Essential reflections versus minimal embeddings

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Recommended Citation
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Communicated by G.M. Kelly
Received 30 April 1984

This note addresses the following question. In a category \( \mathcal{C} \), with a full subcategory \( \mathcal{S} \) which is reflective with each reflection morphism an embedding, and a \( \mathcal{C} \)-object \( A \): when does each \( \mathcal{S} \)-object containing \( A \) contain the \( \mathcal{S} \)-reflection of \( A \)? We prove, under mild hypotheses on \( \mathcal{C} \), the equivalence of (1) The reflection morphism for \( A \) is an essential embedding, (2) Any 'minimal' embedding of \( A \) in an \( \mathcal{S} \)-object is the \( \mathcal{S} \)-reflection of \( A \), (3) 'Always' is the answer to the question above, (4) The reflection functor \( \mathcal{C} \rightarrow \mathcal{C} \) carries embeddings of \( A \) to embeddings. We note that these conditions hold for every \( A \), for every \( \mathcal{S} \), if, in \( \mathcal{C} \), every epic embedding is essential, or if \( \mathcal{C} \) has the Amalgamation Property. Various examples are discussed.

1. Definitions and terminology

Our reference for category theory is [11]. We won't need much, but do presume of the reader some familiarity with reflections.

We shall proceed in somewhat more generality than the Introduction indicates, in order to handle various situations at once:

In a category \( \mathcal{C} \), let \( \mathcal{M} \) be a specified class of monics containing all isomorphisms, with \( \mathcal{M} \circ \mathcal{M} \subseteq \mathcal{M} \). One thinks of \( \mathcal{M} = \) all monics, or, with \( \mathcal{C} \) concrete, \( \mathcal{M} = \) all embeddings [11; p. 262], in particular, for example, for \( \mathcal{C} = \) Tychonoff topological spaces, \( \mathcal{M} = \) all topological embeddings.

\( \mathcal{M} \) is said to be closed under restriction (resp., restriction by \( \mathcal{M} \)) if \( \beta \alpha \in \mathcal{M} \) (resp., \( \beta \alpha \in \mathcal{M} \) and \( \beta \in \mathcal{M} \)) implies \( \alpha \in \mathcal{M} \) (cf. [2]).

Let \( \mathcal{S} \) be a subcategory (always assumed full, isomorphism-closed, nonempty and containing a nonempty object). The \( \mathcal{C} \)-morphism \( f \) is 'to \( \mathcal{S} \)' if \( \text{codomain}(f) \in |\mathcal{S}| \) (the object class of \( \mathcal{S} \)), and '\( \mathcal{M} \)-minimal to \( \mathcal{S} \)' if \( f \) is to \( \mathcal{S} \), \( f \in \mathcal{M} \), and whenever \( f = gh \) with \( g \in \mathcal{M} \cap \mathcal{S} \) and \( h \in \mathcal{M} \), then \( g \) is an isomorphism.

(Regarding this definition, we note: First, the definition of \( \mathcal{M} \)-minimality of \( A \rightarrow X \) is an attempt to capture the idea of \( A \) being an \( \mathcal{M} \)-subobject of \( X \), with no objects of \( \mathcal{S} \) property \( \mathcal{M} \)-between \( A \) and \( X \). Second, in the definition, there is...
no ambiguity about where $g$ is an isomorphism, since $\mathcal{F}$ is full. Third, addressing the condition that $g \in \mathcal{M} \cap \mathcal{F}$, at least in the case $\mathcal{M} =$ all monics: we shall almost always assume that $\mathcal{F}$ is monoreflective, hence epireflective [11; p. 276], and with $\mathcal{F}$ epireflective, an $\mathcal{F}$-morphism $g$ is $\mathcal{F}$-monic iff $\mathcal{M}$-monic (as one can easily prove). Fourth, we shall usually have to assume below at least that $\mathcal{M}$ is closed under restriction by $\mathcal{M}$ and then the condition in the definition that $h \in \mathcal{M}$ becomes redundant.)

Our standing notation for the reflection of an object $A$ into the subcategory $\mathcal{F}$ is $A \xrightarrow{e_A} xA$ (when it exists). When $\mathcal{F}$ is reflective, i.e., each $A$ has a reflection into $\mathcal{F}$, the reflecting functor (left adjoint to the inclusion $\mathcal{C} \to \mathcal{F}$) is $\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{F}$. Reflective $\mathcal{F}$ is called $\mathcal{M}$-reflective if each reflection morphism $e_A$ is in $\mathcal{M}$. Likewise, $\mathcal{M}$-essential-reflective', where a morphism $e$ is called $\mathcal{M}$-essential if $e \in \mathcal{M}$, and $me \in \mathcal{M}$ implies $m \in \mathcal{M}$ (cf. [3]).

