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The Betti numbers of some finite racks

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Abstract

We show that the lower bounds for Betti numbers given in (J. Pure Appl. Algebra 157 (2001) 135) are equalities for a class of racks that includes dihedral and Alexander racks. We confirm a conjecture from the same paper by defining a splitting for the short exact sequence of quandle chain complexes. We define isomorphisms between Alexander racks of certain forms, and we also list the second and third homology groups of some dihedral and Alexander quandles.

MSC: 55N35; 57M99

1. Introduction

In [4], Fenn and Rourke describe a powerful invariant of knots and links, the fundamental rack. Indeed, the fundamental rack completely classifies non-split links in any homology 3-manifold. The categories of racks and quandles are related to the category of groups, and as with the fundamental group, distinguishing racks described by presentations can be as difficult as distinguishing knots. Invariants of racks and quandles, then, yield a new source of invariants of knots and links.

In [1], Carter et al. describe a homology theory for racks and quandles, which they then use to define a family of invariants of knots and links (in higher dimensions as well as the usual knotted circles in S^3). Studying the homology and cohomology groups of racks and quandles, then, opens the door to new ways of distinguishing knots and links.

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We start by recalling some basic definitions. Let X be a non-empty set with a binary operation, which, following Fenn and Rourke, we write as exponentiation: $(a,b)\mapsto a^b$. This allows us to dispense with brackets by using the standard conventions that a^{b^c} means $a^{(b^c)}$ and a^{bc} means $(a^b)^c$. Then X is a *rack* if it satisfies the following two axioms.

(i) For all $a,b\in X$, there is a unique element c of X such that $c^a=b$.
(ii) (The rack identity) For all $a,b,c\in X$, $a^{bc}=a^{cb^c}$.

A *quandle* is a rack satisfying one further axiom.

(iii) (The quandle condition) For all $a\in X$, $a^a=a$.

A rack is *trivial* if $a^b=a$ for all a and b.

By axiom (i), the function $f_a:X\to X$ defined by $f_a(b)=b^a$ is a bijection. For $a,b\in X$, we set $a^{\bar{b}}=f_b^{-1}(a)$. Here $\bar{b}$ does not denote an element of X, but we may identify $\bar{b}$ with the inverse of b in the free group $F(X)$ on X. This allows us to define a (right) action of $F(X)$ on X, and by an *orbit* of X we mean an orbit under this action. The set of orbits of X will be denoted by $\mathcal{O}_X$, and the projection from X to $\mathcal{O}_X$ by π. We regard $\mathcal{O}_X$ as a trivial rack, and then π is a rack homomorphism.

We now define the class of racks that we shall study in Section 3 of this paper. Let X be a finite rack, and $a,b\in X$. Let $N(a,b)$ be the number of elements c of X such that $a^c=b$. Of course, $N(a,b)=0$ if a and b are in different orbits. We say that X *has homogeneous orbits* if, for each orbit ω and each pair of elements a and b of ω, $N(a,b)$ depends only on ω. If this is so, then $|\omega|$ divides $|X|$ for each $\omega\in\mathcal{O}_X$, since for a fixed $a\in\omega$ there are $\sum_{b\in\omega}N(a,b)=|X|$ total actions, and then $N(a,b)=|X|/|\omega|$ for all $a,b\in\omega$; we set $N_\omega=|X|/|\omega|$.

Let us consider some of the standard examples of racks in the light of this definition. Clearly (if uninterestingly), any trivial finite rack has homogeneous orbits. So does any finite conjugation rack conj(G). (Here G is a group, and conj(G) denotes G with the rack operation $g^h=h^{-1}gh$.) Fenn and Rourke use the term *conjugation rack* in a broader sense, to refer to any union of conjugacy classes in a group. In general, these do not have homogeneous orbits (consider $G-\{1\}$); however, any dihedral rack R_n does. (R_n is the set of reflections in the dihedral group of order $2n$.) This is easy to verify directly, and also follows from Proposition 1. Any cyclic rack (except the trivial rack of order 1) does not have homogeneous orbits. (The cyclic rack C_n of order n is the set $\{0,1,\ldots,n-1\}$ with the operation $a^b=a+1 \bmod n$. Here, there is only one orbit, but $N(a,b)=n$ if $b=a+1 \bmod n$, and is 0 otherwise.

As an example of a non-quandle that does have homogeneous orbits, consider a four-element set $X=\{a,b,c,d\}$. We define the operation by specifying the permutations f_x of X: $f_a=f_b$ is the transposition exchanging a and b, and $f_c=f_d$ is the identity. One may check that the rack identity holds, most easily by using the third form of the identity given in [4]; the quandle condition clearly does not. The only non-trivial orbit is $\{a,b\}$, and $N(a,a)=N(a,b)=N(b,a)=N(b,b)=2$.

Next we consider the finite Alexander racks. Let M be any module over the ring $\Lambda=\mathbb{Z}[t,t^{-1}]$ of one-variable Laurent polynomials. Then M may be made into a rack by the operation $a^b = ta + (1-t)b$, and a rack obtained this way is called an *Alexander rack*. For $M = \mathbb{Z}_n[t, t^{-1}]/(t+1)$, the Alexander rack is isomorphic to R_n.

Proposition 1. *Let M be a finite Λ-module, and let $\bar{M}$ be the quotient of M by the submodule $(1-t)M$. When M is considered as an Alexander rack*:
(a) *M has homogeneous orbits*; *and*
(b) *$\mathcal{O}_M$ may be identified with $\bar{M}$.*

Proof. Let $p : M \to \bar{M}$ be the natural map. We have $a^x = a^y$ iff $(1-t)(x-y) = 0$, so for any $a, b \in M$, $N(a,b)$ is either 0 or the order of $\mathrm{Ker}(1-t : M \to M)$. The result will follow once we show that, for $a, b \in M$, the following statements are equivalent:
(1) a and b are in the same orbit;
(2) $p(a) = p(b)$;
(3) $N(a,b) \neq 0$.

