12-1-2004

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Recommended Citation
Long, L. (2004). On Shioda-Inose structures of one-parameter families of K3 surfaces. *Journal of Number Theory, 109* (2), 299-318. https://doi.org/10.1016/j.jnt.2004.06.009

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Journal of Number Theory 109 (2004) 299–318

JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

On Shioda–Inose structures of one-parameter families of K3 surfaces

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Received 26 March 2003; revised 7 April 2004

Abstract

We develop an algorithm to determine a one-parameter family of elliptic curves associated to a one-parameter family of K3 surfaces with generic Picard number 19 by a Shioda–Inose structure. The family of elliptic curves is determined up to an isomorphism and an isogeny. An application to a generalized congruence number problem is also discussed.

Keywords: Elliptic curve; Isogeny; Complex multiplication; K3 surface; Shioda–Inose structure; Picard–Fuchs equation; Monodromy representation

1. Introduction

A K3 surface is a simply connected compact complex smooth surface with trivial canonical bundle. K3 surfaces occupy an important place in the classification of compact complex smooth surfaces. Moreover, since a complex algebraic K3 surface is a two-dimensional Calabi–Yau manifold and Calabi–Yau manifolds play an essential role in Mirror Symmetry arising from String theories in Physics, the study of K3 surfaces has attracted immense attention in recent years. A Kummer surface is a K3 surface and it can be defined in the following way. Given an abelian surface A, there is an involution ι on A sending $x \in A$ to $-x$. The smooth surface $Kum(A)$, obtained by resolving the

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[1] Work supported in part by Grants from NSF DMS 97-29992 and Clay Mathematics institute Liftoff fellowship.

0022-314X/$ - see front matter
doi:10.1016/j.jnt.2004.06.009

16 double points on the quotient space A/ι, is called Kummer surface. In 1977, T. Shioda and H. Inose discovered that when the Picard number, the rank of the Picard group of a K3 surface, obtains its maximal possible value 20, the K3 surface can be realized as a double cover of a Kummer surface [19]. Later, such a particular structure of a K3 surface is referred as a Shioda–Inose structure (we write it as S–I structure in short for convenience) [13]. It is known by a result of Morrison [13] that any K3 surface with Picard number 19 or with Picard number 18 and 17 under some additional conditions also admits a S–I structure. A structure of this kind is useful in the sense that we can reduce the study of a K3 surface to the study of the related abelian surface which is a product of two isogenous elliptic curves for Picard number 20 and 19 cases. By the famous result of Faltings [6], the L-series of elliptic curves over a number field K cannot distinguish isogenous classes of elliptic curves over K. Hence, we can reduce some arithmetic properties of an algebraic K3 surface with Picard number 19 or 20 to the arithmetic properties of certain elliptic curves.

Shioda and Inose further obtain that when the Picard number of an algebraic K3 surface X is 20 the two isogenous elliptic curves associated to X by the S–I structure both have complex multiplications (CM). Hence there is a Grossencharacter χ associated to these two elliptic curves [19]. Because of this additional structure, the surface X has a model defined over a number field K and the L-series of the K3 surface over K is essentially the product of L-series of χ^2 and some Dedekind Zeta functions over K with suitable twists. Theoretically, this character χ can be determined by the Picard group of X. However, some components of the Picard group, the global sections of an elliptic K3 surface with respect to a fixed fibration, are hard to calculate in general. One basic idea to handle this difficulty is to use the S–I structure. More precisely, given an algebraic K3 surface with Picard number greater than 18, we can firstly deform the K3 surface X into a non-trivial one-parameter family X_t whose generic fibers are K3 surfaces with Picard number 19. If we can find a one-parameter family of elliptic curves E_t whose isogenous classes giving rise to a S–I structure of the family X_t, then by calculating the j-invariants of E_t we can identify which fibers have CM and hence the corresponding K3 surfaces will have Picard number 20 and the associated Grossencharacter can also be identified by the j-invariants up to a twisting.

In order to find the suitable family of elliptic curves E_t, we use monodromy representation of the fundamental domain of the parameter space $\mathbb{C}P_t^1$ with some points removed. Moreover, this representation corresponds to an integrable connection known as the Gauss–Manin connection which can be explicitly realized as Picard Fuchs equations. For a non-trivial one-parameter family of K3 surfaces with generic Picard number 19, its Picard Fuchs equation is an degree 3 linear Fuchsian ordinary differential equation which is the symmetric square of a order 2 homogenous linear Fuchsian differential equation by Doran [5]. The goal is to related this order 2 differential equation with the Picard Fuchs equation of elliptic surfaces studied by Stiller [21]. We will need Theorem III.3.3. in Stiller [21] which is formulated here as

Theorem 1. *Two elliptic curves defined over $\mathbb{C}(t)$ with non-constant j-functions are isogenous to each other if and only if after some possible linear transformation of the parameter t, the Picard–Fuchs equations for these two elliptic curves are equivalent.*

The technical condition for "being equivalent" will be defined in Section 3.

To realize the basic idea, we prose an algorithm for finding one-parameter family of elliptic curves whose isogenous classes giving rise to a S–I structure of a family of K3 surfaces with generic Picard 19. This algorithm can be used to construct elliptic K3 surfaces with larger Picard number. As a consequence the groups of global sections for these elliptic surfaces may have larger ranks. It can be also used to prove some identities involving character sums. In a paper of Ahlgren et al. [1], by using pure character sum calculations the authors prove a certain family of elliptic curves E_t is related to a family of K3 surfaces X_t by a S–I structure. But they do not reveal how these families are discovered. In a recent paper [11], the author utilizes this algorithm to explain how to construct explicitly the j-function of E_t once the family X_t in [1] is given.

This algorithm is based on known results in several areas including elliptic surfaces, K3 surfaces, differential equations appeared in [5,13,19,22]. To explain the algorithm, we introduce all the ingredients in a somehow slow but hopeful consistent way. Results are mostly restated so that they can fit together in our language. Proofs will be supplied when they will enlighten the development of the algorithm.

