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SOME CONJUGATION THEOREMS
FOR CERTAIN INVERTIBLE SPACES

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in

The Department of Mathematics

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ABSTRACT

This paper primarily concerns certain groups of homeomorphisms which are associated in a natural way with a variety of spaces, for example spheres, which satisfy a set of axiomatic conditions put forth in Chapter II.

Let us suppose that X is a space of the type in question and that G is an appropriate group of homeomorphisms of X onto itself. In Chapter III we demonstrate the existence of a non-void subcollection \mathcal{D} -- the 'topological dilations' -- of G which is characterized in theorem 1 in the following fashion: suppose $f \in \mathcal{D}$ and $g \in G$, then $g \in \mathcal{D}$ if and only if f is a G -conjugate of g , that is if and only if there exists an element h of G such that $f = hgh^{-1}$.

We proceed then to show in Chapter IV that if f and g are non-identity elements of G , then we may find $\delta, r \in G$ such that the product $(rgr^{-1})(\delta f \delta^{-1}) \in \mathcal{D}$. We then combine this fact with the characterization of \mathcal{D} mentioned above to conclude that each element of \mathcal{D} is a 'universal' element of G in the sense that if $d \in \mathcal{D}$, then any element g of G may be represented as the product of two G -conjugates of d . Furthermore we conclude that if g is not the identity element of G , then g can be represented as the product of three G -conjugates of any non-identity element of G .

Actually the results we obtain are a bit more inclusive than those mentioned above since in certain instances the set of homeomorphisms concerned need not be a group at all.

In Chapter I we mention some examples of the spaces and sets of homeomorphisms to which the conclusions above are applicable. In Chapter V these and other examples are discussed more fully.

CHAPTER I

Preliminary Remarks

In this paper two theorems are proven whose application yields, by way of example, the following:

I. Let S_n denote the n -sphere for $n \geq 1$, and let G denote the group of stable homeomorphisms of S_n onto itself.¹ Let $p \in S_n$ and let q be the point of S_n antipodal to p . We will denote by d an ordinary radial expansion homeomorphism of S_n onto itself which fixes only p and q . (We think of d as the natural extension to S_n of the homeomorphism $d': E_n \rightarrow E_n$ given by $d'(x) = r_0 x$ for some fixed $r_0 \neq 0$.)

If $f, g \in G$, with neither f nor g the identity, then d is the product of a conjugate of f by a conjugate of g , the conjugating homeomorphisms being stable, i.e. there exists elements $\delta, r \in G$ such that $d = (rgr^{-1})(\delta f \delta^{-1})$. From this it immediately follows (corollaries 2, 3, and 4 to theorem 2) that:

- a) any element of G is the product of two conjugates of d ;
- b) if f, g, h , and k are non-identity elements of G , then f is the product of a conjugate of g by a conjugate of

¹ G is the group generated by the collection of all those homeomorphisms of the n -sphere onto itself each of which is the identity on some open set. The term 'stable' used to denote this group appears to be due to Brown and Gluck [3].

h by a conjugate of k; hence, f is the product of three conjugates of any non-identity element of G.

In both a) and b) the conjugating homeomorphisms are themselves elements of G.

II. Let D_n denote the n-cell, $n \geq 2$, and call X' the boundary of D . We take as G the collection of all homeomorphisms of D_n onto itself each of which is not the identity on X' and each also satisfying a stable-like condition. For the homeomorphism d of D_n onto itself we take a naturally defined extension to D_n of the radial expansion homeomorphism defined above on $S_{n-1} = X'$. Then conclusions similar to those in example one follow with 'not the identity on S_n ' replaced by 'not the identity on X' '.

III. Let \mathcal{C} be the Cantor set, realized as the 'middle third' set in $[0,1]$, and denote by G the group of all homeomorphisms of \mathcal{C} onto itself. Define $\{C_i\}_{i=-\infty}^{\infty}$ as follows:

$$\text{for } i \leq 0, C_i = \left[\frac{2}{3^{-i+2}}, \frac{1}{3^{-i+1}} \right] \cap \mathcal{C};$$

$$\text{for } i \geq 1, C_i = \left[\frac{3^i - 1}{3^i}, \frac{3^{i+1} - 2}{3^{i+1}} \right] \cap \mathcal{C}.$$

Then $\{C_i\}_{i=-\infty}^{\infty} \cup \{0\} \cup \{1\} = \mathcal{C}$. Let d be any

homeomorphism of \mathcal{C} onto itself such that $d(0) = 0$, $d(1) = 1$, and $d(C_i) = C_{i+1}$, $-\infty < i < \infty$. We obtain

conclusions like those in example I but here with no restrictions on the homeomorphisms involved other than that they be different from the identity.

IV. The rationals X (or the irrationals) on the line admit a homeomorphism d similar in nature to that in the previous example for the Cantor set. Then here also, for the group G of all homeomorphisms of X onto X , we obtain conclusions like those in the previous example.

These examples, and some additional ones, will be discussed in greater detail in Chapter V.

Conclusion b) in example I, as well as the corresponding conclusion in the other examples, is obtained from corollaries 3 and 4 of theorem 2. These corollaries were obtained previously by R. D. Anderson in [1]. In fact our definition of an A -quadruple is the same as that given by Anderson with the exception that we omit the axiom numbered as (2) in [1].

CHAPTER II

Definitions, Notation, and Axioms

If X is a topological space, if A and B are subsets of X , and if f and g are mappings from X to itself, then we write ' $\text{Cl}(A)$ ' for 'the closure of A in X '; ' $\sim A$ ' for 'the complement of A in X '; ' $B - A$ ' for 'the complement of A in B '; ' A° ' for 'the interior of A in X '; ' $A \subseteq B$ ' for ' A is a subset of B '; and ' $A \subset B$ ' for ' $A \subseteq B$ and $A \neq B$ '. We write ' $f|A$ ' to denote the mapping formed by restricting the domain of f to the set A . The notation ' $f|A = g|A$ ' is often abbreviated as ' $f|A = g$ '. f is said to be supported on A if $f(x) = x$ whenever $x \in \sim A$. ' \bullet ' will denote the identity mapping, and ' \emptyset ' will denote the empty set. Throughout this paper all mappings under consideration will be homeomorphisms of some space X onto itself.

Henceforth we suppose that we have given:

- a) X : a regular, first countable, Hausdorff space.
- b) X' : a non-void, perfect subspace of X (i.e. if $p \in X'$, and if U is a set which is open in X with $p \in U$, then U contains a point of X' other than p).
- c) $(\mathcal{U}, \mathcal{K})$: \mathcal{U} is a basis for X' consisting of sets which are open in X with each element of \mathcal{U} containing a

point of X' ; \mathcal{K} will denote the collection of the closures of the elements of \mathcal{U} .

d) $G(X, X')$: a non-void collection of homeomorphisms of X onto itself, each carrying X' onto X' . $G_{\mathcal{K}}$ will denote the collection of all homeomorphisms of X onto X each carrying X' onto X' and each supported on some element of \mathcal{K} . $G_{\mathcal{K}}^*$ will denote the group generated by $G_{\mathcal{K}}$.

