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A new theory of stochastic integration

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A NEW THEORY OF STOCHASTIC INTEGRATION

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
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by

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Abstract

In this dissertation, we focus mainly on the further study of the new stochastic integral introduced by Ayed and Kuo [1] in 2008. Several properties of this new stochastic integral are obtained. We first introduce the concept of near-martingale for non-adapted stochastic processes. This concept is a generalization of the martingale property for adapted stochastic processes in the Itô theory. We prove a special case of Itô isometry for the stochastic integral of certain instantly independent processes defined in [1]. We obtain some formulas for expressing a new stochastic integral in terms of Itô integrals and Riemann integrals. Several generalized versions of Itô's formula for the new stochastic integral (obtained by Ayed and Kuo in [1] and [2]) are given. We also provide some examples to illustrate the ideas.

Chapter 1

Introduction

The theory of stochastic integration, also known as Itô calculus, was first introduced by Kiyoshi Itô in 1942. The original motivation for Itô was the construction of Markov diffusion processes directly from infinitesimal generators by solving stochastic differential equations [9].

The Itô theory of stochastic integration has been extensively studied and widely applied to many scientific fields. The most well-known application is the Black-Scholes model in finance for which Merton and Scholes won the 1997 Nobel Prize in Economics.

In the Itô theory, the stochastic integral $\int_a^b f(t, \omega) dB(t)$ is only defined for an adapted (or non-anticipating) integrand $f(t)$. It is natural to ask how one can define a stochastic integral for a non-adapted integrand. For instance, in 1976, Itô [7] considered the stochastic integral $\int_0^t B(1) dB(s)$ for $0 \leq t \leq 1$. Note that this integral is not defined as an Itô integral because $B(1)$ is not adapted to the filtration $\{\mathcal{F}_t; 0 \leq t \leq 1\}$ with $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$.

In 1972, Hitsuda [5, 6] defined a stochastic integral for non-adapted integrands and obtained a special case of the Itô's formula for anticipating stochastic integrals. Then his idea was used by Skorokhod [17] in 1975 to define such stochastic integrals via homogeneous chaos expansion. Their stochastic integral can be defined in terms of the white noise differentiation. Since 1972, there have been several approaches to define stochastic integrals for non-adapted integrands, e.g., Buckdahn [3], Léon and Protter [12], Nualart and Pardoux [13], Pardoux and Protter [15].

In 2008, Ayed and Kuo [1, 2] proposed a new approach to define stochastic integrals of non-adapted integrands. They first introduced the class of instantly independent stochastic processes which can be regarded as a counterpart of adapted stochastic processes in the Itô theory. Then they define the stochastic integral of a linear combination of the products of instantly independent and adapted processes to be the limit in probability of its Riemann sum. Their main idea is to use the right endpoints of subintervals as the evaluation points for instantly independent processes in the Riemann sum whereas the left endpoints are still used for adapted processes as in Itô theory.

In this dissertation, we will further study this new Ayed-Kuo integral. In chapter 2, we give a brief review of some basic notions and backgrounds from probability theory.

Chapter 3 is devoted to a brief review of the Itô theory of stochastic integration.

In chapter 4, we describe the idea of the new stochastic integral introduced by Ayed and Kuo [1, 2]. We will describe the motivation of this new idea, the definition and some examples of this new integral, and a special case of the Itô formula for the new stochastic integrals.

In chapter 5, we will present the main results of this dissertation. Several properties of the Ayed-Kuo integral are obtained. We will introduce a new class of stochastic processes called “near martingales” which play an important role for stochastic processes associated with the new stochastic integrals of instantly independent stochastic processes. We will prove the Itô isometry for the new integral of a certain class of instantly independent processes, and several formulas for expressing the new stochastic integral in terms of Itô and Riemann integrals. Finally, we will generalize the Itô’s formula obtained by Ayed and Kuo in [1] and [2].

Chapter 2

Background

In this chapter, we present some basic settings from the probability theory which will be needed in the dissertation. For more details, see [9].

2.1 Brownian Motion

We will begin this section with the definition of stochastic processes.

Definition 2.1. *A stochastic process is a collection of random variables $\{X_t\}_{t \in T}$ defined on a probability space (Ω, \mathcal{F}, P) where T is an index set.*

In other words, a stochastic process can be regarded as a measurable function $X(t, \omega)$ defined on a product space $T \times \Omega$ such that

1. for each $t \in T$, $X(t, \cdot)$ is a random variable,
2. for each fixed $\omega \in \Omega$, $X(\cdot, \omega)$ is a (measurable) function of t (called a sample path).

Remark 2.2. *The index set T usually represents the time. In the discrete case, T is a subset of \mathbb{N} , whereas in the continuous case, T is an interval in \mathbb{R} . (e.g., $T = [0, \infty)$.)*

Remark 2.3. *We sometimes denote $X(t, \omega)$ by $X_t(\omega)$, $X(t)$ or X_t for convenience.*

Example 2.4. *Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent identically distributed random variables. For each $n \in \mathbb{N}$, define S_n to be the random variable*

$$S_n = X_1 + X_2 + \dots + X_n.$$

Then $\{S_n; n \in \mathbb{N}\}$ is a stochastic process and it is called a random walk. (Here, $T = \mathbb{N}$ which is a discrete set, so S_n is an example of a discrete stochastic process.)

One of the well-known and useful continuous stochastic processes in probability theory is the Brownian motion (sometimes called the Wiener process). It is a mathematical model to describe the random movement of a small particle that is suspended in a liquid or gas. It was named after the English botanist, Robert Brown, who observed how pollen grains move irregularly when suspended in water. Nowadays, this mathematical model has some real-world applications in many different fields such as physics and economics. It is also used to model the stock market fluctuations. Moreover, Brownian motion has many well-known properties. For example, it is a Markov process, a Gaussian process, a Lévy process and a martingale.

Let (Ω, \mathcal{F}, P) be a probability space. Below we give the mathematical definition of the Brownian motion.

Definition 2.5. *A stochastic process $\{B(t, \omega); t \geq 0\}$ is called a Brownian motion if it satisfies the following conditions:*

1. $P\{\omega; B(0, \omega) = 0\} = 1$. (We say $B(0) = 0$ almost surely.)
2. For any $0 \leq s < t$, the random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, i.e., for any $a < b$,

$$P\{a \leq B(t) - B(s) \leq b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-x^2/2(t-s)} dx.$$

3. $B(t, \omega)$ has independent increments, that is, for any $0 \leq t_1 < t_2 < \dots < t_n$, the random variables

$$B(t_1), B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1}),$$

are independent.

4. Almost all sample paths of $B(t, \omega)$ are continuous functions, i.e.,

$$P\{\omega; B(\cdot, \omega) \text{ is continuous}\} = 1.$$

Example 2.6. Let $X_{\delta, h}(t)$ be a random walk start at 0 with jumps h and $-h$ equally likely at times $\delta, 2\delta, 3\delta, \dots$ (See Section 1.2 of [9].)

Assume that $h^2 = \delta$. Then, for each $t \geq 0$, it can be shown that the limit

$$B(t) = \lim_{\delta \rightarrow 0} X_{\delta, \sqrt{\delta}}(t)$$

is a Brownian motion.

Example 2.7. In this example, we provide a construction of Brownian motion.

Let $C[0, 1]$ be the Banach space of real-valued continuous functions g on $[0, 1]$ with $g(0) = 0$ equipped with the norm given by $\|g\|_{\infty} = \sup_{t \in [0, 1]} |g(t)|$.

Note that a cylindrical set (or a cylinder set) A of $C[0, 1]$ is a set of the form

$$A = \left\{ g \in C[0, 1] ; (g(t_1), g(t_2), \dots, g(t_n)) \in U \right\} \quad (2.1)$$

where $0 < t_1 < t_2 < \dots < t_n \leq 1$ and $U \in \mathcal{B}(\mathbb{R}^n)$, i.e., U is an element of the Borel σ -field of \mathbb{R}^n .

Let \mathcal{R} denote the collection of all cylindrical subsets of $C[0, 1]$. The collection \mathcal{R} defined in this way is a field, but it is not a σ -field.

Consider the mapping μ_0 defined on \mathcal{R} and given by

$$\mu_0(A) = \int_U \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp \left[-\frac{(u_i - u_{i-1})^2}{2(t_i - t_{i-1})} \right] \right) du_1 du_2 \dots du_n$$

where A is as in Equation (2.1) and $t_0 = u_0 = 0$. It can be easily seen that μ_0 is a finitely additive mapping from the field \mathcal{R} into $[0, 1]$. It was also proved by Norbert Wiener in 1923 that mapping μ_0 is σ -additive. (For more details of the proof, see reference [18]). Thus, by Caratheodory's extension theorem, μ_0 can be extended to a σ -additive measure μ defined on $\sigma(\mathcal{R})$, the σ -field generated by \mathcal{R} . (This extension

is unique since $\mu_0(C[0, 1]) = 1$ and then $\mu_0 : \mathcal{R} \rightarrow [0, 1]$ is σ -finite.) Moreover, it turns out that the σ -field $\sigma(\mathcal{R})$ is the same as the Borel field $\mathcal{B}(C[0, 1])$ of $C[0, 1]$. Thus, $(C[0, 1], \mathcal{B}(C[0, 1]), \mu)$ is a probability space. It is called the Wiener space and the measure μ is called the Wiener measure. Under this construction, we can show that the stochastic process

$$B(t, g) = g(t), \quad 0 \leq t \leq 1, \quad g \in C[0, 1],$$

is, in fact, a Brownian motion. (Here, $\Omega = C[0, 1]$ and $T = [0, 1]$.)

Let $B(t)$ be a Brownian motion. The following theorems are some well-known properties of $B(t)$.

Theorem 2.8. For $s, t \geq 0$, we have $E(B(s)B(t)) = \min\{s, t\}$.

Theorem 2.9. (Translation invariance) For fixed $t_0 \geq 0$, the stochastic process $\tilde{B}(t) = B(t + t_0) - B(t_0)$ is also a Brownian motion.

The above property says that a Brownian motion “restarted” at any moment is also a Brownian motion.

Theorem 2.10. (Rescaling invariance) For any real number $\lambda > 0$, the stochastic process $\tilde{B}(t) = \frac{1}{\sqrt{\lambda}}B(\lambda t)$ is also a Brownian motion.

Theorem 2.11. The sample path of a Brownian motion is nowhere differentiable almost surely.

Theorem 2.12. (Quadratic variation of Brownian motion)

Let $\Delta_n = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$ be a partition of a finite interval $[a, b]$. Then,

$$\sum_{i=1}^n (B(t_i) - B(t_{i-1}))^2 \longrightarrow b - a$$

in $L^2(\Omega)$ as $\|\Delta_n\| = \max_{1 \leq i \leq n} (t_i - t_{i-1})$ tends to 0.

Since we will be using this property of Brownian motion quite often in Chapter 4 and 5, we provide a short proof of this theorem.

Proof. Note that $b - a = \sum_{i=1}^n (t_i - t_{i-1})$. Let

$$\Phi_n = \sum_{i=1}^n \left[(B(t_i) - B(t_{i-1}))^2 - (t_i - t_{i-1}) \right] = \sum_{i=1}^n X_i, \quad (2.2)$$

where $X_i = (B(t_i) - B(t_{i-1}))^2 - (t_i - t_{i-1})$.

To show that $\sum_{i=1}^n (B(t_i) - B(t_{i-1}))^2$ converges to $b - a$ in $L^2(\Omega)$, it suffices to show that $E(\Phi_n^2) \rightarrow 0$ as $\|\Delta_n\| \rightarrow 0$. Thus consider

$$\Phi_n^2 = \sum_{i,j=1}^n X_i X_j = \sum_{i=1}^n X_i^2 + \sum_{i \neq j} X_i X_j.$$

Therefore,

$$E(\Phi_n^2) = \sum_{i=1}^n E(X_i^2) + \sum_{i \neq j} E(X_i X_j). \quad (2.3)$$

For $i \neq j$, we have $E(X_i X_j) = 0$ since $B(t)$ has independent increments and $E(B(t) - B(s))^2 = |t - s|$.

On the other hand, note that $E(B(t) - B(s))^4 = 3|t - s|^2$. Then

$$\begin{aligned} E(X_i^2) &= E \left\{ (B(t_i) - B(t_{i-1}))^4 - 2(t_i - t_{i-1})(B(t_i) - B(t_{i-1}))^2 + (t_i - t_{i-1})^2 \right\} \\ &= 3(t_i - t_{i-1})^2 - 2(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2 \\ &= 2(t_i - t_{i-1})^2. \end{aligned}$$

Therefore, from Equation (2.3), we have

$$\begin{aligned} E(\Phi_n^2) &= \sum_{i=1}^n 2(t_i - t_{i-1})^2 \\ &\leq 2 \|\Delta_n\| \sum_{i=1}^n (t_i - t_{i-1}) \\ &= 2(b - a) \|\Delta_n\| \rightarrow 0 \quad \text{as} \quad \|\Delta_n\| \rightarrow 0. \end{aligned}$$

This shows that Φ_n converges to 0 in $L^2(\Omega)$. Thus from Equation (2.2), we see that $\sum_{i=1}^n (B(t_i) - B(t_{i-1}))^2$ converges to $b - a$ in $L^2(\Omega)$. \square

Moreover, with the similar arguments, one can also prove the following theorem.

Theorem 2.13. *Let $\Delta_n = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$ be a partition of a finite interval $[a, b]$. Then for any $k \geq 3$,*

$$\sum_{i=1}^n (B(t_i) - B(t_{i-1}))^k \longrightarrow 0$$

in $L^2(\Omega)$ as $\|\Delta_n\| = \max_{1 \leq i \leq n} (t_i - t_{i-1})$ tends to 0.

2.2 Conditional Expectation

Here we present an important concept in probability theory called the conditional expectation.

Definition 2.14. *Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^1(\Omega, \mathcal{F})$. That is, $E|X| = \int_{\Omega} |X| dP < \infty$. Suppose \mathcal{G} is a σ -field such that $\mathcal{G} \subset \mathcal{F}$. The conditional expectation of X given \mathcal{G} is defined to be the unique random variable Y (up to P -measure 1) satisfying the following conditions :*

(1) Y is \mathcal{G} -measurable ;

(2) $\int_A Y dP = \int_A X dP$ for all $A \in \mathcal{G}$.

We usually use the notations $E[X|\mathcal{G}]$, $E(X|\mathcal{G})$ or $E\{X|\mathcal{G}\}$ to denote this random variable Y .

Remark 2.15. *The Radon-Nikodym theorem guarantees the existence and uniqueness of the conditional expectation.*

Theorem 2.16. *(Radon-Nikodym Theorem) Let (Ω, \mathcal{F}, P) be a measure space. Let μ be a signed measure on \mathcal{F} . (This means that $\mu : \mathcal{F} \rightarrow (-\infty, \infty)$ is a σ -additive*

measure such that $\mu(\emptyset) = 0$.) Suppose that μ is absolutely continuous with respect to P . (i.e., $\mu(A) = 0$ for any P -null set A .) Then there exists a unique integrable function f such that

$$\mu(A) = \int_A f dP \quad \text{for any } A \in \mathcal{F}.$$

An important difference between $E(X|\mathcal{G})$ and $E(X)$ one should notice is that the conditional expectation $E(X|\mathcal{G})$ is a random variable, whereas the (regular) expectation $E(X)$ is just a real number.

Next, we list some properties of the conditional expectation.

Theorem 2.17. *Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^1(\Omega, \mathcal{F})$. Suppose that \mathcal{G} is a σ -field such that $\mathcal{G} \subset \mathcal{F}$. Then, each of the following properties holds almost surely.*

(a) $E(E[X|\mathcal{G}]) = EX$.

(b) If X is \mathcal{G} -measurable, then $E[X|\mathcal{G}] = X$.

(c) If X and \mathcal{G} are independent, then $E[X|\mathcal{G}] = EX$.

(d) If Y is \mathcal{G} -measurable and $E|XY| < \infty$, then $E[XY|\mathcal{G}] = YE[X|\mathcal{G}]$.

(e) (Tower Property) If \mathcal{H} is a sub- σ -field of \mathcal{G} , then $E[X|\mathcal{H}] = E[E[X|\mathcal{G}]|\mathcal{H}]$.

(f) If $X, Y \in L^1(\Omega)$ and $X \leq Y$, then $E[X|\mathcal{G}] \leq E[Y|\mathcal{G}]$.

(g) $|E[X|\mathcal{G}]| \leq E[|X||\mathcal{G}]$.

(h) $E[aX + bY|\mathcal{G}] = aE[X|\mathcal{G}] + bE[Y|\mathcal{G}]$ for any $a, b \in \mathbb{R}$ and $X, Y \in L^1(\Omega)$.

(i) (Conditional Jensen's inequality) Let $X \in L^1(\Omega)$. Suppose ϕ is a convex function on \mathbb{R} . Then,

$$\phi(E[X|\mathcal{G}]) \leq E[\phi(X)|\mathcal{G}].$$

(j) (Conditional monotone convergence theorem) Let $0 \leq X_1 \leq X_2 \leq \dots \leq X_n \leq \dots$ and assume that $X = \lim_{n \rightarrow \infty} X_n$ in $L^1(\Omega)$. Then

$$E[X|\mathcal{G}] = \lim_{n \rightarrow \infty} E[X_n|\mathcal{G}].$$

(k) (Conditional Fatou's Lemma) Let $X_n \geq 0$, $X_n \in L^1(\Omega)$, $n = 1, 2, \dots$, and assume that $\liminf_{n \rightarrow \infty} X_n \in L^1(\Omega)$. Then

$$E\left[\liminf_{n \rightarrow \infty} X_n \middle| \mathcal{G}\right] \leq \liminf_{n \rightarrow \infty} E[X_n|\mathcal{G}].$$

(l) (Conditional Lebesgue Dominated convergence theorem) Assume that $|X_n| \leq Y$, $Y \in L^1(\Omega)$, and $X = \lim_{n \rightarrow \infty} X_n$ exists almost surely. Then,

$$E[X|\mathcal{G}] = \lim_{n \rightarrow \infty} E[X_n|\mathcal{G}].$$

2.3 Martingales

As mentioned before, one of the good properties of Brownian motion is that it is a martingale. Next, we provide more details about the concept of martingales. We begin with the definition of the filtration and the adaptedness of a stochastic process.

Definition 2.18. Let (Ω, \mathcal{F}, P) be a probability space. Let T be an interval in \mathbb{R} or the set of positive integers. A filtration on T is an increasing family $\{\mathcal{F}_t; t \in T\}$ of sub- σ -fields of \mathcal{F} . In other words, for any $s, t \in T$, the inequality $s \leq t$ implies $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$.

Next, we define the adaptedness of a stochastic process. This concept is very important for the study of the new stochastic integral which we will discuss later in Chapters 4 and 5.

Definition 2.19. A stochastic process X_t , $t \in T$, is said to be adapted to the filtration $\{\mathcal{F}_t; t \in T\}$ if for each $t \in T$, the random variable X_t is \mathcal{F}_t -measurable.

Remark 2.20. We assume that all σ -fields \mathcal{F}_t are complete. This means that if $A \in \mathcal{F}_t$ with $P(A) = 0$, then for any subsets B of A , $B \in \mathcal{F}_t$.

Now we are ready to define a martingale.

Definition 2.21. Let $\{X_t; t \in T\}$ be a stochastic process and $E|X_t| < \infty$ for all $t \in T$. Then X_t is called a martingale with respect to the filtration $\{\mathcal{F}_t; t \in T\}$ if

(1) X_t is adapted to the filtration $\{\mathcal{F}_t; t \in T\}$, and

(2) for $s \leq t$ in T , $E[X_t|\mathcal{F}_s] = X_s$ almost surely.

Remark 2.22. If the filtration is not specified, then the filtration is understood to be the one given by $\mathcal{F}_t = \sigma\{X(s); s \leq t\}$, the σ -field generated by all X_s when $s \leq t$.

