Topological Semigroups of Non-Negative Matrices.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II GENERALITIES</td>
<td>6</td>
</tr>
<tr>
<td>III GROUPS</td>
<td>18</td>
</tr>
<tr>
<td>IV IDEMPOTENTS</td>
<td>25</td>
</tr>
<tr>
<td>V COMMUTATIVE SEMIGROUPS</td>
<td>34</td>
</tr>
<tr>
<td>VI COMPACT SIMPLE SEMIGROUPS</td>
<td>40</td>
</tr>
<tr>
<td>VII BIBLIOGRAPHY</td>
<td>46</td>
</tr>
<tr>
<td>VIII BIOGRAPHY</td>
<td>51</td>
</tr>
</tbody>
</table>
ABSTRACT

This paper is devoted primarily to the study of topological semigroups of non-negative matrices. Usually, these semigroups are also assumed to be compact.

Theorems on matrices and semigroups, which are germane to the paper, are first presented. Attention is focused on the spectrum of a non-negative matrix.

It is first shown that a compact topological group of non-negative matrices is finite, by using the spectral properties of these matrices. From this theorem it follows that a clan (continuum semigroup with unit) of non-negative matrices is contractible. Some results on the existence of I-semigroups in a clan are also given.

Next, the general structure of non-negative idempotents is investigated. As an application of this investigation, the set of non-negative idempotents of a fixed rank and order is shown to be arcwise connected. A similar theorem is obtained for the subset of stochastic idempotents of fixed rank and order.

Commutative semigroups are next studied. The Jordan form of a matrix is used to show that any commutative semigroup of complex matrices is similar to a triangular complex matrix semigroup. This theorem, together with various algebraic and
topological hypotheses, is used to obtain several sets of sufficient conditions that a semigroup be similar to a semigroup of diagonal matrices.

The paper terminates with a chapter concerning topological representations of finite dimensional compact simple semigroups. It is shown that any such entity $S$ in which the idempotents form a subsemigroup has an isomorphic imbedding in the non-negative matrices if and only if the maximal groups of $S$ are finite. An analogous result is proved in which maximal groups are Lie groups and complex matrices are used in place of non-negative real matrices. It is also shown that, if $S$ is simple, if $E$ is connected, and if each maximal group of $S$ is totally disconnected, then $E$ is a subsemigroup of $S$. 
INTRODUCTION

Matrices over a field have appeared throughout the history of topological algebras, as a source of examples, through representation theory, and as separate entities worthy of investigations in themselves. In the latter category lies the copious field of locally compact Lie groups, which is among the more active areas of current mathematical research. Matrix semigroups have not gone unnoticed; papers by the Russian authors Suschkewitsh [26; 27] and Gluskin [12; 13; 14; 15], primarily algebraic in nature, have appeared in this direction. Also to be noted are the papers by Gleason [10; 11] on one-parameter semigroups, and the volume of Hille [38]. The latter work contains an extensive study of normed semigroups.

Moreover, the first investigations of topological matrix semigroups in the prevailing spirit seem to have been made in an unpublished paper by M. J. Etter [7] on stochastic matrices. A matrix is stochastic if it is non-negative and each row sum is one. Etter showed that the stochastic matrices of fixed order form a convex clan, and studied the structure of stochastic idempotents. Following this, Cohen and Collins [5] initiated a study of affine semigroups. A semigroup is affine if it is a convex subset of a real topological

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1 Numbers in brackets refer to the bibliography listed at the end of this dissertation.
linear space, and the left and right translation mappings are affine functions. Among their results is the fact that a finite dimensional affine semigroup with identity and zero is isomorphic to a real matrix semigroup. They also characterize all one and two dimensional compact affine semigroups. Clark [1] has recently abstracted further contributions in this direction.

Matrix representations of semigroups have been treated in considerable detail, notably by Suschkewitsh [25], Clifford [3], and Munn [20; 21]. A complete treatment of the theory of representations of abstract semigroups in its current form is given by Clifford and Preston [35]. The lack of topological results in this direction stems primarily from the loss of the theory of invariant measures; there is no theorem about compact topological semigroups analogous to the Peter-Weyl theorem in compact Lie groups. Indeed, it will be seen in this paper (Chapter 3) that several well known semigroups on an interval cannot occur in a matrix semigroup over the complex numbers. Specifically, it will be shown that the only I semigroup which can occur is the interval \([0,1]\) under real number multiplication.

In the domain of examples, the semigroup of matrices of the form 
\[
\begin{pmatrix}
x & y \\
0 & 1
\end{pmatrix}, \quad x,y \in [0,1], \quad x + y \leq 1,
\]
has been in the folklore of topological semigroups for some time; it may be imbedded in the plane as the point set bounded by the \(x\) and \(y\) axes and the line \(x + y = 1\). There are several other examples of interest given in [5], including one example of a compact real matrix semigroup whose kernel is not convex.
This paper is devoted primarily to the study of topological semigroups of non-negative matrices; some results which are unimproved by the assumption of no negative entries are also included. The additional assumption of compactness (in the sense of bicompactness) is ordinarily also at hand, in order to have available the many theorems of topological semigroups which require this hypothesis.

Chapter two is divided into two parts; in the first part, definitions and theorems in topological semigroups required in this paper are given. Proofs of these theorems are omitted, but referenced. The second part of Chapter two is devoted to the definitions and theorems concerning matrices which are used in the sequel. Since this paper is written from the viewpoint of topological semigroups, rather than matrices, some details are supplied in this section. In particular, a theorem of Frobenius is proved. This theorem states that an irreducible non-negative matrix $M$ has a non-negative spectral element $r$ which dominates the spectrum of $M$, and to which corresponds a positive eigenvector. It is also shown that a non-negative matrix $M$ having a positive eigenvector is similar to the product of a stochastic matrix and the scalar matrix corresponding to the maximal real eigenvalue of $M$. These results motivate the cited papers of Dmitriev, Dynkin, and Kolmogoroff [6] and Karpelevich [16] in which the possible eigenvalues of a non-negative matrix are determined. For reasons to become apparent later, attention is focused on the set of eigenvalues of modulus one in the case that the maximal real eigenvalue of a non-negative matrix is also one. An expository account of the prior results occur in [36].
A topological space $X$ is contractible if the identity mapping of $X$ onto $X$ is homotopic to a constant map. In Chapter three, the results mentioned above are used to prove that a compact group of non-negative matrices is finite, from which it follows that a non-negative matrix clan is contractible. This answers a question raised by Wallace [31]. The results concerning I semigroups, mentioned earlier, are also presented.

Chapter four sets forth structure theorems for non-negative and stochastic idempotents. One result of this part is that a rank $K$ non-negative idempotent matrix is the direct sum of $K$ non-negative rank one idempotents. Theorems of this type are then used to show that the set of stochastic idempotents of a fixed rank is arcwise connected. A similar result is proved for non-negative idempotents of a fixed rank. The former was conjectured in [7].

The fifth chapter treats commutative semigroups of matrices. The principal result of the first part is that a commutative semigroup of complex matrices is similar to a triangular semigroup. This theorem does not seem to occur in the (modern) literature. The proof does not depend on topology, but is an exercise in the use of the Jordan form of a matrix. In the second part, various sufficient conditions for a real matrix semigroup to be diagonal are given. One result in this chapter not requiring commutativity is the following: let $S$ be a connected semigroup of real matrices having an identity $f$ and a zero $e$, whose ranks differ by one; then the convex arc between $e$ and $f$ is also in $S$, and is an I semigroup.
The paper concludes with the study of representations of compact simple semigroups which are finite dimensional. It is shown that any such object $S$ in which the idempotents form a semigroup, and the maximal groups are finite, is isomorphically imbeddable in the non-negative matrices whose order is a function of the dimension of $S$ and the order of a maximal group of $S$. A similar theorem is obtained by replacing "finite" with "Lie" and "non-negative" by "complex" in the previous sentence. If the idempotents are connected, and maximal groups totally disconnected, then the idempotents are shown to form a semigroup. It is conjectured by the author that any finite dimensional compact simple semigroup with finite groups is isomorphically imbeddable in the non-negative matrices of some related order.

The brief mention of cohomology in Chapter three refers to the cohomology theory of Alexander-Kolmogoroff-Wallace-Spanier. A brief treatment of this subject can be found in Wallace's invited address [30].
2.1 Definitions. S, a Hausdorff topological space, is a topological semigroup if there exists a function \( m \) on \( S \times S \) into \( S \) which is jointly continuous and associative; that is \( m[a, m(b, c)] = m[m(a, b), c] \). In the sequel, \( m \) is suppressed and \( m(a, b) \) is written as \( ab \). Conjunction is also used to indicate set multiplication; \( AB = \{ ab \mid a \in A, b \in B \} \). An element \( e \) in \( S \) is idempotent if \( e^2 = ee = e \). A clan is a compact, connected topological semigroup having an element 1 such that \( 1x = x = x1 \) for every \( x \in S \). 1 is called an identity for \( S \). An element \( z \) in \( S \) is a zero for \( S \) if \( zx = z = xz, x \in S \); if \( S \) has a zero \( z \), an element \( x \in S \) is nilpotent if there is a positive integer \( K \) such that \( x^K = z \). A subset \( A \) of \( S \) is a left ideal of \( S \) if \( SA \subseteq A \); right ideals are defined dually. \( A \) is an ideal if it is both a left and a right ideal. \( A \) (left, right) ideal of \( S \) is minimal if it properly contains no (left, right) ideal of \( S \). If \( S \) has a minimal (two-sided) ideal, it is clearly unique; this is false for one-sided ideals. A minimal ideal, if such exists in \( S \), is called the kernel of \( S \), and is denoted by \( K \) without exception. A topological semigroup containing no proper (left, right) ideals is (left, right) simple. A subset \( A \) of \( S \) is a subsemigroup if \( AA \subseteq A \); \( A \) is a subgroup if \( A \) is algebraically a group, and the function \( 0(x) = x^{-1}, x \in A \), is continuous in the relative topology of \( A \). Finally, if \( S, T \) are topological semigroups, then a
function $f$ on $S$ into $T$ is a **homomorphism** if $f(ab) = f(a)f(b)$ for all $a, b \in S$. If $f$ is one to one, it is an **isomorphism**. If, additionally, $f$ is a homeomorphism, then $f$ is an **isomorphism**.

