2016

Global a priori Estimates and Sharp Existence Results for Quasilinear Equations on Nonsmooth Domains.

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GLOBAL A PRIORI ESTIMATES AND SHARP EXISTENCE RESULTS FOR QUASILINEAR EQUATIONS ON NONSMOOTH DOMAINS.

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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May 2016

Acknowledgments

Firstly, I would like to express my sincere gratitude to my adviser Prof. Nguyen Cong Phuc for the continuous support of my Ph.D study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better adviser and mentor for my Ph.D study.

Besides my adviser, I would like to thank Professors W. Adkins, Y. Almog, O. Dasbach, R. Estrada, R. Lipton, J. Madden, F. Neubrander, J. Oxley, R. Litherland, L. Richardson, D. Sage, S. Shipman, P. Sundar and S. Walker at LSU for teaching me mathematics and also being a constant source of encouragement. I would also like to thank Professor Jianhua Chen from the Computer Science department at LSU for agreeing to be the external examiner.

I would like to thank all my friends and colleagues with whom I had the pleasure to interact both mathematically as well as socially both at LSU as well as at previous institutions that I was a part of.

Last but not the least, I would like to thank my family: my parents and my brother for supporting me throughout my life in general. I would specially like to thank my father who has been the inspiration and guiding force for me to be the mathematician I currently am.

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Abstract

This thesis deals obtaining global a priori estimates for quasilinear elliptic equations and sharp existence results for Quasilinear equations with gradient nonlinearity on the right. The main results are contained in Chapters 3, 4, 5 and 6. In Chapters 3 and 4, we obtain global unweighted a priori estimates for very weak solutions below the natural exponent and weighted estimates at the natural exponent. The weights we consider are the well studied Muckenhoupt weights. Using the results obtained in Chapter 4, we obtain sharp existence result for quasilinear operators with gradient type nonlinearity on the right. We characterize the function space which yields such sharp existence results. Finally in Chapter 6, we prove existence of very weak solutions to quasilinear equations below the natural exponent with measure data on the right.

Chapter 1
Introduction

This thesis deals with the regularity properties of solutions to nonlinear elliptic equations of the type

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = -\operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f}$$

in a bounded domain Ω. The quasilinear operator $\mathcal{A}(x, \nabla u)$ (see Chapter 2 for the precise structure) is modeled after the familiar p-Laplace equation given by

$$\begin{aligned} -\Delta_p u \;&:= -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \\ &:= -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left[\sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\right)^2\right]^{\frac{p-2}{2}} \frac{\partial u}{\partial x_i}\right). \end{aligned} \tag{1.1}$$

The kind of operators we consider and the results we prove can be considered as natural analogue of the well developed Calderon Zygmund theory for the linear elliptic equations modeled after the linear Laplace operator:

$$-\Delta u := -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

We shall make precise all the definition and notations in Chapter 2. Before that, let us collect some history of the problem in order to motivate the results in later chapters.

1.1 Motivation - The linear problem

We shall consider the simplest linear equation: the Poisson equation given by

$$-\Delta u = -\operatorname{div}(\mathbf{f}) \qquad \text{in } \mathbb{R}^n \tag{1.2}$$

for any $\mathbf{f} \in C_c^{\infty}(\mathbb{R}^n)$. We say $u \in W^{1,1}(\mathbb{R}^n)$ is a distributional solution to (1.2) if

$$\int_{\mathbb{R}^n} \langle \nabla u, \nabla \phi \rangle \; dx = \int_{\mathbb{R}^n} \langle \mathbf{f}, \nabla \phi \rangle \; dx$$

for all $\phi \in C_c^\infty(\mathbb{R}^n)$. A classical result going back to the work of Calderon and Zygmund asserts that

$$\mathbf{f} \in L^s(\mathbb{R}^n) \quad \implies \quad \nabla u \in L^s(\mathbb{R}^n) \ \text{ for any } 1 < s < \infty.$$

The above implication comes with the quantitative estimate

$$\int_{\mathbb{R}^n} |\nabla u|^s \ dx \leq C(n,s) \int_{\mathbb{R}^n} |\mathbf{f}|^s \ dx. \tag{1.3}$$

It is well known that (1.3) fails when $s = 1$ and $s = \infty$.

The classical proof of (1.3) uses Fourier analysis and singular integral techniques and proceeds as follows: looking at (1.2) in the Fourier space, we get

$$4\pi^2|\zeta|^2\hat{u}(\zeta) = 2\pi i \langle \zeta, \hat{\mathbf{f}}(\zeta)\rangle$$

where $\hat{u}(\zeta) := \int_{\mathbb{R}^n} e^{-2\pi i \langle \zeta, x\rangle} u(x) \ dx$ and $\langle \zeta, \hat{\mathbf{f}}(\zeta)\rangle := \zeta_1\hat{\mathbf{f}}_1(\zeta) + \zeta_2\hat{\mathbf{f}}_2(\zeta) + \cdots + \zeta_n\hat{\mathbf{f}}_n(\zeta)$. Using the simple observation that $\partial_i u(x) = [-2\pi i \zeta_i \hat{u}(\zeta)]^\vee$, we can write

$$\nabla u = (R \otimes R)\mathbf{f} \tag{1.4}$$

where $R \otimes R$ is a matrix with the ij^{th} entry $R_{ij} = R_i \circ R_j$. Here R_i is the i^{th} Riesz transform defined by $R_i(g)(x) = \left(-\frac{i\zeta_i}{|\zeta|}\hat{g}(\zeta)\right)^\vee(x)$. In (1.4), $(R \otimes R)\mathbf{f}$ is to be understood as the matrix $(R \otimes R)$ acting on the vector field $(\mathbf{f})$ by the usual matrix multiplication.

From the theory of Singular Integral operators, we know that

$$\|R_i\|_{L^s(\mathbb{R}^n)\to L^s(\mathbb{R}^n)} \leq C(s) \qquad \text{for any } 1 < s < \infty.$$

Note here that the constant depends only on s and is independent of n (see [4]). This boundedness result applied to (1.4) easily implies the estimate (1.3).

The above boundedness result relied very heavily on the fact that the operator considered in (1.2) was linear and we were studying the problem on the whole space $\mathbb{R}^n$ which enabled the use of Fourier analysis techniques. From these observations, the following questions can now be asked:

- What happens if the consider the problem

$$\begin{cases} -\operatorname{div} A(x)\nabla u = -\operatorname{div} \mathbf{f} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$. What regularity do we need to assume on the coefficient matrix $A(x)$ and the domain Ω?

- If we consider the p-Laplace equation

$$\begin{cases} -\operatorname{div} |\nabla u|^{p-2}\nabla u = -\operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

can one expect estimates of the form (1.3) to hold? What estimates hold true for more general Quasilinear operators of the form $\mathcal{A}(x, \nabla u)$ modelled after the p-laplace equation?

These questions will be looked at in detail in the rest of this thesis.

Before we proceed, we would like to state two examples that highlight some of the hidden difficulties:

1.1.1 Bad Coefficients example

This example is appeared in [45]. In $\mathbb{R}^2$, the function $u(x) = \dfrac{x_1}{\sqrt{|x|}}$ solves the equation $-\text{div}(A(x)\nabla u) = 0$, where

$$A(x) = \frac{1}{4|x|^2}\begin{bmatrix} 4x_1^2 + x_2^2 & 3x_1x_2 \\ 3x_1x_2 & x_1^2 + 4x_2^2 \end{bmatrix}, \qquad x = (x_1, x_2).$$

It is easy to see that $\nabla u \in L^q_{loc}$ for $q < 4$ but $\nabla u \notin L^q(B_1)$ for $q \geq 4$. We see that $A(x)$ is bounded but highly oscillatory near the origin! Hence for obtaining estimates of the form (1.3), we need assume certain regularity on the coefficients of the operator considered.

1.1.2 Bad Domain example

This example appeared in [29]. For $\pi/2 < \theta_0 < \pi$, consider the non convex domain in $\mathbb{R}^2$ defined in polar coordinates by $\Omega = \{(r, \theta) : 0 \leq r \leq 1,\ -\theta_0 \leq \theta \leq \theta_0\}$. Then for

$\lambda = \dfrac{\pi}{2\theta_0} < 1$, let $u(r,\theta) = r^\lambda \cos(\lambda\theta)$ and

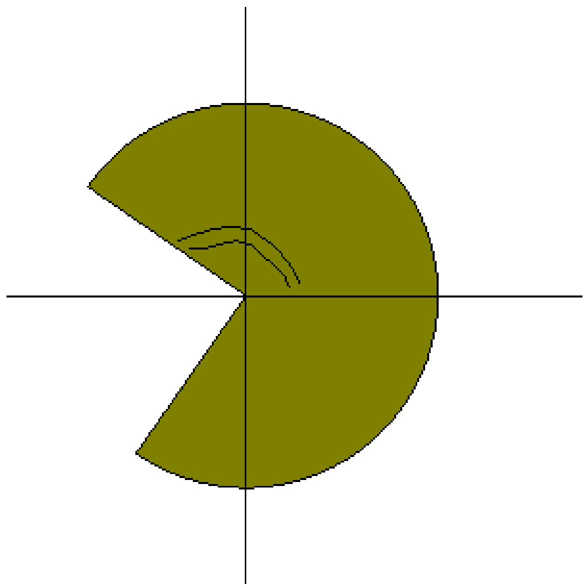
$$v(r,\theta) = u(r,\theta)(1 - r^2),$$

Then it is easy to see that

$$\Delta v = \operatorname{div} \mathbf{f} \quad \text{in } \Omega \qquad\qquad (\text{where } \mathbf{f} := \nabla[-r^2 u(r,\theta)]).$$

Near the origin, it is easy to observe that $|\nabla v| \approx |\nabla u| = \lambda r^{\lambda - 1}$.

Thus for any $q > 4$ we can find a θ_0 such that $|\nabla v| \notin L^q(\Omega_{\theta_0})$



This shows that we need to assume some regularity on the boundary of the domain considered. For example, we cannot expect the estimate of the form (1.3) to hold on all Lipschitz domains.

1.2 Iwaniec Conjectures

T. Iwaniec made the following far reaching conjectures regarding the p-Laplace operator. These conjectures form the main motivation for much of our work in Chapters 3 and 4.

Consider the equation

$$\begin{cases} -\operatorname{div} |\nabla u|^{p-2}\nabla u = -\operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

for any $\mathbf{f} \in L^s(\Omega)$ with $s > \max\{1, p-1\}$. We say that $u \in W_0^{1,s}(\Omega)$ is a weak solution to (1.5) if

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle \, dx = \int_\Omega |\mathbf{f}|^{p-2}\langle \mathbf{f}, \nabla \phi\rangle \, dx$$

for all $\phi \in C_c^\infty(\Omega)$. In the case $s \in (p-1, p)$, we call such solutions as *very weak solutions.*

The following fundamental result is due to T.Iwaniec who established the foundations of the non-linear Calderon Zygmund theory:

Theorem 1.1 ([26])**.** *Let $u \in W^{1,p}(\mathbb{R}^n)$ be a weak solution to* (1.5) *in $\mathbb{R}^n$. Then*

$$\mathbf{f} \in L^s(\mathbb{R}^n, \mathbb{R}^n) \Longrightarrow \nabla u \in L^s(\mathbb{R}^n, \mathbb{R}^n) \quad \textit{for every } s \geq p.$$

The local version of this result is:

Theorem 1.2. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to* (1.5) *in bounded domain Ω. Then*

$$\mathbf{f} \in L^s_{loc}(\Omega, \mathbb{R}^n) \Longrightarrow \nabla u \in L^s_{loc}(\Omega, \mathbb{R}^n) \quad \textit{for every } s \geq p.$$

Moreover, there exists a constant $c = c(n,p,s)$ such that for every ball $B_R \Subset \Omega$, there holds

$$\left(\fint_{B_{R/2}} |\nabla u|^s \, dx \right)^{\frac{1}{s}} \leq c \left(\fint_{B_R} |\nabla u|^p \, dx \right)^{\frac{1}{p}} + \left(\fint_{B_R} |\mathbf{f}|^s \, dx \right)^{\frac{1}{s}}$$

See also [34, 35] for a slightly different form of the local gradient estimate above the natural exponent.

Based on the above two theorems, T.Iwaniec made the following conjecture:

Conjecture 1.3 (T.Iwaniec)**.** The results of Theorem 1.1 and 1.2 should hold for the full range of the exponents $s > \max\{p-1, 1\}$.

The only progress for the case $\max\{p-1, 1\} < s < p$ came independently due to T.Iwaniec and C.Sbordonne in [28] and J.Lewis in [37].

Theorem 1.4 ([28])**.** *Let Ω be a 'regular' domain. Then there exists an $\epsilon = \epsilon_{(n,p)} > 0$ such that for any $s \in (p-\epsilon, p+\epsilon)$ with $p - \epsilon > 1$, and $u \in W_0^{1,s}(\Omega)$ solving* (1.5) *with $\mathbf{f} \in L^s(\Omega)$,* *there holds:*

$$\int_\Omega |\nabla u|^s \, dx \leq C_{(n,p)} \int_\Omega |\mathbf{f}|^s \, dx.$$

See Chapter 3 for the definition of 'regular domain'. The result in [28] is global. The local version was proved using very different techniques by J.Lewis in [37].

In the same spirit as the Conjecture 1.3, the following conjecture was also made by T.Iwaniec:

Conjecture 1.5 (T. Iwaniec [28])**.** Every weak p-harmonic mapping $u \in W^{1,s}_{\text{loc}}(\Omega, \mathbb{R}^n)$ with $\max\{p-1, 1\} < s < p$ solving

$$-\Delta_p u = 0 \qquad \text{in } \Omega$$

belongs to $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R})$.

The only known progress in this direction is due to T. Iwaniec and C. Sbordonne in [28] and independently by John Lewis [37] and the result states:

Theorem 1.6 ([28])**.** *There exists an $\epsilon = \epsilon_{(n,p)} > 0$ such that for any weakly p-harmonic mapping $u \in W^{1,s}_{\text{loc}}(\Omega, \mathbb{R})$ with $s \in (p-\epsilon, p)$ with $p - \epsilon > 1$ belongs to $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R})$.*

In this thesis, we study Conjecture 1.3 from the view of weighted estimates in which we consider weights in the Muckenhoupt class A_p. One of the hallmarks of weighted norm inequalities is the theory of extrapolation developed by Garcia-Cuerva and Rubio de Francia (see [11, 19]).

We show the following scaled version of the extrapolation theorem of Garcia-Cuerva and Rubio de Francia in Chapter 4:

Theorem 1.7. *For a fixed $p > 1$, let $\mathbf{f} \in L^p(\Omega)$ be any given vector field and let $u \in W^{1,p}_0(\Omega)$ be the unique solution to* (1.5)*. Suppose we have*

$$\int_\Omega |\nabla u|^p \, v(x) dx \leq C_{([v]_{\frac{p}{p-1}})} \int_\Omega |\mathbf{f}|^p v(x) \, dx$$

holds for all weights $v \in A_{\frac{p}{p-1}}$*, then for any* $\max\{p-1,1\} < q < \infty$*, there holds:*

$$\int_\Omega |\nabla u|^q \; w(x)dx \leq C_{([w]_{\frac{q}{p-1}})} \int_\Omega |\mathbf{f}|^q w(x)\; dx$$

for all weights $w \in A_{\frac{q}{p-1}}$.

The really nice thing about the above extrapolation theory is that it reduces the problem of obtaining L^q bounds below the natural exponent p to that of obtaining weighted estimates at the natural exponent. While this may appear to provide a simpler approach to Conjecture 1.3, we must emphasis that the difficulty essentially remains the same. In view of the extrapolation theorem, we make the following generalization of Conjecture 1.3 as follows:

Conjecture 1.8. For $p > 1$, let $\mathbf{f} \in L^p(\Omega)$ be a given vector field and denote $u \in W_0^{1,p}(\Omega)$ be the unique solution solving (1.5). Then the following estimate holds for all weights $v \in A_{\frac{p}{p-1}}$:

$$\int_\Omega |\nabla u|^p \; v(x)dx \leq C_{([v]_{\frac{p}{p-1}})} \int_\Omega |\mathbf{f}|^p v(x)\; dx$$

In Chapter 4, we are able to show the estimate

$$\int_\Omega |\nabla u|^p \; v(x)dx \leq C_{([v]_1)} \int_\Omega |\mathbf{f}|^p v(x)\; dx \tag{1.6}$$

holds for all $v \in A_1$. Even though this result is far from being optimal, it nonetheless constitutes an end point estimate to the results proved in [42, 43, 44].

Once we have estimate (1.6), in Chapter 5 we use that to study existence of solutions to problems of the form:

$$\begin{cases} -\Delta_p u = |\nabla u|^p + \sigma & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

where σ is a distribution. This problem has been very well studied over the past several decades (see [12, 42, 43] and the references therein).

In Chapter 3, we prove global estimates similar to those obtained in Theorem 1.4 and 1.6, but with quantifiable conditions on the regularity of the domain and for more general Quasilinear operators satisfying certain natural growth conditions. In Chapter 4, we obtained global weighted estimates analogous to those obtained in Chapter 3. In Chapter 5, we use the results of the preceding chapters to prove some sharp existence results for quasilinear equations of the form (1.7). In Chapter 6, we will show existence of very weak solutions to (1.5) under some mild restrictions on the datum $\mathbf{f}$.

Chapter 2
Preliminaries

We shall now state all the definitions and results that will be used in subsequent chapters.

A *bounded open connected* set $\Omega \Subset \mathbb{R}^n$ is called a domain and we denote the boundary of the domain by $\partial\Omega$.

For any function $u : \Omega \mapsto \mathbb{R}$, we write

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots \frac{\partial u}{\partial x_n} \right) \quad \text{and} \quad |\nabla u| := \left[\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right]^{1/2}$$

We shall use the standard notation $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ for any $x, y \in \mathbb{R}^n$. We shall use the symbol $\Subset$ to denote *compactly contained.*

We will write $B_r(x)$ to denote the Euclidean ball in $\mathbb{R}^n$ centered at x with radius $r > 0$, and $\overline{B_r(x)}$ to be its topological closure under the Euclidean norm.

Definition 2.1. Given any bounded domain Ω, we define the following compact sequence of domains:

$$\Omega_l := \left\{ x \in \Omega,\, d(x, \partial\Omega) > \frac{1}{l} \right\} \qquad \text{for any } l > 0$$

where $d(x, \partial\Omega) := \inf_{y \in \partial\Omega} d(x, y)$ for any $x \in \Omega$.

Notation. *Let $E \subset \mathbb{R}^n$ be any set and given any $\tau > 0$, we denote*

$$E + \tau := \{ y \in \mathbb{R}^n : \, y = x + z \; with \, x \in E \; and \, z \in B_\tau(0) \} .$$

2.1 Assumptions

In this section, we shall collect some definitions and assumptions which will be used in later Chapters.

2.1.1 Assumptions on the operator

The operator that will be considered is denoted by $\mathcal{A}(x, \zeta)$ where $x \in \Omega$ and $\zeta \in \mathbb{R}^n$. This is modelled after the familiar p-Laplace equation (1.1) and satisfies the following properties.