Remark 2.23. If the equality in condition (2) of the definition of martingales is replaced by \geq (or \leq), then X_t is called a submartingale (or supermartingale) with respect to $\{\mathcal{F}_t\}$.

Example 2.24. Let $T = \{1, 2, \dots\}$. Let $\{X_k\}_{k=1}^{\infty}$ be the sequence of independent Bernoulli random variables given by $P(X_k = 1) = P(X_k = -1) = \frac{1}{2}$. Hence, $E(X_k) = 1 \left(\frac{1}{2}\right) + (-1) \left(\frac{1}{2}\right) = 0$ for each k .

Then, the random walk $S_n = X_1 + \dots + X_n$ is a martingale with respect to the filtration $\{\mathcal{F}_n; n \in T\}$ where $\mathcal{F}_n = \sigma(S_m; m \leq n) = \sigma(X_1, X_2, \dots, X_n)$.

Proof. Consider, for any $m < n$ in T ,

$$E(S_n|\mathcal{F}_m) = E\left(\sum_{i=1}^n X_i \middle| \mathcal{F}_m\right) = \sum_{i=1}^m E(X_i|\mathcal{F}_m) + \sum_{j=m+1}^n E(X_j|\mathcal{F}_m). \quad (2.4)$$

For each $1 \leq i \leq m$, X_i is \mathcal{F}_m -measurable which implies that $E(X_i|\mathcal{F}_m) = X_i$. Also, X_j is independent of \mathcal{F}_m for each $m+1 \leq j \leq n$, then $E(X_j|\mathcal{F}_m) = E(X_j)$

for all $m + 1 \leq j \leq n$. Thus Equation (2.4) becomes

$$E(S_n | \mathcal{F}_m) = \sum_{i=1}^m X_i + \sum_{j=m+1}^m E(X_j) = S_m + 0 = S_m.$$

Therefore, S_n is a martingale with respect to the filtration $\{\mathcal{F}_n; n \in T\}$. \square

Example 2.25. *The Brownian motion $B(t)$ is a martingale (as we stated at the beginning of this section).*

Proof. For $s \leq t$, consider

$$\begin{aligned} E(B(t) | \mathcal{F}_s) &= E(B(t) - B(s) + B(s) | \mathcal{F}_s) \\ &= E(B(t) - B(s) | \mathcal{F}_s) + E(B(s) | \mathcal{F}_s). \end{aligned} \quad (2.5)$$

Since $B(t) - B(s)$ is independent of \mathcal{F} and $B(s)$ is \mathcal{F}_s -measurable, Equation (2.5) becomes

$$E(B(t) | \mathcal{F}_s) = E(B(t) - B(s)) + B(s) = 0 + B(s) = B(s).$$

Thus $B(t)$ is a martingale (with respect to $\{\mathcal{F}_t\}$ where $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$). \square

Example 2.26. *Let $B(t)$ be a Brownian motion. Then $B(t)^2$ is a submartingale.*

Proof. For $s \leq t$,

$$\begin{aligned} &E[B(t)^2 | \mathcal{F}_s] \\ &= E\left[(B(t) - B(s) + B(s))^2 | \mathcal{F}_s\right] \\ &= E\left[(B(t) - B(s))^2 | \mathcal{F}_s\right] + 2E[B(s)(B(t) - B(s)) | \mathcal{F}_s] + E[B(s)^2 | \mathcal{F}_s]. \end{aligned} \quad (2.6)$$

Again, since $B(t) - B(s)$ is independent of \mathcal{F} and $B(s)$ is \mathcal{F}_s -measurable, Equation (2.6) becomes

$$\begin{aligned} E [B(t)^2 | \mathcal{F}_s] &= E [(B(t) - B(s))^2] + 2B(s)E[B(t) - B(s) | \mathcal{F}_s] + B(s)^2 \\ &= (t - s) + 2B(s)E[B(t) - B(s)] + B(s)^2 \\ &= (t - s) + 0 + B(s)^2 \\ &\geq B(s)^2. \end{aligned}$$

Thus $B(t)^2$ is a submartingale (with respect to the filtration $\{\mathcal{F}_t\}$ where $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$). □

Chapter 3

The Itô Theory of Stochastic Integration

In this chapter, we review stochastic integral in the Itô theory and its properties. Namely, we consider the integral of the form

$$\int_a^b f(t, \omega) dB(t, \omega)$$

where $B(t)$ is a Brownian motion and f is an adapted stochastic processes. Then, we state the relationship between stochastic integral and its limit of the Riemann sums. Also, the well-known Itô's formula is presented at the end.

Note that the discussion here will be very brief. For more details, the reader can refer to the book [9].

3.1 Itô Integrals

Let $B(t)$ be a fixed Brownian motion, $[a, b]$ be any finite interval on \mathbb{R} , and let $\{\mathcal{F}_t; a \leq t \leq b\}$ be the filtration such that

- (1) for each t , $B(t)$ is \mathcal{F}_t -measurable
- (2) for any $s \leq t$, the random variable $B(t) - B(s)$ is independent of the σ -field \mathcal{F}_s .

Remark 3.1. *The natural Brownian filtration $\mathcal{F}_t^B = \sigma\{B(s); s \leq t\}$ is an example of filtration satisfying those above conditions.*

An *Itô integral* is a stochastic integral of the form

$$I(f) = \int_a^b f(t, \omega) dB(t)$$

where $f(t, \omega)$ is a stochastic process such that $f(t)$ is adapted to the filtration $\{\mathcal{F}_t\}$ and $\int_a^b E(|f(t, \omega)|^2) dt < \infty$.

For convenience, let $L_{\text{ad}}^2([a, b] \times \Omega)$ denote the space of all stochastic processes $f(t, \omega)$, $a \leq t \leq b$, satisfying two conditions above. Note that by Fubini's theorem, the condition $\int_a^b E(|f(t, \omega)|^2) dt < \infty$ can also be stated as

$$E\left(\int_a^b |f(t, \omega)|^2 dt\right) < \infty.$$

In other words, if $f \in L_{\text{ad}}^2([a, b] \times \Omega)$, the stochastic integral $\int_a^b f(t, \omega) dB(t)$ is an Itô integral. For convenience, we sometimes denote it by $\int_a^b f(t) dB(t)$.

For details on how to rigorously define the Itô integrals, the reader should refer to the book [9]. Only the properties of Itô integrals that will be used later will be presented here.

Theorem 3.2. (*Linearity of Itô integral*) For $a, b \in \mathbb{R}$ and $f, g \in L_{\text{ad}}^2([a, b] \times \Omega)$,

$$I(af + bg) = aI(f) + bI(g).$$

The following theorem gives the mean and variance of the Itô integral.

Theorem 3.3. (*Zero expectation and Itô isometry*) Suppose $f \in L_{\text{ad}}^2([a, b] \times \Omega)$. Then the Itô integral $I(f) = \int_a^b f(t) dB(t)$ is a random variable with $E(I(f)) = 0$ and

$$E(|I(f)|^2) = \int_a^b E(|f(t)|^2) dt. \quad (3.1)$$

By this theorem, the map $I : L_{\text{ad}}^2([a, b] \times \Omega) \rightarrow L^2(\Omega)$ is an isometry. Equation (3.1) is often called as the “Itô isometry.”

Example 3.4. Consider $f(t, \omega) = B(t, \omega)^2$. Then $f(t)$ is adapted to the filtration $\{\mathcal{F}_t; a \leq t \leq b\}$ and

$$\int_a^b E|B(t)^2|^2 dt = \int_a^b E|B(t)|^4 dt = \int_a^b 3t^2 dt = b^3 - a^3 < \infty.$$

Thus $f(t) \in L_{ad}^2([a, b] \times \Omega)$, Then $\int_a^b f(t) dB(t) = \int_a^b B(t)^2 dB(t)$ is an Itô integral. Also, by Theorem 3.3, we have that $\int_a^b B(t)^2 dB(t)$ is a random variable with mean 0 and variance $\int_a^b E|B(t)^2|^2 dt = b^3 - a^3$.

Example 3.5. The stochastic integral $\int_0^1 \sqrt{t}e^{B(t)} dB(t)$ is an Itô integral because $\sqrt{t}e^{B(t)}$ is adapted to the filtration $\{\mathcal{F}_t; 0 \leq t \leq 1\}$ and

$$E \left| \sqrt{t}e^{B(t)} \right|^2 = E |te^{2B(t)}| = tE(e^{2B(t)}) = te^{\frac{1}{2}(2)t} = te^{2t},$$

which implies that $\int_0^1 E|\sqrt{t}e^{B(t)}|^2 dt = \int_0^1 te^{2t} dt = 1 + \frac{e^2}{4} < \infty$.

Moreover, by Theorem 3.3, $\int_0^1 \sqrt{t}e^{B(t)} dB(t)$ is a random variable with mean 0 and variance $\int_0^1 E|\sqrt{t}e^{B(t)}|^2 dt = 1 + \frac{e^2}{4}$.

Example 3.6. Consider $f(t) = \text{sgn}(B(t))$. Since

$$\int_a^b E |\text{sgn}(B(t))|^2 dt = \int_a^b E(1) dt = b - a < \infty,$$

it follows that $f(t) = \text{sgn}(B(t)) \in L_{ad}^2([a, b] \times \Omega)$. By Theorem 3.3, the random variable $\int_a^b \text{sgn}(B(t)) dB(t)$ has mean 0 and variance $\int_a^b E |\text{sgn}(B(t))|^2 dt = b - a$.

Suppose that $f \in L_{ad}^2([a, b] \times \Omega)$. Then for any $t \in [a, b]$, $\int_a^t E|f(t)|^2 dt \leq \int_a^b E|f(t)|^2 dt < \infty$. So, $f \in L_{ad}^2([a, t] \times \Omega)$ and the integral $\int_a^t f(s) dB(s)$ is well defined. Consider a stochastic process given by

$$X_t = \int_a^t f(s) dB(s), \quad a \leq t \leq b.$$

Note that, by Theorem 3.3, we have

$$E(|X_t|^2) = E \left| \int_a^t f(s) dB(s) \right|^2 = \int_a^t E|f(s)|^2 ds \leq \int_a^b E|f(s)|^2 ds < \infty.$$

Then, by the Cauchy-Schwarz inequality, $E|X_t| \leq [E(|X_t|^2)]^{1/2} < \infty$. Hence, for each t , the random variable X_t is integrable and we can consider its martingale property.

The next two theorems state the martingale and continuity properties of the stochastic process X_t which is called an *associated stochastic process of f* .

Theorem 3.7. (*Martingale property*) Let $f \in L^2_{ad}([a, b] \times \Omega)$. Then the stochastic process

$$X_t = \int_a^t f(s) dB(s), \quad a \leq t \leq b,$$

is a martingale with respect to the filtration $\{\mathcal{F}_t; a \leq t \leq b\}$.

Example 3.8. The stochastic processes $\int_a^t B(s)^2 dB(s)$ and $\int_a^t \sqrt{s} e^{B(s)} dB(s)$ are martingales with respect to $\{\mathcal{F}_t\}$.

Theorem 3.9. (*Continuity property*) Suppose $f \in L^2_{ad}([a, b] \times \Omega)$. Then the stochastic process

$$X_t = \int_a^t f(s) dB(s), \quad a \leq t \leq b,$$

is continuous, i.e., almost all its sample paths are continuous functions on $[a, b]$.

In particular, consider when $f(t)$ is just a deterministic function (does not depend on ω) in $L^2[a, b]$, the Hilbert space of real valued square integrable functions on $[a, b]$.

Observe that $f(t)$ is $\{\mathcal{F}_t\}$ -adapted and $\int_a^b E |f(t)|^2 dt = \int_a^b |f(t)|^2 dt$ is finite because $f \in L^2[a, b]$.

Therefore, f is also in $L^2_{ad}([a, b] \times \Omega)$. Thus, by the above discussion, we have the following inclusion:

$$L^2[a, b] \subset L^2_{ad}([a, b] \times \Omega).$$

In other words, for any $f \in L^2[a, b]$, the integral $\int_a^b f(t) dB(t)$ is also an Itô integral. Note that when $f \in L^2[a, b]$, we call the stochastic integral $\int_a^b f(t) dB(t)$ a *Wiener integral*.

Example 3.10. Since $f(t) = t \sin\left(\frac{1}{t}\right) \in L^2[0, 1]$, it follows that $\int_0^1 t \sin\left(\frac{1}{t}\right) dB(t)$ is a Wiener integral.

Since all Wiener integrals are Itô integrals, all above properties of Itô integrals (Theorem 3.2 - Theorem 3.9) can also be applied to the Wiener integrals as well.

However, there is a special property of the Wiener integrals that the Itô integrals do not have. For $f(t, \omega) \in L^2_{\text{ad}}([a, b] \times \Omega)$, we know just that the Itô integral $\int_a^b f(t, \omega) dB(t)$ is a random variable with mean 0 and variance $\int_a^b E(|f(t)|^2) dt$. However, if $f \in L^2[a, b]$, we know more than that. The Wiener integral $\int_a^b f(t) dB(t)$ is not any random variable, but a Gaussian random variable with mean 0 and variance $\int_a^b |f(t)|^2 dt$ as stated in the following theorem.

Theorem 3.11. For each $f \in L^2[a, b]$, the Wiener integral $\int_a^b f(t) dB(t)$ is a Gaussian random variable with mean 0 and variance $\|f\|_{L^2[a, b]}^2 = \int_a^b |f(t)|^2 dt$.

Example 3.12. Since $\int_1^2 |t^2|^2 dt = \frac{2^5}{5} - \frac{1}{5} = \frac{31}{5} < \infty$, $f(t) = t^2 \in L^2[1, 2]$. Then by Theorem 3.11, the Wiener integral $\int_1^2 t^2 dB(t)$ is a Gaussian random variable with mean 0 and variance $\int_1^2 t^4 dt = \frac{31}{5}$.

Note that the Itô integral can be extended to a larger class of integrands. Also called Itô integral, it can be defined for stochastic processes $f(t, \omega)$ satisfying the following conditions:

- (a) $f(t)$ is adapted to the filtration $\{\mathcal{F}_t\}$;
- (b) $\int_a^b |f(t)|^2 dt < \infty$ almost surely.

We will use the notation $\mathcal{L}_{\text{ad}}(\Omega, L^2[a, b])$ to denote the space of all stochastic processes $f(t, \omega)$ satisfying conditions (a) and (b) above.

If $f \in L^2_{\text{ad}}([a, b] \times \Omega)$, we have that f is adapted with respect to $\{\mathcal{F}_t\}$ and $E \int_a^b |f(t)|^2 dt = \int_a^b E(|f(t)|^2) dt < \infty$. It follows that $\int_a^b |f(t)|^2 dt < \infty$ almost

surely which implies $f \in \mathcal{L}_{\text{ad}}(\Omega, L^2[a, b])$. Therefore, we have a larger class of integrands $f(t, \omega)$ for the stochastic integral $\int_a^b f(t) dB(t)$. Namely,

$$L_{\text{ad}}^2([a, b] \times \Omega) \subset \mathcal{L}_{\text{ad}}(\Omega, L^2[a, b]).$$

The essential difference between these two spaces is the possible lack of integrability (with respect to ω) of the integrand $f(t, \omega)$ for $f \in \mathcal{L}_{\text{ad}}(\Omega, L^2[a, b])$ which one can see more clearly by the following example.

Example 3.13. Consider the stochastic process $f(t) = e^{B(t)^2}$. Note that

$$\begin{aligned} E(|f(t)|^2) &= E \left[e^{2B(t)^2} \right] \\ &= \int_{-\infty}^{\infty} e^{2x^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\ &= \frac{1}{\sqrt{1-4t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \left(\frac{t}{1-4t}\right)}} e^{-\frac{x^2}{2\left(\frac{t}{1-4t}\right)}} dx \\ &= \begin{cases} \frac{1}{\sqrt{1-4t}} & \text{if } 0 < t < \frac{1}{4}; \\ \infty & \text{if } t \geq \frac{1}{4}. \end{cases} \end{aligned}$$

So $\int_0^1 E|f(t)|^2 dt = \infty$, which implies that $f \notin L_{\text{ad}}^2([0, 1] \times \Omega)$. However, since for almost all fixed ω , $f(t, \omega) = e^{B(t, \omega)^2}$ is a continuous function of t , we have $\int_0^1 |f(t)|^2 dt = \int_0^1 e^{2B(t)^2} dt < \infty$ almost surely. So $f \in \mathcal{L}_{\text{ad}}(\Omega, L^2[0, 1])$.

Hence $f(t) = e^{B(t)^2}$ is an example of stochastic processes such that f belongs to $\mathcal{L}_{\text{ad}}(\Omega, L^2[0, 1])$, but f does not belong to $L_{\text{ad}}^2([0, 1] \times \Omega)$.

As discussed before, the stochastic process $f \in \mathcal{L}_{\text{ad}}(\Omega, L^2[a, b])$ may lack the integrability property. So the stochastic integral $\int_a^b f(t) dB(t)$ is a random variable that may have infinite expectation.

In general, for $f \in \mathcal{L}_{\text{ad}}(\Omega, L^2[a, b])$, the associated stochastic process

$$X_t = \int_a^t f(s) dB(s)$$

may not have finite expectation. Thus it fails to be a martingale. However, we have another concept involving the stopping time called *local martingale* and we have a similar theorem to the Theorem 3.7 stating that $X_t = \int_a^t f(s) dB(s)$ is a local martingale if $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$.

Theorem 3.14. *Let $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$. Then the stochastic process*

$$X_t = \int_a^t f(s) dB(s), \quad a \leq t \leq b,$$

is a local martingale with respect to the filtration $\{\mathcal{F}_t; a \leq t \leq b\}$.

See [9] for more detailed discussions of the stopping times and local martingales.

Example 3.15. *Let $f(t) = e^{B(t)^2}$. We have already seen in Example 3.13 that $f \in \mathcal{L}_{ad}(\Omega, L^2[0, 1])$. So, by Theorem 3.14, we have that the stochastic process*

$$X_t = \int_0^t e^{B(s)^2} dB(s), \quad 0 \leq t \leq 1,$$

is a local martingale but not a martingale.

We also have an analogue of the Theorem 3.9.

Theorem 3.16. *Let $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$. Then the stochastic process*

$$X_t = \int_a^t f(s) dB(s), \quad a \leq t \leq b,$$

has a continuous realization.

3.2 Riemann Sums and Stochastic Integrals

In this section, we will review the relationship between the stochastic integrals of continuous adapted stochastic processes and their convergence of the Riemann-like sums evaluated at the left endpoints of the subintervals.

Note that if $f(t)$ is a continuous $\{\mathcal{F}_t\}$ -adapted process, then $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$. Moreover, its stochastic integral $\int_a^b f(t) dB(t)$ can be expressed as a limit of its Riemann like sums evaluated at the left endpoints.

Theorem 3.17. *Suppose f is a continuous $\{\mathcal{F}_t\}$ -adapted stochastic process. Then $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ and*

$$\int_a^b f(t) dB(t) = \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f(t_{i-1})(B(t_i) - B(t_{i-1})) \quad \text{in probability,}$$

where $\Delta_n = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$ is a partition of the finite interval $[a, b]$ and $\|\Delta_n\| = \max_{1 \leq i \leq n} (t_i - t_{i-1})$.

The proof of this theorem can be found in [9].

More specifically, if $f \in L^2_{ad}(\Omega \times [a, b])$, we know that the Itô integral $\int_a^b f(t) dB(t)$ belongs to $L^2(\Omega)$ and we have an analogue of Theorem 3.17 for $f \in L^2_{ad}(\Omega \times [a, b])$ with a stronger convergence. Namely, the above Riemann sum of f will converge in $L^2(\Omega)$ instead of just in probability, but it still requires a little stronger assumption that $E(f(t)f(s))$ is a continuous function of t and s .

