $W(t)^2 - t$ ,  $0 \leq t \leq T$ , is a  $Q$ -martingale.

In order to show that  $[W(t)^2 - t] \mathcal{E}_h(t)$ ,  $0 \leq t \leq T$ , is a  $P$ -martingale, we first note that by the Itô product formula (Equation 3.8),

$$\begin{aligned} d [W(t)^2 \mathcal{E}_h(t)] &= [dW(t)] W(t) \mathcal{E}_h(t) + W(t) d[W(t) \mathcal{E}_h(t)] + [dW(t)] d[W(t) \mathcal{E}_h(t)] \\ &= [dB(t) - h(t) dt] W(t) \mathcal{E}_h(t) + W(t) [(1 + h(t)W(t)) \mathcal{E}_h(t) dB(t)] \\ &\quad + [1 + h(t)W(t)] \mathcal{E}_h(t) dt \\ &= [2 + h(t)W(t)] W(t) \mathcal{E}_h(t) dB(t) + \mathcal{E}_h(t) dt. \end{aligned}$$

Thus

$$W(t)^2 \mathcal{E}_h(t) = \int_0^t [2 + h(s)W(s)] W(s) \mathcal{E}_h(s) dB(s) + \int_0^t \mathcal{E}_h(s) ds. \quad (4.20)$$

We show the integrand in the first integral on the right belongs to  $L_{ad}^2([0, T] \times \Omega)$ , that is we show the processes  $W(t) \mathcal{E}_h(t)$  and  $h(t)W(t)^2 \mathcal{E}_h(t)$  are in  $L_{ad}^2([0, T] \times \Omega)$ .

First, by using the inequality  $(x + y)^2 \leq 2(x^2 + y^2)$ , we have

$$\begin{aligned} E_P \int_0^T W(t)^2 \mathcal{E}_h(t)^2 dt &\leq E_P \int_0^T 2 \left[ B(t)^2 + \left( \int_0^t h(s) ds \right)^2 \right] \mathcal{E}_h(t)^2 dt \\ &\leq 2 E_P \int_0^T B(t)^2 \mathcal{E}_h(t)^2 dt \\ &\quad + 2 E_P \int_0^T \left( \int_0^t h(s) ds \right)^2 \mathcal{E}_h(t)^2 dt. \end{aligned} \quad (4.21)$$

The first expectation on the right hand side of Inequality 4.21 is finite by condition (d) and Lemma 4.24. For the second expectation, we apply Theorem 2.52 (a) and (b) to get

$$\begin{aligned} &E_P \int_0^T \left( \int_0^t h(s) ds \right)^2 \mathcal{E}_h(t)^2 dt \\ &\leq E_P \int_0^T T \left( \int_0^T h(s)^2 ds \right) \mathcal{E}_h(t)^2 dt \\ &\leq T \left( E_P \int_0^T \left( \int_0^T h(s)^2 ds \right)^2 dt \right)^{\frac{1}{2}} \left( E_P \int_0^T \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (4.22)$$

The second expectation in the right hand side of Inequality 4.22 is finite by condition (d). For the first expectation, by Fubini's theorem we have

$$E_P \int_0^T \left( \int_0^T h(s)^2 ds \right)^2 dt = \int_0^T E_P \left( \int_0^T h(s)^2 ds \right)^2 dt, \quad (4.23)$$

which is finite by condition (c). Thus from Inequality (4.22),

$$E_P \int_0^T \left( \int_0^t h(s) ds \right)^2 \mathcal{E}_h(t)^2 dt < \infty.$$

Applying this to Inequality 4.21, we get

$$E_P \int_0^T W(t)^2 \mathcal{E}_h(t)^2 dt < \infty.$$

Therefore  $W(t)\mathcal{E}_h(t) \in L_{ad}^2([0, T] \times \Omega)$ .

Next we show that  $h(t)W(t)^2\mathcal{E}_h(t)$  is in  $L^2_{ad}([0, T] \times \Omega)$ . By using the inequality  $(x + y)^4 \leq 8(x^4 + y^4)$ , we have

$$\begin{aligned}
E_P \int_0^T h(t)^2 W(t)^4 \mathcal{E}_h(t)^2 dt &= E_P \int_0^T h(t)^2 \left( B(t) - \int_0^t h(s) ds \right)^4 \mathcal{E}_h(t)^2 dt \\
&\leq 8 E_P \int_0^T h(t)^2 \left[ B(t)^4 + \left( \int_0^t h(s) ds \right)^4 \right] \mathcal{E}_h(t)^2 dt \\
&= 8 E_P \int_0^T h(t)^2 B(t)^4 \mathcal{E}_h(t)^2 dt \\
&\quad + 8 E_P \int_0^T h(t)^2 \left( \int_0^t h(s) ds \right)^4 \mathcal{E}_h(t)^2 dt. \tag{4.24}
\end{aligned}$$

By writing  $h(t)^2 B(t)^4 \mathcal{E}_h(t)^2$  as  $(h(t)B(t)^4)(h(t)\mathcal{E}_h(t)^2)$  and by using Theorem 2.52 (a) and (b), we get

$$\begin{aligned}
&E_P \int_0^T h(t)^2 B(t)^4 \mathcal{E}_h(t)^2 dt \\
&\leq E_P \left[ \left( \int_0^T h(t)^2 B(t)^8 dt \right)^{\frac{1}{2}} \left( \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}} \right] \\
&\leq \left( E_P \int_0^T h(t)^2 B(t)^8 dt \right)^{\frac{1}{2}} \left( E_P \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}}, \tag{4.25}
\end{aligned}$$

which is finite by conditions (a) and (b). On the other hand, by using Theorem 2.52 (a), we have

$$\begin{aligned}
\left( \int_0^t h(s) ds \right)^4 &= \left[ \left( \int_0^t (1) h(s) ds \right)^2 \right]^2 \\
&\leq \left[ \left( \int_0^t 1 ds \right) \left( \int_0^t h(s)^2 ds \right) \right]^2 \\
&\leq T^2 \left( \int_0^T h(s)^2 ds \right)^2. \tag{4.26}
\end{aligned}$$

Apply Inequality 4.26 to the second term in Inequality 4.24 to get

$$\begin{aligned}
& E_P \int_0^T h(t)^2 \left( \int_0^t h(s) ds \right)^4 \mathcal{E}_h(t)^2 dt \\
& \leq E_P \int_0^T h(t)^2 \left\{ T^2 \left( \int_0^T h(s)^2 ds \right)^2 \right\} \mathcal{E}_h(t)^2 dt \\
& = T^2 E_P \int_0^T h(t)^2 \left( \int_0^T h(s)^2 ds \right)^2 \mathcal{E}_h(t)^2 dt.
\end{aligned}$$

By writing  $h(t)^2 \left( \int_0^T h(s)^2 ds \right)^2 \mathcal{E}_h(t)^2$  as  $\left[ h(t) \left( \int_0^T h(s)^2 ds \right)^2 \right] \left( h(t) \mathcal{E}_h(t)^2 \right)$  and using Theorem 2.52 (a) and (b), we get

$$\begin{aligned}
& E_P \int_0^T h(t)^2 \left( \int_0^T h(s)^2 ds \right)^2 \mathcal{E}_h(t)^2 dt \\
& \leq E_P \left[ \left( \int_0^T h(t)^2 \left( \int_0^T h(s)^2 ds \right)^4 dt \right)^{\frac{1}{2}} \left( \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}} \right] \\
& \leq \left( E_P \int_0^T h(t)^2 \left( \int_0^T h(s)^2 ds \right)^4 dt \right)^{\frac{1}{2}} \left( E_P \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}} \\
& \leq \left( E_P \left( \int_0^T h(t)^2 dt \right)^5 \right)^{\frac{1}{2}} \left( E_P \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt \right)^{\frac{1}{2}},
\end{aligned}$$

which is finite by conditions (a) and (c). Hence

$$E_P \int_0^T h(t)^2 \left( \int_0^t h(s) ds \right)^4 \mathcal{E}_h(t)^2 dt < \infty. \quad (4.27)$$

Applying Equations 4.25 and 4.27 to Equation 4.24, we get

$$E_P \int_0^T h(t)^2 W(t)^4 \mathcal{E}_h(t)^2 dt < \infty.$$

So  $h(t)W(t)^2\mathcal{E}_h(t) \in L^2_{ad}([0, T] \times \Omega)$ . Therefore we have show that the stochastic integral in Equation 4.20 is a  $P$ -martingale.

Finally we take the conditional expectation of Equation 4.20 to get

$$\begin{aligned}
& E_P [W(t)^2 \mathcal{E}_h(t) | \mathcal{F}_s] \\
&= E_P \left[ \int_0^t [2 + h(u)W(u)] W(u) \mathcal{E}_h(u) dB(u) \middle| \mathcal{F}_s \right] + E_P \left[ \int_0^t \mathcal{E}_h(u) du \middle| \mathcal{F}_s \right] \\
&= \int_0^s [2 + h(u)W(u)] W(u) \mathcal{E}_h(u) dB(u) + E_P \left[ \int_0^s \mathcal{E}_h(u) du \middle| \mathcal{F}_s \right] \\
&\quad + E_P \left[ \int_s^t \mathcal{E}_h(u) du \middle| \mathcal{F}_s \right] \\
&= \int_0^s [2 + h(u)W(u)] W(u) \mathcal{E}_h(u) dB(u) + \int_0^s \mathcal{E}_h(u) du \\
&\quad + E_P \left[ \int_s^t \mathcal{E}_h(u) du \middle| \mathcal{F}_s \right] \\
&= W(s)^2 \mathcal{E}_h(s) + E_P \left[ \int_s^t \mathcal{E}_h(u) du \middle| \mathcal{F}_s \right]. \tag{4.28}
\end{aligned}$$

Since

$$E \left\{ E \left[ \int_s^t \mathcal{E}_h(u) du \middle| \mathcal{F}_s \right] \right\} = E \left\{ \int_s^t \mathcal{E}_h(u) du \right\} = E \left\{ \int_s^t E [\mathcal{E}_h(u) | \mathcal{F}_s] du \right\},$$

it follows that

$$E \left[ \int_s^t \mathcal{E}_h(u) du \middle| \mathcal{F}_s \right] = \int_s^t E [\mathcal{E}_h(u) | \mathcal{F}_s] du = \int_s^t \mathcal{E}_h(s) du = \mathcal{E}_h(s)(t - s).$$

Thus Equation 4.28 becomes

$$E_P [W(t)^2 \mathcal{E}_h(t) | \mathcal{F}_s] = W(s)^2 \mathcal{E}_h(s) + \mathcal{E}_h(s)(t - s),$$

which implies that for any  $s \leq t$ ,

$$E_P [(W(t)^2 - t) \mathcal{E}_h(t) | \mathcal{F}_s] = (W(s)^2 - s) \mathcal{E}_h(s).$$

Thus  $[W(t)^2 - t] \mathcal{E}_h(t)$ ,  $0 \leq t \leq T$  is a  $P$ -martingale. It follows from Equation 4.19 that  $W(t)^2 - t$ ,  $0 \leq t \leq T$  is a  $Q$ -martingale.  $\square$

Now we are ready to look at the “new” Girsanov Theorem. For a comparison, we restate Theorem 4.22.