The non-linearity $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathédory vector valued function, i.e., $\mathcal{A}(x, \xi)$ is measurable in x for every ξ and continuous in ξ for a.e. x. We always assume that $\mathcal{A}(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$. We also require that $\mathcal{A}$ satisfy the following monotonicity and Hölder type conditions: for some $1 < p < \infty$ and $\gamma \in (0, 1]$ there holds

$$\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \zeta), \xi - \zeta \rangle \geq \Lambda_0 (|\xi|^2 + |\zeta|^2)^{\frac{p-2}{2}} |\xi - \zeta|^2 \tag{2.1}$$

$$|\mathcal{A}(x, \xi) - \mathcal{A}(x, \zeta)| \leq \Lambda_1 |\xi - \zeta|^\gamma (|\xi|^2 + |\zeta|^2)^{\frac{p-1-\gamma}{2}} \tag{2.2}$$

for every $(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$ and a.e. $x \in \mathbb{R}^n$. Here Λ_0 and Λ_1 are positive constants. Note that (2.2) and the assumption $\mathcal{A}(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$ implies the following bound

$$|\mathcal{A}(x, \xi)| \leq \Lambda_1 |\xi|^{p-1}. \tag{2.3}$$

We shall introduce the following object which will be used to measure the oscillations of the operator: for any ball $B \Subset \mathbb{R}^n$, we denote

$$\overline{\mathcal{A}}_B(\xi) := \fint_B \mathcal{A}(x, \xi)\, dx = \frac{1}{|B|} \int_B \mathcal{A}(x, \xi)\, dx,$$

and define the following function that measures the oscillation of $\mathcal{A}(\cdot, \xi)$ over B by

$$\Upsilon(\mathcal{A}, B)(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathcal{A}(x, \xi) - \overline{\mathcal{A}}_B(\xi)|}{|\xi|^{p-1}}.$$

To prove the results in Chapter 4 and 5, we need to make further assumptions on the operator as given below:

Definition 2.2. Given two positive numbers γ_0 and R_0, we say that $\mathcal{A}(x, \xi)$ satisfies a $(\gamma_0,\, R_0)$-BMO condition with exponent $\tau > 0$ if

$$[\mathcal{A}]_\tau^{R_0} := \sup_{y \in \mathbb{R}^n,\, 0 < r \leq R_0} \left(\fint_{B_r(y)} \Upsilon(\mathcal{A},\, B_r(y)))(x)^\tau\, dx \right)^{\frac{1}{\tau}} \leq \gamma_0.$$

In the linear case (i.e $p = 2$), where $\mathcal{A}(x, \xi) = A(x)\xi$ for an elliptic matrix A, it is easy to see that

$$\Upsilon(\mathcal{A}, B)(x) \leq |A(x) - \overline{A}_B|$$

holds for a.e. $x \in \mathbb{R}^n$, and thus Definition 2.2 can be viewed as a natural extension of the standard small BMO condition to the nonlinear setting. For general nonlinearities $\mathcal{A}(x,\xi)$ of utmost linear growth, i.e., $p = 2$, the above (γ_0, R_0)-BMO condition was introduced in [8], whereas such a condition for general $p > 1$ appears first in [49]. We remark that the (γ_0, R_0)-BMO condition allows the nonlinearity $\mathcal{A}(x,\xi)$ to have certain discontinuity in x, and it can be used as an appropriate substitute for the Sarason VMO condition (vanishing mean oscillation [53, 7, 8, 20, 27, 47, 54, 59]).

2.1.2 Assumptions on the Domain

We state several assumptions on the Domains that will be needed in later chapters.

Definition 2.3. Given $\gamma_2 \in (0,1)$ and $R_0 > 0$, we say that Ω is a (γ_2, R_0)-Reifenberg flat domain if for every $x_0 \in \partial\Omega$ and every $r \in (0, R_0]$, there exists a system of coordinates $\{y_1, y_2, \ldots, y_n\}$, which may depend on r and x_0, so that in this coordinate system $x_0 = 0$ and

$$B_r(0) \cap \{y_n > \gamma_2 r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\gamma_2 r\}.$$

For more on Reifenberg flat domains and their many applications, we refer to the papers [22, 30, 31, 32, 52, 58]. We mention here that Reifenberg flat domains can be very rough. They include Lipschitz domains with sufficiently small Lipschitz constants (see [58]) and even some domains with fractal boundaries. In particular, all C^1 domains are included in this condition.

Remark 2.4. If Ω is a (γ_2, R_0)-Reifenberg flat domain with $\gamma_2 < 1$, then for any point $x \in \partial\Omega$ and $0 < \rho < R_0(1-\gamma_2)$, there exists a coordinate system $\{z_1, z_2, \cdots, z_n\}$ with the origin 0 at some point in the interior of Ω such that in this coordinate system $x = (0, \ldots, 0, -\gamma_2'\rho)$ and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{(z_1, z_2, \ldots, z_n) : z_n > -2\rho\gamma_2'\},$$

where $\gamma_2' = \gamma_2/(1-\gamma_2)$ and $B_\rho^+(0) := B_\rho(0) \cap \{(z_1, \ldots, z_n) : z_n > 0\}$. Thus, if $\gamma_2 < 1/2$ then

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{(z_1, z_2, \ldots, z_n) : z_n > -4\rho\gamma_2\}.$$

Another type of domain that we need in Chapter 3 are bounded domains Ω whose complement $\Omega^c := \mathbb{R}^n \setminus \Omega$ is uniformly thick with respect to the p-capacity. For details about capacity, see Section 2.4.

Definition 2.5 (Uniform p-thickness)**.** Let $\Omega \Subset \mathbb{R}^n$ be a bounded domain. We say that the complement $\Omega^c := \mathbb{R}^n \setminus \Omega$ is uniformly p-thick for some $1 < p \le n$ with constants $r_0, b > 0$, if the inequality

$$\operatorname{cap}_p(\overline{B_r(x)} \cap \Omega^c, B_{2r}(x)) \ge b \operatorname{cap}_p(\overline{B_r(x)}, B_{2r}(x))$$

holds for any $x \in \partial\Omega$ and $r \in (0, r_0]$.

It is well-known that the class of domains with uniform p-thick complements is very large. They include all domains with Lipschitz boundaries and even those that satisfy Definition 2.3.

2.2 Function Spaces

We shall collect several function spaces that will be used in this thesis.

Definition 2.6. For any domain Ω, by $C_c^\infty(\Omega)$, we mean all infinitely smooth functions $\phi : \Omega \to \mathbb{R}$ such that the set $\overline{\{x \in \Omega : \phi(x) \neq 0\}}$ is compactly contained inside Ω. The overline denotes the topological closure of the set.

Definition 2.7 (Lebesgue Space)**.** For any domain Ω and any $1 \le q < \infty$, we denote $L^q(\Omega)$ to be the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{q,\Omega} := \left(\int_\Omega |u(x)|^q \, dx \right)^{1/q} < \infty.$$

By $L^q_{loc}(\Omega)$, we mean $\|u\|_{q,\Omega'} < \infty$ for all $\Omega' \Subset \Omega$ compactly contained.

In the case $q = \infty$, we have

$$L^\infty(\Omega) := \left\{ u : \Omega \to \mathbb{R} : \|u\|_{\infty,\Omega} := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| < \infty \right\}.$$

Henceforth, for $f \in L^1(B)$ we write

$$\overline{f}_B = \fint_B f(x)dx = \frac{1}{|B|}\int_B f(x)dx.$$

Definition 2.8. We say $u \in C^{0,\alpha}(\Omega)$ for any real $0 < \alpha \leq 1$ if

$$\sup_{x,y\in\Omega,\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

We denote $C^{k.\alpha}(\Omega)$ for any integer $k \geq 0$ to be the function space where all the derivatives upto order k exist and are continuous and $D^k(u) \in C^{0,\alpha}(\Omega)$. Here $D^k(u)$ denotes derivatives of order k.

Even though we never explicitly make use of $C^{0,\alpha}(\Omega)$ spaces in this thesis, several important estimates implicitly make use of $C^{0,\alpha}(\Omega)$ regularity (see [5, 38]).

We now recall the definition of Lorentz space which is an interpolation space that lies inbetween the Lebesgue spaces.

Definition 2.9. The Lorentz space $L(s,t)(\Omega)$, with $0 < s < \infty$ and $0 < t < \infty$ is the set of measurable functions g on Ω such that

$$\|g\|_{L(s,t)(\Omega)} := \left[s\int_0^\infty \alpha^t |\{x\in\Omega : |g(x)| > \alpha\}|^{\frac{t}{s}} \frac{d\alpha}{\alpha}\right]^{\frac{1}{t}} < +\infty$$

For $t = \infty$, the space $L(s,\infty)(\Omega)$ is set to be the Marcinkiewicz space with quasinorm

$$\|g\|_{L(s,\infty)(\Omega)} := \sup_{\alpha>0} \alpha|\{x\in\Omega : |g(x)| > \alpha\}|^{\frac{1}{s}}.$$

It is easy to see that when $t = s$ the Lorentz space $L(s,s)(\Omega)$ is nothing but the Lebesgue space $L^s(\Omega)$. See [19] for more about Lorentz Spaces.

Let us now define the Lorentz-Morrey spaces which are not interpolation spaces.

Definition 2.10. A function $g \in L(s,t)(\Omega)$, $0 < s < \infty$, $0 < t \leq \infty$ is said to belong to the Lorentz-Morrey function space $\mathcal{L}^\theta(s,t)(\Omega)$ for some $0 < \theta \leq n$, if

$$\|g\|_{\mathcal{L}^\theta(s,t)(\Omega)} := \sup_{\substack{0<r\leq \mathrm{diam}(\Omega),\\ z\in\Omega}} r^{\frac{\theta-n}{s}} \|g\|_{L(s,t)(B_r(z)\cap\Omega)} < +\infty.$$

When $\theta = n$, we have $\mathcal{L}^\theta(s,t)(\Omega) = L(s,t)(\Omega)$. Moreover, when $s = t$ the space $\mathcal{L}^\theta(s,t)(\Omega)$ becomes the usual Morrey space based on L^s space.

Remark 2.11. *A basic use of Lorentz spaces is to improve the classical Sobolev Embedding Theorem. For example, if $f \in W^{1,q}$ for some $q \in (1,n)$ then*

$$f \in L(nq/(n-q), q)$$

(see, e.g., [62]), which is better than the classical result

$$f \in L^{nq/(n-q)} = L(nq/(n-q), nq/(n-q))$$

since $L(s,t_1) \subset L(s,t_2)$ whenever $t_1 \leq t_2$. Another use of Lorentz spaces is to capture logarithmic singularities. For example, for any $\beta > 0$ we have

$$\frac{1}{|x|^{n/s}(-\log|x|)^\beta} \in L(s,t)(B_1(0)) \quad \text{if and only if } t > \frac{1}{\beta}.$$

Lorentz spaces have also been used successfully in improving regularity criteria for the full 3D Navier-Stokes system of equations (see, e.g., [55]).

On the other hand, Lorentz-Morrey spaces are neither rearrangement invariant spaces, nor interpolation spaces.

We shall now collect some basic properties of Sobolev spaces.

Definition 2.12 (Weak Derivative)**.** Let α be a multi-index, i.e $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ and suppose that $u, v \in L^1_{loc}(\Omega)$ satisfying

$$\int_\Omega u(x)\partial^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v(x)\phi(x)\, dx \qquad \forall \phi \in C_c^\infty(\Omega),$$

then we say that v is the weak (distributional) derivative of u and denote $\partial^\alpha u$ to be the function v. Here we have set $|\alpha| := \alpha_1 + \ldots + \alpha_n$.

Using this, we can now define Sobolev Spaces as follows:

Definition 2.13. A function u is said to belong to the Sobolev Space $W^{1,p}(\Omega)$ for any $1 \leq p \leq \infty$ if $u \in L^p(\Omega)$ and the weak derivative of u denoted by ∇u exists and also belongs to $L^p(\Omega)$. On this space, we put the norm:

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}.$$

With this norm, the space $W^{1,p}(\Omega)$ becomes a Banach Space.

Remark 2.14. *We denote $W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ under the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. If the boundary $\partial\Omega$ of the domain satisfies some very mild regularity condition (see [13]), then one can actually show that there is a bounded linear operator $T : W^{1,p}(\Omega, H^n) \to L^p(\partial\Omega, H^{n-1})$ and $W_0^{1,p}(\Omega)$ can be identified as the set given by $T^{-1}(0)$. Here H^n is the n-dimensional measure and H^{n-1} is the $n-1$ dimensional surface measure and 0 denotes the zero function in $L^p(\partial\Omega, H^{n-1})$.*

Definition 2.15. Let $\mathbf{f} \in (L^s(\Omega))^n$ be a given vector field for some $1 < s < \infty$. We say $\mu = \operatorname{div}(\mathbf{f})$ is a Radon measure if it satisfies

$$\int_\Omega \varphi(x)\, d\mu(x) = -\int_\Omega \langle \mathbf{f}, \nabla\varphi\rangle\, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

It follows from standard measure theory that for any open set $O \subseteq \Omega$, we have

$$|\mu|(O) = \sup_{\varphi} \left\{ \int_\Omega \langle \mathbf{f}, \nabla\varphi\rangle\, dx : \ \varphi \in C_c^1(O),\ |\varphi| \leq 1 \right\},$$

and for any compact set $K \Subset \Omega$, we have

$$|\mu|(K) = \inf_O \left\{ |\mu|(O) : O \,\text{open and}\, O \Supset K \right\}.$$

Let us define a new function space as follows: (See Section 2.4 for more about Capacities)

Definition 2.16. Define the following seminorm

$$\|\mu\|_{M^{1,p}}^p := \sup_{K \Subset \mathbb{R}^n} \frac{|\mu|(K \cap \Omega)}{\operatorname{cap}_{1,p}(K \cap \overline{\Omega})}$$

and now we define the function space

$$M^{1,p}(\Omega) = \{\mu \,:\, |\mu|(\Omega) < +\infty \quad \text{and} \quad \|\mu\|_{M^{1,p}} < +\infty\}$$

where

$$\mathrm{cap}_{1,p}(K) = \inf\{\|\nabla\phi\|_p \,:\, \phi \geq 1 \text{ on } K, \quad, 0 \leq \phi \leq 1, \quad \phi \in C_c^\infty(\mathbb{R}^n)\}.$$

Remark 2.17. *We have the following characterization of the space $M^{1,p}(\Omega)$. The function space $M^{1,p}(\Omega)$ is the set of all $\phi \in W_0^{1,p}(\Omega)$ for which there exists a constant $C > 0$ (independent of ϕ) such that the following holds:*

$$\int_{K\cap\Omega} |\nabla\phi|^p \, dx \leq C\, \mathrm{cap}_{1,p}(K \cap \overline{\Omega}).$$

We now recall an elementary characterization for functions in Lorentz spaces, which can easily be proved using methods in standard measure theory.

Lemma 2.18. *Assume that $g \geq 0$ is a measurable function in a bounded subset $\Omega \subset \mathbb{R}^n$. Let $\theta > 0$, $\Lambda > 1$ be constants. Then for $0 < s, t < \infty$, we have*

$$g \in L(s,t)(\Omega) \Longleftrightarrow S := \sum_{k\geq 1} \Lambda^{tk} |\{x \in \Omega : g(x) > \theta\Lambda^k\}|^{\frac{t}{s}} < +\infty$$

and moreover the estimate

$$C^{-1}\, S \leq \|g\|^t_{L(s,t)(\Omega)} \leq C\,(|\Omega|^{\frac{t}{s}} + S),$$

holds where $C > 0$ is a constant depending only on θ, Λ, and t. Analogously, for $0 < s < \infty$ and $t = \infty$ we have

$$C^{-1} T \leq \|g\|_{L(s,\infty)(\Omega)} \leq C\,(|\Omega|^{\frac{1}{s}} + T),$$

where T is the quantity

$$T := \sup_{k\geq 1} \Lambda^k |\{x \in \Omega : |g(x)| > \theta\Lambda^k\}|^{\frac{1}{s}}.$$

Analogous to the unweighted case, we have the following more general weighted analogue of Lemma 2.18 whose proof follows in essentially the same way.

Lemma 2.19. *Assume that $g \geq 0$ is a measurable function in a bounded subset $\Omega \subset \mathbb{R}^n$. Let $\theta > 0$, $\Lambda > 1$ be constants, and let w be a weight in $\mathbb{R}^n$, i.e. $w \geq 0$ and $w \in L^1_{loc}(\mathbb{R}^n)$. Then for $0 < q, t < \infty$, we have*

$$g \in L_w(q,\, t)(\Omega) \Longleftrightarrow S := \sum_{k \geq 1} \Lambda^{tk} w(\{x \in \Omega : g(x) > \theta \Lambda^k\})^{\frac{t}{q}} < +\infty.$$

Moreover, there exists a positive constant $C = C_{(\theta, \Lambda, t)} > 0$ such that

$$C^{-1}\, S \leq \|g\|^t_{L_w(q,\, t)(\Omega)} \leq C\,(w(\Omega)^{\frac{t}{q}} + S).$$

Analogously, for $0 < q < \infty$ and $t = \infty$ we have

$$C^{-1} T \leq \|g\|_{L_w(q,\, \infty)(\Omega)} \leq C\,(w(\Omega)^{\frac{1}{q}} + T),$$

where T is the quantity

$$T := \sup_{k \geq 1} \Lambda^k w(\{x \in \Omega : |g(x)| > \theta \Lambda^k\})^{\frac{1}{q}}.$$

2.3 Muckenhoupt Weights

Now we shall collect some properties of weights. In this thesis, we shall only be concerned with Muckenhoupt weights.

Definition 2.20 (Muckenhoupt Weight)**.** By an A_s weight with $1 < s < \infty$, we mean a nonnegative function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that the quantity

$$[w]_s := \sup_B \left(\fint_B w(x)\, dx \right) \left(\fint_B w(x)^{\frac{-1}{s-1}}\, dx \right)^{s-1} < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

For $s = 1$, we say that w is an A_1 weight if

$$[w]_1 := \sup_B \left(\fint_B w(x)\, dx \right) \left\| w^{-1} \right\|_{L^\infty(B)} < +\infty.$$

The quantity $[w]_s$ for $1 \leq s < \infty$, will be referred to as the A_s constant of w. The A_s classes are increasing, i.e., $A_{s_1} \subset A_{s_2}$ whenever $1 \leq s_1 < s_2 < \infty$. A broader class of weights is the A_∞ weights which, by definition, is given by

$$A_\infty = \bigcup_{1\le s<\infty} A_s.$$

The following well known characterization of A_∞ weights will be needed later (see for example [19, Theorem 9.3.3]).

Lemma 2.21. *A weight $w \in A_\infty$ if and only if there are constants $\Xi_0, \Xi_1 > 0$ such that for every ball $B \subset \mathbb{R}^n$ and every measurable subsets E of B, there holds*

$$w(E) \le \Xi_0 \left(\frac{|E|}{|B|}\right)^{\Xi_1} w(B). \tag{2.4}$$

Moreover, if w is an A_s weight with $[w]_s \le \overline{\omega}$ then the constants Ξ_0 and Ξ_1 above can be chosen so that $\max\{\Xi_0, 1/\Xi_1\} \le c(\overline{\omega}, n)$.

In (2.4), the notation $w(E)$ stands for the integral $\int_E w(x)\, dx$, and likewise for $w(B)$. Henceforth, we will refer to (Ξ_0, Ξ_1) as a pair of A_∞ constants of w provided they satisfy (2.4).

Definition 2.22. Let f be locally measurable function and we define Maximal function as:

$$\mathcal{M}(f)(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f| \ \ dx.$$

An important characterization of Muckenhoupt weights is given by the following theorem:

Theorem 2.23. *Let $w \in A_s(\mathbb{R}^n)$ for some $1 < s < \infty$. Then there is a constant $C_{(n,s)}$ such that*

$$\|\mathcal{M}\|_{L^s(w)\to L^s(w)} \le C_{(n,s)}[w]_{A_s}^{\frac{1}{s-1}}.$$

The Muckenhoupt weights also satisfy a very important Reverse Hölder type inequality given in the following Lemma:

Lemma 2.24. *Let $w \in A_s$ for some $1 \le s < \infty$. Then there exists constants C and $\tilde{\gamma} > 0$ that depend only on $n, s, [w]_{A_s}$ such that for every cube Q, we have*

$$\left(\frac{1}{|Q|}\int_Q w(t)^{1+\tilde{\gamma}} \ dt\right)^{\frac{1}{1+\tilde{\gamma}}} \le \frac{C}{|Q|}\int_Q w(t) \ dt$$

An important consequence of the Reverse Hölder property is the following Corollary:

Corollary 2.25. *For any $1 < s < \infty$ and every $w \in A_s$, there is an $\epsilon > 0$ depending only on $[w]_{A_s}, s, n$ such that $w \in A_{\frac{s}{1+\epsilon}}$.*

This corollary will be important for us in Chapter 4.

2.4 Capacity

We shall use the following definition for Capacity:

Definition 2.26. Let $E \Subset \mathbb{R}^n$ be compactly contained set, then the p-capacity of E relative to an open set $\mathcal{O}$ is given by:

$$\operatorname{cap}_p(E, \mathcal{O}) := \inf \left\{ \int_{\mathcal{O}} |\nabla \phi|^p \, dx : \ \phi \in C_c^\infty(\mathcal{O}), \ \phi \geq \chi_E, \ 0 \leq \phi \leq 1 \right\}.$$

The set $\mathcal{O}$ is called the *Reference Domain.*

The following are some properties of the p-capacity:

Theorem 2.27 (see [1] Theorem 2.3)**.** *Let $\mathcal{O}$ be a reference domain, $1 < p \leq n$. Then the following holds:*

- *If $E \subset \mathcal{O}$, then $\operatorname{cap}_p(E) = \inf\{\operatorname{cap}_p(U) : U \subset \mathcal{O}$ open and $E \subset U\}$.*

- *If $E_1 \supset E_2 \supset \cdots$ are compact subsets on $\mathcal{O}$, then*

$$\operatorname{cap}_p\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \operatorname{cap}_p(E_j).$$

- *If $E \Subset \mathcal{O}$ compact, then*

$$\operatorname{cap}_p(E) = \inf \left\{ \int_{\mathcal{O}} |\nabla \phi|^p \, dx : \ \phi \in C_c^\infty(\mathcal{O}), \ u \geq \chi_E \right\}$$

- *If $E_1 \subset E_2 \subset \cdots \subset \mathcal{O}$ are arbitrary sets, then*

$$\operatorname{cap}_p\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \operatorname{cap}_p(E_j).$$

- *We have subadditive property* $\operatorname{cap}_p\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \operatorname{cap}_p(E_j)$ *whenever* $E_1, E_2, \cdots$ *are arbitrary subsets of* $\mathcal{O}$.