Fundamental results concerning the above are due in the main to Wallace [28], Koch [17], and Numakura [22]; see the bibliography in [29]. Semigroups on sets having no mentioned topology have been studied by Suschkewitsh [24], Rees [23], and Clifford [2; 4]. The standard work on abstract semigroups is the book of Clifford and Preston [35]. Let $E$ denote the set of idempotents of a semigroup $S$.

2.2 **The structure of $K$**. If $S$ is a compact semigroup, then $S$ is known to contain minimal ideals of all categories, and $K = \bigcup \{L: L$ is a minimal left ideal $\} = \bigcup \{R: R$ is a minimal right ideal$\} = \bigcup \{H(e): e \in K\}$, where $H(e)$ is the maximal subgroup of $S$ containing $e$. If $L(R)$ is a minimal left (right) ideal, then $L = Se(R = eS)$ for some $e \in K \cap E$. If $a, b \in K$, then there exists $e \in K \cap E$ such that $aS \cap Sb = H(e) = eSe$, and $Se = Sb, eS = aS$. If $y \in eS(eS)$, then the function $F$ defined on $H(e)$ by $F(x) = xy$ ($F(x) = yx$) is an isomorphism of $H(e)$ onto $H(f)$, where $y \in H(f)$. Finally, fix $e \in K \cap E$. Then $K$ is isomorphic to the cartesian product $eS \cap E \times H(e) \times Se \cap E$ under a multiplication defined in [30]. The other results above can be found in the references mentioned in the preceding paragraph.

2.2 **$I$-semigroups**. The topological semigroup structures that an interval (a homeomorphic image of the interval $[0,1]$) can support were among the first characterized. An **$I$-semigroup** (standard thread)
is a topological semigroup $S$ on an interval such that one endpoint is an identity and the other a zero. $I$-semigroups have been characterized by Faucett [8] and Mostert and Shields [18]. Clifford has a series of papers on totally ordered semigroups which is relevant. It is shown in [18] that the only types existant are the following:

(i) $S$ has the multiplication of the real interval $[0,1]$ (type $I_1$).

(ii) $S$ has a multiplication isomorphic to the interval $[1/2,1]$ under the operation $x \circ y = \max\{1/2,xy\}$ (type $I_2$).

(iii) $S$ is idempotent and has a multiplication isomorphic to the interval $[0,1]$ under the operation $x \circ y = \min\{x,y\}$ (type $I_3$).

(iv) $S$ is the union of a collection of semigroups of types $I_1, I_2, I_3$ which meet only at their respective endpoints. In this case, multiplication of elements belonging to different semigroups in the union is given by $x \circ y = \min\{x,y\}$, where the order is that one inherited by mapping $S$ homeomorphically onto $[0,1]$ such that $h(0) = 0$, $h(1) = 1$.

2.4 Definition. Let $S$ be a topological semigroup containing an idempotent $e$. A one-parameter semigroup based at $e$ is a continuous homomorphic image of the additive non-negative real numbers such that $\sigma(0) = e$. A one-parameter subgroup is defined similarly, the domain of $\sigma$ being extended to all real numbers. Such objects
are \textbf{local} if \( \sigma \) is restricted to a (one-sided) neighborhood of 0.

2.5 Definition. A \textit{Lie group} is a topological group whose underlying space is locally euclidean. This definition is chosen because the applications of Lie groups in the sequel do not require any properties of the coordinate transformations mentioned in the standard definition. For further information, see [34] and [41].

In the remainder of this paper, \( S \) denotes a topological semigroup unless specified otherwise. The adjective "topological" will be dropped when no confusion seems likely to arise. \( E \) will ordinarily be the set of idempotents of \( S \). Generally, Roman capital letters represent sets, and lower case letters elements of sets; however, in proofs concerning matrices, it becomes convenient to sometimes specify matrices by Roman capital letters and their entries, and associated vectors, by lower case letters. This ambiguity is sufficiently restricted to prevent confusion.

The definitions of a matrix, matrix multiplication, and the determinant of a matrix are omitted. Matrices are decomposed into block form and multiplied in this manner without further reference. Diagonal blocks are always square submatrices. For further comment on the above remarks, and proofs of the elementary theorems involving matrices, the reader is referred to the works of Albert [33], Halmos [37], and Wedderburn [43].

2.6 Definitions. The \textbf{rank} of a matrix is the dimension of its range; the rank is known to be the number of linearly independent row
(column) vectors in any canonical array representing the matrix.

The **trace** of a matrix is the sum of the elements of its main diagonal. A matrix $M$ is **invertible (regular, non-singular)** if there exists a matrix $Q$ such that $MQ = I = QM$, where $I$ is the identity matrix. $Q$ is denoted by $M^{-1}$. A scalar $x$ is an **eigenvalue** (**characteristic value, spectral value**) of the square matrix $M$ if $\det(M-xI) = 0$. The **spectrum** $S(M)$ of a matrix is the set of all eigenvalues of $M$. A vector $v$ is an eigenvector of a matrix $M$ with respect to $x \in S(M)$ if $M(v) = x(v)$. The number of linearly independent eigenvectors corresponding to a fixed eigenvalue is the **multiplicity** of this eigenvalue. A **similarity transformation** is any inner automorphism generated by an invertible matrix $P$. A **permutation matrix** is one having exactly one 1 in each row and each column, and zeros elsewhere. A **row-column permutation**, or simply a permutation, is a similarity transformation generated by a permutation matrix. A matrix $M$ is **reducible** if there is a permutation carrying $M$ into the block form $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$; otherwise $M$ is **irreducible**. The **general linear group** of order $n$ is the group formed by the invertible matrices of this order. If the scalar field is the real numbers, this group is denoted by $\text{Gl}(n, \mathbb{R})$; if the field is the complex numbers, by $\text{Gl}(n, \mathbb{C})$. These groups are basic examples of Lie groups.

The bounded linear transformations on an $n$-dimensional topological vector space in a natural manner, with norm defined by

$$
\| A \| = \left( \sum_{i, j} (a_{ij})^2 \right)^{1/2}, \text{ where } A = (a_{ij}).
$$
2.7 Theorem [42]. Any locally convex topology on an n-dimensional topological vector space coincides with the topology of the norm given above.

In particular, the topology of euclidean \( \mathbb{R}^n \) space applied to the \( n \times n \) real matrices is locally convex, and is the most convenient for the considerations of this paper. In this frame of reference, it is obvious that matrix multiplication is continuous. These remarks indicate a way in which the results of this paper may be related to normed semigroups.

A fundamental concept in matrices is that of the Jordan form of a matrix. A proof of the following theorem may be found in [36].

2.8 Theorem. Let \( M \) be a complex \( n \times n \) matrix. Then \( M \) is similar to \( M' = \text{diag}(M_1, \ldots, M_k), \) \( M_i = S_i + N_i, \) where \( S_i \) is a scalar matrix and \( N_i \) is a nilpotent matrix having zeros and ones on the first principal subdiagonal and zeros elsewhere. Further, \( S_i \neq S_j \) if \( i \neq j. \)

2.9 Corollary. A rank \( k \) idempotent \( M \) is similar to \( \text{diag}(I_k, 0), \) where \( I_k \) is the \( k \times k \) identity matrix.

Proof: Let \( v \) be an eigenvector of \( M \) for the eigenvalue \( x. \) Then \( x(v) = M(v) = M^2(v) = x^2(v). \) Hence \( x = x^2, \) and \( S(M) \subseteq \{0, 1\}. \) Let \( M' = \text{PMP}^{-1} = \text{diag}(M_1, M_2), \) where \( M_1 = I_k + N_1, M_2 = 0 + N_2, \) with \( N_1 \) and \( N_2 \) nilpotent matrices of the type mentioned in 2.8. Since \( M_1^2 = M_1, M_2^2 = M_2, \) it follows that \( N_1 \) and \( N_2 \) are zero matrices.
2.10 Theorem [34]. The determinant function is a continuous homomorphism of the \( n \times n \) matrices into the scalar field.

2.11 Corollary. Let \( M \) be a matrix, \( S(M) = \{x_1, \ldots, x_k\} \). Let \( x_i \) have multiplicity \( n(i) \). Then \( \det(M) = x_1^{n(1)} x_2^{n(2)} \ldots x_k^{n(k)} \) and \( \text{trace}(M) = n(1)x_1 + \ldots + n(k)x_k \).

2.12 Theorem [33]. Let \( A \) and \( B \) be fields, \( A \subseteq B \). Let \( M \) and \( N \) be \( n \times n \) matrices over \( A \). If \( M \) and \( N \) are similar over \( B \), then they are similar over \( A \).

Proof: In applications, \( A \) will be the real numbers and \( B \) the complex numbers. For simplicity, the theorem is proved in this setting. Let \( M = PNP^{-1} \), \( P \) complex. Then \( P = B + iC \), with \( B, C \) real, \( BN = MB, CN = MC \). Hence, for any integer \( k \), \((B + kC)N = M(B + kC)\). By examining the Jordan form of \( P \), it is easily seen that there must exist a value of \( k \) for which \( B + kC \) is invertible.

In the study of non-negative matrices, the theorem of Frobenius [9] is fundamental. The proof given here is due to Wielandt [32], as presented in [36]. The ordinary inequality symbols, when applied to real matrices, indicate that every entry of the matrix satisfies this inequality. If \( A \) and \( B \) are real matrices, \( A \preceq B \) means \( B-A \geq 0 \). If \( A = (a_{ij}) \), let \( A^+ = (|a_{ij}|) \). If \( x \) is a vector, \( x^+ \) denotes the non-negative vector obtained in the same manner. I always represents the identity matrix and \( V \) the underlying vector space.
2.13 Lemma. If $A \geq 0$ is irreducible of order $n$, then $(I + A)^{n-1} > 0$.

Proof: It suffices to show that, if $y \geq 0$, $y \neq 0$, then

$(I + A)(y)$ has fewer zero coordinates than $y$. For this implies

$(I + A)^{n-1}(y) > 0$, from which it follows that $(I + A)^{n-1} > 0$.