Now $a - a^c = (1-t)(a-c)$, so $p(a) = p(a^c)$, from which it follows that (1) implies (2). If $p(a) = p(b)$, then $b = a + (1-t)c$ for some $c \in M$, which gives $b = a^{a+c}$. Thus (2) implies (3), and trivially (3) implies (1). □

In [5], Fenn et al. associate to each rack X a □-set (a cubical set without degeneracies) as follows. The set of n-cubes is X^n, and the face maps $\partial_i^\varepsilon : X^n \to X^{n-1}$ ($1 \leqslant i \leqslant n$, $\varepsilon = 0$ or 1) are defined by

$$\partial_i^0(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

$$\partial_i^1(x_1, \ldots, x_n) = (x_1^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1}, \ldots, x_n).$$

We follow Carter et al. [1] in denoting the associated chain complex by $C_*^R(X)$, and calling its homology $H_*^R(X)$ the *rack homology* of X. Thus $C_n^R(X)$ is the free abelian group on X^n, and the boundary map $\partial : C_n^R(X) \to C_{n-1}^R(X)$ is defined by $\partial = \sum_{i=1}^n (-1)^i (\partial_i^0 - \partial_i^1)$. Now suppose that X is a quandle, and define $C_n^D(X)$ to be the subgroup of $C_n^R(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some i, $1 \leqslant i < n$. It follows from the quandle condition that $C_*^D(X)$ is a subcomplex of $C_*^R(X)$. The quotient complex is denoted by $C_*^Q(X)$, and its homology $H_*^Q(X)$ is called the *quandle homology* of X. The homology $H_*^D(X)$ of $C_*^D(X)$ is the *degeneration homology* of X. We shall use the convention that in an expression such as $C_n^W(X)$, W may be any one of R, Q or D if X is a quandle, but is always R if not. There are Betti numbers $\beta_n^W(X) = \mathrm{rank}\, H_n^W(X)$. There are also homology and cohomology groups with coefficients in any abelian group G, denoted by $H_n^W(X; G)$ and $H_W^n(X; G)$. For the applications to knot theory (see for example, [1]), the groups of interest are the cohomology groups with coefficients in $\mathbb{Z}_p$ (the integers modulo a prime p), but since these are determined by the integral homology groups, we shall concentrate on the latter. The homology groups in dimensions 0 and 1 are easily computed; see [1, Proposition 3.8]. When the set of orbits of X is regarded as a trivial rack, the chain complex $C_*^W(\mathcal{O}_X)$ has all its boundary maps zero, so $H_n^W(\mathcal{O}_X) = C_n^W(\mathcal{O}_X)$. Thus when X is a finite rack with m orbits, $H_n^W(\mathcal{O}_X)$ ($n \geqslant 1$) is a free abelian group of rank m^n, $m(m-1)^{n-1}$ or $m^n - m(m-1)^{n-1}$ for $W = R$, Q or D, respectively. In [1], it is shown that in this case

$\beta_n^W(X) \geqslant \beta_n^W(\mathcal{O}_X)$. (It is not explicitly stated in [1] that the case $W = R$ holds when X is not a quandle, but this is so by essentially the same proof.) We now state our main result, which shows that these bounds are exact in many cases.

Theorem 2. *Let X be a finite rack with homogeneous orbits. Then $\beta_n^W(X) = \beta_n^W(\mathcal{O}_X)$, and the torsion subgroup of $H_n^W(X)$ is annihilated by $|X|^n$.*

Remark 3. While this paper was in preparation, we learned that Mochizuki has proved an almost identical theorem by a different method ([6, Theorem 1.1]). The main difference in the results is that Mochizuki's theorem applies only to finite Alexander racks.

The case $W = R$ of Theorem 1.1 is proved directly. For the other cases, we need to prove [1, Conjecture 3.11]; this is done in Section 2. Theorem 2 is proved in Sections 3 and 4, we report on some machine calculations of homology groups.

2. Splitting the difference between rack and quandle homology

In this section, X will always denote a quandle. Also, we redefine $C_0^R(X)$ and $C_0^Q(X)$ to be 0. ($C_0^D(X)$ is already 0.) This loses no information, and allows us to avoid treating dimension 0 as a special case at various points. Strictly speaking, we shall be working with the reduced complexes $\tilde{C}_*^R(X)$ and $\tilde{C}_*^Q(X)$, but we abuse notation by leaving off the tildes.

From the short exact sequence

$$0 \to C_*^D(X) \to C_*^R(X) \to C_*^Q(X) \to 0 \tag{1}$$

of chain complexes, we have a long exact sequence

$$\cdots \to H_n^D(X) \to H_n^R(X) \to H_n^Q(X) \to H_{n-1}^D(X) \to \cdots$$

of homology groups. In [1] it is proved (in Proposition 3.9) that the connecting homomorphism $H_n^Q(X) \to H_{n-1}^D(X)$ is the zero map when $n = 3$, and conjectured that this is so for all n; in [3, Theorem 8.2] the case $n = 4$ is proved.

We show that the conjecture is indeed true; in fact we prove more.

Theorem 4. *For any quandle X, the short exact sequence* (1) *is split.*

Remark 5. It is easy to see that, for each n, the sequence

$$0 \to C_n^D(X) \to C_n^R(X) \to C_n^Q(X) \to 0$$

of abelian groups is split, but the obvious splittings are not compatible with the boundary maps.

If $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and $\mathbf{y} = (y_1, \ldots, y_m) \in X^m$, we set $\mathbf{x} * \mathbf{y} = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in X^{n+m}$ and extend bilinearly to get $c * c' \in C_{n+m}^R(X)$ where $c \in C_n^R(X)$ and $c' \in C_m^R(X)$. Also, for $x \in X$ define $\mathbf{x}^x = (x_1^x, \ldots, x_n^x) \in X^n$ and extend linearly to obtain $c^x \in C_n^R(X)$.