**Theorem 4.27.** (Girsanov Theorem) Let  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  and assume that  $E_P[\mathcal{E}_h(t)] = 1$  for all  $t \in [0, T]$ . Then the stochastic process

$$W(t) = B(t) - \int_0^t h(s) ds, \quad 0 \leq t \leq T$$

is a Brownian motion with respect to the probability measure  $Q$  defined by  $dQ = \mathcal{E}_h(T) dP$ , namely  $Q(A) = \int_A \mathcal{E}_h(T) dP$  for  $A \in \mathcal{F}$ .

**Theorem 4.28.** (Girsanov Theorem) Let  $h \in L^2_{ad}([0, T] \times \Omega)$  satisfy the conditions

(a)  $E_P \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt < \infty,$

(b)  $E_P \int_0^T h(t)^2 B(t)^8 dt < \infty,$

(c)  $E_P \left( \int_0^T h(t)^2 dt \right)^5 < \infty,$

(d)  $E_P \int_0^T \mathcal{E}_h(t)^4 dt < \infty.$

Then the stochastic process

$$W(t) = B(t) - \int_0^t h(s) ds, \quad 0 \leq t \leq T$$

is a Brownian motion with respect to the probability measure  $Q$  defined by  $dQ = \mathcal{E}_h(T) dP$ , namely  $Q(A) = \int_A \mathcal{E}_h(T) dP$  for  $A \in \mathcal{F}$ .

*Remark 4.29.* Theorem 4.28 can be generalized into the multidimensional setting.

*Proof.* First note that by the discussion preceding Theorem 4.25, the probability measures  $P$  and  $Q$  are equivalent. Hence  $Q\{W(0) = 0\} = 1$  and  $W(t)$  is a continuous stochastic process. Let  $\{\mathcal{F}_t\}$  be the filtration given by  $\mathcal{F}_t = \sigma\{B(s); 0 \leq s \leq t\}$ ,  $0 \leq t \leq T$ . By Theorem 4.25,  $W(t)$  and  $W(t)^2 - t$  are martingales with respect to  $Q$  and  $\mathcal{F}_t$ . Thus the Doob-Meyer decomposition of  $W(t)^2$  is given by

$$W(t)^2 = [W(t)^2 - t] + t.$$

So  $\langle W \rangle_t = t$  almost surely with respect to  $Q$  for each  $t$ . Hence by the Lévy Characterization Theorem of Brownian motion (Theorem 3.37),  $W(t)$  is a Brownian motion with respect to  $Q$ .  $\square$

## 4.4 Some Examples

Let us now consider some ways the Girsanov theorem may be applied.

*Example 4.30.* Let  $h(t)$  be a deterministic function. The corresponding exponential process is

$$\mathcal{E}_h(t) = e^{\int_0^t h(s) dB(s) - \frac{1}{2} \int_0^t h(s)^2 ds} = e^{-\frac{1}{2} \int_0^t h(s)^2 ds} e^{\int_0^t h(s) dB(s)}.$$

Then

$$E_P(\mathcal{E}_h(t)^4) = e^{-2 \int_0^t h(s)^2 ds} E_P\left(e^{4 \int_0^t h(s) dB(s)}\right) = e^{-2 \int_0^t h(s)^2 ds} e^{2 \int_0^t h(s)^2 ds} = 1,$$

$$(a) \quad E_P \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt = \int_0^T h(t)^2 E(\mathcal{E}_h(t)^4) dt = \int_0^T h(t)^2 dt < \infty.$$

(b) It is a fact that  $E|B(t)|^{2n} \leq C|t|^n$ , where  $C$  is a constant. Thus

$$E_P \int_0^T h(t)^2 B(t)^8 dt = \int_0^T h(t)^2 E_P(B(t)^8) dt \leq \int_0^T C t^4 h(t)^2 dt < \infty.$$

(c)  $E_P\left(\int_0^T h(t)^2 dt\right)^5 < \infty$  since  $h(t)$  is a deterministic function in  $L^2[0, T]$ .

$$(d) \quad E_P \int_0^T \mathcal{E}_h(t)^4 dt = \int_0^T E_P(\mathcal{E}_h(t)^4) dt = \int_0^T 1 dt = T < \infty.$$

So  $h(t)$  satisfies all the conditions in Theorem 4.22. Therefore the stochastic process  $W(t) = B(t) - \int_0^t h(s) ds$ ,  $0 \leq t \leq T$ , for deterministic  $h$  is a Brownian motion with respect to the probability measure defined by  $dQ = e^{\int_0^t h(s) dB(s) - \frac{1}{2} \int_0^t h(s)^2 ds} dP$ .

*Example 4.31.* Consider the function  $h(t) = \text{sgn}(B(t))$ . The corresponding exponential process is

$$\mathcal{E}_h(t) = e^{\int_0^t \text{sgn}(B(s)) dB(s) - \frac{1}{2} \int_0^t 1 ds} = e^{-\frac{1}{2}t} e^{\int_0^t \text{sgn}(B(s)) dB(s)}.$$



In Examples 3.10 and 3.39, we saw that the process  $X_t = \int_0^t \text{sgn}(B(s)) dB(s)$ ,  $0 \leq t \leq T$ , is a Brownian motion with respect to the probability measure  $P$ , with mean 0 and variance  $t$ . So

$$\begin{aligned} E_P [\mathcal{E}_h(t)^4] &= e^{-2t} E_P \left[ e^{4 \int_0^t \text{sgn}(B(s)) dB(s)} \right] \\ &= e^{-2t} \int_{\mathbb{R}} e^{4x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2t}} dx \\ &= e^{-2t} e^{8t} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-4t)^2}{2t}} dx \\ &= e^{6t} \end{aligned}$$

$$(a) \quad E_P \int_0^T (h(t))^2 \mathcal{E}_h(t)^4 dt = \int_0^T E_P [\mathcal{E}_h(t)^4] dt = \int_0^T e^{6t} dt < \infty.$$

$$(b) \quad E_P \int_0^T h(t)^2 B(t)^8 dt = E_P \int_0^T B(t)^8 dt = \int_0^T E_P (B(t)^8) dt < \int_0^T Ct^4 dt < \infty.$$

$$(c) \quad E_P \left( \int_0^T h(t)^2 dt \right)^5 = E_P \left( \int_0^T 1 dt \right)^5 = E_P(T^5) < \infty.$$

$$(d) \quad E_P \int_0^T \mathcal{E}_h(t)^4 dt = \int_0^T E_P(\mathcal{E}_h(t)^4) dt = \int_0^T 1 dt = T < \infty.$$

So  $h(t) = \text{sgn}(B(t))$  satisfies all the conditions in Theorem 4.22. Therefore the stochastic process  $W(t) = B(t) - \int_0^t \text{sgn}(B(s)) ds$ ,  $0 \leq t \leq T$ , is a Brownian motion with respect to the probability measure defined by

$$\begin{aligned} dQ &= e^{\int_0^t h(s) dB(s) - \frac{1}{2} \int_0^t h(s)^2 ds} dP \\ &= e^{-\frac{1}{2}t} e^{\int_0^t h(s) dB(s)} dP. \end{aligned}$$

## 4.5 Comparison of Sufficient Conditions of Girsanov Theorem

In 1960, Girsanov [7] raised the problem of finding a sufficient condition for the exponential process  $\mathcal{E}_h(t)$ ,  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  to be a martingale. Since then many sufficient conditions have been found, for example Novikov [17], Kazamaki [13], Gihman and Skorohod [6], Liptser and Shiryaev [15] and Okada [19]. In this section, we compare some of these conditions for  $h \in L_{ad}^2([0, T] \times \Omega)$ .

Consider a probability space  $(\Omega, \mathcal{F}, P)$ . Throughout this section, the expectation is taken with respect to  $P$  and  $B(t)$  is a Brownian motion with respect to  $P$ . By referring to Theorem 4.6 and Theorem 4.22, we can see that the problem of finding a sufficient condition for the exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$  given by  $h \in L_{ad}^2([0, T] \times \Omega)$  to be a martingale is equivalent to finding sufficient conditions for the validity of the Girsanov Theorem. We restate these two theorems.

**Theorem 4.32.** *Let  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$ . Then the exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$ , given by  $h$  is a martingale if and only if  $E[\mathcal{E}_h(t)] = 1$ , for all  $t \in [0, T]$ .*

**Theorem 4.33.** *Let  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  and assume that  $E[\mathcal{E}_h(t)] = 1$  for all  $t \in [0, T]$ . Then the stochastic process*

$$W(t) = B(t) - \int_0^t h(s) ds, \quad 0 \leq t \leq T$$

*is a Brownian motion with respect to the probability measure  $Q$  defined by  $dQ = \mathcal{E}_h(T) dP$ , namely  $Q(A) = \int_A \mathcal{E}_h(T) dP$  for  $A \in \mathcal{F}$ .*

From Theorem 4.8 and Theorem 4.11, note that for  $h \in L_{ad}^2([0, T] \times \Omega)$ , we have the following sufficient condition:

**Theorem 4.34.** *For  $h \in L_{ad}^2([0, T] \times \Omega)$ , the exponential process  $\mathcal{E}_h(t)$ ,  $0 \leq t \leq T$  is a martingale, if*

$$E \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt < \infty.$$

Now we look at some of the sufficient conditions mentioned at the start of this section.