If X , X' , \mathcal{K} , and $G(X, X')$ are such as to satisfy the following axioms, then $(X, X', \mathcal{K}, G(X, X'))$ will be called an A-quadruple.

- 1) If $K \in \mathcal{K}$, then $Cl(\sim K) \in \mathcal{K}$
- 2) If $K \in \mathcal{K}$, and if $g \in G(X, X')$, then $g(K) \in \mathcal{K}$.
- 3) If $g_1, g_2 \in G(X, X')$, and if $g \in G_{\mathcal{K}}$, then $gg_1, g_1g \in G(X, X')$ and $g_2g_1 \in G_{\mathcal{K}}^*$.
- 4) If $K_1, K_2, K_3 \in \mathcal{K}$, with $K_1 \subset K_3^0 \supset K_2$, then there exists an element $g \in G_{\mathcal{K}}$, supported on K_3 , with $g(K_1) = K_2$.²
- 5) If $g \in G_{\mathcal{K}}^*$, and if $K_1, K_2 \in \mathcal{K}$ with $K_1 \subseteq K_2^0$ and $g(K_1) = K_1$, then there exists an element $g_1 \in G_{\mathcal{K}}$, supported on K_2 , such that $g_1|K_1 = g|K_1$.

²Do not forget that ' \subset ' denotes proper containment.

Axioms 1) and 2) will be frequently used and usually without being cited. Note that in the case where $X = X'$ axioms 1) and 4) assert a sort of invertibility for X .

Axiom 3) implies immediately that if $g \in G_{\mathcal{X}}^*$, and if $g_1 \in G(X, X')$, then $gg_1, g_1g \in G(X, X')$. In view of this 3) also implies that if $g \in G(X, X')$, then $g^{-1} \in G(X, X')$: for $g^2 \in G_{\mathcal{X}}^*$; hence $g^{-2} \in G_{\mathcal{X}}^*$, and therefore $g^{-1} = gg^{-2} \in G(X, X')$.

It now follows that if $g \in G_{\mathcal{X}}^*$ and $K \in \mathcal{K}$, then $g(K) \in \mathcal{K}$. For since $G(X, X') \neq \emptyset$, there exists $g_1 \in G(X, X')$. Then $gg_1^{-1} \in G(X, X')$, and $g_1(K) \in \mathcal{K}$ by axiom 2). Hence, also by axiom 2), $g(K) = gg_1^{-1}(g_1(K)) \in \mathcal{K}$.

Also axiom 3) implies that if $g \in G(X, X')$ and $\delta \in G_{\mathcal{X}}^*$, then $\delta g \delta^{-1} \in G(X, X')$ i.e. the conjugate of any element of $G(X, X')$ by any element of $G_{\mathcal{X}}^*$ is an element of $G(X, X')$.

It follows also that the set $G(X, X')$ and $G_{\mathcal{X}}^*$ are either equal or disjoint; or, in other words, $G(X, X') = G_{\mathcal{X}}^*$ if and only if $\bullet \in G(X, X')$.

Note that since each of $G(X, X')$ and $G_{\mathcal{X}}^*$ consists solely of homeomorphisms which carry X' and X' , any homeomorphism which we may subsequently construct must satisfy this condition if it is to be in either $G(X, X')$ or $G_{\mathcal{X}}^*$.

Henceforth we shall assume that $(X, X', \mathcal{K}, G(X, X'))$ is an A-quadruple.

By way of example, while reading the arguments to follow, one might think of X as the n -cell, $n > 1$, and X' as its boundary.

Definition 1: Suppose $p \in X'$ and $\mathcal{C} = \{C_i\}_{i=t_0}^{\infty} \subseteq \mathcal{X}$.

Then \mathcal{C} is called a null sequence with respect to p or, more briefly, \mathcal{C} is said to be null for p , if:

- a) $C_{i+1} \subset C_i^0$, $i \geq t_0$, and
- b) $\{C_i^0\}_{i=t_0}^{\infty}$ is a neighborhood basis for p .

Of course if $\{C_i\}_{i=t_0}^{\infty}$ is null for p , then $p = \bigcap \{C_i\}_{i=t_0}^{\infty}$.

Null sequences indexed from t_0 to $-\infty$ may be defined in a comparable fashion.

Definition 2: Suppose p and q are distinct points of X' , and suppose $\mathcal{C} = \{C_i\}_{i=-\infty}^{\infty} \subseteq \mathcal{X}$. Then \mathcal{C} is called a dilation structure with respect to the ordered pair (p, q) provided:

- a) $\{C_i\}_{i=0}^{-\infty}$ is a null sequence for p , and
- b) $\{Cl(\sim C_i)\}_{i=0}^{\infty}$ is a null sequence with respect to q .

Definition 3: Suppose $g \in G_{\mathcal{X}}^*$ and suppose $\mathcal{C} = \{C_i\}_{i=-\infty}^{\infty}$ is a dilation structure with respect to (p, q) . If $g(C_i) = C_{i+1}$, $-\infty < i < \infty$, then g is called a topological dilation with carrier \mathcal{C} ; and \mathcal{C} is said to carry g .

Note that if $\{C_i\}_{i=0}^{\infty}$ is null for p , and if $g \in G_{\mathcal{X}}^* \cup G(X, X')$, then $\{g(C_i)\}_{i=0}^{\infty}$ is null for $g(p)$. Hence

if $g \in G_{\lambda}^*$ is a topological dilation with carrier $\mathcal{C} = \{C_i\}_{i=-\infty}^{\infty}$ with respect to (p, q) , and if $\delta \in G_{\lambda}^*$, then

$\delta g \delta^{-1}$ is a topological dilation carried by $\mathcal{C}' = \{\delta(C_i)\}_{i=-\infty}^{\infty}$ with respect to $(\delta(p), \delta(q))$.

CHAPTER III

Topological Dilations

We begin this chapter by proving some lemmas which will be useful throughout the paper. Next we demonstrate the existence of topological dilations and, in fact, may conclude that $G_{\mathcal{X}}^*$ is richly supplied with them. We conclude Chapter III with Theorem 1, which asserts that any two topological dilations are conjugate in $G_{\mathcal{X}}^*$.

Proposition A: Suppose p and q are distinct points of X' . Then there exists a dilation structure with respect to (p, q) .

Proof: Since X is first countable, regular, and Hausdorff, X' is perfect, and \mathcal{U} is a basis, it easily follows that we may find sequences of elements of \mathcal{X} , $\{C_i\}_{i=0}^{\infty}$ and

$\{D_i\}_{i=1}^{\infty}$, null for p and q respectively, with $C_0 \cap D_1 = \emptyset$ and $C_0 \cup D_1 \neq X$. Setting $C_i = Cl(\sim D_i)$ for $i \geq 1$, we see that $\mathcal{C} = \{C_i\}_{i=0}^{\infty}$ is the desired dilation structure.