We shall say that $\mathcal{F}$ has $\mathcal{M}$-intersections if each family $\{B_i \xrightarrow{m_i} B \mid i \in I\} \subseteq \mathcal{M}$ has an intersection $(D, m)$ with $m \in \mathcal{M}$; and $\mathcal{F}$ is closed under $\mathcal{M}$-intersections if whenever $\{X_i \xrightarrow{m_i} X \mid i \in I\} \subseteq \mathcal{M} \cap \mathcal{F}$ has an $\mathcal{M}$-intersection $(D, m)$, then $D \in \mathcal{F}$ (cf. [11]).

Given $A \in \mathcal{M}$, $\mathcal{M}(A, \cdot)$ stands for all morphisms in $\mathcal{M}$, out of $A$, and $\mathcal{M}(A, \mathcal{F})$ stands for all morphisms in $\mathcal{M}$, out of $A$, to $\mathcal{F}$.

2. Proposition. Let $A \xrightarrow{e_A} xA$ be a reflection into $\mathcal{F}$, with $e_A \in \mathcal{M}$. Then $e_A$ is $\mathcal{M}$-minimal to $\mathcal{F}$.

Proof. Suppose $e_A = gh$, with $h \in \mathcal{M}$ and $g \in \mathcal{M} \cap \mathcal{F}$, say $A \xrightarrow{h} X \xrightarrow{g} xA$. Since $X \in \mathcal{F}$, we have $xA \xrightarrow{h} X$ with $h = \mathcal{H}_e_A$. Let $1$ be the identity on $xA$. Then $1 = e_A = e_A = gh = (gh)e_A$, so by uniqueness, $1 = gh$. So $g$ is an $\mathcal{F}$-retraction. Since $g \in \mathcal{M}$, $g$ is $\mathcal{C}$-monic, hence $\mathcal{F}$-monic. Thus $g$ is an $\mathcal{F}$-isomorphism [11; 6.7].

The following is the main observation of this note (among other things telling when the converse to Proposition 2 holds).

3. Theorem. Suppose that $\mathcal{F}$ is $\mathcal{M}$-reflective in $\mathcal{C}$. Suppose further that $\mathcal{C}$ has $\mathcal{M}$-intersections, and that $\mathcal{M}$ is closed under restriction by $\mathcal{M}$. The following conditions on particular $A \xrightarrow{e_A} xA$ are equivalent.

(1) $e_A$ is $\mathcal{M}$-essential.

(2) $e_A$ is, up to isomorphism, the only morphism out of $A$ which is $\mathcal{M}$-minimal to $\mathcal{F}$.

(3) For each $f \in \mathcal{M}(A, \mathcal{F})$, the factorization $f = \mathcal{F}e_A$ has $\mathcal{F} \in \mathcal{M}$.

(4) $\mathcal{X}(A, \cdot) \subseteq \mathcal{M}$.

(5) Any $\mathcal{M}$-minimal morphism out of $A$ to $\mathcal{F}$ is $\mathcal{M}$-essential.

Proof. For (1) $\Rightarrow$ (3), (1) $\Leftrightarrow$ (5), and (3) $\Leftrightarrow$ (4), we only assume that $\mathcal{F}$ is $\mathcal{M}$-reflective:

(1) $\Rightarrow$ (3). By definition of $\mathcal{M}$-essentiality of $e_A$. 
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(3) ⇒ (2). If \( f \) is \( \mathcal{M} \)-minimal to \( \mathcal{I} \), then \( f = \overline{f} e_A \) must have \( \overline{f} \) an isomorphism, whence (2).

(5) ⇒ (1). By Proposition 2.

(1) ⇒ (5). We know (1) ⇒ (2), and clearly (1) & (2) ⇒ (5).

(3) ⇒ (4). Given \( A \xrightarrow{f} B \), we have \( e_B f = (xf)e_A \). If \( f \in \mathcal{M} \), then \( e_B f \in \mathcal{M} \) (since \( e \in \mathcal{M} \)), and thus \( xf \in \mathcal{M} \), assuming (3).

(4) ⇒ (3). Let \( A \rightarrow X \in \mathcal{M} \), with \( X \subseteq \mathcal{I} \). We have \( e_x f = (xf)e_A \), with \( xf \in \mathcal{M} \) by (4). Since \( \mathcal{I} \) is full, \( e_x = 1_X \), whence \( xf = \overline{f} \).

(2) ⇒ (1). Assume that \( \mathcal{I} \) is \( \mathcal{M} \)-reflective, and that \( \mathcal{M} \) is closed under restriction by \( \mathcal{I} \). Let \( xA \xrightarrow{g} B \) have \( ge_A \in \mathcal{M} \). Then \( x(ge_A) \in \mathcal{M} \), by (4). But \( x(ge_A) = e_B g \), by universality. So \( e_B g \in \mathcal{M} \), and since \( e_B \in \mathcal{M} \), we have \( g \in \mathcal{M} \) 'by restriction'.