We have the following comparison result relating p-capacity and the Lebesgue measure:

Remark 2.28 ([1]). *The following properties of the p-capacity holds:*

- *If* $1 < p < n$, *then* $\operatorname{cap}_p(B_r(x)) = n\omega(n)\left(\frac{n-p}{p-1}\right)^{p-1} r^{n-p}$ *where* $\omega(n)$ *is the Lebesgue measure of the unit ball in* $\mathbb{R}^n$.

- $\operatorname{cap}_n(B_r(x), B_1(x)) = n\omega(n)\log^{1-n}(1/r)$ *if* $r < 1$.

The role played by quasicontinuity in the theory of Sobolev spaces is analogous to that played by Lusin's theorem in real analysis. We shall now state the definition of quasicontinuity:

Definition 2.29. A function u is called *p-quasicontinuous*on E if for each $\epsilon > 0$, there exists an open set V with $\operatorname{cap}_p(V) < \epsilon$ such that u restricted to $E \backslash V$ is finite and continuous.

We say that a property holds *p-quasi everywhere* if it holds on all sets having nonzero p-capacity.

We have the following important theorem due to Mazya-Verbitsky:

Theorem 2.30 ([41]). *If* $\gamma \in M_+(\mathbb{R}^n)$ *(positive measure) with* $1 < p < n/p$, *then the following are equivalent*

- *for all compact sets* E, *we have*

$$\gamma(E) \leq Q \operatorname{cap}_{1,p}(E, \mathbb{R}^n).$$

- $\int_E (I_1\gamma)^{p'}(x)\, dx \leq R \operatorname{cap}_{1,p}(E, \mathbb{R}^n)$.

Here $I_1(\gamma) := \int_{\mathbb{R}^n} \frac{d\gamma(y)}{|x-y|^{n-1}}$ for any $x \in \mathbb{R}^n$. An important fact that we will also use is the following proposition which states:

Proposition 2.31 (Proposition 11.3.1 [40]). *For $1 < p < n$, we have that $Q \approx R$ where Q and R are as in Theorem 2.30. The proportionality constant is independent of $\gamma \in M_+(\mathbb{R}^n)$.*

We also have the following important theorem:

Theorem 2.32 (Trace Inequality). *Any measure $\mu \in M^{1,p}(\Omega)$ satisfies the following trace inequality:*

$$\int_\Omega |\phi|^p \, d\mu \leq C_{tr(\|\mu\|_{M^{1,p}})} \int_\Omega |\nabla \phi|^p \, dx \quad \forall \ \phi \in C_c^\infty(\Omega).$$

The converse implication is true if we either consider the whole space $\mathbb{R}^n$ or if we assume μ is compactly supported inside Ω.

Combining Proposition 2.31 and Theorem 2.32, we get the following important remark:

Remark 2.33. *The constant in Theorem 2.32, $C_{tr(\|\mu\|_{M^{1,p}})}$ satisfies the property that given any $\epsilon > 0$, we can find a corresponding $\delta > 0$ such that if*

$$\|\mu\|_{M^{1,p}} < \delta, \quad \textit{then} \ \ C_{tr(\|\mu\|_{M^{1,p}})} < \epsilon.$$

The following very important Lemma gives the connection between weighted estimates and spaces defined in Definition 2.16 (i.e satisfying Theorem 2.32).

Lemma 2.34 ([41]). *Let $1 < p < \infty$ and suppose that a function $\tilde{f} \in L^{p'}_{loc}(\Omega)$ satisfies*

$$\int_K |\tilde{f}|^{p'} \, dx \leq S_0 \, \mathrm{cap}_{1,p}(K) \tag{2.5}$$

and for all weights $w \in A_1$, also satisfies

$$\int_{\mathbb{R}^n} |g|^p w \, dx \leq \tilde{K}(n, p, [w]_1) \int_{\mathbb{R}^n} |\tilde{f}|^{p'} w \, dx, \tag{2.6}$$

then we must have that

$$\int_K |g|^p \, dx \leq C(p, n, \tilde{K}) S_0 \, \mathrm{cap}_{1,p}(K)$$

for all compact sets $K \Subset \Omega$. Note here that $S_0 = \||f|^q\|_{M^{1,p}}$.

2.5 Krylov Sofanov Decomposition

The following technical lemma is a version of the Calderón- Zygmund-Krylov-Safonov decomposition that has been used in [10, 48]. It allows one to work with balls instead of cubes. A proof of this lemma, which uses Lebesgue Differentiation Theorem and the standard Vitali covering lemma, can be found in [7] with obvious modifications to fit the setting here.

The following Calderón-Zygmund decomposition type lemma will be very useful in the later chapters. In the unweighted case various versions of this lemma have been obtained (see, e.g., [10, 60, 7]).

Lemma 2.35. *Assume that $E \subset \mathbb{R}^n$ is a measurable set for which there exist $c_1, r_1 > 0$ such that*

$$|B_t(x) \cap E| \geq c_1 \, |B_t(x)|$$

holds for all $x \in E$ and $0 < t \leq r_1$. Fix $0 < r \leq r_1$ and let $C \subset D \subset E$ be measurable sets for which there exists $0 < \epsilon < 1$ such that

- $|C| < \epsilon \, r^n |B_1|$
- *for all $x \in E$ and $\rho \in (0, r]$, if $|C \cap B_\rho(x)| \geq \epsilon \, |B_\rho(x)|$, then $B_\rho(x) \cap E \subset D$.*

Then we have the estimate

$$|C| \leq (c_1)^{-1} \epsilon \, |D|.$$

Analogous weighted version is stated as follows (see [42] for the proof):

Lemma 2.36. *Let Ω be a (γ, R_0)-Reifenberg flat domain with $\gamma < 1/8$, and let w be an A_∞ weight. Suppose that the sequence of balls $\{B_r(y_i)\}_{i=1}^{L}$ with centers $y_i \in \overline{\Omega}$ and a common radius $r \leq R_0/4$ covers Ω. Let $C \subset D \subset \Omega$ be measurable sets for which there exists $0 < \epsilon < 1$ such that*

1. *$w(C) < \epsilon \, w(B_r(y_i))$ for all $i = 1, \ldots, L$, and*

2. for all $x \in \Omega$ and $\rho \in (0, 2r]$,

$$w(C \cap B_\rho(x)) \geq \epsilon\, w(B_\rho(x)) \quad \Longrightarrow \quad B_\rho(x) \cap \Omega \subset D.$$

Then we have the estimate

$$w(C) \leq B\, \epsilon\, w(D)$$

for a constant B depending only on n and the A_∞ constants of w.

2.6 Some Convergence Results

In this section, we shall collect a few convergence results which will be useful in the rest of this thesis.

Lemma 2.37 (Stability). *Let $\{f_n\} \in L^s(\Omega)$ be a sequence of functions for some $1 < s < \infty$ such that*

$$f_n \to g \text{ a.e. in } \Omega \quad \text{and} \quad f_n \rightharpoonup h \text{ weakly in } L^s(\Omega)$$

then we have $g = h$ a.e in Ω.

Theorem 2.38 (Vitali). *Assume $\mu(X) < \infty$ and $|h_n|^s$ is uniformly integrable, i.e for any $\epsilon > 0$, there is a corresponding $\delta > 0$ such that*

$$\sup_n \int_E |h_n|^s \, d\mu < \epsilon, \qquad \text{whenever} \quad \mu(E) < \delta;$$

and $h_n \to h$ μ-a.e in X and $\int_X |h_n|^s < +\infty$ uniformly, then we must necessarily have that

$$h \in L^s(X) \qquad \text{and} \qquad h_n \to h \quad \text{strongly in} \quad L^s(X).$$

The following elementary result will be repeatedly used:

Proposition 2.39. *Suppose that $h_n \rightharpoonup h$ weakly in L^p and also suppose that $\int_\Omega |h_n|^p \, dx < +\infty$ uniformly independent of n. Given any sequence $g_n \to g$ strongly in $L^{p'}$, then we have that*

$$\lim_{n \to \infty} \int_\Omega h_n g_n \, dx = \int_\Omega hg \, dx.$$

The following theorem was proved in [6] and this plays a very important role in Chapter 5 to prove existence of solutions.

Theorem 2.40 ([6]). *Let $\mathcal{A}(x, \nabla u)$ be a nonlinearity which satisfies (2.1) and (2.2) with $\gamma = 1$ and consider the equation*

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla w_k) = h_k + m_k & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega \end{cases}$$

in $\mathcal{D}'(\Omega)$ and suppose

- $w_k \in W_0^{1,p}(\Omega)$ *and* $w_k \rightharpoonup u$ *weakly in* $W_0^{1,p}(\Omega)$,
- $h_k \in W^{-1,p'}(\Omega)$ *and* $h_k \to h$ *in* $W^{-1,p'}(\Omega)$,
- $m_k \in W^{-1,p'}(\Omega)$ *and* $\|m_k\|_{W^{-1,p'}(\Omega)} \leq C$ *uniformly bounded independent of* k,
- $|\int_K m_k \phi \, dx| \leq C_K \|\phi\|_{L^\infty(\Omega)}$ *for all* $\phi \in \mathcal{D}(\Omega)$ *with* $\operatorname{spt}(\phi) \subset K$.
- $m_k \rightharpoonup m$ *weakly in* $L^1(\Omega)$.

Then the following conclusions hold:

- $\nabla w_k \to \nabla w$ *in* $L^q(\Omega)$ *for any* $q < p$,
- $\nabla w_{k'} \to \nabla w$ *a.e upto a subsequence and*
- w *solves*
$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla w) = h + m & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega \end{cases}$$
in the distributional sense.

The above theorem shows strong convergence of weak solutions for all $q < p$. If we had strong convergence of the gradients for $q = p$, the existence theory would be vastly simplified. All the hard work needed to show existence in Chapters 5 and 6 is to show strong convergence of gradients at $q = p$.

Chapter 3

Global Lorentz and Lorentz-Morrey estimates below the natural exponent

In this Chapter, we are mainly interested in studying the following equation:

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) &= \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.1}$$

For a fixed $\max\{1, p-1\} < s < \infty$, we say $u \in W_0^{1,s}(\Omega)$ is a solution of (3.1), if the following holds:

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla\phi \rangle \, dx = \int_\Omega \langle \mathbf{f}, \nabla\phi \rangle \, dx$$

for all $\phi \in W_0^{1,\frac{s}{s-p+1}}$. When $s < p$, such solutions are called *very weak solutions.*

We shall now state all the assumptions that we need for this chapter:

Hypothesis 3.1 (Assumption on $\mathcal{A}(x,\zeta)$)**.** *We will assume the nonlinearity $\mathcal{A}(x,\zeta)$ satisfies* (2.1) *and* (2.2) *for some $\gamma \in (0,1]$.*

Hypothesis 3.2 (Assumption on Ω)**.** *We will assume that Ω is a bounded domain such that its complement denoted by Ω^c is uniformly p-thick in the sense of Definition 2.5 with constants $r_0, b > 0$..*

We are now ready to state the main results that will proved in this chapter.

3.1 Main Results

Theorem 3.3. *Let $\mathcal{A}$ satisfy all the conditions in Hypothesis 3.1 , and let Ω satisfy all the conditions in Hypothesis 3.2. Then there exists a small $\delta = \delta_{(n,p,\Lambda_0,\Lambda_1,\gamma,b)} > 0$ such that for any very weak solution $u \in W_0^{1,p-2\delta}(\Omega)$ to the boundary value problem* (3.1)*, the following*

This chapter previously appeared in [2]. It is reprinted by permission of Springer (see Page 108) *estimate holds:*

$$\|\nabla u\|_{\mathcal{L}^{\theta}(q,\,t)(\Omega)} \leq C_{(n,p,t,\Lambda_0,\Lambda_1,\gamma,b,\mathrm{diam}(\Omega)/r_0)} \|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,\,t)(\Omega)} \tag{3.2}$$

for all $q \in [p-\delta, p+\delta]$*,* $0 < t \leq \infty$*, and* $\theta \in [p-2\delta, n]$*.*

In the simplest case where $\theta = n$ and $t = q$, Theorem 3.3 yields the following basic Calderón-Zygmund type estimate for solutions of (3.1):

$$\|\nabla u\|_{L^q(\Omega)} \leq C \|\mathbf{f}\|_{L^q(\Omega)} \tag{3.3}$$

for all $q \in [p-\delta, p+\delta]$, provided Ω satisfies Hypothesis 3.2. We observe that inequality (3.3) has been obtained in [28] under stronger conditions on $\mathcal{A}$ and Ω. Namely, on the one hand, a Lipschitz type condition, i.e., $\gamma = 1$ in (2.2), was assumed in [28]. On the other hand, the domain Ω considered [28] was assumed to be regular in the sense that the Calderón-Zygmund type bound

$$\|\nabla v\|_{L^r(\Omega)} \leq C \|\mathbf{f}\|_{L^r(\Omega)} \tag{3.4}$$

holds for *all* $r \in (1, \infty)$ and all solutions to the linear equation

$$\begin{cases} -\Delta v &= -\mathrm{div}\,\mathbf{f} \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

As demonstrated by a counterexample in Subsection 1.1.2 (see also [42]), estimate (3.4), say for large r, generally fails for solutions of (3.5) even for (non-convex) Lipschitz domains. Thus the result of [28] concerning the bound (3.3) does not cover all Lipschitz domains. In this respect, the bound (3.3) is new even for linear equations, where the principal operator is replaced by just the standard Laplacian .

Another new aspect is the following *boundary* higher integrability result for very weak solutions to the associated homogeneous equations.

Theorem 3.4. *Let* $\mathcal{A}$ *satisfy Hypothesis 3.1, and let* Ω *satisfy all the conditions in Hypothesis 3.2. Then there exists a positive number* $\overline{\delta} = \overline{\delta}_{(n,p,\Lambda_0,\Lambda_1,b)}$ *such that the following holds: for* *any* $x_0 \in \partial\Omega$ *and* $R \in (0, r_0/2)$*, if* $w \in W^{1,p-\bar{\delta}}(\Omega \cap B_{2R}(x_0))$ *is a very weak solution to the Dirichlet problem*

$$\begin{cases} \operatorname{div}\mathcal{A}(x, \nabla w) &= 0 \quad \text{in } \Omega \cap B_{2R}(x_0), \\ w &= 0 \quad \text{on } \partial\Omega \cap B_{2R}(x_0), \end{cases} \tag{3.6}$$

then we have $w \in W^{1,p+\bar{\delta}}(\Omega \cap B_R(x_0))$.

A quantitative statement of Theorem 3.4 can be found in Theorem 3.21 below. We notice that, whereas *interior* higher integrability of very weak solutions to the equation $\operatorname{div}\mathcal{A}(x, \nabla w) = 0$ is well-known (see [28, 37]), the boundary higher integrability result has been obtained only for *finite energy solutions*, i.e. solutions which are a priori assumed to be in $w \in W^{1,p}(\Omega \cap B_{2R}(x_0))$ in the paper [33, 46]. The fact that $|\nabla w|$ is allowed to be in $L^{p-\bar{\delta}}$ to begin with plays a crucial role in our proof of Theorem 3.3.

Remark 3.5. *The Hölder type condition* (2.2) *with* $\gamma \in (0, 1]$ *is not needed in Theorem 3.4, while this condition is assumed in Theorem 3.3. As a matter of fact, the proof of Theorem 3.3 requires* (2.2) *only through the use of Corollaries 3.10 and 3.18. Thus by Remark 3.19 below, making use of only* (2.1)*,* (2.3) *and the p-thickness condition as in Theorem 3.3, we still obtain inequality* (3.2) *with a constant* $C = C_{(n,p,q,t,\Lambda_0,\Lambda_1,b,\operatorname{diam}(\Omega)/r_0)}$ *for any finite energy solution* $u \in W_0^{1,p}(\Omega)$ *provided* $q \in (p, p + \delta]$.

3.2 Local interior estimates

In this section, we obtain certain local interior estimates for very weak solutions of (3.1). These include the important comparison estimates below the natural exponent p. We shall make use of the nonlinear Hodge decomposition of [28].

Theorem 3.6 (Nonlinear Hodge Decomposition [28])**.** *Let* $s > 1$*,* $\epsilon \in (-1, s-1)$*, and* $w \in W_0^{1,s}(B)$ *where* $B \subset \mathbb{R}^n$ *is a ball. Then there exist* $\phi \in W_0^{1,\frac{s}{1+\epsilon}}(B)$ *and a divergence free vector field* $\mathcal{H} \in L^{\frac{s}{1+\epsilon}}(B, \mathbb{R}^n)$ *such that*

$$|\nabla w|^{\epsilon}\, \nabla w = \nabla\phi + \mathcal{H}.$$

Moreover, the following estimate holds:

$$\|\mathcal{H}\|_{L^{\frac{s}{1+\epsilon}}(B)} \leq C_{(s,n)}\,|\epsilon|\,\|\nabla w\|^{1+\epsilon}_{L^{s}(B)}.$$

Using the above Hodge decomposition, the authors of [28] obtained gradient L^q regularity below the natural exponent for very weak solutions to certain quasilinear elliptic equations.

Theorem 3.7 ([28])**.** *Suppose that $\mathcal{A}$ satisfies Hypothesis 3.1. Then there exists a constant $\tilde{\delta}_1 = \tilde{\delta}_{1(n,p,\Lambda_0,\Lambda_1,\gamma)}$ with $0 < \tilde{\delta}_1 < \min\{1, p-1\}$ sufficiently small such that the following holds for any $\delta \in (0, \tilde{\delta}_1)$. Let B be a ball and let the vector fields $\tilde{\mathbf{h}}$, $\tilde{\mathbf{f}} \in L^{p-\delta}(B, \mathbb{R}^n)$: Then for any very weak solution $w \in W^{1,p-\delta}_0(B)$ to the equation*

$$\operatorname{div} \mathcal{A}(x, \tilde{\mathbf{h}} + \nabla w) = \operatorname{div} |\tilde{\mathbf{f}}|^{p-2}\tilde{\mathbf{f}} \quad \text{in } B,$$

there holds

$$\int_B |\nabla w(x)|^{p-\delta}\,dx \leq C_{(n,p,\Lambda_0,\Lambda_1,\gamma)} \int_B \left(|\tilde{\mathbf{h}}(x)|^{p-\delta} + |\tilde{\mathbf{f}}(x)|^{p-\delta}\right)\,dx. \tag{3.7}$$

It is worth mentioning that inequality (3.7) was obtained in [28, Theorem 5.1] under a Lipschitz type condition on $\mathcal{A}(x,\cdot)$, i.e., (2.2) was assumed to hold with $\gamma = 1$. We observe that the proof of [28, Theorem 5.1] can easily be modified to obtain (3.7) under the weaker Hölder type condition (2.2) with any $\gamma \in (0,1)$(see also the proof of Theorem 3.17 below.)

We next state a well-known *interior* higher integrability result that was originally obtained in [28] and [37] (see also [46]).

Theorem 3.8 ([28, 37])**.** *Suppose that $\mathcal{A}$ satisfies Hypothesis 3.1, then there exists a constant $\tilde{\delta}_2 = \tilde{\delta}_{2(n,p,\Lambda_0,\Lambda_1)} \in (0, 1/2)$ such that for any very weak solution $w \in W^{1,p-\tilde{\delta}_2}_{\mathrm{loc}}(\tilde{\Omega})$ to the equation*

$$\operatorname{div} \mathcal{A}(x, \nabla w) = 0 \quad \textit{in an open set } \tilde{\Omega}$$

belongs to $W^{1,p+\tilde{\delta}_2}_{\mathrm{loc}}(\tilde{\Omega})$. Moreover, the inequality

$$\left({}_{B_{r/2}(x)} |\nabla w(x)|^{p+\tilde{\delta}_2}\,dx\right)^{\frac{1}{p+\tilde{\delta}_2}} \leq C_{(n,p,\Lambda_0,\Lambda_1)} \left({}_{B_r(x)} |\nabla w(x)|^{p-\tilde{\delta}_2}\,dx\right)^{\frac{1}{p-\tilde{\delta}_2}} \tag{3.8}$$

holds for any ball $B_r(x) \subset \tilde{\Omega}$.