Proposition B: Suppose $K_1, K_2 \in \mathcal{X}$ and suppose $K_1 \subset K_2^{\circ}$. Then there exists $K \in \mathcal{X}$ with $K_1 \subset K^{\circ}$ and $K \subset K_2^{\circ}$.

Proof: Select $x \in K_1^{\circ} \cap X'$ and $E_1 \in \mathcal{X}$ with $x \in E_1^{\circ}$ and $E_1 \subset K_1^{\circ}$. This can be done since X' is perfect. Now select $E_2 \in \mathcal{X}$, with $x \in E_2^{\circ}$, $E_2 \subset E_1^{\circ}$. Using axiom 4), we obtain

$g \in G_{\mathcal{X}}$, supported on K_2 , with $g(K_1) = E_2$. Set $K = g^{-1}(E_1)$.

Lemma 1. Suppose $K \in \mathcal{X}$ and $x_0, y_0 \in (K^0 \cap X')$; suppose also that $\mathcal{C} = \{C_i\}_{i=1}^{\infty}$, $\mathcal{D} = \{D_i\}_{i=1}^{\infty} \subseteq \mathcal{X}$ are null for x_0, y_0 respectively, with $C_1 \subset K^0 \supset D_1$. Then there is a homeomorphism $h \in G_{\mathcal{X}}$, supported on K , such that $h(C_i) = D_i$ for $i \geq 1$.

Proof: By axiom 4) there exists $h_1 \in G_{\mathcal{X}}$, supported on K , such that $h_1(C_1) = D_1$. For $i \geq 2$ we define inductively, by making repeated use of axiom 4), $h_i \in G_{\mathcal{X}}$, supported on D_{i-1} , such that $h_i h_{i-1} \cdots h_1(C_i) = D_i$.

Set $h = \prod_{i=1}^{\infty} h_i$. If $x \neq x_0$, then for some $j \geq 1$, $x \notin C_j$. Then $\prod_{i=1}^j h_i(x) \notin D_j = \prod_{i=1}^j h_i(C_j)$. Since for $i > j$, h_i is supported on D_j , it follows that $h(x) = \prod_{i=1}^j h_i(x)$. From

this, since $h_i \in G_{\mathcal{X}}$ and \mathcal{C} and \mathcal{D} are null for x_0 and y_0 respectively, it follows that h is a homeomorphism supported on K , $h(x_0) = y_0$, and h takes X' onto X' . Hence $h \in G_{\mathcal{X}}$.

As a corollary we obtain lemma 2, which resembles Anderson's telescoping lemma [1].

Lemma 2: Suppose $K_0, K_1 \in \mathcal{X}$ with $K_1 \subset K_0^0$ and $g \in G(X, X')$ with $g(K_0) \subset K_1^0$. Suppose also that $x_0 \in K_1^0 \cap X'$, $\{D_i\}_{i=1}^{\infty}$ is null for x_0 with $D_1 \subset K_1^0$. Then there exists $h \in G_{\mathcal{X}}$,

supported on K_1 , such that $(hg)^i(K_0) = D_i$, for $i \geq 1$.

Proof: Set $C_1 = g(K_0)$ and $C_i = g(D_{i-1})$ for $i \geq 2$. Then $\{C_i\}_{i=1}^{\infty}$ is null for $g(x_0)$. Hence lemma 1 asserts the existence of $h \in G_{\mathcal{X}}$, supported on K_1 , such that $h(C_i) = D_i$, for $i \geq 1$. Hence $(hg)^i(K_0) = D_i$, for $i \geq 1$.

Lemma 3: Any dilation structure carries a topological dilation.

Proof: Suppose $\mathcal{B} = \{B_i\}_{i=-\infty}^{\infty}$ is a dilation structure with

respect to (p, q) . By Proposition B there is an element $K_1 \in \mathcal{X}$ such that $B_0 \subset K_1^{\circ}$, $K_1 \subset B_1^{\circ}$. We now apply lemma 1 with K_1 as K , B_{-i} as C_i , and B_{-i+1} as D_i for $i \geq 1$, and $x_0 = p = y_0$ to obtain $h_1 \in G_{\mathcal{X}}$, supported on K_1 , with $h_1(B_{-i}) = B_{-i+1}$, for $i \geq 1$.

Since $B_0 \subset h_1(B_0^{\circ})$, we may find $K_2 \in \mathcal{X}$ with $B_0 \subset K_2^{\circ}$, $K_2 \subset h_1(B_0^{\circ})$. Then $Cl(\sim h_1(B_0)) \subset \sim K_2$, and $Cl(\sim B_i) \subset \sim K_2$, $i \geq 1$. So we again apply lemma 1 to obtain $h_2 \in G_{\mathcal{X}}$, supported on $Cl(\sim K_2)$ with $h_2(Cl(\sim h_1(B_0))) = Cl(\sim B_1)$, and $h_2(Cl(\sim B_i)) = Cl(\sim B_{i+1})$, $i \geq 1$.

$h = h_2 h_1 \in G_{\mathcal{X}}^*$ is the desired topological dilation.

Corollary: There exists an element $g \in G_{\mathcal{X}}^*$ such that g is a topological dilation.

Proof: Since X' is non-void and perfect, X' contains two distinct points. The corollary now follows from proposition A and lemma 3.

We remarked earlier that if $g \in G_{\mathcal{X}}^*$ is an arbitrary topological dilation, then any conjugate of g by an element

of $G_{\mathcal{X}}^*$ is also a topological dilation. Theorem 1 asserts that the converse is true: that any topological dilation is a conjugate of g .

Theorem 1. Suppose g_1 and g_2 are topological dilations. Then there is an element r of $G_{\mathcal{X}}^*$ such that $g_1 = rg_2r^{-1}$.

Proof: Let $\mathcal{C} = \{C_i\}_{i=-\infty}^{\infty}$ be a dilation structure with respect to (p, q) ; and let $\mathcal{D}' = \{D'_i\}_{i=-\infty}^{\infty}$ be a dilation structure with respect to (p', q') ; with \mathcal{C} and \mathcal{D}' carrying g_2 and g_1 respectively.

We may select a $a \in \sim C_0 \cap X'$, $a \neq p'$. There is an integer $i_0 \leq 0$ such that $a \notin D'_{i_0}$. Hence we may find

$K_1 \in \mathcal{X}$ with $a \in K_1^0$, $K_1 \cap (C_0 \cup D'_{i_0}) = \emptyset$, and $\sim K_1 \neq (C_0 \cup D'_{i_0})$.

Set $D_i = D'_{i+i_0}$, $-\infty < i < \infty$; then $\mathcal{D} = \{D_i\}_{i=-\infty}^{\infty}$ carries g_1 , and $D_0 = D'_{i_0}$.

Since $C_0 \cup D_0 \subset \sim K_1$, we may invoke lemma 1 to obtain $\delta_1 \in G_{\mathcal{X}}$, supported on $Cl(\sim K_1)$, with $\delta_1(C_i) = D_i$, $i \leq 0$.