**Theorem 4.35.** (Novikov) *Let  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  and let*

$$\mathcal{E}_h(t) = e^{\int_0^t h(s) dB(s) - \int_0^t h(s)^2 ds}, \quad 0 \leq t \leq T,$$

be the exponential process given by  $h$ . If

$$E \left[ e^{\frac{1}{2} \int_0^T h(t)^2 dt} \right] < \infty,$$

then  $E[\mathcal{E}_h(T)] = 1$ .

*Proof.* See [17]. □

**Theorem 4.36.** (Kazamaki) *If*

$$E \left[ e^{\frac{1}{2} \int_0^t h(s) dB(s)} \right] < \infty$$

for each  $0 \leq t \leq T$  and  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$ , then  $E[\mathcal{E}_h(T)] = 1$ .

*Proof.* See [13]. □

**Theorem 4.37.** (Gihman and Skorohod) *Suppose that for some number  $\delta > 0$ ,*

$$E \left[ e^{(1+\delta) \int_0^T h(t)^2 dt} \right] < \infty$$

for  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$ . Then  $E[\mathcal{E}_h(T)] = 1$ .

*Proof.* See [6], [10]. □

**Theorem 4.38.** (Liptser and Shiryaev) *Suppose that for some number  $\delta > 0$ ,*

$$E \left[ e^{(\frac{1}{2}+\delta) \int_0^T h(t)^2 dt} \right] < \infty$$

for  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$ . Then  $E[\mathcal{E}_h(T)] = 1$ .

*Proof.* See [15]. □

**Theorem 4.39.** (Gihman and Skorohod) *If there exists  $\alpha > 0$  such that for each  $t$  with  $t + \alpha \leq T$ ,*

$$E \left[ e^{(1+\delta) \int_t^{t+\alpha} h(s)^2 ds} \right] < \infty$$

for  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$  and some  $\delta > 0$ , then  $E[\mathcal{E}_h(T)] = 1$ .

*Proof.* See [10]. □

**Corollary 4.40.** *Suppose that there exists  $\varepsilon > 0$  and a constant  $C > 0$  such that*

$$E \left[ e^{\varepsilon h(t)^2} \right] \leq C$$

*for each  $t \in [0, T]$  and  $h \in \mathcal{L}_{ad}(\Omega, L^2[0, T])$ , then  $E[\mathcal{E}_h(T)] = 1$ .*

*Proof.* Fix  $\lambda > 1$  and choose a finite partition  $\{t_j\}$  of  $[0, T]$  such that  $\lambda t_1, \lambda(t_2 - t_1), \dots, \lambda(t_T - t_{n-1})$  are all less than  $\varepsilon$ . Then by Jensen's inequality (Theorem 2.53 (a)), for each  $j$  we have

$$\begin{aligned} E \left( e^{\left[ \lambda \int_{t_j}^{t_{j+1}} h(s)^2 ds \right]} \right) &\leq E \left( e^{\left[ \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} \varepsilon h(s)^2 ds \right]} \right) \\ &\leq E \left( \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} e^{\varepsilon h(s)^2} ds \right) \\ &= \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} E \left[ e^{\varepsilon h(s)^2} \right] ds \\ &\leq C. \end{aligned}$$

Therefore the conclusion follows from Theorem 4.39. □

**Theorem 4.41.** (Kallianpur) *Suppose that  $\int_0^t h(s)^2 ds$  is locally bounded. That is for every  $t > 0$ , there exists a constant  $C > 0$  such that*

$$\int_0^t h(s)^2 ds \leq C \quad \text{almost surely.}$$

*Then  $E[\mathcal{E}_h(T)] = 1$ .*

*Proof.* See [10]. □

*Remark 4.42.* This theorem is also true as a corollary of Theorem 4.35 since

$$E \left[ e^{\frac{1}{2} \int_0^T h(t)^2 dt} \right] \leq E \left[ e^{\frac{1}{2} C} \right] < \infty.$$

In Chapter 3, we saw that  $L_{ad}^2([0, T] \times \Omega) \subset \mathcal{L}_{ad}(\Omega, L^2[0, T])$ . So we can summarize some of the preceding sufficient conditions for  $h \in L_{ad}^2([0, T] \times \Omega)$  as follows:

**Theorem 4.43.** *For  $h \in L_{ad}^2([0, T] \times \Omega)$ , we get  $E[\mathcal{E}_h(T)] = 1$  under any one of the following:*

- (a)  $E \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt < \infty$ .
- (b)  $E \left[ e^{\frac{1}{2} \int_0^T h(t)^2 dt} \right] < \infty$ .
- (c)  $E \left[ e^{\frac{1}{2} \int_0^T h(t) dB(t)} \right] < \infty$ .
- (d)  $E \left[ e^{(1+\delta) \int_0^T h(t)^2 dt} \right] < \infty$ , for some  $\delta > 0$ .
- (e)  $E \left[ e^{(1+\delta) \int_t^{t+\alpha} h(s)^2 ds} \right] < \infty$ , for  $\alpha > 0$ ,  $0 \leq t \leq T$  and some  $\delta > 0$ .
- (f)  $E \left[ e^{(\frac{1}{2}+\delta) \int_0^T h(t)^2 dt} \right] < \infty$ , for some  $\delta > 0$ .
- (g)  $E \left[ e^{\varepsilon h(t)^2} \right] \leq C$ , for some  $\varepsilon > 0$  and constant  $C$ .

**Theorem 4.44.** *From Theorem 4.43, we have the following implications.*

1. (b)  $\Rightarrow$  (c).
2. (b)  $\Rightarrow$  (a) if  $E[h(t)^4] < \infty$ .
3. (d)  $\Rightarrow$  (b).
4. (d)  $\Rightarrow$  (c).
5. (d)  $\Rightarrow$  (e).
6. (d)  $\Rightarrow$  (f).
7. (f)  $\Rightarrow$  (b).

8. (f)  $\Rightarrow$  (c).

9. (g)  $\Rightarrow$  (e).

*Proof.*

1. Since

$$e^{\frac{1}{2} \int_0^T h(t) dB(t)} = e^{\frac{1}{2} \int_0^T h(t) dB(t) - \frac{1}{4} \int_0^T h(t)^2 dt} e^{\frac{1}{4} \int_0^T h(t)^2 dt},$$

we have by Schwarz's inequality (Theorem 2.52 (b)) that

$$\begin{aligned} E \left[ e^{\frac{1}{2} \int_0^T h(t) dB(t)} \right] &= E \left[ e^{\frac{1}{2} \int_0^T h(t) dB(t) - \frac{1}{4} \int_0^T h(t)^2 dt} e^{\frac{1}{4} \int_0^T h(t)^2 dt} \right] \\ &\leq \left( E \left[ e^{\int_0^T h(t) dB(t) - \frac{1}{2} \int_0^T h(t)^2 dt} \right] \right)^{\frac{1}{2}} \left( E \left[ e^{\frac{1}{2} \int_0^T h(t)^2 dt} \right] \right)^{\frac{1}{2}} \\ &\leq \left( E \left[ e^{\frac{1}{2} \int_0^T h(t)^2 dt} \right] \right)^{\frac{1}{2}} \\ &< \infty. \end{aligned}$$

Thus (b) implies (c).

2. First we show that  $E[\mathcal{E}_h(t)^8] < \infty$ . Note that if  $h \in L_{ad}^2([0, T] \times \Omega)$ , then  $16h \in L_{ad}^2([0, T] \times \Omega)$ . So

$$\begin{aligned} E[\mathcal{E}_h(t)^8] &= E \left[ e^{8 \int_0^T h(t) dB(t) - 4 \int_0^T h(t)^2 dt} \right] \\ &= E \left[ e^{\frac{1}{2} \int_0^T 16h(t) dB(t) - \frac{1}{4} \int_0^T (16h(t))^2 dt} e^{\frac{15}{64} \int_0^T (16h(t))^2 dt} \right] \\ &\leq \left( E \left[ e^{\int_0^T 16h(t) dB(t) - \frac{1}{2} \int_0^T (16h(t))^2 dt} \right] \right)^{\frac{1}{2}} \left( E \left[ e^{\frac{15}{32} \int_0^T (16h(t))^2 dt} \right] \right)^{\frac{1}{2}}. \end{aligned}$$

Since  $E \left[ e^{\int_0^T 16h(t) dB(t) - \frac{1}{2} \int_0^T (16h(t))^2 dt} \right] \leq 1$ , apply Hölder's inequality with  $p = \frac{16}{15}$  and  $q = 16$  to get

$$E \left[ e^{\frac{15}{32} \int_0^T (16h(t))^2 dt} \right] \leq \left( E \left[ e^{\frac{1}{2} \int_0^T (16h(t))^2 dt} \right] \right)^{\frac{15}{16}}.$$

We thus have

$$E[\mathcal{E}_h(t)^8] \leq \left( E \left[ e^{\frac{1}{2} \int_0^T (16h(t))^2 dt} \right] \right)^{\frac{15}{32}} < \infty.$$

Now if  $E[h(t)^4] < \infty$ , we have

$$E[h(t)^2 \mathcal{E}_h(t)^4] \leq (E[h(t)^4])^{\frac{1}{2}} (E[\mathcal{E}_h(t)^8])^{\frac{1}{2}} < \infty.$$

Thus  $E \int_0^T h(t)^2 \mathcal{E}_h(t)^4 dt = \int_0^T E[h(t)^2 \mathcal{E}_h(t)^4] dt < \infty$ . This shows that (b) implies (a).

3. Since  $e^{\frac{1}{2} \int_0^T h(t)^2 dt} \leq e^{(1+\delta) \int_0^T h(t)^2 dt}$ , the implication follows.
4. Since (d) implies (b) and (b) implies (c), the implication follows.
5. Since  $e^{(1+\delta) \int_t^{t+\alpha} h(s)^2 ds} \leq e^{(1+\delta) \int_0^T h(s)^2 ds}$  for  $t+\alpha \leq T$ , the implication follows.
6. Since  $e^{(\frac{1}{2}+\delta) \int_0^T h(t)^2 dt} \leq e^{(1+\delta) \int_0^T h(t)^2 dt}$ , the implication follows.
7. Since  $e^{\frac{1}{2} \int_0^T h(t)^2 dt} \leq e^{(\frac{1}{2}+\delta) \int_0^T h(t)^2 dt}$ , the implication follows.
8. Since (f) implies (b) and (b) implies (c), the implication follows.
9. See Corollary 4.40.