Without loss of generality, suppose $y = \begin{pmatrix} u \\ 0 \end{pmatrix}$, $(I + A)(y) = \begin{pmatrix} v \\ 0 \end{pmatrix}$,

with $u,v$ positive vectors of the same dimension. Let $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$,

with the column dimension of $B$ equal to that of $u$. Then

$\begin{pmatrix} v \\ 0 \end{pmatrix} = (I + A) \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} B & C \\ D & E \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix}$. Hence $D(u) = 0$;

since $u > 0, D = 0$. This contradicts the irreducibility of $A$.

2.14 Theorem (Frobenius). If $A \geq 0$ is irreducible of order $n$, then there exists a positive real eigenvalue $r$ of $A$. If $x \in S(A)$, then $|x| \leq r$. There is a positive eigenvector $z$ corresponding to $r$, unique to within scalar multiples.

Proof: Let $C = \{x = (x_i) \in V : (x_i) \geq 0, x \neq 0\}$. For $x \in C$, let $A(x) = w = (w_i)$, $r_x = \min\{w_i/x_i : x_i \neq 0\}$, $r = \max\{r_x : x \geq 0\}$.

Note $r_x = \max\{t : t(x) \leq A(x)\}$. Define $f$ on $C$ into the non-negative real numbers by $f(x) = r_x$. To prove that $r$ is well defined, it must be shown that $f$ is bounded. By the continuity of division, the minimum function, and the operator $A$, $f$ is continuous on the positive cone of $V$. Further, for any positive real number $t$, $r_{tx} = r_x$. Let $M$ be the unit sphere of $V$,

$M = \{x = (x_i) : x_1^2 + \ldots + x_n^2 = 1\}$. In virtue of the preceding remark,
\[ f(C) = f(M \cap C) \]. Note that \( M \cap C \) is a compact set. Let
\[ N = (I + A)^{n-1}(M \cap C) \]. By 2.13, \( N \) is a compact subset of the
positive cone. Hence \( f \) is bounded on \( N \). For \( x \in M \cap C \),
let \( y = (I + A)^{n-1}(x) \). Since \( r_x(x) \leq A(x) \), and \((I + A)^{n-1} > 0\),
it follows that \( r_x(y) \leq A(y) \), and thus \( r_x \leq r_y \). Therefore \( f \) is
bounded on \( C \) and, since \( f \) is continuous on \( N \), there exists a
vector \( z = (z_1) \in N \) such that \( r = f(z) = r_z \). It remains to be
shown that \( r \) and \( z \) have the desired properties.

Let \( Z = \{ v \in C : f(v) = r \} \). Vectors in \( Z \) are titled \textit{extremal} vectors.
Let \( u \in C \), \( u > 0 \). By irreducibility \( A \) has no zero row; hence
\( r_u > 0 \), and therefore \( r > 0 \). Fix \( v \in Z \), and let \( x = (I + A)^{n-1}(v) \).
By 2.13, \( x > 0 \). Now \( r(v) \leq A(v) \); if \( r(v) \neq A(v) \), then
\( r(x) < A(x) \). By the maximality of \( r \), this is impossible. Hence
\( A(v) = r(v) \). Therefore \( x = (I + A)^{n-1}(v) = (1 + r)^{n-1}(v) \), and
\( v > 0 \). Hence \( z > 0 \).

Now suppose there exist \( y \in V \), \( y \neq 0 \), \( t \in S(A) \) such that \( A(y) = t(y) \).
Then \( |t| (y^+) = (ty)^+ = (Ay)^+ \leq A(y^+) \); therefore \( |t| \leq r_{y^+} \leq r \).
Note that, if \( t = r \) in the above argument, then \( y^+ \in Z \); hence no
non-zero eigenvector of \( A \) can have a zero coordinate. It follows
from this fact that \( Z \) is one-dimensional. This completes the proof.

2.15 Corollary [36]. If \( A \geq 0 \), then there exists a non-negative
real eigenvalue \( r \) of \( A \) such that \( |t| \leq r \) for all \( t \in S(A) \). There
is a non-negative eigenvector \( y \) corresponding to \( r \).

Proof: If \( A \) is reducible, it can be permuted into the form
\[
\begin{pmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{pmatrix}
\]

; this can be continued until \(A\) is in block triangular form \((A_{ij})\), with each \(A_{ii}\) an irreducible matrix and \(A_{ij} = 0\) if \(j > i\). The matrices \(A_{ii}\) may be triangulated independently; hence \(S(A) = U\{S(A_{ii})\}\). Let \(r_i\) be the maximal real eigenvalue of \(A_{ii}\); clearly \(r = \max\{r_i\}\). Note that \(r > 0\) unless \(A\) is nilpotent. Let \(z\) be a positive eigenvector corresponding to \(r\) for, say, \(A_{11}\). Then by filling in remaining coordinates with zeros, a non-negative eigenvector of \(A\) for \(r\) is easily constructed.

Recall that a non-negative matrix \(A\) is stochastic if each row sum of \(A\) is one. Let \(M_n = \{x: \text{there exists an } n \times n \text{ stochastic matrix } A \text{ such that } x \in S(A)\}\). Karpelevich [16] has shown that \(M_n\) is a curvilinear polygon with roots of unity as vertices, and has set forth the equations of the bounding curves. Dmitriev, Dynkin, and Kolmogoroff [6] showed that, for \(r > 0\), the set \(rM_n = \{x: \text{there exists } A > 0 \text{ with maximal real eigenvalue } r \text{ such that } x \in S(A)\}\).

Applications in this paper (see 3.3) arise in the case 
\(r = 1, |x| = 1\). The foregoing discussion shows that \(x^k = 1, k \leq n\), where \(M\) is the order of the matrix \(A\). The remainder of this chapter is devoted to an outline of the proof of this fact.

2.16 Lemma [6]. Let \(A = (a_{ij})\) be a stochastic matrix. If \(x \in S(A)\), then there exists a convex \(k\)-angular polygon \(Q\) of complex numbers such that \(xQ \subseteq Q\).

Proof: Let \(x \in S(A)\), and let \(z = (z_1)\) be an eigenvector of \(A\) belonging to \(x\). Let \(Q\) be the convex hull of \(\{z_1\}\). Since
A(z) = x(z), it follows that \( \sum_j a_{ij}z_j = xz_i, \) \( i = 1, \ldots, n. \) Now \( \sum_j a_{ij} = 1, \) hence \( xz_i \in Q, i = 1, \ldots, n. \) Since the numbers \( xz_i \) are the vertices of \( xQ, \) clearly \( xQ \subseteq Q. \)

2.17 Corollary. Let \( A, x, Q, k \) be as in 2.16. If \( |x| = 1, \) then \( x^j = 1 \) for some \( j \leq k. \)

Proof: \( xQ \subseteq Q \) by 2.16. Let \( z \) be a point of maximum modulus in \( Q; \) \( z \) is clearly a vertex of \( Q. \) Since \( |x| = 1, |xz| = |z|; \) hence \( xz \) is also a vertex of \( Q. \) It follows that \( x^t z \) is a vertex of \( Q, t = 1, \ldots, k. \) Since \( Q \) has only \( k \) vertices, these numbers are not all distinct; the result now follows.

2.18 Theorem [36]. If \( A \geq 0 \) is an irreducible non-negative matrix of order \( n \) with maximal real eigenvalue \( r > 0, \) then \( A \) is similar to the product of a stochastic matrix and the scalar matrix \( rI. \)

Proof: By 2.14, there exists a positive eigenvector \( z = (z_i) \) corresponding to \( r. \) Let \( P = \text{diag}(z_1, \ldots, z_n). \) Note \( P \) is invertible. Define \( Z = r^{-1}P^{-1}AP; \) computation shows \( Z \) is stochastic.

2.19 Theorem. Let \( A \) be a non-negative \( n \times n \) matrix with maximal real eigenvalue \( 1. \) Let \( x \in S(A), |x| = 1. \) Then \( x^k = 1 \) for some \( k \leq n. \)

Proof: If \( A \) is irreducible, the proof is immediate on applying 2.18 and 2.17. Otherwise, \( A \) can be permuted into block triangular form as in 2.16, \( A = (A_{ij}) \) with each \( A_{ii} \) irreducible. Let \( x \in S(A_{ii}); \)
since $|x| = 1$, the maximal real eigenvalue of $A_{11}$ must be 1.

By applying the proof of the previous case, the theorem is proved.
Throughout the remainder of this paper, $N_n$ denotes the set of $n \times n$ non-negative matrices.

The study of compact semigroups in $N_n$ begins with the study of compact groups in $N_n$. In this chapter it is proved that such objects are finite. From this it follows that any clan is $N_n$ is contractible.

3.1 Lemma. Let $H(e)$ be a compact topological group of $n \times n$ complex matrices. Then $H(e)$ is a Lie group.

Proof: Define $f$ on $H(e)$ into $\text{Gl}(n, \mathbb{C})$ by $f(x) = x + I - e$, $x \in H(e)$. $f$ is clearly an isomorphism. $f(H(e))$ is therefore a closed subgroup of $\text{Gl}(n, \mathbb{C})$, hence a Lie group [41]. Therefore $H(e)$ is a Lie group.

3.2 Lemma. Let $X \in G$, a compact subgroup of $\text{Gl}(n, \mathbb{C})$. If $\lambda \in S(X)$, then $|\lambda| = 1$.

Proof: Since the determinant function is a continuous homomorphism of $\text{Gl}(n, \mathbb{C})$ onto the multiplicative group of non-zero complex numbers, $\text{det}(G)$ is a compact subgroup of the unit circle. Hence $1 = |\text{det}X| = |\lambda_1 \lambda_2 \cdots \lambda_n|$, $\lambda \in S(X)$. Clearly, if $|\lambda_1| \leq 1$ for each $i$, then $|\lambda_i| = 1$ for each $i$. Let $P \in \text{Gl}(n, \mathbb{C})$ such that $A = PXP^{-1}$ is triangular, diagonal $A = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. The group $PGP^{-1}$ is compact, therefore contained in a bounded subset
of complex $n^2$ space. Since diagonal $A^k = (\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k)$, and for each $k$, $A^k \in \text{PGP}^{-1}$, it follows that, for each $i$, 
$$|\lambda_i^k| \leq 1,$$
and $3.2$ is proved.