Theorem 3.18. *Suppose $f \in L^2_{ad}([a, b] \times \Omega)$ and assume that $E(f(t)f(s))$ is a continuous function of t and s . Then,*

$$\int_a^b f(t) dB(t) = \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f(t_{i-1})(B(t_i) - B(t_{i-1})) \quad \text{in } L^2(\Omega),$$

where $\Delta_n = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$ and $\|\Delta_n\| = \max_{1 \leq i \leq n} (t_i - t_{i-1})$.

The proof of this Theorem can also be found in [9].

Example 3.19. *We will evaluate the integral $\int_0^t B(s) dB(s)$, for $t \geq 0$.*

Let $\Delta_n = \{0 = s_0 < s_1 < \dots < s_{n-1} < s_n = t\}$ be a partition of the interval $[0, t]$.

Note that $B(s) \in L^2_{ad}([0, t] \times \Omega)$ since $B(t)$ is adapted and $\int_0^t E(B(s))^2 ds = \int_0^t s ds = \frac{t^2}{2} < \infty$ for any fixed $t \geq 0$. Consider

$$E(B(t)B(s)) = \min\{t, s\} = \frac{t + s - |t - s|}{2},$$

which is a continuous function of s and t . Then Theorem 3.18 provides that $\int_0^t B(s) dB(s)$ is the L^2 -limit of the summation

$$\sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1})).$$

To find its limit in $L^2(\Omega)$, consider

$$B(s_i)^2 = (B(s_i) - B(s_{i-1}))^2 + 2B(s_{i-1})(B(s_i) - B(s_{i-1})) + B(s_{i-1})^2$$

which can be rewritten as

$$B(s_{i-1})(B(s_i) - B(s_{i-1})) = \frac{1}{2}B(s_i)^2 - \frac{1}{2}B(s_{i-1})^2 - \frac{1}{2}(B(s_i) - B(s_{i-1}))^2$$

Thus, we have

$$\begin{aligned} & \sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1})) \\ &= \frac{1}{2} \sum_{i=1}^n (B(s_i)^2 - B(s_{i-1})^2) - \frac{1}{2} \sum_{i=1}^n (B(s_i) - B(s_{i-1}))^2 \\ &= \frac{1}{2} (B(t)^2 - B(0)^2) - \frac{1}{2} \sum_{i=1}^n (B(s_i) - B(s_{i-1}))^2. \end{aligned}$$

By the quadratic variation of the Brownian motion (Theorem 2.12), we note that the last summation converges to $t - 0$ in $L^2(\Omega)$. So, we conclude that

$$\sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1})) \longrightarrow \frac{1}{2} (B(t)^2 - t) \quad \text{in } L^2(\Omega) \quad \text{as } \|\Delta_n\| \rightarrow 0.$$

Therefore,

$$\int_0^t B(s) dB(s) = \frac{1}{2} (B(t)^2 - t).$$

3.3 Itô's Formula

In ordinary calculus, we deal with deterministic functions. One of the useful rules for differentiation that we use very often is the chain rule.

The chain rule states that if f and g are differentiable, then its composite function $f \circ g$ is also differentiable and has derivative given by

$$\frac{d}{dt}(f \circ g)(t) = \frac{d}{dt}f(g(t)) = f'(g(t))g'(t)$$

which can be rewritten in the integral form as

$$f(g(t)) - f(g(a)) = \int_a^t f'(g(s))g'(s) ds. \quad (3.2)$$

On the other hand, in Itô calculus, we deal with the stochastic processes which are random functions. For example, if $g(t)$ is the stochastic process $B(t)$, then we have the composite function $f(B(t))$. Note that almost all sample paths of the Brownian motion $B(t)$ are nowhere differentiable, so the expression $\frac{d}{dt}f(B(t)) = f'(B(t))B'(t)$ has absolutely no meaning.

However, by rewriting $B'(s) ds$ as the integrator $dB(s)$ in the Itô integral, we have a similar integral version of the chain rule in the Itô calculus. It is called *Itô's formula* (or sometimes called *Itô's lemma*). Later, we will see that this Itô's formula can be generalized to some larger classes of stochastic processes.

Here, we will present several versions of Itô formula without the proofs. For more details about their proofs, the reader may look at [9].

Let $B(t)$, $a \leq t \leq b$, be a Brownian motion. We begin with the simplest form of the Itô formula.

Theorem 3.20. *Let f be a C^2 -function. That is, f is twice differentiable and f'' is continuous. Then*

$$f(B(t)) - f(B(a)) = \int_a^t f'(B(s)) dB(s) + \frac{1}{2} \int_a^t f''(B(s)) ds \quad (3.3)$$

where the first integral on the right hand side is an Itô integral as defined in section 3.1 and the second integral is a Riemann integral for each sample path of $B(s)$.

Remark 3.21. Compared to Equation (3.2), the Itô formula in Equation (3.3) has an extra term $\frac{1}{2} \int_a^t f''(B(s)) ds$. This is a consequence of the nonzero quadratic variation of the Brownian motion which is a main point distinguishing Itô calculus from the ordinary calculus.

Example 3.22. Let $f(x) = x^2$. Then, by Theorem 3.20, we get

$$B(t)^2 - B(a)^2 = 2 \int_a^t B(s) dB(s) + (t - a).$$

Thus we have

$$\int_a^t B(s) dB(s) = \frac{1}{2} (B(t)^2 - B(a)^2 - (t - a)). \quad (3.4)$$

When putting $a = 0$, Equation (3.4) reduces to

$$\int_0^t B(s) dB(s) = \frac{1}{2} (B(t)^2 - t) \quad (3.5)$$

which is exactly the same as what we obtained in Example 3.19.

Example 3.23. We will evaluate the integral $\int_0^t B(s)^4 dB(s)$. To evaluate this integral by using Itô's formula, we let $f(x) = \frac{x^5}{5}$ (so that we will have $f'(x) = x^4$). Hence, by Theorem 3.20, we have

$$\frac{B(t)^5}{5} - 0 = \int_0^t B(s)^4 dB(s) + \frac{1}{2} \int_0^t 4B(s)^3 ds.$$

Thus, we have

$$\int_0^t B(s)^4 dB(s) = \frac{1}{5} B(t)^5 - 2 \int_0^t B(s)^3 ds.$$

Next, we give a more generalized Itô's formula. Namely, we consider a function $f(t, x)$ of two variables t and x . We set $x = B(t)$ to get a stochastic process $f(t, B(t))$. Observe that t appears in two places, one as a variable of f and the other one is in the Brownian motion. For the first t , we can apply the ordinary calculus. However, we need the Itô calculus to take care of the second t in $B(t)$. The following theorem is a slightly generalized Itô's formula for the function $f(t, x)$.

Theorem 3.24. Let $f(t, x)$ be a continuous function with continuous derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$. Then

$$\begin{aligned} f(t, B(t)) - f(a, B(a)) \\ = \int_a^t \frac{\partial f}{\partial x}(s, B(s)) dB(s) + \int_a^t \left(\frac{\partial f}{\partial t}(s, B(s)) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, B(s)) \right) ds. \end{aligned}$$

Again, the first integral is the Itô integral and the second integral is the Riemann integral for each sample path of the Brownian motion $B(t)$.

Example 3.25. We will find $\int_0^t sB(s) dB(s)$ by using Itô's formula. First, let $f(t, x) = \frac{tx^2}{2}$ (so that we will have $\frac{\partial f}{\partial x}(t, x) = tx$). Then

$$\frac{\partial f}{\partial x}(t, x) = tx, \quad \frac{\partial^2 f}{\partial x^2}(t, x) = t, \quad \frac{\partial f}{\partial t}(t, x) = \frac{x^2}{2}.$$

By Theorem 3.24, we have

$$\begin{aligned} \frac{tB(t)^2}{2} - 0 &= \int_0^t sB(s) dB(s) + \int_0^t \left(\frac{B(s)^2}{2} + \frac{1}{2}(s) \right) ds \\ &= \int_0^t sB(s) dB(s) + \frac{1}{2} \int_0^t B(s)^2 ds + \frac{1}{2} \left(\frac{t^2}{2} \right). \end{aligned}$$

Thus, we obtain

$$\int_0^t sB(s) dB(s) = \frac{1}{2}tB(t)^2 - \frac{1}{4}t^2 - \frac{1}{2} \int_0^t B(s)^2 ds.$$

Example 3.26. Let $f(t, x) = x^3 - t$. Note that

$$\frac{\partial f}{\partial x}(t, x) = 3x^2, \quad \frac{\partial^2 f}{\partial x^2}(t, x) = 6x, \quad \frac{\partial f}{\partial t}(t, x) = -1.$$

Thus, by theorem 3.24, we have that

$$\begin{aligned} (B(t)^3 - t) - (B(a)^3 - a) &= \int_a^t 3B(s)^2 dB(s) + \int_a^t \left(-1 + \frac{1}{2}(6B(s)) \right) ds \\ &= 3 \int_a^t B(s)^2 dB(s) - (t - a) + 3 \int_a^t B(s) ds \end{aligned}$$

which can be simplified to

$$\int_a^t B(s)^2 dB(s) = \frac{1}{3} (B(t)^3 - B(a)^3) - \int_a^t B(s) ds.$$

When putting $a = 0$, we obtain

$$\int_0^t B(s)^2 dB(s) = \frac{B(t)^3}{3} - \int_0^t B(s) ds. \quad (3.6)$$

Before we move to the most general form of Itô's formula, let us introduce a special class of stochastic processes which is called "Itô process."

Recall that $\mathcal{L}_{ad}(\Omega, L^2[a, b])$ is the space consisting of all \mathcal{F}_t -adapted stochastic processes $f(t)$ such that $\int_a^b |f(t)|^2 dt < \infty$ almost surely.

We also denote by $\mathcal{L}_{ad}(\Omega, L^1[a, b])$ the class of all \mathcal{F}_t -adapted stochastic processes $f(t)$ such that $\int_a^b |f(t)| dt < \infty$ almost surely.

Definition 3.27. *An Itô process is a stochastic process of the form*

$$X_t = X_a + \int_a^t f(s) dB(s) + \int_a^t g(s) ds, \quad a \leq t \leq b, \quad (3.7)$$

where X_a is \mathcal{F}_a -measurable, $f \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$, and $g \in \mathcal{L}_{ad}(\Omega, L^1[a, b])$.

A more convenient way to write Equation (3.7) is the following "stochastic differential"

$$dX_t = f(t) dB(t) + g(t) dt, \quad a \leq t \leq b. \quad (3.8)$$

One must note that the expression above has no meaning because of the non-differentiability of the Brownian paths. The differential form in (3.8) is just a symbolic expression for the integral equation in (3.7).

Example 3.28. *Let θ be a C^2 -function on $[a, b]$. By the Itô formula (Equation (3.3)), we have*

$$\theta(B(t)) = \theta(B(a)) + \int_a^t \theta'(B(s)) dB(s) + \int_a^t \frac{1}{2} \theta''(B(s)) ds, \quad a \leq t \leq b.$$

Then, $X_t = \theta(B(t))$ is an Itô process. In other words, every C^2 -function of the Brownian motion is an Itô process.

The following theorem is the most general form of the Itô formula. Namely, it is the Itô formula for functions of Itô processes.

Theorem 3.29. *Let X_t be an Itô process given by*

$$X_t = X_a + \int_a^t f(s) dB(s) + \int_a^t g(s) ds, \quad a \leq t \leq b.$$

Suppose that $\theta(t, x)$ is a continuous function with continuous derivatives $\frac{\partial \theta}{\partial t}$, $\frac{\partial \theta}{\partial x}$, and $\frac{\partial^2 \theta}{\partial x^2}$. Then $\theta(t, X_t)$ is also an Itô process and

$$\begin{aligned} \theta(t, X_t) = & \theta(a, X_a) + \int_a^t \frac{\partial \theta}{\partial x}(s, X_s) dB(s) \\ & + \int_a^t \left[\frac{\partial \theta}{\partial t}(s, X_s) + \frac{\partial \theta}{\partial x}(s, X_s)g(s) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(s, X_s)f(s)^2 \right] ds. \end{aligned} \quad (3.9)$$

One of the best ways to get Equation (3.9) is through the Taylor expansion in the stochastic differential form and the following useful “Itô table” :

\times	$dB(t)$	dt
$dB(t)$	dt	0
dt	0	0

TABLE 3.1. Itô table.

First, we apply the Taylor expansion of $\theta(t, X_t)$ to the first order for dt and to the second order for dX_t to get

$$d\theta(t, X_t) = \frac{\partial \theta}{\partial t}(t, X_t) dt + \frac{\partial \theta}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(t, X_t) (dX_t)^2. \quad (3.10)$$

Then, apply the above Itô table to get $(dX_t)^2 = f(t)^2 dt$. Finally, we obtain

$$\begin{aligned} d\theta(t, X_t) &= \frac{\partial \theta}{\partial t}(t, X_t) dt + \frac{\partial \theta}{\partial x}(t, X_t) \left(f(t) dB(t) + g(t) dt \right) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(t, X_t) (f(t)^2 dt) \\ &= \frac{\partial \theta}{\partial t}(t, X_t) f(t) dB(t) + \left(\frac{\partial \theta}{\partial t}(t, X_t) + \frac{\partial \theta}{\partial x}(t, X_t) g(t) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(t, X_t) f(t)^2 \right) dt. \end{aligned}$$

Example 3.30. Let $f \in \mathcal{L}_{ad}(\Omega, L^2[0, 1])$. Consider an Itô process

$$X_t = \int_0^t f(s)dB(s) - \frac{1}{2} \int_0^t f(s)^2 ds, \quad 0 \leq t \leq 1.$$

We will show that $\tilde{Y}_t = e^{X_t}$ is a solution of the stochastic differential equation

$$\begin{cases} dY_t = f(t)Y_t dB(t), & 0 \leq t \leq 1, \\ Y_0 = 1. \end{cases} \quad (3.11)$$

First note that $\tilde{Y}_0 = e^{X_0} = e^0 = 1$. Now, we apply the Taylor expansion as in Equation (3.10) to $\theta(x) = e^x$ to the second order and use the Itô table to get

$$\begin{aligned} d\tilde{Y}_t &= d\theta(X_t) \\ &= \theta'(X_t) dX_t + \frac{1}{2}\theta''(X_t) (dX_t)^2 \\ &= e^{X_t} \left(f(t) dB(t) - \frac{1}{2}f(t)^2 dt \right) + \frac{1}{2}e^{X_t} (f(t)^2 dt) \\ &= e^{X_t} f(t) dB(t) \\ &= f(t)\tilde{Y}_t dB(t). \end{aligned}$$

Thus we conclude that $\tilde{Y}_t = e^{X_t}$ is a solution of the stochastic differential equation (3.11).

Moreover, the Itô formula can be generalized to the multi-dimensional case. We will not precisely state the formula here. For those interested in the multi-dimensional Itô's formula, see the book [9].

With a special case of the multi-dimensional Itô's formula, the following useful *Itô product formula* is obtained.

Let X_t and Y_t be two Itô processes defined for $a \leq t \leq b$. Then the integral form of the Itô product formula is given by

$$X_t Y_t = X_a Y_a + \int_a^t Y_s dX_s + \int_a^t X_s dY_s + \int_a^t dX_s dY_s. \quad (3.12)$$

The Itô product formula can also be expressed in the differential form as

$$d(X_t Y_t) = Y_t dX_t + X_t dY_t + (dX_t)(dY_t) . \quad (3.13)$$

Note that the Itô product formula in Equation (3.13) looks similar to the product rule in the ordinary calculus, but it contains the extra term $(dX_t)(dY_t)$.

More specifically, if X_t and Y_t are given by

$$dX_t = f(t) dB(t) + \xi(t) dt \quad \text{and} \quad dY_t = g(t) dB(t) + \eta(t) dt , \quad a \leq t \leq b.$$

Then by Equation (3.12) and Itô table, we have

$$\begin{aligned} X_t Y_t &= X_a Y_a + \int_a^t Y_s \left(f(s) dB(s) + \xi(s) ds \right) + \int_a^t X_s \left(g(s) dB(s) + \eta(s) ds \right) \\ &\quad + \int_a^t \left(f(s) dB(s) + \xi(s) ds \right) \left(g(s) dB(s) + \eta(s) ds \right) \\ &= X_a Y_a + \int_a^t f(s) Y_s dB(s) + \int_a^t \xi(s) Y_s ds \\ &\quad + \int_a^t g(s) X_s dB(s) + \int_a^t \eta(s) X_s ds + \int_a^t f(s) g(s) ds \\ &= X_a Y_a + \int_a^t \left(f(s) Y_s + g(s) X_s \right) dB(s) \\ &\quad + \int_a^t \left(\xi(s) Y_s + \eta(s) X_s + f(s) g(s) \right) ds. \end{aligned}$$

Chapter 4

The New Stochastic Integral

In Chapter 3, all stochastic integrals $I(f) = \int_a^b f(t) dB(t)$ are only defined for $\{\mathcal{F}_t\}$ -adapted stochastic processes $f(t)$. In this chapter, we will explain Ayed and Kuo's new idea to define the new stochastic integral. This new integral can be regarded as an extension of Itô integral because it can also be defined for some non-adapted integrands. More details about this new idea can be found in [1] and [2].

Throughout this chapter, let $B(t)$, $t \geq 0$, be a Brownian motion and let $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$.

4.1 Motivation

We begin this section with some history of anticipating stochastic integration. Then we review some previous approaches to define the stochastic integrals for non-adapted integrands before Ayed and Kuo obtain this new idea.

In 1976, Itô raised a very interesting question in the International Symposium on Stochastic Differential Equations at Kyoto. The question was how one should define $\int_0^1 B(1) dB(t)$ which is not an Itô integral because the integrand $B(1)$ is not adapted to $\{\mathcal{F}_t; t \geq 0\}$?

In [7], Itô provided his idea to define this anticipating stochastic integral. His idea is to enlarge the filtration so that the integrand $B(1)$ is adapted. Namely, let $\mathcal{G}_t = \sigma\{\mathcal{F}_t, B(1)\}$; the σ -field generated by \mathcal{F}_t and $B(1)$. By doing this, the stochastic process $B(t)$ is no longer a Brownian motion with respect to the new filtration $\{\mathcal{G}_t\}$.

However, the process $B(t)$ can still be decomposed as

$$B(t) = \left(B(t) - \int_0^t \frac{B(1) - B(u)}{1 - u} du \right) + \int_0^t \frac{B(1) - B(u)}{1 - u} du$$

which shows that $B(t)$ is a quasimartingale with respect to the filtration $\{\mathcal{G}_t\}$.

Then he finally defines $\int_0^1 B(1) dB(t)$ as a stochastic integral with respect to a quasimartingale. Then the result of this approach is

$$\int_0^1 B(1) dB(t) = B(1)^2. \quad (4.1)$$

Note that this stochastic integral fails to have zero expectation, one of the important properties of Itô integrals.

After that, there were many approaches to define stochastic integrals with non-adapted integrands. One of the notable approaches is via the white noise theory. For those who are interested in white noise theory, see the book [8] for more details.

Although the white noise approach provides an extension of stochastic integral to the non-adapted integrands, there are several disadvantages of this approach to anticipating stochastic integration:

- It requires too much complicated background from white noise theory.
- The computation of the white noise integral involving the S -transform is usually very difficult, even for very simple examples.
- There is no available characterization theorem for generalized functions to be realized as random variables in $L^p(\mathcal{S}(\mathbb{R}), \mu)$ for some $p > 1$, i.e., we do not know when a white noise integral is a random variable or when it is just a generalized function.
- It lacks probabilistic interpretation, e.g., it is unknown how to deal with convergence in probability in terms of the S -transform.

More details of an extension of Itô integrals to anticipating stochastic processes via this approach can be found in Chapter 13 of [8].

Nevertheless, in 2008, Ayed and Kuo introduced a new approach to define stochastic integrals of anticipating integrands. Their idea is not only much simpler than the white noise approach, but it also has probabilistic interpretation. We provide more details about their idea in the next section.