□

# Chapter 5

## Application to Finance

In the previous chapter, we proved the Girsanov theorem for stochastic processes  $h \in L_{ad}^2([0, T] \times \Omega)$  satisfying some new conditions in term of moments. In this chapter, we look at an application of the Girsanov theorem in finance. In particular, with these new conditions, we show the nonexistence of an arbitrage in a market. Then we demonstrate a simplified version of the Black-Scholes model. Throughout this chapter, we consider the probability space  $(\Omega, \mathcal{F}, P)$  and  $B(t)$  is a Brownian motion with respect to  $P$ , unless otherwise stated.

### 5.1 Background from the Theory of Finance

We begin by introducing some definitions and terms in finance theory.

Let  $B_1(t), B_2(t), \dots, B_m(t)$  be  $m$  independent Brownian motions defined on a probability space  $(\Omega, \mathcal{F}, P)$ . Let the filtration  $\{\mathcal{F}_t; t \geq 0\}$  be given by  $\mathcal{F}_t = \sigma\{B_j(s); 1 \leq j \leq m, s \leq t\}$ .

**Definition 5.1.** A *market* is an  $\mathbb{R}^{n+1}$ -valued Itô process

$$X(t) = (X^{(0)}(t), X^{(1)}(t), \dots, X^{(n)}(t)), \quad 0 \leq t \leq T,$$

with the components specified by

$$dX^{(0)}(t) = \rho(t)X^{(0)}(t) dt, \quad X^{(0)}(0) = 1; \quad (5.1)$$

$$dX^{(i)}(t) = \mu_i(t) dt + \sum_{j=1}^m \sigma_{ij}(t) dB_j(t), \quad 1 \leq i \leq n. \quad (5.2)$$

where the adapted stochastic processes  $\rho(t)$ ,  $\mu_i(t)$  and  $\sigma_{ij}(t)$  satisfy the conditions that for  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ ,

$$\int_0^T (|\rho(t)| + |\mu_i(t)| + |\sigma_{ij}(t)|^2) dt < \infty, \quad \text{almost surely.}$$



We usually interpret  $X^{(0)}(t)$  as the unit price of the *safe* investment (e.g., bond or saving in a bank account) and  $X^{(i)}(t)$  as the unit price of the  $i$ th *risky* investment (e.g., shares).

From Equation 5.1, we can derive  $X^{(0)}(t)$  as follows:

$$\begin{aligned} dX^{(0)}(t) &= \rho(t)X^{(0)}(t) dt \\ dX^{(0)}(t) - \rho(t)X^{(0)}(t) dt &= 0 \\ e^{-\int_0^t \rho(s) ds} [dX^{(0)}(t) - \rho(t)X^{(0)}(t) dt] &= 0 \\ d[e^{-\int_0^t \rho(s) ds} X^{(0)}(t)] &= 0 \\ e^{-\int_0^t \rho(s) ds} X^{(0)}(t) &= k \\ X^{(0)}(t) &= ke^{\int_0^t \rho(s) ds}. \end{aligned}$$

Since  $X^{(0)}(0) = 1$ , we get  $k = 1$ . So  $X^{(0)}(t) = e^{\int_0^t \rho(s) ds}$ .

**Definition 5.2.** A *portfolio* is a stochastic process  $\theta(t) = (\theta_0(t), \theta_1(t), \dots, \theta_n(t))$ ,  $0 \leq t \leq T$ , where  $\theta_i(t)$ 's are  $\mathcal{F}_t$ -adapted stochastic processes.

*Remark 5.3.*  $\theta_i(t)$ 's may not be Itô processes.

We interpret  $\theta_i(t)$  as the number of units of the  $i$ th investment.

**Definition 5.4.** The *value* of a portfolio  $\theta(t)$  in a market  $\{X(t)\}$  is given by

$$V^\theta(t) = \sum_{i=0}^n \theta_i(t) X^{(i)}(t) = \theta(t) \cdot X(t),$$

where “ $\cdot$ ” is the dot product.

**Definition 5.5.** A portfolio  $\theta(t)$  is called *self-financing* if its value  $V^\theta(t)$  satisfies

$$V^\theta(t) = V^\theta(0) + \int_0^t \theta(s) \cdot dX(s),$$

which can be written in the stochastic differential form as

$$dV^\theta(t) = \theta(t) \cdot dX(t).$$

We interpret a self-financing portfolio as a system where there is no money being brought in or taken out from it at any time.

*Example 5.6.* Consider the market  $X(t) = (1, B(t))$  and the portfolio  $\theta(t) = (1, 1)$ . Then the value  $V^\theta(t)$  of  $\theta(t)$  is  $V^\theta(t) = 1 + B(t)$ . So  $dV^\theta(t) = dB(t)$ . Also  $dX(t) = (0, dB(t))$ . Thus

$$\theta(t) \cdot dX(t) = dB(t) = dV^\theta(t).$$

By Definition 5.5, the portfolio  $\theta(t) = (1, 1)$  is self-financing in the market  $X(t) = (1, B(t))$ .

*Example 5.7.* Consider the market  $X(t) = (1, B(t))$  but now with the portfolio  $\theta(t) = (1, t)$ . Then the value  $V^\theta(t)$  of  $\theta(t)$  is  $V^\theta(t) = 1 + tB(t)$ . So  $dV^\theta(t) = B(t)dt + tdB(t)$ . On the other hand,  $dX(t) = (0, dB(t))$ . So

$$\theta(t) \cdot dX(t) = tdB(t) \neq dV^\theta(t).$$

Therefore, the portfolio  $\theta(t) = (1, t)$  is not self-financing in the market  $X(t) = (1, B(t))$ .

**Theorem 5.8.** *If the stochastic processes  $\theta_1(t), \dots, \theta_n(t)$  are given, then there exists  $\theta_0(t)$  such that the portfolio  $\theta(t) = (\theta_0(t), \theta_1(t), \dots, \theta_n(t))$  is self-financing.*

*Proof.* We need to find  $\theta_0(t)$  such that  $dV^\theta(t) = \theta(t) \cdot dX(t)$ . Note that

$$\begin{aligned} dV^\theta(t) &= \theta(t) \cdot dX(t) \\ &= \theta_0(t) dX^{(0)}(t) + \sum_{i=1}^n \theta_i(t) dX^{(i)}(t) \\ &= \theta_0(t) \rho(t) X^{(0)}(t) dt + \sum_{i=1}^n \theta_i(t) dX^{(i)}(t) \end{aligned}$$

Hence

$$\begin{aligned}
V^\theta(t) &= V^\theta(0) + \int_0^t \theta_0(s)\rho(s)X^{(0)}(s) ds + \sum_{i=1}^n \int_0^t \theta_i(s) dX^{(i)}(s) \\
\theta_0(t)X^{(0)}(t) + \sum_{i=1}^n \theta_i(t) dX^{(i)}(t) &= V^\theta(0) + \int_0^t \theta_0(s)\rho(s)X^{(0)}(s) ds \\
&\quad + \sum_{i=1}^n \int_0^t \theta_i(s) dX^{(i)}(s). \tag{5.3}
\end{aligned}$$

Let  $Y_0(t) = \theta_0(t)X^{(0)}(t)$ , then Equation 5.3 becomes

$$Y_0(t) = V^\theta(0) + \int_0^t \rho(s)Y_0(s) ds + \sum_{i=1}^n \int_0^t \theta_i(s) dX^{(i)}(s) - \sum_{i=1}^n \theta_i(t) dX^{(i)}(t).$$

Now by writing  $dA(t) = \sum_{i=1}^n \theta_i(s) dX^{(i)}(s) - d(\sum_{i=1}^n \theta_i(t) dX^{(i)}(t))$ , we have

$$\begin{aligned}
dY_0(t) &= \rho(t)Y_0(t) dt + dA(t) \\
dY_0(t) - \rho(t)Y_0(t) dt &= dA(t) \\
e^{-\int_0^t \rho(s) ds} (dY_0(t) - \rho(t)Y_0(t) dt) &= e^{-\int_0^t \rho(s) ds} dA(t) \\
d\left(e^{-\int_0^t \rho(s) ds} Y_0(t)\right) &= e^{-\int_0^t \rho(s) ds} dA(t) \\
d\theta_0(t) &= e^{-\int_0^t \rho(s) ds} dA(t) \\
\theta_0(t) &= \theta_0(0) + \int_0^t e^{-\int_0^s \rho(u) du} dA(s).
\end{aligned}$$

□

**Definition 5.9.** A self-financing portfolio  $\theta(t)$  is called *admissible* if there exists a constant  $K > 0$  such that

$$V^\theta(t, \omega) \geq -K, \quad \text{for almost all } (t, \omega) \in [0, T] \times \Omega_0,$$

where  $P(\Omega_0) = 1$ , namely  $V^\theta(t, \omega)$  is bounded below for all  $t$  and almost surely.

*Example 5.10.* Consider the portfolio  $\theta(t) = \left(-tB(t)^2 + \int_0^t B(s)^2 ds, B(t)^2\right)$  in the market  $X(t) = (1, t)$ . Then the value  $V^\theta(t)$  of  $\theta(t)$  is

$$V^\theta(t) = -tB(t)^2 + \int_0^t B(s)^2 ds + tB(t)^2 = \int_0^t B(s)^2 ds.$$

Hence  $dV^\theta(t) = B(t)^2 dt$ . On the other hand, since  $dX(t) = (0, dt)$ , we have

$$\theta(t) \cdot dX(t) = B(t)^2 dt = dV^\theta(t).$$

This shows that  $\theta(t)$  is self-financing. Also  $\theta(t)$  is admissible because  $V^\theta(t) = \int_0^t B(s)^2 ds$  is always bounded below by  $-K$ , where  $K$  is any positive constant.

*Example 5.11.* Now consider the portfolio  $\theta(t) = \left(-tB(t) + \int_0^t B(s) ds, B(t)\right)$  in the market  $X(t) = (1, t)$ . Then the value  $V^\theta(t)$  of  $\theta(t)$  is

$$V^\theta(t) = -tB(t) + \int_0^t B(s) ds + tB(t) = \int_0^t B(s) ds.$$

Hence  $dV^\theta(t) = B(t) dt$ . On the other hand, since  $dX(t) = (0, dt)$ , we have

$$\theta(t) \cdot dX(t) = B(t) dt = dV^\theta(t).$$

This shows that  $\theta(t)$  is self-financing. However  $\theta(t)$  is not admissible because  $V^\theta(t) = \int_0^t B(s) ds$  is not bounded below.

