Local stabilization of nonlinear systems through the reduction model approach

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Local Stabilization of Nonlinear Systems Through the Reduction Model Approach

Frederic Mazenc and Michael Malisoff

Abstract—We study a general class of nonlinear systems with input delays of arbitrary size. We adapt the reduction model approach to prove local asymptotic stability of the closed loop input delayed systems, using feedbacks that may be nonlinear. Our Lyapunov-Krasovskii functionals make it possible to determine estimates of the basins of attraction for the closed loop systems.

Index Terms—Delay, nonlinear, reduction model, stabilization.

I. INTRODUCTION

12 The reduction model approach is a well-known stabilization technique for systems with input delays. It originated in [1] and has been studied in many works, e.g., [2]–[6]. It is effective for stabilizing linear time-invariant systems with arbitrarily long pointwise or distributed input delays. However, the approach does not directly apply to nonlinear systems; it is extended by introducing an extra dynamic (which gives the ‘state predictor’) whose initial condition is given by an implicit equation (as is done in [7]–[9], and [6, Chapt. 6, p. 128]), and only a few recent works adapt it to time varying systems [10]. This is a limitation, because many systems are nonlinear and lead to the stabilization of time varying nonlinear systems when a trajectory has been computed. Moreover, the work [11] is limited to globally Lipschitz nonlinear systems, and it has a restriction on the size of the delays. See also [12] and [13] for stabilization of nonlinear systems with arbitrarily long input delays when the systems have special structures, and [14] for compensation of arbitrarily long input delays under input sampling based on prediction.

29 These remarks motivate our work. We show that the reduction model approach can be used to locally asymptotically stabilize a large family of nonlinear time varying systems of the form $\dot{x}(t) = A(t)x(t) + B(t)u(t - \tau) + F(t, x(t))$, with arbitrarily long constant known input delays $\tau$, where $F$ is of order 2 in $x$ at the origin.

36 Our key assumption is the stabilizability of a linear approximation of the closed loop system at 0. Under this assumption, the result seems intuitively obvious. However, to the best of the authors’ knowledge, it has never been rigorously established. In particular, the stability of the closed loop system we obtain cannot be proven by applying the Hartman-Grobman theorem, which only applies to ordinary differential equations; see [15, Chapt. 1]. One of the crucial benefits offered by our result is that it yields asymptotically stable closed loop systems for which one can determine a suitable subset of the basin of attraction of the closed loop systems. This information is valuable, because it gives a guarantee that some solutions converge to the origin. We estimate the basin of attraction by building a Lyapunov-Krasovskii functional. It is different from the one in [16], but can be combined with it to establish ISS results. See also [17] for estimates of the basins of attraction for time invariant nonlinear systems with 48 predictor feedbacks, under an ISS assumption on the closed loop 49 systems with undelayed controllers. The predictor feedbacks in [17] 50 can be implemented using numerical methods but are totally different 51 from ours, so our work can be viewed as complementary to [17]. 52 Our work is mainly a methodological development, rather than a 53 specific real-world application or experiments. However, input delays 54 naturally arise from measurement and transport phenomena, and our 55 assumptions are very general, so we anticipate that our work can 56 have considerable benefits when applied to mechanical systems where 57 latencies commonly occur.

68 The rest of this note is organized as follows. We give our definitions in Section II. In Section III, we show how the class of systems we study naturally arises in tracking problems. We state our main result in Section IV, and we prove it in Section V. In Section VI, we discuss a large class of examples where the estimates of the basins of attraction become arbitrarily large when the input delays converge to zero. In Section VII, we illustrate our result in a worked out example. We 69 conclude in Section VIII with a summary of our findings.

II. DEFINITIONS AND NOTATION

67 We let $n \in \mathbb{N}$ be arbitrary and $I_n$ denote the identity matrix in $\mathbb{R}^{n \times n}$, and $\cdot$ be the usual Euclidean norm of matrices and vectors. 68 For square matrices $M_1$ and $M_2$ of the same size, we write $M_1 \geq M_2$ to mean that $M_1 - M_2$ is nonnegative definite. For each integer $r \geq 1$, let $C_r$ denote the set of all functions whose partial derivatives up to order $r$ exist and are continuous, and $C^\infty$ denotes the set of all $32$ continuous functions, when the domains and ranges are clear from the context. When we want to emphasize the domains and ranges, we use $C^r(U, V)$ to denote the set of all $C^r$ functions having domain $U$ and 76 range $V$, where $U$ and $V$ are suitable subsets of Euclidean spaces. For 77 all constant $\tau \geq 0$ and any continuous function $g : [-\tau, \infty) \to \mathbb{R}^n$ and all $t \geq 0$, we define the function $g_t : \mathbb{R}^n \to \mathbb{R}^n$ for each $39$ $\theta \in [-\tau, 0]$, i.e., the translation operator. Let $\mathcal{K}_{\infty}$ be the set of all $C_0$ functions $\gamma : [0, \infty) \to [0, \infty)$ such that $\gamma(0) = 0$ and $\gamma$ is strictly increasing and unbounded. Given subsets $S_1$ and $S_2$ of Euclidean spaces, we say that a function $J : S_1 \times S_2 \to \mathbb{R}$ is locally Lipschitz $83$ with respect to its second argument provided for each compact set $E \subseteq S_2$, there is a constant $L_E$ such that $|J(p, x) - J(p, y)| \leq L_E|x - y|$ for all $p \in S_1$ and all $x, y \in E$. We say that $J$ is strictly increasing in its second argument provided the function $Y(x) = J(p, x)$ is strictly increasing for each $p \in S_1$; we define strictly increasing and 88 nondecreasing in either argument in a similar way. We say that $J$ has order 2 in $y$ at the origin provided there is a continuous function $\alpha$ such that $|J(p, y)| \leq |y|^2 \alpha(|y|)$ for all $(p, y) \in S_1 \times S_2$. We 89 sometimes omit arguments of functions when the arguments are clear from the context.