4.2 New Ideas of Ayed and Kuo

In this section, we describe the idea of Ayed and Kuo on how to define the new stochastic integrals. Namely, this new stochastic integral is defined for a special class of anticipating stochastic processes. Inspired by Itô's original idea to define $\int_0^1 B(1) dB(t)$, their idea came from a very simple observation. Instead of decomposing the integrator $B(t)$ as in Itô's idea, they focused on decomposing the integrands. The following are some simple examples that demonstrate the decompositions.

Example 4.1. *The anticipating stochastic process $B(1)$ can be decomposed as*

$$B(1) = (B(1) - B(t)) + B(t).$$

Example 4.2. *Consider another anticipating stochastic process $B(1)^2$. This anticipating integrand $B(1)^2$ can be decomposed as*

$$B(1)^2 = (B(1) - B(t))^2 + 2B(t)(B(1) - B(t)) + B(t)^2. \quad (4.2)$$

Example 4.3. *For $n \in \mathbb{N}$, it follows from the binomial theorem that $B(1)^n$ can be decomposed as*

$$B(1)^n = (B(1) - B(t) + B(t))^n = \sum_{k=1}^n \binom{n}{k} (B(1) - B(t))^k B(t)^{n-k}.$$

Example 4.4. *The stochastic process $e^{B(1)}$ can be written as*

$$e^{B(1)} = e^{B(1)-B(t)} e^{B(t)}. \quad (4.3)$$

By above examples, every decomposition is a linear combination of products of an adapted part (e.g., $B(t)$, $B(t)^k$, $e^{B(t)}$) and an anticipating part with a special property (e.g., $B(1) - B(t)$, $(B(1) - B(t))^k$, $e^{B(1)-B(t)}$). This special property inspired Ayed and Kuo to define a new class of stochastic processes called “instantly independent.”

Definition 4.5. *A stochastic process $\varphi(t)$, $a \leq t \leq T$, is said to be instantly independent with respect to the filtration $\{\mathcal{F}_t\}$ if $\varphi(t)$ and \mathcal{F}_t are independent for each t .*

Example 4.6. *For $0 \leq t \leq 1$, the following stochastic processes are all instantly independent with respect to $\{\mathcal{F}_t\}$: (i) $B(1) - B(t)$, (ii) $[B(1) - B(t)]^n$; $n \in \mathbb{N}$, (iii) $e^{B(1)-B(t)}$, and (iv) $\int_t^1 h(s)dB(s)$ where $h(s)$ is a deterministic function in $L^2[0, 1]$.*

A useful property of this class of stochastic processes can be seen in Lemma 4.7.

Lemma 4.7. *If a stochastic process $\varphi(t)$ is both adapted and instantly independent with respect to the filtration $\{\mathcal{F}_t\}$, then $\varphi(t)$ must be a deterministic function.*

By this lemma, one can regard this class of instantly independent stochastic processes as a counterpart to the class of adapted stochastic processes in the Itô theory.

By the above examples, we see that many anticipating stochastic processes can be decomposed into a sum of the products of Itô parts (adapted processes) and counterparts (instantly independent processes). This turns out to be a key idea of

Ayed and Kuo approach to define the new stochastic integral. Namely, to use their idea to evaluate an anticipating stochastic integral, one needs to

1. keep the filtration $\{\mathcal{F}_t\}$ and the integrator $B(t)$ as the same, but
2. decompose the integrand into a sum of the products of adapted stochastic processes and instantly independent stochastic processes.
3. evaluate each stochastic integral of a product of an adapted stochastic process and an instantly independent stochastic process.

Hence the next question we need to answer is how one can define a stochastic integral of a product of an adapted process and an instantly independent process.

Recall from Theorem 3.17 in the previous chapter that the stochastic integral of a continuous $\{\mathcal{F}_t\}$ -adapted process can be defined as the limit in probability of its Riemann-like sum evaluated at the left endpoints.

Another key idea of this new approach is motivated by Theorem 3.17. Namely, Ayed and Kuo define the stochastic integral of a stochastic process which is a product of an adapted stochastic process and an instantly independent stochastic process as the limit in probability of its Riemann sum with fixed evaluation points. More specifically, they use the left endpoints as the evaluation points for the adapted parts and evaluate the instantly independent parts at the right endpoints of the subintervals.

Definition 4.8. *For an adapted stochastic process $f(t)$ and an instantly independent stochastic process $\varphi(t)$, the new stochastic integral of $f(t)\varphi(t)$ is defined to be the limit*

$$I(f\varphi) = \int_a^b f(t)\varphi(t) dB(t) = \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1}))$$

provided that the limit exists in probability.

Remark 4.9. Note that, by the above definition, if $f(t)$ is continuous and $\varphi(t) = 1$,

$$I(f) = \int_a^b f(t) dB(t) = \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f(t_{i-1})(B(t_i) - B(t_{i-1})).$$

We see that this new integral reduces to the Itô integral according to Theorem 3.17 of the previous chapter. This is why it can be regarded as an extension of Itô integral.

In general, for a stochastic process $F(t) = \sum_{k=1}^N f_k(t)\varphi_k(t)$ where $f_k(t)$'s are adapted and $\varphi_k(t)$'s are instantly independent, its stochastic integral $\int_a^b F(t) dB(t)$ is defined as follows.

Definition 4.10. Let $F(t)$ be a stochastic process given by

$$F(t) = \sum_{k=1}^N f_k(t)\varphi_k(t),$$

where $f_k(t)$'s are adapted and $\varphi_k(t)$'s are instantly independent. Then the stochastic integral of $F(t)$ is defined to be

$$\int_a^b F(t) dB(t) = \sum_{k=1}^N \int_a^b f_k(t)\varphi_k(t) dB(t),$$

where, for each k , the integral $\int_a^b f_k(t)\varphi_k(t) dB(t)$ is defined as in Definition 4.8.

Next, let us provide some examples to show how this idea works. More examples can be found in [1] and [2]. First, we begin with the simplest case when $f(t) = 1$.

Example 4.11. We will find $\int_0^1 (B(1) - B(t)) dB(t)$.

Let $\Delta_n = \{0 = t_0 < t_1 < t_2 < \dots < t_n = 1\}$ be a partition of the interval $[0, 1]$. Here, $\varphi(t) = B(1) - B(t)$ is instantly independent. Thus, on each subinterval $[t_{i-1}, t_i]$, we take the right endpoint t_i as the evaluation point to form a Riemann like sum. So the stochastic integral $\int_0^1 (B(1) - B(t)) dB(t)$ is defined to be the

following limit in probability

$$\begin{aligned}
& \int_0^1 (B(1) - B(t)) dB(t) \\
&= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (B(1) - B(t_i))(B(t_i) - B(t_{i-1})) \\
&= B(1) \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (B(t_i) - B(t_{i-1})) - \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n B(t_i)(B(t_i) - B(t_{i-1})) \\
&= B(1)^2 - \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n \left\{ [B(t_i) - B(t_{i-1})] + B(t_{i-1}) \right\} (B(t_i) - B(t_{i-1})) \\
&= B(1)^2 - \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (B(t_i) - B(t_{i-1}))^2 - \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n B(t_{i-1})(B(t_i) - B(t_{i-1})) \\
&= B(1)^2 - 1 - \int_0^1 B(t) dB(t).
\end{aligned}$$

The last integral $\int_0^1 B(t) dB(t)$ is an Itô integral since $B(t)$ is adapted. Hence, by the decomposition in Example 4.1 and Definition 4.10, we have

$$\begin{aligned}
\int_0^1 B(1) dB(t) &= \int_0^1 (B(1) - B(t)) dB(t) + \int_0^1 B(t) dB(t) \\
&= \left[B(1)^2 - 1 - \int_0^1 B(t) dB(t) \right] + \int_0^1 B(t) dB(t) \\
&= B(1)^2 - 1.
\end{aligned}$$

Note that by using this new approach, the stochastic integral $\int_0^1 B(1) dB(t)$ is not the same as the one obtained by using Itô's idea in Equation (4.1). However, it coincides with the Hitsuda-Skorokhod integral defined via the white noise approach. (See Example 13.14 of [8].) In general, the new stochastic integrals have expectation 0 as what we have for the Itô integral in Theorem 3.3. This property is presented in Theorem 4.12 below.

Theorem 4.12. *Let $f(t)$, $a \leq t \leq b$, be adapted and let $\varphi(t)$, $a \leq t \leq b$, be instantly independent. Then the new stochastic integral of $f(t)$ and $\varphi(t)$ has zero expectation.*

That is,

$$E(I(f\varphi)) = E\left(\int_a^b f(t)\varphi(t) dB(t)\right) = 0.$$

The proof of this theorem can be seen in [1].

Below we provide more examples of direct computations of the new integrals.

Example 4.13. We will evaluate the stochastic integral $\int_0^t B(s)(B(1)-B(s)) dB(s)$ for $0 \leq t \leq 1$. For $0 \leq t \leq 1$, $f(t) = B(t)$ is adapted and $\varphi(t) = B(1) - B(t)$ is instantly independent. Let $\Delta_n = \{0 = s_0 < s_1 < s_2 < \dots < s_n = t\}$ be a partition of the interval $[0, t]$. By definition, the stochastic integral $\int_0^t B(s)(B(1) - B(s))dB(s)$ is the limit in probability of the following summation

$$\begin{aligned} & \sum_{i=1}^n B(s_{i-1})(B(1) - B(s_i))(B(s_i) - B(s_{i-1})) \\ &= \sum_{i=1}^n B(s_{i-1})B(1)(B(s_i) - B(s_{i-1})) - \sum_{i=1}^n B(s_{i-1})B(s_i)(B(s_i) - B(s_{i-1})) \\ &= B(1) \sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1})) \\ & \quad - \sum_{i=1}^n B(s_{i-1})\{B(s_i) - B(s_{i-1}) + B(s_{i-1})\}(B(s_i) - B(s_{i-1})) \\ &= B(1) \sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1})) \\ & \quad - \sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1}))^2 - \sum_{i=1}^n B(s_{i-1})^2(B(s_i) - B(s_{i-1})). \end{aligned}$$

Note that, as $\|\Delta_n\| \rightarrow 0$, the above three summations converge in probability to $\int_0^t B(s) dB(s)$, $\int_0^t B(s) ds$ and $\int_0^t B(s)^2 dB(s)$, respectively. Therefore,

$$\int_0^t B(s)(B(1)-B(s))dB(s) = B(1) \int_0^t B(s) dB(s) - \int_0^t B(s) ds - \int_0^t B(s)^2 dB(s). \quad (4.4)$$

Since both $\int_0^t B(s) dB(s)$ and $\int_0^t B(s)^2 dB(s)$ are Itô integrals, we can apply the regular Itô's formula (Theorem 3.20) to evaluate these two integrals. (See Equation

(3.5) in Example 3.22 and Equation (3.6) in Example 3.26.) So, for $0 \leq t \leq 1$, we have

$$\begin{aligned}
& \int_0^t B(s)(B(1) - B(s)) dB(s) \\
&= B(1) \left[\frac{1}{2} (B(t)^2 - t) \right] - \int_0^t B(s) ds - \left[\frac{B(t)^3}{3} - \int_0^t B(s) ds \right] \\
&= \frac{1}{2} B(1)(B(t)^2 - t) - \frac{B(t)^3}{3}.
\end{aligned}$$

Note that when putting $t = 1$, this stochastic integral becomes

$$\int_0^1 B(s)(B(1) - B(s)) dB(s) = \frac{1}{2} B(1)(B(1)^2 - 1) - \frac{B(1)^3}{3} = \frac{1}{6} [B(1)^3 - 3B(1)]$$

which still coincides with the Hitsuda-Skorokhod integral defined in the white noise approach. (See Example 2.5 in Chapter 2 of [14].)

Example 4.14. We will evaluate the integral $\int_0^t (B(1) - B(s))^2 dB(s)$ for $0 \leq t \leq 1$. Let $\Delta_n = \{0 = s_0 < s_1 < s_2 < \dots < s_n = t\}$ be a partition of the interval $[0, t]$. For convenience, let $\Delta B_i := B(s_i) - B(s_{i-1})$. Then, by definition, the integral $\int_0^t (B(1) - B(s))^2 dB(s)$ is the limit in probability of the summation

$$\begin{aligned}
& \sum_{i=1}^n \left\{ (B(1) - B(s_i))^2 \right\} (B(s_i) - B(s_{i-1})) \\
&= \sum_{i=1}^n \{ B(1)^2 - 2B(1)B(s_i) + B(s_i)^2 \} (B(s_i) - B(s_{i-1})) \\
&= B(1)^2 \sum_{i=1}^n (B(s_i) - B(s_{i-1})) - 2B(1) \sum_{i=1}^n B(s_i) \Delta B_i + \sum_{i=1}^n B(s_i)^2 \Delta B_i \\
&= B(1)^2 [B(t) - B(0)] - 2B(1) \sum_{i=1}^n \{ B(s_i) - B(s_{i-1}) + B(s_{i-1}) \} \Delta B_i \\
&\quad + \sum_{i=1}^n \{ B(s_i) - B(s_{i-1}) + B(s_{i-1}) \}^2 \Delta B_i
\end{aligned}$$

$$\begin{aligned}
&= B(1)^2 B(t) - 2B(1) \sum_{i=1}^n (B(s_i) - B(s_{i-1}))^2 - 2B(1) \sum_{i=1}^n B(s_{i-1}) \Delta B_i \\
&\quad + \sum_{i=1}^n \left\{ (B(s_i) - B(s_{i-1}))^2 + 2B(s_{i-1})(B(s_i) - B(s_{i-1})) + B(s_{i-1})^2 \right\} \Delta B_i \\
&= B(1)^2 B(t) - 2B(1) \sum_{i=1}^n (B(s_i) - B(s_{i-1}))^2 - 2B(1) \sum_{i=1}^n B(s_{i-1}) \Delta B_i \\
&\quad + \sum_{i=1}^n (\Delta B_i)^3 + 2 \sum_{i=1}^n B(s_{i-1}) (\Delta B_i)^2 + \sum_{i=1}^n B(s_{i-1})^2 \Delta B_i.
\end{aligned}$$

Note that, as $\|\Delta_n\| \rightarrow 0$, the above five summations converge in probability to t , $\int_0^t B(s) dB(s)$, 0 , $\int_0^t B(s) ds$ and $\int_0^t B(s)^2 dB(s)$, respectively. So, for $0 \leq t \leq 1$, we have

$$\begin{aligned}
\int_0^t (B(1) - B(s))^2 dB(s) &= B(1)^2 B(t) - 2B(1)t - 2B(1) \int_0^t B(s) dB(s) \\
&\quad + 2 \int_0^t B(s) ds + \int_0^t B(s)^2 dB(s). \tag{4.5}
\end{aligned}$$

Also, by Equation (3.5) and Equation (3.6), Equation (4.5) becomes

$$\begin{aligned}
\int_0^t (B(1) - B(s))^2 dB(s) &= B(1)^2 B(t) - 2B(1)t - 2B(1) \left[\frac{1}{2} (B(t)^2 - t) \right] \\
&\quad + 2 \int_0^t B(s) ds + \left[\frac{B(t)^3}{3} - \int_0^t B(s) ds \right] \\
&= B(1)^2 B(t) - B(1)B(t)^2 + \int_0^t B(s) ds + \frac{B(t)^3}{3}.
\end{aligned}$$

Example 4.15. We will evaluate the stochastic integral $\int_0^t B(1)^2 dB(s)$ for $t \geq 0$. First consider when $0 \leq t \leq 1$, we decompose the integrand $B(1)^2$ as in Equation (4.2):

$$B(1)^2 = (B(1) - B(t))^2 + 2B(t)(B(1) - B(t)) + B(t)^2.$$

Then, by Definition 4.10, we have

$$\begin{aligned}
\int_0^t B(1)^2 dB(s) &= \int_0^t (B(1) - B(s))^2 dB(s) \\
&\quad + 2 \int_0^t B(s)(B(1) - B(s)) dB(s) + \int_0^t B(s)^2 dB(s).
\end{aligned}$$

From Equation (4.4) and Equation (4.5), we get

$$\begin{aligned} & \int_0^t B(1)^2 dB(s) \\ &= \left[B(1)^2 B(t) - 2B(1)t - 2B(1) \int_0^t B(s) dB(s) + 2 \int_0^t B(s) ds + \int_0^t B(s)^2 dB(s) \right] \\ & \quad + 2 \left[B(1) \int_0^t B(s) dB(s) - \int_0^t B(s) ds - \int_0^t B(s)^2 dB(s) \right] + \int_0^t B(s)^2 dB(s). \end{aligned}$$

Therefore,

$$\int_0^t B(1)^2 dB(s) = B(1)^2 B(t) - 2B(1)t, \quad 0 \leq t \leq 1.$$

Next, consider when $t > 1$. Note that $B(1)^2$ is \mathcal{F}_t -measurable for all $t \geq 1$. Then for each $t > 1$, $\int_1^t B(1)^2 dB(s)$ is an Itô integral. So

$$\int_1^t B(1)^2 dB(s) = B(1)^2 \int_1^t dB(s) = B(1)^2 (B(t) - B(1)).$$

Therefore, when $t > 1$, we have

$$\begin{aligned} \int_0^t B(1)^2 dB(s) &= \int_0^1 B(1)^2 dB(s) + \int_1^t B(1)^2 dB(s) \\ &= [B(1)^3 - 2B(1)] + B(1)^2 (B(t) - B(1)) \\ &= B(1)^2 B(t) - 2B(1). \end{aligned}$$

Thus

$$\int_0^t B(1)^2 dB(s) = \begin{cases} B(1)^2 B(t) - 2B(1)t & \text{if } 0 \leq t \leq 1; \\ B(1)^2 B(t) - 2B(1) & \text{if } t > 1. \end{cases}$$

Example 4.16. Consider the integral $\int_0^t e^{B(2)} dB(s)$ for $t \geq 0$.

When $0 \leq t \leq 2$, $e^{B(2)}$ can be rewritten as in Equation (4.3)

$$e^{B(2)} = e^{B(2)-B(t)} e^{B(t)}.$$

Let $\Delta_n = \{0 = s_0 < s_1 < s_2 < \dots < s_n = t\}$ be a partition of the interval $[0, t]$. Then, by definition, the integral $\int_0^t e^{B(2)} dB(s)$ is the limit in probability of the following summation

$$\begin{aligned}
& \sum_{i=1}^n e^{B(s_{i-1})} e^{B(2)-B(s_i)} (B(s_i) - B(s_{i-1})) \\
&= e^{B(2)} \sum_{i=1}^n e^{B(s_{i-1})} e^{-B(s_i)} (B(s_i) - B(s_{i-1})) \\
&= e^{B(2)} \sum_{i=1}^n e^{-(B(s_i)-B(s_{i-1}))} (B(s_i) - B(s_{i-1})) \\
&= e^{B(2)} \sum_{i=1}^n \left\{ 1 - (B(s_i) - B(s_{i-1})) + \frac{1}{2} (B(s_i) - B(s_{i-1}))^2 \right. \\
&\quad \left. - o((B(s_i) - B(s_{i-1}))^2) \right\} (B(s_i) - B(s_{i-1})) \\
&= e^{B(2)} \left\{ \sum_{i=1}^n (B(s_i) - B(s_{i-1})) - \sum_{i=1}^n (B(s_i) - B(s_{i-1}))^2 \right. \\
&\quad \left. + \frac{1}{2} \sum_{i=1}^n (B(s_i) - B(s_{i-1}))^3 - \frac{1}{6} \sum_{i=1}^n (B(s_i) - B(s_{i-1}))^4 + \dots \right\} \\
&= e^{B(2)} \left\{ B(t) - \sum_{i=1}^n (B(s_i) - B(s_{i-1}))^2 + \frac{1}{2} \sum_{i=1}^n (B(s_i) - B(s_{i-1}))^3 + \dots \right\}.
\end{aligned}$$

Note that, as $\|\Delta_n\| \rightarrow 0$, the first summation converges to t , and all other summations converge to 0 in probability. Hence we have

$$\int_0^t e^{B(2)} dB(s) = e^{B(2)} (B(t) - t), \quad 0 \leq t \leq 2.$$

Next, consider when $t > 2$. Since $e^{B(2)}$ is \mathcal{F}_t -measurable for all $t \geq 2$. Then for each $t > 2$, $\int_2^t e^{B(2)} dB(s)$ is an Itô integral and so

$$\int_2^t e^{B(2)} dB(s) = e^{B(2)} \int_2^t dB(s) = e^{B(2)} (B(t) - B(2)).$$

Therefore, for $t > 2$, we have

$$\begin{aligned}\int_0^t e^{B(2)} dB(s) &= \int_0^2 e^{B(2)} dB(s) + \int_2^t e^{B(2)} dB(s) \\ &= e^{B(2)}(B(2) - 2) + e^{B(2)}(B(t) - B(2)) \\ &= e^{B(2)}(B(t) - 2).\end{aligned}$$

That is,

$$\int_0^t e^{B(2)} dB(s) = \begin{cases} e^{B(2)}(B(t) - t) & \text{if } 0 \leq t \leq 2; \\ e^{B(2)}(B(t) - 2) & \text{if } t > 2. \end{cases}$$

4.3 Simple Itô's Formula for the New Stochastic Integral

In [1], Ayed and Kuo provide a version of Itô's formula for the new integral. We give this in the following theorem.

