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Relations Between Classes of Ideals in an Integral Domain.

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RELATIONS BETWEEN CLASSES OF IDEALS IN AN INTEGRAL DOMAIN

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ABSTRACT

This paper is a study of relations between selected classes of ideals in a domain D with unit. An ideal A in D is a valuation ideal if it is the intersection of D with an ideal of a valuation ring R_v containing D and contained in the quotient field of D. In this definition, if ''valuation'' is replaced by Prüfer or Dedekind, we obtain definitions for Prüfer ideal or Dedekind ideal, respectively.

It is shown, in the first chapter, that every Prüfer ideal in D is primary if and only if there exists only one proper prime ideal in D, and that the set of Prüfer ideals in D coincides with the set of primary ideals in D if and only if D is a rank one valuation ring. The condition that prime ideals of D are chained is necessary and sufficient for Prüfer ideals to be semi-primary; furthermore, the set of Prüfer ideals coincides with the set of semi-primary ideals if and only if D is a valuation ring. It is also shown that Prüfer ideals are powers of prime ideals if and only if the domain D is contained in only one valuation ring, it being P-adic for some prime ideal P of D.

In the second chapter, it is shown that D is a Dedekind domain if every proper ideal of D is a Dedekind ideal, not necessarily for the same Dedekind domain. Several

necessary and sufficient conditions are given for D to be an almost Dedekind domain; one of these conditions is that 1) primary ideals are Dedekind ideals, and 2) proper prime ideals in D are maximal. This paper is concluded by studying the prime ideal structure in D when semi-primary ideals are valuation ideals and when primary ideals are powers of prime ideals.

INTRODUCTION

In this paper D will always denote an integral domain with unit, and K will denote the quotient field of D. An ideal A of D will be called a proper ideal if A is distinct from (0) and D. ''Prüfer domain'' will mean a domain in which each finitely generated nonzero ideal is invertible. A domain in which each proper quotient ring is a discrete, rank one valuation ring will be called an ''almost Dedekind domain.'' All valuations encountered in this paper will be valuations of K which are non-negative on D, hence all valuation rings encountered will contain D and be contained in K. These valuations lead to a special class of ideals in D which are called valuation ideals [11;340].[1]

Definition I

An ideal A of D is a valuation ideal if it is the intersection of D with an ideal of a valuation ring V which contains D.

Note

The following equivalent definition will be used more often

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[1]Pairs of numbers in brackets refer to correspondingly numbered references in the Selected Bibliography and page numbers, respectively. A single number in a bracket refers to the correspondingly numbered reference in the Selected Bibliography.

in this paper: An ideal A of D is a valuation ideal if there exists a valuation ring V containing D and contained in K such that $A \cdot V \cap D = A$.

Definition II

An ideal A of D will be called a Prüfer ideal, or an almost Dedekind ideal, or a Dedekind ideal if there exists a Prüfer domain J, or an almost Dedekind domain J, or a Dedekind domain J, respectively, such that $D \subset J \subset K$ and $A \cdot J \cap D = A$.

Notation

Let $\mathcal{V}$ denote the set of valuation ideals of D, $\mathcal{P}$ the set of Prüfer ideals of D, $\mathcal{A}$ the set of almost Dedekind ideals of D, $\mathcal{Q}$ the set of primary ideals of D, $\mathcal{S}$ the set of semi-primary ideals of D (i.e., ideals having prime radical), and $\mathcal{PP}$ the set of prime power ideals of D. The terminology used will in general be that of Zariski and Samuel [10], [11]. In particular, the symbol ''$\subset$'' will indicate containment while ''$<$'' will indicate proper containment.

Statement of Problem

In a paper by Gilmer and Ohm [5], a study was made of the relationships between the set $\mathcal{V}$ of valuation ideals of D and the set $\mathcal{Q}$ of primary ideals of D. In this paper we will investigate relationships between these and the other classes of ideals mentioned above. We note that the following containments are always true; (1) $\mathcal{V} \subset \mathcal{P}$, (2) $\mathcal{A} \subset \mathcal{P}$, (3) $\mathcal{PP} \subset \mathcal{S}$, and (4) $\mathcal{V} \subset \mathcal{S}$. Containments (1) and (2)

follow since every valuation ring, respectively almost Dedekind domain, is a Prüfer domain; (3) is obvious and (4) is a result of [11;342].

The following three theorems are results of Gilmer and Ohm [5].

Theorem I

A domain J is a Prüfer domain if and only if it satisfies any of the following equivalent statements:

a) Every nonzero finitely generated ideal of J is invertible.

b) J_p is a valuation ring for every prime ideal P of J.

c) Whenever $A \neq (0)$, B, C are ideals such that A is finitely generated and $AB = AC$, then $B = C$.

d) Every ideal of J is complete (see definition, [11;347]).

e) Every ideal A of J, $A \neq J$, is an intersection of valuation ideals.

Theorem II

In a domain D, $\mathcal{V} \subset \mathcal{Q}$ if and only if every proper prime ideal of D is maximal.

Definition III

A domain D is said to satisfy the ascending chain condition for prime ideals provided any strictly ascending chain of prime ideals $P_1 \subset P_2 \subset P_3 \subset \cdots$ is finite.

Theorem III

Let D be a domain which satisfies the a.c.c. for prime ideals. If $\mathcal{Q} \subset \mathcal{V}$, then D is a Prüfer domain (and conversely).

Note

The following properties of extended and contracted ideals, listed in Zariski and Samuel [10;219], will be useful. Suppose J is a domain such that $D \subset J \subset K$; let A, B denote ideals of D, and R, S ideals of J, then:

1) If $R \subset S$, then $R \cap D \subset S \cap D$; if $A \subset B$, then $A \cdot J \subset B \cdot J$.
2) $(R \cap S) \cap D = (R \cap D) \cap (S \cap D)$.
3) $(RS) \cap D \supset (R \cap D) \cdot (S \cap D)$; $(AB) \cdot J = (A \cdot J) \cdot (B \cdot J)$.
4) $(\sqrt{R} \cap D) = \sqrt{R \cap D}$.

The following theorem, also from Zariski and Samuel [10;228], will be used frequently.

Theorem IV

Let P be a prime ideal in a ring R. The mapping $A \longrightarrow A \cdot R_P$ establishes a 1-1 correspondence between the set of prime (primary) ideals of R contained in P and the set of all prime (primary) ideals in R_P.

CHAPTER I

In this chapter, our attention is focused principally on relationships between the classes of ideals $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{S}$, and $\mathcal{PP}$ in a domain D. Necessary and sufficient conditions are given in order that $\mathcal{P} \subset \mathcal{S}$, $\mathcal{P} = \mathcal{S}$, $\mathcal{P} \subset \mathcal{Q}$, $\mathcal{P} = \mathcal{Q}$, and $\mathcal{P} \subset \mathcal{PP}$.

Theorem 1.1

Let $V_1, V_2, \ldots, V_n$ be valuation rings containing D and contained in the quotient field K of D, let $J = \bigcap_{i=1}^{n} V_i$, and denote by M_i the unique maximal ideal in V_i for $i = 1, \ldots, n$. Then J is a Prüfer domain and if $V_k \not\subset V_j$ for any $(k \neq j)$, then the maximal ideals of J are $P_i = M_i \cap J$ and $V_i = J_{p_i}$ for $i = 1, \ldots, n$.

Proof: Suppose $V_k \subset V_1$ for some $k \neq 1$, then $J = \bigcap_{i=1}^{n} V_i = \bigcap_{i=2}^{n} V_i$. We may therefore assume, without loss of generality, that $V_k \not\subset V_j$ for any $(k \neq j)$. An ideal A of J is maximal if and only if $A = M_i \cap J = P_i$ for some i, and $V_i = J_{p_i}$ for each i [8;38]. If Q is any prime ideal of J, then $Q \subset P_j$ for some j, hence $J_Q \supset J_{p_j}$. Therefore J_Q is a valuation ring, whence J is a Prüfer domain by Theorem I. This completes the proof.

Theorem 1.2

$\mathcal{P} \subset \mathcal{Q}$ in a domain D if and only if there exists only one

proper prime ideal in D (which is maximal, of course).