(2) ⇒ (3) will conclude the proof of the theorem, and will use all the hypotheses. We isolate

4. Lemma. Suppose that \( \mathcal{E} \) has, and \( \mathcal{I} \) is closed under \( \mathcal{M} \)-intersections, and suppose that \( \mathcal{M} \) is closed under restriction by \( \mathcal{I} \). Then, each \( f \in \mathcal{M} \), to \( \mathcal{I} \), has a factorization \( f = \overline{f} e \), with \( \overline{f} \in \mathcal{M} \) and \( e \) \( \mathcal{M} \)-minimal to \( \mathcal{I} \).

Proof. (A standard method: see [11; 17.8]) Given \( B \xrightarrow{f} X \in \mathcal{M} \), with \( X \subseteq \mathcal{I} \), let \( D \xrightarrow{f} X \) be the intersection of

\[
\mathcal{I} = \left\{ X' \xrightarrow{m} X \mid m \in \mathcal{M} \cap \mathcal{I}, f = mf' \text{ for some } f' \right\}.
\]

By hypothesis, \( D \subseteq \mathcal{I} \) and \( \overline{f} \in \mathcal{M} \). The universal property of \( \overline{f} \) provides \( B \xrightarrow{e} D \) with \( f = \overline{f} e \), and \( e \in \mathcal{M} \) 'by restriction'.

To show \( e \) \( \mathcal{M} \)-minimal to \( \mathcal{I} \), suppose that \( e = gh \) with \( g \in \mathcal{M} \cap \mathcal{I} \). Then \( f = \overline{f} gh \), and \( \overline{f} g \in \mathcal{M} \cap \mathcal{I} \). Thus, \( \overline{f} g \in \mathcal{I} \). Let \( X' = \text{domain}(\overline{f} g) \). The definition of intersection provides \( D \xrightarrow{f} X' \) with \( (\overline{f} g) \mu = \overline{f} \). Since \( \overline{f} \in \mathcal{M} \), \( \overline{f} \) is monic, whence \( g \mu = 1_D \). So \( g \) is a retraction, already monic, thus an isomorphism. \( \square \)

(2) ⇒ (3). Assume all the hypotheses of the theorem. Since \( \mathcal{I} \) is \( \mathcal{M} \)-reflexive, it is monoreflective, and thus epireflective [11; 36.3], and thus closed under all intersections [11; 20.3 and 36.16]. So the lemma applies. Now, given \( f \in \mathcal{M}(A, \mathcal{I}) \), we have \( f = \overline{f} e \) per the lemma. By (2), \( e 'is' e_A \).

That concludes the proof of the theorem. \( \square \)

The next three propositions indicate the applicability of Theorem 3.

5. Proposition. Let \( \mathcal{M} = \text{all monics} \). If every epic monic is (mono-) essential, then every monoreflective \( \mathcal{I} \) is essential-reflective.

Proof. Monoreflective = epireflective [11; 36.3]. \( \square \)
6. **Proposition.** Suppose that $\mathcal{M}$ is closed under restriction. Then, each of the following implies the next.

1. $\mathcal{C}$ has finite products, and the category $(|\mathcal{C}|, \mathcal{M})$ is injectively-complete.
2. The category $(|\mathcal{C}|, \mathcal{M})$ has the Amalgamation Property.
3. Each $\mathcal{M}$-reflective subcategory of $\mathcal{C}$ is $\mathcal{M}$-essential-reflective.

**Proof.** (1) $\Rightarrow$ (2) is a slight variant of [14; 3.4], which see.

(2) $\Rightarrow$ (3). Let $m \colon A \to B \in \mathcal{M}$. By AP, there are $m_1, m_2 \in \mathcal{M}$ with $m_1 m = m_2 e_A$. Let $C = \text{codomain}(m_1)$. There is $a$ with $a(xm) = e m_2$. This is in $\mathcal{M}$, so 'by restriction', $xm \in \mathcal{M}$. Thus, (3)(5) holds, hence also 3(1).

A generalization of Proposition 5, under a weakening of the hypothesis 6(2) is

7. **Proposition.** (See [2; 2.1].) If $\mathcal{M}$ is closed under restriction, and 'transferable', then each $\mathcal{C}$-epic in $\mathcal{M}$ is $\mathcal{M}$-essential, and hence each $\mathcal{M}$-reflective $\mathcal{F}$ is $\mathcal{M}$-essential-reflective.