This paper is arranged in the following order. We first discuss the Shioda–Inose structures of K3 surfaces in Section 2. Section 3 is devoted to order 2 ordinary linear Fuchsian differential equations. Picard–Fuchs equations of one-parameter families of elliptic curves and M_n-polarized K3 surfaces will be studied in Sections 4 and 5 separately while we focus mainly on the elliptic curve case. In Section 6 we propose the algorithm. It follows by an application to a generalized congruent number problem in Section 7. As a consequence we give a new proof of the following result which has been obtained before by Fine [7]:

Proposition 2. *Every natural number occurs as the area of a rational triangle.*

2. Shioda–Inose structures for K3 surfaces

We will recall some relevant results from literature in this section.

Let X be a complex K3 surface. The cohomology group $H^2(X, \mathbb{Z})$ is a lattice whose bilinear form is given by the cup-product. Let $NS(X)$ denote the *Néron–Severi group* of X, which is the group of divisors of X modulo algebraic equivalence (for K3 surfaces, algebraic equivalence is the same as linear equivalence, hence $NS(X)$ is the same as the Picard group of X). For a K3 surface, $NS(X)$ is torsion free and can be considered as a sublattice of $H^2(X, \mathbb{Z})$. Its orthogonal complement in $H^2(X, \mathbb{Z})$ is called the *transcendental lattice* of X, denoted by T_X.

Firstly studied by Shioda and Inose [19], and later further developed by Morrison [13], Peters [15], and Dolgachev [4], a K3 surface X is said to admit a *Shioda–Inose structure* if there is an involution i on X satisfying $i^*(\omega) = \omega$ for every $\omega \in H^{2,0}(X)$, and a rational quotient map $\pi : X \longrightarrow Y$ such that Y is a Kummer surface, and π_* induces a lattice isometry $T_X(2) \cong T_Y$ on the transcendental lattices, where $T_X(2)$ is the lattice whose bilinear form is 2 times the bilinear form of T_X.

Furthermore, the lattice $H^2(X,\mathbb{Z})$ is, unimodular in the sense that the determinant of the matrix of its bilinear form is ± 1 by the Poincaré duality, even by Wu's formula [25], with signature (3, 19) by the Hodge index theory. By the classification of even unimodular lattices [2], the K3 lattice $H^2(X,\mathbb{Z})$ is isometric to $E_8(-1)^2 \oplus U^3$, where U is the rank 2 even unimodular hyperbolic lattice with signature (1, 1) and $E_8(-1)$ is the unique negatively definite even unimodular lattice of rank 8. Let $M = \mathbb{Z}\mathbf{v}$ be a primitive sublattice of U such that $\langle \mathbf{v}, \mathbf{v}\rangle = 2n$ for some positive integer n. The orthogonal complement of M in $E_8(-1)^2 \oplus U^2$, denoted by M_n, is isometric to

$$E_8(-1)^2 \oplus U \oplus \mathbb{Z}\mathbf{u}\,, \tag{1}$$

satisfying that $\langle \mathbf{u}, \mathbf{u}\rangle = -2n$. The lattice M_n is of rank 19. An *M_n-polarized K3 surface* is a pair (X, j) where X is a K3 surface and $j : M_n \hookrightarrow NS(X)$ is a primitive lattice embedding.

Let $\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z}) \mid c \equiv 0 \bmod n \right\}$ be a level n congruence subgroup of $PSL_2(\mathbb{Z})$.

Let $F_n = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix} \in PSL_2(\mathbb{R})$ be the *Fricke involution*, which is in the normalizer of $\Gamma_0(n)$. The *Fricke modular group of level n* is the subgroup of $PSL_2(\mathbb{R})$ generated by $\Gamma_0(n)$ and F_n, and is denoted by $\Gamma_0(n)^+$. Let $\mathfrak{H}$ denote the Poincaré upper half plane $\{\tau \in \mathbb{C} \mid Im(\tau) > 0\}$.

Theorem 3 (*Dolgachev [4]*). *Let M_n and $\Gamma_0(n)^+$ be defined as above. The coarse moduli space of* K3 *surfaces polarized by M_n is isomorphic to the orbit space $X_0(n) := \mathfrak{H}/\Gamma_0(n)^+$.*

Given an M_n-polarized K3 surface X, the corresponding point τ in $\mathfrak{H}$ by the isomorphism of Dolgachev is called the *period point* of X.

Theorem 4 (*Dolgachev [4]*). *Suppose X* is an *M_n-polarized* K3 *surface with period $\tau \in \mathfrak{H}$. Let $E_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and $E'_\tau = \mathbb{C}/(\mathbb{Z}+(-1/(n\tau))\mathbb{Z})$ be the corresponding pair of n-isogenous elliptic curves. There exists a canonical involution ι* on *X such that $X/(\iota) = (E_\tau \times E'_\tau)/(\pm 1)$ birationally.*

3. Order 2 ordinary linear Fuchsian differential equations

3.1. General theory

Let L be an order 2 differential operator given by

$$L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t),\ \ p(t), q(t) \in \mathbb{C}(t). \tag{2}$$

The *projective normal form* of L is

$$\bar{L} = \frac{d^2}{dt^2} + \left(q(t) - \frac{1}{2}p'(t) - \frac{1}{4}p(t)^2\right). \tag{3}$$

where $p'(t) := \frac{d}{dt}p(t)$. Note that the coefficient of $\frac{d}{dt}$ at $\bar{L}$ is 0. If u is a solution of $Lu = 0$ then $u \exp(\int \frac{p}{2})$ is the solution of $\bar{L}u = 0$. A point a is called a *singular point* of (2) if $p(t)$ or $q(t)$ has a pole at a. A differential operator with only regular singular points is called a *Fuchsian* differential equation. We only consider Fuchsian equations in this article. Let Σ be the set of singular points of (2). Let t_0 be a point in $\mathbb{C}P^1 - \Sigma$. There is a representation of the fundamental group

$$\rho : \pi(\mathbb{C}P^1 - \Sigma, t_0) \longrightarrow GL_2(\mathbb{C})$$

induced by the action of the fundamental group acting on the local solutions of $Lu = 0$ around t_0 [26]. The image of the fundamental group in $GL_2(\mathbb{C})$ (resp. $SL_2(\mathbb{C})$) is called the *monodromy group* (resp. *projective monodromy group*) of (2) (resp. (3)). The conjugacy class of the monodromy group in $GL_2(\mathbb{C})$ (resp. the projective monodromy group in $SL_2(\mathbb{C})$) is determined by the operator (2) (resp. (3)). Let $a \in \Sigma$, Let C be a loop in $\mathbb{C}P^1 - \Sigma$ starting and ending at t_0 so that no other singular points besides a lie inside C. We call the matrix $\rho(\{C\}) \in GL_2(\mathbb{C})$ the *local monodromy matrix* of (2) at a.