Note that $\partial(c * y) = \partial(c) * y + (-1)^{n+1}(c - c^y)$. Next we define homomorphisms $\alpha_n : C_n^R(X) \to C_n^R(X)$ by setting

$$\alpha_n(\mathbf{x}) = x_1 * (x_2 - x_1) * (x_3 - x_2) * \ldots * (x_n - x_{n-1})$$

on n-tuples and extending linearly to all of $C_n^R(X)$. Then α_1 is the identity on $C_1^R(X)$ and the maps α_n satisfy the recursive relationship

$$\alpha_{n+1}(\mathbf{x} * x) = \alpha_n(\mathbf{x}) * x - \alpha_n(\mathbf{x}) * x_n.$$

We also define homomorphisms $\beta_n : C_n^R(X) \to C_{n+1}^R(X)$ by $\beta_n(\mathbf{x}) = \alpha_n(\mathbf{x}) * x_n$. Then, for any $c \in C_n^R(X)$ and $x \in X$ we have

$$\alpha_{n+1}(c * x) = \alpha_n(c) * x - \beta_n(c).$$

Lemma 6. *The homomorphisms $\alpha_n : C_n^R(X) \to C_n^R(X)$ form a chain map $\alpha : C_*^R(X) \to C_*^R(X)$.*

Proof. Note first that for $\mathbf{x} \in X^n$ and $y \in X$ we have $\alpha_n(\mathbf{x}^y) = \alpha_n(\mathbf{x})^y$. We prove that $\partial\alpha_n = \alpha_{n-1}\partial$ by induction on $n \geqslant 2$. For $n = 2$ we have $\alpha_2(x, y) = (x, y) - (x, x)$, so since (x, x) is a cycle, $\partial\alpha_2(x, y) = \partial(x, y) = \alpha_1\partial(x, y)$. Suppose then that the result is true for some $n \geqslant 2$, and let $\mathbf{x} \in X^n$ and $y \in X$. We compute

$$\begin{aligned}
\partial\alpha_{n+1}(\mathbf{x} * y) &= \partial(\alpha_n(\mathbf{x}) * y) - \partial(\alpha_n(\mathbf{x}) * x_n) \\
&= \partial\alpha_n(\mathbf{x}) * y + (-1)^{n+1}(\alpha_n(\mathbf{x}) - \alpha_n(\mathbf{x})^y) \\
&\quad - \partial\alpha_n(\mathbf{x}) * x_n - (-1)^{n+1}(\alpha_n(\mathbf{x}) - \alpha_n(\mathbf{x})^{x_n}) \\
&= \alpha_{n-1}\partial(\mathbf{x}) * y - \alpha_{n-1}\partial(\mathbf{x}) * x_n + (-1)^n\alpha_n(\mathbf{x}^y) \\
&\quad - (-1)^n\alpha_n(\mathbf{x}^{x_n})
\end{aligned}$$

and

$$\begin{aligned}
\alpha_n\partial(\mathbf{x} * y) &= \alpha_n(\partial(\mathbf{x}) * y + (-1)^{n+1}(\mathbf{x} - \mathbf{x}^y)) \\
&= \alpha_{n-1}\partial(\mathbf{x}) * y - \beta_{n-1}\partial(\mathbf{x}) - (-1)^n\alpha_n(\mathbf{x}) + (-1)^n\alpha_n(\mathbf{x}^y).
\end{aligned}$$

Hence $\partial\alpha_{n+1}(\mathbf{x} * y) = \alpha_n\partial(\mathbf{x} * y)$ iff

$$\alpha_{n-1}\partial(\mathbf{x}) * x_n + (-1)^n\alpha_n(\mathbf{x}^{x_n}) = \beta_{n-1}\partial(\mathbf{x}) + (-1)^n\alpha_n(\mathbf{x}). \tag{2}$$

Now, for $1 \leqslant i < n$ and $\varepsilon = 0$ or 1, $\partial_i^\varepsilon(\mathbf{x})$ is an element of X^{n-1} with last entry x_n, so $\alpha_{n-1}\partial_i^\varepsilon(\mathbf{x}) * x_n = \beta_{n-1}\partial_i^\varepsilon(\mathbf{x})$. Further,

$$\alpha_{n-1}\partial_n^0(\mathbf{x}) * x_n - \beta_{n-1}\partial_n^0(\mathbf{x}) = \alpha_n(\partial_n^0(\mathbf{x}) * x_n) = \alpha_n(\mathbf{x})$$

and

$$\alpha_{n-1}\partial_n^1(\mathbf{x}) * x_n - \beta_{n-1}\partial_n^1(\mathbf{x}) = \alpha_n(\partial_n^1(\mathbf{x}) * x_n) = \alpha_n(\mathbf{x}^{x_n}).$$

(The last step here uses the quandle condition.) It follows that

$$\alpha_{n-1}\partial(\mathbf{x}) * x_n - \beta_{n-1}\partial(\mathbf{x}) = (-1)^n(\alpha_n(\mathbf{x}) - \alpha_n(\mathbf{x}^{x_n})),$$

proving Eq. (2), and with it the lemma. □

Proof of Theorem 4. We show that the chain map $C_*^R(X) \to C_*^R(X)$ sending c to $c - \alpha(c)$ is a projection onto the subcomplex $C_*^D(X)$. We must prove the following two statements.
(a) If $c \in C_n^D(X)$ then $\alpha_n(c) = 0$.
(b) If $c \in C_n^R(X)$ then $c - \alpha_n(c) \in C_n^D(X)$.

That (a) holds is clear from the closed form of α_n, since every term of $c \in C_n^D(X)$ has $x_i = x_{i+1}$ for some i and hence each term in $\alpha_n(c)$ is multiplied by $x_{i+1} - x_i = 0$.

As for (b), this is clear for $n = 1$, so suppose that it holds for some $n \geqslant 1$ and take $\mathbf{x} \in X^n$ and $y \in X$. Then

$$\begin{aligned}\mathbf{x} * y - \alpha_{n+1}(\mathbf{x} * y) - \mathbf{x} * x_n &= (\mathbf{x} - \alpha_n(\mathbf{x})) * y - (\mathbf{x} - \alpha_n(\mathbf{x})) * x_n \\ &\in C_{n+1}^D(X).\end{aligned}$$

Since $\mathbf{x} * x_n$ is in $C_{n+1}^D(X)$ by the inductive hypothesis, so is $\mathbf{x} * y - \alpha_{n+1}(\mathbf{x} * y)$, and (b) follows. □

We shall denote the free abelian group on a set A by $\mathbb{Z}[A]$. (This is consistent with the usage $\mathbb{Z}[G]$ for a group ring.) It is shown in [1, Proposition 3.9] that $H_2^D(X) \cong \mathbb{Z}[\mathcal{O}_X]$. Combining this with Theorem 4 gives the first assertion of the next theorem; for the second we need some lemmas.

Theorem 7. *For any quandle X, we have*

$$H_2^R(X) \cong H_2^Q(X) \oplus \mathbb{Z}[\mathcal{O}_X]$$

and

$$H_3^R(X) \cong H_3^Q(X) \oplus H_2^Q(X) \oplus \mathbb{Z}[\mathcal{O}_X^2].$$

Let $C_n^L(X)$ be the subgroup of $C_n^D(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some i, $2 \leqslant i < n$. (We use the letter L because the degeneracy occurs late in these n-tuples.) Note that $C_n^L(X) = 0$ for $n < 3$.