8. **Examples.** We illustrate the foregoing with various $\mathcal{C}, \mathcal{M}$ 'from nature' where: Every $\mathcal{M}$-reflective $\mathcal{F}$ is $\mathcal{M}$-essential; or, only some $\mathcal{M}$-reflective $\mathcal{F}$ are $\mathcal{M}$-essential; or, no $\mathcal{M}$-reflective $\mathcal{F}$ is $\mathcal{M}$-essential, while for $\mathcal{F}$, some objects $A$ have $\mathcal{M}$-essential $e_A$. We omit almost all technical details.

(1) Proposition 6 applies to various classes of algebras with the Amalgamation Property with $\mathcal{M}$ = all monics (1-1 homomorphisms). For example, torsion-free abelian groups is injectively-complete, while lattice-ordered abelian groups is not, but has the AP; Proposition 5 also applies to these. See [9] for various monoreflectives.

(2) Let Arch (resp. Arch$_1$) be archimeean l-groups (resp., with weak unit). An ad hoc argument verifies Proposition 5 (while AP fails). Various monoreflectives in Arch$_1$ are described in [8], and a special case of Theorem 3 appears in [10]. Not much is known about Arch.

(3) Let Unif be separated uniform spaces with uniformly continuous maps. We consider several $\mathcal{M}$'s:

- All monics: Monic means 1-1, monoreflective is equivalent to $\mathcal{H}$-reflective (see below), and there is no monic-essential map which is not onto.

- $\mathcal{E}$ = all uniform embeddings: Epic means 'dense image', and epireflective $\mathcal{F}$ is $\mathcal{E}$-reflective iff $|\mathcal{F}|$ contains all complete spaces. Using properties of completion, one can show that each epic embedding is $\mathcal{E}$-essential, so each $\mathcal{E}$-reflective $\mathcal{F}$ is $\mathcal{E}$-essential-reflective, as in the proof of Proposition 5. (3)(3) for $\mathcal{H}$ = complete spaces is the familiar statement that each complete space containing $A$ contains the completion of $A$.) While each space embeds in an injective space, 6(1) and 6(2) fail.

- $\mathcal{F}$ = all uniformly continuous homeomorphisms (into): Epireflective $\mathcal{F}$ is $\mathcal{F}$-reflective (resp., onto-reflective) iff $|\mathcal{F}|$ contains all compact (resp., precompact) spaces. Any homeomorphism which is onto is $\mathcal{F}$-essential, whence each onto-$\mathcal{F}$-
reflective \( \mathcal{F} \) is \( \mathcal{N} \)-essential-reflective. These are probably all: Neither \( \mathcal{F} = \) complete spaces, nor \( \mathcal{F} = \) compact spaces, is \( \mathcal{N} \)-essential-reflective.

(4) Let \( \text{Tych} = \) Tychonoff topological spaces. Monic means 1-1 (and epic means dense image), monoreflective is equivalent to \( \delta \)-reflective (see below), and there is no monic-essential map which is not onto.

Consider \( \mathcal{E} = \) all topological embeddings. Epireflective \( \mathcal{F} \) is \( \delta \)-reflective iff \( |\mathcal{F}| \) contains all compact spaces [12]. One conjectures that there is no \( \mathcal{E} \)-essential-reflective \( \mathcal{F} \); but for most \( \mathcal{F} \) there will be \( \mathcal{A} \)'s with \( e_{\mathcal{A}} \) \( \mathcal{E} \)-essential: for \( \mathcal{F} = \) compact spaces, for which the reflector is the Stone–Čech compactification \( \beta \), Theorem 3 describes the almost-compact spaces of Doss and Hewitt (see [4]).

(5) Let \( \text{Alex} = \) separated Alexandroff spaces (also called zero-set spaces) with the natural morphisms. See [7] for a survey. The situation here is intermediate between Tych and Unif.

Consider \( \mathcal{E} = \) all embeddings, and the \( \mathcal{E} \)-reflective subcategories of: compact spaces, with reflector \( \beta \) (constructed in [1]); realcompact spaces, say \( \mathcal{R} \), with reflector \( \nu \) (constructed in [6]). It is shown in [6] that \( \mathcal{R} \) satisfies 3(3) for every \( \mathcal{A} \), and [5] notes that \( \mathcal{R} \) satisfies 3(1) for every \( \mathcal{A} \). One can show easily that \( \mathcal{A} \rightarrow \beta \mathcal{A} \) is \( \delta \)-essential iff \( \nu \mathcal{A} = \beta \mathcal{A} \), i.e., \( \mathcal{A} \) is pseudocompact (in some contrast to Tych).

Further, one conjectures that \( \mathcal{F} \) is \( \delta \)-reflective iff \( \mathcal{F} \supseteq \mathcal{R} \).

Acknowledgements

We acknowledge a debt to [10], where appears a special case of part of the above result, which was developed to answer a particular example of the question above, in lattice-ordered algebra.

References