3.3 Theorem. Let $H(E)$ be a compact group in $N_n$. Then $H(E)$ is finite.

Proof: Since $H(E)$ is a Lie group, the identity component $C$ of $H(E)$ is open [41]. It is therefore sufficient to prove that $H(E)$ is totally disconnected. If $C \not= E$, then $C$ has a non-trivial one parameter group [41], hence elements of infinite order. The proof is then completed by contradiction when it is shown that every element of $H(E)$ has finite order.

Let $X \in H(E)$. By 2.9 and 2.12, there exists $B \in \text{Gl}(n, \mathbb{R})$ such that

$$BEB^{-1} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

where rank $E$ is assumed equal to $k$. Since

$$BEB^{-1}$$
is an identity for $XB^{-1}$, $XB^{-1} = \begin{pmatrix} X_k & 0 \\ 0 & 0 \end{pmatrix}$, where $X_k$ is a rank $k$ real $k \times k$ matrix. Let $f$ be the isomorphism of $BH(E)B^{-1}$ into $\text{Gl}(n, \mathbb{R})$ defined by $f(BXB^{-1}) = BX^{-1} + I - BEB^{-1}$.

Since $f(BH(E)B^{-1})$ is isomorphic to $H(E)$, it suffices to find an integer $m$ such that $f(BXB^{-1})^m = f(BEB^{-1}) = I$.

Assume $k < n$. Note $S(X) = S(BXB^{-1}) = S(f(BXB^{-1})) \cup \{0\}$. For if $\lambda \in S(BXB^{-1})$, $\lambda \neq 0$, then $\det(X_k - \lambda I_k) = 0$. Hence

$$\det(f(BXB^{-1}) - \lambda I) = (1 - \lambda)^{n-k} \cdot \det(X_k - \lambda I_k) = 0,$$

and $\lambda \in S(f(BXB^{-1}))$. Conversely, if $\lambda \neq 1$ and $\lambda \in S(f(BXB^{-1}))$, then $\lambda \in S(BXB^{-1})$. Finally, by 3.2, $\lambda \in S(f(BXB^{-1}))$ gives $|\lambda| = 1$;
therefore $\lambda \in S(BXB^{-1})$, $\lambda \neq 0$ also yields $|\lambda| = 1$. Since $X \in N_n$, by 2.15, $1 \in S(BXB^{-1})$, and $S(BXB^{-1}) = S(f(BXB^{-1})) \cup \{0\}$.

By 2.19, $S(BXB^{-1}) \subseteq \{ \lambda : \lambda^t = 1, t \leq n \} \cup \{0\}$. If $k = n$, a similar argument can be given. In either event,

$$S(f(BXB^{-1})) \subseteq \{ \lambda : \lambda^t = 1, t \leq n \}.$$ Let $P \in GL(n, C)$ such that $D = Pf(BXB^{-1})P^{-1}$ is lower triangular and diagonal $D = \{ \lambda_1, ..., \lambda_n \}$. Note $\lambda_i \in S(f(BXB^{-1}))$, $i = 1, ..., n$. Let $m$ be the least common multiple $\{ t_i : \lambda_i^t = 1, t_i \leq n \}$. Then diagonal $D^m = \{ 1, 1, ..., 1 \}$.

Now if $j = i-1$, then $(D^m)^{ij} = p^*(D^m)^{ij}$. Hence, by the compactness of $Pf(BH(E)B^{-1})P^{-1}, (D^m)^{ij} = 0, j = i-1$. By a straightforward induction, it follows that $(D^m)^{ij} = 0, j < i, i = 1, ..., n$. Hence $D^m = I$, and therefore $f(BXB^{-1})$ has order $\leq m$, which completes the proof.

3.4 Conjecture. On the separable Hilbert space $H$ of real square-summable sequences, call a bounded linear operator $A$ non-negative if it maps the non-negative cone of $H$ into itself. If $G$ is a compact topological group of non-negative operators on $H$ (in, say, the norm topology), then $G$ is totally disconnected.

3.5 Corollary. Let $S$ be a continuum semigroup in $N_n$. Then $K \subseteq E$.

Proof: Fix $e \in E \cap K$. Then $eSe = H(e)$ (see 2.2). By 3.3, $H(e)$ is finite. Since $eSe$ is a continuum, it is degenerate, hence $H(e) = e$. Since $K = U\{H(e) : e \in K\}$, the corollary follows.

A topological space is acyclic if $H^n(S) = 0$, $n \geq 0$, assuming reduced
groups in dimension zero. Clearly contractible spaces are acyclic.

If $S$ is a clan, it is known [29] that $H^0(S) = H^0(eSe)$ for $e \in K$, $n \geq 0$. If, also, $S \subseteq N_n$, then by 3.5, $H^n(S) = H^n(eSe) = H^n({e}) = 0$.

Hence $S$ is acyclic. The stronger result that $S$ is contractible also holds and will be demonstrated later in this chapter.

The following lemma is due to Gluskin [12], and was rediscovered by the author.

3.6 Lemma. Let $S$ be a $n \times n$ complex matrix semigroup. Let $e, f \in E$ and $f \in eSe$. If $f \neq e$, then $\text{rank } f < \text{rank } e$.

Proof: Suppose $\text{rank } e = r, e \neq f$. Choose $v$ such that

$$v = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$ Then $vf^{-1} = \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}$, since $e$ is an

for $f, g$ is an $r \times r$ complex matrix, and $g^2 = g$. Since

$\text{rank } vf^{-1} = \text{rank } f$, it suffices to show $\det(g) = 0$. If not, then

$g$ is an idempotent in $\text{Gl}(r, \mathbb{C})$, hence $g = I_r$. But this implies

$f = e$, contrary to assumption. Therefore $\det(g) = 0$ and $\text{rank } f < r$.

3.7 Lemma. Let $S$ be a clan in which, for each $e \in E, H(e)$ is totally disconnected. Suppose also that there exists a neighborhood $V$ of $1$ such that $V \cap E = \{1\}$. Then there is an $I$-semigroup in $S$ based at $1$.

Proof: It is well known [19] that the existence of the neighborhood $V$ above is sufficient to insure a local one-parameter semigroup $\sigma([0,1])$ in $V$ such that $\sigma(0) = 1$, $\sigma(a) \notin H(1)$, $0 < a \leq 1$, and if $\sigma(a) = \sigma(b)g$, $g \in H(1)$, then $a = b$ and $g = 1$. In the
same paper, it is shown that $\sigma^{-}$ can be extended to a full
one-parameter semigroup by defining $\sigma(t) = \sigma(1) \sigma(t-1)$ for
t $\in [1,2]$ and proceeding inductively. Now the closure of
$\sigma([0,\infty))$ is a commutative clan, hence its kernel is a (connected)
group [17], and therefore a single point $z$. It follows by a theorem
of Koch [17] that this clan has exactly 2 idempotents and is an
I-semigroup.

2.8 Theorem. Let $S$ be a non-degenerate clan in $N_n$. Then $S$
contains a standard thread from 1 to $K$.

Proof: By 3.6, there exists a neighborhood $V$ of 1 containing no
other idempotents; this follows from the fact that the rank of an
idempotent equals its trace, see 2.9. By 3.3, each $H(e)$ is
finite. Applying 3.7, there exists an I-semigroup from 1 to $e \in E$.
By 3.6, $\text{rank}(e) < \text{rank}(1)$. $eSe$ is a subclan with unit $e$. If
$e \notin K$, the above argument produces an I-semigroup from $e$ to $f \in E$,
$\text{rank}(f) < \text{rank}(e)$. After finitely many repetitions of this
procedure, an idempotent $z$ of minimal rank in $S$ is obtained. Clearly
$z \in K$; otherwise an idempotent of smaller rank can be obtained as
before. The union of the I-semigroups so obtained is the desired
standard thread.

2.9 Lemma. Let $S$ be a clan containing a standard thread $T$ from
1 to $e \in K$, and let $K \subseteq E$. Then $S$ is contractible.

Proof: Define $F:S \rightarrow S$ by $F(x,t) = t \times t$. Then
$F(x,1) = x$, and $F(x,e) = exe = e$, for each $x \in S$.  

3.10 Corollary. Let $S$ be a clan $\subset N_n$. Then $S$ is contractible.

By 3.6, it is clear that the I-semigroups composing the standard thread of 3.8 cannot be semigroups of type $I_3$ (see 2.3). The next lemma is well known and gives more information on this subject.

3.11 Lemma. Let $A$ be a nilpotent $n \times n$ complex matrix. Then there exists $m \leq n$ such that $A^m = 0$.

Proof: The Jordan form of $A$ is strictly lower triangular; hence $A^n = 0$.

3.12 Corollary. Let $S$ be an I-semigroup of complex $n \times n$ matrices. Then $S$ is the union of semigroups of type $I_1$.

Proof: Let $z$ be the zero of $S$. Define $f$ on $S$ into the complex $n \times n$ matrices by $f(x) = x - z$. $f(S)$ is isomorphic to $S$, and $f(z) = 0$. By 3.6, $S$ contains no subsemigroup of type $I_3$; by 3.11, $f(S)$, and hence $S$, cannot contain a subsemigroup of type $I_2$.

The corollary now follows by the remarks in 2.3.

For convenience, the results of the latter part of this chapter are now summarized.

3.13 Theorem. Let $S$ be a clan, $S \subset N_n$. Then

(i) $K \subset E$.

(ii) There exists a standard thread of at most

$n$ I-semigroups of type $I_1$ from $1$ to $e \in K$.