$\{\delta_1(Cl(\sim C_i))\}_{i=1}^{\infty}$ is null for $\delta_1(q)$, and $\delta_1(Cl(\sim C_0)) = Cl(\sim D_0)$. Hence we may again apply lemma 1, this time to the sequences $\{\delta_1(Cl(\sim C_i))\}_{i=1}^{\infty}$ and $\{Cl(\sim D_i)\}_{i=1}^{\infty}$, to

obtain $\delta_2 \in G_{\mathcal{X}}$, supported on $Cl(\sim D_0)$, with

$$\delta_2 \delta_1(Cl(\sim C_i)) = Cl(\sim D_i), \quad i \geq 1.$$

Set $\delta = \delta_2 \delta_1 \in G_{\mathcal{X}}^*$. $\delta(C_i) = D_i$, $-\infty < i < \infty$, and

$\delta(p) = p'$, $\delta(q) = q'$. Hence \mathcal{D} carries the topological dilation $g = \delta g_2 \delta^{-1}$.

We may think of the construction of δ as the first step in the conjugation procedure to change g_2 into g_1 , since it yields the topological dilation g which is carried by the same structure that carries g_1 . Of course pointwise g need not equal g_1 . To achieve pointwise equality a further conjugation is necessary, which in the event that D_0 is open and closed is quite simple. We proceed to that case:

Set $B_i = (\text{boundary of } D_i) = D_i - D_i^0$. Since $g^i(B_0) = B_i$, $-\infty < i < \infty$, $B_i = \emptyset$ if and only if $B_0 = \emptyset$. If $B_0 = \emptyset$, define α as follows:

$$\alpha(p') = p', \quad \alpha(q') = q';$$

$$\text{if } x \in D_{i+1} - D_i, \quad \alpha(x) = g_1^i g^{-i}(x), \quad -\infty < i < \infty.$$

Then α is a homeomorphism of X onto itself taking X' onto X' . Since α takes D_0 onto itself, and since $B_0 = \emptyset$, if we define $\alpha_1 = \alpha$ on D_0 and $\alpha_1 = e$ on $\sim D_0$; and $\alpha_2 = \alpha$ on $\sim D_0$ and $\alpha_2 = e$ on D_0 , then $\alpha_1, \alpha_2 \in G_{\mathcal{K}}$ and $\alpha = \alpha_2 \alpha_1 \in G_{\mathcal{K}}^*$. Since $g_1 = \alpha g \alpha^{-1}$, if we set $r = \alpha \delta$, then $g_1 = r g_2 r^{-1}$.

It is in the remaining case that axiom 5) finds its only employment. Thus we might observe now, if the reader hasn't already done so, that in case \mathcal{K} consists of sets which are open and closed in X axiom 5) may be omitted.

Now suppose $B_0 \neq \emptyset$.

By Proposition B there exists $K_1 \in \mathcal{X}$ with $D_0 \subset K_1^0$, $K_1 \subset D_1^0$. Set $\alpha_1 = g_1 g^{-1} \in G_{\mathcal{X}}^*$; α_1 takes D_0 onto D_1 , $-\infty < i < \infty$. By axiom 5) there exists $\bar{\alpha}_1 \in G_{\mathcal{X}}$, supported on $Cl(\sim K_1)$ such that $\bar{\alpha}_1|_{Cl(\sim D_1)} = \alpha_1$. In particular, $\bar{\alpha}_1$ takes D_0 onto D_1 and $\bar{\alpha}_1|_{B_1} = \alpha_1$.

Proceeding inductively, for $i \geq 2$ we may select $K_i \in \mathcal{X}$ with $D_{i-1} \subset K_i^0$, $K_i \subset D_i^0$. Set $\alpha_i = g_1^i g^{-i} (\bar{\alpha}_{i-1} \cdot \bar{\alpha}_{i-2} \cdots \bar{\alpha}_1)^{-1}$. $\alpha_i \in G_{\mathcal{X}}^*$ and α_i takes D_{i-1} onto D_i , $-\infty < j < \infty$. By axiom 5) there is $\bar{\alpha}_i \in G_{\mathcal{X}}$, supported on $Cl(\sim K_i)$, such that $\bar{\alpha}_i|_{Cl(\sim D_i)} = \alpha_i$. In particular, $\bar{\alpha}_i$ takes D_{i-1} onto D_i and $\bar{\alpha}_i|_{B_i} = \alpha_i$.

Set $\alpha' = \prod_{i=1}^{\infty} \bar{\alpha}_i$. α' is a homeomorphism, supported on $Cl(\sim K_1)$, and taking X' onto X' and X onto X ; hence $\alpha' \in G_{\mathcal{X}}$.

Observe that $\alpha'|_{B_i} = g_1^i g^{-i}$, $i \geq 0$; and α' takes D_i onto D_i , $-\infty < i < \infty$.

Now take $K_{-1} \in \mathcal{X}$ with $D_{-1} \subset K_{-1}^0$, $K_{-1} \subset D_0^0$. Set $\alpha_{-1} = g_1^{-1} g$. $\alpha_{-1} \in G_{\mathcal{X}}^*$, and α_{-1} takes D_{-1} onto D_0 , $-\infty < i < \infty$. As before, from axiom 5) we obtain $\bar{\alpha}_{-1} \in G_{\mathcal{X}}$, supported on K_{-1} , with $\bar{\alpha}_{-1}|_{D_{-1}} = \alpha_{-1}$. In particular, $\bar{\alpha}_{-1}$ takes D_{-1} onto D_0 , and $\bar{\alpha}_{-1}|_{B_{-1}} = \alpha_{-1}$.

Again proceeding inductively for $i \geq 2$, select $K_{-i} \in \mathcal{X}$ with $D_{-i} \subset K_{-i}^0$, $K_{-i} \subset D_{-i+1}^0$. Set $\alpha_{-i} = g_1^{-i} g^i (\bar{\alpha}_{-i+1} \bar{\alpha}_{-i+2} \cdots \bar{\alpha}_{-1})^{-1}$. Then α_{-i} takes D_{-i} onto D_{-i+1} , $-\infty < j < \infty$, and $\alpha_{-i} \in G_{\mathcal{X}}^*$. We use axiom 5) to obtain

$\bar{\alpha}_{-i} \in G_{\mathcal{X}}$, supported on K_{-i} , with $\bar{\alpha}_{-i}|_{D_{-i}} = \alpha_{-i}$. And $\bar{\alpha}_{-i}$ takes D_j onto D_j , and $\bar{\alpha}_{-i}|_{B_{-i}} = \alpha_{-i}$.

Set $\bar{\alpha} = \prod_{i=1}^{\infty} \bar{\alpha}_{-i}$. $\bar{\alpha} \in G_{\mathcal{X}}$, is supported on K_{-1} , takes D_i onto D_i , $-\infty < i < \infty$, and $\bar{\alpha}|_{B_{-i}} = g_1^{-i} g_1^i$, $i \geq 0$.

Now set $\alpha = \bar{\alpha} \cdot \alpha' \in G_{\mathcal{X}}^*$. Since $K_{-1} \cap Cl(\sim K_1) = \emptyset$, α takes D_i onto D_i and $\alpha|_{B_i} = g_1^i g_1^{-i}$, $-\infty < i < \infty$.