For any $a \in \mathbb{C}P^1$, the *characteristic equation* of L at a is

$$s(s-1) + p_0 s + q_0 = 0, \tag{4}$$

where when $a \neq \infty$,

$$p_0 = (t-a)p(t)|_{t=a} \text{ and } q_0 = (t-a)^2 q(t)|_{t=a}; \tag{5}$$

when $a = \infty$,

$$p_0 = -\frac{1}{t}p\left(\frac{1}{t}\right)\bigg|_{t=0} \text{ and } q_0 = \frac{1}{t^2}q\left(\frac{1}{t}\right)\bigg|_{t=0}. \tag{6}$$

The *characteristic exponents* r_1 and r_2 are the roots of the characteristic equation (4).

In particular, the characteristic exponents r_1 and r_2 of any projective normalized operator (3) at any base point a satisfy $r_1 + r_2 = 1$. For simplicity, in the case when $r_1, r_2 \in \mathbb{Q}$ and $r_1 + r_2 = 1$, we write them as

$$r_1 = \frac{x+y}{2x},\ r_2 = \frac{x-y}{2x} \tag{7}$$

for some coprime integers x, y.

Let L as above given by (2), $a \in \Sigma$, and r_1 and r_2 be the characteristic exponents of L at a. By the Frobenius Method [26], the Jordan normal form of the local monodromy matrix of (2) at a is

$$\begin{pmatrix} e^{2\pi i r_1} & 1 \\ 0 & e^{2\pi i r_2} \end{pmatrix} \text{ or } \begin{pmatrix} e^{2\pi i r_1} & 0 \\ 0 & e^{2\pi i r_2} \end{pmatrix},$$

depending on the fact that locally around a the equation $Lu = 0$ has logarithmic solutions or not.

3.2. Twisted equivalence

Let $a(t)$ be a non-constant algebraic function of t and L be a Fuchsian differential operator given by (2). Let

$$L^a := \frac{d^2}{dt^2} + \left(2\frac{a'}{a} + p\right)\frac{d}{dt} + \left(\frac{a''}{a} + \frac{a'}{a}p + q\right) \tag{8}$$

be the operator L twisted by a. It is easy to check that

$$aL^a\left(\frac{u}{a}\right) = Lu\,. \tag{9}$$

Hence, if u satisfies $Lu = 0$ then $\frac{u}{a}$ is a solution of $L^a u = 0$. We say that two linear Fuchsian differential operators L and L' are *twisted equivalent* if there is an algebraic function $a(t)$ such that L' is a scalar multiple of L^a.

Use the notations as above. At each point $t_0 \in \mathbb{C}$, let $a(t) = (t - t_0)^e \cdot h(t)$ for some local holomorphic function $h(t)$ around t_0. If *r_1 and r_2 are the characteristic exponents of $Lu = 0$* at *t_0, then the characteristic exponents of $L^a u = 0$ at t_0 are $r_1 - e$ and $r_2 - e$.*

Proof. If the characteristic equation for $Lu = 0$ at t_0 is as (4) with the values of p_0 and q_0 determined by (5) or (6), then by easy calculation the characteristic equation of $L^a u = 0$ at t_0 is given by

$$s(s-1) + (2e + p_0)s + e^2 - e + p_0 e + q_0 = 0. \qquad \square \tag{10}$$

It is easy to see that two order 2 linear Fuchsian differential operators have the same projectively normal form with normalized leading coefficient 1 if and only if they are twisted equivalent.

In particular, if L and L' are twisted equivalent by a rational function, then at any point t_0 the local monodromy matrices of L and L' have the same Jordan normal form by Lemma 3.2. Hence the monodromy groups associated to L and L' are the same up to a conjugation in $GL_2(\mathbb{C})$. In this case, we say that L and L' are *equivalent*.

3.3. Changing of variables

Let L be an order 2 ordinary Fuchsian differential operator in variable s. If s is a non-constant differentiable function of t, then after changing variable from s to t and dividing the leading coefficient, we get a new operator. We denote the projective normal form of this new operator by L_t and simply called it the *projectively normalized operator in t of L*.

Lemma 5. *Let $L = \frac{d^2}{dx^2} + q(s)$ be a Fuchsian operator, where $q(s)$ is a rational function of s. Suppose $s = f(t)$ is a non-constant rational function of t. Let L_t be the projectively normalized operator in t of L. If s_0 is a regular singular point of L with two rational characteristic exponents $r_1 = \frac{x+y}{2x}$ and $r_2 = \frac{x-y}{2x}$, $x, y \in \mathbb{Z}$, and $t_0 \in \mathbb{C}$ such that t_0 is a zero of $f(t) - s_0 = 0$ of multiplicity n, then one of the following two cases will occur*:

(1) *if $n = 1$ or $\frac{n^2-1}{4} - n^2 r_1 r_2 \neq 0$, then t_0 is a regular singular point of L_t with characteristic exponents $\frac{x-ny}{2x}$ and $\frac{x+ny}{2x}$*;
(2) *if $\frac{n^2-1}{4} - n^2 r_1 r_2 = 0$ and $n \geqslant 2$, then t_0 is not a singular point of L_t.*