Remark 3.9. *We notice that Theorem 3.8 was obtained in [28] under a homogeneity condition on $\mathcal{A}(x,\cdot)$, i.e., $\mathcal{A}(x,\lambda\xi) = |\lambda|^{p-2}\lambda\mathcal{A}(x,\xi)$ for all $x, \xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. This condition has been removed in [16]. Moreover, the proof of Theorem 3.8 in [28] uses inequality* (3.7) *and thus requires the Hölder type condition* (2.2)*. As a matter of fact, following the method of [37], one can prove interior higher integrability under only conditions* (2.1) *and* (2.3)*. For details see, e.g., [46, Theorem 9.4].*

A consequence of Theorems 3.7 and 3.8 is the following important existence result.

Corollary 3.10 ([28])**.** *Under Hypothesis 3.1, let $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are as in Theorems 3.7 and 3.8, respectively and let $B \subset \mathbb{R}^n$ be a ball. For any function $w_0 \in W^{1,p-\delta}(B)$, with $\delta \in (0, \min\{\tilde{\delta}_1, \tilde{\delta}_2\})$, there exists a very weak solution $w \in w_0 + W^{1,p-\delta}(B)$ to the equation* $\operatorname{div} \mathcal{A}(x, \nabla w) = 0$ *such that*

$$\int_B |\nabla w(x)|^{p-\delta}\, dx \leq C_{(n,p,\Lambda_0,\Lambda_1,\gamma)} \int_B |\nabla w_0(x)|^{p-\delta}\, dx.$$

We shall need to prove versions of Theorems 3.7 and Corollary 3.10 for domains satisfying Hypothesis 3.2. These new results will be obtained later in Theorem 3.17 and Corollary 3.18. A version of Theorem 3.8 upto the boundary of a domain whose complement is uniformly p-thick will also be obtained in Theorem 3.21 below.

Next, for each ball $B_{2R} = B_{2R}(x_0) \Subset \Omega$ and for any $\delta \in (0, \min\{\tilde{\delta}_1, \tilde{\delta}_2\})$ with $\tilde{\delta}_1$ and $\tilde{\delta}_2$ as in Theorems 3.7 and 3.8, respectively, we define $w \in u + W_0^{1,p-\delta}(B_{2R})$ as a very weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } B_{2R}, \\ w &= u \quad \text{on } \partial B_{2R}. \end{cases} \tag{3.9}$$

The existence of w is ensured by Corollary 3.10. *We mention that the uniqueness of w is still unknown.* Moreover, by Theorem 3.8 we have that $w \in W^{1,p}_{\mathrm{loc}}(B_{2R})$. Thus it follows from the standard interior Hölder continuity of solutions that we have the following decay estimates.

Lemma 3.11 (Theorem 7.7 in [18]). *Let w be as in* (3.9)*. Then there exists a constant $\beta_0 = \beta_{0(n,p,\Lambda_0,\Lambda_1)} \in (0, 1/2]$ such that*

$$\left(\fint_{B_\rho(z)} |w - \overline{w}_{B_\rho(z)}|^p \, dx \right)^{\frac{1}{p}} \leq C \left(\frac{\rho}{r}\right)^{\beta_0} \left(\fint_{B_r(z)} |w - \overline{w}_{B_r(z)}|^p \, dx \right)^{\frac{1}{p}}$$

for any $z \in B_{2R}(x_0)$ with $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$. Moreover, there holds

$$\left(\fint_{B_\rho(z)} |\nabla w|^p \, dx \right)^{\frac{1}{p}} \leq C \left(\frac{\rho}{r}\right)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^p \, dx \right)^{\frac{1}{p}} \tag{3.10}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$.

Using the higher integrability result of Theorem 3.8, inequality (3.10) can be further ameliorated as in the following lemma. We notice that this kind of result can be proved by means of a covering/interpolation argument as demonstrated in [18, Remark 6.12].

Lemma 3.12. *Let w be as in* (3.9)*. There exists a $\beta_0 = \beta_{0(n,p,\Lambda_0,\Lambda_1)} \in (0, 1/2]$ such that for any $t \in (0, p]$ there holds*

$$\left(\fint_{B_\rho(z)} |\nabla w|^t \, dx \right)^{\frac{1}{t}} \leq C_{(n,p,\Lambda_0,\Lambda_1,t)} \left(\frac{\rho}{r}\right)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^t \, dx \right)^{\frac{1}{t}}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$.

We shall now prove the following comparison estimate with exponents below the natural exponent.

Lemma 3.13. *Under Hypothesis 3.1, let $\delta \in (0, \min\{\tilde{\delta}_1, \tilde{\delta}_2\})$, where $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are as in Theorems 3.7 and 3.8, respectively. With $\mathbf{f} \in L^{p-\delta}(\Omega)$, for any $u \in W_0^{1,p-\delta}(\Omega)$ solving* (3.1) *and any $w \in u + W_0^{1,p-\delta}(B_{2R})$ solving* (3.9)*, we have the following:*

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx \lesssim \delta^{\frac{p-\delta}{p-1}} \fint_{B_{2R}} |\nabla u|^{p-\delta} \, dx + \fint_{B_{2R}} |\mathbf{f}|^{p-\delta} \, dx$$

if $p \geq 2$ and

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx \lesssim \delta^{p-\delta} \fint_{B_{2R}} |\nabla u|^{p-\delta} \, dx + \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta} \, dx \right)^{p-1} \left(\fint_{B_{2R}} |\nabla u|^{p-\delta} \, dx \right)^{2-p}$$

if $1 < p < 2$. Here we assume $B_{2R} \Subset \Omega$.

Proof. Let δ be as in the hypothesis. Applying Theorem 3.6 with $s = p - \delta$ and $\epsilon = -\delta$, we have

$$|\nabla u - \nabla w|^{-\delta}(\nabla w - \nabla u) = \nabla\phi + \mathcal{H}$$

in B_{2R}. Here $\phi \in W_0^{1,\frac{p-\delta}{1-\delta}}(B_{2R})$ and $\mathcal{H}$ is a divergence free vector field with

$$\|\mathcal{H}\|_{L^{\frac{p-\delta}{1-\delta}}(B_{2R})} \lesssim \delta \, \|\nabla u - \nabla w\|_{L^{p-\delta}(B_{2R})}^{1-\delta}. \tag{3.11}$$

Using ϕ as a test function in (3.1) and (3.9), we have

$$\begin{aligned} I &:= \int_{B_{2R}} \langle \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla w), \nabla w - \nabla u \rangle |\nabla w - \nabla u|^{-\delta} \, dx, \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.12}$$

where we have set

$$\begin{aligned} I_1 &:= \int_{B_{2R}} \langle \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla w), \mathcal{H} \rangle \, dx, \\ I_2 &:= \int_{B_{2R}} |\mathbf{f}|^{p-2} \langle \mathbf{f}, \nabla w - \nabla u \rangle |\nabla w - \nabla u|^{-\delta} \, dx, \\ I_3 &:= -\int_{B_{2R}} |\mathbf{f}|^{p-2} \langle \mathbf{f}, \mathcal{H} \rangle \, dx. \end{aligned}$$

Applying the monotonicity condition (2.1), we have

$$I \gtrsim \int_{B_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla u - \nabla w|^{2-\delta} \, dx.$$

Thus when $p \geq 2$ we can bound I from below using the triangle inequality

$$I \gtrsim \int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx. \tag{3.13}$$

For $1 < p < 2$, we have by Hölder's inequality with exponents $\dfrac{2-\delta}{p-\delta}$ and $\dfrac{2-\delta}{2-p}$, and Corollary 3.10 that

$$\begin{aligned} \int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx &= \int_{B_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{(p-\delta)(p-2)}{(2-\delta)2} + \frac{(\delta-p)(p-2)}{(2-\delta)2}} |\nabla u - \nabla w|^{p-\delta} \, dx, \\ &\lesssim \left(\int_{B_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla u - \nabla w|^{2-\delta} \, dx \right)^{\frac{p-\delta}{2-\delta}} \times \\ &\qquad \times \left(\int_{B_{2R}} |\nabla u|^{p-\delta} \, dx \right)^{\frac{2-p}{2-\delta}}. \end{aligned}$$

This gives, when $1 < p < 2$, that

$$\int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim I^{\frac{p-\delta}{2-\delta}} \left(\int_{B_{2R}} |\nabla u|^{p-\delta}\, dx \right)^{\frac{2-p}{2-\delta}}. \tag{3.14}$$

We shall estimate I_1 from above by making use of Hölder's inequality along with (2.3), (3.11), and Corollary 3.10 to obtain

$$\begin{aligned} |I_1| &\leq \Lambda_1 \int_{B_{2R}} (|\nabla u|^{p-1} + |\nabla w|^{p-1}) |\mathcal{H}|\, dx, \\ &\lesssim \delta \left(\int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}} \left(\int_{B_{2R}} |\nabla u|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}}. \end{aligned} \tag{3.15}$$

We estimate I_2 from above by using Hölder's inequality to obtain

$$|I_2| \leq \left(\int_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}} \left(\int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}}. \tag{3.16}$$

Finally, for I_3, we combine Hölder's inequality with (3.11) and obtain

$$\begin{aligned} |I_3| &\leq \int_{B_{2R}} |\mathbf{f}|^{p-1} |\mathcal{H}|\, dx, \\ &\lesssim \delta \left(\int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}} \left(\int_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}}. \end{aligned} \tag{3.17}$$

At this point, combining estimates (3.15), (3.16), (3.17) with (3.12) and (3.13) we get the desired estimate when $p \geq 2$:

$$\int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim \delta^{\frac{p-\delta}{p-1}} \int_{B_{2R}} |\nabla u|^{p-\delta}\, dx + \int_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx.$$

Likewise, for $1 < p < 2$, combining the estimates (3.15), (3.16), (3.17) with (3.12) and (3.14), we have

$$\begin{aligned} \int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim \Bigg\{ &\delta \left(\int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}} \left(\int_{B_{2R}} |\nabla u|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}} \\ &+ \left(\int_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}} \left(\int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}} \\ &+ \delta \left(\int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}} \left(\int_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}} \Bigg\}^{\frac{p-\delta}{2-\delta}} \times \\ &\times \left(\int_{B_{2R}} |\nabla u|^{p-\delta}\, dx \right)^{\frac{2-p}{2-\delta}}. \end{aligned}$$

Simplifying the above inequality, we get the desired estimate for the case $1 < p < 2$:

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim \delta^{p-\delta} \fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx + \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx \right)^{p-1} \left(\fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx \right)^{2-p}.$$

This completes the proof of Lemma 3.13. □

3.3 Local boundary estimates

We now extend the results of the previous section upto the boundary of a domain satisfying Hypothesis 3.2. While the approach of [28] via nonlinear Hodge decomposition could be used upto the boundary of the domain, it requires that the boundary be "sufficiently regular". This regularity assumption is unfortunately not easy to quantify. To overcome the roughness of the domain boundary, we shall employ the Lipschitz truncation method introduced in [37]. Here some of the ideas of [61] and the pointwise Hardy inequality obtained in [21] will be useful for our purpose. On the other hand, it should be noted that the approach of this section could be modified to obtain, e.g., the local interior comparison estimate (Lemma 3.13) that was previously derived by means of the nonlinear Hodge decomposition.

Beside the standard boundedness property of the Maximal function $\mathcal{M}$ on L^s spaces, we also use the following property. Given a non-zero function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and a number $\beta \in (0,1)$, there holds $\mathcal{M}(f)^\beta \in A_1$ with $[\mathcal{M}(f)^\beta]_{A_1} \leq C(n,\beta)$. Moreover, if β is away from 1, say $\beta \leq 0.9$, then $[\mathcal{M}(f)^\beta]_{A_1} \leq C(n)$ independent of β (see, e.g., [57] p. 229).

Lemma 3.14. *Let $\tilde{\Omega}$ is a bounded domain whose complement is uniformly p-thick with constants r_0 and $b > 0$. There exists a $\delta_0 = \delta_{0(n,p,b)} \in (0, 1/2)$ such that the following holds for any $\delta \in (0, \delta_0/2)$. Let $v \in W_0^{1,p-\delta}(\tilde{\Omega})$, $v \not\equiv 0$, and extend v by zero outside $\tilde{\Omega}$. Define*

$$g(x) = \max\left\{ \mathcal{M}(|\nabla v|^q)^{1/q}(x), \frac{|v(x)|}{d(x, \partial\tilde{\Omega})} \right\},$$

where $q \in (p - \delta_0, p - 2\delta]$ and $d(x, \partial\tilde{\Omega})$ is the distance of x from $\partial\tilde{\Omega}$. Then we have $g \simeq \mathcal{M}(|\nabla v|^q)^{1/q}$ a.e. in $\mathbb{R}^n$ and

$$\int_{\tilde{\Omega}} g^{p-\delta}\, dx \lesssim \int_{\tilde{\Omega}} |\nabla v|^{p-\delta}\, dx. \tag{3.18}$$

Moreover, the function $g^{-\delta} \in A_{p/q}$ *with* $[g^{-\delta}]_{A_{p/q}} \le C_{(n,p,b)}$.

Proof. As $\tilde{\Omega}^c$ is uniformly p-thick, it is also uniformly p_0-thick for some $1 < p_0 < p$ with $p_0 = p_{0(n,p,b)}$ (see [36]). Moreover, there exists a constant $\delta_0 = \delta_{0(n,p,b)} \in (0, 1/2)$ with $p - \delta_0 \ge p_0$ such that for $q \in (p - \delta_0, p - 2\delta]$, where $\delta \in (0, \delta_0/2)$, the pointwise Hardy inequality

$$\frac{|v(x)|}{d(x, \partial\tilde{\Omega})} \lesssim \mathcal{M}(|\nabla v|^q)^{1/q}(x)$$

holds for a.e. $x \in \tilde{\Omega}$ (see [21]). It follows that $g(x) \simeq \mathcal{M}(|\nabla v|^q)^{1/q}(x)$ for a.e. $x \in \mathbb{R}^n$. Thus by the boundedness of the Hardy-Littlewood maximal function $\mathcal{M}$ we obtain inequality (3.18). Moreover, for any ball $B \subset \mathbb{R}^n$ we have

$$\begin{aligned}
&\fint_B g^{-\delta}\, dx \left(\fint_B g^{\frac{\delta q}{p-q}}\, dx \right)^{\frac{p-q}{q}} \\
&\quad \lesssim \fint_B \mathcal{M}(|\nabla v|^q)^{-\delta/q}\, dx \left(\fint_B \mathcal{M}(|\nabla v|^q)^{\frac{\delta}{p-q}}\, dx \right)^{\frac{p-q}{q}} \\
&\quad \lesssim \left\{ \inf_{y \in B} \mathcal{M}(|\nabla v|^q)(y) \right\}^{-\delta/q} \left\{ \inf_{y \in B} \mathcal{M}(|\nabla v|^q)(y) \right\}^{\delta/q} \\
&\quad \le C.
\end{aligned}$$

Here we used that the function $\mathcal{M}(|\nabla v|^q)^{\frac{\delta}{p-q}}$ is an A_1 weight since $\dfrac{\delta}{p-q} \le 1/2 < 1$ (see [57] p. 229). □

We now present an extension lemma which can be found in [61].

Lemma 3.15. *Let* $v \in W_0^{1,s}(\tilde{\Omega})$, $s \ge 1$, *where* $\tilde{\Omega}$ *is a bounded domain and let* $\lambda > 0$. *Extend* v *by zero outside* $\tilde{\Omega}$ *and set*

$$F_\lambda(v, \tilde{\Omega}) = \left\{ x \in \tilde{\Omega} : \mathcal{M}(|\nabla v|^s)^{1/s}(x) \le \lambda, |v(x)| \le \lambda d(x, \partial\tilde{\Omega}) \right\}, \tag{3.19}$$

where $d(x, \partial\tilde{\Omega})$ *is the distance of* x *from* $\partial\tilde{\Omega}$. *Then there exists a* $c\lambda$*-Lipschitz function* v_λ *defined on* $\mathbb{R}^n$ *with* $c = c_{(n)} \ge 1$ *and the following properties:*

- $v_\lambda(x) = v(x)$ *and* $\nabla v_\lambda(x) = \nabla v(x)$ *for a.e.* $x \in F_\lambda$;

- $v_\lambda(x) = 0$ *for every* $x \in \tilde{\Omega}^c$; *and*

- $|\nabla v_\lambda(x)| \le c_{(n)}\lambda$ *for a.e.* $x \in \mathbb{R}^n$.

Proof. Given the hypothesis of the lemma, there exists a set $N \subset \mathbb{R}^n$ with $|N| = 0$ such that

$$|v(x) - v(y)| \le c\,|x-y|[\mathcal{M}(|\nabla v|^s)^{1/s}(x) + \mathcal{M}(|\nabla v|^s)^{1/s}(y)] \tag{3.20}$$

holds for every x, $y \in \mathbb{R}^n \setminus N$. The proof of inequality (3.20) is due to L. I. Hedberg which can be found in [24]. It is then easy to show that $v|_{(F_\lambda \setminus N)\cup\tilde{\Omega}^c}$ is a $c\lambda$-Lipschitz continuous function for some $c(n) \ge 1$. Indeed, in the case when $x, y \in F_\lambda \setminus N$, then by using (3.19) in (3.20), we see that

$$\begin{aligned} |v(x) - v(y)| &\le c\,|x-y|[\mathcal{M}(|\nabla v|^q)^{1/q}(x) + \mathcal{M}(|\nabla v|^q)^{1/q}(y)] \\ &\le 2c\,\lambda|x-y|. \end{aligned}$$

On the other hand, if $x \in F_\lambda \setminus N$ but $y \in \tilde{\Omega}^c$, by making use of (3.19), we observe that

$$|v(x) - v(y)| = |v(x)| \le \lambda d(x, \partial\tilde{\Omega}) \le \lambda|x-y|.$$

We can now extend $v|_{F_\lambda \setminus N)\cup\tilde{\Omega}^c}$ to a Lipschitz continuous function v_λ on the whole $\mathbb{R}^n$ with the same Lipschitz constant by the classical Kirszbraun-McShane extension theorem (see, e.g., [14, p. 80]). This extension satisfies all the properties highlighted in this lemma. $\square$

We next state a generalized Sobolev-Poincaré's inequality which was originally obtained by V. Maz'ya [39, Sec. 10.1.2]. See also [33, Sec. 3.1] and [1, Corollary 8.2.7].

Theorem 3.16. *Let B be a ball and $\phi \in W^{1,s}(B)$ be s-quasicontinuous (see Definition 2.29) with $s > 1$ and let $\kappa = n/(n-s)$ if $1 < s < n$ and $\kappa = 2$ if $s = n$. Then there exists a constant $c_{(n,s)} > 0$ such that*

$$\left(\fint_B |\phi|^{\kappa s}\,dx\right)^{\frac{1}{\kappa s}} \le c_{(n,s)}\left(\frac{1}{\mathrm{cap}_s(N(\phi), 2B)}\int_B |\nabla\phi|^s\,dx\right)^{\frac{1}{s}},$$

where $N(\phi) = \{x \in B : \phi(x) = 0\}$.

The following estimate with exponents below the natural one has been known only for regular domains (see [28]). Here, for the first time, it is obtained for domains with p-thick complements. We state the below theorem in a slightly more general form.