(iii) $S$ is contractible.
In closing, note that the compactness in 3.3 is essential; the positive $n \times n$ diagonal matrices form a group isomorphic to the direct sum of $n$ copies of the multiplicative group of positive real numbers. The non-negativity is also clearly necessary; the circle group can be represented isomorphically by the real matrices of the form

\[
\begin{pmatrix}
a & b \\
-b & a
\end{pmatrix}, \quad a^2 + b^2 = 1.
\]
IDEMPOTENTS

In this chapter, structure theorems for non-negative and stochastic idempotents are given. As an application of these theorems, it is shown that the set of stochastic idempotents of order $n$ and fixed arbitrary rank $k$ is arcwise connected, as well as the set of non-negative idempotents of order $n$ and fixed rank.

4.1 Lemma. Let $E = (e_{ij}) \in \mathbb{N}_n$, $E = E^2$, $e_{ii} = 0$. Then either $e_{ij} = 0$, $j = 1, \ldots, n$ or $e_{ji} = 0$, $j = 1, \ldots, n$.

Proof: Since permutations preserve non-negativity, it may be assumed that $e_{11} = 0$. Then for each $k = 1, \ldots, n$, either $e_{1k} = 0$ or $e_{k1} = 0$. By permuting rows and columns, it may be assumed that $e_{11} > 0$, $i = 2, \ldots, t$; $e_{1i} = 0$, $i = t + 1, \ldots, n$. Write

$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$, with $E_{11}$ a $t \times t$ submatrix. The arrangement of the first row of $E$ and non-negativity yield $E_{12} = 0$. Now rewrite $E$ as

$\begin{pmatrix} 0 & F_{12} & 0 \\ 0 & F_{22} & 0 \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$, with $E_{11} = \begin{pmatrix} 0 & F_{12} \\ 0 & F_{22} \end{pmatrix}$.

$E_{21} = [F_{31} \ F_{32}]$, $F_{33} = E_{22}$, and order $F_{22} = t-1$. Note that $F_{12}$ and $F_{31}$ are, respectively, row and column matrices. Now

$F_{12} F_{22} = F_{12}^2 = F_{22}$, and $F_{32} = F_{31} F_{12} + F_{32} F_{22} + F_{33} F_{22}$.

Multiplying the latter equality on the right by $F_{22}$ yields
F_{31}F_{12} = 0. Since all pair products $e_{pl}e_{lq}$, $t + 1 \leq p \leq n$,
$2 \leq q \leq t$ Compose the matrix $F_{31}F_{12}$, either $e_{pl} = 0, p = t + 1, \ldots, n$,
or $e_{lq} = 0, q = 2, \ldots, t$. Hence either $F_{31} = 0$ or $F_{12} = 0$,
and the lemma is proved.

If $E$ in 4.1 is also stochastic, then clearly $E$ cannot have zero
rows; hence $e_{ii} = 0$ implies $e_{ji} = 0, j = 1, \ldots, n$. This case was
done by Etter [7].

4.2 Lemma. Let $E = E^2 \in N_n$, and suppose $E$ has a positive row
(column). Then rank $E = 1$.

Proof: If $E$ has order 1, then $E = (1)$, and the lemma is obvious;
suppose the theorem has been proved for idempotents of order less than
$n$, and let $E$ have order $n$. By using a permutation, it may be
assumed that the first row of $E$ is positive, $e_{1j} > 0, j = 1, \ldots, n$.
If $e_{ii} = 0$, then by 4.1 $e_{ij} = 0, j = 1, \ldots, n$. By a permutation,
$E$ may be written as \begin{pmatrix} E_{11} & E_{12} \\ 0 & 0 \end{pmatrix}, with $E_{11}$ possessing all
positive diagonal elements. If $E_{11}$ has order less than $n$, then by
the inductive hypothesis rank $E = \text{rank } E_{11} = 1$. Otherwise, it may
be assumed that $e_{ii} > 0, i = 1, \ldots, n$.

Assume that the first $k$ rows of $E$ are all of the positive rows
of $E$. Write $E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$, with $E_{11}$ a $k \times k$ submatrix.

Note $E_{21} = 0$; for if $e_{ij} > 0$, $i > k$, $j \leq k$, then $e_{ij} > 0$,
using the idempotency, $j = 1, \ldots, n$, which is impossible. Thus
\[ E^2_{11} = E_{11}E_{22} = E_{22}^2. \] Since \( E_{12} = E_{11}E_{12} + E_{12}E_{22} \), it follows that \( E_{11}E_{12}E_{22} = 0 \). But \( E_{11}E_{12} = E_{12} > 0 \); hence \( E_{22} = 0 \), contrary to the previous paragraph. Therefore \( E > 0 \), and is irreducible. By 2.14, the range of \( E \) can have only one linearly independent positive vector. Since the canonical unit vectors are mapped into positive vectors, the dimension of the range of \( E \) is one; hence \( E \) has rank 1.

**4.3 Corollary.** Let \( E = (e_{ij}) \) be a stochastic idempotent having a positive row. Then all rows of \( E \) are identical.

**Proof:** By 4.2, rank \( E = 1 \). Since the rows of \( E \) are proportional, and each row sum is one, these rows must be identical. Note this also shows any rank one stochastic matrix is idempotent.

If \( E \) is a rank \( k \) idempotent, then \( E \) is similar to \( \text{diag}(I_k, 0) \) by 2.9, and is therefore the direct sum of \( k \) idempotents of rank 1.

The next theorem shows that, if \( E \in \mathbb{N}_n \), then each summand may also be chosen to be non-negative.

**4.4 Theorem.** If \( E = E^2 \in \mathbb{N}_n \), rank \( E = k \), then \( E \) is the direct sum of \( k \) non-negative idempotents of rank 1.

**Proof:** If \( E \) has rank 1, there is nothing to be proved. Suppose the theorem has been proved for idempotents of rank \( k-1 \). Let \( E \) have rank \( k \). Without loss of generality, assume \( e_{ii} > 0, i = 1, \ldots, t \), \( e_{ii} = 0, i = t + 1, \ldots, n \). If \( t = n \), then rank \( E = 1 \) by 4.1; assume \( t < n \). Write \( E = \begin{pmatrix} E_{11} & E_{12} \\ E_{22} & E_{22} \end{pmatrix} \), with \( t \) the order of
As in 4.1, it follows immediately that $E_{12} = 0$. Hence

$$E_{11}^2 = E_{11}, \quad E_{22}^2 = E_{22}, \quad \text{and} \quad E_{21} = E_{21}E_{11} + E_{22}E_{21}. \quad \text{Let}$$

$$F = \begin{pmatrix} E_{11} & 0 \\ E_{21}E_{11} & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 \\ E_{22}E_{21} & E_{22} \end{pmatrix}. \quad \text{Then} \quad E = F + G,$$

$$F^2 = F, \quad G^2 = G, \quad \text{and} \quad FG = 0. \quad \text{To get} \quad GF = 0, \quad \text{note that}$$

$$E_{22}E_{21}E_{11} = 0; \quad \text{this is obtained by multiplying the equality}$$

$$E_{21} = E_{21}E_{11} + E_{22}E_{21} \quad \text{on the right by} \quad E_{11}. \quad \text{By 4.1, rank} \quad E_{11} = 1;$$

since trace $F = \text{trace} \quad E_{11}$, rank $F = 1$, rank $G = k - 1$. The theorem now follows from the inductive hypothesis.

4.5 Theorem. If $E = (e_{ij})$ is a rank $k$ idempotent in $\mathbb{M}_n$, and if $e_{ii} > 0$, $i = 1, \ldots, n$, then $E$ is permutable to the super diagonal form $(E_{11}, \ldots, E_{pp})$, with each $E_{ii}$ a rank 1 idempotent. If $E$ is stochastic, then each $E_{ii}$ is stochastic.

Proof: Assuming the obvious inductive hypothesis, let $E$ be a rank $k$ non-negative idempotent. As in the argument of 4.4, $E$ can be permuted to the form

$$E = \begin{pmatrix} E_{11} & 0 \\ E_{22} & E_{22} \end{pmatrix}, \quad \text{with} \quad E_{11} \quad \text{a rank 1 idempotent.}$$

Again, $E_{22}E_{21}E_{11} = 0$. Let $E_{11} = (a_{ij})$, $E_{21} = (b_{ij})$, $E_{22} = (c_{ij})$. Then $[E_{22}E_{21}E_{11}]_{ij}$ is zero, and contains a summand of the form

$$c_{ii}b_{ij}a_{jj}. \quad \text{Since} \quad c_{ii}a_{jj} > 0, \quad b_{ij} = 0. \quad \text{Therefore} \quad E_{21} = 0,$$

and $E = \text{diag}(E_{11}, E_{22})$. On applying the inductive hypothesis to $E_{22}$, the theorem is proved. The second conclusion is obvious.

It is known [1] that the set of idempotents of fixed rank over a
full real matrix algebra of order $n$ is arcwise connected. The
next theorems show that the set of stochastic idempotents of fixed
rank is arcwise connected, as well as the set of non-negative
idempotents of fixed rank. Methods developed earlier in this chapter
are utilized in the proofs of these theorems. Denote by $X^n_k$ the set
of rank $k$ stochastic idempotents of order $n$, and by $Y^n_k$ the set
of rank $k$ non-negative idempotents of order $n$. Clearly $X^n_k \subseteq Y^n_k$.
It is easily seen that if $A,B \in X^n_k (Y^n_k)$, with either $AB = B, BA = A$
or $AB = A, BA = B$, then the convex gull of $A$ and $B$ lies within
$X^n_k (Y^n_k)$. This fact will be used in the sequel.

The set $X^1_k$ consists of the matrix (1) and is therefore arcwise
connected; suppose $X^m_k$ has been proved arcwise connected for
$k \leq m < n$. Clearly $k$ may be assumed less than $n$. $X^n_k$ will be
proved arcwise connected by a sequence of lemmas which are now
outlined. Let $X_i = \{ A \in X^n_k : a_{i1} = 0 \}$. $X^{n-1}_k$ will be homeomorphically
imbedded in $X_i$ for each $i = 1, ..., n$. By use of the inductive
hypothesis, each $X_i$ is shown to be arcwise connected. Next, using
an imbedding of $X^{n-1}_{k-1}$ in $X^n_k$, each pair $X_i, X_j$ is shown to be
arcwise connected in $X^n_k$. Finally, any idempotent of $X^n_k$ not
belonging to $\bigcup X_i$ is shown to be connected to this set by an arc
in $X^n_k$. This will complete the proof.