Proof: If there exists only one proper prime ideal in D, then every ideal in D is primary, hence Prüfer ideals are primary. Conversely, if Prüfer ideals are primary, then valuation ideals are primary and hence the proper prime ideals of D are maximal by Theorem II. Suppose P_1 and P_2 are distinct maximal ideals of D. There exist valuation rings V_1 and V_2 containing D with maximal ideals M_1 and M_2 respectively, such that $M_1 \cap D = P_1$ and $M_2 \cap D = P_2$. If $V_1 \subset V_2$, then $M_2 \cap V_1 \subset M_1$ since M_1 is maximal in V_1 and hence $P_2 = (M_2 \cap V_1) \cap D \subset M_1 \cap D = P_1$. Since $P_2 \not\subset P_1$ we see that $V_1 \not\subset V_2$ and in a similar manner $V_2 \not\subset V_1$. We set $V_1 \cap V_2 = J$, then by Theorem I.1, J is a Prüfer domain and $M_1 \cap J = Q_1$ and $M_2 \cap J = Q_2$ are the only maximal ideals in J. Furthermore, Q_1 and Q_2 are distinct since neither V_1 nor V_2 is contained in the other. We have

$$P_1 \cap P_2 = (M_1 \cap D) \cap (M_2 \cap D) =$$
$$\{(M_1 \cap J) \cap D\} \cap \{(M_2 \cap J) \cap D\} =$$
$$(Q_1 \cap D) \cap (Q_2 \cap D) = (Q_1 \cap Q_2) \cap D.$$

But $(Q_1 \cap Q_2) \cap D$ is a Prüfer ideal, hence $\sqrt{P_1 \cap P_2} = P_1 \cap P_2$ is maximal. Therefore $P_1 \subset P_2$ or $P_2 \subset P_1$. This contradiction establishes the converse.

Lemma 1.3

If A is a Prüfer ideal in D, then A is the intersection of valuation ideals.

Proof: Let J denote a Prüfer domain such that $A \cdot J \cap D = A$. If $\mathcal{M}$ denotes the set of maximal ideals in J, then $J = \bigcap_{M \in \mathcal{M}} J_M$ [11;94]. Also, $A \cdot J = \bigcap_{M \in \mathcal{M}} A \cdot J_M$ [11;94], where each $A \cdot J_M$ is a valuation ideal since J_M is a valuation ring for each prime ideal M of J. Therefore $A = A \cdot J \cap D = (\bigcap_{M \in \mathcal{M}} A \cdot J_M) \cap D = \bigcap_{M \in \mathcal{M}} (A \cdot J_M \cap D)$ where each $A \cdot J_M \cap D$ is a valuation ideal in D.

Lemma 1.4

If every ideal A of D, $A \neq D$, is a Prüfer ideal, then D is a Prüfer domain.

Proof: By Lemma 1.3, every ideal $A \neq D$ is the intersection of valuation ideals, hence D is a Prüfer domain by Theorem I-(e).

Theorem 1.5

$\mathcal{P} = \mathcal{Q}$ if and only if D is a rank one valuation ring.

Proof: Let D be a rank one valuation ring, then D has only one proper prime ideal, it being maximal. Therefore $\mathcal{P} \subset \mathcal{Q}$ by Theorem 1.2. Since every ideal of D is a Prüfer ideal, $\mathcal{Q} \subset \mathcal{P}$, whence $\mathcal{P} = \mathcal{Q}$. Conversely, suppose $\mathcal{P} = \mathcal{Q}$. By Theorem 1.2, there exists only one proper prime ideal M in D; therefore every ideal of D is primary, hence Prüfer. Now D is a Prüfer domain, by Lemma 1.4, and thus a rank one valuation ring since $D = D_M$.

Theorem 1.6

In a domain D, $\mathcal{P} \subset \mathcal{S}$ if and only if the prime ideals of D are chained.

Proof: Suppose $\mathcal{P} \subset \mathcal{S}$ and let P_1 and P_2 denote arbitrary prime ideals of D. Let V_1 and V_2 be valuation rings containing D with maximal ideals M_1 and M_2 respectively, such that $M_1 \cap D = P_1$ and $M_2 \cap D = P_2$. If $V_1 \subset V_2$, then $M_2 \cap V_1 \subset M_1$, hence $P_2 = (M_2 \cap V_1) \cap D \subset M_1 \cap D = P_1$. If $V_2 \subset V_1$, then $P_1 \subset P_2$ by a similar argument. If $V_1 \not\subset V_2$, $V_2 \not\subset V_1$, then $J = V_1 \cap V_2$ is a Prüfer domain with maximal ideals $Q_1 = M_1 \cap J$ and $Q_2 = M_2 \cap J$, by Theorem 1.1. $(Q_1 \cap Q_2) \cap D = P_1 \cap P_2$ is a Prüfer ideal in D, thus $\sqrt{P_1 \cap P_2} = P_1 \cap P_2$ is prime. If $P_1 \not\subset P_2$ and $P_2 \not\subset P_1$, there is an element $x \in P_1$, $x \notin P_2$ and an element $y \in P_2$, $y \notin P_1$ such that $xy \in P_1 \cap P_2$. Since neither x nor y is an element of $P_1 \cap P_2$, we have contradicted the fact that $P_1 \cap P_2$ is prime. Therefore $P_1 \subset P_2$ or $P_2 \subset P_1$ and hence prime ideals in D are chained. Conversely, if the prime ideals of D are chained, every ideal of D has prime radical, hence $\mathcal{P} \subset \mathcal{S}$.

Corollary 1.7

If $\mathcal{P} = \mathcal{V}$ in a domain D, then the prime ideals of D are chained.

Proof: Since $\mathcal{P} = \mathcal{V}$ implies $\mathcal{P} \subset \mathcal{S}$, the corollary follows by applying the above theorem.

Theorem 1.8

In a domain D, $\mathcal{P} = \mathcal{S}$ if and only if D is a valuation ring.

Proof: If $\mathcal{P} = \mathcal{S}$, the prime ideals of D are chained by Theorem 1.6. Hence every ideal of D has prime radical and

is therefore a Prüfer ideal. By Lemma 1.4, D is a Prüfer domain. Since the prime ideals are chained, D has only one maximal ideal M. Then $D = D_M$ is a valuation ring. The converse is obvious.

Theorem 1.9

If the valuation rings containing D are chained, then $\mathcal{V} = \mathcal{P}$.

Proof: Let A be any Prüfer ideal in D and let J be a Prüfer domain with the property that $A \cdot J \cap D = A$. Since J lies between D and its quotient field K, the valuation rings containing J are also chained. Suppose M and N are maximal ideals in J and $M \neq N$, then J_M and J_N are valuation rings containing J, whence $J_M \subset J_N$ or $J_N \subset J_M$. Suppose $J_M \subset J_N$, then $N \cdot J_N \cap J_M \subset M \cdot J_M$ and therefore $N \subset M$. This contradicts the maximality of N since $M \neq N$, thus $J_M \not\subset J_N$. Likewise, $J_N \not\subset J_M$, hence J has only one maximal ideal, say M. Therefore $J = J_M$ is a valuation ring and hence $A \cdot J \cap D = A$ is a valuation ideal. The containment $\mathcal{V} \subset \mathcal{P}$ is always true so the proof is complete.

Corollary 1.10

If the valuation rings containing D are chained, then the prime ideals of D are chained (i.e., $\mathcal{P} \subset \mathcal{S}$).

Proof: Apply Theorem 1.9 and Corollary 1.7.

Remark

Example 1.30 shows that the converse of the above corollary is false.

Theorem 1.11

In a domain D, $\mathcal{P} = \mathcal{S}$ if and only if $\mathcal{P} = \mathcal{V} = \mathcal{S}$.

Proof: If $\mathcal{P} = \mathcal{S}$, then D is a valuation ring by Theorem 1.8, hence $\mathcal{P} = \mathcal{V} = \mathcal{S}$. The converse is obvious.

Theorem 1.12

If D satisfies the ascending chain condition for prime ideals, then $\mathcal{S} \subset \mathcal{V}$ implies D is a Prüfer domain.

Proof: Since $\mathcal{S} \subset \mathcal{V}$ implies $\mathcal{Q} \subset \mathcal{U}$, the theorem follows from Theorem III.