Theorem 4.17. *Let $f(x)$ and $\varphi(x)$ be C^2 -functions and $\theta(x, y) = f(x)\varphi(y - x)$.*

Then the following equality holds for $a \leq t \leq T$,

$$\begin{aligned}\theta(B(t), B(T)) &= \theta(B(a), B(T)) + \int_a^t \frac{\partial \theta}{\partial x}(B(s), B(T)) dB(s) \\ &\quad + \int_a^t \left\{ \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(B(s), B(T)) + \frac{\partial^2 \theta}{\partial x \partial y}(B(s), B(T)) \right\} ds. \quad (4.6)\end{aligned}$$

Observe that the last integral of Equation (4.6) contains both the Itô correction term $\frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(B(s), B(T))$ for the Itô part and the correction term $\frac{\partial^2 \theta}{\partial x \partial y}(B(s), B(T))$ for the counterpart.

The following is the corollary of Itô's formula in the case of $t > T$.

Corollary 4.18. *Let $f(x)$ and $\varphi(x)$ be C^2 -functions and $\theta(x, y) = f(x)\varphi(y - x)$.*

Then the following equality holds for $t > T$,

$$\begin{aligned}\theta(B(t), B(T)) &= \theta(B(a), B(T)) + \int_a^t \frac{\partial \theta}{\partial x}(B(s), B(T)) dB(s) \\ &\quad + \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(B(s), B(T)) ds + \int_a^T \frac{\partial^2 \theta}{\partial x \partial y}(B(s), B(T)) ds.\end{aligned}$$

Note that the correction term $\frac{\partial^2 \theta}{\partial x \partial y}(B(s), B(T))$ due to the counterpart holds only up to T . The proof of Theorem 4.17 and Corollary 4.18 can be found in [1].

The following example shows how one can use the above Itô's formula to evaluate the new stochastic integral.

Example 4.19. We will evaluate the integral $\int_0^t e^{B(2)} dB(s)$ for $t \geq 0$ by using the above Itô formula. Let $\theta(x, y) = xe^y$ so that we have $\frac{\partial \theta}{\partial x}(x, y) = e^y$. Then we have

$$\frac{\partial \theta}{\partial x}(x, y) = e^y, \quad \frac{\partial^2 \theta}{\partial x^2}(x, y) = 0, \quad \frac{\partial^2 \theta}{\partial x \partial y}(x, y) = e^y.$$

Note that $\theta(x, y) = xe^y = xe^x e^{y-x} = f(x)\varphi(y-x)$ where $f(x) = xe^x$ and $\varphi(x) = e^x$ are C^2 -functions. Therefore, by Itô's formula in Theorem 4.17, we have that for $0 \leq t \leq 2$,

$$\begin{aligned} B(t)e^{B(2)} &= B(0)e^{B(2)} + \int_0^t e^{B(2)} dB(s) + \int_0^t \left\{ \frac{1}{2}(0) + e^{B(2)} \right\} ds \\ &= 0 + \int_0^t e^{B(2)} dB(s) + e^{B(2)} \int_0^t ds. \end{aligned}$$

which implies

$$\int_0^t e^{B(2)} dB(s) = B(t)e^{B(2)} - e^{B(2)}t = e^{B(2)}(B(t) - t), \quad 0 \leq t \leq 2.$$

For $t > 2$, we can apply the Corollary 4.18, that is

$$\begin{aligned} B(t)e^{B(2)} &= B(0)e^{B(2)} + \int_0^t e^{B(2)} dB(s) + \int_0^t \frac{1}{2}(0) ds + \int_0^2 e^{B(2)} ds \\ &= 0 + \int_0^t e^{B(2)} dB(s) + e^{B(2)} \int_0^2 ds. \end{aligned}$$

So

$$\int_0^t e^{B(2)} dB(s) = B(t)e^{B(2)} - 2e^{B(2)} = e^{B(2)}(B(t) - 2), \quad t \geq 2.$$

Therefore,

$$\int_0^t e^{B(2)} dB(s) = \begin{cases} e^{B(2)}(B(t) - t) & \text{if } 0 \leq t \leq 2; \\ e^{B(2)}(B(t) - 2) & \text{if } t > 2. \end{cases}$$

Observe that we obtain the same result as by computing the integral directly from the definition in Example 4.16.

Chapter 5

Some Properties of the New Stochastic Integral

In this chapter, we present the main results of this dissertation. In fact, these results are collected from my joint works with Kuo and Szozda during my last 2 years of study. These selected results are the description of a new class of stochastic processes called “near-martingales,” the Itô isometry of a special case of stochastic integrals of instantly stochastic processes, some formulas for the computation of the new integrals and some generalized Itô formulas for the new integrals which are much easier to use than the one given by Ayed and Kuo in Theorem 4.17.

5.1 Near-Martingales

In Itô theory, one of the well-known and widely used property of a stochastic process is the “martingale property.” Recall from Definition 2.21 that a martingale with respect to the filtration $\{\mathcal{F}_t\}$ is an $\{\mathcal{F}_t\}$ -adapted stochastic process with $E|X_t| < \infty$ and $E(X_t|\mathcal{F}_s) = X_s$ for any $s \leq t$. So we see that this property makes sense only for adapted stochastic processes.

In the new theory, we use mainly instantly independent stochastic processes and products of an adapted process and an instantly independent process which are not adapted in general. Therefore, it is natural to ask if we shall have a similar property to that martingale property in this new theory which still makes sense for non-adapted stochastic processes.

Obviously, in order to obtain such similar property, the first assumption that X_t is $\{\mathcal{F}_t\}$ -adapted must be removed and consider only the assumption $E(X_t|\mathcal{F}_s) = X_s$ for any $s \leq t$.

Suppose $E(X_t|\mathcal{F}_s) = X_s$ for any $s \leq t$. Note that the condition (1) in the definition of the conditional expectations (Definition 2.14) implies the \mathcal{F}_s -measurability of X_s for all s . This means that the $\{\mathcal{F}_t\}$ -adaptedness of X_t also follows from the condition (2) $E(X_t|\mathcal{F}_s) = X_s$ of the Definition 2.21.

However, this condition $E(X_t|\mathcal{F}_s) = X_s$ for any $s \leq t$ can be rewritten as

$$E\{X_t - X_s|\mathcal{F}_s\} = 0, \quad \forall s \leq t. \quad (5.1)$$

Observe that, the expression in Equation (5.1) still makes sense for non-adapted stochastic processes including instantly independent stochastic processes in the new theory. Therefore, this property in Equation (5.1) becomes our new concept called “near-martingale.”

Remark 5.1. *Note that condition $E(X_t|\mathcal{F}_s) = X_s$ for any $s \leq t$ implies Equation (5.1), but the converse is not true in general.*

Definition 5.2. *Let X_t be a stochastic process with $E|X_t| < \infty$ for all t . Then X_t is a near-martingale with respect to a filtration $\{\mathcal{F}_t\}$ if*

$$E\{X_t - X_s|\mathcal{F}_s\} = 0, \quad \forall s \leq t. \quad (5.2)$$

Remark 5.3. *By Remark 5.1, every martingale is a near-martingale.*

Remark 5.4. *If X_t is a near-martingale with respect to the filtration $\{\mathcal{F}_t\}$ and is adapted to $\{\mathcal{F}_t\}$, then it is a martingale with respect to $\{\mathcal{F}_t\}$. This is the reason why we call this class of stochastic processes near-martingales.*

Next, we prove a theorem providing the necessary and sufficient conditions for an instantly independent process to be a near martingale.

Theorem 5.5. *Suppose $\varphi(t)$ is instantly independent with respect to a filtration $\{\mathcal{F}_t\}$ and $E|\varphi(t)| < \infty$ for all t . Then $\varphi(t)$ is a near-martingale with respect to $\{\mathcal{F}_t\}$ if and only if $E[\varphi(t)] = E[\varphi(s)]$ for all s and t , i.e., $E[\varphi(t)]$ is constant.*

Proof. Let $s \leq t$. By the tower property of conditional expectation, we have

$$\begin{aligned} E\{\varphi(t) - \varphi(s)|\mathcal{F}_s\} &= E\{\varphi(t)|\mathcal{F}_s\} - E\{\varphi(s)|\mathcal{F}_s\} \\ &= E\{E[\varphi(t)|\mathcal{F}_t]|\mathcal{F}_s\} - E\{\varphi(s)|\mathcal{F}_s\}. \end{aligned} \quad (5.3)$$

Since $\varphi(t)$ is instantly independent with respect to $\{\mathcal{F}_t\}$, Equation (5.3) becomes

$$\begin{aligned} E\{\varphi(t) - \varphi(s)|\mathcal{F}_s\} &= E\{E[\varphi(t)]|\mathcal{F}_s\} - E[\varphi(s)] \\ &= E[\varphi(t)] - E[\varphi(s)]. \end{aligned}$$

Therefore, we conclude that $\varphi(t)$ is a near-martingale with respect to $\{\mathcal{F}_t\}$ if and only if $E[\varphi(t)] = E[\varphi(s)]$ for all s and t . \square

The following are some simple examples of stochastic processes that are near-martingales and some stochastic processes that are not near-martingales with respect to the filtration $\{\mathcal{F}_t; a \leq t \leq T\}$ where $\mathcal{F}_t = \sigma\{B(s); a \leq s \leq t\}$.

Example 5.6. For each $k \in \mathbb{N}$, the stochastic process

$$X_t = (B(T) - B(t))^{2k+1}, \quad a \leq t \leq T,$$

is instantly independent to $\{\mathcal{F}_t; a \leq t \leq T\}$ and $E\left[(B(T) - B(t))^{2k+1}\right] = 0$.

Therefore, by Theorem 5.5, $X_t = (B(T) - B(t))^{2k+1}$ is a near-martingale with respect to $\{\mathcal{F}_t\}$.

Example 5.7. On the other hand, for $a \leq t \leq T$, $E\left[(B(T) - B(t))^2\right] = T - t$ which is not a constant. Thus it follows from Theorem 5.5 that the instantly independent stochastic process $Y_t = (B(T) - B(t))^2$ is not a near-martingale with respect to the filtration $\{\mathcal{F}_t\}$.

However, for $a \leq t \leq T$, the stochastic processes $\tilde{Y}_t = (B(T) - B(t))^2 - (T - t)$ and $\tilde{\tilde{Y}}_t = (B(T) - B(t))^2 + t$ are near-martingales with respect to $\{\mathcal{F}_t\}$ since their expectations are constants not depending on t .

Example 5.8. *The instantly independent stochastic process $Z_t = e^{(B(T)-B(t))}$ is not a near-martingale with respect to $\{\mathcal{F}_t\}$ since $E[e^{(B(T)-B(t))}] = e^{\frac{1}{2}(T-t)}$ is not a constant, but the stochastic process $\tilde{Z}_t = e^{(B(T)-B(t))-\frac{1}{2}(T-t)}$ is a near-martingale with respect to $\{\mathcal{F}_t\}$ (since $E[\tilde{Z}_t] = 1$).*

Next, consider the case of a product of an adapted process and an instantly independent process. We also have a theorem giving sufficient conditions for this product of stochastic processes to be a near-martingale.

Theorem 5.9. *Let $\{\mathcal{F}_t\}$ be a filtration. Assume that $f(t)$ and $\varphi(t)$ are stochastic processes such that*

1. $f(t)$ is a martingale with respect to $\{\mathcal{F}_t\}$;
2. $\varphi(t)$ is instantly independent with respect to $\{\mathcal{F}_t\}$ and $E[\varphi(t)]$ is constant;
3. $E|f(t)\varphi(t)| < \infty$ for all t .

Then $\theta(t) = f(t)\varphi(t)$ is a near-martingale with respect to $\{\mathcal{F}_t\}$.

Remark 5.10. *Observe that*

1. *by Theorem 5.5, condition (2) implies that $\varphi(t)$ is a near-martingale,*
2. *Theorem 5.9 fails to be true if we just assume that $\varphi(t)$ is a near-martingale without the instantaneous independence of $\varphi(t)$.*

Proof. Let $s \leq t$. Since $f(s)$ is measurable with respect to \mathcal{F}_s , we have

$$\begin{aligned} E[\theta(t) - \theta(s)|\mathcal{F}_s] &= E[f(t)\varphi(t) - f(s)\varphi(s)|\mathcal{F}_s] \\ &= E[f(t)\varphi(t)|\mathcal{F}_s] - E[f(s)\varphi(s)|\mathcal{F}_s] \\ &= E[f(t)\varphi(t)|\mathcal{F}_s] - f(s)E[\varphi(s)|\mathcal{F}_s]. \end{aligned}$$

Since $\mathcal{F}_s \subset \mathcal{F}_t$, we have $E[f(t)\varphi(t)|\mathcal{F}_s] = E[E[f(t)\varphi(t)|\mathcal{F}_t]|\mathcal{F}_s]$. Thus we have

$$E[\theta(t) - \theta(s)|\mathcal{F}_s] = E\left[E[f(t)\varphi(t)|\mathcal{F}_t]|\mathcal{F}_s\right] - f(s)E[\varphi(s)] \quad (5.4)$$

Also, by the \mathcal{F}_t -measurability of $f(t)$ and the instantaneous independence of $\varphi(t)$, Equation (5.4) becomes

$$\begin{aligned} E[\theta(t) - \theta(s)|\mathcal{F}_s] &= E\left[f(t)E[\varphi(t)|\mathcal{F}_t]|\mathcal{F}_s\right] - f(s)E[\varphi(s)] \\ &= E\left[f(t)E[\varphi(t)|\mathcal{F}_s]\right] - f(s)E[\varphi(s)] \\ &= E[\varphi(t)]E[f(t)|\mathcal{F}_s] - f(s)E[\varphi(s)]. \end{aligned}$$

Note that $f(t)$ is a martingale with respect to $\{\mathcal{F}_t\}$ and $E[\varphi(t)]$ is constant. Therefore,

$$E[\theta(t) - \theta(s)|\mathcal{F}_s] = E[\varphi(t)]f(s) - f(s)E[\varphi(s)] = 0.$$

Hence the stochastic process $\theta(t) = f(t)\varphi(t)$ is a near-martingale with respect to the filtration $\{\mathcal{F}_t\}$. \square

Example 5.11. *The stochastic process $B(t)(B(T) - B(t))$, $a \leq t \leq T$, is a near-martingale with respect to $\{\mathcal{F}_t\}$.*

Pardoux and Protter (in [15]) have defined backward and forward filtrations as follows.

Definition 5.12. 1. *We will say that a family $\{\mathcal{G}_t\}$ of sub- σ -fields is a forward filtration if $\mathcal{G}_s \subseteq \mathcal{G}_t$ for $0 \leq s \leq t$;*

2. *We will say that a family $\{\mathcal{G}^{(t)}\}$ of sub- σ -fields is a backward filtration if $\mathcal{G}^{(s)} \supseteq \mathcal{G}^{(t)}$ for $0 \leq s \leq t$.*

Remark 5.13. *The usual and widely used definition of the filtration in Definition 2.18 is actually the forward filtration in this sense.*

By dealing with the backward filtrations, we can also define a near-martingale with respect to a backward filtration as one can see in Definition 5.14.

Definition 5.14. Let $E|X_t| < \infty$ for all t . We say that X_t is a near-martingale with respect to a **backward filtration** $\{\mathcal{F}^{(t)}\}$ if

$$E\{X_t - X_s \mid \mathcal{F}^{(t)}\} = 0, \quad \forall s \leq t. \quad (5.5)$$

With the similar arguments in the proof of Theorem 5.5 and Theorem 5.9, we can also prove the following two theorems.

Theorem 5.15. Suppose $\varphi(t)$ is instantly independent with respect to a **backward filtration** $\{\mathcal{F}^{(t)}\}$ and $E|\varphi(t)| < \infty$ for all t . Then $\varphi(t)$ is a near-martingale with respect to the backward filtration $\{\mathcal{F}^{(t)}\}$ if and only if $E[\varphi(t)] = E[\varphi(s)]$ for all s and t , i.e., $E[\varphi(t)]$ is constant.

Proof. Let $s \leq t$. Consider

$$E\{\varphi(t) - \varphi(s) \mid \mathcal{F}^{(t)}\} = E\{\varphi(t) \mid \mathcal{F}^{(t)}\} - E\{\varphi(s) \mid \mathcal{F}^{(t)}\}. \quad (5.6)$$

Since $\mathcal{F}^{(t)} \subset \mathcal{F}^{(s)}$, then by the tower property, Equation (5.6) becomes

$$\begin{aligned} E\{\varphi(t) - \varphi(s) \mid \mathcal{F}^{(t)}\} &= E\{\varphi(t) \mid \mathcal{F}^{(t)}\} - E\{E[\varphi(s) \mid \mathcal{F}^{(s)}] \mid \mathcal{F}^{(t)}\} \\ &= E[\varphi(t)] - E\{E[\varphi(s)] \mid \mathcal{F}^{(t)}\} \\ &= E[\varphi(t)] - E[\varphi(s)]. \end{aligned}$$

Therefore, we conclude that $\varphi(t)$ is a near-martingale with respect to the backward filtration $\{\mathcal{F}^{(t)}\}$ if and only if $E[\varphi(t)] = E[\varphi(s)]$ for all s and t . \square

Theorem 5.16. Let $\{\mathcal{F}^{(t)}\}$ be a backward filtration. Assume that

1. $f(t)$ is a martingale with respect to $\{\mathcal{F}^{(t)}\}$;

2. $\varphi(t)$ is instantly independent with respect to $\{\mathcal{F}^{(t)}\}$ and $E[\varphi(t)]$ is constant;
3. $E|f(t)\varphi(t)| < \infty$ for all t .

Then $\theta(t) = f(t)\varphi(t)$ is a near-martingale with respect to $\{\mathcal{F}^{(t)}\}$.

Remark 5.17. The condition (1) $f(t)$ is a martingale with respect to the backward filtration $\{\mathcal{F}^{(t)}\}$ means that $f(t)$ is $\mathcal{F}^{(t)}$ -measurable for all t and $E[f(s)|\mathcal{F}^{(t)}] = f(t)$ for any $s \leq t$.