4.6 Lemma. For each $i$, $X^{n-1}_k$ is homeomorphically imbedded in $X_i$.

Proof: Let $A \in X^{n-1}_k$, define $f(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $f(A) \in N_n$. 
Decompose A as \( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \), with \( A_{11} \) a square submatrix of order \( i - 1 \). Let \( P_i \) be the permutation matrix such that

\[
P_i^{-1} f(A) P_i^{-1} = \begin{pmatrix} A_{11} & 0 & A_{12} \\ 0 & 0 & 0 \\ 0 & 0 & A_{21} \end{pmatrix}.
\]

Fix \( t \neq i \), and let Q be the invertible matrix such that, for any matrix M, \( QM \) is altered from M by the addition of the \( t \)th row to the \( i \)th row. Then

\[
QP_i^{-1} f(A) P_i^{-1} = \begin{pmatrix} A_{11} & 0 & A_{12} \\ H & 0 & J \\ A_{21} & 0 & A_{22} \end{pmatrix},
\]

with \( H = (a_{t1} \ldots a_{ti-1}) \), \( J = (a_{t(i+1)} \ldots a_{tn}) \). The matrix \( QP_i^{-1} f(A) P_i^{-1} \) is clearly stochastic, with trace equal to trace \( A = k \). Furthermore,

\[
[QP_i^{-1} f(A) P_i^{-1}] [QP_i^{-1} f(A) P_i^{-1}] = QP_i^{-1} f(A) P_i^{-1}.
\]

Hence \( QP_i^{-1} f(A) P_i^{-1} \in X_i \), and the mapping defined by \( g_i(A) = QP_i^{-1} f(A) P_i^{-1} \) is a homeomorphism of \( X_k^{n-1} \) into \( X_i \).

4.7 Lemma. For each \( i \), \( X_i \) is arcwise connected.

Proof: Let \( B \in X_i \). Decompose \( B \) as \( \begin{pmatrix} B_{11} & 0 & B_{13} \\ B_{21} & 0 & B_{23} \\ B_{31} & 0 & B_{33} \end{pmatrix} \), with

\( B_{11} \) a submatrix of order \( i - 1 \) and the zeros representing the \( i \)th column. Clearly \( B' = \begin{pmatrix} B_{11} & B_{13} \\ B_{31} & B_{33} \end{pmatrix} \in X_k^n \). Let \( A = g_i(B') \in X_i \).
Now $BA = B$ and $AB = A$. Hence the line between $B$ and $A$ lies wholly within $X_1$. By the inductive hypothesis, $g_i(X_{n-1}^k)$ is arcwise connected, and it has been shown that any element of $X_i$ can be connected to $g_i(X_{n-1}^k)$ by an arc in $X_i$. Hence the lemma is proved.

4.8 Lemma. For any $i,j,X_i$ and $X_j$ are connected by an arc in $X_k^n$.

Proof: It is no loss of generality to assume $n \geq 3$; for the rank 1, order 2 stochastic idempotents are precisely the convex hull of the matrices \[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\], and hence are arcwise connected. Under this assumption, fix an integer $q, q \neq i, j$. For convenience, assume $q > i, j$. Let $A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix} \in X_{n-1}^l$, with $A_{11}$ a submatrix of order $q - 1$. Define $h:x_{k-1}^{n-1} \rightarrow x_k^n$ by

$$h(A) = \begin{pmatrix}
A_{11} & 0 & A_{12} \\
0 & 1 & 0 \\
A_{21} & 0 & A_{22}
\end{pmatrix}$$

$h$ is clearly a homeomorphic imbedding of $x_{k-1}^{n-1}$ into $x_k^n$. Letting $[x_{k-1}^{n-1}]_i$ and $[x_{k-1}^{n-1}]_j$ correspond in $x_{k-1}^{n-1}$ to the sets $X_i$ and $X_j$ in $x_k^n$, choose $C_1 \in [x_{k-1}^{n-1}]_i$, $C_2 \in [x_{k-1}^{n-1}]_j$. By the inductive hypothesis, $C_1$ and $C_2$ are connected by an arc in $x_{k-1}^{n-1}$; hence $h(C_1)$ and $h(C_2)$ are connected by an arc in $x_k^n$. Since $h(C_1) \in X_i$, $h(C_2) \in X_j$, the lemma is proved when $q > i, j$. If $j = n$, the argument is similar.
4.2 Lemma. Let $A \in X^n_k$, $a_{ii} > 0$, $i = 1, \ldots, n$. Then $A$ can be connected to some $X^i_1$ by an arc in $X^n_k$.

Proof: By 4.5, there exists a permutation matrix $Q$ such that $QAQ^{-1} = \text{diag}(A_{11}, \ldots, A_{kk})$, with each $A_{ii}$ a rank 1 stochastic idempotent. Since $k < n$ is assumed, there is an $i$ such that $a_{ii} < 1$. Assume $a_{ii} \in A_{jj}$ in $QAQ^{-1}$. $A_{jj}$ has order at least 2.

Let $C \in X^n_k$ be the idempotent $\text{diag}(A_{11}, \ldots, A_{j-1}, c_{jj}, A_{j+1, j+1}, \ldots, A_{kk})$, with $C_{jj}$ a rank 1 stochastic matrix of the same order as $A_{jj}$, but having $c_{ii}$, the diagonal entry corresponding to $a_{ii}$, equal to zero. $C_{jj}$ may be constructed as a column of ones, different from the column containing $c_{ii}$, with all other entries zero. Since $A_{jj}$ and $C_{jj}$ have order less than $n$, they are connected by an arc of rank one stochastic idempotents. Hence $QAQ^{-1}$ and $C$ are arcwise connected in $X^n_k$, and $C \in X^n_1$. Consequently, $A$ and $Q^{-1}CQ$ are arcwise connected in $X^n_k$, and $Q^{-1}CQ \in X^n_j$ for some $j$. This completes the proof.

4.10 Theorem. $X^n_k$ is arcwise connected.

Proof: Apply 4.6, 4.7, 4.8, and 4.9.

4.11 Corollary. $Y^n_k$ is arcwise connected.

Proof: It is first shown that $Y^n_1$ is arcwise connected; this property is immediate in $X^n_1$ since this set is convex (see 4.3). Let $A \in Y^n_1$; recall trace $A = 1$. Assume $a_{kk} > 0$. Let
\[ a_{ik} = t_i a_{kk}, \ i = 1, \ldots, n. \] Since the rows of \( A \) are proportional, \( a_{ii} = t_i a_{ki}, \) and \( l = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} t_i a_{ki}. \) Let \( B = (b_{ij}) \) be the rank one idempotent defined by \( b_{ik} = t_i, \ b_{ij} = 0, \ j \neq k. \)

Then \( AB = B, \ BA = A, \) and therefore the line between \( A \) and \( B \) is in \( Y_1^n. \) Let \( C = (c_{ij}) \) be the rank 1 stochastic idempotent defined by \( c_{ik} = 1, \ i = 1, \ldots, n, \ c_{ij} = 0, \ j \neq k. \) Then \( BC = B, \ CB = C, \) and thus the line between \( B \) and \( C \) lies inside \( Y_1^n. \) Therefore \( A \) is connected to \( X_1^n \) by an arc in \( Y_1^n, \) which proves \( Y_1^n \) is arcwise connected.

Let \( Y_i = \{ A \in Y_k^n : a_{ii} = 0 \}. \) Note \( X_1 \subseteq Y_i. \) Lemmas similar to 4.6, 4.7, and 4.8 are easily derived to prove \( \bigcup Y_i \) is arcwise connected. Using 4.5 and the arcwise connectedness of \( Y_1^n, \) a lemma analogous to 4.9 can be proved to complete the proof of the corollary. The arguments establishing these lemmas are omitted because of their close relation to arguments given in 4.6 - 4.9.

4.12 Conjectures. The rank 2, order 3 stochastic idempotents can be seen to form a simple closed curve. Does \( X_k^n \) form a manifold whose dimension is a function of \( n \) and \( k? \) If so, what, topologically, is \( Y_k^n \)? To what connectedness conditions are the various sets of non-negative, bounded linear idempotent operators on separable Hilbert space subject?
Commutative matrices have been examined in detail; an excellent bibliography of pertinent papers is given in \[43\]. In this chapter, using the Jordan form (2.8) of a matrix and a theorem of Kaplansky, it is first shown that any commutative semigroup of \( n \times n \) complex matrices is similar to a semigroup of lower triangular matrices (matrices which are zero above the main diagonal). Following this, commutative semigroups of real matrices are studied. For a proof of the first theorem, see \[40\].

5.1 Theorem (Kaplansky). Let \( S \) be a multiplicative semigroup of \( n \times n \) matrices over a division ring, consisting of nilpotent elements. Then \( S \) is similar to a semigroup of strictly lower triangular matrices (matrices which are zero on and above the main diagonal).

5.2 Lemma. Let \( A \) be an \( n \times n \) complex matrix in Jordan form, 
\[
A = \text{diag}(A_{11}, \ldots, A_{kk}),
\]
with each \( \lambda_i \) a scalar and each \( N_{ii} \) a lower triangular nilpotent having ones and zeros on the first principal subdiagonal and zeros elsewhere, and 
\[
\lambda_i \neq \lambda_j \text{ if } i \neq j.
\]
Let \( B = (B_{ij}) \) be an \( n \times n \) complex matrix decomposed in the same dimensions as \( A \). If \( AB = BA \), then 
\[
B_{ij} = 0, \text{ if } i \neq j; \text{ that is, } B \text{ is in super-diagonal form,}
\]
\[
B = \text{diag}(B_{11}, \ldots, B_{kk}).
\]
Proof: Fix $i = p$, $j = q$. Since $AB = BA$, it follows that
\[ \lambda_k B_{pq} + N_{pp} B_{pq} = \lambda_q B_{pq} + B_{pq} N_{qq}, \]
whence $(\lambda_p - \lambda_q) B_{pq} = B_{pq} N_{qq} - N_{pp} B_{pq}$. Let $B_{pq} = (b_{ij})$, $i = 1, \ldots, r$; $j = 1, \ldots, s$, and let $N_{pp} = (n_{ij})$, $N_{qq} = (n_{ij})$. By the prior equality,
\[ (\lambda_1 - \lambda_j)b_{lj} = b_{lj+1} + b_{lj}, \quad j < s, \quad \text{and} \quad (\lambda_1 - \lambda_s)b_{ls} = 0. \]
Since $\lambda_1 \neq \lambda_s$, $b_{ls} = 0$, and hence $b_{lj} = 0$, $j = 1, \ldots, s$. Assume the first $k - 1$ rows of $B_{pq}$ have been proved identically zero. Then
\[ (\lambda_k - \lambda_j)b_{kj} = 0, \quad \text{from which it follows that} \quad b_{kj} = 0, \quad j = 1, \ldots, s. \]
Therefore $B_{pq} = 0$.