Theorem 1.13

Let M be a multiplicative system in D and A an ideal of D such that $A \cap M = \emptyset$. Let D_1 be a domain containing D such that $A \cdot D_1 \cap D = A$. Let $D_1^* = (D_1)_M$, $D^* = D_M$, and $A^* = A \cdot D_M$. Then $A^* = A^* \cdot D_1^* \cap D^*$.

Proof: $A^* \subset A^* \cdot D_1^* \cap D^*$ is clear. Suppose $x \in A^* \cdot D_1^* \cap D^* = A \cdot D_1^* \cap D^*$. Then $x = a(d/m) = r/n$ where $d \in D_1$, $a \in A$, $r \in D$, and $m,n \in M$. Therefore $nad = rm$ and $nad \in A \cdot D_1$, $rm \in D$, so $rm \in A \cdot D_1 \cap D = A$. Since $m,n \in M$, then $1/mn \in D^*$, hence $rm(1/mn) = r/n \in A \cdot D^* = A^*$. Therefore $A^* = A^* \cdot D_1^* \cap D^*$.

Corollary 1.14

Let M be a multiplicative system in D and let A be a Dedekind ideal such that $A \cap M = \emptyset$. Then $A \cdot D_M$ is also a Dedekind ideal.

Proof: If J is a Dedekind domain with the property that $A \cdot J \cap D = A$, then J_M is also a Dedekind domain, and by the above theorem, $A \cdot D_M = A \cdot J_M \cap D_M$. Therefore $A \cdot D_M$ is a Dedekind ideal.

Remark

The above corollary remains valid if Dedekind is replaced by either Prüfer, valuation, or almost Dedekind.

Theorem 1.15

If every prime ideal is the radical of an almost Dedekind ideal, then prime ideals are almost Dedekind ideals and conversely.

Proof: If P is a prime ideal of D, let A be an almost Dedekind ideal having radical P. There exists an almost Dedekind domain J such that $A \cdot J \cap D = A$, hence $\sqrt{A \cdot J} \cap D = \sqrt{A} = P$. Therefore P is an almost Dedekind ideal. The converse is obvious.

Remark

It is easy to show, by using the remark following Corollary 1.14, that if a proper prime ideal P of D is an almost Dedekind ideal, then there exists a discrete, rank one valuation ring V such that the center of V in D is P (and conversely). Example 1.31 provides a domain with a maximal ideal M such that M is not an almost Dedekind ideal.

Theorem 1.16

If the prime ideals of D are almost Dedekind ideals, then

$\mathcal{A} \subset \mathcal{I}$ if and only if the prime ideals of D are chained.

Proof: If the prime ideals of D are chained, then $\mathcal{P} \subset \mathcal{I}$ by Theorem 1.6, hence $\mathcal{A} \subset \mathcal{I}$. Conversely, if $\mathcal{A} \subset \mathcal{I}$, let $P \neq Q$ be arbitrary proper prime ideals of D. Let J and J' be almost Dedekind domains with the property that $P \cdot J \cap D = P$ and $Q \cdot J' \cap D = Q$. By Theorem 1.13, we have $P \cdot D_M = P \cdot J_M \cap D_M$ and $Q \cdot D_N = Q \cdot J'_N \cap D_N$, where $M = D - P$ and $N = D - Q$. Now $P \cdot D_M$ is the unique maximal ideal in D_M so there exists a maximal ideal R of J_M, which contains $P \cdot J_M$, such that $R \cap D_M = P \cdot D_M$. Therefore $R \cap D = (R \cap D_M) \cap D = P \cdot D_M \cap D = P$. Similarly, there exists a maximal ideal S in J'_N, which contains $Q \cdot J'_N$, such that $S \cap D = Q$. J_M and J'_N contain J and J' respectively, thus are almost Dedekind domains [9;8]. Therefore $(J_M)_R$ and $(J'_N)_S$ are discrete, rank one valuation rings and unequal since $P \neq Q$. If either $(J_M)_R \subset (J'_N)_S$ or $(J'_N)_S \subset (J_M)_R$, the theorem would be proved since then $Q \subset P$ or $P \subset Q$. We assume that neither containment holds, hence $T = (J_M)_R \cap (J'_N)_S$ is a Prüfer domain with exactly two maximal ideals, by Theorem 1.1. Furthermore, T has no other proper prime ideals since both $(J_M)_R$ and $(J'_N)_S$ are discrete, rank one valuation rings and $T_{(R \cap T)} = (J_M)_R$ and $T_{(S \cap T)} = (J'_N)_S$, by Theorem 1.1. By definition, T is then an almost Dedekind domain, hence $(R \cap T) \cap D = P$, $(S \cap T) \cap D = Q$, and $\{(R \cap T) \cap (S \cap T)\} \cap D = P \cap Q$ are almost Dedekind ideals. Since $\mathcal{A} \subset \mathcal{I}$, then $P \cap Q \in \mathcal{A}$ implies $P \subset Q$ or $Q \subset P$. This completes the proof.

Corollary 1.17

If the proper prime ideals of D are almost Dedekind ideals, then $\mathcal{A} \subset \mathcal{I}$ if and only if $\mathcal{P} \subset \mathcal{I}$.

Proof: Apply Theorems 1.6 and 1.16.

Theorem 1.18

If $\mathcal{A} = \mathcal{I}$ in a domain D, then D is a valuation ring.

Proof: The prime ideals of D are almost Dedekind ideals since $\mathcal{I} \subset \mathcal{A}$. Then $\mathcal{A} \subset \mathcal{I}$ implies prime ideals are chained, by Theorem 1.16. Therefore, every ideal in D has prime radical, in particular $\mathcal{P} \subset \mathcal{I}$. Now $\mathcal{A} = \mathcal{I}$ implies $\mathcal{I} \subset \mathcal{P}$, hence $\mathcal{P} = \mathcal{I}$. By Theorem 1.8, D is then a valuation ring.

Remark

Example 1.31 shows that the converse of the above theorem is false.

Corollary 1.19

If $\mathcal{A} = \mathcal{I}$ in a domain D, then $\mathcal{A} = \mathcal{I} = \mathcal{P} = \mathcal{V}$ (and conversely).

Proof: The equality $\mathcal{P} = \mathcal{I}$ is shown in the above theorem. The equality $\mathcal{V} = \mathcal{I}$ then follows from Theorem 1.11.

Proposition 1.20

A Prüfer domain J is a valuation ring if and only if the prime ideals of J are chained.

Proof: If J is a valuation ring, all the ideals of J are chained. On the other hand, if the prime ideals are

chained, there exists only one maximal ideal M in J. Then $J = J_M$ is a valuation ring.

Corollary 1.21

If J is a Prüfer domain, then $\mathcal{V} = \mathcal{P}$ if and only if the prime ideals of J are chained (i.e., $\mathcal{P} \subset \mathcal{S}$).

Proof: If $\mathcal{V} = \mathcal{P}$, the prime ideals are chained by Corollary 1.7. Conversely, if the prime ideals of D are chained, J is a valuation ring by the above proposition, hence $\mathcal{V} = \mathcal{P}$.

The following theorem due to Phillips [9;4] will be useful.

Theorem 1.22

J is an almost Dedekind domain if and only if each ideal of J, with prime radical, is a prime power.

Proposition 1.23

If $\mathcal{S} = \mathcal{PP}$ in a domain D, then $\mathcal{Q} = \mathcal{V} = \mathcal{S} = \mathcal{PP}$.

Proof: By the above theorem, D is an almost Dedekind domain, hence the equality $\mathcal{Q} = \mathcal{V}$ follows from Theorems II and III. Every ideal in $\mathcal{S}$ has maximal radical, thus is primary, hence $\mathcal{Q} = \mathcal{S}$.

The following theorem and corollary due to Gilmer [2] will also be useful here.

Theorem 1.24

Let D be an integral domain with identity. Let $\mathcal{PP}$ be the set of prime power ideals of D, $\mathcal{V}$ the set of valuation

ideals of D, and let K be the quotient field of D. $V \subset PP$ if and only if the following conditions hold:

i) If P is a nonzero proper prime ideal of D, $\bigcap_{n=0}^{\infty} P^n = (0)$ and the function $v_P : D - \{0\} \longrightarrow Z$ (non-negative integers) defined by $v_P(x) = i$ if $x \in P^i - P^{i+1}$ can be extended to a valuation of K.

ii) Every valuation of K, finite on D, is isomorphic to some v_P.