III. MOTIVATION: TRACKING PROBLEM

94 In this section, we explain how the problem of tracking a trajectory may lead to the problem we solve in the next section. Consider a time $96$ varying nonlinear system

$$\dot{x}(t) = g(t, x(t)) + B(t)u(t - \tau)$$

(1)

where the state $x$ is valued in $\mathbb{R}^n$, the control $u$ is valued in $\mathbb{R}^p$, $98 \tau \geq 0$ is a known constant delay, $g = (g_1, g_2, \ldots, g_n)^T$ is a nonlinear function of class $C^2$, and $B$ is a continuous function. The dimensions $n$ and $p$ are arbitrary. We assume that (1) is forward complete for all measurable locally essentially bounded choices for $\mu$, so $\xi(t)$ is

99 $\geq 0$. The control $u$ is valued in $\mathbb{R}^p$, $98 \tau \geq 0$ is a known constant delay, $g = (g_1, g_2, \ldots, g_n)^T$ is a nonlinear function of class $C^2$, and $B$ is a continuous function. The dimensions $n$ and $p$ are arbitrary. We assume that (1) is forward complete for all measurable locally essentially bounded choices for $\mu$, so $\xi(t)$ is

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102
holds for all positive constant. In the case of (7), we can take $(\text{the fundamental solution associated with})$

locally Lipschitz with respect to $x = \xi - \xi_i$ should be asymptotically stable.

By admissible, we mean that there is a known continuous function $\mu$ where $\mu > 0.$

where $\alpha(t, t) = \mathcal{E}$ and $F(t, x) = \mathcal{E}^2 \sin(x),$ so Assumption 1 holds 144 with $h(\tau) = 1.$ To ease the readability of our technical assumptions, 145 we will explain how the example (7) satisfies our assumptions, after 146 we introduce each of our three assumptions. Our next assumption is: 147

Assumption 2: There are a continuous function $K : [0,\infty)^2 \to \mathbb{R}^{\mathbb{R}^\mathbb{R}^\mathbb{R}}$, a nondecreasing continuous function $k : [0,\infty) \to (0,\infty), an 149 everywhere positive definite and symmetric function $Q : [0,\infty)^2 \to \mathbb{R}^{\mathbb{R}^\mathbb{R}^\mathbb{R}}$ of class $C^1$ with respect to its first argument, and continuous 151 functions $q_i : [0,\infty) \to (0,\infty)$ for $i = 1, 2, 3$ such that $\int K(t, \tau) \leq 152 k(\tau)$ for all $(t, \tau) \in [0,\infty)^2,$ and such that with the choices $H(t, \tau) = 153 \alpha(t) + \lambda(t, t + \tau) B(t + \tau) K(t, \tau)$ and $R(t, \tau, s) = s^2 Q(t, \tau,s),$ the 154 following two conditions are satisfied for all $\tau \geq 0$: (i) Along 155 all trajectories of $\dot{s}(t) = H(t, \tau)s(t),$ we have $\dot{R}(t, \tau, s(t)) \leq 156 -q_1(t) R(t, \tau, s(t))$ and (ii) the bounds

$q_2(t) I_n \leq Q(t, \tau)$ and $|Q(t, \tau)| \leq q_3(\tau) \quad$ (8)

are satisfied for all $t \geq 0.$

Assumption 2 holds for (7) as well. In fact, by choosing $K(t, \tau) = 159 -2\mathcal{E}^2,$ we obtain $H(t, \tau) = 1 - c^{-2} 2\mathcal{E} = -1,$ so Assumption 2 is 160 satisfied with $Q(t, \tau) = 1/2$, $q_1(\tau) = 2$, $q_2(\tau) = q_3(\tau) = 1/2$, and 161 $k(\tau) = 2\mathcal{E}^2.$ Finally, we assume:

Assumption 3: There are two continuous functions $f_1$ and $f_2$ that 163 are locally Lipschitz with respect to their last argument, and continuos functions $\alpha_1$ and $\alpha_2$, such that

\[ F(t, x) = \lambda(t, t + \tau) B(t + \tau) f_1(t, \tau, x) + F(t, x) \quad \] (9)

\[ |f_1(t, \tau, x)| \leq |x|^2 \alpha_1(\tau, |x|^2) \] and

\[ |f_2(t, x)| \leq |x|^2 \alpha_2(|x|) \] (10)

for all $t \in \mathbb{R}, \tau \geq 0, \text{ and } x \in \mathbb{R}^n.$ Also, $|\beta_3(\tau, m) = m \alpha_3(\tau, m)|$ 166 is strictly increasing and unbounded in $m$, and $|\beta_3(\tau, m) = m \alpha_3(\tau, m)|$ is 167 nonlinearly increasing in $m.$ Finally, there are continuous functions $\theta_1 : [0, \infty)^2 \to 168 [0, \infty), \text{ and } \theta_2 : [0, \infty) \to (0, \infty)$ such that

\[ |\alpha_1(\tau, b + c) - \alpha_1(\tau, c)| \leq \theta_1(\tau, b + c) \] (11)

\[ |\alpha_2(b + c) - \alpha_2(c)| \leq \theta_2(b + c) \] (12)

are satisfied for all $b \geq 0$ and $c \geq 0.$

To see why (7) satisfies Assumption 3, note that for (7), the fact 171 $\lambda(t, t + \tau) B(t + \tau)$ is invertible implies that one can choose $f_2 = 0$ and 172 $f_1(t, \tau, x) = \mathcal{E}^2 x^2 \sin(x).$ Then we can satisfy Assumption 3 for (7) 173 by taking $\alpha_1(\tau, m) = \mathcal{E}^2$ and $\alpha_2(m) = 0$ for all $m$ and $\tau.$