Setting $\bar{g} = \alpha g \alpha^{-1}$, we observe that $\bar{g}|_{B_i} = g_1$, $-\infty < i < \infty$.

Finally, we define β as follows:

$$\beta(p') = p', \quad \beta(q') = q';$$

$$\text{if } x \in D_{i+1} - D_i, \quad \beta(x) = g_1^i \bar{g}^{-i}(x), \quad -\infty < i < \infty.$$

β is a homeomorphism of X onto itself, carrying X' onto X' . Also $\beta|_{B_i} = e$, $-\infty < i < \infty$. And since, in addition, β takes D_0 onto D_0 , if we define $\beta_1 = \beta$ on D_0 and $\beta_1 = e$ on $\sim D_0$; and $\beta_2 = \beta$ on $\sim D_0$ and $\beta_2 = e$ on D_0 , then $\beta_1, \beta_2 \in G_{\mathcal{X}}$; hence $\beta = \beta_2 \beta_1 \in G_{\mathcal{X}}^*$. And since $g_1 = \beta \bar{g} \beta^{-1}$, if we set $r = \beta \alpha \delta \in G_{\mathcal{X}}^*$, we have $g_1 = r g_2 r^{-1}$.

CHAPTER IV

The Principal Theorem; Its Corollaries

In this chapter we prove the principal theorem of the paper and deduce some corollaries.

Theorem 2: Suppose f and g are elements of $G(X, X')$ neither of which is the identity on X' . Then there exists elements h_1 and h_2 of $G^*_{\mathcal{X}}$ such that $(h_2gh_2^{-1})(h_1fh_1^{-1})$ is a topological dilation. Furthermore h_1 and h_2 may be chosen as topological dilations.

Proof: We distinguish two cases: Case 1: At least one of f and g , say f , is not of period two on X' . Case 2: Both f and g are of period two on X' .

Case 1: Since f is not of period two on X' , neither f nor g is the identity on X' , and X' is perfect, we may find elements $C_1, D_1, A_0,$ and B_0 of \mathcal{X} such that the sets $\{C_1, D_1, A_0, B_0, f(A_0), f^{-1}(A_0), g(B_0)\}$ are mutually disjoint and such that the sets $\{C_1, D_1, A_0, B_0, f(A_0), f^{-1}(A_0), g^{-1}(B_0)\}$ are mutually disjoint.

Select $c \in C_1^0 \cap X', d \in D_1^0 \cap X', a \in A_0^0 \cap X',$ and $b \in B_0^0 \cap X'$. We now take sequences of elements of $\mathcal{X} - \{C_i\}_{i=1}^{\infty}, \{D_i\}_{i=1}^{\infty}, \{A_i\}_{i=0}^{-\infty},$ and $\{B_i\}_{i=0}^{-\infty}$ -null for $c, d, a,$ and b respectively.

Set $A_i = Cl(\sim C_i)$, $B_i = Cl(\sim D_i)$, $i \geq 1$. Then $\mathcal{A} = \{A_i\}_{i=-\infty}^{\infty}$ and $\mathcal{B} = \{B_i\}_{i=-\infty}^{\infty}$ are dilation structures with respect to (a,c) and (b,d) respectively. By lemma 3 \mathcal{A} and \mathcal{B} respectively carry topological dilations δ and r . Set $\mathcal{P} = (rgr^{-1})(\delta f \delta^{-1})$. $\mathcal{P} \in G_X^*$ by axiom 3).

Now a remark about what is to follow: It is true that $\mathcal{P}(A_1) \subset A_1^0$ and $\mathcal{P}^{-1}(C_1) \subset C_1^0$ ($A_1 = Cl(\sim C_1)$). Hence $\mathcal{P}^i(A_1) \subset \mathcal{P}^{i-1}(A_1^0)$ for $i \geq 1$, and $\mathcal{P}^{-i}(C_1) \subset \mathcal{P}^{-i+1}(C_1^0)$, $i \geq 1$. Therefore if $\{\mathcal{P}^i(A_1)\}_{i=0}^{\infty}$ were null for a point of X' and if $\{\mathcal{P}^{-i}(C_1)\}_{i=0}^{\infty}$ were null for a point of X' , then it is apparent that \mathcal{P} would then be a topological dilation. Therefore, in order to achieve this we are going to multiply r and δ by appropriate elements of G .

Consider $\mathcal{P}(A_1)$:

$$\delta^{-1}(A_1) = A_0$$

$$f(A_0) \subset A_1^0 - A_0$$

$$\delta(A_1^0 - A_0) = A_2^0 - A_1 = C_1^0 - C_2$$

$$\text{Hence } \delta f \delta^{-1}(A_1) \subset C_1^0 - C_2$$

$$r^{-1}(C_1) \subset B_0^0 - B_{-1}$$

$$g(B_0) \subset B_1^0 - B_0$$

$$r(B_1^0 - B_0) = B_2^0 - B_1 = D_1^0 - D_2$$

Hence $\mathcal{P}(A_1) \subset D_1^0 - D_2$. Set $K_1 = rg(B_0)$. $\mathcal{P}(A_1) \subset K_1^0$, $K_1 \subset D_1^0 - D_2$. Select $x_1 \in K_1^0 \cap X'$ and a sequence of

elements of \mathcal{X} , $(T_i)_{i=1}^{\infty}$, null for x_1 , with $T_1 \subset K_1^{\circ}$. We now apply lemma 2, with A_1 as K_0 and with β as g , to obtain $\alpha_1 \in G_{\mathcal{X}}$, supported on K_1 , with $(\alpha_1 \beta)^i(A_1) = T_i$, $i \geq 1$.

Now set $\bar{\beta} = (\delta f^{-1} \delta^{-1})(\alpha_1 r g^{-1} r^{-1} \alpha_1^{-1})$ and consider $\bar{\beta}(C_1)$.

Since $K_1 \cap C_1 = \emptyset$, $rg^{-1}r^{-1}\alpha_1^{-1}(C_1) = rg^{-1}r^{-1}(C_1)$.

$$r^{-1}(C_1) \subset B_0^{\circ}$$

$$g^{-1}(B_0) \subset B_1^{\circ} - B_0$$

$$r(B_1^{\circ} - B_0) = B_2^{\circ} - B_1 = D_1^{\circ} - D_2.$$

So $rg^{-1}r^{-1}(C_1) \subset D_1^{\circ} - D_2$. Since $K_1 \subset D_1^{\circ} - D_2$,

$$\alpha_1 rg^{-1}r^{-1}(C_1) \subset D_1^{\circ} - D_2 \subset D_1^{\circ}.$$

$$\delta^{-1}(D_1) \subset A_0^{\circ}; \text{ and } f^{-1}(A_0) \subset A_1^{\circ} - A_0;$$

$$\delta(A_1^{\circ} - A_0) = A_2^{\circ} - A_1 = C_1^{\circ} - C_2.$$

Hence $\bar{\beta}(C_1) \subset C_1^{\circ} - C_2$.