5.3 Theorem. If $S$ is a commutative semigroup of $n \times n$ complex matrices, then $S$ is similar to a semigroup of lower triangular matrices.

Proof: The theorem is proved by induction on the order of the matrices composing $S$. If $n = 1$, there is nothing to prove. Suppose the theorem has been proved for semigroups with matrices of order $k < n$, and let $S$ be a commutative semigroup of $n \times n$ complex matrices. If there exists $A \in S$ such that $A$ has $j > 1$ distinct eigenvalues, then by 5.2 $S$ can be decomposed into super diagonal form, $S = \text{diag}(S_1, \ldots, S_j)$. Each $S_i$ is a commutative semigroup of order less than $n$, hence is similar to a semigroup of lower triangular matrices. Since the semigroups $S_p, S_q$, $p \neq q$, are independent of one another, it follows that $S$ is similar to a semigroup of lower triangular matrices.
It remains to dispose of the case in which each $A \in S$ has a unique eigenvalue. By referring to the Jordan form of $A$, it can be seen that each $A \in S$ decomposes into the sum of a scalar (the eigenvalue of $A$) and a nilpotent, $A = \lambda + N$. Let

$$\mathcal{N} = \{N : \text{for some } A \in S, \ e S(A), A = \lambda + N\}.$$ 

If $M, N \in \mathcal{N}$, then $MN = NM$; for, let $A = \lambda + N, B = \alpha + M$. Then

$$\lambda \alpha + \lambda M + \alpha N + MN = AB = BA = \alpha \lambda + \lambda M + \alpha N + MN,$$

hence $MN = NM$. Let $\mathcal{N}'$ be the semigroup generated by $\mathcal{N}$. Since elements of $\mathcal{N}$ commute, $\mathcal{N}'$ consists of nilpotent elements. Hence, by 5.1, there exists $P$ such that $P \mathcal{N}' P^{-1}$, and therefore $P \mathcal{N} P^{-1}$, is in strictly lower triangular form. Finally, if $A = \lambda + N \in S$, then $PAP^{-1} = \lambda + PNP^{-1}$, which is lower triangular. Hence $PSP^{-1}$ is lower triangular.

**5.4 Corollary.** Let $S$ be as in 5.3. If there is an $A \in S$ having $n$ distinct eigenvalues, then $S$ is similar to a semigroup of diagonal matrices. If, furthermore, the matrices in $S$ are real and $A$ has a real spectrum, then $S$ is similar to a semigroup of real diagonal matrices.

Proof: The Jordan form of $A$ is diagonal; by 5.2 it follows that $S$ can be diagonalized. The latter part of the corollary follows from the fact that the Jordan form of $A$ is real, and (2.12).

Note that a semigroup of $n \times n$ diagonal matrices is isomorphic to a subsemigroup of the Cartesian product of $n$ copies of the scalar field under coordinate multiplication. Sufficient conditions, different from those in 5.4, for a semigroup of complex (real)
matrices to be similar to a diagonal semigroup are now investigated.

**5.5 Lemma.** Let \( \{E_i\}, i = 1, \ldots, k \) be \( n \times n \) complex (real) idempotents such that \( E_i E_j = E_j E_i = E_i, i \leq j \), rank \( E_i = r(i) \).

Then there exists a complex (real) matrix \( P \) such that

\[
P E_i P^{-1} = \begin{pmatrix} I_{r(i)} & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, k.
\]

**Proof:** By 3.5, if \( i < j \), then \( r(i) < r(j) \). The lemma is proved by induction on \( n \). If \( n = 1 \), there is nothing to prove. Assume the lemma has been proved for matrices of order less than \( n \). As remarked above, \( E_k \) is the idempotent of maximal rank \( r(k) \) in the set \( \{E_i\} \). From 2.9, there exists a complex (real) matrix \( P \) such that

\[
P E_k P^{-1} = \begin{pmatrix} I_{r(k)} & 0 \\ 0 & 0 \end{pmatrix}.
\]

Since \( E_k \) is an identity for \( E_i \), \( i = 1, \ldots, k \), it follows that

\[
P E_i P^{-1} = \begin{pmatrix} X_i & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( X_i \) is some \( r(k) \times r(k) \) submatrix. If \( r(k) < n \), then the system \( P E_i P^{-1} \) is isomorphic to a system of dimension less than \( n \), and the lemma is proved, otherwise, \( E_k = I_n \), in which case the above argument is applied to \( E_{k-1} \). Since \( r(k - 1) < n \), the inductive hypothesis can be applied to the system \( \{E_i\}, i = 1, \ldots, k - 1 \), to complete the proof of the lemma.

**5.6 Corollary.** Let \( S \) be a commutative semigroup of complex (real) \( n \times n \) matrices with zero \( E_1 \), rank \( E_1 = s \); identity \( E_k \), rank \( E_k = s + k - 1 \). Suppose, also, that \( S \) contains a system of \( n \times n \) matrices with zero \( E_1 \), rank \( E_1 = s \); identity \( E_k \), \( k \) idempotents \( \{E_i\} \) as in 5.5, between \( E_1 \) and \( E_k \) with
rank $E_1 = s + i - 1$. Then $S$ is isomorphic to a diagonal semigroup of $(k - 1) \times (k - 1)$ complex (real) matrices.

Proof: Let $f$ be the function defined by $f(X) = X - E_1$, $X \in S$. It is clear that $f$ is an isomorphism of $S$ into the $n \times n$ matrices such that rank $f(E_1) = i - 1$. Let $P$ be the matrix of 5.5 such that $Pf(E_1)P^{-1} = \begin{pmatrix} I_{i-1} & 0 \\ 0 & 0 \end{pmatrix}$. Now $Pf(S)P^{-1}$ has identity $\begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}$; hence if $X \in S$, then $Pf(X)P^{-1} = \begin{pmatrix} X' & 0 \\ 0 & 0 \end{pmatrix}$, where $X'$ is a $(k - 1) \times (k - 1)$ submatrix. Since $X'$ must commute with $I_{i-1}$, $i = 1, \ldots, k$, it follows by direct computation that $X'$ must be diagonal. The details are omitted. The function $f$ and the similarity generated by $P$ are clearly continuous.

Note that, if $S$ satisfies the hypotheses of 5.6 and is, in addition, compact, then the diagonal entries of $S$ are bounded above in modulus by 1. Hence, in the complex case, $S$ is a subsemigroup of the Cartesian product of unit discs; in the real case, $S$ is a subsemigroup of the Cartesian product of real intervals $[-1, 1]$. If $S$ has stochastic entries, the full interval $[-1, 1]$ may still be realized, as is shown by the 2 2 example \[ \begin{pmatrix} x & 1 - x \\ x & 1 - x \end{pmatrix} \], $x \in [0, 1]$.

5.7 Corollary. If $S$ is a semigroup of $n \times n$ real matrices, and if $S$ has a zero $E$ and an identity $F$ whose ranks differ by one, then $S$ is isomorphic to a subsemigroup of the real numbers, and hence commutative. If $S$ is also connected, then $S$ contains the
convex arc between $E$ and $F$.

Proof: In the argument establishing 5.6, the commutativity is not needed when there are less than 3 idempotents in the chain. Hence $S$ is isomorphic to a semigroup of $1 \times 1$ real matrices, which are essentially the real numbers. If $S$ is connected, then the isomorphic copy of $S$ must contain the arc between (1) and (0).

Since the function $f$ and the similarity generated by $P$ of 5.6 are affine mappings, $S$ must contain the convex arc from $E$ to $F$.

If the ranks of $E$ and $F$ differ by more than one, then $S$ need not even be diagonal. Indeed, the mapping

$$f(x) = \begin{pmatrix} x & 0 \\ (-\ln x)x & x \end{pmatrix} \quad x \in (0,1], \quad f(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is an isomorphic imbedding of the unit interval demonstrating this fact.
COMPACT SIMPLE SEMIGROUPS

The theorems in this chapter are involved with topological representations of finite dimensional compact simple semigroups. For a thorough treatment concerning representations of algebraic semigroups, see [35]. The characterization of compact simple semigroups mentioned in 2.2 and treated in [30] is assumed without further reference.

6.1 Theorem. Let $S$ be a compact, simple, idempotent semigroup contained in euclidean $n$-space. Then $S$ is isomorphically imbeddable in $\mathbb{N}_{2n+2}$.

Proof: By the compactness, $S$ is bounded. Hence there exists a homeomorphism of $\mathbb{E}^n$ carrying $S$ into the non-negative cone of $\mathbb{E}^n$. Assume this has been done. For each $y = (y_1) \in S$, define $A(y)$ to be the $(n+1) \times (n+1)$ non-negative real matrix having $1, y_1, \ldots, y_n$ as its first row and zeros elsewhere. Further, let $B(y)$ be the transpose of $A(y)$. Note, for every $y \in S$, $A(y)$ and $B(y)$ are idempotents. Fix $x \in S$. For $y \in xS$, let

$$f(y) = \begin{pmatrix} A(y) & 0 \\ 0 & B(x) \end{pmatrix}.$$ 

$f(xS)$ is clearly a homeomorphic image of $xS$, since $A$ is a homeomorphism. Now, multiplication in $xS$ is right trivial; that is $yz = z, zy = y$ for all $y, z \in xS$. Since $A(y)A(z) = A(z)A(y) = A(y)$, and $B(x)$ is an idempotent, it follows that $f(y)f(z) = f(z)$ and $f(z)f(y) = f(y)$. Hence $f$.
maps $xS$ isomorphically into $N_{2n+2}$. By defining

$$f(z) = \begin{pmatrix} A(x) & 0 \\ 0 & B(z) \end{pmatrix}, \quad z \in Sx,$$

it is easily seen that $f$ is a homeomorphism of $Sx \cup xS$ into $N_{2n+2}$, which is an isomorphism on $xS$ and $Sx$ separately. It remains to extend $f$ over all of $S$.