**Definition 5.12.** An admissible portfolio  $\theta(t)$  is an *arbitrage* in a market  $X(t)$ ,  $0 \leq t \leq T$ , if the corresponding value  $V^\theta(t)$  satisfies the conditions

$$V^\theta(0) = 0, \quad V^\theta(T) \geq 0, \quad P\{V^\theta(T) > 0\} > 0.$$

*Example 5.13.* Consider the portfolio  $\theta(t) = \left(-tB(t)^2 + \int_0^t B(s)^2 ds, B(t)^2\right)$  in the market  $X(t) = (1, t)$ . By Example 5.10,  $\theta(t)$  is an admissible portfolio. Since  $V^\theta(t) = \int_0^t B(s)^2 ds$ , we have  $V^\theta(0) = 0$  and  $V^\theta(T) = \int_0^T B(s)^2 ds \geq 0$ . So  $\theta(t)$  is an arbitrage.

*Example 5.14.* Consider the portfolio  $\theta(t) = \left(-\frac{1}{2}B(t)^2 - \frac{1}{2}t, B(t)\right)$  in the market  $X(t) = (1, B(t))$ . Then the value  $V^\theta(t)$  of  $\theta(t)$  is

$$V^\theta(t) = \left(-\frac{1}{2}B(t)^2 - \frac{1}{2}t\right) + B(t)^2 = \frac{1}{2}(B(t)^2 - t).$$

Hence  $dV^\theta(t) = \frac{1}{2} [(2B(t)dB(t) + dt) - dt] = B(t)dB(t)$ . On the other hand, since  $dX(t) = (0, dB(t))$ , we have  $\theta(t) \cdot dX(t) = B(t)dB(t)$ . This shows that  $\theta(t)$  is self-financing. Since  $V^\theta(t) = \frac{1}{2} (B(t)^2 - t) \geq -\frac{1}{2}T$ , it follows that  $\theta(t)$  is admissible. We have that  $V^\theta(0) = 0$ . However  $V^\theta(T) = \frac{1}{2} (B(T)^2 - T) \not\geq 0$  almost surely, so  $\theta(t)$  is not an arbitrage.

**Definition 5.15.** A market  $X(t) = (X^{(0)}(t), X^{(1)}(t), \dots, X^{(n)}(t))$  is *normalized* if  $X^{(0)}(t) = 1$ . A *normalization* of a market  $X(t) = (X^{(0)}(t), X^{(1)}(t), \dots, X^{(n)}(t))$  is the market

$$\tilde{X}(t) = \left( 1, \frac{X^{(1)}(t)}{X^{(0)}(t)}, \dots, \frac{X^{(n)}(t)}{X^{(0)}(t)} \right) = \frac{1}{X^{(0)}(t)} X(t).$$

**Theorem 5.16.** *Suppose the portfolio  $\theta(t)$  is self-financing in a market  $X(t)$ , then it is also self-financing in the normalized market  $\tilde{X}(t)$ .*

*Proof.* First note that since

$$\tilde{X}(t) = \frac{1}{X^{(0)}(t)} X(t) = \xi(t)X(t),$$

where  $\xi(t) = (X^{(0)}(t))^{-1} = e^{-\int_0^t \rho(s) ds}$ , we have  $d\tilde{X}(t) = \xi(t) [dX(t) - \rho(t)X(t) dt]$ .

Let  $\tilde{V}^\theta(t)$  be the value of  $\theta(t)$  in  $\tilde{X}(t)$ . Then

$$\tilde{V}^\theta(t) = \theta(t) \cdot \tilde{X}(t) = \theta(t) \cdot \xi(t)X(t) = \xi(t) [\theta(t) \cdot X(t)] = \xi(t)V^\theta(t).$$

So by the Itô product formula (Equation 3.8),

$$\begin{aligned} d\tilde{V}^\theta(t) &= \xi(t) dV^\theta + V^\theta d\xi(t) + [d\xi(t)][dV^\theta(t)] \\ &= \xi(t) \theta(t) \cdot dX(t) - \theta(t) \cdot X(t) \rho(t) \xi(t) dt + 0 \\ &= \theta(t) \cdot [\xi(t) \{dX(t) - \rho(t)X(t) dt\}] \\ &= \theta(t) \cdot d\tilde{X}(t). \end{aligned}$$

Therefore,  $\theta(t)$  is self-financing in  $\tilde{X}(t)$ . □

**Theorem 5.17.** *If the portfolio  $\theta(t)$  is admissible in the market  $X(t)$ , then it is admissible in the normalized market  $\tilde{X}(t)$ .*

*Proof.* The values of  $\theta(t)$  in the markets  $X(t)$  and  $\tilde{X}(t)$  are, respectively,

$$V^\theta(t) = \theta(t) \cdot X(t) \quad \text{and} \quad \tilde{V}^\theta(t) = \theta(t) \cdot \tilde{X}(t).$$

Note that since  $\tilde{X}(t) = \xi(t)X(t)$ , where  $\xi(t) = e^{-\int_0^t \rho(s) ds} > 0$ , we can write

$$\tilde{V}^\theta(t) = \theta(t) \cdot \xi(t)X(t) = \xi(t)[\theta(t) \cdot X(t)] = \xi(t)V^\theta(t).$$

But  $V^\theta(t) \geq -K$  (by the admissibility of  $\theta(t)$  in  $X(t)$ ), thus we have

$$\tilde{V}^\theta(t) = \xi(t)V^\theta(t) \geq -K,$$

i.e.,  $\tilde{V}^\theta(t)$  is admissible in  $\tilde{X}(t)$ . □

**Theorem 5.18.** *If the portfolio  $\theta(t)$  is an arbitrage in the market  $X(t)$ , then it is also an arbitrage in the normalized market  $\tilde{X}(t)$ .*

*Proof.* Consider  $\tilde{V}^\theta(t) = e^{-\int_0^t \rho(s) ds} V^\theta(t)$ , the value of  $\theta(t)$  in the market  $\tilde{X}(t)$ . By Theorem 5.17,  $\theta(t)$  is admissible in  $\tilde{X}(t)$ . Then

(a)  $\tilde{V}^\theta(0) = 0,$

(b)  $\tilde{V}^\theta(T) = e^{-\int_0^T \rho(s) ds} V^\theta(T) > 0$  almost surely,

(c)  $P \left\{ \tilde{V}^\theta(T) > 0 \right\} = P \left\{ V^\theta(T) > 0 \right\} > 0.$

So  $\theta(t)$  is an arbitrage in the normalized market  $\tilde{X}(t)$ . □

## 5.2 Nonexistence of an Arbitrage

According to Definition 5.12, given that a portfolio  $\theta(t)$  is an arbitrage means there is an increase in the value of the portfolio from time  $t = 0$  to time  $t = T$  almost surely, and a strictly positive increase with positive probability. So  $\theta(t)$  generates a profit without any risk of losing money. This clearly contradicts the real life situation in finance. So how can we decide if a given market  $X(t)$  allows an arbitrage or not? The following gives a simple but useful result.

**Lemma 5.19.** *If  $Y(t)$  is a local martingale with respect to a probability measure  $Q$  and  $Y(t)$  is bounded below, then  $Y(t)$  is a supermartingale.*

*Proof.* By the definition of a local martingale (Definition 2.46), there exists an increasing sequence  $\{\tau_n\}$  of stopping times such that  $Y(t \wedge \tau_n)$  is a martingale, i.e.,

$$E_Q [Y(t \wedge \tau_n) | \mathcal{F}_s] = Y(s \wedge \tau_n) \quad s \leq t.$$

By letting  $n \rightarrow \infty$ , we get

$$\liminf E_Q [Y(t \wedge \tau_n) | \mathcal{F}_s] = \liminf Y(s \wedge \tau_n) = Y(s).$$

Since  $Y(t)$  is bounded below, by Fatou's lemma for conditional expectation ([1], Theorem 5.5.6 (b), page 223),

$$E_Q [Y(t) | \mathcal{F}_s] = E_Q [\liminf \{Y(t \wedge \tau_n) | \mathcal{F}_s\}] \leq \liminf E_Q [Y(t \wedge \tau_n) | \mathcal{F}_s] = Y(s).$$

So  $Y(t)$  is a supermartingale. □

**Lemma 5.20.** *Suppose there exists a probability measure  $Q$  on the filtration  $\{\mathcal{F}_t\}$  such that  $Q$  is equivalent to  $P$  and the normalized market  $\tilde{X}(t)$  is a local martingale with respect to  $Q$ . Then the market  $X(t)$  has no arbitrage.*

*Proof.* Suppose  $\theta(t)$  is an arbitrage in  $X(t)$ . Then  $\theta(t)$  is also an arbitrage in  $\tilde{X}(t)$  by Theorem 5.18.

Let  $\tilde{V}^\theta(t) = \theta(t) \cdot \tilde{X}(t)$  be the value of  $\theta(t)$  in  $\tilde{X}(t)$ . Since an arbitrage is self-financing and  $\tilde{V}^\theta(0) = 0$ , we have  $d\tilde{V}^\theta(t) = \theta(t) \cdot d\tilde{X}(t)$  and thus

$$\tilde{V}^\theta(t) = \int_0^t \theta(s) \cdot d\tilde{X}(s).$$

Hence  $\tilde{V}^\theta(t)$  is a local martingale with respect to  $Q$ . Also, by the admissibility of  $\theta(t)$ ,  $\tilde{V}^\theta(t)$  is bounded below, i.e., there exists  $K > 0$  such that  $\tilde{V}^\theta(t, \omega) \geq -K$  for almost all  $t \in [0, T]$ ,  $\omega \in \Omega$ . By Lemma 5.19,  $\tilde{V}^\theta(t)$  is a supermartingale with respect to  $Q$ . So  $E_Q[\tilde{V}^\theta(t)] \leq E_Q[\tilde{V}^\theta(0)] = 0$ .