Proof. Let $s \leq t$. Since $f(t)$ is measurable with respect to \mathcal{F}_t , we have

$$\begin{aligned} E[\theta(t) - \theta(s)|\mathcal{F}^{(t)}] &= E[f(t)\varphi(t) - f(s)\varphi(s)|\mathcal{F}^{(t)}] \\ &= f(t)E[\varphi(t)|\mathcal{F}^{(t)}] - E[f(s)\varphi(s)|\mathcal{F}^{(t)}]. \end{aligned} \quad (5.7)$$

Since $\mathcal{F}^{(t)} \subset \mathcal{F}^{(s)}$, we have $E[f(s)\varphi(s)|\mathcal{F}^{(t)}] = E[E[f(s)\varphi(s)|\mathcal{F}^{(s)}]|\mathcal{F}^{(t)}]$. So Equation (5.7) becomes

$$\begin{aligned} E[\theta(t) - \theta(s)|\mathcal{F}^{(t)}] &= f(t)E[\varphi(t)|\mathcal{F}^{(t)}] - E[E[f(s)\varphi(s)|\mathcal{F}^{(s)}]|\mathcal{F}^{(t)}] \\ &= f(t)E[\varphi(t)] - E[f(s)E[\varphi(s)|\mathcal{F}^{(s)}]|\mathcal{F}^{(t)}] \\ &= f(t)E[\varphi(t)] - E[f(s)E[\varphi(s)]|\mathcal{F}^{(t)}] \\ &= f(t)E[\varphi(t)] - E[\varphi(s)]E[f(s)|\mathcal{F}^{(t)}]. \end{aligned}$$

Note that $f(t)$ is a martingale with respect to the backward filtration $\{\mathcal{F}^{(t)}\}$ and $E[\varphi(t)]$ is constant. Therefore,

$$E[\theta(t) - \theta(s)|\mathcal{F}_s] = f(t)E[\varphi(t)] - E[\varphi(s)]f(t) = 0.$$

Hence the stochastic process $\theta(t) = f(t)\varphi(t)$ is a near-martingale with respect to the backward filtration $\{\mathcal{F}^{(t)}\}$. □

Now, another theorem in Itô theory related to the martingale property that we should consider is the theorem about the martingale property of the associated

stochastic process $X_t = \int_a^t f(s) dB(s)$ which is defined for $f \in L_{\text{ad}}^2(\Omega \times [a, b])$. (See Theorem 3.7.)

In the new theory, we can also define these stochastic processes associated with the new integrals. Namely, for continuous functions f and φ , we define X_t to be the stochastic process

$$X_t = \int_a^t f(B(s))\varphi(B(T) - B(s)) dB(s), \quad a \leq t \leq T, \quad (5.8)$$

and we will prove, under some conditions, that X_t is a near-martingale with respect to the (forward) filtration $\{\mathcal{F}_t\}$ where $\mathcal{F}_t = \sigma\{B(s); a \leq s \leq t\}$. (See Theorem 5.18)

Moreover, we introduce another associated stochastic process defined by

$$Y^{(t)} = \int_t^T f(B(s))\varphi(B(T) - B(s)) dB(s), \quad a \leq t \leq T, \quad (5.9)$$

and, under the same conditions, we can also show that the stochastic process $Y^{(t)}$ is a near-martingale with respect to the same filtration.

For $a \leq t \leq T$, let $\mathcal{F}_t = \sigma\{B(s); a \leq s \leq t\}$. The next two theorems establish the near martingale property (with respect to the filtration $\{\mathcal{F}_t\}$) of X_t and $Y^{(t)}$.

Theorem 5.18. *Let $f(x)$ and $\varphi(x)$ be continuous functions. Let*

$$X_t = \int_a^t f(B(s))\varphi(B(T) - B(s)) dB(s), \quad a \leq t \leq T,$$

and assume that $E|X_t| < \infty$ for all $a \leq t \leq T$. Then X_t is a near-martingale with respect to the forward filtration $\{\mathcal{F}_t\}$.

Proof. Let $a \leq s \leq t \leq T$. Consider

$$X_t - X_s = \int_s^t f(B(s))\varphi(B(T) - B(s)) dB(s).$$

Let $\Delta_n = \{s = t_0 < t_1 < \dots < t_{n-1} < t_n = t\}$ be a partition of the interval $[s, t]$, and for convenience, write $\Delta B_i = B(t_i) - B(t_{i-1})$ for each $1 \leq i \leq n$. Thus

$$\begin{aligned}
E[X_t - X_s | \mathcal{F}_s] &= E \left[\int_s^t f(B(s)) \varphi(B(T) - B(s)) dB(s) \mid \mathcal{F}_s \right] \\
&= E \left[\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f(B(t_{i-1})) \varphi(B(T) - B(t_i)) \Delta B_i \mid \mathcal{F}_s \right] \\
&= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n E \left[f(B(t_{i-1})) \varphi(B(T) - B(t_i)) \Delta B_i \mid \mathcal{F}_s \right]. \quad (5.10)
\end{aligned}$$

To show that $E[X_t - X_s | \mathcal{F}_s] = 0$, it suffices to show that $E \left[f(B(t_{i-1})) \varphi(B(T) - B(t_i)) \Delta B_i \mid \mathcal{F}_s \right] = 0$ for each $1 \leq i \leq n$. By the fact that $f(B(t_{i-1}))$ and ΔB_i are \mathcal{F}_{t_i} -measurable and $\varphi(B(T) - B(t_i))$ is independent of \mathcal{F}_{t_i} , we have

$$\begin{aligned}
&E \left[f(B(t_{i-1})) \varphi(B(T) - B(t_i)) \Delta B_i \mid \mathcal{F}_s \right] \\
&= E \left[E \left[f(B(t_{i-1})) \varphi(B(T) - B(t_i)) \Delta B_i \mid \mathcal{F}_{t_i} \right] \mid \mathcal{F}_s \right] \\
&= E \left[f(B(t_{i-1})) \Delta B_i E \left[\varphi(B(T) - B(t_i)) \mid \mathcal{F}_{t_i} \right] \mid \mathcal{F}_s \right] \\
&= E \left[f(B(t_{i-1})) \Delta B_i E \left[\varphi(B(T) - B(t_i)) \right] \mid \mathcal{F}_s \right] \\
&= E \left[\varphi(B(T) - B(t_i)) \right] E \left[f(B(t_{i-1})) \Delta B_i \mid \mathcal{F}_s \right].
\end{aligned}$$

We apply the tower property of conditional expectations one more time. Also, by the fact that $f(B(t_{i-1}))$ is \mathcal{F}_{t_i} -measurable and ΔB_i is independent of $\mathcal{F}_{t_{i-1}}$, we have that for each $1 \leq i \leq n$,

$$\begin{aligned}
&E \left[f(B(t_{i-1})) \varphi(B(T) - B(t_i)) \Delta B_i \mid \mathcal{F}_s \right] \\
&= E \left[\varphi(B(T) - B(t_i)) \right] E \left[E \left[f(B(t_{i-1})) \Delta B_i \mid \mathcal{F}_{t_{i-1}} \right] \mid \mathcal{F}_s \right] \\
&= E \left[\varphi(B(T) - B(t_i)) \right] E \left[f(B(t_{i-1})) E \left[\Delta B_i \mid \mathcal{F}_{t_{i-1}} \right] \mid \mathcal{F}_s \right] \\
&= E \left[\varphi(B(T) - B(t_i)) \right] E \left[f(B(t_{i-1})) E \left[\Delta B_i \right] \mid \mathcal{F}_s \right] \\
&= E \left[\varphi(B(T) - B(t_i)) \right] E \left[\Delta B_i \right] E \left[f(B(t_{i-1})) \mid \mathcal{F}_s \right] \\
&= 0.
\end{aligned}$$

Thus, by Equation (5.10), we conclude that X_t is a near-martingale with respect to $\{\mathcal{F}_t\}$. □

Theorem 5.19. *Let $f(x)$ and $\varphi(x)$ be continuous functions. Let*

$$Y^{(t)} = \int_t^T f(B(s))\varphi(B(T) - B(s)) dB(s), \quad a \leq t \leq T,$$

and assume that $E|Y^{(t)}| < \infty$ for all $a \leq t \leq T$. Then $Y^{(t)}$ is a near-martingale with respect to the forward filtration $\{\mathcal{F}_t\}$.

Proof. Note that for $a \leq s < t \leq T$, we have

$$\begin{aligned} Y^{(t)} - Y^{(s)} &= \int_t^T f(B(s))\varphi(B(T) - B(s)) dB(s) - \int_s^T f(B(s))\varphi(B(T) - B(s)) dB(s) \\ &= - \int_s^t f(B(s))\varphi(B(T) - B(s)) dB(s) \\ &= -(X_t - X_s), \end{aligned}$$

where X_t is as in Theorem 5.18. Therefore,

$$E[Y^{(t)} - Y^{(s)} | \mathcal{F}_s] = E[-(X_t - X_s) | \mathcal{F}_s] = -E[X_t - X_s | \mathcal{F}_s] = 0.$$

Thus $Y^{(t)}$ is a near-martingale with respect to $\{\mathcal{F}_t\}$. □

Now, we turn our attention to the backward filtration. For $a \leq t \leq T$, let

$$\mathcal{F}^{(t)} = \sigma \{B(T) - B(s); t \leq s \leq T\},$$

the σ -field generated by $B(T) - B(s)$ for all $t \leq s \leq T$.

Obviously, $\{\mathcal{F}^{(t)}; a \leq t \leq T\}$ is a backward filtration. It can also be shown that the stochastic process X_t defined in Equation (5.8) and $Y^{(t)}$ defined in Equation (5.9) are both near-martingales with respect to this backward filtration $\{\mathcal{F}^{(t)}\}$.

Theorem 5.20. *Let $f(x)$ and $\varphi(x)$ be continuous functions. Then the stochastic process $X_t = \int_a^t f(B(s))\varphi(B(T) - B(s)) dB(s)$, $a \leq t \leq T$, is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.*

Proof. Let $a \leq s < t \leq T$ and $\|\Delta_n\| = \{s = t_0 < t_1 < \dots < t_{n-1} < t_n = t\}$ be a partition of the interval $[s, t]$. Then, as in the proof of Theorem 5.18, we have

$$E[X_t - X_s | \mathcal{F}^{(t)}] = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n E\left[f(B(t_{i-1}))\varphi((B(T) - B(t_i))\Delta B_i | \mathcal{F}^{(t)}]\right]. \quad (5.11)$$

and it is enough to show that for each $1 \leq i \leq n$,

$$E[f(B(t_{i-1}))\varphi(B(T) - B(t_i))\Delta B_i | \mathcal{F}^{(t)}] = 0.$$

Also note that ΔB_i is $\mathcal{F}^{(t_{i-1})}$ -measurable since

$$\Delta B_i = B(t_i) - B(t_{i-1}) = (B(T) - B(t_{i-1})) - (B(T) - B(t_i)) \in \mathcal{F}^{(t_{i-1})}.$$

By the tower property of conditional expectations and $\mathcal{F}^{(t_{i-1})}$ -measurability of $\varphi(B(T) - B(t_i))$ and ΔB_i , we have

$$\begin{aligned} & E[f(B(t_{i-1}))\varphi(B(T) - B(t_i))\Delta B_i | \mathcal{F}^{(t)}] \\ &= E\left[E[f(B(t_{i-1}))\varphi(B(T) - B(t_i))\Delta B_i | \mathcal{F}^{(t_{i-1})}] | \mathcal{F}^{(t)}\right] \\ &= E[\varphi(B(T) - B(t_i))\Delta B_i E[f(B(t_{i-1})) | \mathcal{F}^{(t_{i-1})}] | \mathcal{F}^{(t)}]. \end{aligned} \quad (5.12)$$

Note that for each $s > t_{i-1}$, $B(T) - B(s)$ is independent of the σ -field $\mathcal{F}_{t_{i-1}}$. This implies the independence of the σ -fields $\mathcal{F}^{(t_{i-1})}$ and $\mathcal{F}_{t_{i-1}}$. Since $f(B(t_{i-1}))$ is $\mathcal{F}_{t_{i-1}}$ -measurable, it follows that $f(B(t_{i-1}))$ is independent of $\mathcal{F}^{(t_{i-1})}$. Then Equation (5.12) becomes

$$\begin{aligned} & E[f(B(t_{i-1}))\varphi(B(T) - B(t_i))\Delta B_i | \mathcal{F}^{(t)}] \\ &= E[\varphi(B(T) - B(t_i))\Delta B_i E[f(B(t_{i-1}))] | \mathcal{F}^{(t)}]. \end{aligned}$$

Since $E[f(B(t_{i-1}))]$ is just a deterministic function, then it is $\mathcal{F}^{(t)}$ -measurable.

Therefore,

$$\begin{aligned}
& E[f(B(t_{i-1}))\varphi(B(T) - B(t_i))\Delta B_i|\mathcal{F}^{(t)}] \\
&= E[f(B(t_{i-1}))] E[\varphi(B(T) - B(t_i))\Delta B_i|\mathcal{F}^{(t)}] \\
&= E[f(B(t_{i-1}))] E\left[E[\varphi(B(T) - B(t_i))\Delta B_i|\mathcal{F}^{(t_i)}]|\mathcal{F}^{(t)}\right] \\
&= E[f(B(t_{i-1}))] E\left[\varphi(B(T) - B(t_i))E[\Delta B_i|\mathcal{F}^{(t_i)}]|\mathcal{F}^{(t)}\right]. \tag{5.13}
\end{aligned}$$

Since, for each $1 \leq i \leq n$, $\mathcal{F}^{(t_i)}$ is generated by all $B(T) - B(s)$ such that $t_i \leq s \leq T$, it follows that $\Delta B_i = B(t_i) - B(t_{i-1})$ is independent of $\mathcal{F}^{(t_i)}$. Thus Equation (5.13) becomes

$$\begin{aligned}
& E[f(B(t_{i-1}))\varphi(B(T) - B(t_i))\Delta B_i|\mathcal{F}^{(t)}] \\
&= E[f(B(t_{i-1}))] E\left[\varphi(B(T) - B(t_i))E[\Delta B_i|\mathcal{F}^{(t)}]\right] \\
&= E[f(B(t_{i-1}))] E[\Delta B_i] E[\varphi(B(T) - B(t_i))|\mathcal{F}^{(t)}] \\
&= 0.
\end{aligned}$$

Therefore, by Equation (5.11), we conclude that X_t is a near-martingale with respect to the backward filtration $\{\mathcal{F}^{(t)}\}$. \square

Theorem 5.21. *Let $f(x)$ and $\varphi(x)$ be continuous functions. The stochastic process $Y^{(t)} = \int_t^T f(B(s))\varphi(B(T) - B(s)) dB(s)$ defined for $a \leq t \leq T$ is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.*

Proof. From the proof of Theorem 5.19, we have already seen that $Y^{(t)} - Y^{(s)} = -(X_t - X_s)$. Also, in the proof of Theorem 5.20, we have that $E[X_t - X_s|\mathcal{F}^{(t)}] = 0$. Therefore,

$$E[Y^{(t)} - Y^{(s)}|\mathcal{F}^{(t)}] = E[-(X_t - X_s)|\mathcal{F}^{(t)}] = -E[X_t - X_s|\mathcal{F}^{(t)}] = 0.$$

Thus $Y^{(t)}$ is a near-martingale with respect to the backward filtration $\{\mathcal{F}^{(t)}\}$. \square

5.2 Itô Isometry of the New Stochastic Integral

Recall that in Theorem 3.3 of Section 3.1, we have seen the zero expectation and the Itô isometry of Itô integrals. Comparing these properties with the new integral, we have already seen that the zero expectation of the new integrals still holds in Theorem 4.17 of Chapter 4.

In this section, we investigate the Itô isometry of the new integral. Unfortunately, we still do not have the Itô isometry for the general case of new stochastic integrals yet. We only have the Itô isometry for a special class of the new integrals. More specifically, we will prove the Itô isometry for the new stochastic integral of an instantly independent process of the form $\varphi(B(T) - B(t))$ where $\varphi(x)$ is an analytic function on \mathbb{R} .

We need the following two lemmas in order to prove the above Itô isometry.

Lemma 5.22. *For $a \leq t \leq T$, we have*

$$\begin{aligned}
 & E \left[\left(\int_a^t (B(T) - B(s))^m dB(s) \right) \left(\int_a^t (B(T) - B(s))^n dB(s) \right) \right] \\
 &= \begin{cases} \frac{(m+n-1)!!}{\frac{m+n}{2}+1} \left((T-a)^{\frac{m+n}{2}+1} - (T-t)^{\frac{m+n}{2}+1} \right) & \text{if } m+n \text{ is even;} \\ 0 & \text{if } m+n \text{ is odd.} \end{cases} \quad (5.14)
 \end{aligned}$$

Proof. Let τ, σ be any positive real numbers. Using the Taylor expansion of e^x , we have the following equality

$$e^{\tau(B(T)-B(t))} = \sum_{m=0}^{\infty} \frac{(B(T) - B(t))^m \tau^m}{m!}.$$

So

$$\int_a^t e^{\tau(B(T)-B(s))} dB(s) = \sum_{m=0}^{\infty} \frac{\tau^m}{m!} \int_a^t (B(T) - B(s))^m dB(s). \quad (5.15)$$

Consider $E \left[\left(\int_a^t e^{\tau(B(T)-B(s))} dB(s) \right) \left(\int_a^t e^{\sigma(B(T)-B(s))} dB(s) \right) \right]$. Then, by Equation (5.15), we can express this expectation as

$$\begin{aligned}
& E \left[\left(\int_a^t e^{\tau(B(T)-B(s))} dB(s) \right) \left(\int_a^t e^{\sigma(B(T)-B(s))} dB(s) \right) \right] \\
&= E \left[\left(\sum_{m=0}^{\infty} \frac{\tau^m}{m!} \int_a^t (B(T) - B(s))^m dB(s) \right) \left(\sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \int_a^t (B(T) - B(s))^n dB(s) \right) \right] \\
&= E \left[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\tau^m \sigma^n}{n! m!} \left(\int_a^t (B(T) - B(s))^m dB(s) \right) \left(\int_a^t (B(T) - B(s))^n dB(s) \right) \right] \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\tau^m \sigma^n}{n! m!} E \left[\left(\int_a^t (B(T) - B(s))^m dB(s) \right) \left(\int_a^t (B(T) - B(s))^n dB(s) \right) \right].
\end{aligned} \tag{5.16}$$

Next, we will use the Itô formula for the new integrals to express this expectation in another way. That is, we first apply Theorem 4.17 to $\theta(x, y) = -\frac{e^{\tau(y-x)}}{\tau}$ to obtain

$$\int_a^t e^{\tau(B(T)-B(s))} dB(s) = \frac{1}{\tau} e^{\tau(B(T)-B(a))} - \frac{1}{\tau} e^{\tau(B(T)-B(t))} - \frac{\tau}{2} \int_a^t e^{\tau(B(T)-B(s))} ds. \tag{5.17}$$

By Equation (5.17), $E \left[\left(\int_a^t e^{\tau(B(T)-B(s))} dB(s) \right) \left(\int_a^t e^{\sigma(B(T)-B(s))} dB(s) \right) \right]$ can also be expressed as the following

$$\begin{aligned}
& E \left[\left(\int_a^t e^{\tau(B(T)-B(s))} dB(s) \right) \left(\int_a^t e^{\sigma(B(T)-B(s))} dB(s) \right) \right] \\
&= E \left[\left(\frac{1}{\tau} e^{\tau(B(T)-B(a))} - \frac{1}{\tau} e^{\tau(B(T)-B(t))} - \frac{\tau}{2} \int_a^t e^{\tau(B(T)-B(s))} ds \right) \right. \\
&\quad \left. \times \left(\frac{1}{\sigma} e^{\sigma(B(T)-B(a))} - \frac{1}{\sigma} e^{\sigma(B(T)-B(t))} - \frac{\sigma}{2} \int_a^t e^{\sigma(B(T)-B(s))} ds \right) \right].
\end{aligned} \tag{5.18}$$