Returning to the general system (5), it follows from Assumptions 175 2–3 that for any constant $\tau \geq 0$ and:

\[ \alpha_3(\tau, m) = \frac{q_3(\tau)}{\sqrt{q_3(\tau)}} \alpha_2(m) + 2a \alpha_1(\tau, m), \] (13)

where $a$ is any constant such that

\[ 0 < a \leq \frac{q_1(\tau) q_2(\tau)}{8k(\tau)} \] (14)

there are unique positive values $v_1(\tau)$ and $v_2(\tau)$ (which also depend 178 on $a$) such that

\[ v_1(\tau) \alpha_3 \left( \tau, \frac{4}{\sqrt{q_3(\tau)}} v_1^2(\tau) \right) = \frac{q_1(\tau) q_2(\tau)}{16} \] and

\[ v_2(\tau) \alpha_3 \left( \tau, -\frac{4h^2(\tau)}{a^2} v_2^2(\tau) \right) = \frac{a^2}{4\mathcal{E}^2 h^2(\tau)}. \] (16)
The existence of unique values $v_1(\tau)$ and $v_2(\tau)$ follows because 
$$\beta_2(\tau, m)$$ is strictly increasing and unbounded in $m$ and $\beta_1(m)$ is 
nondecreasing, so $m_{\alpha_2}(\tau, m^2)$ is strictly increasing and unbounded 
in $m$. The choice of $\alpha_3$ in (13) will become clear when we prove: 

**Theorem 1:** Let $\tau > 0$ be any constant and Assumptions 1–3 
hold. Let $a$ be any constant satisfying (14), and set $v(\tau) = 
\min\{v_1(\tau), v_2(\tau)\}$ where $v_1$ and $v_2$ are as above. Then, for each 
187 initial function $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$ satisfying 
$$\sqrt{q(\tau)} \int_{-\tau}^{0} \lambda(0, m + \tau) B(m + \tau) \phi_u(m) dm 
+ a \int_{-\tau}^{0} \gamma(m + 2\tau) \phi_u(m) dm < v(\tau)$$ 
(17) 

188 the unique solution of (5), in closed loop with 
$$u(t) = -f_1(t, \tau, x(t)) + K(t, \tau) \int_{-\tau}^{t} \lambda(t, m + \tau) B(m + \tau) u(m) dm$$ 
(18) 

189 converges to 0 as $t \to \infty$. Moreover, (18) locally asymptotically 
190 stabilizes (5) to 0. \hfill \Box 

**Remark 1:** We comment that our control (18) agrees with the 
191 standard predictor controller in the timedelay invariant case where 
192 $f_1 = f_2 = 0$ and $A$ and $B$ are constant. The extra term $-f_1(t, \tau, x(t))$ 
193 is used to compensate part of the nonlinearity of the system (5). 
194 Assumption 2 is a generalization of the standard assumption that 
(19) $(A, B)$ is a stabilizable pair, which is the special case of Assumption 
197 where $\tau = 0$, $A$ and $B$ are constant, and where $K$ and $Q$ can also be 
199 taken to be constant. However, we allow the delay $\tau > 0$ to be as large 
199 as we want. On the other hand, since the $q_i$’s are continuous positive 
200 valued functions of the delay, they have positive upper and lower 
201 bounds over all $\tau \in [0, \tau_M]$ for any constant $\tau_M$. Also, the function 
202 $k$ from Assumption 2 is nondecreasing in $\tau$. Hence, if we are only 
203 concerned with a bounded set $[0, \tau_M]$ of possible values for $\tau$, then we 
204 can assume in Assumption 2 that the $q_i$’s and $k$ are all positive constant 
205 quantities, by replacing them by the constants $\min\{q_i(\tau) : 0 < \tau \leq \tau_M\}$, 
206 $\max\{q_i(\tau) : 0 < \tau \leq \tau_M\}$, $\max\{q_i(\tau) : 0 \leq \tau \leq \tau_M\}$, and 
207 without relabeling. These observations will be key to our proof in 
208 Section VI that for important special cases, our estimate of the domain 
209 of attraction becomes arbitrarily large when $\tau \to 0^+$. \hfill \Box 

**Remark 2:** Assumptions 1–2 always hold when $A$ and $B$ are con- 
210 stant provided $B$ is stabilizable. Indeed, in that case $\lambda(t, t_0) = 
211 e^{(t-t_0) \lambda}$, so the stabilizability of $(A, B)$ is equivalent to the stabiliz- 
212 ability of $(A, \lambda(t, t + \tau) B)$. Also, when the $\alpha_i$’s are $C^1$, the existence 
214 of functions $\theta_i$ satisfying the requirement from Assumption 3 follows 
215 from the Mean Value Theorem, since Assumption 2 only requires 
216 (11), (12) for nonnegative $b$’s and $c$’s. Since $F$ is order of 2 in $x$ 
217 at 0, we can always satisfy Assumption 3 with $f_1 = 0$ and $f_2 = F$. 
218 However, these choices may lead to a conservative estimate of the size 
219 of the basin of attraction; see the example in Section VII. Our use 
220 of a feedback control with distributed terms is motivated by the facts 
221 that $\tau$ is arbitrary and $\xi(t) = A(t) \xi(t)$ may be exponentially unstable. 
222 In general, the explicit expression for $\lambda$ is unknown, but it can be 
223 computed in many important cases. This is the case in particular if 
224 $A$ is constant or $n = 1$. We illustrate Theorem 1 in Section VII. \hfill \Box 

**Remark 3:** In conjunction with our local asymptotic stability result, 
226 we have boundedness of the control from Theorem 1, along all of the 
227 closed loop trajectories. 