Proof. Suppose $L_t = \frac{d^2}{dt^2} + q'(t)$. By formal calculation and the above assumptions, we can get that the characteristic equation of L_t at t_0 is

$$s(s-1) - \left(\frac{n^2-1}{4} - n^2 r_1 r_2\right) = 0\,. \tag{11}$$

By a Theorem of Fuchs, [26, Section 2.4], L_t is also a Fuchsian equation. We can further check that if $n > 1$ and $\frac{n^2-1}{4} - n^2 r_1 r_2 = 0$ then $t = t_0$ is not a pole of $q'(t)$. Hence it is not a singular point of L_t. □

3.4. Symmetric square

The symmetric square of the operator L (2) is an order 3 operator given by

$$Sym^2(L) := \frac{d^3}{dt^3} + 3p(t)\frac{d^2}{dt^2} + (2p(t)^2 + 4q(t) + p'(t))\frac{d}{dt} + (4p(t)q(t) + 2q'(t))\,. \tag{12}$$

It satisfies that the product of any two solutions of $Lu = 0$ is a solution of $Sym^2(L)u = 0$.

We can define, in a similar manner, the characteristic equation of (12). It will be a cubic equation in one variable. The solutions of this equation are again called the characteristic exponents of (12). If r_1 and r_2 are the characteristic exponents of (2) at t_0, then $2r_1$, $2r_2$ and $r_1 + r_2$ will be the 3 characteristic exponents of (12) at t_0.

4. Picard–Fuchs equations of one-parameter families of elliptic curves

For a one-parameter family of algebraic varieties, a Picard–Fuchs differential equation is a differential equations satisfied by the periods of the differential top forms of the fibers. By the work of P. A. Griffiths, Picard–Fuchs equations are Fuchsian equations.

4.1. One-parameter family of elliptic curves

There is a one-to-one correspondence between one-parameter families of elliptic curves with non-constant j-functions and smooth minimal elliptic surfaces $f : X \longrightarrow \mathbb{C}P^1$ with non-constant functional invariants [12]. We will always assume in this article that every elliptic surface is smooth and minimal in the sense that it does not contain any rational curves with self-intersection number -1.

In this paper, any one-parameter families of elliptic curves will be given in the following Weierstrass form:

$$E_t : y^2 = 4x^3 + b_2(t)x^2 + b_4(t)x + b_6(t),\ b_i(t) \in \mathbb{C}(t) \tag{13}$$

with the natural elliptic fibration

$$\begin{array}{rcl} f : \{(x, y, t) | y^2 = 4x^3 + b_2(t)x^2 + b_4(t)x + b_6(t)\} & \longrightarrow & \mathbb{C}P^1 \\ (x,\ y,\ t) & \longmapsto & t\,. \end{array}$$

The j-function of E_t is a rational function which characterizes isomorphism classes of E_t in the algebraic closure of $\mathbb{C}(t)$. It is also called the functional invariant of corresponding elliptic surfaces. We will always assume that the j-functions of the elliptic curves we consider are not constant functions.

We use two notations to denote an elliptic surface. When we use E_t we emphasize the elliptic fibration. We also use X (in short for $(X,\ f)$) to denote the total space. Let Σ be the finite set of points in $\mathbb{C}P^1$ such that the corresponding fibers are of genus different from 1. We adapt Kodaira's notation [8] for special fibers of the smooth minimal model of the corresponding elliptic surface. The *Mordell–Weil group* of E_t is the group of global sections of E_t with respect to the fibration f. When the j-function is not a constant, by a theorem of Mordell this groups is a finitely generated abelian group [9].

Let

$$\omega_t := \frac{dx}{y} = \frac{dx}{\sqrt{4x^3 + b_2(t)x^2 + b_4(t)x + b_6(t)}} \tag{14}$$

denote the differential 1-form on E_t.

4.2. Monodromy representations and Picard–Fuchs equations of elliptic curves

Let t_0 be a fixed point in $\mathbb{C}P^1 - \Sigma$. Let C be a loop in $\mathbb{C}P^1 - \Sigma$ starting with and ending at t_0. Choose any basis γ_1 and γ_2 for $H_1(f^{-1}(t_0), \mathbb{Z})$. Let these two cycles change analytically along C. They end up with a new basis on $H_1(f^{-1}(t_0), \mathbb{Z})$, and let $M(C)$ be the matrix transforming the old basis to the new basis. The monodromy representation of the fundamental group is given by

$$\begin{aligned} \rho : \pi(\mathbb{C}P^1 - \Sigma, t_0) &\longrightarrow SL_2(\mathbb{Z}) \\ C &\longmapsto M(C). \end{aligned}$$

For a one-parameter family of elliptic curves with non-constant j-function, the periods given formally by $\int \omega_t$ satisfy a homogenous order 2 ordinary linear Fuchsian equation

$$Lu = \left(\frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)\right)u = 0$$

(it is unique up to a scalar in $\mathbb{C}(t)$), called the *Picard–Fuchs equation* of the family. In fact, L is the order 2 operator which makes $L\omega_t$ an exact differential in x. For any one-parameter family of elliptic curves given by (13), its Picard–Fuchs equation is given explicitly by the j-function and the λ-function of the family.

For further details, the reader is referred to [21,22].

4.3. Isogeny

Let $E^{(1)}$ and $E^{(2)}$ be two elliptic curves defined over a certain field K which may be a number field, a finite field, or a function field, etc. An isogeny between $E^{(1)}$ and $E^{(2)}$ over K is a non-trivial over K morphism

$$\phi : E^{(1)} \longrightarrow E^{(2)}.$$

For an elliptic curve E defined over a number field K, the set of isogeny classes of E over K is finite [16]. Deligne extended the above result and obtained that the set of isomorphism classes of the monodromy representation

$$\rho : \pi_1(\mathbb{C}P^1 - \Sigma, t_0) \longrightarrow GL(H_1(f^{-1}(t_0), \mathbb{Z}) \otimes \mathbb{Q})$$

is finite [3].

The following theorem corresponds to Theorem III.3.1 in [21]. To enlighten the computation that follows we give a proof here.