On the other hand, since  $\theta(t)$  is an arbitrage, we have  $\tilde{V}^\theta(T) \geq 0$   $P$ -almost surely and  $P\{\tilde{V}^\theta(T) > 0\} > 0$ . Hence  $\tilde{V}^\theta(T) \geq 0$   $Q$ -almost surely and  $Q\{\tilde{V}^\theta(T) > 0\} > 0$  because  $Q$  is equivalent to  $P$ . So  $E_Q[\tilde{V}^\theta(T)] > 0$ , which is a contradiction. Therefore the market  $\tilde{X}(t)$  has no arbitrage, likewise for  $X(t)$ .  $\square$

In the next theorem, we give a sufficient condition for the nonexistence of an arbitrage in a market.

Let  $\rho(t)$ ,  $\mu_i(t)$  and  $\sigma_{ij}(t)$  be processes as in Definition 5.1. We write  $\mu(t) = (\mu_1(t), \dots, \mu_n(t))$  and let  $\sigma(t)$  be the  $(n \times m)$ -matrix with  $ij$ th entries  $\sigma_{ij}$ . Also let  $\hat{X}(t)$  to be  $\hat{X}(t) = (X^{(1)}(t), \dots, X^{(n)}(t))$ . So we can write Equation 5.2 as

$$d\hat{X}(t) = \mu(t) dt + \sigma(t) dB(t). \tag{5.4}$$

**Theorem 5.21.** *Suppose that there exists an  $(m \times 1)$ -column vector valued  $\mathcal{F}_t$ -adapted stochastic process  $h(t)$  satisfying the following conditions:*

- (a)  $\sigma(t, \omega) h(t, \omega) = \rho(t, \omega) \hat{X}(t) - \mu(t, \omega)$  for almost all  $(t, \omega) \in [0, T] \times \Omega$ ,
- (b)  $E \int_0^T |h(t)|^2 \mathcal{E}_h(t)^4 dt < \infty$ ,



$$(c) \ E \int_0^T |h(t)|^2 |B(t)|^8 dt < \infty,$$

$$(d) \ E \left( \int_0^T |h(t)|^2 dt \right)^5 < \infty,$$

$$(e) \ E \int_0^T \mathcal{E}_h(t)^4 dt < \infty,$$

where  $B(t)$  is an  $m$ -dimensional Brownian motion and  $\mathcal{E}_h(t)$  is the exponential process

$$\mathcal{E}_h(t) = e^{\int_0^t h(s) \cdot dB(s) - \frac{1}{2} \int_0^t |h(s)|^2 ds}, \quad 0 \leq t \leq T.$$

Then the market  $X(t)$  has no arbitrage.

*Proof.* By Theorem 5.18, we can assume that the market  $X(t)$  is normalized, namely  $\rho(t) = 0$ .

Define the probability measure  $Q$  given by  $dQ = \mathcal{E}_h(T) dP$ . Then  $Q$  is equivalent to  $P$ . By conditions (b) to (e), we can apply Theorem 4.28 (in the multi-dimensional setting) to get that the process

$$W(t) = B(t) - \int_0^t h(s) ds, \quad 0 \leq t \leq T,$$

is an  $m$ -dimensional Brownian motion with respect to  $Q$ . By Equation 5.4, we have

$$\begin{aligned} d\widehat{X}(t) &= \mu(t) dt + \sigma(t) dB(t) \\ &= \mu(t) dt + \sigma(t)[dW(t) + h(t)dt] \\ &= \sigma(t)dW(t) + [\mu(t) + \sigma(t)h(t)] dt \\ &= \sigma(t)dW(t) + \rho(t)\widehat{X}(t) dt \\ &= \sigma(t)dW(t) \quad (\rho(t) = 0). \end{aligned} \tag{5.5}$$

So  $\widehat{X}(t) = \widehat{X}(0) + \int_0^t \sigma(s) dW(s)$ . Thus  $\widehat{X}(t)$  is a local martingale with respect to  $Q$ . By Lemma 5.20,  $X(t)$  has no arbitrage.  $\square$

*Example 5.22.* Consider a normalized market given by

$$dX(t) = (0, dt + dB_1(t) + 2 dB_2(t), -dt + dB_1(t) + dB_2(t))$$

where  $B_1(t)$  and  $B_2(t)$  are independent Brownian motions.

In this case, we have

$$\sigma(t) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mu(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So by the system equation  $\sigma(t)h(t) = -\mu(t)$  with  $h(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$ , we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Solving this we get  $h(t) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ . By Example 4.30, this process  $h(t)$  satisfies conditions (b) to (e) in Theorem 5.21. So  $X(t)$  has no arbitrage.

## 5.3 Black-Scholes Model

In the preceding section, we showed the nonexistence of an arbitrage in a market. In this section, we demonstrate a simplified version of the Black-Scholes model. The Black-Scholes model was developed in the early 70's by Fischer Black and Myron Scholes, based on earlier research by Edward Thorpe, Paul Samuelson and Robert C. Merton. The Black-Scholes model gives a very useful formula for pricing call options.

**Definition 5.23.** A lower bounded  $\mathcal{F}_T$ -measurable random variable  $\Phi$  is called a *T-claim*. A T-claim  $\Phi$  is said to be *attainable* in a market  $X_t$ ,  $0 \leq t \leq T$  if there

exist a real number  $r$  and an admissible portfolio  $\theta(t)$  such that

$$\Phi = V^\theta(T) = r + \int_0^T \theta(t) \cdot dX(t) \quad \text{almost surely.} \quad (5.6)$$

If such a portfolio  $\theta(t)$  exists, it is called a *hedging portfolio* for  $\Phi$ .

By definition, a  $T$ -claim is attainable if there exists a real number  $r$  such that if we start our fortune with  $r$ , then we can find an admissible portfolio  $\theta(t)$  which generates a value  $V^\theta(T)$  at time  $T$  which equals  $\Phi$  almost surely.

Let  $\tilde{V}^\theta(t)$  be the value of the admissible portfolio  $\theta(t)$  in the normalized market  $\tilde{X}(t) = (X^{(0)}(t))^{-1}X(t) = \xi(t)X(t)$ . By the self-financing property of  $\theta(t)$ ,

$$\tilde{V}^\theta(t) = r + \int_0^t \theta(s) \cdot d\tilde{X}(s) = r + \int_0^t \xi(s)\hat{\theta}(s) \cdot d\hat{X}(s), \quad (5.7)$$

because  $d\tilde{V}^\theta(t) = \xi(t)dV^\theta(t) = \xi(t)d\hat{V}_\theta$ . From Equation 5.5, we have  $d\hat{X}(t) = \sigma(t)dW_h(t)$ , where  $W_h(t) = B(t) - \int_0^t h(s)ds$ . So Equation 5.7 becomes

$$\tilde{V}^\theta(t) = r + \int_0^t \xi(s)\hat{\theta}(s) \cdot (\sigma(t)dW_h(t)). \quad (5.8)$$

By Theorem 3.17,  $V^\theta(t)$  is a local martingale with respect to  $Q$ . For the sake of integrability, the portfolio  $\theta(t)$  in Equation 5.7 is always assumed to have the property that the associated stochastic process  $\tilde{V}^\theta(t)$  in Equation 5.7 is actually a martingale with respect to  $Q$ .

**Definition 5.24.** A market  $X(t)$ ,  $0 \leq t \leq T$  is said to be *complete* if every  $T$ -claim  $\Phi$  is attainable.

The next theorem gives a condition for a market  $X(t)$ ,  $0 \leq t \leq T$  to be complete.

**Theorem 5.25.** Let  $X(t)$ ,  $0 \leq t \leq T$  be a market specified by  $\rho(t)$ ,  $\mu(t)$  and  $\sigma(t)$  as in Definition 5.1. Assume that there exists a process  $h(t)$  such that

$$(a) \quad \sigma(t, \omega)h(t, \omega) = \rho(t, \omega)\hat{X}(t, \omega) - \mu(t, \omega),$$

$$(b) \ E \int_0^T |h(t)|^2 \mathcal{E}_h(t)^4 dt < \infty,$$

$$(c) \ E \int_0^T |h(t)|^2 |B(t)|^8 dt < \infty,$$

$$(d) \ E \left( \int_0^T |h(t)|^2 dt \right)^5 < \infty,$$

$$(e) \ E \int_0^T \mathcal{E}_h(t)^4 dt < \infty.$$

In addition, assume that  $\sigma\{W_h(s); 0 \leq s \leq t\} = \sigma\{B(s); 0 \leq s \leq t\}$ ,  $0 \leq t \leq T$ , where  $W_h(t) = B(t) - \int_0^t h(s) ds$ . Then  $X(t)$ ,  $0 \leq t \leq T$  is complete if  $\sigma(t)$  has a left inverse for almost all  $(t, \omega) \in T \times \Omega$ , i.e., there exists an  $(m \times n)$ -matrix valued adapted stochastic process  $L(t, \omega)$  such that

$$L(t, \omega) \sigma(t, \omega) = I_m, \quad \text{almost everywhere,}$$

where  $I_m$  is the  $m \times m$  identity matrix.

*Remark 5.26.* Conditions (a)-(e) guarantee that the market  $X(t)$  has no arbitrage (Theorem 5.21).

*Proof.* Let  $\Phi$  be a  $T$ -claim. We need to find a real number  $r$  and an admissible portfolio  $\theta(t)$  such that

$$\Phi = V^\theta(T) = r + \int_0^T \theta(t) \cdot dX(t). \quad (5.9)$$

By Equation 5.8, we have

$$\begin{aligned} \xi(T) \Phi &= \xi(T) V^\theta(T) = \tilde{V}^\theta(T) \\ &= r + \int_0^T \xi(t) \hat{\theta}(t) \cdot (\sigma(t) dW_h(t)), \end{aligned} \quad (5.10)$$

where  $\xi(t) = (X^{(0)}(t))^{-1} = e^{-\int_0^t \rho(s) ds}$ . Thus we can first find  $r$  and  $\hat{\theta}(t)$  such that Equation 5.10 holds. Then by Theorem 5.8, we can find  $\theta_0(t)$  to get an admissible  $\theta(t)$  satisfying Equation 5.9.