Set $K_2 = \delta f^{-1}(A_0)$. Then $\bar{\beta}(C_1) \subset K_2^{\circ}$, $K_2 \subset C_1^{\circ} - C_2$.

Note that $K_2 \cap \delta f(A_0) = \emptyset$.

Select $x_2 \in K_2^{\circ} \cap X'$ and a sequence of elements of \mathcal{X} , $(T_i)_{i=-1}^{\infty}$ null for x_2 , with $T_{-1} \subset K_2^{\circ}$. We again apply

lemma 2, this time with C_1 as K_0 and with $\bar{\beta}$ as g , to obtain $\alpha_2 \in G_{\mathcal{X}}$, supported on K_2 , with $(\alpha_2 \bar{\beta})^i(C_1) = T_{-i}$, $i \geq 1$.

Set $d = (\alpha_1 r g r^{-1} \alpha_1^{-1})(\alpha_2 \delta f \delta^{-1} \alpha_2^{-1})$ and observe

that if $d^i(A_1) = T_i$ and $d^{-i}(C_1) = T_{-i}$ for $i \geq 1$, then d is a topological dilation carried by the dilation structure $\{T'_i\}_{i=-\infty}^{\infty}$ with respect to (x_2, x_1) , where $T'_i = T_i$, $i \leq -1$, $T'_0 =$

C_1 , $T'_i = Cl(\sim T_i)$, $i \geq 1$.

Consider $d(A_1)$:

Since $K_2 \cap A_1 = \emptyset$, $\delta f \delta^{-1} \alpha_2^{-1}(A_1) = \delta f \delta^{-1}(A_1) = \delta f(A_0)$. Since $K_2 \cap \delta f(A_0) = \emptyset$, $\alpha_2 \delta f(A_0) = \delta f(A_0) \subset C_1$. But $K_1 \cap C_1 = \emptyset$; so $\alpha_1^{-1} \delta f(A_0) = \delta f(A_0)$. Hence $d(A_1) = \alpha_1 r g r^{-1} \delta f \delta^{-1}(A_1) = \alpha_1 \theta(A_1) = T_1$. Since for $i \geq 1$, $T_i \subset A_1$, $d(T_i) = \alpha_1 \theta(T_i) = T_{i+1}$. Therefore $d^i(A_1) = T_i$, $i \geq 1$.

Now consider $d^{-1}(C_1) = \alpha_2 \delta f^{-1} \delta^{-1} \alpha_2^{-1} \alpha_1 r g^{-1} r^{-1} \alpha_1^{-1}(C_1)$:

As seen earlier, $\alpha_1 r g^{-1} r^{-1} \alpha_1^{-1}(C_1) \subset D_1^0$. Since $K_2 \cap D_1 = \emptyset$, $\alpha_2^{-1} \alpha_1 r g^{-1} r^{-1} \alpha_1^{-1}(C_1) = \alpha_1 r g^{-1} r^{-1} \alpha_1^{-1}(C_1)$. Hence $d^{-1}(C_1) = \alpha_2 \delta f^{-1} \delta^{-1} \alpha_1 r g^{-1} r^{-1} \alpha_1^{-1}(C_1) = \alpha_2 \bar{\theta}(C_1) = T_{-1}$. And since for $i \geq 1$, $T_{-i} \subset C_1$, $d^{-1}(T_{-i}) = \alpha_2 \bar{\theta}(T_{-i}) = T_{-i-1}$. Hence $d^{-1}(C_1) = T_{-1}$, and hence d is a topological dilation.

Furthermore, since α_1 is supported on $K_1 \subset D_1^0 - D_2$ and α_2 is supported on $K_2 \subset C_1^0 - C_2$, $h_1 = \alpha_2 \delta$ and $h_2 = \alpha_1 r$ are topological dilations carried by \mathcal{A} and \mathcal{B} respectively; and $d = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$.

In the above we had set $K_2 = \delta f^{-1}(A_0)$ and had then remarked that $K_2 \cap \delta f(A_0) = \emptyset$. It was this which enabled

us to construct α_1 independently of α_2 . Of course the construction of α_2 was dependent on α_1 ; however we defined $K_1 = \text{rg}(B_0)$ and had we been able to assert that $K_1 \cap \text{rg}^{-1}(B_0) = \emptyset$ we could have constructed α_1 and α_2 independently of each other. But, since in the event that g was of period two on X' , $K_1 \cap \text{rg}^{-1}(B_0) \neq \emptyset$, no such assertion was possible. Had f also been of period two on X' then neither α_1 nor α_2 could have been constructed independently of the other. For this reason Case 2 requires a slightly modified procedure.

Case 2. Since neither f nor g is the identity on X' , we may select elements of \mathcal{X} , C_1 , D_0 , A_0 , and B_0 , such that the sets $\{C_1, D_0, A_0, B_0, f(A_0), g(B_0)\}$ are mutually disjoint and the sets $\{C_1, D_0, A_0, B_0, f^{-1}(A_0), g^{-1}(B_0)\}$ are mutually disjoint.

Now select $c \in C_1^0 \cap X'$, $d \in D_0^0 \cap X'$, $a \in A_0^0 \cap X'$, and $b \in B_0^0 \cap X'$. As before we may take sequences of elements of $\mathcal{X} - (C_i)_{i=1}^{\infty}$, $(D_i)_{i=0}^{\infty}$, $(A_i)_{i=0}^{-\infty}$, and $(B_i)_{i=c}^{-\infty}$ - null for c , d , a , and b respectively.

We now set $A_i = \text{Cl}(\sim C_i)$, $B_i = \text{Cl}(\sim D_i)$ for $i \geq 1$ and observe that $\mathcal{A} = (A_i)_{i=-\infty}^{\infty}$ and $\mathcal{B} = (B_i)_{i=-\infty}^{\infty}$ are dilation structures with respect to (a, c) and (b, d) respectively and hence, by lemma 3, carry topological dilations δ_1 and r_1 respectively.

$\delta_1^{-1}(D_0) \subset A_0^0 - A_{-1}$; select $x_1 \in \delta_1^{-1}(D_0^0) \cap X'$. Since $f^{-1}(x_1) = f(x_1) \in f(A_0^0)$, we may find $K_1 \in \mathcal{K}$ with $x_1 \in K_1^0$, $K_1 \subset \delta_1^{-1}(D_0^0)$ and $f^{-1}(K_1) \subset f(A_0^0)$.

Now $\delta_1(K_1) \subset D_0^0 \supset D_1$. Hence, by axiom 4), there is $\alpha_1 \in G_{\mathcal{X}}$, supported on D_0 , such that $\alpha_1 \delta_1(K_1) = D_1$.

Set $\delta = \alpha_1 \delta_1$. δ is a topological dilation, carried by \mathcal{A} , and $\delta(K_1) = D_1$.