Multiply two expressions under the expectation, we will obtain 9 terms. Then it is straightforward to compute each of the expectations term by term. Combining all

together, we have that the expectation in Equation (5.18) is simplified to be

$$\begin{aligned}
& \frac{2}{(\tau + \sigma)^2} \left(e^{\frac{1}{2}(T-a)(\tau+\sigma)^2} - e^{\frac{1}{2}(T-t)(\tau+\sigma)^2} \right) \\
&= \frac{2}{(\tau + \sigma)^2} \left[\sum_{n=0}^{\infty} \frac{(T-a)^n}{2^n n!} (\tau + \sigma)^{2n} - \sum_{n=0}^{\infty} \frac{(T-t)^n}{2^n n!} (\tau + \sigma)^{2n} \right] \\
&= \sum_{n=1}^{\infty} \frac{(T-a)^n - (T-t)^n}{2^{n-1} n!} (\tau + \sigma)^{2(n-1)} \\
&= \sum_{n=0}^{\infty} \frac{(T-a)^{n+1} - (T-t)^{n+1}}{2^n (n+1)!} (\tau + \sigma)^{2n}. \tag{5.19}
\end{aligned}$$

For convenience, let $C_n = \frac{(T-a)^{n+1} - (T-t)^{n+1}}{2^n (n+1)!}$. Then, we apply the binomial theorem to $(\tau + \sigma)^{2n}$. Then the right-hand side of Equation (5.19) can be rewritten as

$$\sum_{n=0}^{\infty} C_n (\tau + \sigma)^{2n} = \sum_{n=0}^{\infty} C_n \sum_{k=0}^{2n} \binom{2n}{k} \sigma^k \tau^{2n-k} = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} C_n \binom{2n}{k} \sigma^k \tau^{2n-k}.$$

Next we switch the order of the summations to get

$$\sum_{n=0}^{\infty} \sum_{k=0}^{2n} C_n \binom{2n}{k} \sigma^k \tau^{2n-k} = \sum_{k=0}^{\infty} \sum_{n=\lceil \frac{k}{2} \rceil}^{\infty} C_n \binom{2n}{k} \sigma^k \tau^{2n-k}.$$

Then we split the summation over k into two parts. Namely, k are odd numbers and k are even numbers. Hence we have the following

$$\begin{aligned}
& \sum_{k=0}^{\infty} \sum_{n=\lceil \frac{k}{2} \rceil}^{\infty} C_n \binom{2n}{k} \sigma^k \tau^{2n-k} \\
&= \sum_{k=\text{odd}} \sum_{n=\lceil \frac{k}{2} \rceil}^{\infty} C_n \binom{2n}{k} \sigma^k \tau^{2n-k} + \sum_{k=\text{even}} \sum_{n=\lceil \frac{k}{2} \rceil}^{\infty} C_n \binom{2n}{k} \sigma^k \tau^{2n-k} \\
&= \sum_{j=0}^{\infty} \sum_{n=\lceil \frac{2j+1}{2} \rceil}^{\infty} C_n \binom{2n}{2j+1} \sigma^{2j+1} \tau^{2n-2j-1} + \sum_{j=0}^{\infty} \sum_{n=\lceil \frac{2j}{2} \rceil}^{\infty} C_n \binom{2n}{2j} \sigma^{2j} \tau^{2n-2j} \\
&= \sum_{j=0}^{\infty} \left[\sum_{n=j+1}^{\infty} C_n \binom{2n}{2j+1} \sigma^{2j+1} \tau^{2n-2j-1} + \sum_{n=j}^{\infty} C_n \binom{2n}{2j} \sigma^{2j} \tau^{2n-2j} \right]
\end{aligned}$$

Now we make some substitutions to the two summations in the square bracket.

Namely, let $m = 2n - 2j - 1$ for the first summation and let $m = 2n - 2j$ for

the second summation. Once we have done this, we will see that m will run from $m = 1, 3, 5, \dots$ (all positive odd numbers) in the first summation. On the other hand, m in the second summation will run from $m = 0, 2, 4, \dots$ (all nonnegative even numbers). So the above summation becomes

$$\begin{aligned}
& \sum_{j=0}^{\infty} \left[\sum_{n=j+1}^{\infty} C_n \binom{2n}{2j+1} \sigma^{2j+1} \tau^{2n-2j-1} + \sum_{n=j}^{\infty} C_n \binom{2n}{2j} \sigma^{2j} \tau^{2n-2j} \right] \\
&= \sum_{j=0}^{\infty} \left[\sum_{m=\text{odd}} C_{\lceil \frac{m+2j+1}{2} \rceil} \binom{m+2j+1}{2j+1} \sigma^{2j+1} \tau^m + \sum_{m=\text{even}} C_{\lceil \frac{m+2j}{2} \rceil} \binom{m+2j}{2j} \sigma^{2j} \tau^m \right] \\
&= \sum_{j=0}^{\infty} \sum_{m=\text{odd}} C_{\frac{m+2j+1}{2}} \binom{m+2j+1}{2j+1} \sigma^{2j+1} \tau^m + \sum_{j=0}^{\infty} \sum_{m=\text{even}} C_{\frac{m+2j}{2}} \binom{m+2j}{2j} \sigma^{2j} \tau^m \\
&= \sum_{n=\text{odd}} \sum_{m=\text{odd}} C_{\frac{m+n}{2}} \binom{m+n}{n} \sigma^n \tau^m + \sum_{n=\text{even}} \sum_{m=\text{even}} C_{\frac{m+n}{2}} \binom{m+n}{n} \sigma^n \tau^m.
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
& E \left[\left(\int_a^t e^{\tau(B(T)-B(s))} dB(s) \right) \left(\int_a^t e^{\sigma(B(T)-B(s))} dB(s) \right) \right] \\
&= \sum_{n=\text{odd}} \sum_{m=\text{odd}} \binom{m+n}{n} C_{\frac{m+n}{2}} \tau^m \sigma^n + \sum_{n=\text{even}} \sum_{m=\text{even}} \binom{m+n}{n} C_{\frac{m+n}{2}} \tau^m \sigma^n. \quad (5.20)
\end{aligned}$$

By comparing the coefficients of $\tau^m \sigma^n$ in Equation (5.20) with those in Equation (5.16), we have

$$\begin{aligned}
& E \left[\left(\int_a^t (B(T) - B(s))^m dB(s) \right) \left(\int_a^t (B(T) - B(s))^n dB(s) \right) \right] \\
&= \begin{cases} n!m! \binom{m+n}{n} C_{\frac{m+n}{2}} & \text{if } m, n \text{ are both odd OR both even;} \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \frac{(m+n)! \left((T-a)^{\frac{m+n}{2}+1} - (T-t)^{\frac{m+n}{2}+1} \right)}{2^{\frac{m+n}{2}} \left(\frac{m+n}{2} + 1 \right)!} & \text{if } m+n \text{ is even;} \\ 0 & \text{if } m+n \text{ is odd.} \end{cases} \\
&= \begin{cases} \frac{(m+n-1)!!}{\frac{m+n}{2} + 1} \left((T-a)^{\frac{m+n}{2}+1} - (T-t)^{\frac{m+n}{2}+1} \right) & \text{if } m+n \text{ is even;} \\ 0 & \text{if } m+n \text{ is odd.} \end{cases}
\end{aligned}$$

Thus we complete the proof of Lemma 5.22. \square

Lemma 5.23. *For any $a \leq t \leq T$ and $m, n \in \{0, 1, 2, \dots\}$ we have*

$$\begin{aligned} & E \left[\left(\int_a^t (B(T) - B(s))^n dB(s) \right) \left(\int_a^t (B(T) - B(s))^m dB(s) \right) \right] \\ &= \int_a^t E \left[(B(T) - B(s))^{n+m} \right] ds. \end{aligned} \quad (5.21)$$

Proof. First, recall that if X is normally distributed with mean 0 and variance σ^2 , the k -th moment of X is given by

$$E(X^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \sigma^k (k-1)!! & \text{if } k \text{ is even.} \end{cases}$$

Since $B(T) - B(s)$ is normally distributed with mean 0 and variance $T - s$, $E(B(T) - B(s))^{n+m}$ is given by

$$E(B(T) - B(s))^{n+m} = \begin{cases} 0 & \text{if } m+n \text{ is odd;} \\ (T-s)^{\frac{m+n}{2}} (m+n-1)!! & \text{if } m+n \text{ is even.} \end{cases}$$

Therefore,

$$\begin{aligned} & \int_a^t E \left[(B(T) - B(s))^{n+m} \right] ds \\ &= \begin{cases} (m+n-1)!! \int_a^t (T-s)^{\frac{m+n}{2}} ds & \text{if } m+n \text{ is even;} \\ 0 & \text{if } m+n \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{(m+n-1)!!}{\frac{m+n}{2} + 1} \left((T-a)^{\frac{m+n}{2}+1} - (T-t)^{\frac{m+n}{2}+1} \right) & \text{if } m+n \text{ is even;} \\ 0 & \text{if } m+n \text{ is odd.} \end{cases} \end{aligned}$$

which is exactly the right-hand side of Equation (5.14) in Lemma 5.22. Therefore, Equation (5.21) holds. \square

Now, we are ready to prove the Itô isometry that we have mentioned before.

Theorem 5.24. Suppose that $\varphi(x)$ is an analytic function on \mathbb{R} such that

$\int_a^T E[\varphi(B(T) - B(t))]^2 dt < \infty$ and $\int_a^T \varphi(B(T) - B(t)) dB(t)$ exists. Then for $a \leq t \leq T$,

$$E \left[\left(\int_a^t \varphi(B(T) - B(s)) dB(s) \right)^2 \right] = \int_a^t E[\varphi(B(T) - B(s))]^2 ds. \quad (5.22)$$

Proof. Here, we will give only an informal derivation of Equation (5.22). Because of the analyticity of φ , for $x \in \mathbb{R}$, $\varphi(x)$ can be expanded as

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} x^n,$$

where $\varphi^{(n)}(x)$ is the n -th derivative of $\varphi(x)$. We first consider the left-hand side of Equation (5.22).

$$\begin{aligned} & E \left[\int_a^t \varphi(B(T) - B(s)) dB(s) \right]^2 \\ &= E \left[\int_a^t \left(\sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} (B(T) - B(s))^n \right) dB(s) \right]^2 \\ &= E \left[\sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \int_a^t (B(T) - B(s))^n dB(s) \right]^2 \\ &= E \left[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{\varphi^{(n)}(0)\varphi^{(m)}(0)}{n!m!} \right. \right. \\ &\quad \left. \left. \times \left(\int_a^t (B(T) - B(s))^n dB(s) \right) \left(\int_a^t (B(T) - B(s))^m dB(s) \right) \right\} \right] \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{\varphi^{(n)}(0)\varphi^{(m)}(0)}{n!m!} \right. \\ &\quad \left. \times E \left[\left(\int_a^t (B(T) - B(s))^n dB(s) \right) \left(\int_a^t (B(T) - B(s))^m dB(s) \right) \right] \right\}. \quad (5.23) \end{aligned}$$

On the other hand, consider the right-hand side of Equation (5.22).

$$\begin{aligned}
& \int_a^t E [\varphi(B(T) - B(s))]^2 ds \\
&= \int_a^t E \left[\sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} (B(T) - B(s))^n \right]^2 ds \\
&= \int_a^t E \left[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varphi^{(n)}(0)\varphi^{(m)}(0)}{n!m!} (B(T) - B(s))^{n+m} \right] ds \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varphi^{(n)}(0)\varphi^{(m)}(0)}{n!m!} \int_a^t E [(B(T) - B(s))^{n+m}] ds. \tag{5.24}
\end{aligned}$$

So, Equation (5.22) holds by Lemma 5.23. \square

Example 5.25. We will find the mean and variance of the integral $\int_0^1 e^{B(1)-B(t)} dB(t)$.

Note that the new integral always has mean zero by Theorem 4.12 in Chapter 4.

Thus,

$$E \left[\int_0^1 e^{B(1)-B(t)} dB(t) \right] = 0.$$

Also, since $\varphi(x) = e^x$ is analytic on \mathbb{R} , by Theorem 5.24, its variance is given by

$$\begin{aligned}
E \left[\left(\int_0^1 e^{B(1)-B(s)} dB(s) \right)^2 \right] &= \int_0^1 E [e^{B(1)-B(s)}]^2 ds \\
&= \int_0^1 E [e^{2(B(1)-B(s))}] ds \\
&= \int_0^1 e^{\frac{1}{2}(2)^2(1-s)} ds \\
&= \int_0^1 e^{2(1-s)} ds \\
&= \frac{1}{2} (e^2 - 1).
\end{aligned}$$

5.3 Some Formulas for the Integral Computation

In this section, we will prove two theorems to express the new integral of certain stochastic processes in terms of Itô integrals and Riemann integrals.

Theorem 5.26. Let $f(t)$ be an adapted stochastic process with respect to $\{\mathcal{F}_t\}$ and let $\varphi(x)$ be an analytic function on \mathbb{R} . Assume that $\int_a^T f(t)\varphi(B(T) - B(t)) dB(t)$ exists. Then for $a \leq t \leq T$,

$$\begin{aligned} & \int_a^t f(s)\varphi(B(T) - B(s)) dB(s) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\varphi^{(n)}(B(T))}{n!} \left(\int_a^t f(s)B(s)^n dB(s) + n \int_a^t f(s)B(s)^{n-1} ds \right), \end{aligned} \quad (5.25)$$

where $\varphi^{(n)}(x)$ is the n -th derivative of $\varphi(x)$.

Proof. Let $a \leq t \leq T$ and $\Delta = \{a = s_0 < s_1 < s_2 < \dots < s_n = t\}$ be a partition of the interval $[a, t]$. Also, for convenience, let $\Delta B_i = B(s_i) - B(s_{i-1})$. By using the definition of the new stochastic integral and the series expansion of $\varphi(x)$, we have

$$\begin{aligned} & \int_a^t f(s)\varphi(B(T) - B(s)) dB(s) \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(s_{i-1})\varphi(B(T) - B(s_i))\Delta B_i \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left[f(s_{i-1}) \left\{ \sum_{m=0}^{\infty} \frac{\varphi^{(m)}(0)}{m!} (B(T) - B(s_i))^m \right\} \Delta B_i \right] \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left[f(s_{i-1}) \left\{ \sum_{m=0}^{\infty} \frac{\varphi^{(m)}(0)}{m!} \sum_{k=0}^m \binom{m}{k} (-1)^k B(T)^{m-k} B(s_i)^k \right\} \Delta B_i \right] \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left[f(s_{i-1}) \left\{ \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{\varphi^{(m)}(0)}{(m-k)!k!} (-1)^k B(T)^{m-k} (B(s_{i-1}) + \Delta B_i)^k \right\} \Delta B_i \right] \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left[f(s_{i-1}) \left\{ \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{\varphi^{(m)}(0)}{(m-k)!k!} (-1)^k B(T)^{m-k} \right. \right. \\ & \quad \left. \left. \times \left(B(s_{i-1})^k + kB(s_{i-1})^{k-1}\Delta B_i + \dots + (\Delta B_i)^k \right) \right\} \Delta B_i \right] \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left[f(s_{i-1}) \left\{ \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{\varphi^{(m)}(0)}{(m-k)!k!} (-1)^k B(T)^{m-k} \right. \right. \\ & \quad \left. \left. \times \left(B(s_{i-1})^k \Delta B_i + kB(s_{i-1})^{k-1}(\Delta B_i)^2 + \dots + (\Delta B_i)^{k+1} \right) \right\} \right] \end{aligned} \quad (5.26)$$

Since all the terms with $(\Delta B_i)^n$, $n \geq 3$, converge to 0 in probability, Equation (5.26) becomes

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left[f(s_{i-1}) \left\{ \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{\varphi^{(m)}(0)}{(m-k)!k!} (-1)^k B(T)^{m-k} \right. \right. \\ \left. \left. \times \left(B(s_{i-1})^k \Delta B_i + k B(s_{i-1})^{k-1} (\Delta B_i)^2 \right) \right\} \right] \quad (5.27)$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(-1)^k \varphi^{(m)}(0)}{(m-k)!k!} B(T)^{m-k} \left[\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(s_{i-1}) B(s_{i-1})^k \Delta B_i \right. \\ \left. + k \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(s_{i-1}) B(s_{i-1})^{k-1} (\Delta B_i)^2 \right] \\ = \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(-1)^k \varphi^{(m)}(0)}{(m-k)!k!} B(T)^{m-k} \left(\int_a^t f(s) B(s)^k dB(s) + k \int_a^t f(s) B(s)^{k-1} ds \right). \quad (5.28)$$

Next we change the order of summation. The right hand side of Equation (5.28) becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\int_a^t f(s) B(s)^k dB(s) + k \int_a^t f(s) B(s)^{k-1} ds \right) \sum_{m=k}^{\infty} \frac{\varphi^{(m)}(0)}{(m-k)!} B(T)^{m-k} \\ = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\int_a^t f(s) B(s)^k dB(s) + k \int_a^t f(s) B(s)^{k-1} ds \right) \varphi^{(k)}(B(T)) \\ = \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{(n)}(B(T))}{n!} \left(\int_a^t f(s) B(s)^n dB(s) + n \int_a^t f(s) B(s)^{n-1} ds \right).$$

Thus, the formula in Equation (5.25) is obtained, and the proof is complete. \square

Example 5.27. Suppose $f(t)$ is an $\{\mathcal{F}_t\}$ -adapted stochastic process. Let us evaluate the stochastic integral $\int_0^t f(s) e^{B(1)-B(s)} dB(s)$ when $0 \leq t \leq 1$.

First, let $\varphi(x) = e^x$. Of course, φ is analytic on \mathbb{R} and $\varphi^{(n)}(x) = e^x$ for every $n = 0, 1, 2, \dots$

Therefore, by the formula in Theorem 5.26, for $0 \leq t \leq 1$, we have

$$\begin{aligned}
& \int_0^t f(s)e^{B(1)-B(s)} dB(s) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{e^{B(1)}}{n!} \left(\int_0^t f(s)B(s)^n dB(s) + n \int_0^1 f(s)B(s)^{n-1} dt \right) \\
&= e^{B(1)} \left(\int_0^t \left[f(s) \sum_{n=0}^{\infty} \frac{(-B(s))^n}{n!} \right] dB(s) - \int_0^t \left[f(s) \sum_{n=1}^{\infty} \frac{(-B(s))^{n-1}}{(n-1)!} \right] dt \right) \\
&= e^{B(1)} \left(\int_0^s f(s)e^{-B(s)} dB(s) - \int_0^t f(s)e^{-B(s)} ds \right).
\end{aligned}$$

Thus, for $0 \leq t \leq 1$, we have the expression

$$\int_0^t f(s)e^{B(1)-B(s)} dB(s) = e^{B(1)} \left(\int_0^t f(s)e^{-B(s)} dB(s) - \int_0^t f(s)e^{-B(s)} ds \right).$$

The formula (5.25) in Theorem 5.26 is used to evaluate the new stochastic integral of a product of an adapted stochastic process and an instantly independent stochastic process of the form $\varphi(B(T) - B(t))$ for $a \leq t \leq T$ where φ is an analytic function on \mathbb{R} . In fact, by using the same theorem, we are able to prove another formula to evaluate the anticipating stochastic integral of the form

$$\int_0^t f(s)\varphi(B(T)) dB(s), \quad a \leq t \leq T,$$

where $f(t)$ is an adapted stochastic process and $\varphi(x)$ is an analytic function on \mathbb{R} .

Theorem 5.28. *Let $f(t)$ be adapted with respect to $\{\mathcal{F}_t\}$ and let $\varphi(x)$ be an analytic function on \mathbb{R} . Assume that $\int_a^T f(t)\varphi(B(T)) dB(t)$ exists. Then for $a \leq t \leq T$,*

$$\begin{aligned}
& \int_a^t f(s)\varphi(B(T)) dB(s) \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{(n+k)}(B(T))}{n!k!} \left(\int_a^t f(s)B(s)^{n+k} dB(s) + n \int_a^t f(s)B(s)^{n+k-1} ds \right)
\end{aligned} \tag{5.29}$$

where $\varphi^{(n)}$ is the n -th derivative of φ .