V. PROOF OF THEOREM 1 
228 
Throughout the proof, we consider any solution of (5) in closed loop 229 with (18) for any initial condition satisfying the requirements (17) of 230 Theorem 1, and any constant delay $\tau \geq 0$. 

First Part: New Representation of the System: Let $t_e$ be any positive 
232 real number or $\infty$ such that the solution is defined over $[-\tau, t_e]$. Such 
233 a $t_e > 0$ exists, because the dynamics (5) grows linearly in $x$ in any 234 bounded open neighborhood of $x(0)$. Later we show that $t_e$ can always 
235 be taken to be $\infty$ for all of the trajectories we are considering. We 236 introduce the operators 
$$z(t) = x(t) + \Gamma(t, u_t),$$ 
(19) 

In all of what follows, we assume that $t \in [0, t_e)$ is arbitrary, unless 238 otherwise noted, and we omit some of the arguments of the time 239 derivatives when they are clear, so $\Gamma(t)$ means $(d/dt) \Gamma(t, u_t)$. Then 240 the properties of the fundamental matrix give $\Gamma(t) = A(t) \Gamma(t, u_t) + 241 \lambda(t, t + \tau) B(t + \tau) u(t) - B(t) u(t - \tau)$. Using the formula (5) and 242 our decomposition (9) for $F(t, x)$, we obtain 
$$z(t) = A(t) z(t) + \lambda(t, t + \tau) B(t + \tau) [u(t) + f_1(t, \tau, x(t))] 
+ f_2(t, x(t)).$$ 
(20) 

Also, our feedback (18) satisfies $u(t) = -f_1(t, \tau, x(t)) + 244 K(t, \tau) z(t)$. Consequently, in terms of our function $H$ from 245 Assumption 2, (20) becomes 
$$z(t) = H(t, \tau) z(t) + f_2(t, x(t)).$$ 
(21) 

Assumption 2 ensures global asymptotic stability of the linearizations 
247 $z(t) = H(t, \tau) z(t)$ of (21) at 0. Moreover, the equality 
$$x(t) = z(t) - \int_{-\tau}^{t} \lambda(t, m + \tau) B(m + \tau) u(m) dm$$ 
(22) 

is satisfied. 

Second Part: Decay Conditions: We study the stability of the closed 249 loop system using its representation as (21) coupled with (22). We 250 introduce the operator 
$$\mathcal{E}(u_t) = \frac{1}{\Gamma(t)} \int_{-\tau}^{t} \lambda(m + 2\tau) \|u(m)\| dm.$$ 
(23) 

Observe for later use that 
$$\int_{-\tau}^{t} \|u(m)\| dm \leq \mathcal{E}(u_t) \leq 2 \int_{-\tau}^{t} \|u(m)\| dm.$$ 
(24) 

Then, for all $t \geq 0$, we have 
$$\mathcal{E}(t) \leq 2 \|u(0)\| - \frac{1}{\Gamma(t)} \int_{-\tau}^{t} \|u(m)\| dm.$$ 
(25) 

Also, we can use the upper bound on $f_1$ from (10), the bound for 255 $K$ given in Assumption 2 and the formula $u(t) = -f_1(t, \tau, x(t)) + 256 K(t, \tau) z(t)$ to get $|u(t)| \leq k(\tau) |z(t)| + |x(t)|^2 \alpha_1(\tau, |x(t)|^2)$. More- 
257 over, (8) implies that for all $t \geq 0$ and all $z \in \mathbb{R}^n$, we have $q_2(\tau) |z|^2 \leq 258...
Combining the last two estimates with (25) gives

\[ \dot{\mathbf{x}}(t) \leq -\frac{1}{\tau} \int_{t-\tau}^{t} |u(m)| \, dm + \frac{2k(\tau)}{\sqrt{q_2(\tau)}} \sqrt{R(t, \tau, z(t))} \]

\[ + 2 |x(t)|^2 \alpha_1(\tau, |x(t)|^2). \]  

(27)

264 From (26), we deduce that

\[ \dot{R}(t) \leq -q_1(\tau) R(t, \tau, z(t)) \]

\[ + 2q_3(\tau) \sqrt{\frac{R(t, \tau, z(t))}{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2). \]  

(29)

265 Consider the family of functions \( S_\varepsilon(t, \tau, z) = \sqrt{R(t, \tau, z)} + \varepsilon - \sqrt{z} \) parameterized by the constant \( \varepsilon \in [0, 1] \) and let \( S = S_0 \). Since \( R \)

is of class \( C^1 \) with respect to \( t \) and \( z \), it follows that for all \( \varepsilon \in (0, 1) \), \( S_\varepsilon \)

is of class \( C^1 \) with respect to \( t \) and \( z \), while \( S \) is only 269 continuous. Also, (29) and Lemma 1 in the Appendix (applied with the 270 choice \( r = R(t, \tau, z) \)) give

\[ \dot{S}_\varepsilon(t) \leq -q_1(\tau) \frac{R(t, \tau, z(t))}{2\sqrt{R(t, \tau, z(t))}} + \varepsilon \]

\[ + q_3(\tau) \sqrt{\frac{R(t, \tau, z(t))}{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2) \]

\[ \leq -q_1(\tau) S(t, \tau, z(t)) \]

\[ + q_3(\tau) \sqrt{\frac{R(t, \tau, z(t))}{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2) \]

\[ + \frac{q_3(\tau)}{2} \varepsilon \frac{\tau}{2} [1 + S(t, \tau, z(t))]. \]  

(30)

271 along all trajectories of (21).