Lemma 8. *The subgroups $C_n^L(X)$ form a subcomplex of $C_*^D(X)$.*

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n)$ have $x_i = x_{i+1}$ for some i with $2 \leqslant i < n$. Since $\partial_i^\varepsilon(\mathbf{x}) = \partial_{i+1}^\varepsilon(\mathbf{x})$ for $\varepsilon = 0$ or 1 (and, as for any $\mathbf{x} \in X^n$, $\partial_1^0(\mathbf{x}) = \partial_1^1(\mathbf{x})$), we have

$$\partial(\mathbf{x}) = \sum_{j=2}^{i-1}(-1)^j(\partial_j^0(\mathbf{x}) - \partial_j^1(\mathbf{x})) + \sum_{j=i+2}^{n}(-1)^j(\partial_j^0(\mathbf{x}) - \partial_j^1(\mathbf{x})). \qquad (3)$$

Fix j and ε, and set $\mathbf{y}=(y_1,\dots,y_{n-1})=\partial_j^{\varepsilon}(\mathbf{x})$. If $i=2$, the first sum in (3) is empty. If $i>2$ and $2\leqslant j\leqslant i-1$, $y_{i-1}=y_i$, so $\mathbf{y}\in C_{n-1}^L(X)$. For $i+2\leqslant j\leqslant n$, $y_i=y_{i+1}$, so again $\mathbf{y}\in C_{n-1}^L(X)$, and it follows that $\partial(\mathbf{x})\in C_{n-1}^L(X)$. □

Lemma 9. *There is an isomorphism of chain complexes* $C_*^D(X)\cong C_{*-1}^Q(X)\oplus C_*^L(X)$.

Proof. We let $i:C_*^D(X)\to C_*^R(X)$ and $j:C_*^L(X)\to C_*^D(X)$ be the inclusions. Define $r:C_{*-1}^R(X)\to C_*^D(X)$ by

$$r_n(x_1,x_2,\dots,x_{n-1})=(x_1,x_1,x_2,\dots,x_{n-1})$$

for $n\geqslant 2$. (For $n\leqslant 1$ the groups involved are 0.) A straightforward computation shows that r is a chain map. Since $r(C_{*-1}^D(X))\leqslant C_*^L(X)$, r induces $s:C_{*-1}^D(X)\to C_*^L(X)$.

Now r is injective, $C_*^D(X)$ is generated by $\mathrm{Im}(r)$ and $C_*^L(X)$, and $\mathrm{Im}(r)\cap C_*^L(X)=\mathrm{Im}(r\circ i)=\mathrm{Im}(j\circ s)$. Hence, there is a short exact sequence

$$0\to C_{*-1}^D(X)\xrightarrow{\phi}C_{*-1}^R(X)\oplus C_*^L(X)\xrightarrow{\psi}C_*^D(X)\to 0,$$

where $\phi(c)=(i(c),-s(c))$ and $\psi(d,e)=r(d)+j(e)$. By Theorem 1, there is an isomorphism of chain complexes $\chi:C_{*-1}^R(X)\to C_{*-1}^Q(X)\oplus C_{*-1}^D(X)$ such that, for $c\in C_{*-1}^D(X)$, $\chi i(c)=(0,c)$. Then $C_*^D(X)$ is isomorphic to the cokernel of

$$(\chi\oplus\mathrm{id})\circ\phi:C_{*-1}^D(X)\to C_{*-1}^Q(X)\oplus C_{*-1}^D(X)\oplus C_*^L(X).$$

But, for $c\in C_{*-1}^D(X)$, $(\chi\oplus\mathrm{id})(\phi(c))=(0,c,-s(c))$, so this cokernel is isomorphic as a chain complex to $C_{*-1}^Q(X)\oplus C_*^L(X)$, and we are done. □

We denote the homology of $C_*^L(X)$ by $H_*^L(X)$.

Lemma 10. *For any quandle* X, $H_3^L(X)\cong\mathbb{Z}[\mathcal{O}_X^2]$.

Proof. A basis for $C_3^L(X)$ consists of all elements of X^3 of the form (x,y,y), and these are all cycles. The group $C_4^L(X)$ is generated by all elements of X^4 of one of the forms (x,y,y,z) and (x,z,y,y), and we have

$$\partial(x,y,y,z)=(x,y,y)-(x^z,y^z,y^z)$$

and

$$\partial(x,z,y,y)=(x,y,y)-(x^z,y,y).$$

It follows that $H_3^L(X)$ is free abelian, with a basis consisting of the equivalence classes of triples (x,y,y) under the equivalence relation $\sim$ generated by

$$(x,y,y)\sim(x^z,y,y)\sim(x^z,y^z,y^z)\quad\text{for all } x,y,z\in X.$$

Given $x,y,z\in X$, let w be the element of X such that $w^z=x$. Then $(w,y,y)\sim(w^z,y,y)=(x,y,y)$ and $(w,y,y)\sim(w^z,y^z,y^z)=(x,y^z,y^z)$, so $(x,y,y)\sim(x,y^z,y^z)$. It follows that $(x,y,y)\sim(x',y',y')$ iff $\pi(x)=\pi(x')$ and $\pi(y)=\pi(y')$, so the set of equivalence classes of $\sim$ may be identified with $\mathcal{O}_X^2$. □

The second assertion of Theorem 7 follows immediately from Theorem 4 and Lemmas 9 and 10.

3. Proof of Theorem 2

In this section, X is a rack with homogeneous orbits, and $\mathbf{x} = (x_1, \dots, x_n)$ is an element of X^n ($n >= 0$). Define $\phi_n^j : C_n^R(X) \to C_n^R(X)$ by

$$\phi_n^j(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{for } j = 0, \\ \sum_{\mathbf{y} \in X^j} (x_1^{y_1}, \dots, x_j^{y_j}, x_{j+1}, \dots, x_n) & \text{for } 1 \leqslant j \leqslant n, \\ |X|^{j-n} \phi_n^n(\mathbf{x}) & \text{for } j > n \end{cases}$$

and $D_n^j : C_n^R(X) \to C_{n+1}^R(X)$ by

$$D_n^j(\mathbf{x}) = \begin{cases} \sum_{\mathbf{y} \in X^j} (x_1^{y_1}, \dots, x_{j-1}^{y_{j-1}}, x_j, y_j, x_{j+1}, \dots, x_n) & \text{for } 1 \leqslant j \leqslant n, \\ 0 & \text{for } j > n. \end{cases}$$

Note that $D_n^1 = \sum_{y \in X} (x_1, y, x_2, \dots, x_n)$.