Since $r_1 g(B_0) \subset D_1^0$, $\delta^{-1} r_1 g(B_0) \subset K_1^0$. Select $x_2 \in \delta^{-1} r_1 g(B_0^0) \cap X'$. Now, since $f^{-1}(x_2) = f(x_2) \in f(K_1^0)$, we may select inductively a sequence of elements of \mathcal{K} , $\{K_i\}_{i=2}^{\infty}$, null for x_2 , with $K_2 \subset \delta^{-1} r_1 g(B_0^0)$ and $f^{-1}(K_i) \subset f(K_{i-1}^0)$, $i \geq 2$.

Since $\delta f(A_0) \subset C_1^0 - C_2$, we have $r_1^{-1} \delta f(A_0) \subset B_0^0 - B_{-1}$. Set $x_3 = r_1^{-1} \delta f(x_2) \in r_1^{-1} \delta f(K_1^0)$, $i \geq 1$. And of course $x_3 \in X'$.

Now since $g^{-1}(x_3) = g(x_3) \in g r_1^{-1} \delta f(K_1^0)$, we may find $E_1 \in \mathcal{X}$ such that $x_3 \in E_1^0$, $E_1 \subset r_1^{-1} \delta f(K_1^0)$, and $g^{-1}(E_1) \subset g r_1^{-1} \delta f(K_1^0)$. Select inductively a sequence of elements of \mathcal{X} , $\{E_i\}_{i=2}^{\infty}$, null for x_3 , $E_2 \subset E_1^0$, such that $g^{-1}(E_i) \subset g r_1^{-1} \delta f(K_i)$, $i \geq 1$.

Since $r_1(E_i) \subset \delta f(A_0^0) \supset \delta f^{-1}(K_i)$ for $i \geq 1$, we may employ lemma 1 to obtain $\alpha_2 \in G_{\mathcal{X}}$, supported on $\delta f(A_0)$, such that $\alpha_2 r_1(E_{i+1}) = \delta f^{-1}(K_i)$, $i \geq 1$.

We now set $r = \alpha_2 r_1$. r is a topological dilation, carried by \mathcal{B} , and $r(E_{i+1}) = \delta f^{-1}(K_i)$, $i \geq 1$. Since α_2 is supported on $\delta f(A_0) \subset C_1^0$, we have also that $r/\sim C_1 = r_1$. In particular, $g^{-1}(E_i) \subset gr^{-1}\delta f(K_i)$, $i \geq 1$.

Define $\beta \in G_{\mathcal{X}}$ as : $\beta = (rgr^{-1})(\delta f \delta^{-1})$. Consider $\beta(A_1)$:

$$\delta^{-1}(A_1) = A_0; \text{ so}$$

$$\delta f \delta^{-1}(A_1) = \delta f(A_0). \text{ Since } \alpha_2 \text{ is supported on}$$

$$\delta f(A_0), r^{-1}\delta f(A_0) = r_1^{-1}\delta f(A_0) \subset B_0^0.$$

$$r_1 g(B_0) \subset D_1^0 - D_2. \text{ Since } \alpha_2/D_1 = e, r_1 g(B_0) = rg(B_0).$$

$$\text{Hence, } \beta(A_1) = r_1 gr_1^{-1} \delta f \delta^{-1}(A_1) \subset r_1 g(B_0^0).$$

$$\text{Since, for } i \geq 2, K_i \subset \delta^{-1}r_1 g(B_0^0), \delta(K_i) \subset r_1 g(B_0^0).$$

Hence we may apply lemma 2 to obtain $\alpha_3 \in G_{\mathcal{X}}$, supported on $r_1 g(B_0) = rg(B_0)$, such that $(\alpha_3 \beta)^i(A_1) = \delta(K_{i+1})$, $i \geq 1$.

Set $d = (\alpha_3 rgr^{-1} \alpha_3^{-1})(\delta f \delta^{-1})$ and observe that if $\{d^i(A_1)\}_{i=0}^{\infty}$ is null for $\delta(x_2)$ and if $\{d^i(C_1)\}_{i=0}^{\infty}$ is null

for $\delta f(x_2) = \delta f^{-1}(x_2)$, then d is a topological dilation carried by $\{T_i\}_{i=-\infty}^{\infty}$ with respect to $(\delta f(x_2), \delta(x_2))$ where

$$T_i = d^i(C_1) \text{ for } i \leq 0, \text{ and } T_i = Cl(\sim d^i(A_1)) \text{ for } i \geq 1.$$

Consider $d(A_1)$:

Since $\delta f \delta^{-1}(A_1) \subset C_1$ and α_3 is supported on

$$rg(B_0) \subset D_1, \alpha_3^{-1} \delta f \delta^{-1}(A_1) = \delta f \delta^{-1}(A_1). \text{ Hence } d(A_1) =$$

$\alpha_3 r g r^{-1} \delta f \delta^{-1}(A_1) = \alpha_3 \theta(A_1) = \delta(K_2)$. Since, for $i \geq 2$, $\delta(K_1) \subset D_1 \subset A_1^0$, $d(\delta(K_1)) = \alpha_3 \theta(\delta(K_1)) = \delta(K_{1+1})$. Hence $d^i(A_1) = \delta(K_{1+i})$, $i \geq 1$. Since $(K_1)_{i=2}^\infty$ is null for x_2 , $(\delta(K_1))_{i=2}^\infty$ is null for $\delta(x_2)$. Therefore $(d^i(A_1))_{i=0}^\infty$ is null for $\delta(x_2)$.

Consider now $d^{-1}(C_1) = \delta f^{-1} \delta^{-1} \alpha_3 r g^{-1} r^{-1} \alpha_3^{-1}(C_1)$:

Since α_3 is the identity on C_1 , $r g^{-1} r^{-1} \alpha_3^{-1}(C_1) = r g^{-1} r^{-1}(C_1)$.

$$r^{-1}(C_1) \subset B_0^0$$

$$g^{-1} r^{-1}(C_1) \subset B_1^0 - B_0$$

$$r g^{-1} r^{-1}(C_1) \subset D_1^0 - D_2 \subset D_1^0. \text{ Since } \alpha_3 \text{ takes } D_1 \text{ onto}$$

itself,

$$\alpha_3 r g^{-1} r^{-1}(C_1) \subset D_1^0. \text{ Hence}$$

$$d^{-1}(C_1) \subset \delta f^{-1} \delta^{-1}(D_1^0). \text{ But since } \delta^{-1}(D_1) = K_1,$$

$$d^{-1}(C_1) \subset \delta f^{-1}(K_1^0) \subset C_1^0.$$

$$\text{Consider } d^{-2}(C_1) = d^{-1}(d^{-1}(C_1)) \subset d^{-1}(\delta f^{-1}(K_1^0)):$$

$$\text{Since } \alpha_3 \text{ is the identity on } \delta f^{-1}(K_1) \subset C_1^0,$$

$$r^{-1} \alpha_3^{-1} \delta f^{-1}(K_1^0) = r^{-1} \delta f^{-1}(K_1^0) = E_2^0. \text{ Since}$$

$$g^{-1}(E_2) \subset g r^{-1} \delta f(K_2), \text{ } r g^{-1}(E_2^0) \subset r g r^{-1} \delta f(K_2^0) =$$

$$r g r^{-1} \delta f \delta^{-1}(\delta(K_2^0)) = \theta(\delta(K_2^0)). \text{ Hence}$$

$$d^{-2}(C_1) \subset \delta f^{-1} \delta^{-1} \alpha_3 \theta(\delta(K_2^0)) = \delta f^{-1} \delta^{-1}(\delta(K_2^0)) =$$

$$\delta f^{-1}(K_3^0). \text{ An inductive repetition of the above}$$

yields that

$d^{-1}(C_1) \subset \delta f^{-1}(K_{2i-1}^0)$, $i \geq 1$. Since $d^{-1}(C_1) \subset C_1^0$, we conclude that $\{d^i(C_1)\}_{i=0}^{-\infty}$ is null for $\delta f^{-1}(x_2) = \delta f(x_2)$.