Theorem 3.17. *Suppose that $\mathcal{A}$ satisfies Hypothesis 3.1 and $\tilde{\Omega}$ satisfy Hypothesis 3.2, then there exists a constant $\delta_1 = \delta_{1(n,p,b,\Lambda_0,\Lambda_1,\gamma)} \in (0, \delta_0/2]$, with δ_0 as in Lemma 3.14, such that the following holds for any $\delta \in (0, \delta_1)$: Given any vector fields $\mathbf{h}, \mathbf{f} \in L^{p-\delta}(\tilde{\Omega})$ and any very weak solution $w \in W_0^{1,p-\delta}(\tilde{\Omega})$ to equation*

$$\operatorname{div} \mathcal{A}(x, \mathbf{h} + \nabla w) = \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f}, \tag{3.21}$$

there holds

$$\int_{\tilde{\Omega}} |\nabla w|^{p-\delta}\, dx \le C_{(n,p,b,\Lambda_0,\Lambda_1,\gamma)} \int_{\tilde{\Omega}} \left(|\mathbf{h}(x)|^{p-\delta} + |\mathbf{f}(x)|^{p-\delta}\right) dx. \tag{3.22}$$

Proof. As $\tilde{\Omega}^c$ is uniformly p-thick, it is also uniformly p_0-thick for some $1 < p_0 < p$. Let $\delta_0 \in (0, 1/2)$, with $p - \delta_0 \ge p_0$, be as in Lemma 3.14. Let $\delta \in (0, \delta_0/2)$ and q be such that $p - \delta_0 < q \le p - 2\delta < p - \delta$. Defining

$$g(x) := \max\left\{\mathcal{M}(|\nabla w|^q)^{1/q}(x), \frac{|w(x)|}{d(x, \partial\tilde{\Omega})}\right\},$$

then it follows from Lemma 3.14 that

$$\int_{\tilde{\Omega}} g(x)^{p-\delta}\, dx \lesssim \int_{\tilde{\Omega}} |\nabla w|^{p-\delta}\, dx. \tag{3.23}$$

We now apply Lemma 3.15 with $s = q$ and $v = w$, to get a global $c\lambda$-Lipschitz function v_λ such that $v_\lambda \in W_0^{1,\frac{p-\delta}{1-\delta}}(\tilde{\Omega})$. Using v_λ as a test function in (3.21) together with (2.3) we have

$$\begin{aligned}
&\int_{\tilde{\Omega}\cap F_\lambda} \langle \mathcal{A}(x, \nabla w), \nabla v_\lambda\rangle\, dx - \int_{\tilde{\Omega}\cap F_\lambda} |\mathbf{f}|^{p-2}\langle \mathbf{f}, \nabla v_\lambda\rangle\, dx \\
&- \int_{\tilde{\Omega}\cap F_\lambda} \langle \mathcal{A}(x, \nabla w) - \mathcal{A}(x, \mathbf{h} + \nabla w), \nabla v_\lambda\rangle\, dx \\
&\quad = -\int_{\tilde{\Omega}\cap F_\lambda^c} \langle \mathcal{A}(x, \mathbf{h} + \nabla w), \nabla v_\lambda\rangle\, dx + \int_{\tilde{\Omega}\cap F_\lambda^c} |\mathbf{f}|^{p-2}\langle \mathbf{f}, \nabla v_\lambda\rangle\, dx \\
&\quad \lesssim \lambda \int_{\tilde{\Omega}\cap F_\lambda^c} |\mathbf{h} + \nabla w|^{p-1}\, dx + \lambda \int_{\tilde{\Omega}\cap F_\lambda^c} |\mathbf{f}|^{p-1}\, dx,
\end{aligned} \tag{3.24}$$

where $F_\lambda := F_\lambda(w, \tilde{\Omega}) = \{x \in \tilde{\Omega} : g(x) \leq \lambda\}$. Multiplying equation (3.24) by $\lambda^{-(1+\delta)}$ and integrating from 0 to ∞ with respect to λ, we get

$$\begin{aligned} I_1 - I_2 - I_3 := \int_0^\infty \lambda^{-(1+\delta)} & \int_{\tilde{\Omega}\cap F_\lambda} \langle \mathcal{A}(x, \nabla w), \nabla v_\lambda\rangle \, dx \, d\lambda \\ & - \int_0^\infty \lambda^{-(1+\delta)} \int_{\tilde{\Omega}\cap F_\lambda} |\mathbf{f}|^{p-2}\langle \mathbf{f}, \nabla v_\lambda\rangle \, dx \, d\lambda \\ & - \int_0^\infty \lambda^{-(1+\delta)} \int_{\tilde{\Omega}\cap F_\lambda} \langle \mathcal{A}(x, \nabla w) - \mathcal{A}(x, \mathbf{h} + \nabla w), \nabla v_\lambda\rangle \, dx \, d\lambda \\ \lesssim \int_0^\infty \lambda^{-\delta} & \int_{\tilde{\Omega}\cap F_\lambda^c} \left(|\mathbf{h} + \nabla w|^{p-1} + |\mathbf{f}|^{p-1}\right) \, dx \, d\lambda =: I_4. \end{aligned}$$

We now continue with the following estimates for I_j, $j = 1, 2, 3, 4$.

Estimate for I_1 from below: Note that we have $\nabla v_\lambda = \nabla w$ a.e. on F_λ. Thus by changing the order of integration and using (2.1), we get

$$\begin{aligned} I_1 &= \int_{\tilde{\Omega}} \int_{g(x)}^\infty \lambda^{-(1+\delta)} \langle \mathcal{A}(x, \nabla w), \nabla w\rangle \, d\lambda \, dx \\ &= \frac{1}{\delta} \int_{\tilde{\Omega}} g(x)^{-\delta} \langle \mathcal{A}(x, \nabla w), \nabla w\rangle \, dx \\ &\gtrsim \frac{1}{\delta} \int_{\tilde{\Omega}} g(x)^{-\delta} |\nabla w|^p \, dx. \end{aligned} \tag{3.25}$$

By Hölder's inequality, we have

$$\int_{\Omega_{2\rho}} |\nabla w|^{p-\delta} \, dx \lesssim \left(\int_{\Omega_{2\rho}} |\nabla w|^p g(x)^{-\delta} \, dx\right)^{\frac{p-\delta}{p}} \left(\int_{\Omega_{2\rho}} g(x)^{p-\delta} \, dx\right)^{\frac{\delta}{p}},$$

and then by making use of (3.23), we obtain the estimate

$$\int_{\Omega_{2\rho}} |\nabla w|^{p-\delta} \, dx \lesssim \int_{\Omega_{2\rho}} |\nabla w|^p g(x)^{-\delta} \, dx. \tag{3.26}$$

Now we combine (3.25) with (3.26) and get

$$I_1 \gtrsim \frac{1}{\delta} \int_{\tilde{\Omega}} |\nabla w|^{p-\delta} \, dx. \tag{3.27}$$

Estimate for I_2 from above: Again by changing the order of integration and making use of Young's inequality, we get

$$\begin{aligned}
I_2 &= \int_{\tilde{\Omega}} \int_{g(x)}^{\infty} \lambda^{-(1+\delta)} |\mathbf{f}|^{p-2} \langle \mathbf{f}, \nabla w \rangle \, d\lambda \, dx \\
&= \frac{1}{\delta} \int_{\tilde{\Omega}} g(x)^{-\delta} |\mathbf{f}|^{p-2} \langle \mathbf{f}, \nabla w \rangle \, dx \\
&\leq \frac{1}{\delta} \int_{\tilde{\Omega}} |\mathbf{f}|^{p-1} \, |\nabla w|^{1-\delta} \, dx \\
&\leq \frac{c_{(\epsilon)}}{\delta} \int_{\tilde{\Omega}} |\mathbf{f}|^{p-\delta} \, dx + \frac{\epsilon}{\delta} \int_{\tilde{\Omega}} |\nabla w|^{p-\delta} \, dx
\end{aligned}$$

for any $\epsilon > 0$. Here we used that $g^{-\delta} \leq |\nabla w|^{-\delta}$ a.e. in $\tilde{\Omega}$ in the first inequality.

Estimate for I_3 from above: Likewise, changing the order of integration and making use of Young's inequality along with the Hölder type condition (2.2), we get

$$\begin{aligned}
I_3 &= \int_{\tilde{\Omega}} \int_{g(x)}^{\infty} \lambda^{-(1+\delta)} \langle \mathcal{A}(x, \nabla w) - \mathcal{A}(x, \mathbf{h} + \nabla w), \nabla w \rangle \, d\lambda \, dx \\
&\lesssim \frac{1}{\delta} \int_{\tilde{\Omega}} g(x)^{-\delta} |\mathbf{h}|^{\gamma} \left(|\mathbf{h}|^{p-1-\gamma} + |\nabla w|^{p-1-\gamma} \right) |\nabla w| \, dx \\
&\lesssim \frac{c_{(\epsilon)}}{\delta} \int_{\tilde{\Omega}} |\mathbf{h}|^{p-\delta} \, dx + \frac{\epsilon}{\delta} \int_{\tilde{\Omega}} |\nabla w|^{p-\delta} \, dx
\end{aligned}$$

for any $\epsilon > 0$.

Estimate for I_4 from above: Changing the order of integration and applying Young's inequality along with estimate (3.23), we get

$$\begin{aligned}
I_4 &= \int_{\tilde{\Omega}} \int_{0}^{g(x)} \lambda^{-\delta} (|\mathbf{h} + \nabla w|^{p-1} + |\mathbf{f}|^{p-1}) \, d\lambda \, dx \\
&= \frac{1}{1-\delta} \int_{\tilde{\Omega}} g(x)^{1-\delta} (|\mathbf{h} + \nabla w|^{p-1} + |\mathbf{f}|^{p-1}) \, dx \\
&\lesssim \int_{\tilde{\Omega}} |\nabla w|^{p-\delta} \, dx + \int_{\tilde{\Omega}} (|\mathbf{h}|^{p-\delta} + |\mathbf{f}|^{p-\delta}) \, dx.
\end{aligned} \tag{3.28}$$

Combining estimates (3.27)-(3.28) and recalling that $I_1 - I_2 - I_3 \lesssim I_4$, we have

$$\begin{aligned}
\int_{\tilde{\Omega}} |\nabla w|^{p-\delta} dx \quad &\leq \quad c_1 (c_{(\epsilon)} + \delta) \int_{\tilde{\Omega}} |\mathbf{f}|^{p-\delta} dx + c_1 (2\epsilon + \delta) \int_{\tilde{\Omega}} |\nabla w|^{p-\delta} \, dx + \\
&\quad + c_1 (c_{(\epsilon)} + \delta) \int_{\tilde{\Omega}} |\mathbf{h}|^{p-\delta} \, dx
\end{aligned}$$

for a constant c_1 independent of ϵ and δ.

We now choose $\epsilon = 1/(4c_1)$ and $\delta_1 = \min\{1/(4c_1), \delta_0/2\}$ in the last inequality to obtain estimate (3.22) for any $\delta \in (0, \delta_1)$. □

Once we have the a priori estimate (3.22) and the interior higher integrability result from Theorem 3.8, the following existence result follows by using techniques employed in the proof of [28, Theorem 2].

Corollary 3.18. *Suppose that $\mathcal{A}$ satisfies Hypothesis 3.1 and let $\tilde{\Omega}$ satisfy Hypothesis 3.2 and $\delta \in (0, \min\{\delta_1, \tilde{\delta}_2\})$, with δ_1 as in Theorem 3.17 and $\tilde{\delta}_2$ as in Theorem 3.8. Then given any $w_0 \in W^{1,p-\delta}(\tilde{\Omega})$, there exists a very weak solution $w \in w_0 + W_0^{1,p-\delta}(\tilde{\Omega})$ to the equation* $\mathrm{div}\mathcal{A}(x, \nabla w) = 0$ *such that*

$$\int_{\tilde{\Omega}} |\nabla w|^{p-\delta} dx \leq C_{(n,p,b,\Lambda_0,\Lambda_1,\gamma)} \int_{\tilde{\Omega}} |\nabla w_0|^{p-\delta} dx.$$

Remark 3.19. It is well-known that in the case $\delta = 0$, Corollary 3.18 and Corollary 3.10 hold as long as $\mathcal{A}$ satisfies (2.1) and (2.3), i.e., the condition (2.2) holding with $\gamma \in (0, 1)$ is not needed. Moreover, the so-obtained solution w is unique in this case, whereas uniqueness remains unknown in the case $\delta > 0$. We also notice that Corollary 3.18 has been known earlier but only for more regular domains (see [28]).

In what follows, we shall only consider Ω to be a bounded domain satisfying Hypothesis 3.2. Fix $x_0 \in \partial\Omega$ and choose $R > 0$ such that $2R \leq r_0$ and denote $\Omega_{2R} = \Omega_{2R}(x_0) = \Omega \cap B_{2R}(x_0)$. With some $\delta \in (0, \min\{1, p-1\})$, we consider the following Dirichlet problem:

$$\begin{cases} \mathrm{div}\mathcal{A}(x, \nabla w) &= 0 \quad \text{in } \Omega_{2R}, \\ w &= 0 \quad \text{on } \partial\Omega \cap B_{2R}(x_0). \end{cases}$$

Definition 3.20. A function $w \in W^{1,p-\delta}(\Omega_{2R})$ is called a very weak solution to (3.6) if its zero extension from $\Omega_{2R}(x_0)$ to $B_{2R}(x_0)$ belongs to $W^{1,p-\delta}(B_{2R}(x_0))$ and for all $\varphi \in W_0^{1,\frac{p-\delta}{1-\delta}}(\Omega_{2R})$, we have

$$\int_{\Omega_{2R}} \mathcal{A}(x, \nabla w) \cdot \nabla\varphi \, dx = 0.$$

In the following theorem we obtain a higher integrability result for equation (3.6), which gives a boundary analogue of Theorem 3.8, and hence Theorem 3.4. We shall follow the Lipschitz truncation method of [37] that was used to treat the interior case; see also [46, Theorem 9.4]. Here to deal with the boundary case we use an idea from [61].

Theorem 3.21. *Suppose that $\mathcal{A}$ satisfies Hypothesis 3.1 and Ω satisfy Hypothesis 3.2, then there exists a constant $\delta_2 = \delta_{2(n,p,b,\Lambda_0,\Lambda_1)} > 0$ sufficiently small such that if $w \in W^{1,p-\delta_2}(\Omega_{2R})$ is a very weak solution to equation* (3.6)*, then $w \in W^{1,p+\delta_2}(\Omega_R)$. Moreover, if we extend w by zero from Ω_{2R} to B_{2R}, then the estimate*

$$\left(\fint_{\frac{1}{2}B} |\nabla w|^{p+\delta_2}\, dx \right)^{\frac{1}{p+\delta_2}} \leq C_{(n,p,b,\Lambda_0,\Lambda_1)} \left(\fint_{7B} |\nabla w|^{p-\delta_2}\, dx \right)^{\frac{1}{p-\delta_2}}$$

holds for all balls B such that $7B \subset B_{2R}$.

Proof. Let $z \in \partial\Omega \cap B_{2R}(x_0)$ be a boundary point and let $\rho > 0$ be such that $B_{2\rho}(z) \subset B_{2R}(x_0)$. We now set $\Omega_{2\rho} = \Omega_{2\rho}(z) = \Omega \cap B_{2\rho}(z)$ and observe that $\Omega_{2\rho} \subset \Omega_{2R}(x_0)$.

As Ω^c is uniformly p-thick, it is also uniformly p_0-thick for some $1 < p_0 < p$. The same is also true for $\Omega^c_{2\rho}$. Let $\delta_0 \in (0, 1/2)$, with $p - \delta_0 \geq p_0$, be as in Lemma 3.14 with $\tilde{\Omega} = \Omega_{2\rho}$. Let $\delta \in (0, \delta_0/2)$ and q be such that $p - \delta_0 < q \leq p - 2\delta < p - \delta$.

Suppose now that $w \in W^{1,p-\delta}(\Omega_{2R}(x_0))$ is a solution of (3.6). Extending w to $B_{2\rho} = B_{2\rho}(z)$ by zero we have $w \in W^{1,p-\delta}(B_{2\rho})$. Let $\phi \in C_c^\infty(B_{2\rho})$ with $0 \leq \phi \leq 1$, $\phi \equiv 1$ on B_ρ and $|\nabla\phi| \leq 4/\rho$. Define $\bar{w} = \phi w$ and g to be the function

$$g(x) = \max\left\{ \mathcal{M}(|\nabla \bar{w}|^q)^{1/q}(x), \frac{|\bar{w}(x)|}{d(x, \partial\Omega_{2\rho})} \right\}.$$

Then it follows from Lemma 3.14 that

$$\int_{\Omega_{2\rho}} g^{p-\delta}\, dx \lesssim \int_{\Omega_{2\rho}} |\nabla \bar{w}|^{p-\delta}\, dx. \tag{3.29}$$

We now apply Lemma 3.15 with $s = q$, $\tilde{\Omega} = \Omega_{2\rho}$ and $v = \bar{w}$, to get a global $c\lambda$-Lipschitz function v_λ such that $v_\lambda \in W_0^{1,\frac{p-\delta}{1-\delta}}(\Omega_{2\rho})$. Using v_λ as a test function in (3.6) together with

(2.3) we have

$$\int_{\Omega_{2\rho}\cap F_\lambda} \langle \mathcal{A}(x,\nabla w), \nabla v_\lambda\rangle \, dx = -\int_{\Omega_{2\rho}\cap F_\lambda^c} \langle \mathcal{A}(x,\nabla w), \nabla v_\lambda\rangle \, dx$$
$$\lesssim \lambda \int_{\Omega_{2\rho}\cap F_\lambda^c} |\nabla w|^{p-1} \, dx, \tag{3.30}$$

where $F_\lambda := F_\lambda(\bar{w}, \Omega_{2\rho}) = \{x \in \Omega_{2\rho} : g(x) \le \lambda\}$. Multiply equation (3.30) by $\lambda^{-(1+\delta)}$ and integrate from 0 to ∞ with respect to λ, we then get

$$\begin{aligned} I_1 := & \int_0^\infty \lambda^{-(1+\delta)} \int_{\Omega_{2\rho}\cap F_\lambda} \langle \mathcal{A}(x,\nabla w), \nabla v_\lambda\rangle \, dx \, d\lambda \\ \lesssim & \int_0^\infty \lambda^{-\delta} \int_{\Omega_{2\rho}\cap F_\lambda^c} |\nabla w|^{p-1} \, dx \, d\lambda \\ = & \int_{\Omega_{2\rho}} \int_0^{g(x)} \lambda^{-\delta} d\lambda \, |\nabla w|^{p-1} \, d\lambda \, dx \\ = & \frac{1}{1-\delta} \int_{\Omega_{2\rho}} g(x)^{1-\delta} |\nabla w|^{p-1} \, dx \end{aligned}$$

where the first equality follows by Fubini's Theorem. Thus after applying Young's inequality and using (3.29), we obtain

$$\begin{aligned} I_1 & \lesssim \int_{\Omega_{2\rho}} |\nabla w|^{p-\delta} \, dx + \int_{\Omega_{2\rho}} |\nabla \bar{w}|^{p-\delta} \, dx \\ & \lesssim \int_{\Omega_{2\rho}} (|\nabla w|^{p-\delta} + |w/\rho|^{p-\delta}) \, dx \\ & \lesssim \int_{B_{2\rho}} |\nabla w|^{p-\delta} \, dx. \end{aligned} \tag{3.31}$$

Here the last inequality follows from Theorem 3.16 since $w = 0$ on $\Omega^c \cap B_{2\rho}$.

Our next goal is to estimate I_1 from below. To this end, changing the order of integration and noting that $\nabla v_\lambda = \nabla \bar{w}$ a.e. on F_λ, we can write

$$\begin{aligned} I_1 & = \int_{\Omega_{2\rho}} \int_{g(x)}^\infty \lambda^{-(1+\delta)} d\lambda \ \langle \mathcal{A}(x,\nabla w), \nabla \bar{w}\rangle \, dx \\ & = \frac{1}{\delta} \int_{\Omega_{2\rho}} g(x)^{-\delta} \langle \mathcal{A}(x,\nabla w), \nabla \bar{w}\rangle \ dx. \end{aligned}$$

To continue we set

$$D_1 = \left\{x \in \Omega_{2\rho} \setminus \Omega_\rho : \mathcal{M}(|\nabla \bar{w}|^q)^{1/q} \le \delta \mathcal{M}(|\nabla w|^q \chi_{\Omega_{2\rho}})^{1/q}\right\},$$

$$D_2 = \Omega_{2\rho} \setminus (\Omega_\rho \cup D_1),$$

and note that $w = \bar{w}$ on Ω_ρ. Thus it follows from (2.1) and (2.3) that

$$
\begin{aligned}
\delta\, I_1 &\geq \Lambda_0 \int_{\Omega_\rho} g^{-\delta} |\nabla w|^p \, dx + \int_{D_1} g(x)^{-\delta} \langle \mathcal{A}(x, \nabla w), \nabla \bar{w} \rangle \; dx \\
&\qquad + \int_{D_2} g(x)^{-\delta} \langle \mathcal{A}(x, \nabla w), \nabla \phi \rangle w \; dx \\
&\geq \Lambda_0 \int_{\Omega_\rho} g^{-\delta} |\nabla w|^p \, dx - \Lambda_1 \int_{D_1} g^{-\delta} |\nabla w|^{p-1} |\nabla \bar{w}| \, dx \\
&\qquad - \frac{4\Lambda_1}{\rho} \int_{D_2} g^{-\delta} |\nabla w|^{p-1} |w| \, dx \\
&=: I_2 - I_3 - I_4.
\end{aligned} \tag{3.32}
$$

Combining (3.31) and (3.32), we obtain

$$
I_2 \lesssim I_3 + I_4 + \delta \int_{B_{2\rho}} |\nabla w|^{p-\delta} \, dx. \tag{3.33}
$$

We now consider the following estimates for I_2, I_3, and I_4.