Corollary 1.25

Using the notation of the above theorem, if $V \subset PP$, then $V = PP$ and D is one-dimensional.

Theorem 1.26

In a domain D, $P \subset PP$ if and only if D is contained in only one valuation ring, it being P-adic for some prime ideal P of D.

Proof: If $P \subset PP$, then $V \subset PP$, hence $V = PP$ and D is one-dimensional by Corollary 1.25. Furthermore, $P \subset PP$ implies $P \subset J$, thus the prime ideals of D are chained, by Theorem 1.6. As a consequence, D has only one proper prime ideal, and since every valuation of K, finite on D, is P-adic for some prime ideal P of D, by Theorem 1.24, then there exists only one valuation ring J between D and K. Conversely, if D is contained in only one valuation ring J, it being P-adic for some prime ideal P of D, then D has a unique proper prime ideal. Therefore $V \subset PP$, by Theorem 1.24, and $V = P$ by Theorem 1.9. Hence $P \subset PP$ and

the proof of the theorem is complete.

Corollary 1.27

If $\mathcal{P} \subset \mathcal{PP}$ in a domain D, then $\mathcal{P} = \mathcal{V} = \mathcal{PP}$ and $\mathcal{Q} = \mathcal{S}$.

Proof: Since $\mathcal{P} \subset \mathcal{PP}$ implies $\mathcal{V} \subset \mathcal{PP}$, Corollary 1.25 then gives us $\mathcal{V} = \mathcal{PP}$, hence $\mathcal{PP} \subset \mathcal{P}$ and therefore $\mathcal{P} = \mathcal{PP} = \mathcal{V}$. $\mathcal{Q} = \mathcal{S}$ since D has only one proper prime ideal.

Theorem 1.28

A necessary and sufficient condition that D be a discrete, rank one valuation ring is that D be integrally closed and $\mathcal{P} \subset \mathcal{PP}$.

Proof: By Theorem 1.26, D is contained in only one valuation ring J, it being P-adic for some proper prime ideal P of D, hence J is discrete and rank one. The intersection of all valuation rings containing D, and non-negative on D, is the integral closure of D in K, hence D = J since D is integrally closed. Conversely, if D is a discrete, rank one valuation ring, every proper ideal is a power of the maximal ideal, thus $\mathcal{P} \subset \mathcal{PP}$, and of course every valuation ring is integrally closed.

Note

The following example shows that integral closure is necessary in the statement of the above theorem. Clearly, integral closure alone is not sufficient since a Dedekind domain with more than one maximal ideal is not a discrete, rank one valuation ring.

Example 1.29

Consider the domain $D = \{m/p + n/q\sqrt{5} \mid m,n,p,q$ are rational integers and p,q are odd$\}$. The quotient field of D is $K = R(\sqrt{5})$, where R denotes the field of rational numbers. Let S denote the ring of algebraic integers in K, then $S = \{a + b(1 + \sqrt{5})/2 \mid a,b$ are rational integers$\}$, since $\{1, (1 + \sqrt{5})/2\}$ forms an integral basis of K [6;33]. Now S is a Dedekind domain, hence every valuation v of K, non-negative on D, is a P-adic valuation for some prime ideal P of S, and the value group of v is the additive group of integers. Furthermore, the valuation ring corresponding to v is the quotient ring S_P of S with respect to the proper prime ideal P in S [11;39]. Therefore, every valuation ring contained in K and containing D, such that the corresponding valuation is non-negative on D, is discrete. Now $(2)\cdot S$ is a prime ideal of S [6;66], hence $S_{(2)\cdot S}$ is a discrete, rank one valuation ring. It is easy to show that D is contained in $S_{(2)\cdot S}$, but not equal. Now $1/q \in D$ for every rational prime $q \neq 2$, hence there are no other P-adic valuations of K, P a prime ideal of S, (and therefore no other valuations of K), which are non-negative on D, since a prime ideal P of S must contain one and only one prime rational integer. Therefore, $S_{(2)\cdot S}$ is the integral closure of D in K. It follows that $(2)\cdot S_{(2)\cdot S} \cap D = M = (2, 1 + \sqrt{5})\cdot D$ and that D has no other proper prime ideals. Now $\bigcap_{n=0}^{\infty} M^n = (0)$ and the function $v_M : D - \{0\} \longrightarrow Z$, (the non-negative integers), by $v_M(x) = i$ if $x \in M^i - M^{i+1}$ extends to coincide with the

(2)·S-adic valuation of K. By Theorem 1.24, $\mathcal{V} \subset \mathcal{PP}$ and hence $\mathcal{V} = \mathcal{PP}$ by Corollary 1.25. Since Prüfer domains are integrally closed in their quotient fields, any Prüfer domain containing D must contain $S_{(2)\cdot S}$, hence be equal to $S_{(2)\cdot S}$ since there are no rings between $S_{(2)\cdot S}$ and K. Therefore $\mathcal{Q} = \mathcal{P} = \mathcal{V} = \mathcal{PP}$, but D is not a valuation ring. Furthermore, (2)·D is M-primary but not a power of M, hence $\mathcal{PP}$ is contained in $\mathcal{Q}$, but not equal.

Example 1.30

Let $D = \{a/b + c/d\sqrt{17} \mid a,b,c,d$ are rational integers, and b,d are odd$\}$. The quotient field of D is $K = R(\sqrt{17})$ where R denotes the field of rational numbers and $S = \{a + b(1 + \sqrt{17})/2 \mid a,b$ are rational integers$\}$ is the set of algebraic integers in $R(\sqrt{17})$ [6;33]. Since every valuation of K, non-negative on D, is P-adic for some prime ideal P of S [11;39], we see that each prime ideal P of S, for which there exists such a valuation, must contain the ideal (2)·S. Let $P = \left(2, (1 + \sqrt{17})/2\right)$, and let $Q = \left(2, (1 - \sqrt{17})/2\right)$, then (2)·S = P·Q where both P and Q are maximal ideals of S. Then P and Q are the only proper prime ideals of S containing (2)·S, hence $S_P \neq S_Q$ are the only valuation rings containing D whose corresponding valuations are non-negative on D. Now $J = S_P \cap S_Q$ is a Dedekind domain (see Corollary 2.8 of Chapter II) and the only proper prime ideals of J are $P' = P\cdot S_P \cap J$ and $Q' = Q\cdot S_Q \cap J$. We have $P' \neq Q'$, but $P' \cap D = Q' \cap D = (2, 1 + \sqrt{17})\cdot D = M$, and M is the only proper prime ideal

of D. Also, this prime ideal M does not determine a M-adic valuation of K. The domain D is not a valuation ring, yet the prime ideals of D are chained, that is, $P \subset I$.

Note

We note that both Example 1.29 and Example 1.30 provide a domain D such that the only valuation rings containing D, whose corresponding valuations are non-negative on D, are discrete and rank one, yet D is not an almost Dedekind domain. Further, Example 1.29 has the property that the valuation rings containing D are chained.

Example 1.31

Let I denote the set of integers, J the field of rational numbers, and let x,y denote algebraically independent indeterminants over J. Define a function ω from J[x,y] to $I[\sqrt{2}]$ by $\omega(\sum a_{ij}x^iy^j) = \min_{a_{ij}\neq 0}(i + j\sqrt{2})$. Extend ω to J(x,y) by $\omega(f/g) = \omega(f) - \omega(g)$ where f and g are polynomials in J[x,y]. Then ω is a valuation of J(x,y) and the set V of all rational functions $h \in J(x,y)$, such that $\omega(h)$ is non-negative, is a valuation ring. The value group of ω is dense in the set of real numbers, hence V is a non-discrete, rank one valuation ring. The maximal ideal M of V is idempotent, thus M is not an almost Dedekind ideal.

CHAPTER II

In this chapter, we give a necessary and sufficient condition, in terms of Dedekind ideals, for a domain D to be Dedekind. We also present several necessary and sufficient conditions for D to be an almost Dedekind domain and then study the prime ideal structure of D when $\mathcal{I} \subset \mathcal{V}$ and when $\mathcal{Q} \subset \mathcal{PP}$.