We have homomorphisms of graded groups $\phi^j = (\phi_n^j)_{n=0}^{\infty} : C_*^R(X) \to C_*^R(X)$ for $j \geqslant 0$ and $D^j = (D_n^j)_{n=0}^{\infty} : C_*^R(X) \to C_{*+1}^R(X)$ for $j \geqslant 1$. We will show through a series of lemmas that D^j is a chain homotopy carrying ϕ^j to $|X|\phi^{j-1}$, and hence each ϕ^j is chain homotopic to $|X|^j$ times the identity. Note that this also implies ϕ^j is a chain map.

Lemma 11. *Let G be an abelian group. Then if $g : X \to G$ is a function we have*

$$\sum_{y \in X} g(x^y) = \sum_{y \in X} g(x^{yw})$$

for any word $w \in F(X)$ in the free group on X.

Proof. As y runs over X, x^y runs over $\pi(x)$, taking on each value $N_{\pi(x)}$ times. Thus

$$\sum_{y \in X} g(x^y) = N_{\pi(x)} \sum_{z \in \pi(x)} g(z).$$

The automorphism $f_w : X \to X$ given by $f_w(x) = x^w$ is in particular a bijection and carries $\pi(x)$ to itself, so the restriction $f|_{\pi(x)}$ is also a bijection.

Hence the sum

$$\sum_{y \in X} g(x^{yw}) = \sum_{y \in X} g(f_w(x^y)) = N_{\pi(x)} \sum_{z \in \pi(x)} g(z) = \sum_{y \in X} g(x^y). \qquad \square$$

Lemma 12. *Let G be an abelian group. Then if $g : X \to G$ is a function, we have*

$$\sum_{y \in X} g(x^y) = \sum_{y \in X} g(x^{wy})$$

for any word $w \in F(X)$ in the free group on X.

Proof. Since $\pi(x^w) = \pi(x)$, we have

$$\sum_{y\in X} g(x^y) = N_{\pi(x)}\left(\sum_{z\in\pi(x)} g(z)\right) = N_{\pi(x^w)}\left(\sum_{z\in\pi(x^w)} g(z)\right) = \sum_{y\in X} g(x^{wy}). \qquad \square$$

Lemma 13. *For* $1 \leqslant i \leqslant j \leqslant n$, $\partial_i^0 D_n^j(\mathbf{x}) = \partial_i^1 D_n^j(\mathbf{x})$.

Proof. For $i < j$, we have

$$\partial_i^0 D_n^j(\mathbf{x}) = \partial_i^0\left(\sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,x_{j+1},\ldots,x_n)\right)$$
$$= \sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{i-1}^{y_{i-1}},x_{i+1}^{y_{i+1}},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,\ldots,x_n)$$

and

$$\partial_i^1 D_n^j(\mathbf{x}) = \partial_i^1\left(\sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,x_{j+1},\ldots,x_n)\right)$$
$$= \sum_{\mathbf{y}\in X^j}(x_1^{y_1x_i^{y_i}},\ldots,x_{i-1}^{y_{i-1}x_i^{y_i}},x_{i+1}^{y_{i+1}},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,\ldots,x_n).$$

For $i = j$ we have

$$\partial_j^0 D_n^j(\mathbf{x}) = \sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},y_j,\ldots,x_n)$$

and

$$\partial_j^1 D_n^j(\mathbf{x}) = \sum_{\mathbf{y}\in X^j}(x_1^{y_1x_j},\ldots,x_{j-1}^{y_{j-1}x_j},y_j,\ldots,x_n).$$

Applying Lemma 11 $i-1$ times, the sums agree as required. $\square$

Lemma 14. *For* $1 \leqslant i \leqslant j < n$, $D_{n-1}^j\partial_i^0(\mathbf{x}) = D_{n-1}^j\partial_i^1(\mathbf{x})$.

Proof. For $1 \leqslant i \leqslant j$,

$$D_{n-1}^j\partial_i^0(\mathbf{x}) = D_{n-1}^j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$
$$= \sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{i-1}^{y_{i-1}},x_{i+1}^{y_i},\ldots,x_j^{y_{j-1}},x_{j+1},y_j,x_{j+2},\ldots,x_n)$$

and

$$D_{n-1}^j\partial_i^1(\mathbf{x}) = D_{n-1}^j(x_1^{y_1},\ldots,x_{i-1}^{y_{i-1}},x_{i+1},\ldots,x_n)$$
$$= \sum_{\mathbf{y}\in X^j}(x_1^{x_iy_1},\ldots,x_{i-1}^{x_iy_{i-1}},x_{i+1}^{y_i},\ldots,x_j^{y_{j-1}},x_{j+1},y_j,x_{j+2},\ldots,x_n).$$

Applying Lemma 12 $i-1$ times, the sums agree as required. $\square$

Lemma 15. *For* $1 \leqslant j \leqslant n$, $\partial^0_{j+1} D^j_n(\mathbf{x}) = |X|\phi^{j-1}_n(\mathbf{x})$.

Proof.

$$\begin{aligned}\partial^0_{j+1} D^j_n(\mathbf{x}) &= \partial^0_{j+1}\left(\sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,x_{j+1},\ldots,x_n)\right)\\ &= \sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j,x_{j+1},\ldots,x_n)\\ &= \sum_{y_j\in X}\phi^{j-1}_n(\mathbf{x})\\ &= |X|\phi^{j-1}_n(\mathbf{x}). \qquad \square\end{aligned}$$

Lemma 16. *For* $1 \leqslant j \leqslant n$, $\partial^1_{j+1} D^j_n(\mathbf{x}) = \phi^j_n(\mathbf{x})$.