Estimate for I_2 from below: Recall that by Lemma 3.14, $g^{-\delta} \in A_{p/q}$. Thus by the boundedness of $\mathcal{M}$ we have

$$
I_2 = \Lambda_0 \int_{\Omega_\rho} g(x)^{-\delta} |\nabla w|^p \, dx \gtrsim \int_{B_\rho} g(x)^{-\delta} \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{p/q} \, dx. \tag{3.34}
$$

On the other hand, for $x \in B_{\rho/2}$, there holds

$$
\begin{aligned}
\mathcal{M}(|\nabla \bar{w}|^q)^{1/q}(x) &\leq \sup_{\substack{x \in B' \\ B' \subset B_\rho}} \left(\fint_{B'} |\nabla \bar{w}|^q dy \right)^{1/q} + \sup_{\substack{x \in B' \\ B' \cap B_\rho^c \neq \emptyset}} \left(\fint_{B'} |\nabla \bar{w}|^q dy \right)^{1/q} \\
&\leq \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{1/q}(x) + c \left(\fint_{B_{2\rho}} |\nabla \bar{w}|^q dy \right)^{1/q},
\end{aligned}
$$

where we have used that $\bar{w} = w$ on B_ρ and $w = 0$ on $\Omega^c \cap B_\rho$. Also, recall that $\bar{w}$ is zero outside $B_{2\rho}$. By Theorem 3.16 we find

$$
\fint_{B_{2\rho}} |\nabla \bar{w}|^q dy \leq \fint_{B_{2\rho}} |\nabla w|^q dy + \frac{c}{\rho^q} \fint_{B_{2\rho}} |w|^q dy \leq c \fint_{B_{2\rho}} |\nabla w|^q dy,
$$

which gives

$$
\begin{aligned}
g(x) &\leq c\, \mathcal{M}(|\nabla \bar{w}|^q)^{1/q}(x) \\
&\leq c_1 \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{1/q}(x) + c_2 \left(\fint_{B_{2\rho}} |\nabla w|^q dy \right)^{1/q}
\end{aligned}
$$

for all $x \in B_{\rho/2}$. Here recall from Lemma 3.14 that $g \simeq \mathcal{M}(|\nabla \bar{w}|^q)^{1/q}$ a.e. in $\mathbb{R}^n$.

Letting now

$$G = \left\{ x \in B_{\rho/2} : c_1 \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{1/q}(x) \geq c_2 \Big(\fint_{B_{2\rho}} |\nabla w|^q dy \Big)^{1/q} \right\},$$

then for every $x \in G$ we have

$$g(x) \leq 2c_1 \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{1/q}(x). \tag{3.35}$$

Combining (3.34) and (3.35) we can estimate I_2 from below by

$$\begin{aligned} I_2 &\geq c \int_G \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{-\delta/q} \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{p/q}\, dx \\ &\geq c \int_{B_{\rho/2}} |\nabla w|^{p-\delta}\, dx - c_0 \rho^n \left(\fint_{B_{2\rho}} |\nabla w|^q\, dx \right)^{\frac{p-\delta}{q}}. \end{aligned} \tag{3.36}$$

Estimate for I_3 from above: By the definition of D_1 and the boundedness of the maximal function $\mathcal{M}$, we have

$$\begin{aligned} I_3 &= \Lambda_1 \int_{D_1} g^{-\delta} |\nabla w|^{p-1} |\nabla \bar{w}|\, dx \\ &\lesssim \int_{D_1} \mathcal{M}(|\nabla \bar{w}|^q)^{\frac{1-\delta}{q}} |\nabla w|^{p-1}\, dx \\ &\lesssim \delta^{1-\delta} \int_{\Omega_{2\rho}} \mathcal{M}(|\nabla w|^q \chi_{\Omega_{2\rho}})^{\frac{1-\delta}{q}} |\nabla w|^{p-1}\, dx \\ &\lesssim \delta^{1-\delta} \int_{\Omega_{2\rho}} |\nabla w|^{p-\delta}\, dx. \end{aligned}$$

Estimate for I_4 from above: By the definition of D_2 we have

$$\begin{aligned} I_4 &= \frac{4\beta}{\rho} \int_{D_2} g^{-\delta} |\nabla w|^{p-1} |w|\, dx \\ &\lesssim \frac{1}{\rho} \int_{D_2} \mathcal{M}(|\nabla \bar{w}|^q)^{-\delta/q} |\nabla w|^{p-1} |w|\, dx \\ &\lesssim \frac{\delta^{-\delta}}{\rho} \int_{D_2} \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{(p-1-\delta)/q}\, |w|\, dx. \end{aligned}$$

With this and making use of Young's inequality, we find, for any $\epsilon > 0$,

$$\begin{aligned} I_4 &\lesssim \epsilon \int_{\Omega_{2\rho}} \mathcal{M}(|\nabla w|^q \chi_{\Omega_{2\rho}})^{\frac{p-\delta}{q}}\, dx + \frac{c_{(\epsilon)}}{\rho^{p-\delta}} \int_{B_{2\rho}} |w|^{p-\delta}\, dx \\ &\lesssim \epsilon \int_{\Omega_{2\rho}} |\nabla w|^{p-\delta}\, dx + c_{(\epsilon)} \rho^n \left(\fint_{B_{2\rho}} |\nabla w|^q\, dx \right)^{\frac{p-\delta}{q}}. \end{aligned} \tag{3.37}$$

Here the last inequality follows from the boundedness of $\mathcal{M}$ and Theorem 3.16 provided δ_0 is sufficiently small so that $nq/(n-q) \geq p$.

Collecting all of the estimates in (3.33), (3.36)-(3.37) we obtain

$$\begin{aligned}\int_{B_{\rho/2}} |\nabla w|^{p-\delta}\,dx \lesssim (1+c_{(\epsilon)})\rho^n \Big(\fint_{B_{2\rho}} |\nabla w|^q\,dx\Big)^{\frac{p-\delta}{q}} \\ +\ (\delta+\delta^{1-\delta}+\epsilon)\int_{B_{2\rho}} |\nabla w|^{p-\delta}\,dx.\end{aligned} \tag{3.38}$$

Recall that the balls in (3.38) are centered at $z \in \partial\Omega \cap B_{2R}(x_0)$ and we have $B_{2\rho} = B_{2\rho}(z) \subset B_{2R}(x_0)$. Let $x_1 \in B_{2R}(x_0)$ and $\rho > 0$ be such that we have $B_{7\rho}(x_1) \subset B_{2R}(x_0)$ and assume for now that $B_\rho(x_1) \cap \partial\Omega \neq \emptyset$. Choosing any $z \in \partial\Omega \cap B_\rho(x_1)$ such that $|x_1 - z| = d(x_1, \partial\Omega)$, we have $|x_1 - z_0| \leq \rho$ and thus

$$B_{\rho/2}(x_1) \subset B_{3\rho/2}(z) \subset B_{6\rho}(z) \subset B_{7\rho}(x_1).$$

With this, applying (3.38) we have

$$\begin{aligned}\int_{B_{\rho/2}(x_1)} |\nabla w|^{p-\delta}\,dx \lesssim (1+c_{(\epsilon)})\rho^n \Big(\fint_{B_{7\rho}(x_1)} |\nabla w|^q\,dx\Big)^{\frac{p-\delta}{q}} \\ +\ (\delta+\delta^{1-\delta}+\epsilon)\int_{B_{7\rho}(x_1)} |\nabla w|^{p-\delta}\,dx.\end{aligned} \tag{3.39}$$

At this point, choosing δ and ϵ small enough in (3.39) we arrive at

$$\fint_{B_{\rho/2}(x_1)} |\nabla w|^{p-\delta}\,dx \leq c\Big(\fint_{B_{7\rho}(x_1)} |\nabla w|^q\,dx\Big)^{\frac{p-\delta}{q}} + \frac{1}{2}\fint_{B_{7\rho}(x_1)} |\nabla w|^{p-\delta}\,dx.$$

On the other hand, from the interior higher integrability bound (3.8) in Theorem 3.8 it follows that the last inequality also holds with any ball $B_{7\rho}(x_1) \subset B_{2R}(x_0)$ such that $B_\rho(x_1) \subset \Omega$, as long as we further restrict $\delta_0 \in (0, \tilde{\delta}_2)$ so that $q > p - \tilde{\delta}_2$. Here $\tilde{\delta}_2$ is as in Theorem 3.8.

Now using the well-known Gehring's lemma (see [17, p. 122]; see also [15, 46]) and a simple covering argument, we get the desired higher integrability upto the boundary. $\square$

We now set

$$\delta_3 = \min\{\delta_1, \tilde{\delta}_2, \delta_2\}$$

with $\delta_1, \tilde{\delta}_2$, and δ_2 as in Theorems 3.17, 3.8, and 3.21, respectively. For $\delta \in (0, \delta_3)$ and $u \in W_0^{1,p-\delta}(\Omega)$, we let $w \in W^{1,p-\delta}(\Omega_{2R}(x_0))$ be a very weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) = 0 & \text{in } \Omega_{2R}(x_0), \\ w \in u + W_0^{1,p-\delta}(\Omega_{2R}(x_0)). \end{cases} \tag{3.40}$$

The existence of such a w is now ensured by Corollary 3.18. Moreover, since we have higher integrability upto the boundary from Theorem 3.21, we can now obtain the boundary versions of Lemmas 3.11 and 3.12 (see Lemmas 3.7 and 3.8 in [51]).

Lemma 3.22 ([51])**.** *Let $u \in W_0^{1,p-\delta}(\Omega)$, with $\delta \in (0, \delta_3)$, and let w be a very weak solution of (3.40). Then there exists $\beta_0 = \beta_{0(n,p,b,\Lambda_0,\Lambda_1)} \in (0, 1/2]$ such that*

$$\left(\fint_{B_\rho(z)} |w|^p \, dx \right)^{\frac{1}{p}} \leq C_{(n,p,b,\Lambda_0,\Lambda_1)} \left(\frac{\rho}{r} \right)^{\beta_0} \left(\fint_{B_r(z)} |w|^p \, dx \right)^{\frac{1}{p}}$$

for any $z \in \partial\Omega$ with $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$. Moreover, there holds

$$\left(\fint_{B_\rho(z)} |\nabla w|^p \, dx \right)^{\frac{1}{p}} \leq C_{(n,p,b,\Lambda_0,\Lambda_1)} \left(\frac{\rho}{r} \right)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^p \, dx \right)^{\frac{1}{p}}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$.

Lemma 3.23 ([51])**.** *Let $u \in W_0^{1,p-\delta}(\Omega)$, with $\delta \in (0, \delta_3)$, and let w be a very weak solution of (3.40). Then there exists a $\beta_0 = \beta_{0(n,p,b,\Lambda_0,\Lambda_1)} \in (0, 1/2]$ such that for any $t \in (0, p]$ there holds*

$$\left(\fint_{B_\rho(z)} |\nabla w|^t \, dx \right)^{\frac{1}{t}} \leq C_{(n,p,b,t,\Lambda_0,\Lambda_1)} \left(\frac{\rho}{r} \right)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^t \, dx \right)^{\frac{1}{t}}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$.

We now prove the boundary analogue of Lemma 3.13.

Lemma 3.24. *Under Hypothesis 3.1 and Hypothesis 3.2, let $u \in W_0^{1,p-\delta}(\Omega)$ where $\delta \in (0, \min\{\delta_1, \tilde{\delta}_2\})$ with δ_1 and $\tilde{\delta}_2$ as in Theorems 3.17 and 3.8, respectively, be a very weak solution to (3.1) with $\mathbf{f} \in L^{p-\delta}(\Omega)$. Let $w \in u + W_0^{1,p-\delta}(\Omega_{2R})$ where $\Omega_{2\rho} = \Omega_{2R}(x_0)$ with*

$x_0 \in \partial\Omega$ and $2R \le r_0$, be a very weak solution to (3.40). Then after extending $\mathbf{f}$ and u by zero outside Ω and w by u outside Ω_{2R}, we have

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim \delta^{\frac{p-\delta}{p-1}} \fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx + \fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx$$

if $p \ge 2$ and

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim \delta^{p-\delta} \fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx + \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx\right)^{p-1} \left(\fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx\right)^{2-p}$$

if $1 < p < 2$.

Proof. Let $\delta \in (0, \min\{\delta_1, \tilde{\delta}_2\})$. Then $\delta \in (0, \delta_0/2)$ with δ_0 as in Lemma 3.14. Let $q \in (p - \delta_0, p - 2\delta]$ and define g to be the function

$$g(x) = \max\left\{\mathcal{M}(|\nabla u - \nabla w|^q)^{1/q}(x), \frac{|u(x) - w(x)|}{d(x, \partial\Omega_{2R})}\right\}.$$

Then it follows from Lemma 3.14 with $\tilde{\Omega} = \Omega_{2R}$ that

$$\int_{\Omega_{2R}} g^{p-\delta}\, dx \lesssim \int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx. \tag{3.41}$$

Also, by Theorem 3.17 we have

$$\int_{\Omega_{2R}} |\nabla w|^{p-\delta}\, dx \lesssim \int_{\Omega_{2R}} |\nabla u|^{p-\delta}\, dx. \tag{3.42}$$

We now apply Lemma 3.15 with $s = q$, $\tilde{\Omega} = \Omega_{2R}$ and $v = u - w$, to get a global $c\lambda$-Lipschitz function $v_\lambda \in W_0^{1,\frac{p-\delta}{1-\delta}}(\Omega_{2R})$. Using v_λ as a test function in (3.1) and (3.40) along with (2.3), we obtain

$$\begin{aligned}
&\int_{\Omega_{2R} \cap F_\lambda} \langle \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla w), \nabla v_\lambda \rangle\, dx - \int_{\Omega_{2R} \cap F_\lambda} |\mathbf{f}|^{p-2} \langle \mathbf{f}, \nabla v_\lambda \rangle\, dx \\
&\quad = \int_{\Omega_{2R} \cap F_\lambda^c} \langle \mathcal{A}(x, \nabla w) - \mathcal{A}(x, \nabla u), \nabla v_\lambda \rangle\, dx + \int_{\Omega_{2R} \cap F_\lambda^c} |\mathbf{f}|^{p-2} \langle \mathbf{f}, \nabla v_\lambda \rangle\, dx \\
&\quad \lesssim \lambda \int_{\Omega_{2R} \cap F_\lambda^c} \left(|\mathbf{f}|^{p-1} + |\nabla u|^{p-1} + |\nabla w|^{p-1}\right)\, dx,
\end{aligned}$$

where $F_\lambda := F_\lambda(u-w,\Omega_{2R}) = \{x \in \Omega_{2R} : g(x) \le \lambda\}$. Multiplying the above equation by $\lambda^{-(1+\delta)}$ and integrating from 0 to ∞ with respect to λ, we then get

$$\begin{aligned} I_1 - I_2 := & \int_0^\infty \int_{\Omega_{2R}\cap F_\lambda} \lambda^{-(1+\delta)} \langle \mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w), \nabla u - \nabla w\rangle \, dx\, d\lambda \\ & - \int_0^\infty \int_{\Omega_{2R}\cap F_\lambda} \lambda^{-(1+\delta)} |\mathbf{f}|^{p-2} \langle \mathbf{f}, \nabla u - \nabla w\rangle \, dx\, d\lambda \\ & \lesssim \int_0^\infty \int_{\Omega_{2R}\cap F_\lambda^c} \lambda^{-\delta} \left(|\mathbf{f}|^{p-1} + |\nabla u|^{p-1} + |\nabla w|^{p-1}\right) dx\, d\lambda =: I_3. \end{aligned}$$

We now proceed with the following estimates for I_1, I_2, and I_3.

Estimate for I_1 from below: By changing the order of integration and making use of (2.1), we get

$$\begin{aligned} I_1 &= \int_{\Omega_{2R}} \int_{g(x)}^\infty \lambda^{-(1+\delta)} \langle \mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w), \nabla u - \nabla w\rangle \, d\lambda\, dx \\ &= \frac{1}{\delta} \int_{\Omega_{2R}} g(x)^{-\delta} \langle \mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w), \nabla u - \nabla w\rangle \, dx \\ &\gtrsim \frac{1}{\delta} \int_{\Omega_{2R}} g(x)^{-\delta} \left(|\nabla u|^2 + |\nabla w|^2\right)^{\frac{p-2}{2}} |\nabla u - \nabla w|^2 \, dx. \end{aligned} \tag{3.43}$$

We now consider separately the case $p \ge 2$ and $1 < p < 2$.

Case i: For $p \ge 2$, by using (3.41) along with Hölder's inequality, we obtain

$$\begin{aligned} \int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx &\le \left(\int_{\Omega_{2R}} g^{-\delta} |\nabla u - \nabla w|^p \, dx \right)^{\frac{p-\delta}{p}} \left(\int_{\Omega_{2R}} g^{p-\delta} \, dx \right)^{\frac{\delta}{p}} \\ &\le \left(\int_{\Omega_{2R}} g^{-\delta} |\nabla u - \nabla w|^2 (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} \, dx \right)^{\frac{p-\delta}{p}} \times \\ &\quad \times \left(\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx \right)^{\frac{\delta}{p}}. \end{aligned}$$

Simplifying the above expression and substituting into (3.43), we get

$$I_1 \gtrsim \frac{1}{\delta} \int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx. \tag{3.44}$$

Case ii: For $1 < p < 2$, we use the following equality

$$\begin{aligned} |\nabla u - \nabla w|^{p-\delta} = & \left[(|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla u - \nabla w|^2 g^{-\delta} \right]^{\frac{p-\delta}{2}} \times \\ & \times \left(|\nabla u|^2 + |\nabla w|^2\right)^{\frac{(p-\delta)(2-p)}{4}} g^{\frac{p-\delta}{2}\delta}. \end{aligned} \tag{3.45}$$

Integrating (3.45) over Ω_{2R} and making use of Hölder's inequality with exponents $\dfrac{2}{p-\delta}, \dfrac{2}{2-p}$ and $\dfrac{2}{\delta}$, we get

$$\begin{aligned}\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx \le \left(\int_{\Omega_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-\delta}{2}}\,dx\right)^{\frac{2-p}{2}} \left(\int_{\Omega_{2R}} g(x)^{p-\delta}\,dx\right)^{\frac{\delta}{2}} \times \\ \times \left(\int_{\Omega_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla u - \nabla w|^2 g(x)^{-\delta}\,dx\right)^{\frac{p-\delta}{2}}.\end{aligned} \tag{3.46}$$

Combining (3.41) and (3.42) into (3.46) and then simplifying we get

$$\begin{aligned}\left(\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx\right)^{1-\frac{\delta}{2}} \lesssim \left(\int_{\Omega_{2R}} |\nabla u|^{p-\delta}\,dx\right)^{\frac{2-p}{2}} \times \\ \times \left(\int_{\Omega_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla u - \nabla w|^2 g(x)^{-\delta}\,dx\right)^{\frac{p-\delta}{2}}.\end{aligned}$$

Using this in (3.43), we arrive at

$$I_1 \gtrsim \frac{1}{\delta} \left(\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx\right)^{\frac{2-\delta}{p-\delta}} \left(\int_{\Omega_{2R}} |\nabla u|^{p-\delta}\,dx\right)^{\frac{p-2}{p-\delta}}. \tag{3.47}$$

Estimate for I_2 from above: By changing the order of integration, we get

$$\begin{aligned} I_2 &= \int_{\Omega_{2R}} \int_{g(x)}^{\infty} \lambda^{-(1+\delta)} |\mathbf{f}|^{p-2} \langle \mathbf{f}, \nabla u - \nabla w\rangle\,d\lambda\,dx \\ &= \frac{1}{\delta} \int_{\Omega_{2R}} g(x)^{-\delta} |\mathbf{f}|^{p-2} \langle \mathbf{f}, \nabla u - \nabla w\rangle\,dx \\ &\le \frac{1}{\delta} \int_{\Omega_{2R}} g(x)^{-\delta} |\mathbf{f}|^{p-1} |\nabla u - \nabla w|\,dx. \end{aligned} \tag{3.48}$$

Since $|\nabla u(x) - \nabla w(x)| \le g(x)$ for a.e. x, by using Hölder's inequality in (3.48), we have

$$\begin{aligned} I_2 &\le \frac{1}{\delta} \int_{\Omega_{2R}} |\nabla u - \nabla w|^{-\delta} |\mathbf{f}|^{p-1} |\nabla u - \nabla w|\,dx \\ &\le \frac{1}{\delta} \left(\int_{\Omega_{2R}} |\mathbf{f}|^{p-\delta}\,dx\right)^{\frac{p-1}{p-\delta}} \left(\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx\right)^{\frac{1-\delta}{p-\delta}}. \end{aligned} \tag{3.49}$$

Estimate for I_3 from above: By changing the order of integration, we get

$$\begin{aligned} I_3 &= \int_{\Omega_{2R}} \int_0^{g(x)} \lambda^{-\delta} \left(|\mathbf{f}|^{p-1} + |\nabla u|^{p-1} + |\nabla w|^{p-1}\right)\,d\lambda\,dx \\ &= \frac{1}{1-\delta} \int_{\Omega_{2R}} g(x)^{1-\delta} \left(|\mathbf{f}|^{p-1} + |\nabla u|^{p-1} + |\nabla w|^{p-1}\right)\,dx. \end{aligned}$$

Thus Hölder's inequality along with (3.41) and Theorem 3.17 then yield

$$I_3 \lesssim \left(\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}} \left(\int_{\Omega_{2R}} |\mathbf{f}|^{p-\delta} + |\nabla u|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}}. \tag{3.50}$$

As $I_1 - I_2 \lesssim I_3$, we can now combine estimates (3.49) and (3.50), along with (3.44) in the case $p \geq 2$ or (3.47) in the case $1 < p < 2$ to obtain the desired bounds. □

Remark 3.25. Henceforth, unless otherwise stated, we shall always assume that $0 < \delta < \min\{\tilde{\delta}_1, \tilde{\delta}_2, \delta_1, \delta_2\}$, where $\tilde{\delta}_1, \tilde{\delta}_2, \delta_1$, and δ_2 are as in Theorems 3.7, 3.8, 3.17, and 3.21, respectively.