Note that  $\xi(T) \Phi$  is measurable with respect to  $\mathcal{F}_T^{W_h}$ , due to the assumption that  $\sigma\{W_h(s); 0 \leq s \leq t\} = \sigma\{B(s); 0 \leq s \leq t\}$ ,  $0 \leq t \leq T$ . Hence  $\xi(T) \Phi$  belongs to  $L^2(\mathcal{F}_T^{W_h})$ . By applying Theorem 3.15 to  $\xi(T) \Phi$ , we obtain a stochastic process  $f(t) \in L^2_{ad}([0, T] \times \Omega)$  such that

$$\xi(T) \Phi = E \{ \xi(T) \Phi \} + \int_0^T f(t) dW_h(t). \quad (5.11)$$

By comparing Equations 5.10 and 5.11, we get

$$r = E \{ \xi(T) \Phi \}$$

and  $\widehat{\theta}(t)$  is the solution of the equation

$$\xi(t) \widehat{\theta}(t) \cdot (\sigma(t) v) = f(t) \cdot v, \quad \forall v \in \mathbb{R}^m.$$

This is equivalent to the matrix equation

$$\xi(t) \widehat{\theta}(t)^* \sigma(t) = f(t)^*,$$

where  $\widehat{\theta}(t)^*$  denotes the *transpose* of  $\widehat{\theta}(t)$ . Equivalently,

$$\sigma(t)^* \widehat{\theta}(t) = X^{(0)}(t) f(t). \quad (5.12)$$

By hypothesis, there exists an  $(m \times n)$ -matrix valued stochastic process  $L(t)$  such that  $L(t) \sigma(t) = I_m$ . Hence  $\sigma(t)^* L(t)^* = I_m$ . Thus if  $\widehat{\theta}(t) = X^{(0)} L(t)^* f(t)$ , then

$$\sigma(t)^* \widehat{\theta}(t) = \sigma(t)^* (X^{(0)} L(t)^* f(t)) = X^{(0)}(t) \sigma(t)^* L(t)^* f(t) = X^{(0)}(t) f(t).$$

This shows that  $\widehat{\theta}(t) = X^{(0)} L(t)^* f(t)$  is a solution of Equation 5.12.

Finally by Theorem 5.8, we can find  $\theta_0(t)$  such that  $\theta(t) = (\theta_0(t), \widehat{\theta}(t))$  is a hedging portfolio for the  $T$ -claim  $\Phi$ . Therefore the market  $X(t)$  is complete.  $\square$

*Example 5.27.* Consider a market  $X(t) = (X_0(t), X_1(t), X_2(t), X_3(t))$  satisfying  $X_0(t) = 1$  and

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \\ dX_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}.$$

Then  $\rho(t) = 0$ ,  $\mu(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $\sigma(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ . Note that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

namely there is a left inverse for  $\sigma(t)$ . Also from the equation  $\sigma(t)h(t) = \rho(t)\widehat{X}(t) - \mu(t)$ , we can get  $h(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . Since  $h(t)$  is a constant, Conditions (b)-(e) in Theorem 5.25 are satisfied. Therefore by Theorem 5.25, the market  $X(t)$  is complete.

**Definition 5.28.** A (European) *option* on a  $T$ -claim  $\Phi$  is a guarantee to pay the amount  $\Phi$  at time  $t = T$ .

It is natural to raise the question: what is the “price” that one is willing to pay or to sell for an option at time  $t = 0$ ? Suppose a buyer pays an amount  $y$  for an option. With this initial fortune (debt)  $-y$ , the buyer wishes that he could hedge to time  $T$  a value  $V_{-y}^\theta(T)$  such that

$$V_{-y}^\theta(T) + \Phi \geq 0, \quad \text{almost surely.}$$

This is equivalent to saying that the buyer can hedge a portfolio  $\theta(t)$  such that

$$-y + \int_0^T \theta(t) dX(t) + \Phi \geq 0, \quad \text{almost surely.}$$

So the maximal “price” a buyer is willing to pay for an option at time  $t = 0$  is

$$\mathcal{P}_b(\Phi) = \sup \left\{ y; \exists \theta(t) \text{ such that } -y + \int_0^T \theta(t) dX(t) + \Phi \geq 0 \text{ a.s.} \right\}.$$

On the other hand, suppose a seller receives the “price”  $z$  for this guarantee. Then the seller can use this as the initial value in an investment strategy. With this initial value  $z$ , the seller wishes he could hedge to time  $T$  a value  $V_z^\theta(T)$  such that

$$V_z^\theta(T) \geq \Phi \quad \text{almost surely.}$$

This is equivalent to saying that the seller can hedge a portfolio  $\psi(t)$  such that

$$z + \int_0^T \psi(t) dX(t) \geq \Phi \quad \text{almost surely.}$$

So the minimal “price” a seller is willing to accept for an option at time  $t = 0$  is

$$\mathcal{P}_s(\Phi) = \inf \left\{ z; \exists \psi(t) \text{ such that } z + \int_0^T \psi(t) dX(t) \geq \Phi \text{ a.s.} \right\}.$$

In general,  $\mathcal{P}_b(\Phi) \leq \mathcal{P}_s(\Phi)$ . In fact we have the following:

**Theorem 5.29.** *Let  $X(t)$ ,  $0 \leq t \leq T$  be a market specified by  $\rho(t)$ ,  $\mu(t)$  and  $\sigma(t)$  as in Definition 5.1. Suppose there exists a process  $h(t)$  such that*

$$(a) \quad \sigma(t, \omega) h(t, \omega) = \rho(t, \omega) \widehat{X}(t, \omega) - \mu(t, \omega),$$

$$(b) \quad E \int_0^T |h(t)|^2 \mathcal{E}_h(t)^4 dt < \infty,$$

$$(c) \quad E \int_0^T |h(t)|^2 |B(t)|^8 dt < \infty,$$

$$(d) \quad E \left( \int_0^T |h(t)|^2 dt \right)^5 < \infty,$$

$$(e) \quad E \int_0^T \mathcal{E}_h(t)^4 dt < \infty.$$

Then for any  $T$ -claim  $\Phi$ ,

$$\text{essinf}(\Phi) \leq \mathcal{P}_b(\Phi) \leq E_Q[\xi(T)\Phi] \leq \mathcal{P}_s(\Phi) \leq \infty,$$

where  $Q$  is the probability measure given by  $dQ = e^{\int_0^T h(t) dB(t) + \frac{1}{2} \int_0^T h(t)^2 dt} dP$  and  $\xi(t) = (X^{(0)}(t))^{-1} = e^{-\int_0^t \rho(s) ds}$ .

*Proof.* By definition,  $\text{essinf}(\Phi) = \sup\{b \in \mathbb{R}; P(\{\Phi < b\}) = 0\}$ . If  $x \in \text{essinf}(\Phi)$ , then  $\Phi \geq x$  almost surely, thus  $-x \geq -\Phi$  almost surely. By taking the portfolio  $\theta(t) = 0$ , we get that

$$x \in \left\{ y; \exists \theta(t) \text{ such that } -y + \int_0^T \theta(t) dX(t) + \Phi \geq 0 \text{ a.s.} \right\}.$$

So  $\text{essinf}(\Phi) \leq \mathcal{P}_b(\Phi)$ .

Suppose  $y \in \mathbb{R}$  and there exists  $\theta(t)$  such that  $-y + \int_0^T \theta(t) dX(t) \geq -\Phi$  almost surely. This is equivalent to

$$-y + \int_0^T \xi(t) \hat{\theta}(t) \sigma(t) dW_h(t) \geq -\xi(T)\Phi,$$

because  $dV^\theta(t) = \xi(t) d\tilde{V}^\theta(t) = \xi(t) \hat{\theta}(t) d\hat{X}(t)$  and  $d\hat{X}(t) = \sigma(t) dW_h(t)$ . By taking the expectation with respect to  $Q$ , we have  $-y + 0 \geq -E_Q[\xi(T)\Phi]$ , i.e.,

$$y \leq E_Q[\xi(T)\Phi]. \quad (5.13)$$

Since this is true for any  $y$  satisfying Inequality 5.13, by taking the supremum we get  $\mathcal{P}_b(\Phi) \leq E_Q[\xi(T)\Phi]$ .

Suppose  $z \in \mathbb{R}$  and there exists  $\psi(t)$  such that  $z + \int_0^T \psi(t) dX(t) \geq \Phi$  almost surely. This is equivalent to

$$z + \int_0^T \xi(t) \hat{\psi}(t) \sigma(t) dW_h(t) \geq \xi(T)\Phi.$$

By taking the expectation with respect to  $Q$ , we have  $z + 0 \geq E_Q[\xi(T)\Phi]$ , i.e.,

$$z \geq E_Q[\xi(T)\Phi]. \quad (5.14)$$



Since this is true for any  $z$  satisfying Inequality 5.14, by taking the infimum we get  $\mathcal{P}_s(\Phi) \geq E_Q [\xi(T) \Phi]$ , provided such  $z$  and  $\psi(t)$  exist. If no such  $z$  and  $\psi(t)$  exist, then  $\mathcal{P}_s(\Phi) = \infty > E_Q [\xi(T) \Phi]$ .  $\square$

**Definition 5.30.** The *price* of a  $T$ -claim  $\Phi$  is said to exist if  $\mathcal{P}_b(\Phi) = \mathcal{P}_s(\Phi)$ . The common value, denoted by  $\mathcal{P}(\Phi)$  is called the *price* of  $\Phi$  at time  $t = 0$ .

In addition to the conditions in Theorem 5.29, if the market is complete and  $E_Q [\xi(T) \Phi] < \infty$ , then the price of a  $T$ -claim  $\Phi$  exists.

**Theorem 5.31.** Let  $X(t)$ ,  $0 \leq t \leq T$  be a complete market specified by  $\rho(t)$ ,  $\mu(t)$  and  $\sigma(t)$  as in Definition 5.1. Suppose there exists a process  $h(t)$  such that

$$(a) \quad \sigma(t, \omega) h(t, \omega) = \rho(t, \omega) \widehat{X}(t, \omega) - \mu(t, \omega),$$

$$(b) \quad E \int_0^T |h(t)|^2 \mathcal{E}_h(t)^4 dt < \infty,$$

$$(c) \quad E \int_0^T |h(t)|^2 |B(t)|^8 dt < \infty,$$

$$(d) \quad E \left( \int_0^T |h(t)|^2 dt \right)^5 < \infty,$$

$$(e) \quad E \int_0^T \mathcal{E}_h(t)^4 dt < \infty.$$

Moreover if  $E_Q [\xi(T) \Phi] < \infty$  for a  $T$ -claim  $\Phi$  in  $X(t)$ , then

$$\mathcal{P}(\Phi) = E_Q [\xi(T) \Phi],$$

where  $Q$  is the probability measure given by  $dQ = e^{\int_0^T h(t) dB(t) + \frac{1}{2} \int_0^T h(t)^2 dt} dP$  and  $\xi(t) = (X^{(0)}(t))^{-1} = e^{-\int_0^t \rho(s) ds}$ .