272 Third Part: Lyapunov-Krasovskii Functionals: Let us consider the 273 family of functions

\[ V_\varepsilon(t, z, u_t) = a \Xi(u_t) + S_\varepsilon(t, \tau, z) \]  

(31)

274 where the constant \( a \) satisfies (14) and we omit the argument \( \tau \) in \( V_\varepsilon \)

275 to simplify the notation. Then, (27) and (30) give

\[ \dot{V}_\varepsilon(t) \leq \left( \frac{2ak(\tau)}{\sqrt{q_2(\tau)}} \frac{q_3(\tau)}{2} \right) S(t, \tau, z(t)) \]

\[ + \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2) \]

\[ - \frac{a}{\tau} \int_{t-\tau}^{t} |u(m)| \, dm + 2a |x(t)|^2 \alpha_1(\tau, |x(t)|^2) \]

\[ + \frac{q_3(\tau)}{2} \varepsilon \frac{\tau}{2} [1 + S(t, \tau, z(t))]. \]  

(32)

Since \( a \) satisfies (14), we get

\[ \dot{V}_\varepsilon(t) \leq -\frac{q_1(\tau)}{4} S(t, \tau, z(t)) + |x(t)|^2 \alpha_3(\tau, |x(t)|^2) \]

\[ - \frac{a}{\tau} \int_{t-\tau}^{t} |u(m)| \, dm + \frac{q_3(\tau)}{2} \varepsilon \frac{\tau}{2} [1 + S(t, \tau, z(t))]. \]  

(33)

Since \( \alpha_3 \) was defined in (13), we get

Next, we find a suitable upper bound on the term \( \int_{t-\tau}^{t} |u(m)| \, dm \) from (33). Our formula (22) for \( x(t) \), 279 Assumption 1, and our bound (26) on \( |z| \) give

\[ |x(t)| \leq |z(t)| + h(\tau) \int_{t-\tau}^{t} |u(m)| \, dm \]

\[ \leq \frac{1}{\sqrt{q_2(\tau)}} S(t, \tau, z(t)) + h(\tau) \int_{t-\tau}^{t} |u(m)| \, dm. \]  

(34)

Recall that our monotonicity properties of \( \beta_3 \) and \( \beta_4 \) from Assumption 281 3 imply that \( m \alpha_3(\tau, m^2) \) is strictly increasing as a function of \( m \) 282 for each \( \tau \). Therefore, by separately considering the cases where \( S(t, \tau, z(t))/\sqrt{q_2(\tau)} \leq h(\tau) \int_{t-\tau}^{t} |u(m)| \, dm \) and where the reverse 284 inequality holds, we get

\[ |x(t)|^2 \alpha_3(\tau, |x(t)|^2) \]

\[ \leq \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \alpha_3(\tau, \frac{1}{q_2(\tau)} S^2(t, \tau, z(t))) \]

\[ + 4h^2(\tau) \left[ \int_{t-\tau}^{t} |u(m)| \, dm \right]^2 \alpha_3(\tau, 4h^2(\tau) \left[ \int_{t-\tau}^{t} |u(m)| \, dm \right]^2). \]  

(35)

We can combine this inequality with (33) to get

\[ \dot{V}_\varepsilon(t) \leq \left[ \frac{q_1(\tau)}{4} + \frac{4}{q_2(\tau)} S(t, \tau, z(t)) \alpha_3 \right] \]

\[ \times \left( \tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \right) \]

\[ \times \left( \tau, 4h^2(\tau) \left[ \int_{t-\tau}^{t} |u(m)| \, dm \right]^2 \right) \]

\[ \times \int_{t-\tau}^{t} |u(m)| \, dm + \frac{q_3(\tau)}{2} \varepsilon \frac{\tau}{2} [1 + S(t, \tau, z(t))]. \]  

(36)
287 Also, \( V_c(t, z(t), u_t) \geq \sqrt{R(t)} \) for all \( t \in [0, t_2] \), \( V_c(t, z(t), u_t) \geq \frac{\epsilon}{2} \) for all \( t \in [0, t_2] \), by (24). Since \( m \alpha_3(\tau, m^2) \) is increasing in \( m \) for each \( \tau \), it follows that:

\[
S(t, \tau, z(t)) \alpha_3 \left( \tau, \frac{4}{q_3(\tau)} S^2(t, \tau, z(t)) \right) \leq \left[ \sqrt{V_c(t, z(t), u_t) + \sqrt{T}} \right] \alpha_3 \times \left( \frac{4}{q_3(\tau)} [2V_c(t, z(t), u_t) + \sqrt{T}] + \frac{4}{q_3(\tau)} V_c^2(t, z(t), u_t) \right).
\]