Proof. First note that φ is analytic, so for any $x, y \in \mathbb{R}$, we have the expression $\varphi(x + y) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(x)}{k!} y^k$. Therefore, we can write

$$\begin{aligned} & \int_a^t f(s)(B(T)) dB(s) \\ &= \int_a^t f(s) \left(\sum_{k=0}^{\infty} \frac{\varphi^{(k)}(B(T) - B(s))}{k!} B(s)^k \right) dB(s) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_a^t f(s) B(s)^k \varphi^{(k)}(B(T) - B(s)) dB(s). \end{aligned} \quad (5.30)$$

Since $f(s)B(s)^k$ is adapted and $\varphi^{(k)}$ is still analytic on \mathbb{R} , we apply Theorem 5.26 for the case $f(s)B(s)^k$ and $\varphi^{(k)}(B(T) - B(s))$ to the integral in the right-hand side of (5.30). Finally, it becomes

$$\begin{aligned} & \int_a^t f(s)\varphi(B(T)) dB(s) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} (-1)^n \frac{(\varphi^{(k)})^{(n)}(B(T))}{n!} \left[\int_a^t f(s)B(s)^k B(s)^n dB(s) \right. \\ & \quad \left. + n \int_a^t f(s)B(s)^k B(s)^{n-1} ds \right] \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{(n+k)}(B(T))}{n!k!} \left(\int_a^t f(s)B(s)^{n+k} dB(s) + n \int_a^t f(s)B(s)^{n+k-1} ds \right). \end{aligned}$$

□

Example 5.29. Suppose $f(t)$ is $\{\mathcal{F}_t\}$ -adapted. For $0 \leq t \leq 2$, let us evaluate the integral $\int_0^t f(s)e^{B(2)} dB(s)$ by using the formula in Theorem 5.28.

Let $0 \leq t \leq 2$. As in Example 5.27, let $\varphi(x) = e^x$. Then $\varphi^{(n+k)}(x) = e^x$ for all non-negative integers n, k . By Theorem 5.28, we have

$$\begin{aligned} & \int_0^t f(s)e^{B(2)} dB(s) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n \frac{e^{B(2)}}{n!k!} \left(\int_0^t f(s)B(s)^{n+k} dB(s) + n \int_0^t f(s)B(s)^{n+k-1} ds \right). \end{aligned}$$

That is,

$$\begin{aligned}
& \int_0^t f(s) e^{B(s)} dB(s) \\
&= e^{B(t)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_0^t \left[f(s) B(s)^n \sum_{k=0}^{\infty} \frac{B(s)^k}{k!} \right] dB(s) \right. \\
&\quad \left. + n \int_0^t \left[f(s) B(s)^{n-1} \sum_{k=0}^{\infty} \frac{B(s)^k}{k!} \right] ds \right) \\
&= e^{B(t)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_0^t \left[f(s) B(s)^n e^{B(s)} \right] dB(s) + n \int_0^t \left[f(s) B(s)^{n-1} e^{B(s)} \right] ds \right) \\
&= e^{B(t)} \left(\int_0^t \left[f(s) e^{B(s)} \sum_{n=0}^{\infty} \frac{(-B(s))^n}{n!} \right] dB(s) \right. \\
&\quad \left. - \int_0^t \left[f(s) e^{B(s)} \sum_{n=1}^{\infty} \frac{(-B(s))^{n-1}}{(n-1)!} \right] ds \right) \\
&= e^{B(t)} \left(\int_0^t f(s) dB(s) - \int_0^t f(s) ds \right).
\end{aligned}$$

Thus for $0 \leq t \leq 2$, we have the equality

$$\int_0^t f(s) e^{B(s)} dB(s) = e^{B(t)} \left(\int_0^t f(s) dB(s) - \int_0^t f(s) ds \right).$$

A special case of the above equation, with $f(t) = 1$, is given in Example 4.16 of Chapter 4.

5.4 Generalized Itô's Formula for the New Stochastic Integral

In this section, we generalize the Itô formula for the new stochastic integral obtained by Ayed and Kuo (Theorem 4.17) in Chapter 4.

Recall that Theorem 4.17 states if $\theta(x, y) = f(x)\varphi(y - x)$ where f and φ are C^2 -functions on \mathbb{R} , then for $a \leq t \leq T$,

$$\begin{aligned}
& d\theta(B(t), B(T)) \\
&= \frac{\partial \theta}{\partial x}(B(t), B(T)) dB(t) + \left(\frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(B(t), B(T)) + \frac{\partial^2 \theta}{\partial x \partial y}(B(t), B(T)) \right) dt.
\end{aligned}$$

The assumption that $\theta(x, y)$ must be in the form $f(x)\varphi(y-x)$ makes the theorem somewhat difficult to apply. Let us show this on a very simple example below.

Example 5.30. We will evaluate the integral $\int_0^t B(1) dB(t)$ for $0 \leq t \leq 1$ by using Theorem 4.17.

Here we want to have $\frac{\partial \theta}{\partial x}(x, y) = y$. So we let $\theta(x, y) = xy$. We see that $\theta(x, y)$ is not of the form $f(x)\varphi(y-x)$, so we are not able to apply Theorem 4.17 to this function. However, it can be rewritten as

$$\theta(x, y) = xy = x(y-x) + x^2 = \theta_1(x, y) + \theta_2(x, y),$$

where $\theta_1(x, y) = x(y-x)$ and $\theta_2(x, y) = x^2$.

With $x = B(t)$ and $y = B(1)$, it follows that

$$d\theta(B(t), B(1)) = d\theta_1(B(t), B(1)) + d\theta_2(B(t), B(1))$$

That is,

$$d(B(t)B(1)) = d\left(B(t)(B(1) - B(t))\right) + d(B(t)^2). \quad (5.31)$$

Now, $\theta_1(x, y) = x(y-x)$ is of the form $f(x)\varphi(y-x)$. We can apply Theorem 4.17 to θ_1 . Since

$$\frac{\partial \theta_1}{\partial x}(x, y) = y - 2x, \quad \frac{\partial^2 \theta_1}{\partial x^2}(x, y) = -2, \quad \frac{\partial^2 \theta_1}{\partial x \partial y}(x, y) = 1,$$

by Theorem 4.17, we have

$$\begin{aligned} d\left(B(t)(B(1) - B(t))\right) &= (B(1) - 2B(t)) dB(t) + \left(\frac{1}{2}(-2) + 1\right) dt \\ &= (B(1) - 2B(t)) dB(t). \end{aligned} \quad (5.32)$$

Note that we can compute $d(B(t)^2)$ easily by using the regular Itô's formula in Theorem 3.20 with $f(x) = x^2$. So

$$d(B(t)^2) = 2B(t) dB(t) + \frac{1}{2}(2) dt = 2B(t) dB(t) + dt. \quad (5.33)$$

From Equation (5.32) and Equation (5.33), Equation (5.31) becomes

$$\begin{aligned}
 d(B(t)B(1)) &= d\left(B(t)(B(1) - B(t))\right) + d(B(t)^2) \\
 &= (B(1) - 2B(t)) dB(t) + 2B(t) dB(t) + dt \\
 &= B(1) dB(t) + dt.
 \end{aligned}$$

Thus

$$B(t)B(1) = \int_0^t B(1) dB(s) + \int_0^t ds = \int_0^t B(1) dB(s) + t,$$

which implies that

$$\int_0^t B(1) dB(s) = B(1)B(t) - t, \quad 0 \leq t \leq 1.$$

We see that it requires many steps to evaluate a simple stochastic integral. So it is a good idea if we can adjust the form of $\theta(x, y)$ in the assumption to make the theorem easier to use. The two theorems below provide Itô's formula for functions $\theta(x, y)$ of the form $f(x)\varphi(y)$. We not only adjust the form of $\theta(x, y)$, but we also allow x and y to be more general stochastic processes. Namely, instead of $B(t)$, we can let x to be an Itô integral. Also, instead of $B(T)$, we allow y to be a stochastic process $\int_t^T h(s) dB(s)$ or a random variable $\int_a^T h(s) dB(s)$, where h is a deterministic function. The first theorem is a generalized Itô's formula for the case when the first variable x is evaluated by an Itô integral X_t and the second variable y is evaluated by $Y^{(t)} = \int_t^T h(s) dB(s)$.

Theorem 5.31. *Let $X_t = \int_a^t \xi(s) dB(s)$, $a \leq t \leq T$, where $\xi(t) \in L^2_{ad}([a, T] \times \Omega)$ and let $Y^{(t)} = \int_t^T h(s) dB(s)$, $a \leq t \leq T$, where $h(t) \in L^2[a, T]$.*

Let $\theta(x, y) = f(x)\varphi(y)$ where f and φ are both C^2 -functions on \mathbb{R} . Then for $a \leq t \leq T$,

$$\begin{aligned} \theta(X_t, Y^{(t)}) &= \theta(X_a, Y^{(a)}) + \int_a^t \frac{\partial \theta}{\partial x}(X_s, Y^{(s)}) dX_s + \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(X_s, Y^{(s)}) (dX_s)^2 \\ &\quad + \int_a^t \frac{\partial \theta}{\partial y}(X_s, Y^{(s)}) dY^{(s)} - \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial y^2}(X_s, Y^{(s)}) (dY^{(s)})^2, \end{aligned} \quad (5.34)$$

or equivalently,

$$\begin{aligned} &\theta(X_t, Y^{(t)}) - \theta(X_a, Y^{(a)}) \\ &= \int_a^t \frac{\partial \theta}{\partial x}(X_s, Y^{(s)}) \xi(s) dB(s) + \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(X_s, Y^{(s)}) \xi^2(s) ds \\ &\quad - \int_a^t \frac{\partial \theta}{\partial y}(X_s, Y^{(s)}) h(s) dB(s) - \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial y^2}(X_s, Y^{(s)}) h^2(s) ds. \end{aligned} \quad (5.35)$$

With $\xi(t) = 1$ and $h(t) = 1$, X_t becomes $B(t)$ and $Y^{(t)}$ becomes $B(T) - B(t)$.

We have the following corollary.

Corollary 5.32. Let $\theta = f(x)\varphi(y)$ where f and φ are both C^2 -functions on \mathbb{R} .

Then for $a \leq t \leq T$,

$$\begin{aligned} &\theta(B(t), B(T) - B(t)) - \theta(B(a), B(T) - B(a)) \\ &= \int_a^t \frac{\partial \theta}{\partial x}(B(s), B(T) - B(s)) dB(s) + \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(B(s), B(T) - B(s)) ds \\ &\quad - \int_a^t \frac{\partial \theta}{\partial y}(B(s), B(T) - B(s)) dB(s) - \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial y^2}(B(s), B(T) - B(s)) ds. \end{aligned} \quad (5.36)$$

Example 5.33. Let us evaluate the integral $\int_0^t B(s)(B(1) - B(s)) dB(s)$ for $0 \leq t \leq 1$ by using the above corollary.

First, we let $\theta(x, y) = \frac{x^2}{2}y$ so that we have $\frac{\partial \theta}{\partial x}(x, y) = xy$. It follows that

$$\frac{\partial^2 \theta}{\partial x^2}(x, y) = y, \quad \frac{\partial \theta}{\partial y}(x, y) = \frac{x^2}{2}, \quad \frac{\partial^2 \theta}{\partial y^2}(x, y) = 0.$$

Therefore, by Corollary 5.32, we have

$$\begin{aligned} & \frac{1}{2}B(t)^2(B(1) - B(t)) - \frac{1}{2}B(0)^2(B(1) - B(0)) \\ &= \int_0^t B(s)(B(1) - B(s)) dB(s) + \frac{1}{2} \int_0^t (B(1) - B(s)) ds - \int_0^t \frac{B(s)^2}{2} dB(s) - 0. \end{aligned}$$

That is, for $0 \leq t \leq 1$,

$$\begin{aligned} & \int_0^t B(s)(B(1) - B(s)) dB(s) \\ &= \frac{1}{2}B(t)^2(B(1) - B(t)) - \frac{1}{2} \int_0^t (B(1) - B(s)) ds + \frac{1}{2} \int_0^t B(s)^2 dB(s) \\ &= \frac{1}{2}B(t)^2(B(1) - B(t)) - \frac{1}{2} \left[B(1) \int_0^t ds - \int_0^t B(s) ds \right] + \frac{1}{2} \int_0^t B(s)^2 dB(s). \end{aligned} \tag{5.37}$$

Note from Example 3.26 that $\int_0^t B(s)^2 dB(s) = \frac{B(t)^3}{3} - \int_0^t B(s) ds$, so Equation (5.37) becomes

$$\begin{aligned} & \int_0^t B(s)(B(1) - B(s)) dB(s) \\ &= \frac{1}{2}B(1)B(t)^2 - \frac{1}{2}B(t)^3 - \frac{1}{2}tB(1) + \frac{1}{2} \int_0^t B(s) ds + \frac{1}{2} \left[\frac{B(t)^3}{3} - \int_0^t B(s) ds \right] \\ &= \frac{1}{2}B(1)B(t)^2 - \frac{1}{2}tB(1) - \frac{B(t)^3}{3} \\ &= \frac{1}{2}B(1)(B(t)^2 - t) - \frac{B(t)^3}{3}. \end{aligned}$$

Observe that this result coincides with the one obtained by the direct computation in Example 4.13.

Furthermore, using an idea similar to the one in the above example, we can compute the integral $\int_a^t f(B(s))(B(T) - B(s)) dB(s)$, $a \leq t \leq T$, for any C^2 -function $f(x)$ on \mathbb{R} with an antiderivative $F(x)$ by simply letting $\theta(x, y) = yF(x)$.

The next theorem is another generalized version of Itô's formula for the new stochastic integral. The difference between this version and the previous one (Theorem 5.31) is the evaluation at the second variable of function θ . Instead of being

evaluated by a stochastic process $Y^{(t)} = \int_t^T h(s) dB(s)$, we evaluate the variable y of $\theta(x, y)$ by the random variable $Y^{(a)} = \int_a^T h(s) dB(s)$.

Theorem 5.34. *Let $X_t = \int_a^t \xi(s) dB(s)$, $a \leq t \leq T$, where $\xi(t) \in L^2_{ad}([a, T] \times \Omega)$ and let $Y^{(t)} = \int_t^T h(s) dB(s)$, $a \leq t \leq T$, where $h(t) \in L^2[a, T]$.*

Let $\theta(x, y) = f(x)\varphi(y)$ where f is C^2 -function on \mathbb{R} and φ is an analytic function on \mathbb{R} . Then for $a \leq t \leq T$,

$$\begin{aligned} \theta(X_t, Y^{(a)}) &= \theta(X_a, Y^{(a)}) + \int_a^t \frac{\partial \theta}{\partial x}(X_s, Y^{(a)}) dX_s + \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(X_s, Y^{(a)}) (dX_s)^2 \\ &\quad - \int_a^t \frac{\partial^2 \theta}{\partial x \partial y}(X_s, Y^{(a)}) (dX_s)(dY^{(s)}), \end{aligned}$$

or equivalently,

$$\begin{aligned} &\theta(X_t, Y^{(a)}) - \theta(X_a, Y^{(a)}) \\ &= \int_a^t \frac{\partial \theta}{\partial x}(X_s, Y^{(a)}) \xi(s) dB(s) + \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(X_s, Y^{(a)}) \xi^2(s) ds \\ &\quad + \int_a^t \frac{\partial^2 \theta}{\partial x \partial y}(X_s, Y^{(a)}) \xi(s) h(s) ds, \end{aligned}$$

Again, with $\xi(t) = 1$ and $h(t) = 1$, we also have the following corollary.

Corollary 5.35. *Let $\theta = f(x)\varphi(y)$ where f is a C^2 -function on \mathbb{R} and φ is an analytic function on \mathbb{R} . Then for $a \leq t \leq T$,*

$$\begin{aligned} &\theta(B(t), B(T) - B(a)) - \theta(B(a), B(T) - B(a)) \\ &= \int_a^t \frac{\partial \theta}{\partial x}(B(s), B(T) - B(a)) dB(s) + \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(B(s), B(T) - B(a)) ds \\ &\quad + \int_a^t \frac{\partial^2 \theta}{\partial x \partial y}(B(s), B(T) - B(a)) ds. \end{aligned}$$

One of the very useful application of Corollary 5.35 above is that we can easily evaluate the stochastic integral of the form

$$\int_0^t g(B(T)) dB(s), \quad 0 \leq t \leq T,$$

where g is an analytic function on \mathbb{R} .

Namely, we let $\theta(x, y) = xg(y)$ so that we will have $\frac{\partial\theta}{\partial x}(x, y) = g(y)$. It follows that

$$\frac{\partial^2\theta}{\partial x^2}(x, y) = 0, \quad \frac{\partial^2\theta}{\partial x\partial y}(x, y) = g'(y).$$

Then by Corollary 5.35 with $a = 0$, for $0 \leq t \leq T$ we have

$$B(t)g(B(T)) - B(0)g(0) = \int_a^t g(B(T)) dB(s) + 0 + \int_a^t g'(B(T)) ds.$$

This implies that for $0 \leq t \leq T$,

$$\begin{aligned} \int_0^t g(B(T)) dB(s) &= B(t)g(B(T)) - g'(B(T)) \int_0^t ds \\ &= B(t)g(B(T)) - t g'(B(T)). \end{aligned} \quad (5.38)$$

Example 5.36. Let us use Equation (5.38), to evaluate the integral $\int_0^t B(1)^2 dB(t)$ for $0 \leq t \leq 1$. We let $g(x) = x^2$ and $T = 1$. By Equation (5.38), we obtain

$$\int_0^t B(1)^2 dB(t) = B(t) (B(1)^2) - t(2B(1)) = B(1)^2 B(t) - 2B(1)t, \quad 0 \leq t \leq 1,$$

which is exactly the same result as in Example 4.15. Note that this method is much easier than the direct computation in Example 4.15 because we do not need to compute as many integrals as in Example 4.15.

Moreover, Corollary 5.35 can also be used to evaluate the stochastic integral of the form

$$\int_0^t \psi(B(s))g(B(T)) dB(s), \quad 0 \leq t \leq T,$$

given that g is analytic on \mathbb{R} and ψ is a C^2 -function with antiderivative Ψ .

This can be done by letting $\theta(x, y) = \Psi(x)g(y)$. Then it follows that

$$\frac{\partial\theta}{\partial x} = \psi(x)g(y), \quad \frac{\partial^2\theta}{\partial x^2}(x, y) = \psi'(x)g(y), \quad \frac{\partial^2\theta}{\partial x\partial y}(x, y) = \psi(x)g'(y).$$

Therefore, by the corollary 5.35 with $a = 0$ and $0 \leq t \leq T$, we have

$$\begin{aligned} & \Psi(B(t))g(B(T)) \\ &= \Psi(B(0))g(B(T)) + \int_0^t \psi(B(s))g(B(T)) dB(s) \\ & \quad + \frac{1}{2} \int_0^t \psi'(B(s))g(B(T)) ds + \int_0^t \psi(B(s))g'(B(T)) ds. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} & \int_0^t \psi(B(s))g(B(T)) dB(s) \\ &= \Psi(B(t))g(B(T)) - \Psi(B(0))g(B(T)) - \frac{g(B(T))}{2} \int_0^t \psi'(B(s)) ds \\ & \quad - \int_0^t \psi(B(s))g'(B(T)) ds \\ &= g(B(T)) \left[\Psi(B(t)) - \Psi(B(0)) - \frac{1}{2} \int_0^t \psi'(B(s)) ds \right] - \int_0^t \psi(B(s))g'(B(T)) ds \\ &= g(B(T)) \int_0^t \psi(B(s)) dB(s) - \int_0^t \psi(B(s))g'(B(T)) ds, \end{aligned}$$

where the last equality follows from the regular Itô's formula in Theorem 3.20.

Note that a specific case when $g(x) = x$ and $T = 1$ is provided in Example 2.7 of [1]. Namely,

$$\int_0^t B(1)\psi(B(s)) dB(s) = B(1) \int_0^t \psi(B(s)) dB(s) - \int_0^t \psi(B(s)) ds, \quad 0 \leq t \leq 1.$$

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Vita

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