Proof.

$$\begin{aligned}\partial^1_{j+1} D^j_n(\mathbf{x}) &= \partial^1_{j+1}\left(\sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,x_{j+1},\ldots,x_n)\right)\\ &= \sum_{\mathbf{y}\in X^j}(x_1^{y_1 y_j},\ldots,x_{j-1}^{y_{j-1} y_j},x_j^{y_j},x_{j+1},\ldots,x_n)\\ &= \sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j^{y_j},x_{j+1},\ldots,x_n)\\ &= \phi^j_n(\mathbf{x})\end{aligned}$$

by $j-1$ applications of Lemma 11. $\square$

Lemma 17. *For* $1 \leqslant j < i \leqslant n$, $D^j_{n-1}\partial^0_i(\mathbf{x}) = \partial^0_{i+1} D^j_n(\mathbf{x})$.

Proof.

$$\begin{aligned}D^j_{n-1}\partial^0_i(\mathbf{x}) &= D^j_{n-1}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\\ &= \sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,x_{j+1},\ldots,x_{i-1},x_{i+1},\ldots,x_n)\end{aligned}$$

and

$$\begin{aligned}\partial^0_{i+1} D^j_n(\mathbf{x}) &= \partial^0_{i+1}\left(\sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,x_{j+1},\ldots,x_n)\right)\\ &= \sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,x_{j+1},\ldots,x_{i-1},x_{i+1},\ldots,x_n). \qquad \square\end{aligned}$$

Lemma 18. *For* $1 \leqslant j < i \leqslant n$, $D^j_{n-1}\partial^1_i(\mathbf{x}) = \partial^1_{i+1}D^j_n(\mathbf{x})$.

Proof.

$$\begin{aligned} D^j_{n-1}\partial^1_i(\mathbf{x}) &= D^j_{n-1}(x_1^{x_i},\ldots,x_{i-1}^{x_i},x_{i+1},\ldots,x_n) \\ &= \sum_{\mathbf{y}\in X^j}(x_1^{x_i y_1},\ldots,x_{j-1}^{x_i y_{j-1}},x_j^{x_i},y_j,x_{j+1}^{x_i},\ldots,x_{i-1}^{x_i},x_{i+1},\ldots,x_n) \\ &= \sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j^{x_i},y_j,x_{j+1}^{x_i},\ldots,x_{i-1}^{x_i},x_{i+1},\ldots,x_n) \end{aligned}$$

and

$$\begin{aligned} \partial^1_{i+1}D^j_n(\mathbf{x}) &= \partial^1_{i+1}\left(\sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j,y_j,x_{j+1},\ldots,x_n)\right) \\ &= \sum_{\mathbf{y}\in X^j}(x_1^{y_1 x_i},\ldots,x_{j-1}^{y_{j-1}x_i},x_j^{x_i},y_j^{x_i},x_{j+1}^{x_i},\ldots,x_{i-1}^{x_i},x_{i+1},\ldots,x_n), \\ &= \sum_{\mathbf{y}\in X^j}(x_1^{y_1},\ldots,x_{j-1}^{y_{j-1}},x_j^{x_i},y_j^{x_i},x_{j+1}^{x_i},\ldots,x_{i-1}^{x_i},x_{i+1},\ldots,x_n) \end{aligned}$$

by $j-1$ applications of Lemmas 11 and 12. But these sums agree as the set $\{y_j^{x_i}\,|\,y_j\in X\}$ is the image of $\{y_j\,|\,y_j\in X\}$ under the bijection f_{x_i}. □

Putting all this together, we have

Proposition 19. *For* $j \geqslant 1$, $D^j : C^R_*(X) \to C^R_{*+1}(X)$ *is a chain homotopy from* ϕ^j *to* $|X|\phi^{j-1}$.

Proof. We need to show that

$$\partial_{n+1}D^j_n(\mathbf{x}) + D^j_{n-1}\partial_n(\mathbf{x}) = \pm(\phi^j_n(\mathbf{x}) - |X|\phi^{j-1}_n(\mathbf{x})).$$

For $j > n$, we have $D^j_n = D^j_{n+1} = 0$, while

$$\phi^j_n(\mathbf{x}) = |X|^{j-n}\phi^n_n(\mathbf{x}) = |X|(|X|^{(j-1)-n}\phi^n_n(\mathbf{x})) = |X|\phi^{j-1}_n(\mathbf{x})$$

as required.

For $j = n$, we have $D^j_{n-1} = 0$ and

$$\begin{aligned} \partial_{n+1}D^j_n(\mathbf{x}) &= \sum_{i=1}^{n+1}(-1)^i(\partial^0_i D^n_n(\mathbf{x}) - \partial^1_i D^n_n(\mathbf{x})) \\ &= \sum_{i\leqslant n}(-1)^i(\partial^0_i D^n_n(\mathbf{x}) - \partial^1_i D^n_n(\mathbf{x})) \\ &\quad + (-1)^{n+1}(\partial^0_{n+1}D^n_n(\mathbf{x}) - \partial^1_{n+1}D^n_n(\mathbf{x})). \end{aligned}$$

By Lemma 13, the first sum adds to zero, and by Lemmas 15 and 16 we have

$$\partial_{n+1}D_n^j(\mathbf{x}) = (-1)^{n+1}(|X|\phi_n^{n-1}(\mathbf{x}) - \phi_n^n(\mathbf{x}))$$

as required.

For $j < n$,

$$\begin{aligned}\partial_{n+1}D_n^j(\mathbf{x}) &= \sum_{i=1}^{n+1}(-1)^i(\partial_i^0 D_n^j(\mathbf{x}) - \partial_i^1 D_n^j(\mathbf{x}))\\ &= \sum_{i\leqslant n}(-1)^i(\partial_i^0 D_n^j(\mathbf{x}) - \partial_i^1 D_n^j(\mathbf{x}))\\ &\quad + (-1)^{j+1}(\partial_{n+1}^0 D_n^j(\mathbf{x}) - \partial_{n+1}^1 D_n^j(\mathbf{x}))\\ &\quad + \sum_{i=j+2}^{n+1}(-i)^i(\partial_i^0 D_n^j(\mathbf{x}) + \partial_i^1 D_n^j(\mathbf{x}))\end{aligned}$$

which, by Lemmas 13, 15 and 16 as above yields

$$\begin{aligned}\partial_{n+1}D_n^j(\mathbf{x}) &= (-1)^{j+1}(|X|\phi_n^{j-1}(\mathbf{x}) - \phi_n^j(\mathbf{x}))\\ &\quad + \sum_{i=j+2}^{n+1}(-i)^i(\partial_i^0 D_n^j(\mathbf{x}) + \partial_i^1 D_n^j(\mathbf{x}))\end{aligned}$$