290 We now apply (11), (12), with \( b = (4/q_3(\tau))(2V_c(t, z(t), u_t) + \sqrt{T} + \epsilon) \) and \( c = 4V_c^2(t, z(t), u_t) + \sqrt{T} \), and use the fact that \( \epsilon \leq \sqrt{T} \leq 2^\epsilon \) for all \( \epsilon \in [0, 1] \), to find a continuous positive valued and non-decreasing function \( \varphi_c(\epsilon) \) (also depending on \( \tau \), but independent of \( \epsilon \))

294 such that

\[
\dot{V}_c(t) \leq -\frac{\epsilon}{4} + \frac{4}{q_3(\tau)} V_c(t, z(t), u_t) \times \alpha_3 \left( \tau, \frac{4}{q_3(\tau)} V_c^2(t, z(t), u_t) \right) S(t, \tau, z(t)) + \frac{1}{2a^2} \left[ -\frac{a^2}{\tau} + 4H^2(\tau)V_c(t, z(t), u_t) \times \alpha_3 \left( \tau, 4H^2(\tau) \frac{V_c^2(t, z(t), u_t)}{a^2} \right) \right] \int_{t-\tau}^{t} |u(m)| \, dm + \frac{\epsilon}{4} \varphi_c(\epsilon) (V_c(t, z(t), u_t)).
\]

295 Next, recall that our assumption (17) implies that \( \sqrt{q_3(\tau)|z(0)|} + (a/\tau) \int_{t-\tau}^{t} |u(m)| \, dm < \nu(\tau) \), where \( \nu(\tau) = \min \{ \nu_1(\tau), 297 \nu_2(\tau) \} \) as before. Then (8) from Assumption 2 gives \( V_0(0, z(0), u_0) < 298 (a/\tau) \nu(\tau) \). Since \( c_1 + c_2 \leq \sqrt{c_1 + \sqrt{c_2}} \) holds for all nonnegative constants \( c_1 \) and \( c_2 \), we know that \( V_0(0, z(0), u_0) \leq V_0(0, 0(0), u_0) \leq \sqrt{\nu(\tau)} \) for all \( \epsilon \in [0, 1] \). It follows that \( V_0(0, 0(0), u_0) \leq V_0(0, 0(0), u_0) \leq \sqrt{\nu(\tau)} \) holds for all \( \epsilon \in [0, 1] \), where \( \nu(\tau) = \max \{ V_0(0, 0(0), u_0) + \nu(\tau) \} / 2 \geq 0 \). Then 302 \( \nu(\tau) \geq \nu(\tau) \).

303 Set \( \tau_0 = (\nu(\tau) + \nu(\tau))/2 \). Since \( m \alpha_3(\tau, m^2) \) is strictly increasing in \( \nu(\tau) \), and since \( \nu(\tau) < \nu(\tau) = \min \{ \nu_1(\tau), \nu_2(\tau) \} \), it follows from our 305 conditions (15), (16) on \( v_1(\tau) \) and \( v_2(\tau) \) that the constants:

\[
\begin{align*}
\bar{p}_1 &= \frac{q_1(\tau)}{4} - \frac{4}{q_3(\tau)} \nu_0 \alpha_0 \left( \tau, \frac{4}{q_3(\tau)} \nu_0^2 \right), \\
\bar{p}_2 &= \frac{a^2}{\tau} - 4H^2(\tau) \nu_0 \alpha_0 \left( \tau, 4H^2(\tau) \nu_0^2 \right) \end{align*}
\]

306 are positive for all \( \tau > 0 \). Fix any value of \( \epsilon \in (0, 1] \) satisfying

\[
\epsilon \in \left( 0, \frac{\min \{ p_1, p_2 \} \nu}{4 \varphi_c(\epsilon) \max \{ a^2, 1 \}} \right)^4.
\]

307 where the left endpoint is omitted because we need \( \epsilon > 0 \).

308 Next, we prove by contradiction that \( V_c(t, z(t), u_t) \leq \nu(\tau) \) for all \( t \geq t_0 \). Assume that this property does not hold. Then, since \( \bar{p}_0 > \nu(\tau) \) and \( V_0(0, 0(0), u_0) \leq \nu(\tau) \), we cannot find a \( t_2 > 0 \) such that \( 311 V_c(t, z(t), u_t) \leq \nu(\tau) \) for all \( t \in [0, t_2] \) and \( V_c(t_2, z(t_2), u_2) > \nu(\tau) \). Set \( 312 t_2 = \inf \{ t \leq t_2 : V_c(t, z(t), u_t) \leq \nu(\tau) \} \) for all \( t \in [0, t_2] \). Since \( 313 t \rightarrow V_c(t, z(t), u_t) \) is continuous, we get \( V_c(t, z(t), u_t) \leq \nu(\tau, p_0) \) for all \( t \in [t_2, t_3] \), \( V_c(t_3, z(t_3), u_{t_3}) = \nu(\tau, p_0) \) for all \( t \geq t_3 \).
VI. Arbitrarily Large Domains of Attraction

Theorem 1 applies for all $\tau > 0$. On the other hand, consider the special case where $f_2 = 0$ in the decomposition (9) of $F$. Then, setting $\tau = 0$ in (9) and in our control (18) produces the uniformly globally asymptotically stable closed loop system $\dot{x}(t) = [A(t) + B(t)K(t,0)]x(t)$ from Assumption 2. This suggests that the domain of attraction should become arbitrarily large as $\tau \to 0^+$ when $f_2 = 0$.