Proposition 3.26. *There exists $A = A_{(n,p,b,\Lambda_0,\Lambda_1,\gamma)} > 1$ sufficiently large so that the following holds for any $T > 1$ and any $\lambda > 0$: fix a ball $B_0 = B_{R_0}$ and assume that for some ball $B_\rho(y)$ with $\rho \leq \min\{r_0, 2R_0\}/26$, we have*

$$B_\rho(y) \cap B_0 \cap \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) \leq \lambda\} \cap \{\mathcal{M}(\chi_{4B_0}|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}} \leq \epsilon(T)\lambda\} \neq \emptyset,$$

with $\epsilon(T) = T^{\frac{-2\delta}{p-\delta}\max\left\{1, \frac{1}{p-1}\right\}}$; then there holds

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > AT\lambda\} \cap B_\rho(y)| < H\,|B_\rho(y)|, \tag{3.51}$$

where

$$H = H(T) = T^{-(p+\delta)} + \delta^{(p-\delta)\min\left\{1, \frac{1}{p-1}\right\}}.$$

Proof. By hypothesis, there exists $x_0 \in B_\rho(y) \cap B_0$ such that for any $r > 0$, we have

$$ {}_{B_r(x_0)} \chi_{4B_0}|\nabla u|^{p-\delta}\, dx \leq \lambda^{p-\delta} \tag{3.52}$$

and

$$ {}_{B_r(x_0)} \chi_{4B_0}|\mathbf{f}|^{p-\delta}\, dx \leq [\epsilon(T)\lambda]^{p-\delta}. \tag{3.53}$$

Since $8\rho \leq R_0$, we have $B_{23\rho}(y) \subset B_{24\rho}(x_0) \subset 4B_0$. We now claim that for $x \in B_\rho(y)$, there holds

$$\mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})(x) \leq \max\left\{\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|^{p-\delta})(x), 3^n\lambda^{p-\delta}\right\}. \tag{3.54}$$

Indeed, for $r \leq \rho$ we have $B_r(x) \cap 4B_0 \subset B_{2\rho}(y) \cap 4B_0 = B_{2\rho}(y)$ and thus

$$\fint_{B_r(x)} \chi_{4B_0} |\nabla u|^{p-\delta}\, dz = \fint_{B_r(x)} \chi_{B_{2\rho}(y)} |\nabla u|^{p-\delta}\, dz,$$

whereas for $r > \rho$ we have $B_r(x) \subset B_{3r}(x_0)$ from which, by making use of (3.52), yields

$$\fint_{B_r(x)} \chi_{4B_0} |\nabla u|^{p-\delta}\, dz \leq 3^n \fint_{B_{3r}(x_0)} \chi_{4B_0} |\nabla u|^{p-\delta}\, dz \leq 3^n \lambda^{p-\delta}.$$

We now restrict A to the range $A \geq 3^{\frac{n}{p-\delta}}$. Then in view of (3.54) we see that in order to obtain (3.51), it is enough to show that

$$|\{\mathcal{M}(\chi_{B_{2\rho}(y)} |\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda\} \cap B_\rho(y)| < H\, |B_\rho(y)|. \tag{3.55}$$

Moreover, since $|\nabla u| = 0$ outside Ω, the later inequality trivially holds provided $B_{4\rho}(y) \subset \mathbb{R}^n \backslash \Omega$, thus it is enough to consider (3.55) for the case $B_{4\rho}(y) \subset \Omega$ and the case $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$.

Let us first consider the interior case: $B_{4\rho}(y) \subset \Omega$. Let $w = u + W_0^{1,p-\delta}(B_{4\rho})(y)$ be a solution, obtained from Corollary 3.10, to the problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } B_{4\rho}(y), \\ w &= u \quad \text{on } \partial B_{4\rho}(y). \end{cases}$$

By the weak type $(1,1)$ estimate for the maximal function, we have

$$\begin{aligned} &|\{\mathcal{M}(\chi_{B_{2\rho}(y)} |\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda\} \cap B_\rho(y)| \\ &\quad \leq |\{\mathcal{M}(\chi_{B_{2\rho}(y)} |\nabla w|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda/2\} \cap B_\rho(y)| \\ &\qquad + |\{\mathcal{M}(\chi_{B_{2\rho}(y)} |\nabla u - \nabla w|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda/2\} \cap B_\rho(y)| \\ &\quad \lesssim (AT\lambda)^{-(p+\delta)} \int_{B_{2\rho}(y)} |\nabla w|^{p+\delta}\, dx + (AT\lambda)^{-(p-\delta)} \int_{B_{2\rho}(y)} |\nabla u - \nabla w|^{p-\delta}\, dx. \end{aligned} \tag{3.56}$$

On the other hand, applying Theorem 3.8, we get

$$\fint_{B_{2\rho}(y)} |\nabla w|^{p+\delta}\, dx \lesssim \left(\fint_{B_{4\rho}(y)} |\nabla u|^{p-\delta}\, dx \right)^{\frac{p+\delta}{p-\delta}} + \left(\fint_{B_{4\rho}(y)} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{p+\delta}{p-\delta}},$$

whereas by (3.52)-(3.53) and Lemma 3.13 there holds

$$\fint_{B_{4\rho}(y)} |\nabla u|^{p-\delta}\, dx \lesssim \fint_{B_{5\rho}(x_0)} |\nabla u|^{p-\delta}\, dx \lesssim \lambda^{p-\delta},$$

and

$$\begin{aligned} \fint_{B_{4\rho}(y)} |\nabla u - \nabla w|^{p-\delta}\, dx &\lesssim \delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} \lambda^{p-\delta} + [\epsilon(T)^{\min\{1,p-1\}}\lambda]^{p-\delta} \\ &\lesssim \lambda^{p-\delta}\Big[\delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} + T^{-2\delta}\Big], \end{aligned} \tag{3.57}$$

where we used $B_{4\rho}(y) \subset B_{5\rho}(x_0)$ and the definition of $\epsilon(T)$.

Combining (3.56)-(3.57) we now obtain

$$\begin{aligned} &|\{\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda\} \cap B_\rho(y)| \\ &\quad\lesssim |B_\rho(y)|(AT)^{-(p+\delta)}\left[1 + \delta^{(p+\delta)\min\left\{1,\frac{1}{p-1}\right\}} + T^{-2\delta\frac{p+\delta}{p-\delta}}\right] \\ &\qquad + |B_\rho(y)|(AT)^{-(p-\delta)}\left[\delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} + T^{-2\delta}\right] \\ &\quad\lesssim |B_\rho(y)|\, A^{-(p-\delta)}T^{-(p+\delta)} + |B_\rho(y)|\, A^{-(p-\delta)}\delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} \end{aligned}$$

since $A, T > 1$ and $\delta \in (0, 1)$.

At this point, we can take A sufficiently large to get the desired estimates in the interior case $B_{4\rho}(y) \subset \Omega$.

We now look at the boundary case when $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$. Let us recall that $u \in W_0^{1,p-\delta}(\Omega)$ and let $y_0 \in \partial\Omega$ be a boundary point such that $|y - y_0| = \mathrm{dist}(\mathrm{y}, \partial\Omega)$. Define $w \in u + W_0^{1,p-\delta}(\Omega_{32\rho}(y_0))$ as a solution to the problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } \Omega_{32\rho}(y_0), \\ w &= u \quad \text{on } \partial\Omega_{32\rho}(y_0). \end{cases}$$

Here we first extend u to be zero on $\mathbb{R}^n \setminus \Omega$ and we then extend w to be u on $\mathbb{R}^n \setminus \Omega_{16\rho}(y_0)$. Since

$$B_{28\rho}(y) \subset B_{32\rho}(y_0) \subset B_{36\rho}(y) \subset B_{37\rho}(x_0) \subset 4B_0,$$

we then obtain by making use of Theorem 3.21,

$$
\begin{aligned}
\Big(\fint_{B_{2\rho}(y)} |\nabla w|^{p+\delta}\,dx\Big)^{\frac{p-\delta}{p+\delta}} &\lesssim \fint_{B_{28\rho}(y)} |\nabla w|^{p-\delta} dx \\
&\lesssim \fint_{B_{37\rho}(x_0)} |\nabla u|^{p-\delta}\,dx + \fint_{B_{32\rho}(y_0)} |\nabla u - \nabla w|^{p-\delta}\,dx.
\end{aligned}
$$

Now using (3.52)-(3.53) and Lemma 3.24 in (3.56), we obtain the desired estimate in the boundary case. □

The above proposition can be restated in the following way.

Proposition 3.27. *There exists a constant $A = A_{(n,p,b,\Lambda_0,\Lambda_1,\gamma)} > 1$ such that the following holds for any $T > 1$ and any $\lambda > 0$: Let $u \in W_0^{1,p-\delta}(\Omega)$ be a solution of (3.1) with $\mathcal{A}$ satisfying Hypothesis 3.1. Fix a ball $B_0 = B_{R_0}$, and suppose that for some ball $B_\rho(y)$ with $\rho \leq \min\{r_0, 2R_0\}/26$ we have*

$$
|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > AT\lambda\} \cap B_\rho(y)| \geq H\,|B_\rho(y)|,
$$

then there holds

$$
B_\rho(y) \cap B_0 \subset \{\mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > \lambda\} \cup \{\mathcal{M}(\chi_{4B_0}|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}} > \epsilon(T)\lambda\}.
$$

Here $\epsilon(T)$ and $H = H(T)$ are as defined in Proposition 3.26.

We can now apply Lemma 2.35 and the previous proposition to get the following result.

Lemma 3.28. *There exists a constant $A = A_{(n,p,b,\Lambda_0,\Lambda_1,\gamma)} > 1$ such that the following holds for any $T > 2$. Let u be a solution of (3.1) and let B_0 be a ball of radius R_0. Fix a real number $0 < r \leq \min\{r_0, 2R_0\}/26$ and suppose that there exists $N > 0$ such that*

$$
|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N\}| < H\,r^n|B_1|. \tag{3.58}
$$

Then for any integer $k \geq 0$ there holds

$$
\begin{aligned}
&|\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^{k+1}\}| \\
&\quad \leq c_{(n)}\,H\,|\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^{k}\}| \\
&\qquad + c_{(n)}\,|\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \epsilon(T)N(AT)^{k}\}|.
\end{aligned}
$$

Here $\epsilon(T)$ and $H = H(T)$ are as defined in Proposition 3.26.

Proof. Let A be as in Proposition 3.27 and set

$$C = \{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^{k+1}\}, \quad D = D_1 \cap B_0,$$

where D_1 is the union

$$D_1 = \{\mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^k\} \cup \{\mathcal{M}(\chi_{4B_0}|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \epsilon(T)N(AT)^k\},$$

with $\epsilon(T)$ and H being as defined in Proposition 3.26.

Since $AT > 1$ the assumption (3.58) implies that $|C| < H\, r^n |B_1|$. Moreover, if $x \in B_0$ and $\rho \in (0, r]$ such that $|C \cap B_\rho(x)| \geq H\, |B_\rho(x)|$, then using Proposition 3.27 with $\lambda = N(AT)^k$ we have

$$B_\rho(x) \cap B_0 \subset D.$$

Thus the hypotheses of Lemma 2.35 are satisfied with $E = B_0$ and $\epsilon = H \in (0, 1)$. This yields

$$\begin{aligned} |C| &\leq c(n)\, H\, |D| \\ &\leq c(n)\, H\, |\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^k\}| \\ &\quad + c(n)\, |\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \epsilon(T)N(AT)^k\}| \end{aligned}$$

as desired. $\square$

Using Lemma 3.28, we can now obtain a gradient estimate in Lorentz spaces over every ball centered in the domain.

Theorem 3.29. *Let all the Hypothesis in 3.1 and 3.2 be satisfied, then, with δ as in Remark 3.25, for any $p - \delta/2 \leq q \leq p + \delta/2$, $0 < t \leq \infty$ and for any very weak solution solution $u \in W_0^{1,p-\delta}(\Omega)$ to (3.1), there holds*

$$\|\nabla u\|_{L(q,t)(B_0)} \leq C|B_0|^{\frac{1}{q}} \|\nabla u\|_{L^{p-\delta}(4B_0)} [\min\{r_0, 2R_0\}]^{\frac{-n}{p-\delta}} + C\|\mathbf{f}\|_{L(q,t)(4B_0)}.$$

Here the constant $C = C_{(n,p,t,\gamma,\Lambda_0,\Lambda_1,b)}$ and $B_0 = B_{R_0}(z_0)$ is any ball with $z_0 \in \Omega$ and $R_0 > 0$.

Proof. Let B_0 be a ball of radius $R_0 > 0$ and set $r = \min\{r_0, 2R_0\}/26$. As usual we set u and $\mathbf{f}$ to be zero in $\mathbb{R}^n \setminus \Omega$. In what follows we consider only the case $t \neq \infty$ as for $t = \infty$ the proof is similar. Moreover, to prove the theorem, we may assume that

$$\|\nabla u\|_{L^{p-\delta}(B_0)} \neq 0.$$

For $T > 2$ to be determined, we claim that there exists $N > 0$ such that

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N\}| < H\, r^n |B_1|.$$

with $H = H(T)$ being as in Proposition 3.26. To see this, we first use the weak type $(1, 1)$ estimate for the maximal function to get

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N\}| < \frac{C_{(n)}}{N^{p-\delta}} \int_{4B_0} |\nabla u|^{p-\delta}\, dx.$$

Then we choose $N > 0$ so that

$$\frac{C_{(n)}}{N^{p-\delta}} \int_{4B_0} |\nabla u|^{p-\delta}\, dx = H\, r^n\, |B_1|. \tag{3.59}$$

Let A and $\epsilon(T)$ be as in Proposition 3.26. For $0 < t < \infty$, we now consider the sum

$$S = \sum_{k=1}^{\infty} (AT)^{tk} |\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u/N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > (AT)^k\}|^{\frac{t}{q}}.$$

By Lemma 2.18, we have

$$C^{-1}\, S \leq \left\| \mathcal{M}(\chi_{4B_0}|\nabla u/N|^{p-\delta})^{\frac{1}{p-\delta}} \right\|^t_{L(q,\,t)(B_0)} \leq C\, (|B_0|^{\frac{t}{q}} + S).$$

We next evaluate S by making use of Lemma 3.28 as follows:

$$\begin{aligned} S &\leq c \sum_{k=1}^{\infty} (AT)^{tk} \Big\{ H\, |\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u/N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > (AT)^{k-1}\}| \\ &\qquad + |\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\mathbf{f}/N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \epsilon(T)(AT)^{k-1}\}| \Big\}^{\frac{t}{q}} \\ &\leq c\, (AT)^t H^{\frac{t}{q}} (S + |B_0|^{\frac{t}{q}}) + c\, \|\mathcal{M}(\chi_{4B_0}|\mathbf{f}/N|^{p-\delta})^{\frac{1}{p-\delta}}\|^t_{L(q,\,t)(B_0)}. \end{aligned}$$

At this point we choose T large enough and δ small so that

$$c\, (AT)^t H^{\frac{t}{q}} = c\, (AT)^t \left(T^{-(p+\delta)} + \delta^{(p-\delta)\min\left\{1, \frac{1}{p-1}\right\}} \right)^{\frac{t}{q}} \leq 1/2.$$

This is possible as $q \le p + \delta/2$, and moreover, T can be chosen to be independent of q. We then obtain

$$S \lesssim |B_0|^{\frac{t}{q}} + \|\mathcal{M}(\chi_{4B_0}|\mathbf{f}/N|^{p-\delta})^{\frac{1}{p-\delta}}\|^t_{L(q,\,t)(B_0)}.$$

Now applying the boundedness property of the maximal function $\mathcal{M}$ and recalling N from (3.59), we finally get

$$\begin{aligned}\|\nabla u\|_{L(q,t)(B_0)} &\lesssim |B_0|^{\frac{1}{q}} N + \|\mathbf{f}\|_{L(q,t)(4B_0)} \\ &\lesssim |B_0|^{\frac{1}{q}} \|\nabla u\|_{L^{p-\delta}(4B_0)} r^{\frac{-n}{p-\delta}} + \|\mathbf{f}\|_{L(q,t)(4B_0)}.\end{aligned}$$

□

3.4 Proof of Main Theorem

We are now ready to prove the main result of this chapter.

Proof of Theorem 3.3. Let $\delta > 0$ be as in Remark 3.25, and let $B_0 = B_{R_0}(z_0)$, where $z_0 \in \Omega$ and $0 < R_0 \le \mathrm{diam}(\Omega)$. We shall prove the theorem with $\delta/2$ in place of δ. Hence, we assume that $p - \delta/2 \le q \le p + \delta/2$, $\theta \in [p-\delta, n]$, and $u \in W_0^{1,p-\delta}(\Omega)$. By Theorem 3.29, we have

$$\begin{aligned}\|\nabla u\|_{L(q,t)(B_0)} &\lesssim |B_0|^{\frac{1}{q}} \|\nabla u\|_{L^{p-\delta}(4B_0)} \left[\min\{r_0, 2R_0\}\right]^{-n/(p-\delta)} + \|\mathbf{f}\|_{L(q,t)(4B_0)} \\ &\lesssim |B_0|^{\frac{1}{q}} \|\nabla u\|_{L^{p-\delta}(4B_0)} \left[\min\{r_0, 2R_0\}\right]^{-n/(p-\delta)} + R_0^{\frac{n-\theta}{q}} \|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)},\end{aligned} \tag{3.60}$$

where the second inequality follows from just the definition of Morrey spaces.

To continue we consider the following two cases.

Case (i). $\dfrac{r_0}{8} < R_0 \le \mathrm{diam}(\Omega)$: By using (3.60) and the inequality

$$\begin{aligned}\int_{4B_0} |\nabla u|^{p-\delta}\, dx &\le C \int_{\Omega} |\mathbf{f}|^{p-\delta}\, dx \\ &\le C\, \mathrm{diam}(\Omega)^{n - \frac{n(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{L(q,t)(\Omega)} \\ &\le C\, \mathrm{diam}(\Omega)^{n - \frac{\theta(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^{\theta}(q,t)(\Omega)},\end{aligned}$$

which follows from Theorem 3.17 and Hölder's inequality, we get

$$\begin{aligned}\|\nabla u\|_{L(q,t)(B_0)} &\lesssim R_0^{n/q}\|\nabla u\|_{L^{p-\delta}(4B_0)} r_0^{-n/(p-\delta)} + R_0^{\frac{n-\theta}{q}}\|\mathbf{f}\|_{\mathcal{L}^\theta(q,t)(\Omega)} \\ &\lesssim R_0^{n/q}\mathrm{diam}(\Omega)^{-\theta/q}[\mathrm{diam}(\Omega)/r_0]^{\frac{n}{(p-\delta)}}\|\mathbf{f}\|_{\mathcal{L}^\theta(q,t)(\Omega)} + R_0^{\frac{n-\theta}{q}}\|\mathbf{f}\|_{\mathcal{L}^\theta(q,t)(\Omega)} \\ &\lesssim R_0^{\frac{n-\theta}{q}}\|\mathbf{f}\|_{\mathcal{L}^\theta(q,t)(\Omega)}\left\{[\mathrm{diam}(\Omega)/r_0]^{\frac{n}{(p-\delta)}}+1\right\}.\end{aligned} \tag{3.61}$$

Case (ii). $0 < R_0 \leq \min\left\{\frac{r_0}{8}, \mathrm{diam}(\Omega)\right\}$: From (3.60), we have

$$\|\nabla u\|_{L(q,t)(B_0)} \lesssim R_0^{n/q}\|\nabla u\|_{L^{p-\delta}(4B_0)} R_0^{-n/(p-\delta)} + \|\mathbf{f}\|_{L(q,t)(B_0)}. \tag{3.62}$$

We next aim to estimate the first term on the right-hand side of (3.62). To that end, let $r \in (0, r_0]$. If $B_{r/4}(z_0) \subset \Omega$ we let $w \in u + W_0^{1,p-\delta}(B_{r/5}(z_0))$ solve

$$\begin{cases} \mathrm{div}\ \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } B_{r/5}(z_0), \\ w &= u \quad \text{on } \partial B_{r/5}(z_0). \end{cases}$$

Otherwise, i.e., $B_{r/4}(z_0) \cap \partial\Omega \neq \emptyset$, we let $w \in u + W_0^{1,p-\delta}(\Omega_{r_0/2}(x_0))$ be a solution to

$$\begin{cases} \mathrm{div}\ \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } \Omega_{r/2}(x_0), \\ w &= u \quad \text{on } \partial\Omega_{r/2}(x_0). \end{cases}$$

Here $x_0 \in \partial\Omega \cap B_{r/4}(z_0)$ is chosen so that $|z_0 - x_0| = \mathrm{dist}(z_0, \partial\Omega)$, and thus it follows that $B_{r_0/5}(z_0) \Subset B_{r/2}(x_0) \subset B_{3r/4}(z_0)$. The existence of w follows from Corollary 3.10 or Corollary 3.18. In any case, by Lemmas 3.12 and 3.23 for any $0 < \rho \leq r/5$ we have

$$\int_{B_\rho(z_0)} |\nabla w|^{p-\delta}\, dx \lesssim (\rho/r)^{n+(p-\delta)(\beta_0-1)} \int_{B_{r/5}(z_0)} |\nabla w|^{p-\delta}\, dx,$$

where $\beta_0 = \beta_{0(n,p,b,\Lambda_0,\Lambda_1)} \in (0, 1/2]$ is the smallest of those found in Lemmas 3.12 and 3.23.