Now,

$$\begin{aligned}D_{n-1}^j\partial_n(\mathbf{x}) &= \sum_{i=1}^{n}(-1)^i(D_{n-1}^j\partial_i^0(\mathbf{x}) - D_{n-1}^j\partial_i^1(\mathbf{x}))\\ &= \sum_{i=1}^{j}(-1)^i(D_{n-1}^j\partial_i^0(\mathbf{x}) - D_{n-1}^j\partial_i^1(\mathbf{x}))\\ &\quad + \sum_{i=j+1}^{n}(-1)^i(D_{n-1}^j\partial_i^0(\mathbf{x}) - D_{n-1}^j\partial_i^1(\mathbf{x})).\end{aligned}$$

The first sum is zero by Lemma 14 and applying Lemmas 17 and 18 we get

$$D_{n-1}^j\partial_n(\mathbf{x}) = \sum_{i=j+1}^{n}(-1)^i(\partial_{i+1}^0 D_n^j(\mathbf{x}) - \partial_{i+1}^1 D_n^j(\mathbf{x})).$$

Reindexing this sum by replacing $i+1$ with i', we have

$$D_{n-1}^j\partial_n(\mathbf{x}) = \sum_{i'=j+2}^{n+1}(-1)^{i'+1}(\partial_{i'}^0 D_n^j(\mathbf{x}) - \partial_{i'}^1 D_n^j(\mathbf{x})),$$

so that

$$\partial_{n+1}D_n^j(\mathbf{x}) + D_{n-1}^j\partial_n(\mathbf{x}) = (-1)^{j+1}(|X|\phi_n^{j-1}(\mathbf{x}) - \phi_n^j(\mathbf{x}))$$

as required. □

Proof of Theorem 2. We deal first with the case $W = R$. There is a chain map $\pi^R : C_*(X) \to C_*(\mathcal{O}_X)$ induced by the projection of X onto its orbit rack. In [1, Lemma 4.2], it is proved that for $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \mathcal{O}_X^n$, the element $\sum_{z_j \in \omega_j, j=1,\ldots,n} (z_1, \ldots, z_n)$ of $C_n^R(X)$ is a cycle. Since the boundary maps in $C_*(\mathcal{O}_X)$ are all zero, this means that we can define a chain map $\psi : C_*^R(\mathcal{O}_X) \to C_*^R(X)$ by setting

$$\psi_n(\boldsymbol{\omega}) = \left(\prod_{i=1}^n N_{\omega_i}\right) \sum_{z_j \in \omega_j, j=1,\ldots,n} (z_1, \ldots, z_n).$$

This is almost the same as the chain map used in [1, Theorem 4.1]. Then, for $\mathbf{x} \in X^n$,

$$\begin{aligned}\psi_n \pi_n^R(\mathbf{x}) &= \left(\prod_{i=1}^n N_{\pi(x_i)}\right) \sum_{z_j \in \pi(x_j), j=1,\ldots,n} (z_1, \ldots, z_n) \\ &= \sum_{\mathbf{y} \in X^n} (x_1^{y_1}, \ldots, x_n^{y_n}) \\ &= \phi_n^n(\mathbf{x}).\end{aligned}$$

Hence, by Proposition 19, the induced map $\psi_* \pi_*^R : H_n^R(X) \to H_n^R(X)$ is multiplication by $|X|^n$. It follows, since $H_n^R(\mathcal{O}_X)$ is free abelian, that the torsion subgroup of $H_n^R(X)$ is equal to $\operatorname{Ker} \pi_*^R$ and is annihilated by $|X|^n$, and that $\beta_n^R(X) \leqslant \beta_n^R(\mathcal{O}_X)$. Since the reverse inequality was proved in [1], the proof in the case of rack homology is complete.

When X is a quandle, the other two cases follow from the case just proved [1, Theorems 4, 4.1]. □

4. Computations

In [2] (Table 1), the cohomology groups $H_Q^n(X; \mathbb{Z}_p)$ of some Alexander racks are given for $n = 2$ or 3 and the first few primes p. These racks are of the form $\Lambda_n/(h)$, and the number m of orbits is easily computed from Proposition 1(b).

For $X = \Lambda_3/(t^2+t+1)$, $m=3$, so according to Theorem 2, the dimension of $H_Q^2(X; \mathbb{Z}_p)$ should be 6 for $p \neq 3$, while the value in [2] is 3 in these cases. This led the first author to write a C program to check the computations. Apart from $\Lambda_3/(t^2+t+1)$, where the recomputation gave the same values as for $\Lambda_9/(t-4)$, the results agreed with one exception, for $X = \Lambda_3/(t^2 - t + 1)$. Here [2] has $\dim H_Q^2(X; \mathbb{Z}_3) = 0$, while the recomputation yields $\dim H_Q^2(X; \mathbb{Z}_3) = 1$. The value 1 is in agreement with [6, Corollary 2.4]. It turns out that the disagreement is due to typographical errors in [2], and the values just given are the ones computed by Carter et al.

A variant of this program computes the integral homology of racks; we present in Table 1 the results of some calculations. In view of Theorem 7, we give only the quandle homology, though the program has been run to compute rack homology with the results expected from Theorem 7. As in [2], the racks considered are non-trivial, of order at most 9, and of the form $\Lambda_n/(h)$ where h is a monic polynomial whose constant term is a unit in $\mathbb{Z}_n$. The list of racks is different from that in [2] in two ways. First,