Proposition 6. *Suppose $E^{(1)}$ and $E^{(2)}$ are two elliptic curves given by equations of the form of* (13) *over $\mathbb{C}(t)$ with non-constant j-functions. If there is an isogenous* *map* $\phi : E^{(1)} \longrightarrow E^{(2)}$, *then the Picard–Fuchs equations satisfied by* $E^{(1)}$ *and* $E^{(2)}$ *respectively are equivalent.*

Proof. Let $\omega_t^{(1)}$ and $\omega_t^{(2)}$ defined as (14) be holomorphic 1-forms on $E^{(1)}$ and $E^{(2)}$, respectively. If L is the Picard–Fuchs operator of $E^{(2)}$ then $L\omega_t^{(2)} = df$, for some function f, is an exact differential. Let $\phi^*\omega_t^{(2)}$ be the pull-back of $\omega_t^{(2)}$ on $E^{(1)}$, which is also a differential on $E^{(1)}$. Hence

$$\phi^*\omega_t^{(2)} = a(t)\omega_t^{(1)},$$

for some $a = a(t) \in \mathbb{C}(t)$.

$$aL^a\omega_t^{(1)} = aL^a\frac{1}{a}\phi^*\omega_t^{(2)} = L\phi^*\omega_t^{(2)} = \phi^*L\omega_t^{(2)} = \phi^*df = d\phi^*f$$

is exact. Hence L^a is a Picard–Fuchs operator for $E_t^{(1)}$. □

Let A be an element in an algebraic closure $\bar{K}$ of K, then the quadratic twisting of E by A is the elliptic curve defined over $\bar{K}$ by the equation

$$E^A : Ay^2 = 4x^3 + b_2x^2 + b_4x + b_6. \tag{15}$$

By Lemma 3.2, if $L = \frac{d^2}{dt^2} + p\frac{d}{dt} + q$ is the Picard–Fuchs operator for E, then the twisted operator $L^{\sqrt{A}}$ (defined in (8)) is the Picard–Fuchsian operator for E^A.

Theorem 7 (*Stiller [21, Theorem III.3.3]*). *Let* $f_1 : E^{(1)} \longrightarrow \mathbb{C}P^1$, $f_2 : E^{(2)} \longrightarrow \mathbb{C}P^1$ *be two elliptic curves over* $\mathbb{C}(t)$ *and assume that* j*-invariants are not constant. If the Picard–Fuchs equations of these two families are equivalent, then up to an isomorphism, there exists an isogeny* ϕ *from* $E^{(1)}$ *to* $E^{(2)}$.

If E is defined over K, we can choice a defining equation for E to be

$$E : y^2 = 4x^3 + b_2x^2 + b_4x + b_6 \tag{16}$$

such that b_i's are in the ring of integers $\mathcal{O}_K$ of K. Such an equation is called a *minimal* equation if there does not exist any non-unit element $a \in \mathcal{O}_K$ such that $a^2|b_2,\ a^4|b_4,\ a^6|b_6$.

Remark 8. Two elliptic curves over $\mathbb{C}(t)$ are isomorphic if after linear transformations of the base curves (if necessary) the elliptic curves have the same minimal Weierstrass equations over $\mathbb{C}[t]$.

Corollary 9 (*Stiller [21, Theorem III.3.7]*). *Let* $E^{(1)}$ *and* $E^{(2)}$ *be two elliptic curves with non-constant* j*-functions defined over* $\mathbb{C}(t)$ *satisfying that the projective normal* *forms of their Picard–Fuchs equations are the same. Then after a possible linear transform of the base variable t, $E^{(1)}$ is isogenous to $E^{(2)}$ up to a quadratic twisting by an element in an algebraic closure of $\mathbb{C}(t)$.*

4.4. Local monodromy of projectively normalized Picard–Fuchs equations

For any one-parameter family of elliptic curves in the Weierstrass form (13) with non-constant j-function, we consider the projective normal form of its Picard–Fuchs equation. One method is to use Stiller's "pulling back by the j-map" approach [22]. This method is based on the following fact: consider any family of elliptic curves parameterized by the j-function, then the projective normal form of the Picard–Fuchs operator associated to this family is

$$L = \frac{d^2}{dj^2} + \frac{j^2 - 1968j + 2654208}{4j^2(j-1728)^2}\,. \tag{17}$$

For simplicity, we denote the rational function $\frac{j^2-1968j+2654208}{4j^2(j-1728)^2}$ by $Q(j)$.

Remark 10. The operator (17) has 3 regular singular points, namely 0, 1728 and ∞. The characteristic exponents of L at these regular singular points are $\frac{1}{3}$ and $\frac{2}{3}$, $\frac{1}{4}$ and $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{1}{2}$, respectively.

Apply Lemma 5 to the operator (17), we can obtain the following Theorem which is a restatement of Proposition 4.9 [5].

Theorem 11. *Suppose E_t is a one-parameter family of elliptic curves with non-constant j-function $j(t)$ given by a Weierstrass Eq. (13). Let L_t be the projective normal form of the Picard–Fuchs operator of E_t. We have*

(1) *if t_0 is a zero of $j(t)$ with multiplicity n, then the characteristic exponents of L_t at t_0 are $\frac{n+3}{6}$, $\frac{-n+3}{6}$;*
(2) *if t_0 is a zero of $j(t) - 1728$ with multiplicity n, then the characteristic exponents of L_t at t_0 are $\frac{n+2}{4}$, $\frac{-n+2}{4}$;*
(3) *if t_0 is a pole of $j(t)$ with multiplicity n, then the characteristic exponents of L_t at t_0 are $\frac{1}{2}$, $\frac{1}{2}$.*

Proof. Let $\frac{x+y}{2x}$ and $\frac{x-y}{2x}$ be the characteristic exponents of (17) around the singular points 0, 1728, or ∞. By Remark 10, we have

$$(x, y) = \begin{cases} (3, 1) & \text{for case 1,} \\ (2, 1) & \text{for case 2,} \\ (2, 0) & \text{for case 3.} \end{cases}$$

The Theorem follows from Lemma 5. □

By Theorem 11 and the Tate algorithm [23], we derive the following conclusion, which has already been stated in some other literatures like [14].