Our next theorem implies that this is indeed the case. We will assume that the functions $q_1$ and $k$ from Assumption 2 are constant, so we omit their arguments $\tau$. This is not restrictive, since now we only need to consider $\tau$'s on a bounded interval; see Remark 1.

Corollary 1: Let Assumptions 1–3 hold with $f_2 = 0$ and the $q_i$'s and $k$ all constant. Then for each constant $v_\ast > 0$, we can find values $a \in (0, q_1 \sqrt{h}/(8k))$ and $\tau_M > 0$ (both depending on $v_\ast$) such that:

For each initial condition $(\phi_0, \phi_0) \in C^0([0, \tau_M], \mathbb{R}^n \times \mathbb{R}^p)$ satisfying

$$
\sqrt{q_1(\tau)} \phi_\ast(0) + \int_0^\tau \lambda(0, m + \tau)B(m + \tau)\phi_\ast(m)\,dm
+ a \tau \int_0^\tau (m + 2\tau) |\phi_\ast(m)|\,dm < v_\ast,
$$

(48)

and each constant delay $\tau \in (0, \tau_M)$, the trajectory of (5) in closed loop with (18) converges to 0 as $t \to \infty$.

Proof: We set $\alpha_2 = 0$, so we have $\alpha_3 = 2\alpha_1$. Then (15), (16) become

$$
v_1(\tau)\alpha_1 \left(\tau, \frac{4}{q_2}v_1^2(\tau)\right) = \frac{q_1\sqrt{q_2}}{32a} \quad \text{and} \quad (50)
$$

$$
v_2(\tau)\alpha_1 \left(\tau, \frac{4h^2(\tau)}{a^2}v_2^2(\tau)\right) = \frac{a}{8\pi h^4(\tau)}.
$$

(49)

530 For each constant $\tau_M > 0$, Assumption 3 provides a function $\overline{\tau}$ of 535 class $K_\infty$ such that $m\alpha_1(\tau, m^2) \leq \overline{\tau}(m)$ for all $\tau \in (0, \tau_M)$ and 536 $m \geq 0$. Then, replacing $\alpha_1(\tau, m^2)$ in (49) by $\overline{\tau}(m)/m$ gives

$$
\overline{\tau} \left(\frac{4}{q_2}v_1(\tau)\right) = \frac{q_1\sqrt{q_2}}{16a} \quad \text{and} \quad (51)
$$

$$
\overline{\tau} \left(\frac{2h(\tau)}{a}v_2(\tau)\right) = \frac{1}{4\pi h^4(\tau)}.
$$

(50)

537 For all $\tau \in (0, \tau_M)$. Our proof of Theorem 1 shows that the conclusions 538 of that theorem remain true when $v_1(\tau)$ and $v_2(\tau)$ are defined to be the 539 solutions of (50). Therefore

$$
v_1(\tau) = \frac{\sqrt{q_2}}{2} \frac{1}{\overline{\tau}^{-1}} \left(\frac{q_1\sqrt{q_2}}{16a}\right) \quad \text{and} \quad (52)
$$

$$
v_2(\tau) = \frac{a}{2h(\tau)} \frac{1}{\overline{\tau}^{-1}} \left(\frac{1}{4\pi h^4(\tau)}\right).$$

(51)

360 Also, when $\tau$ is sufficiently small, the choice

$$
a = \frac{1}{\sqrt{\overline{\tau}^{-1}} \left(\frac{1}{4\pi h^4(\tau)}\right)}
$$

will satisfy our requirements (14) on $a$, because (52) converges to 0 as $\tau \to 0^+$ and because we are now assuming that the $q_i$'s and $k$ are 362 positive constants. Then (51) become

$$
v_1(\tau) = \frac{\sqrt{q_2}}{2} \frac{1}{\overline{\tau}^{-1}} \left(\frac{q_1\sqrt{q_2}}{16}\right) \sqrt{\frac{1}{4\pi h^4(\tau)}} \quad \text{and} \quad (53)
$$

$$
v_2(\tau) = \frac{1}{2h(\tau)} \frac{1}{\overline{\tau}^{-1}} \left(\frac{1}{4\pi h^4(\tau)}\right).$$

(50)

Therefore, both $v_1(\tau)$ and $v_2(\tau)$ converge to $\infty$ when $\tau \to 0^+$. It 364 follows that $v(\tau) \to \infty$ as $\tau \to 0^+$, so we can satisfy (48) for small 365 enough $\tau > 0$ by choosing $\tau$ such that $v(\tau) > v_\ast$. The corollary now 366 follows from Theorem 1.

VII. Illustrative Example

We illustrate Theorem 1 using the 1 dimensional system from (7), 369 which is

$$
\dot{x}(t) = x(t) + u(t - \tau) + lx^2(t) \sin(x(t))
$$

(54)

where $u \in \mathbb{R}$ is the input, $\tau$ is a positive constant delay, and $l$ is 371 a positive constant. This system is not globally Lipschitz in the 372 state $x$. With the notation of the previous sections, we have $A = 1, 373 B = 1, \lambda(t, t_0) = e^{t-t_0},$ and $F(t, x) = lx^2 \sin(x)$. As we saw in 374 Section IV, (54) satisfies our assumptions with $h(\tau) = 1, K(t, \tau) = 375 \frac{2\pi e^\tau}{\pi}, Q(\tau) = 1/2, q_1(\tau) = 2, q_2(\tau) = q_3(\tau) = 1/2, k(\tau) = 2e^\tau, 376 f_2 = 0, f_1(t, \tau, x) = le^x \sin(x), \alpha_1(\tau, m) = le^x$ and $\alpha_2(m) = 0$. 377 According to (14), the inequalities $0 < a \leq 1/(8\sqrt{2}e^\tau)$ have to be 378 satisfied and, by the expression of $\alpha_3$ in (13), $\alpha_3(\tau, m) = 2ae^\tau$. 379