*Proof.* Let  $\Phi$  be a  $T$ -claim. By the completeness of the market  $X(t)$ , there exists  $r_\psi \in \mathbb{R}$  and a portfolio  $\psi(t)$  such that

$$\Phi = r_\psi + \int_0^T \psi(t) dX(t), \quad \text{almost surely.}$$

This is equivalent to saying

$$\xi(T)\Phi = r_\psi + \int_0^T \xi(t)\widehat{\psi}(t) \cdot \sigma(t) dW_h(t).$$

By taking the expectation with respect to  $Q$ , we have

$$E_Q [\xi(T)\Phi] = r_\psi.$$

But  $r_\psi \in \{y; \exists \psi(t) \text{ such that } y + \int_0^T \psi(t) dX(t) \geq \Phi \text{ almost surely}\}$ , hence

$$E_Q [\xi(T)\Phi] \geq \mathcal{P}_s(\Phi).$$

Together with  $E_Q [\xi(T)\Phi] \leq \mathcal{P}_s(\Phi)$  from Theorem 5.29, we get  $E_Q [\xi(T)\Phi] = \mathcal{P}_s(\Phi)$ .

Similarly, by the completeness of  $X(t)$ , there exists  $r_\theta \in \mathbb{R}$  and a portfolio  $\theta(t)$  such that

$$-r_\theta + \int_0^T \theta(t) dX(t) = -\Phi, \quad \text{almost surely.}$$

This is equivalent to saying

$$-\xi(T)\Phi = -r_\theta + \int_0^T \xi(t)\widehat{\theta}(t) \cdot \sigma(t) dW_h(t).$$

By taking the expectation with respect to  $Q$ , we have

$$-E_Q [\xi(T)\Phi] = -r_\theta$$

$$E_Q [\xi(T)\Phi] = r_\theta.$$

But  $r_\theta \in \{x; \exists \theta(t) \text{ such that } -x + \int_0^T \theta(t) dX(t) + \Phi \geq 0, \text{ almost surely}\}$ , thus

$$E_Q [\xi(T)\Phi] \leq \mathcal{P}_b(\Phi).$$

Together with  $\mathcal{P}_b(\Phi) \leq E_Q [\xi(T)\Phi]$  from Theorem 5.29, we get  $E_Q [\xi(T)\Phi] = \mathcal{P}_b(\Phi)$ .

□

Now we explain a simplified version of Black-Scholes model. Suppose a market  $X(t) = (X^{(0)}(t), X^{(1)}(t))$  is given by

$$\begin{cases} dX^{(0)}(t) = \rho(t)X^{(0)}(t) dt, & X^{(0)}(0) = 1; \\ dX^{(1)}(t) = \alpha(t)X^{(1)}(t) dB(t) + \beta(t)X^{(1)}(t) dt, & X^{(1)}(0) = x_1. \end{cases} \quad (5.15)$$

We can get the solution for the equations in Equation 5.15 as follows:

$$\begin{aligned} dX^{(0)}(t) &= \rho(t)X^{(0)}(t) dt \\ dX^{(0)}(t) - \rho(t)X^{(0)}(t) dt &= 0 \\ e^{-\int_0^t \rho(s) ds} [dX^{(0)}(t) - \rho(t)X^{(0)}(t) dt] &= 0 \\ d[e^{-\int_0^t \rho(s) ds} X^{(0)}(t)] &= 0 \\ e^{-\int_0^t \rho(s) ds} X^{(0)}(t) &= k \\ X^{(0)}(t) &= ke^{\int_0^t \rho(s) ds} \\ X^{(0)}(t) &= e^{\int_0^t \rho(s) ds}, \quad (\text{because } X^{(0)}(0) = 1) \end{aligned}$$

and

$$\begin{aligned} dX^{(1)}(t) &= \alpha(t)X^{(1)}(t) dB(t) + \beta(t)X^{(1)}(t) dt \\ dX^{(1)}(t) - \alpha(t)X^{(1)}(t) dB(t) - \beta(t)X^{(1)}(t) dt &= 0 \\ e^{-\int_0^t \beta(s) ds - \int_0^t \alpha(s) dB(s) + \frac{1}{2} \int_0^t \alpha(s)^2 ds} \{dX^{(1)}(t) - \alpha(t)X^{(1)}(t) dB(t) - \beta(t)X^{(1)}(t) dt\} &= 0 \\ d \left\{ e^{-\int_0^t \beta(s) ds - \int_0^t \alpha(s) dB(s) + \frac{1}{2} \int_0^t \alpha(s)^2 ds} X^{(1)}(t) \right\} &= 0 \\ e^{-\int_0^t \beta(s) ds - \int_0^t \alpha(s) dB(s) + \frac{1}{2} \int_0^t \alpha(s)^2 ds} X^{(1)}(t) &= k \\ X^{(1)}(t) &= ke^{\int_0^t \beta(s) ds + \int_0^t \alpha(s) dB(s) - \frac{1}{2} \int_0^t \alpha(s)^2 ds} \\ X^{(1)}(t) &= x_1 e^{\int_0^t \beta(s) ds + \int_0^t \alpha(s) dB(s) - \frac{1}{2} \int_0^t \alpha(s)^2 ds}, \end{aligned} \quad (5.16)$$

where the last equality follows since  $X^{(1)}(0) = x_1$ .

Observe that the market  $X(t)$  is specified by  $\rho(t)$ ,  $\mu(t) = \beta(t) X^{(1)}(t)$  and  $\sigma(t) = \alpha(t) X^{(1)}(t)$ . Thus the equation in condition (a) of Theorem 5.21 becomes

$$\alpha(t)X^{(1)}(t)h(t) = \rho(t)X^{(1)}(t) - \beta(t)X^{(1)}(t),$$

which gives the solution for  $h(t)$  to be

$$h(t) = \frac{\rho(t) - \beta(t)}{\alpha(t)}. \quad (5.17)$$

**Theorem 5.32.** (Black-Scholes) *Suppose a market  $X(t) = (X^{(0)}(t), X^{(1)}(t))$  is given by*

$$dX^{(0)}(t) = \rho(t)X^{(0)}(t) dt, \quad X^{(0)}(0) = 1,$$

$$dX^{(1)}(t) = \alpha(t)X^{(1)}(t) dB(t) + \beta(t)X^{(1)}(t) dt, \quad X^{(1)}(0) = x_1.$$

*Assume that*

$$(a) \ E \int_0^T |h(t)|^2 \mathcal{E}_h(t)^4 dt < \infty,$$

$$(b) \ E \int_0^T |h(t)|^2 |B(t)|^8 dt < \infty,$$

$$(c) \ E \left( \int_0^T |h(t)|^2 dt \right)^5 < \infty,$$

$$(d) \ E \int_0^T \mathcal{E}_h(t)^4 dt < \infty,$$

*where  $h(t) = \frac{\rho(t) - \beta(t)}{\alpha(t)}$ .*

*Suppose  $\rho(t)$  and  $\alpha(t)$  are deterministic functions in  $L^1[0, T]$  and  $L^2[0, T]$ , respectively and the  $T$ -claim  $\Phi$  is of the form  $\Phi = F(X^{(1)}(T))$ . Then the price at time  $t = 0$  of  $\Phi$  is given by*

$$\mathcal{P}(\Phi) = \xi(T) \frac{1}{\sqrt{2\pi} \|\alpha\|} \int_{-\infty}^{\infty} F \left( x_1 e^{x + \int_0^T (\beta(t) - \frac{1}{2}\alpha(t)^2) dt} \right) e^{-\frac{x^2}{2\|\alpha\|^2}} dx,$$

*where  $\|\alpha\|^2 = \int_0^T \alpha(t)^2 dt$  and  $\xi(T) = (X^{(0)}(T))^{-1} = e^{-\int_0^T \rho(t) dt}$ .*

*Proof.* By Conditions (a)-(d) and Theorem 5.21, the market  $X(t)$  has no arbitrage. Then by Theorem 5.25, the market is complete. So by Theorem 5.31, the price of the  $T$ -claim  $\Phi$  at time  $t = 0$  is given by

$$\mathcal{P}(\Phi) = E_Q [\xi(T)\Phi].$$

Since  $\rho(t)$  is deterministic, it follows that  $\mathcal{P}(\Phi) = \xi(T)E_Q(\Phi)$ . By the hypothesis  $\Phi = F(X^{(1)}(t))$  and Equation 5.16, we have

$$\begin{aligned} \mathcal{P}(\Phi) &= \xi(T) E_Q(\Phi) \\ &= \xi(T) E_Q [F(X^{(1)}(T))] \\ &= \xi(T) E_Q \left[ F \left( x_1 e^{\int_0^T \alpha(t) dB(t) + \int_0^T (\beta(t) - \frac{1}{2}\alpha(t)^2) dt} \right) \right]. \end{aligned}$$

Note that since  $\alpha(t)$  is deterministic, the integral  $\int_0^T \alpha(t) dB(t)$  is a Wiener integral with mean zero and variance  $\|\alpha\|^2 = \int_0^T \alpha(t)^2 dt$  (Theorem 3.2). Thus

$$\begin{aligned} \mathcal{P}(\Phi) &= \xi(T) E_Q \left[ F \left( x_1 e^{\int_0^T \alpha(t) dB(t) + \int_0^T (\beta(t) - \frac{1}{2}\alpha(t)^2) dt} \right) \right] \\ &= \xi(T) \frac{1}{\sqrt{2\pi} \|\alpha\|} \int_{-\infty}^{\infty} F \left( x_1 e^{x + \int_0^T (\beta(t) - \frac{1}{2}\alpha(t)^2) dt} \right) e^{-\frac{x^2}{2\|\alpha\|^2}} dx. \end{aligned}$$

□

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# Vita

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