Corollary 12. (1) *Let t_0 be a zero of j of multiplicity $n \geqslant 0$.*

(a) *If $n \equiv 1 \bmod 3$, then the local monodromy for the fiber E_{t_0} is either* $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. *Hence the special fiber E_{t_0} is of the type II or IV^*.*

(b) *If $n \equiv 2 \bmod 3$, then the local monodromy for the fiber E_{t_0} is either* $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. *Hence the special fiber E_{t_0} is of the type II^* or IV.*

(c) *If $n \equiv 0 \bmod 3$, the special fiber E_{t_0} is of the type I_0 or I_0^*.*

(2) *Let t_0 be a zero of $j - 1728$ of multiplicity $n \geqslant 0$.*

(a) *If $n \equiv 1 \bmod 2$, then the local monodromy for the fiber E_{t_0} is either* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. *Hence the special fiber E_{t_0} is of the type III or III^*.*

(b) *If $n \equiv 0 \bmod 2$, E_{t_0} is of the type I_0 or I_0^*.*

(3) *If t_0 is a pole of j of multiplicity $n \geqslant 1$, then local monodromy for the special fiber E_{t_0} is* $\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. *The E_{t_0} is of the type I_b or I_b^*.*

5. Picard–Fuchsian equations of M_n-polarized K3 surfaces

Let X_t be a one-parameter family of M_n-polarized K3 surfaces such that the image of X_t on $\mathfrak{H}$ via the Dolgachev isomorphism 3 is a not single point. Let ω_t be a holomorphic 2-form on X_t. Then the formal periods $\int \omega_t$ satisfy an order 3 ordinary linear Fuchsian differential equation in t, called the *Picard–Fuchs equation* of X_t. It satisfies the following property.

Theorem 13 (*Doran [5]*). *The Picard–Fuchs equation of a family of M_n-polarized* K3 *surfaces is the symmetric square of an order* 2 *homogenous linear Fuchsian ordinary differential equation.*

If up to a change of variables if necessary the order 2 differential equation can be realized as the Picard–Fuchs equation of a one-parameter family of elliptic curves, then by Theorem 7, this family of elliptic curves is associated to the K3 family by a S–I structure.

6. An algorithm

The following algorithm is developed to construct a one-parameter family of elliptic curves associated to a one-parameter family of K3 surfaces with generic Picard number

19 by a Shioda–Inose structure. The family of elliptic curves is determined up to an isomorphism and an isogeny.

Suppose we have a one-parameter family of K3 surfaces X_t with generic Picard number. This family lies in the moduli space of a M_n-polarized K3 surfaces with $n = \det T_X$ for a general fiber. We assume that the image of X_t on $\mathfrak{H}$ via the Dolgachev isomorphism 3 is a not single point.

(1) Calculate the Picard–Fuchs equation L_3 of X_t. This can be done for example by formal calculation on the holomorphic 2-form ω_t for the K3 surface X_t.
(2) Calculate the order 2 ordinary linear differential equation L satisfy $L_3 = Sym^2(L)$.
(3) Calculate the projectively normalized operator $\bar{L}$ of L.
(4) With the guidance of Lemma 5, suitably choose the variable $t = t(s)$ to be a rational function of s such that the characteristic exponents of projectively normalized operator $\bar{L}_s$ in s of $\bar{L}$ satisfy the conditions of Remark 10 and Theorem 11. We then construct a suitable j-function $j(s)$ which satisfies all the local monodromy conditions imposed by Corollary 12. Now we pick

$$E'_s : \; y^2 + xy = x^3 - \frac{36}{j(s) - 1728}x - \frac{1}{j(s) - 1728}.$$

It is a family of elliptic curves with j-function to be exactly $j(s)$.
(5) Choose a suitable algebraic equation $A(s)$ such that the Picard–Fuchs equation of the twisted family $E_s = (E'_s)^A$ coincides with the order 2 operator L with variable in s.

By Theorem 7, a S–I structure of X_s is given by isogenous class of E_s up to an isomorphism.

We also note that two elliptic curves defined over a number field K will have the same j-invariant if there are differed by a quadratic twisting and will have the same L-function if there are isogenous over K. Hence the algorithm will apply to solve certain arithmetic problems. For example, it is interesting to find points on $\mathbb{C}P^1$ such that the corresponding fibers X_s have Picard number 20.

7. An application

An application of this algorithm to find the j-function of the family of elliptic curves giving rise to a S–I structure of the family of K3 surfaces studied by Ahlgren et al. [1] has been discussed in a separate paper [11]. We now consider another application in a generalized congruence number problem.

A natural number n is called a (classical) *congruent number* if it is the area of a right rational triangle. For instance, 6 is a congruent number since (3,4,5) is a Pythagorean triple. For a survey on congruent number problems and their variants in dimensions 1 and 2, refer to a paper of Topp and Yui. [24].

In this section, the question we are concerned is formulated as follows: *Given a square-free natural number n and a real number α $(0 < \alpha < \pi)$. Is there any rational* *triangle with an angle* α *whose area is* n? When $\alpha = \pi/2$, it is the classical congruent number problem.

Let n be the area of the rational triangle with the lengths of the sides x, y, z and α the angle between x and y. Then we have the following identities:

$$xy \sin(\alpha) = 2n\,, \tag{18}$$

$$x^2 + y^2 - 2xy\cos(\alpha) = z^2\,. \tag{19}$$

Hence $\sin(\alpha), \cos(\alpha) \in \mathbb{Q}$ and there is a positive rational number $t \in \mathbb{Q}^{\times}$, such that

$$\sin(\alpha) = \frac{2t}{t^2+1}, \quad \cos(\alpha) = \frac{t^2-1}{t^2+1}. \tag{20}$$

For a given $t \in \mathbb{Q}^{\times}$, a square-free natural number n is called a *t-congruent number* if there is a positive rational triple (x, y, z) satisfying the equations

$$xy\frac{2t}{t^2+1} = 2n, \tag{21}$$

$$x^2 + y^2 - 2xy\frac{t^2-1}{t^2+1} = z^2. \tag{22}$$

When $t = 1$, this is the classical congruent number problem.