Choosing

$$
a = \frac{1}{8\sqrt{2}e^\tau}
$$

(55)

we can straightforwardly derive an estimate of the basin of attraction 381 from Theorem 1 by using $v = \min\{v_1, v_2\}$, where

$$
v_1(\tau) = \frac{1}{2\sqrt{2l}}
$$

(56)

and

$$
v_2(\tau) = \frac{1}{64\sqrt{2\pi e^\tau}}
$$

(57)

which converge to $\infty$ as $l \to 0$ for each $\tau > 0$. On the other hand, 384 when $\tau \in (0, 1]$, we can take

$$
a = \frac{\sqrt{\tau}}{8\sqrt{2}e^\tau}
$$

(58)

to obtain the values

$$
v_1(\tau) = \frac{1}{2\sqrt{2l}}
$$

(59)

and

$$
v_2(\tau) = \frac{1}{64\sqrt{2\pi e^\tau}}
$$

(60)

so $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ converges to $\infty$ as $l$ converges to zero 388 for fixed $\tau > 0$, or as $\tau$ converges to zero for fixed $l$, so the basin 389 of attraction becomes arbitrarily large. This gives convergence of the 390 closed loop solution to 0.
If, on the other hand, we had chosen, \(f_1 = 0\) and \(f_2(t, x) = 393 \, t x^2 \sin(x)\), then one could choose \(\alpha_1 = c_0\) for any constant \(c_0 > 0\) and \(\alpha_2(m) = l\). This gives \(\alpha_3(\tau, m) = 2ac_0 + (1/\sqrt{2})l\). Then the 395 corresponding solutions of (15), (16) with the choice
\[
\alpha = \frac{1}{8\sqrt{2}c} \tag{61}
\]
396 satisfy
\[
v_1(\tau) \leq \frac{\sqrt{2}}{16l} \tag{62}
\]
397 and
\[
v_2(\tau) \leq \frac{1}{256\sqrt{2}c^2 \tau l} \tag{63}
\]
398 which would mean that \(v(\tau) = \min\{v_1(\tau), v_2(\tau)\}\) does not converge 399 to \(\infty\) as \(\tau\) goes to zero. Thus, the choice \(f_1 = 0\) and \(f_2(t, x) = 400 \, t x^2 \sin(x)\) is conservative.

VIII. Conclusion

Stabilization of nonlinear systems with input delays is a central 402 problem that has been studied by many authors using model reduction, 403 prediction, and other methods. Here we adapted the reduction model 404 approach to the problem of locally asymptotically stabilizing the origin 405 of time varying nonlinear systems with pointwise input delays. Our 406 method of proof makes it possible to determine an estimate of the basin 407 of attraction. The result can be adapted to the case where the delay in 408 the input is distributed. Our results can be combined with those of [5] 410 and [10].

Appendix

Technical Lemma

We used the following to get (30) in the second part of the proof of 414 Theorem 1:

Lemma 1: Let \(\varepsilon \in (0, 1]\) be a positive real number. Then
\[
-\frac{r}{\sqrt{\tau + \varepsilon}} \leq -\sqrt{\tau + \varepsilon} \left[1 + \sqrt{\tau}\right] \tag{64}
\]
416 holds for all \(\tau \geq 0\).

Proof: Let \(\tau \geq 0\) be given. We first prove that
\[
\frac{r}{\sqrt{\tau + \varepsilon}} \geq \frac{1}{\sqrt{1 + \varepsilon}} \sqrt{\tau - \varepsilon - \varepsilon^2} \tag{65}
\]
418 If \(\sqrt{\tau}/(\sqrt{1 + \varepsilon}) - \varepsilon^2/4 \leq 0\), then (65) is satisfied. On the 419 other hand, if \(\sqrt{\tau}/(\sqrt{1 + \varepsilon}) - \varepsilon^2/4 \geq 0\), then \(\tau \geq (1 + \sqrt{\varepsilon})\varepsilon + \varepsilon r + r\). Consequently, 420 it follows that \((\sqrt{\varepsilon} + 1)r \geq (1 + \sqrt{\varepsilon})\varepsilon + r \geq \varepsilon r + r\). Consequently, 421 \(r/(\tau + \varepsilon) \geq 1/(\sqrt{\varepsilon} + 1)\). Taking the square root, and then multiply- 422 ing through by \(\sqrt{\tau}\), we obtain
\[
\sqrt{\frac{1}{\tau + \varepsilon}} \geq \frac{\sqrt{\tau}}{\sqrt{\varepsilon} + 1} \tag{66}
\]
Therefore, (65) holds in both cases. Next, observe that (65) implies that
\[
-\frac{r}{\sqrt{\tau + \varepsilon}} \leq -\frac{1}{\sqrt{1 + \varepsilon}} \left[1 + \sqrt{\tau + \varepsilon}\right] \tag{67}
\]
424 Hence, the relation \(\sqrt{b + c} \leq \sqrt{b} + \sqrt{c}\) for nonnegative values \(b, c\) and 425 \(\varepsilon\) gives \(-r/\sqrt{\tau + \varepsilon} \leq -\sqrt{\tau + \varepsilon + \varepsilon} \tag{67}\). This gives the 426 conclusion. □

References

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