Suppose $w \notin xS \cup Sx$. Since $S = SxS = (Sx)(xS), w$ can be written uniquely as $zy$, with $z = Sx \cap wS$ and $y = xS \cap Sw$. Define $f(w) = f(z)f(y) = \begin{pmatrix} A(y) & 0 \\ 0 & B(z) \end{pmatrix}$. Since $w$ is represented uniquely as $zy$, $z \in xS$, $y \in Sx$, it follows that $f$ is well-defined.

That $f$ is one to one is immediate from the fact that different elements of $S$ have different coordinates. $S$ being compact, it suffices to show that $f$ is continuous to complete the proof. Let $\{w_n\}$ be a sequence in $S$ converging to $w = zy$. Each $w_n$ may be represented uniquely as $z_n y_n$, with $z_n \in Sx, y_n \in xS$. Now, $xS, Sx$ are compact; therefore there exist convergent subsequences $\{y_{n(i)}\}, \{z_{n(i)}\}$, having the same indices, such that $\{y_{n(i)}\}$ converges to $a \in xS$, $\{z_{n(i)}\}$ converges to $b \in Sx$. By continuity of multiplication in $S$, $\{z_{n(i)}y_{n(i)}\}$ converges to $ba$. Hence $w = ba$, and by the uniqueness of representation, $b = z$ and $a = y$. Hence $\{z_{n(i)}\}$ converges to $z$ and $\{y_{n(i)}\}$ converges to $y$. Now, $f$ has been shown to be a homeomorphism on $Sx \cup xS$. Therefore $f(z_{n(i)})$ converges to $f(z)$ and $f(y_{n(i)})$ to $f(y)$. Since $f(w_{n(i)}) = f(z_{n(i)}y_{n(i)}) = f(z_{n(i)})f(y_{n(i)})$, and $f(w) = f(zy) = f(z)f(y)$, and matrix multiplication is continuous, it
follows that $f(v_n(i))$ converges to $f(w)$, and $f$ is continuous. Note that uniqueness of representation is essential to this proof.

6.2 **Corollary.** If $S$ is a compact, simple, idempotent, $n$-dimensional semigroup, then $S$ is isomorphically imbeddable in $N_{4n+4}$.

Proof: It is well known [39] that an $n$-dimensional space is topologically imbeddable in $E^{2n+1}$.

6.3 **Corollary.** Let $S$ be a compact simple semigroup in $E^n$ such that $E$, the idempotents of $S$, form a semigroup. Then:

(i) $S$ is isomorphic to a non-negative matrix semigroup if, and only if, each maximal group of $S$ is finite;

(ii) $S$ is isomorphic to a complex matrix semigroup if, and only if, each maximal group of $S$ is a Lie group.

Proof: If $S$ is imbeddable in the non-negative matrices, then each maximal group is finite by 3.3. Conversely, fix $x \in E$ as in 6.1. Since $H(x)$ is finite, it is isomorphic to a subgroup of $S_k$, the permutation group on $k$ elements, for some $k$. $S_k$, in turn, is isomorphic to the group of $k \times k$ permutation matrices $P_k$ (see 2.6). Let $g$ be the composite isomorphism imbedding $H(x)$ in $P_k$. Since $H(x)$ is finite, $g$ is trivially continuous. If $t \in H(x)$, define $h(t) = \begin{pmatrix} f(x) & 0 \\ 0 & g(t) \end{pmatrix}$, where $f$ is the function on $E$ defined in 6.1. To extend $h$ to $xS$, let $y \in xS \cap E$. Then $H(y) = H(x)y$, and this right translation mapping
is an isomorphism. Let $h(y) = \begin{pmatrix} f(y) & 0 \\ 0 & g(x) \end{pmatrix}$. If $s \in H(y)$, then $s = ty$, $t \in H(x)$, and this representation of $s$ is unique. Define

$$h(s) = h(t) h(y) = \begin{pmatrix} f(y) & 0 \\ 0 & g(t) \end{pmatrix}.$$  

The $h$ is clearly an isomorphism of $xS$ into the non-negative matrices of dimension $2n + 2 + k$. $h$ is now extendable to $Sx$ in the same manner as in 6.1.

To extend $h$ over all of $S$, let $w \in E$, $w \notin xS \cup Sx$. Let $z = Sx \cap E \cap wS$, $y = xS \cap E \cap Sw$. Then $zH(x)y = H(w)$, and the mapping this defined is an isomorphism. Let $p \in H(w)$, $p = zty$, $t \in H(x)$. Define $h(p) = h(z)h(t)h(y)$. By the uniqueness of this representation of $p$, the extension of $h$ is well defined. Now if $p, q \in H(w)$, $p = zty$, $q = zsy$, $s, t \in H(x)$, then $h(p)h(q) = h(z)h(t)h(y)h(z)h(s)h(y)$. Since $E$ is a semigroup, $yz = z$; hence $h(y)h(x) = h(yz) = h(x)$, and the above expression collapses to $h(z)h(ts)h(y) = h(ztsy) = h(pq)$, so that $h$ is a homomorphism. The remaining properties of $h$ are developed exactly as in 6.1.

If $S$ is imbeddable in the complex matrices of some dimension, then each maximal group of $S$ is isomorphic to a compact group of complex matrices, hence is a Lie group (3.1 and [34]). Conversely, suppose $H(x)$ is a compact Lie group. Then there exists a faithful representation, that is an isomorphism, of $H(x)$ into $Gl(k, \mathbb{C})$ [34]. On replacing the mapping $g$ of the previous paragraph by the faithful representation mentioned above, it is easily seen that the analogous definition of $h$ yields the desired imbedding.
Note that right (left) simple compact semigroups in $E^n$ with the proper types of maximal groups satisfy the hypotheses of 5.3, and hence have topological representations in the complex matrices.

6.4 Lemma. Let $S$ be a simple semigroup with $E$ connected, each $H(e)$ totally disconnected. Then $E$ is a subsemigroup of $S$.

Proof: Let $C$ be the component of $S$ containing $E$. Since $C^2$ is connected and meets $C$, it follows that $C^2 \subseteq C$, hence $C$ is a subsemigroup. It remains to show $C = E$. Let $e \in E$. Then $H(e) \cap C = eSe \cap C = eCe = \{e\}$, since $eCe$ is connected. This completes the proof.

6.5 Corollary. Let $S$ be a finite-dimensional compact simple semigroup with $E$ connected, each $H(e)$ finite. Then $S$ is isomorphically imbeddable in the non-negative matrices of some order.

6.6 Conjecture. Any finite dimensional compact simple semigroup with finite maximal groups is imbeddable in the non-negative matrices of some order.

6.7 Theorem. Let $S$ be a compact simple semigroup in $N^n$. Let $S$ contain a diagonal idempotent. Then $E$ is a semigroup.

Proof: By a row-column permutation, it may be assumed that

\[
\begin{pmatrix}
I_k & 0 \\
0 & 0
\end{pmatrix} \in S, \text{ with } k \text{ the rank of } S. \text{ Call this element } e.
\]

If $f \in eS \cap E, g \in Se \cap E$, then by computation, $f = \begin{pmatrix}
I_k & F \\
0 & 0
\end{pmatrix}$.
\[
g = \begin{pmatrix} I_k & 0 \\ G & 0 \end{pmatrix}, \text{ with } F, G \text{ non-negative matrices of dimensions } k \times (n-k) \text{ and } (n-k) \times k \text{ respectively. Now}
\]
\[
fg = \begin{pmatrix} I_k + FG & 0 \\ 0 & 0 \end{pmatrix} \in H(e). \text{ If } (fg)^t = e, \text{ then } (I_k + FG)^t = I_k.
\]

By non-negativity, \(FG = 0\), hence \(fg = e\). It follows that the product of any pair of idempotents is an idempotent, hence \(E\) is a semigroup.

The idempotents of a simple semigroup of non-negative matrices need not form a semigroup. However, any counterexample must have at least two distinct minimal left ideals, two distinct minimal right ideals, and groups of order at least two, a total of at least eight elements.

The following example is one of this type.

6.8. Example. Let \(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). Let
\[
E = \begin{pmatrix} I & J \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}, G = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & 0 \\ J & I \end{pmatrix},
\]
\[
X = \begin{pmatrix} J & I \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} J & J \\ 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 \\ I & J \end{pmatrix}, W = \begin{pmatrix} 0 & 0 \\ J & J \end{pmatrix}.
\]

Then \(H(E) = \{E,X\}, H(F) = \{F,Y\}, H(G) = \{G,W\}, H(K) = \{K,Z\}\).

Let \(S = H(E) \cup H(F) \cup H(G) \cup H(K)\). Then \(S\) is a simple semigroup with \(ES = FS, SE = SK\). Furthermore, \(FK = X\), which shows the idempotents of \(S\) do not form a semigroup. For non-negative matrices of order less than 4, it may be shown by computation that the idempotents must form a semigroup.
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Dennison Robert Brown was born in New Orleans, Louisiana, on May 17, 1934. He was educated in the public schools of New Orleans, Lake Charles, and Baton Rouge, Louisiana, and of Winnetka, Illinois. He attended Duke University upon graduation from high school, receiving the B. S. degree in physics from that institution in 1955. From June of that year until May, 1958, he served in the United States Navy, and currently holds the rank of Lieutenant, USNR.

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EXAMINATION AND THESIS REPORT

Candidate: Dennison Robert Brown

Major Field: Mathematics

Title of Thesis: Topological Semigroups of Non-negative Matrices

Approved:

[Signatures]

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination: May 10, 1963