Using identities (21) and (22), we have

$$(x-y)^2 = z^2 - 4\frac{n}{t}\,, \tag{23}$$

$$(x+y)^2 = z^2 + 4nt\,. \tag{24}$$

Combining (23) and (24) to obtain

$$(x^2-y^2)^2 = (z^2 - 4\frac{n}{t})(z^2+4nt)\,. \tag{25}$$

Multiply both hand sides of (25) by z^2, we have

$$(zx^2 - zy^2)^2 = z^2\left(z^2 - 4\frac{n}{t}\right)(z^2+4nt)\,. \tag{26}$$

Putting $X = z^2/4, Y = (zx^2 - zy^2)/8$ into (26), we obtain

$$Y^2 = X\left(X - \frac{n}{t}\right)(X+nt)\,. \tag{27}$$

Proposition 14. *Fix a positive rational number* t, *and let* n *be a square-free natural number. Then* n *is a* t-*congruent number, if and only if both* $\frac{n}{t}$, t^2+1 *are rational squares or the elliptic curve*

$$E_{n,t} : y^2 = x(x - n/t)(x + nt)$$

has a non-trivial rational point, i.e. its y-*coordinate is not zero.*

Proof. If n is a t-congruent number, then there exists $(X, Y, Z) \in \mathbb{Q}_+^3$, such that

$$X^2 + Y^2 - 2XY\frac{t^2-1}{t^2+1} = Z^2, \;\; XY\frac{2t}{t^2+1} = 2n. \tag{28}$$

It is easy to check that

$$P = (x, y) = \left(\left(\frac{Z}{2}\right)^2, \frac{(X^2-Y^2)Z}{8}\right) \in E_{n,t}(\mathbb{Q}).$$

If $X^2 = Y^2$, then we know by the above two equations (28)

$$\frac{n}{t} = \left(\frac{Z}{2}\right)^2, \;\; t^2 + 1 = \left(\frac{XZ}{2}\right)^2.$$

Hence both of them are rational squares. Otherwise $y = \frac{(X^2-Y^2)Z}{8}$ is not zero.

Conversely, if $\frac{n}{t}$, t^2+1 are rational squares, then

$$Z = 2\sqrt{\frac{n}{t}}, \; X = Y = \sqrt{\frac{n(t^2+1)}{t}}$$

will be the lengths of the three sides of a rational square with the area n. If $P = (x, y) \in E_{n,t}(\mathbb{Q})$, with $y \neq 0$, then the triple

$$X = \left|\frac{x^2 - n^2 - nx(1/t - t)}{y}\right|, \;\; Y = n\left|\frac{x(t + 1/t)}{y}\right|, \;\; Z = \frac{x^2 + n^2}{|y|}$$

is a rational solution of equation (27). □

For example, take $t = 4$, we have

$$E_{2,4} : y^2 = x(x - 1/2)(x + 8).$$

The Mordell–Weil group $MW(E_{2,4}(\mathbb{Q}))$ has rank 1, hence 2 is a 4-congruent number.

Consider the transformation $(X, Y) \mapsto (\frac{x}{n}, \frac{yt}{n})$. Eq. (27) becomes

$$y^2 = xnt(xt-1)(x+t)\,. \tag{29}$$

Assume that n is a linear function of t, i.e. $n = a(t+s)$. We further assume $a = 1$. For generic choices of s, the smooth minimal model of the elliptic surface

$$\begin{array}{rccl} f : K_s = \{(x, y, t) \mid y^2 = x(t+s)t(xt-1)(x+t)\} & \longrightarrow & \mathbb{C}P^1 \\ (x, y, z) & \longmapsto & t \end{array}$$

is a K3 surface. We are interested in finding some particular values of s such that K_s has non-trivial global section over $\mathbb{Q}(t)$.

The j-function of K_s is

$$j = 256\frac{(t^2+t+1)^3(t^2-t+1)^3}{t^4(t^2+1)^2}.$$

For a generic s, K_s has 5 special fibers at the points $0,\ i,\ -i,\ -s,\ \infty$ of the types $I_4^*,\ I_2,\ I_2,\ I_0^*,\ I_4$, respectively. By the Shioda–Tate formula [18, Corollary1.5], the generic Picard number is 19.

If we choose the value of s suitably such that the fiber K_s is a K3 surface with Picard number 20, then the Mordell–Weil rank of K_s with fibration over t-line may be 1. If the Mordell–Weil rank is indeed 1, we can give a family solutions of Eq. (29) as

$$x = x(t),\ \ y = y(t),\ \ n = n(t).$$

Since t is a linear function of n,

$$x = x(n),\ \ y = y(n),\ \ t = t(n)\,.$$

If $x(t),\ \ y(t) \in \mathbb{Q}(t)$, by Proposition 14, every positive integer n can be realized as the area of a rational triangle. Motivated by this idea, we consider the S–I structure of this family K_s.

We follow Stienstra's method to determine the Picard–Fuchs equation for the family of K3 surfaces. The method applies to families of K3 surfaces satisfying certain conditions imposed in his Theorem [20, Theorem 2].