Table 1
Some quandle homology groups

X	$H_2^Q(X)$	$H_3^Q(X)$
R_3	0	$\mathbb{Z}_3$
R_4	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^4$
R_5	0	$\mathbb{Z}_5$
R_6	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3^2$
R_7	0	$\mathbb{Z}_7$
R_8	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_8^2$
R_9	0	$\mathbb{Z}_9$
$\Lambda_5/(t-2)$	0	0
$\Lambda_5/(t-3)$	0	0
$\Lambda_7/(t-2)$	0	0
$\Lambda_7/(t-3)$	0	0
$\Lambda_7/(t-4)$	0	0
$\Lambda_7/(t-5)$	0	0
$\Lambda_8/(t-5)$	$\mathbb{Z}^{12} \oplus \mathbb{Z}_2^4$	$\mathbb{Z}^{36} \oplus \mathbb{Z}_2^{24}$
$\Lambda_9/(t-2)$	0	$\mathbb{Z}_3$
$\Lambda_9/(t-4)$	$\mathbb{Z}^6 \oplus \mathbb{Z}_3^3$	$\mathbb{Z}^{12} \oplus \mathbb{Z}_3^{12}$
$\Lambda_9/(t-5)$	0	$\mathbb{Z}_3$
$\Lambda_2/(t^2+t+1)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$
$\Lambda_3/(t^2+1)$	$\mathbb{Z}_3$	$\mathbb{Z}_3^3$
$\Lambda_3/(t^2-1)$	$\mathbb{Z}^6$	$\mathbb{Z}^{12} \oplus \mathbb{Z}_3^3$
$\Lambda_3/(t^2-t+1)$	$\mathbb{Z}_3$	$\mathbb{Z}_3 \oplus \mathbb{Z}_9$
$\Lambda_3/(t^2+t-1)$	0	0
$\Lambda_3/(t^2-t-1)$	0	0
$\Lambda_2/(t^3+1)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^6 \oplus \mathbb{Z}_4^2$
$\Lambda_2/(t^3+t^2+1)$	0	$\mathbb{Z}_2$
$\Lambda_2/(t^3+t+1)$	0	$\mathbb{Z}_2$
$\Lambda_2/(t^3+t^2+t+1)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_8^2$

we have included $\Lambda_3/(t^2-t-1)$ and $\Lambda_2/(t^3+t^2+t+1)$. Second, it turns out that $R_4 \simeq \Lambda_2/(t^2+1)$, $\Lambda_9/(t-4) \simeq \Lambda_9/(t-7) \simeq \Lambda_3/(t^2+t+1)$, and $R_8 \simeq \Lambda_8/(t-3)$ (where $\simeq$ denotes rack-isomorphism), and we have omitted all but the first of each isomorphism class. That $R_4 \simeq \Lambda_2/(t^2+1)$ and $\Lambda_9/(t-4) \simeq \Lambda_9/(t-7)$ was noted in [2]. The other isomorphisms were discovered by a brute-force computation, and that remains the only assurance we have that the racks we have listed are all distinct. The existence of all these isomorphisms follows from the next two propositions.

Proposition 20. *If k is coprime to n then $\Lambda_{n^2}/(t-(kn+1)) \simeq \Lambda_n/((t-1)^2)$.*

Proof. We identify $\Lambda_{n^2}/(t-(kn+1))$ with $\mathbb{Z}_{n^2}$ under the operation $a^b=(kn+1)a-knb$. There is a short exact sequence of abelian groups

$$0 \to \mathbb{Z}_n \xrightarrow{\alpha} \mathbb{Z}_{n^2} \xrightarrow{\beta} \mathbb{Z}_n \to 0,$$

where $\alpha(1)=n$ and $\beta(1)=1$. Note that for $a\in\mathbb{Z}_{n^2}$, $\alpha^{-1}(na)=\beta(a)$. Let $\gamma:\mathbb{Z}_n\to\mathbb{Z}_{n^2}$ be a function such that $\beta\gamma=\mathrm{id}$, and define $\delta:\mathbb{Z}_{n^2}\to\mathbb{Z}_n$ by $\delta(a)=\alpha^{-1}(a-\gamma\beta(a))$. The function $\mathbb{Z}_{n^2}\to\mathbb{Z}_n^2$ sending a to $(\beta(a),\delta(a))$ is a bijection. Now define $f:\mathbb{Z}_{n^2}\to\Lambda_n/((t-1)^2)$ by $f(a)=k\beta(a)+(t-1)\delta(a)$; because k is coprime to n, f is also a bijection. We have, for $a,b\in\mathbb{Z}_{n^2}$, $\beta(a^b)=\beta(a)$ and

$$\begin{aligned}\delta(a^b)&=\alpha^{-1}((kn+1)a-knb-\gamma\beta(a))\\&=\alpha^{-1}(kna)-\alpha^{-1}(knb)+\alpha^{-1}(a-\gamma\beta(a))\\&=k\beta(a)-k\beta(b)+\delta(a).\end{aligned}$$

Hence

$$\begin{aligned}f(a^b)&=k\beta(a)+(t-1)(k\beta(a)-k\beta(b)+\delta(a))\\&=kt\beta(a)+(t-1)\delta(a)+k(1-t)\beta(b).\end{aligned}$$

On the other hand,

$$\begin{aligned}f(a)^{f(b)}&=t(k\beta(a)+(t-1)\delta(a))+(1-t)(k\beta(b)+(t-1)\delta(b))\\&=kt\beta(a)+(t-1)\delta(a)+k(1-t)\beta(b),\end{aligned}$$

so f is the desired isomorphism. □

Proposition 21. *If n is divisible by 4 then $R_{2n}\simeq\Lambda_{2n}/(t-(n-1))$.*

Proof. Here, the underlying sets of both racks are naturally identified with $\mathbb{Z}_{2n}$. We use a^b for the rack operation in R_{2n}, and $a^{[b]}$ for that in $\Lambda_{2n}/(t-(n-1))$. Thus, for $a,b\in\mathbb{Z}_{2n}$,

$$a^b=2b-a$$

and

$$a^{[b]}=(n-1)a+(2-n)b.$$

Define functions ε and f from $\mathbb{Z}_{2n}$ to itself by

$$\varepsilon(a)=\begin{cases}0, & \text{if } a\equiv 0 \text{ or } 1 \pmod 4,\\ n, & \text{if } a\equiv 2 \text{ or } 3 \pmod 4\end{cases}$$

and $f(a)=a+\varepsilon(a)$. Since $f(a)\equiv a\pmod 4$, f is an involution. Since $a^b\equiv a\pmod 2$, we have that $\varepsilon(a^b)=\varepsilon(a)$ iff $a^b\equiv a\pmod 4$, which in turn is equivalent to $a\equiv b\pmod 2$. Since ε only takes on the values 0 and n, this implies that $\varepsilon(a^b)=\varepsilon(a)+n(a-b)$. Hence

$$\begin{aligned}f(a^b)&=2b-a+\varepsilon(a)+n(a-b)\\&=(n-1)a+\varepsilon(a)+(2-n)b.\end{aligned}$$

On the other hand,

$$f(a)^{[f(b)]} = (n-1)(a+\varepsilon(a)) + (2-n)(b+\varepsilon(b))$$
$$= (n-1)a + \varepsilon(a) + (2-n)b,$$

so f is the desired isomorphism. □

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