Theorem 2.1

If A is a Dedekind ideal in D, then A can be written as a finite intersection of primary Dedekind ideals, where each primary ideal in the representation is a valuation ideal.

Proof: Let J be a Dedekind domain having the property that $A \cdot J \cap D = A$, then $A \cdot J = P_1^{e_1} \cap \ldots \cap P_n^{e_n}$, where P_i, for $i = 1,2,\ldots,n$, is a prime ideal of J and each e_i is a positive integer. Then $A = A \cdot J \cap D =$ $(P_1^{e_1} \cap D) \cap \ldots \cap (P_n^{e_n} \cap D)$, where, for each i, $P_i^{e_i} \cap D$ is a primary Dedekind ideal in D for $P_i \cap D$. Furthermore, each $P_i^{e_i} \cap D$ is a valuation ideal since $P_i^{e_i}$ is a valuation ideal in J.

Theorem 2.2

If B and C are almost Dedekind (Dedekind) ideals of D for the same almost Dedekind (Dedekind) domain J, and if A is

any proper ideal of D such that AB = AC, then B = C.

Proof: If AB = AC, then $(AB)\cdot J = (AC)\cdot J$, hence $(A\cdot J)\cdot(B\cdot J) = (A\cdot J)\cdot(C\cdot J)$. Therefore $B\cdot J = C\cdot J$ since the cancellation law for ideals is valid in an almost Dedekind (Dedekind) domain [4], but then $B = B\cdot J \cap D = C\cdot J \cap D = C$ since both B and C are almost Dedekind (Dedekind) ideals for J.

Corollary 2.3

If every pair of proper ideals of D are almost Dedekind ideals for some almost Dedekind domain J, then D is an almost Dedekind domain (and conversely).

Proof: If A, B, and C are ideals of D such that AB = AC and $A \neq (0)$, then B = C by the previous theorem. Therefore the cancellation law for ideals is valid in D, hence D is an almost Dedekind domain [4].

Theorem 2.4

If prime ideals are Dedekind ideals, then each prime ideal of D is the contraction of a prime ideal of a Dedekind domain containing D.

Proof: Let P be a prime ideal of D and J a Dedekind domain such that $P\cdot J \cap D = P$. In J, $P\cdot J$ factors as $P\cdot J = Q_1^{e_1}\cdots Q_n^{e_n}$, where each Q_i is a prime ideal of J and e_i is a positive integer for each i, and $\sqrt{P\cdot J} = Q_1 \cap \ldots \cap Q_n = Q_1\cdots Q_n$. Therefore, $\sqrt{P\cdot J} \cap D = \sqrt{P\cdot J \cap D} = \sqrt{P} = P = (Q_1 \cap \ldots \cap Q_n) \cap D = (Q_1 \cap D) \cap \ldots \cap (Q_n \cap D)$,

thus $P \supset (Q_1 \cap D)\cdots(Q_n \cap D)$ implies $P \supset Q_j \cap D$ for some $1 \leq j \leq n$. Hence $P = Q_j \cap D$ since $P \subset Q_i \cap D$ for all i.

Remark

Example 1.31 provides a domain with maximal ideal M such that M is not a Dedekind ideal.

Corollary 2.5

Suppose proper prime ideals are Dedekind ideals, and let P be an arbitrary proper prime ideal of D. Then there exists a rank one, discrete valuation ring V such that $D \subset V \subset K$ and $P{:}V \cap D = P$.

Proof: Let J be a Dedekind domain such that $P \cdot J \cap D = P$. By the previous theorem, there exists a prime ideal Q in J such that $Q \cap D = P$, hence J_Q is a rank one, discrete valuation ring. Now $P \cdot J_Q \subset Q \cdot J_Q$, thus
$P \subset P \cdot J_Q \cap D \subset Q \cdot J_Q \cap D = (Q \cdot J_Q \cap J) \cap D = Q \cap D = P$.
Therefore $P = P \cdot J_Q \cap D$.

We state here a theorem due to Phillips [9;9].

Theorem 2.6

J is an almost Dedekind domain if and only if the following conditions hold:

1) J is a Prüfer domain.
2) Proper prime ideals of J are maximal.
3) J contains no proper idempotent prime ideal.

Theorem 2.7

If proper prime ideals are Dedekind ideals, and if $P \neq Q$

are proper prime ideals of D, then there exists a Dedekind domain J containing D and having prime ideals P' and Q' such that $P' \cap D = P$ and $Q' \cap D = Q$. Furthermore, P' and Q' are the only prime ideals of J.

Proof: Let $P \neq Q$ be arbitrary proper prime ideals of D. By Theorem 2.4, there exist prime ideals M and N in Dedekind domains R and S respectively, such that $M \cap D = P$ and $N \cap D = Q$. Furthermore, $R_M = V$ and $S_N = W$ are discrete, rank one valuation rings with maximal ideals $M \cdot V$ and $N \cdot W$ respectively. If $V \subset W$, then $N \cdot W \cap V \subset M$, hence $N \cdot W \cap V = M$ since $N \cdot W \cap V$ is a nonzero prime ideal in V. However, this implies $P = Q$, thus neither of these valuation rings is contained in the other. We let $V \cap W = J$, then by Theorem 1.1, J is a Prüfer domain with exactly two maximal ideals, namely $M \cdot V \cap J = P'$ and $N \cdot W \cap J = Q'$, and also $V = J_{P'}$ and $W = J_{Q'}$. J has no proper prime ideals other than P' and Q' since V and W are discrete, rank one valuation rings, hence J is an almost Dedekind domain. Since P' and Q' are maximal ideals and unequal, there exists $x \in P'$, $x \notin Q'$, hence $\sqrt{(x)} = P'$. Therefore, (x) is a power of P' since J is an almost Dedekind domain, by Theorem 1.22. Let $(x) = (P')^n$ for some $n \geq 1$. The ideal (x) is invertible, hence P' is invertible [10;272]. In an analogous manner, we can show that Q' is invertible, hence J is a Dedekind domain since every proper prime ideal of J is invertible [1;33]. To complete the proof, we have $P' \cap D = (M \cdot V \cap J) \cap D = M \cdot V \cap D = (M \cdot V \cap R) \cap D = M \cap D = P$, and similarly $Q' \cap D = Q$.

Corollary 2.8

Let $V_1, V_2, \ldots, V_n$ denote rank one, discrete valuation rings containing D and contained in the quotient field K of D, then $\bigcap_{i=1}^{n} V_i$ is a Dedekind domain. Furthermore, if $V_k \not\subset V_j$ for any $(k \neq j)$, then the intersection has exactly n proper prime ideals.

Proof: The corollary is proved in the previous theorem when n = 2. The proof is similar for any finite number of such valuation rings.

Theorem 2.9

If every proper ideal of D is a Dedekind ideal (not necessarily for the same Dedekind domain), then D is a Dedekind domain.

Proof: If A and B are proper ideals of D, then there exist Dedekind domains J_1 and J_2 such that $A \cdot J_1 \cap D = A$ and $B \cdot J_2 \cap D = B$. In J_1, we have $A \cdot J_1 = P_1^{e_1} \cdots P_n^{e_n}$ where P_i is a prime ideal of J_1, for each i. Let $J_1^* = \bigcap_{i=1}^{n} (J_1)_{P_i}$. We wish to show that $A \cdot J_1 = (A \cdot J_1) \cdot J_1^* \cap J_1$. Clearly $A \cdot J_1 \subset (A \cdot J_1) \cdot J_1^* \cap J_1$. Now $A \cdot J_1 = P_1^{e_1} \cdots P_n^{e_n} = P_1^{e_1} \cap \ldots \cap P_n^{e_n} = \{(P_1^{e_1} \cdots P_n^{e_n}) \cdot (J_1)_{P_1} \cap J_1\} \cap \ldots \cap \{(P_1^{e_1} \cdots P_n^{e_n}) \cdot (J_1)_{P_n} \cap J_1\} = \{(A \cdot J_1) \cdot (J_1)_{P_1} \cap J_1\} \cap \ldots \cap \{(A \cdot J_1) \cdot (J_1)_{P_n} \cap J_1\} = \bigcap_{i=1}^{n} \{(A \cdot J_1) \cdot (J_1)_{P_i} \cap J_1\} = \{\bigcap_{i=1}^{n} (A \cdot J_1) \cdot (J_1)_{P_i}\} \cap J_1 \supset (A \cdot J_1) \cdot \{\bigcap_{i=1}^{n} (J_1)_{P_i}\} \cap J_1 = (A \cdot J_1) \cdot J_1^* \cap J_1$. These two