Hence d is a topological dilation. Since α_3 is supported on $\text{rg}(B_0) \subset D_1^0 - D_2$, $\alpha_3 r$ is a topological dilation carried by \mathcal{B} . If we set $h_1 = \delta$, $h_2 = \alpha_3 r$, then $d = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$.

Let $d \in G_{\mathcal{X}}^*$ be an arbitrary but fixed topological dilation:

Corollary 1: If $f, g \in G(X, X')$, neither f nor g being the identity on X' , then d is the product of a conjugate of f by a conjugate of g , the conjugating homeomorphisms being elements, of $G_{\mathcal{X}}^*$.

Proof: By theorem 2 there exist $h_1, h_2 \in G_{\mathcal{X}}^*$ such that $\bar{d} = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$, where \bar{d} is a topological dilation.

But by theorem 1 there exists $h_3 \in G_{\mathcal{X}}^*$ such that $h_3 d h_3^{-1} = \bar{d}$. Hence $h_3 d h_3^{-1} = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$. Therefore $d = h_3^{-1}(h_2 g h_2^{-1})(h_1 f h_1^{-1})h_3 = (h_3^{-1} h_2 g h_2^{-1} h_3)(h_3^{-1} h_1 f h_1^{-1} h_3)$.

Corollary 2: If $f \in G(X, X')$ and either $f|_{X'} \neq e$ or $f = e$ and if $d \in G(X, X')$, then f is a product of two conjugates of d , the conjugating homeomorphisms being elements of $G_{\mathcal{X}}^*$.

Proof: If $f|_{X'} \neq e$, then by corollary 1, with d^{-1} as g , we have $d = (h_2 d^{-1} h_2^{-1})(h_1 f h_1^{-1})$ for some $h_2, h_1 \in G_{\mathcal{X}}^*$. Hence $(h_2 d h_2^{-1}) d = h_1 f h_1^{-1}$; or $h_1^{-1}(h_2 d h_2^{-1}) d h_1 = f$.

Therefore $(h^{-1}h_2dh_2^{-1}h_1)(h_1^{-1}dh_1) = f$.

If $f = e$, then notice that d^{-1} is a topological dilation, and hence, for some $h_1 \in G_X^*$, $h_1dh_1^{-1} = d^{-1}$. Then $d(h_1dh_1^{-1}) = dd^{-1} = e = f$.

Corollary 3: Suppose f, g, h , and k are elements of $G(X, X')$ and none of which are the identity on X' . Then f is the product of a conjugate of g by a conjugate of h by a conjugate of k , the conjugating elements being in G_X^* .

Proof: By corollary 1 $(h_1fh_1^{-1})(h_2g^{-1}h_2^{-1}) = d = (h_3kh_3^{-1})(h_4hh_4^{-1})$ for some $h_1, h_2, h_3, h_4 \in G_X^*$. Then $h_1fh_1^{-1} = (h_3kh_3^{-1})(h_4hh_4^{-1})(h_2gh_2^{-1})$. Hence $f = (h_1^{-1}h_3kh_3^{-1}h_1)(h_1^{-1}h_4hh_4^{-1}h_1)(h_1^{-1}h_2gh_2^{-1}h_1)$.

Corollary 4: Suppose $f, g \in G(X, X')$ and $f|X' \neq e \neq g|X'$. Then f is the product of three conjugates of g , the conjugating elements being in G_X^* .

Proof: Corollary 3 with $h = k = g$.

CHAPTER V

Examples

- (1) Let $X = X' = S_n$, the n -sphere for $n \geq 1$. Let $G(X, X') = G_{\mathcal{K}}^*$ = the group of stable homeomorphisms of S_n onto itself. \mathcal{K} is the collection of images of a geometric n -cell under the elements of $G_{\mathcal{K}}^*$.

That axioms 1), 2), and 3) are satisfied in this case is evident. The proof that axioms 4) and 5) obtain may be found in [3]. The homeomorphism d mentioned in connection with this example in Chapter I is easily seen to be a topological dilation. (And owing to Theorem 1 any topological dilation, as defined, is a 'stable conjugate' of d .)

- (2) Take $X, X', \mathcal{K}, G_{\mathcal{K}}^*$ as above; let $G(X, X')$ be the set of all homeomorphisms which are the product of some geometric orientation-reversing involution h and an element of $G_{\mathcal{K}}^*$. Since $d \notin G(X, X')$, corollary 2 does not obtain here. However $dh \in G(X, X')$; so we may substitute for corollary 2 the following: if $f \in G(X, X')$, then f is a product of a conjugate of d by a conjugate of dh ; the conjugating homeomorphisms, of course, being in $G_{\mathcal{K}}^*$.

- (3) Let $X = S_n$, $n \geq 1$. Let X' be a countable dense subset of X , and let $G_{\mathcal{K}}^* = G(X, X')$ be those stable

homeomorphisms which take X' onto X' . \mathcal{K} is the collection of images of a geometric n -cell under elements of $G^*_\mathcal{K}$.

- (4) $X = D_n =$ the n -cell, $n \geq 2$. $X' =$ the boundary of D_n . $G_\mathcal{K}$ is the set of all homeomorphisms of D_n onto itself each of which is the identity on some neighborhood of some point of X' . $G(X, X') = G^*_\mathcal{K}$ and \mathcal{K} is the collection of images of a geometric half-cell under the elements of $G^*_\mathcal{K}$.

Axioms 1), 2), and 3) are obviously satisfied, and axioms 4) and 5) follow from arguments like those given in [3].

- (5) Let $X = X' =$ the rationals (or irrationals) on the line. Let $G^*_\mathcal{K} = G(X, X')$ = the collection of all homeomorphisms of X onto itself; and let \mathcal{K} be the collection of all non-void, proper open and closed subsets of X .

See [1] for a further discussion of the above examples.

In case the annulus conjecture turns out to be true, then in (1) we may take $G^*_\mathcal{K} = G(X, X')$ to be the collection of all orientation preserving homeomorphisms; and in (2) we may take $G(X, X')$ to be all the orientation reversing homeomorphisms.

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Major Field: Mathematics

Title of Thesis: Some Conjugation Theorems for Certain Invertible Spaces

Approved:

R. D. Anderson

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