Rewrite K_s into the following form such that the right-hand side is a homogenous polynomial in $X,\ T,\ Z$ as

$$K_s : Y^2 = X(T+sZ)T(XT-Z^2)(X+T). \tag{30}$$

For generic $s \in \mathbb{Z}$, the Artin–Mazur functor $H^2(X_s, \hat{G}_{m, X_s})$ is a formal group over $\mathbb{Z}$ of dimension 1. The coefficients of logarithm $l(\tau)$ of this formal group can be determined explicitly by the formula given in [20, Theorem 2]. In this case we get that

$$l(\tau) = \sum_{m \geqslant 1} m^{-1} \left(\sum_{k \geqslant 0} \sum_{k} \binom{\frac{p-1}{2}}{2k} \binom{\frac{p-1}{2}}{k}^2 (-s^2)^k \right) \tau^m.$$

In the Appendix of Stienstra paper, the logarithm is also interpreted as an integral [20, (A3)], hence is a formal period. It follows that the coefficients of $l(\tau)$ in our case, as functions of s, satisfy the Picard–Fuchs equations of the family X_s. Moreover for any odd prime p,

$$\sum_{k \geqslant 0} \binom{\frac{p-1}{2}}{2k} \binom{\frac{p-1}{2}}{k}^2 (-s^2)^k \equiv \sum_{k \geqslant 0} \frac{(\frac{1}{2})_{2k} (\frac{1}{2})_k (\frac{1}{2})_k}{(2k)!(k!)^2} (-s^2)^k \bmod p,$$

where for any $a \in \mathbb{C}$ and a positive integer k, $(a)_k = a(a+1) \cdots (a+k-1)$. It indicates that $\sum_{k} \frac{(\frac{1}{2})_{2k} (\frac{1}{2})_k (\frac{1}{2})_k}{(2k)!(k!)^2} (-s^2)^k$ is a local holomorphic solution around 0 of the Picard–Fuchs equation of the family of K3 surfaces with parameter s.

Let

$$_2F_1(a_1, a_2; b_1; x) = \sum_{n \geqslant 0} \frac{(a_1)_n (a_2)_n}{(b_1)_n (1)_n} x^n,$$

$$_3F_2(a_1, a_2, a_3; b_1, b_2; x) = \sum_{n \geqslant 0} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n (1)_n} x^n,$$

where $a_i, \ b_i \in \mathbb{C}$, $(a)_n = a(a+1) \cdots (a+n-1)$, and b_i's are not $0, -1, -2, \ldots$.

The radius of convergence of the hypergeometric series $_2F_1$ and $_3F_2$ is 1, unless one of the a_i's is a non-positive integer.

In terms of hypergeometric series identity [10]

$$\sum_{k \geqslant 0} \frac{(\frac{1}{2})_{2k} (\frac{1}{2})_k (\frac{1}{2})_k}{(2k)!(k!)^2} (-s^2)^k = {}_3F_2\left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}; 1, 1; -s^2\right) = {}_2F_1\left(\frac{1}{8}, \frac{3}{8}; 1; -s^2\right)^2.$$

In fact, the symmetric square of the differential operator L satisfied by $_2F_1(\frac{1}{8}, \frac{3}{8}; 1; -s^2)^2$ is the Picard–Fuchs operator of the family of K3 surfaces X_s.

The projective normal form $\bar{L}$ of L is

$$\bar{L} = \frac{d^2}{ds^2} + \frac{-5s^2 + 3s^4 + 4}{16s^2(1+s^2)^2}. \tag{31}$$

Unfortunately, by checking the ramification information, it is easy to prove that there is no elliptic curve defined over $\mathbb{C}(s)$ whose projectively normalized Picard–Fuchs operator is given by (31). Now we search for a rational function to replace s such that on one hand side we can minimize the number of singular points and on the other we make all the singular points correspond to special fibers of the types I_b or I_b^* by Corollary 12. It turns out that if we let $s = \frac{u^4-1}{2u^2}$. Then the projective normal form of the order 2 ordinary operator satisfied by ${}_2F_1(\frac{1}{8}, \frac{3}{8}; 1; -\frac{(u^4-1)^2}{4u^4})$ is

$$\frac{d^2}{du^2} + 4\frac{u^2}{(u^4-1)^2}\,. \tag{32}$$

We note that only the linear transformation $u \longrightarrow -u$ will leave the equation invariant.

Any elliptic curves over $\mathbb{C}(u)$ whose Picard–Fuchs operator has the normal form (32) will have 4 special fibers at $1, -1, i, -i$ respectively of the types I_b or I_b^* for $b > 0$.

It turns out that the elliptic modular surface associated to the congruence subgroup $\Gamma(2) \bigcap \Gamma_0(4)$ using the notations of [17] with a suitable base parameter satisfies the above conditions. Under this base parameter, say u, the functional invariant of this surface is

$$j(u) = \frac{64(3u^2+1)^3(u^2+3)^3}{(1+u^2)^2(u^2-1)^4}\,. \tag{33}$$

The linear transformation $u \longrightarrow -u$ leaves operator (32) as well as the j-function (33) invariant. By Corollary 9, for any algebraic number u such that $j(u)$ is the j-invariant of an elliptic curve with complex multiplication, then $X_{s(u)}$ has Picard number 20. In particular, the j-invariant of the elliptic curve with complex multiplication by the ring $\mathbb{Z}[\sqrt{-5}]$ is $632,000 + 282,880\sqrt{-5}$. In this case,

$$u^2 = \frac{-1 \pm \sqrt{-5}}{2},\ s = -\frac{1}{2}.$$

As what we expected, the Mordell–Weil group of

$$K_{-\frac{1}{2}} : y^2 = x\left(t - \frac{1}{2}\right)t(xt-1)(x+t)$$

is of rank 1, and the generator of its non-torsion part is

$$(x,\ y) = \left(\frac{t}{2t-1},\ \frac{t^2(t-1)}{2t-1} \right). \tag{34}$$

Proposition 15. *Every natural number can be realized as the area of a rational triangle.*

Proof. By parameterization (34), we can get that for any $n \neq 2$, the rational triangle with three sides parameterized as

$$n - \frac{1}{2},\ \frac{n(n^2+n+5/4)}{(n+1/2)(n-1/2)},\ \frac{5/2n^2+1/2n+1/8}{(n+1/2)(n-1/2)}$$

will have area n. For $n = 2$, we have seen that it is a 4-congruent number. □

Note that a similar parameterization have also been obtained by and Chowla [7] and later independently by Cohen (communicated to Yui and Long).

Acknowledgments

The author would like to express her sincerest gratitude to Wenching Winnie Li, Noriko Yui, Erico Bombieri, Pierre Deligne, Tomohide Terasoma, Scott Ahlgren, Ken Ono, David Penniston, Charles F. Doran, and Jan Stienstra for discussions and suggestions.

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