containments give $A \cdot J_1 = (A \cdot J_1) \cdot J_1^* \cap J_1$. In a similar manner we get a Dedekind domain J_2^* such that $B \cdot J_2 = (B \cdot J_2) \cdot J_2^* \cap J_2$. Let $J = J_1^* \cap J_2^*$, then J is a Dedekind domain since it is the intersection of a finite number of discrete, rank one valuation rings, by Corollary 2.8. Furthermore, $A = A \cdot J_1 \cap D = \{(A \cdot J_1) \cdot J_1^* \cap J_1\} \cap D = A \cdot J_1^* \cap D \supset A \cdot J \cap D$, and $A \subset A \cdot J \cap D$ is clear, thus $A = A \cdot J \cap D$. In a similar manner, $B \cdot J \cap D = B$. Now D is an almost Dedekind domain by Corollary 2.3. If A is an arbitrary proper ideal of D, then A can be expressed as $A = Q_1 \cap \ldots \cap Q_n$, where each Q_i is primary, by Theorem 2.1. We may assume that $Q_k \not\subset Q_j$ for any $(k \neq j)$. Now since D is an almost Dedekind domain, each Q_i is a prime power, by Theorem 1.22. For each i, we let $Q_i = M_i^{e_i}$, where M_i is a maximal ideal of D and e_i is a positive integer; then $A = M_1^{e_1} \cap \ldots \cap M_n^{e_n} = M_1^{e_1} \cdots M_n^{e_n}$ since the $M_i^{e_i}$ are pairwise comaximal [10;177]. Therefore A is the product of prime ideals, hence D is a Dedekind domain.

Theorem 2.10

A domain D is an almost Dedekind domain if and only if the following conditions hold:

a) Primary ideals of D are Dedekind ideals.

b) Proper prime ideals of D are maximal.

Proof: If we assume conditions a) and b) above, we will show that conditions 1), 2), and 3) of Theorem 2.6 are valid. Condition b) above is the same as condition 2).

Condition 3) is satisfied, for if $P = P^2$ for some proper prime ideal P of D, then P = D, by Theorem 2.2. For condition 1), we let P be an arbitrary proper prime ideal of D and form the quotient ring D_P. Then $P \cdot D_P$ is the only proper prime of D_P since proper prime ideals of D are maximal. Furthermore, if Q is any P-primary ideal in D, then $Q \cdot D_P$ is primary in D_P and is also a Dedekind ideal, by Corollary 1.14. Thus every proper ideal in D_P is primary for $P \cdot D_P$ and is also a Dedekind ideal, hence every proper ideal in D_P is the intersection of valuation ideals, by Theorem 2.1. Then D_P is a Prüfer domain by Theorem I-(e). But, D_P has a unique maximal ideal, so $D_P = (D_P)_{P \cdot D_P}$ is a valuation ring. Therefore, D is a Prüfer domain since D_P is a valuation ring for each proper prime ideal P in D, by Theorem I-(b). This shows that condition 1) is valid, hence D is an almost Dedekind domain. Conversely, if D is an almost Dedekind domain, we need only to show the validity of condition a). If P is an arbitrary proper prime ideal of D, then D_P is a Dedekind domain, and if Q is any P-primary ideal in D, we have $Q = Q \cdot D_P \cap D$. Hence condition a) is satisfied.

Corollary 2.11

If primary ideals are rank one, discrete valuation ideals, and if proper prime ideals are maximal, then D is an almost Dedekind domain and conversely.

Proof: Every rank one, discrete valuation ring is a

Dedekind domain, thus D is an almost Dedekind domain by the previous theorem. The converse also follows from the previous theorem, for if P is any proper prime ideal of D, then D_P is a rank one, discrete valuation ring, whence P-primary ideals are rank one, discrete valuation ideals.

For the proofs in the remainder of this chapter, it will be convenient to state here the following Lemmas 2.12, 2.13, 2.14 and Theorem 2.15 due to Gilmer and Ohm [5].

Lemma 2.12

Let D be a domain, and let A be an ideal of D such that A^n is a valuation ideal for all n. Then $B = \bigcap_{n=1}^{\infty} A^n$ is prime.

Lemma 2.13

Let P be a prime ideal of a valuation ring V, and let A be the intersection of the primary ideals belonging to P. Then A is prime, and there exists no prime ideal P_1 such that $A \subset P_1 \subset P$.

Lemma 2.14

Let M be a prime ideal of a domain D, and suppose there exists a prime ideal $P \subset M$ such that there is no prime ideal P_1 with $P \subset P_1 \subset M$. Then P is the intersection of the M-primary ideals of D which contain P.

Theorem 2.15

Let M be a prime ideal of a domain D, and suppose every M-primary ideal is a valuation ideal. If there exists a prime ideal $P \subset M$ such that there is no prime ideal P_1

with $P < P_1 < M$, then P is unique (and is, in fact, the intersection of all M-primary ideals).

Theorem 2.16

If $P_1 > P_2$ are prime ideals of D, then there exist prime ideals P and P* such that $P_1 \supset P > P^* \supset P_2$ and there are no prime ideals properly between P and P*.

Proof: We may assume there exist proper prime ideals between P_1 and P_2, for otherwise the theorem is trivially satisfied. Since $P_1 > P_2$, we select $x \in P_1$, $x \notin P_2$, and consider the ideal $(P_2, (x))$. We have $P_1 \supset (P_2, (x)) > P_2$, hence P_1 contains a prime ideal P such that P is a minimal prime belonging to $(P_2, (x))$ [7;107]. Now we have $P_1 \supset P \supset (P_2, (x)) > P_2$. We assume there exist prime ideals between P and P_2, but that the theorem is not satisfied by any of these prime ideals. Now consider the set S of all strictly increasing chains of prime ideals properly between P_2 and P. If C_α and C_β are elements of S, we order $C_\alpha \leq C_\beta$ if every prime ideal in the chain C_α is in the chain C_β. If S_1 is an arbitrary totally ordered subset of S, denote by C_1 the chain of prime ideals having the property that if P_α is a prime ideal in any element of S_1, then P_α is a prime ideal in C_1. Then C_1 is an upper bound of S_1 and thus every totally ordered subset of S has an upper bound in S. Therefore, by Zorn's Lemma, S contains maximal chains. We consider then a maximal chain in S and examine the union $\bigcup_\alpha P'_\alpha$ of the elements of this chain. Now $\bigcup_\alpha P'_\alpha$ is a prime ideal and properly contains each member

of the chain, hence $\bigcup_{\alpha} P'_{\alpha} = P$. But $x \in P$, so we must have $x \in P'_{\beta}$ for some prime ideal P'_{β} in this chain. We have $P \supset P'_{\beta} \supset (P_2, (x))$, hence $P = P'_{\beta}$ since P is a minimal prime of $(P_2, (x))$. This contradicts the fact that $P > P'_{\beta}$, therefore there must exist a prime ideal, containing P_2 and contained in P, which satisfies the theorem.

Lemma 2.17

If R is a valuation ring and P is a prime ideal of R such that P is the only P-primary ideal of R, then $P = P^2$. Furthermore, if $\{P_{\alpha}\}$ denotes the set of prime ideals properly contained in P, then $P = \bigcup_{\alpha} P_{\alpha}$.

Proof: Suppose $P \neq P^2$, then $\bigcap_{n=1}^{\infty} P^n = P^*$ is a prime ideal since each P^n is a valuation ideal, by Lemma 2.12. Thus $P > P^*$ and if P_1 is any prime ideal with the property that $P \supset P_1 \supset P^*$, then either $P^n \subset P_1$ or $P^n \supset P_1$, for each integer $n > 0$. We consider the following two cases; either 1) $P_1 \supset P^n$ for some n, or 2) $P^n \supset P_1$ for all n. In case 1), $P = \sqrt{P^n} \subset P_1$ implies $P = P_1$; and in case 2), $P_1 = P^*$ since then $P_1 \subset \bigcap_{n=1}^{\infty} P^n = P^*$. Therefore, if $P \supset P_1 > P^*$, then $P = P_1$ and there are no prime ideals properly between P and P^*. Then P^* is the intersection of all P-primary ideals in R, by Lemma 2.13, but this contradicts the hypothesis that P is the only P-primary ideal of R. Hence $P = P^2$. For the second part of the lemma, we consider two cases; either 1) there exists a prime ideal P_1

such that $P_1 < P$ and there are no prime ideals properly between P_1 and P_2, or 2) there exists no prime ideal satisfying case 1). If case 1) holds, then P_1 is the intersection of the P-primary ideals, by Theorem 2.15, hence $P_1 = P$ and therefore case 1) cannot hold. Since case 2) must hold, we let $\{P_\alpha\}$ denote the set of all prime ideals of R which are properly contained in P. All the ideals of R are chained, hence $\bigcup_\alpha P_\alpha$ is a prime ideal and properly contains each P_α, thus $P = \bigcup_\alpha P_\alpha$.

Theorem 2.18

If $\mathcal{PP} \subset \mathcal{V}$ and if P is any prime ideal of D, then $\bigcap_{n=1}^{\infty} P^n = P^*$ is a prime ideal. Furthermore, if $\{Q_\alpha\}$ denotes the set of P-primary ideals of D, then $P^* \supset \bigcap_\alpha Q_\alpha$.

Proof: The first part of the theorem is a special case of Lemma 2.12. If P is an idempotent prime ideal, then $\bigcap_{n=1}^{\infty} P^n = P^* = P \supset \bigcap_\alpha Q_\alpha$. If P is not idempotent, let n denote an arbitrary positive integer larger than one. P^n is a valuation ideal by hypothesis, so there exists a valuation ring $R_V \supset D$ such that $P^n \cdot R_V \cap D = P^n$. Furthermore, $\sqrt{P^n \cdot R_V} = P_V$ is a prime ideal of R_V and $P_V \cap D = P$. Thus every P_V-primary ideal of R_V contracts to a P-primary ideal of D. In R_V, we have either 1) $P^n \cdot R_V$ contains a P_V-primary ideal, or 2) $P^n \cdot R_V$ contains no P_V-primary ideal. If case 2) holds, then $P^n \cdot R_V \subset \bigcap_\beta \bar{Q}_\beta$, where $\{\bar{Q}_\beta\}$ denotes the set of all P_V-primary ideals of R_V. But $\bigcap_\beta \bar{Q}_\beta$ is a prime ideal by Lemma 2.13, thus $P_V = \sqrt{P^n \cdot R_V} \subset \bigcap_\beta \bar{Q}_\beta$

implies $P_v = \bigcap_\beta \bar{Q}_\beta$. Then P_v is the only P_v-primary ideal in R_v and thus $P_v = P_v^2$, by the previous lemma. Since $\sqrt{P^n \cdot R_v} = P_v$, we have $P^n \cdot R_v$ contains every prime ideal which is properly contained in P_v. If $\{\bar{P}_\alpha\}$ denotes the set of prime ideals of R_v which are properly contained in P_v, then $P^n \cdot R_v \supset \bigcup_\alpha \bar{P}_\alpha$. But $P_v = \bigcup_\alpha \bar{P}_\alpha$ by the previous lemma, hence $P^n \cdot R_v = P_v$. Now $P^n = P^n \cdot R_v \cap D = P_v \cap D = P$ shows P is idempotent, thus case 2) cannot hold. Therefore case 1) holds and thus $P^n \cdot R_v \cap D = P^n$ contains a P-primary ideal. The integer n is arbitrary, thus P^n contains a P-primary ideal for every positive integer n, and hence $P^* \supset \bigcap_\alpha Q_\alpha$ where $\{Q_\alpha\}$ denotes the set of all P-primary ideals of D.

Theorem 2.19

If $\mathcal{J} \subset \mathcal{V}$ and $P \supset P^*$ are prime ideals of D such that there are no prime ideals properly between them, then either $\bigcap_{n=1}^{\infty} P^n = P$ or $\bigcap_{n=1}^{\infty} P^n = P^*$.

Proof: If P is an idempotent prime ideal, then $\bigcap_{n=1}^{\infty} P^n = P$. If P is not idempotent, then $\bigcap_{n=1}^{\infty} P^n$ is a prime ideal by the above theorem. Furthermore, $P^* = \bigcap_\alpha Q_\alpha$ where $\{Q_\alpha\}$ denotes the set of all P-primary ideals in D, by Theorem 2.15. But the previous theorem also states $\bigcap_{n=1}^{\infty} P^n \supset \bigcap_\alpha Q_\alpha$, so we have $P \supset \bigcap_{n=1}^{\infty} P^n \supset \bigcap_\alpha Q_\alpha = P^*$. Therefore $P^* = \bigcap_{n=1}^{\infty} P^n$.

Corollary 2.20

If $\mathcal{J} \subset \mathcal{V}$ and $P_1 \supset P_2$ are prime ideals of D, then $P_1^n \supset P_2$ for all n.

Proof: By Theorem 2.16, there exist prime ideals P and P* such that $P_1 \supset P \rangle P^* \supset P_2$, and such that there exist no prime ideals properly between P and P*. By the previous theorem, we have either $\bigcap_{n=1}^{\infty} P^n = P$ or $\bigcap_{n=1}^{\infty} B^n = P^*$, thus $P_1^n \supset P^{n*} \rangle P^* \supset P_2$ for all n.

Theorem 2.21

If $\mathcal{A} \subset \mathcal{V}$ and P is a prime ideal of D with the property that if P' is any prime ideal such that $P \rangle P'$ then there is a prime ideal properly contained between P and P', then $P = P^2$.

Proof: By the previous corollary, $P^n \rangle P'$ for all n, hence $\bigcap_{n=1}^{\infty} P^n \supset P'$. But $\bigcap_{n=1}^{\infty} P^n$ is a prime ideal, by Theorem 2.18, and contains every prime ideal P' which is properly contained in P. We have, therefore, $P \supset \bigcap_{n=1}^{\infty} P^n$ and there are no prime ideals properly contained between these two prime ideals, hence $P = \bigcap_{n=1}^{\infty} P^n$ and thus $P = P^2$.

Proposition 2.22

Let $\mathcal{A} \subset \mathcal{V}$ and suppose there are no proper idempotent prime ideals in D. Let P be a proper prime ideal such that there exists a prime ideal P* with the property that $P \rangle P^*$ and there are no prime ideals properly contained between P and P*. Then $\bigcap_{n=1}^{\infty} P^n = P^*$ and if $\bar{P}$ is any prime ideal such that $P \rangle \bar{P}$, then $P^* \supset \bar{P}$.

Proof: The equality $\bigcap_{n=1}^{\infty} P^n = P^*$ follows from Theorem 2.19.

By Corollary 2.20, we have $P^n \rangle \overline{P}$ for all n, hence $P^* = \bigcap_{n=1}^{\infty} P^n \supset \overline{P}$.

Theorem 2.23

If $\mathcal{J} \subset \mathcal{V}$ and there are no proper idempotent prime ideals in D, then the ascending chain condition for prime ideals is valid in D.

Proof: Let $P_1 \langle P_2 \langle P_3 \langle \cdots$ be an ascending chain of prime ideals in D; then $\bigcup_i P_i = P$ is also prime. If this chain is not finite, then $P \rangle P_j$ for each j, hence $P^2 \rangle P_j$ for each j, by Corollary 2.20. Therefore $P^2 \supset \bigcup_i P_i$ and thus $P = P^2$. This contradiction establishes the ascending chain condition for prime ideals in D.

Theorem 2.24

If $\mathcal{J} \subset \mathcal{V}$ and if there are no idempotent prime ideals in D, then D is a Prüfer domain.

Proof: By the above theorem, the ascending chain condition for prime ideals in D is valid, hence the theorem follows from Theorem 1.12.

Theorem 2.25

If semi-primary ideals in D are rank one valuation ideals and if the ascending chain condition for prime ideals is valid, then D is a one-dimensional Prüfer domain (and conversely).

Proof: Since D is a Prüfer domain, by Theorem 1.12, let A denote an ideal of D with prime radical P. Then there

exists a rank one valuation ring R_v containing D such that $A \cdot R_v \cap D = A$ and also $\sqrt{A \cdot R_v} \cap D = P$. Let P_v denote the unique prime ideal of R_v, then $\sqrt{A \cdot R_v} = P_v$, hence $A \cdot R_v$ is P_v-primary in R_v, and A is P-primary. Therefore every semi-primary ideal of D is primary, so $\mathcal{S} = \mathcal{Q}$, and hence proper prime ideals of D are maximal [3;1274].

Theorem 2.26

If semi-primary ideals in D are rank one, discrete valuation ideals, then D is an almost Dedekind domain (and conversely).

Proof: Since prime ideals are rank one, discrete valuation ideals, there are no proper idempotent prime ideals in D and therefore the ascending chain condition for prime ideals is valid in D, by Theorem 2.23. By the previous theorem, D is a one-dimensional Prüfer domain, hence proper prime ideals are maximal. Since rank one, discrete valuation ideals are Dedekind ideals, the theorem follows by applying Theorem 2.10.

Theorem 2.27

If $\mathcal{Q} \subset \mathcal{PP}$ in D, and P $\supset$ P* are two prime ideals of D such that D $\neq$ P and there are no prime ideals properly between P and P*, then P^n is primary for every n and $P^* = \bigcap_{n=1}^{\infty} P^n$.

Proof: We have P* is the intersection of the P-primary ideals of D which contain P* by Lemma 2.14, thus $P^* \supset \bigcap_{n=1}^{\infty} P^n$. Suppose m is the smallest positive integer

such that P^m is not P-primary. Then $P^{m-1} \cdot D_P \cap D = P^{m-1}$ is P-primary and $P^m \cdot D_P \cap D$ is P-primary, but $P^m < P^m \cdot D_P \cap D$. Therefore, $P^m \cdot D_P \cap D = P^{m-1}$ since $\mathcal{Q} \subset \mathcal{PP}$, but this means $P^m \cdot D_P = P^{m-1} \cdot D_P$ so $(P \cdot D_P)^m = (P \cdot D_P)^{m-1}$. As a result, $(P \cdot D_P)^{m-1} = (P \cdot D_P)^k$ for any positive integer $k \geq m-1$, hence there are no P-primary ideals properly contained in P^{m-1}. This implies, however, that $P^* = P^s$ where $1 < s \leq m-1$, but P^s is not a prime ideal for $s > 1$. This contradiction implies that there is no smallest positive integer m such that P^m is not P-primary, hence P^n is primary for every n. Furthermore, if $P^m \not\supset P^*$, then $P^{m+i} \not\supset P^*$ for every $i \geq 0$, thus $P^* = P^j$ for some $1 < j < m$. However, P^j is not a prime ideal for $j > 1$, hence $P^m > P^*$ for every m and so $\bigcap_{n=1}^{\infty} P^n \supset P^*$. This, with our earlier containment, gives $P^* = \bigcap_{n=1}^{\infty} P^n$.

Theorem 2.28

If $\mathcal{Q} \subset \mathcal{PP}$ in D and $P_1 > P_2$ are prime ideals of D, then $P_1^n > P_2$ for every positive integer n.

Proof: By Theorem 2.16, there exist prime ideals P and P* such that $P_1 \supset P > P^* \supset P_2$ and there are no prime ideals properly between P and P*. From the previous theorem, we get $P^n > \bigcap_{m=1}^{\infty} P^m = P^*$ for every n, hence $P_1^n \supset P^n > P^* \supset P_2$ for every positive integer n.

Theorem 2.29

If $\mathcal{Q} \subset \mathcal{PP}$ in D and P is a proper prime ideal, then $P = P^2$

if and only if P is the union of a chain of prime ideals P_α such that $P \supset P_\alpha$.

Proof: Suppose $P = \bigcup_\alpha P_\alpha$ and $P \supset P_\alpha$ for each prime ideal P_α. We have $P^2 \supset P_\alpha$ by the previous theorem and this is true for every P_α in the chain, hence $P^2 \supset \bigcup_\alpha P_\alpha$ and therefore $P = P^2$. Conversely, suppose that $\bigcup_\alpha P_\alpha \subset P$ for every chain of prime ideals P_α with the property that $P \supset P_\alpha$. By Zorn's Lemma, there exists a maximal chain of prime ideals P'_α such that $P \supset P'_\alpha$, hence there are no prime ideals properly contained between the prime ideal $\bigcup_\alpha P'_\alpha$ and P. By Theorem 2.27, $\bigcup_\alpha P'_\alpha = \bigcap_{n=1}^{\infty} P^n$, thus $P^2 \subset P$ and the converse is proved.

Theorem 2.30

A necessary and sufficient condition that $\mathfrak{Q} \subset \mathfrak{P}\mathfrak{P}$ in D is for $\mathfrak{Q} = \mathfrak{P}\mathfrak{P}$.

Proof: Suppose $\mathfrak{Q} \subset \mathfrak{P}\mathfrak{P}$ and let P denote an arbitrary prime ideal of D. We may assume that $P \neq P^2$, for otherwise the theorem is trivial. By the previous theorem, P is not the union of a chain of prime ideals P_α such that $P \supset P_\alpha$, thus $\bigcup_\alpha P_\alpha \subset P$ for any such chain. By Zorn's Lemma, there exists a prime ideal $P^* \subset P$ such that there are no prime ideals properly between P^* and P. Then P^n is primary for every n, by Theorem 2.27, and hence $\mathfrak{P}\mathfrak{P} \subset \mathfrak{Q}$ since P is an arbitrary prime ideal of D.

Theorem 2.31

Let $\mathfrak{N}$ denote the set of ideals A in D such that there

exists a prime ideal P of D for which D - P is prime to A, that is, $A \cdot D_P \cap D = A$. Then $\mathcal{A} \subset \mathcal{P}\mathcal{P}$ if and only if $\mathcal{A} = \mathcal{P}\mathcal{P}$.

Proof: Since $\mathcal{Q} \subset \mathcal{A}$, we have $\mathcal{Q} = \mathcal{P}\mathcal{P}$ by the previous theorem, hence $\mathcal{A} = \mathcal{P}\mathcal{P}$.

Lemma 2.32

The powers of any proper almost Dedekind ideal intersect in (0).

Proof: Let A be an almost Dedekind ideal and let J denote an almost Dedekind domain with the property $A \cdot J \cap D = A$. Let M be any maximal ideal of J which contains A. Then $A \cdot J \subset A \cdot J_M \subset M \cdot J_M$ and $\bigcap_{n=1}^{\infty} (A \cdot J)^n \subset \bigcap_{n=1}^{\infty} (M \cdot J_M)^n = (0)$ since J_M is a Dedekind domain. Now, since $A^n \subset (A \cdot J)^n \cap D$, it follows that $\bigcap_{n=1}^{\infty} A^n = (0)$.

Theorem 2.33

If $\mathcal{Q} \subset \mathcal{P}\mathcal{P}$ in D and proper prime ideals are almost Dedekind ideals, then D is an almost Dedekind domain (and conversely).

Proof: Let P be an arbitrary proper prime ideal of D. Now $\bigcap_{n=1}^{\infty} P^n = (0)$, by the above lemma, and therefore P is a minimal proper prime of D, by Theorem 2.28. As a result, every proper prime ideal of D is minimal and therefore maximal, hence semi-primary ideals are primary and thus prime powers. Then D is an almost Dedekind domain by Theorem 1.22.

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BIOGRAPHY

Robert Henry Cranford was born in Columbia, Tennessee, on September 10, 1935. He attended the public schools of Columbia, where he graduated from high school in 1953. In the fall of that year he entered Middle Tennessee State College and received his B.S. degree in the spring of 1957. He entered Louisiana State University in the summer of 1957 as a graduate assistant and received his M.S. degree in May, 1959. From October, 1959, until September, 1961, he served in the U.S. Army, and currently holds the rank of 1st Lieutenant, USAR.

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