Hence, when $p \geq 2$, we get from Lemmas 3.13 and 3.24 that

$$\begin{aligned}\int_{B_\rho(z_0)} |\nabla u|^{p-\delta} &\lesssim \int_{B_\rho(z_0)} |\nabla w|^{p-\delta}\, dx + \int_{B_\rho(z_0)} |\nabla u - \nabla w|^{p-\delta}\, dx \\ &\lesssim \left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} \int_{B_{r/5}(z_0)} |\nabla w|^{p-\delta}\, dx + \int_{B_{r/5}(z_0)} |\nabla u - \nabla w|^{p-\delta}\, dx \\ &\lesssim \left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} \int_{B_{r/5}(z_0)} |\nabla w|^{p-\delta}\, dx + \\ &\qquad + \delta^{\frac{p-\delta}{p-1}} \int_{B_{3r/4}(z_0)} |\nabla u|^{p-\delta}\, dx + \int_{B_{3r/4}(z_0)} |\mathbf{f}|^{p-\delta}\, dx.\end{aligned}$$

Similarly, in the case $1 < p < 2$, using Lemmas 3.13 and 3.24 and Young's inequality we find, for any $\epsilon > 0$,

$$\int_{B_\rho(z_0)} |\nabla u|^{p-\delta} \lesssim \left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} \int_{B_{r/5}(z_0)} |\nabla w|^{p-\delta}\, dx + \\ + (\delta^{p-\delta} + \epsilon) \int_{B_{3r/4}(z_0)} |\nabla u|^{p-\delta}\, dx + C_{(\epsilon)} \int_{B_{3r/4}(z_0)} |\mathbf{f}|^{p-\delta}\, dx.$$

Therefore, if we denote by

$$\phi(\rho) = \int_{B_\rho(z_0)} |\nabla u|^{p-\delta}\, dx,$$

then we have

$$\phi(\rho) \lesssim \left[\left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} + \delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} + \epsilon \right] \phi\left(\frac{3r}{4}\right) + C_{(\epsilon)} \int_{B_{3r/4}(z_0)} |\mathbf{f}|^{p-\delta}\, dx, \tag{3.63}$$

which holds for all $\epsilon > 0$ and $\rho \in (0, r/5]$. By enlarging the constant if necessary, we see that (3.63) actually holds for all $\rho \in (0, 3r/4]$.

On the other hand, by Hölder's inequality there holds

$$\int_{B_{3r/4}(z_0)} |\mathbf{f}|^{p-\delta}\, dx \lesssim r^{n-\frac{n(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{L(q,t)(B_{3r/4}(z_0))} \lesssim r^{n-\frac{\theta(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^\theta(q,t)(\Omega)},$$

and thus (3.63) yields

$$\phi(\rho) \lesssim \left[\left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} + \delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} + \epsilon \right] \phi(3r/4) + C_{(\epsilon)}\, r^{n-\frac{\theta(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^\theta(q,t)(\Omega)} \tag{3.64}$$

for all $\rho \in (0, 3r/4]$. Since $\theta \in [p-\delta, n]$ and $q \in [p-\delta/2, p+\delta/2]$, we have

$$0 \le n - \frac{\theta(p-\delta)}{q} < n + (p-\delta)(\beta_0 - 1), \tag{3.65}$$

as long as we restrict $\delta < 2p\beta_0/(1+\beta_0)$. Note that the constant hidden in $\lesssim$ in (3.64) depends only on $n, p, \Lambda_0, \Lambda_1, \gamma$, and b. Thus using (3.64) and (3.65), we can now apply Lemma 3.4 from [23] to obtain a $\overline{\delta} = \overline{\delta}_{(n,p,\Lambda_0,\Lambda_1,\gamma,b)} > 0$ such that

$$\phi(\rho) \lesssim \left(\frac{\rho}{r}\right)^{n-\frac{\theta(p-\delta)}{q}} \phi(3r/4) + \rho^{n-\frac{\theta(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^\theta(q,t)(\Omega)}$$

provided we further restrict $\delta < \overline{\delta}$. Since this estimate holds for all

$$0 < \rho \leq 3r/4 \leq 3r_0/4,$$

we can choose $\rho = 4R_0 \leq \dfrac{r_0}{2}$ and $r = r_0$ to arrive at

$$\phi(4R_0) \lesssim \left(\frac{R_0}{r_0}\right)^{n-\frac{\theta(p-\delta)}{q}} \phi(3r_0/4) + R_0^{n-\frac{\theta(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^\theta(q,t)(\Omega)}. \tag{3.66}$$

Substituting (3.66) into (3.62), we find

$$\begin{aligned} \|\nabla u\|_{L(q,t)(B_0)} &\lesssim R_0^{\frac{n-\theta}{q}} r_0^{\frac{\theta}{q}-\frac{n}{p-\delta}} \|\nabla u\|_{L^{p-\delta}(\Omega)} + R_0^{\frac{n-\theta}{q}} \|\mathbf{f}\|_{\mathcal{L}^\theta(q,t)(\Omega)} \\ &\lesssim R_0^{\frac{n-\theta}{q}} r_0^{\frac{\theta}{q}-\frac{n}{p-\delta}} \|\mathbf{f}\|_{L^{p-\delta}(\Omega)} + R_0^{\frac{n-\theta}{q}} \|\mathbf{f}\|_{\mathcal{L}^\theta(q,t)(\Omega)}, \end{aligned} \tag{3.67}$$

where we used Theorem 3.17 in the last inequality. Thus using Hölder's inequality in (3.67) we get

$$\|\nabla u\|_{L(q,t)(B_0)} \lesssim R_0^{\frac{n-\theta}{q}} \|\mathbf{f}\|_{\mathcal{L}^\theta(q,t)(\Omega)} \left\{ \left(\operatorname{diam}(\Omega)/r_0\right)^{\frac{n}{p-\delta}-\frac{\theta}{q}} + 1 \right\}. \tag{3.68}$$

Finally, combining the decay estimates (3.61) and (3.68) for $\|\nabla u\|_{L(q,t)(B_0)}$ in both cases we arrive at the desired Morrey space estimate. $\square$

Chapter 4
Global Weighted estimates in Lorentz Spaces

One of the main goals of this Chapter is to obtain global gradient weighted estimates of the form

$$\int_{\Omega} |\nabla u|^p w dx \leq C \int_{\Omega} |\mathbf{f}|^p w dx \tag{4.1}$$

for weights w in the Muckenhoupt class A_1 and for solutions u to the nonhomogeneous nonlinear boundary value problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

We shall state all the assumptions that we need for this chapter:

Hypothesis 4.1 (Assumption on $\mathcal{A}(x,\zeta)$)**.** *We will assume the nonlinearity $\mathcal{A}(x,\zeta)$ satisfies* (2.1) *and* (2.2)*. Along with this, we will also assume that $\mathcal{A}$ satisfies* (γ, R_0)*-BMO (see Definition 2.2) condition as quantified in Theorem 4.3.*

Hypothesis 4.2 (Assumption on Ω)**.** *We assume that Ω is a (γ, R_0)-Reifenberg flat domain (see Definition 2.3) for some (γ, R_0) as quantified in Theorem 4.3.*

4.1 Main Theorems

We are now ready to state the main results proved in this chapter.

Theorem 4.3. *Suppose that $\mathcal{A}$ satisfies Hypothesis 4.1 . Let $t \in (0, \infty]$, $q \geq p$, and let w be an $A_{q/p}$ weight. There exist constants $\tau = \tau_{(n,p,\Lambda_0,\Lambda_1)} > 1$ and $\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,q,[w]_\infty)} > 0$ such that the following holds. If $u \in W_0^{1,p}(\Omega)$ is a solution of* (4.2) *in a* (γ, R_0)*-Reifenberg flat*

This chapter previously appeared in [3]. It is reprinted by permission of Springer (see Page 109) *domain* Ω *with* $[\mathcal{A}]_\tau^{R_0} \le \gamma$*, then one has the estimate*

$$\|\nabla u\|_{L_w(q,t)(\Omega)} \le C_{(n,p,\Lambda_0,\Lambda_1,q,t,[w]_{q/p},\mathrm{diam}(\Omega)/R_0)} \|\mathbf{f}\|_{L_w(q,t)(\Omega)}.$$

Remark 4.4. *By Remark 4.18 below and Lemma 2.21, it follows that if* $\overline{\omega}$ *is an upper bound for* $[w]_{q/p}$*, i.e.,* $[w]_{q/p} \le \overline{\omega}$*, then the constants* C *and* γ *above can be chosen to depend on* $\overline{\omega}$ *instead of* $[w]_{q/p}$ *or* $[w]_\infty$*.*

Theorem 4.3 follows from Theorem 4.16 below (applied with $M = q$) and the boundedness property of the Hardy-Littlewood maximal function on weighted spaces. Its main contribution is the end-point case $q = p$, which yields inequality (4.1) for all A_1 weights w as proposed earlier. The case $q > p$ has been obtained in [43, 44] but the proofs in those papers can only yield a weak-type bound at the end-point $q = p$.

Theorem 4.16 also yields the following gradient estimate below the natural exponent for *very weak* solutions.

Theorem 4.5. *Suppose that* $\mathcal{A}$ *satisfies Hypothesis 4.1 and let* $\theta_0 \in (0, n]$ *be a fixed number. Then there exist* $\tau = \tau_{(n,p,\Lambda_0,\Lambda_1)} > 1$*,* $\delta = \delta_{(n,p,\theta_0,\Lambda_0,\Lambda_1)} > 1$*, and* $\gamma = \gamma_{(n,p,\theta_0,\Lambda_0,\Lambda_1)} > 0$ *such that the following holds: If* $u \in W_0^{1,p-\delta}(\Omega)$ *is a very weak solution of* (3.1) *in a* $(\gamma,\ R_0)$*-Reifenberg flat domain* Ω *with* $[\mathcal{A}]_\tau^{R_0} \le \gamma$ *and* $\mathbf{f} \in \mathcal{L}^\theta(q,t)(\Omega, \mathbb{R}^n)$*, then there holds:*

$$\|\nabla u\|_{\mathcal{L}^\theta(q,t)(\Omega)} \le C_{(n,p,q,t,\theta_0,\Lambda_0,\Lambda_1,\mathrm{diam}(\Omega)/R_0)} \|\mathbf{f}\|_{\mathcal{L}^\theta(q,t)(\Omega)} \tag{4.3}$$

for all $q \in (p - \delta, p]$*,* $0 < t \le \infty$ *and* $\theta_0 \le \theta \le n$*.*

The proof of Theorem 4.5 follows by first applying Theorem 4.16 (with $M = p$ in Theorem 4.16) and the weight functions

$$w(x) = \min\{|x - z|^{-n+\theta-\rho}, r^{-n+\theta-\rho}\},$$

for any $z \in \Omega$ and $r \in (0, \mathrm{diam}(\Omega)]$ and a fixed $\rho \in (0, \theta)$. Note that w is an A_1 weight with its A_1 constant $[w]_1$ being bounded from above by a constant independent of z and r. See also Remark 4.18. The rest of the proof then follows verbatim as in that of [43, Theorem 2.3].

Remark 4.6. *We mention that the sub-natural bound* (4.3) *was obtained in Theorem 3.3 (see also [2]) but with the restriction* $\theta \in [p-2\delta, n]$, *and in [28] with* $\theta = n$, *i.e., for pure Lebesgue spaces only. Note also that the super-natural case* $q > p$ *has been obtained in [43, 44].*

4.2 Interior estimates

Let $u \in W_0^{1,p-\delta}(\Omega)$ for some $\delta \in (0, \min\{1, p-1\})$ be a very weak solution to the equation

$$\operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f}$$

in a domain Ω. For each ball $B_{2R} = B_{2R}(x_0) \Subset \Omega$, we let $w \in u + W_0^{1,p-\delta}(B_{2R})$ be a very weak solution to the problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) = 0 & \text{in } B_{2R} \\ w = u & \text{on } \partial B_{2R}. \end{cases} \tag{4.4}$$

For sufficiently small δ, the existence of such w follows from the result of [28, Theorem 2]. Lemma 3.13 tells more on the integrability property of w and its relation to u by means of a comparison estimate.

Now with u as in (3.1) and w as in (4.4), we further define another function $v \in w + W_0^{1,p}(B_R)$ as the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \overline{\mathcal{A}}_{B_R}(\nabla v) = 0 & \text{in } B_R, \\ v = w & \text{on } \partial B_R, \end{cases} \tag{4.5}$$

where $B_R = B_R(x_0)$. This equation makes sense since we have good regularity for w as a consequence of Theorem 3.8. We shall now prove another useful interior difference estimate.

Lemma 4.7. *Under Hypothesis 4.1, let* $\delta \in (0, \tilde{\delta}_2)$, *where* $\tilde{\delta}_2$ *is as in Theorem 3.8 and let* w *and* v *be as in* (4.4) *and* (4.5). *For* $\tau = \dfrac{p}{\delta_0}\dfrac{(p+\delta_0)}{(p-1)}$, *there exists a constant* $C = C_{(n,p,\Lambda_0,\Lambda_1)}$ *such that*

$$\fint_{B_R} |\nabla v - \nabla w|^{p-\delta}\, dx \le C\Big(\fint_{B_R} \Upsilon(\mathcal{A}, B_R)(x)^{\tau}\, dx \Big)^{\min\{p-\delta, \frac{p-\delta}{p-1}\}/\tau} \Big(\fint_{B_{2R}} |\nabla w|^{p-\delta}\, dx \Big).$$

Proof. Using (2.1) and the fact that both v and w are solutions, we have

$$\begin{aligned}
{}_{B_R}(|\nabla v|^2+|\nabla w|^2)^{\frac{p-2}{2}}|\nabla w-\nabla v|^2\,dx &\lesssim {}_{B_R}\langle \overline{\mathcal{A}}_{B_R}(\nabla w)-\overline{\mathcal{A}}_{B_R}(\nabla v),\nabla w-\nabla v\rangle\,dx\\
&= C\ {}_{B_R}\langle \overline{\mathcal{A}}_{B_R}(\nabla w)-\mathcal{A}(x,\nabla w),\nabla w-\nabla v\rangle\,dx\\
&\lesssim {}_{B_R}\Upsilon(\mathcal{A},B_R)(x)|\nabla w|^{p-1}|\nabla w-\nabla v|\,dx.
\end{aligned}$$

Using Hölder's inequality with exponents p, $\dfrac{p+\delta_0}{p-1}$, and τ we get

$$\begin{aligned}
&{}_{B_R}(|\nabla v|^2+|\nabla w|^2)^{\frac{p-2}{2}}|\nabla w-\nabla v|^2\,dx\\
&\qquad\lesssim \left({}_{B_R}\Upsilon(\mathcal{A},B_R)(x)^\tau\,dx\right)^{\frac{1}{\tau}}\left({}_{B_R}|\nabla w|^{p+\delta_0}\,dx\right)^{\frac{p-1}{p+\delta_0}}\left({}_{B_R}|\nabla w-\nabla v|^p\,dx\right)^{\frac{1}{p}}\\
&\qquad\lesssim \left({}_{B_R}\Upsilon(\mathcal{A},B_R)(x)^\tau\,dx\right)^{\frac{1}{\tau}}\left({}_{B_{2R}}|\nabla w|^{p-\delta}\,dx\right)^{\frac{p-1}{p-\delta}}\left({}_{B_R}|\nabla w-\nabla v|^p\,dx\right)^{\frac{1}{p}},
\end{aligned}\tag{4.6}$$

where the last inequality follows from (3.8) of Theorem 3.8.

Thus for $p\geq 2$, using pointwise estimate

$$|\nabla w-\nabla v|^p\leq (|\nabla v|^2+|\nabla w|^2)^{\frac{p-2}{2}}|\nabla w-\nabla v|^2,$$

we find

$$\left({}_{B_R}|\nabla w-\nabla v|^p\,dx\right)^{\frac{p-1}{p}}\lesssim \left({}_{B_R}\Upsilon(\mathcal{A},B_R)^\tau\,dx\right)^{\frac{1}{\tau}}\left({}_{B_{2R}}|\nabla w|^{p-\delta}\,dx\right)^{\frac{p-1}{p-\delta}}.$$

By Hölder's inequality this yields the desired estimate in the case $p\geq 2$.

For $1<p<2$ we write

$$|\nabla v-\nabla w|^p=(|\nabla v|^2+|\nabla w|^2)^{\frac{(p-2)p}{4}}|\nabla w-\nabla v|^p(|\nabla v|^2+|\nabla w|^2)^{\frac{(2-p)p}{4}},$$

and apply Hölder's inequality with exponents $\dfrac{2}{p}$ and $\dfrac{2}{2-p}$ to obtain

$$
\begin{aligned}
\fint_{B_R} |\nabla w - \nabla v|^p \, dx \leq & \left(\fint_{B_R} (|\nabla v|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla w - \nabla v|^2 \, dx \right)^{\frac{p}{2}} \times \\
& \times \left(\fint_{B_R} (|\nabla v|^2 + |\nabla w|^2)^{\frac{p}{2}} \, dx \right)^{\frac{2-p}{2}} \\
\lesssim & \left(\fint_{B_R} \Upsilon(\mathcal{A}, B_R)^\tau \, dx \right)^{\frac{p}{2\tau}} \left(\fint_{B_{2R}} |\nabla w|^{p-\delta} \, dx \right)^{\frac{(p-1)p}{(p-\delta)2}} \times \\
& \times \left(\fint_{B_R} |\nabla w - \nabla v|^p \, dx \right)^{\frac{1}{2}} \left(\fint_{B_R} |\nabla w|^p \, dx \right)^{\frac{2-p}{2}}.
\end{aligned}
$$

Here we used (4.6) and the easy energy bound

$$
\int_{B_R} |\nabla v|^p dx \leq c \int_{B_R} |\nabla w|^p dx
$$

in the last inequality. Using (3.8) of Theorem 3.8 yields

$$
\fint_{B_R} |\nabla w - \nabla v|^p \, dx \lesssim \left(\fint_{B_R} \Upsilon(\mathcal{A}, B_R)^\tau \, dx \right)^{\frac{p}{\tau}} \left(\fint_{B_{2R}} |\nabla w|^{p-\delta} \, dx \right)^{\frac{p}{p-\delta}}.
$$

Now an application of Hölder's inequality gives the desired estimate. □

Corollary 4.8. *Under Hypothesis 4.1, let* $\tau = \dfrac{p}{\tilde{\delta}_2} \dfrac{(p+\tilde{\delta}_2)}{(p-1)}$ *and* $\delta \in (0, \tilde{\delta}_2)$, *where* $\tilde{\delta}_2$ *is as in Theorem 3.8. Then for any* $\epsilon > 0$, *there exists* $\gamma = \gamma_{(\epsilon)} > 0$ *such that if* $u \in W_0^{1,p-\delta}(\Omega)$ *is a very weak solution of* (3.1) *satisfying*

$$
\fint_{B_{2R}} |\nabla u|^{p-\delta} \, dx \leq 1, \qquad \fint_{B_{2R}} |\mathbf{f}|^{p-\delta} \, dx \leq \gamma^{p-\delta} \text{ and } \qquad \fint_{B_R} \Upsilon(\mathcal{A}, B_R)^\tau dx \leq \gamma^\tau,
$$

for a ball $B_{2R} \Subset \Omega$, *then there exists* $v \in W^{1,p}(B_R) \cap W^{1,\infty}(B_{R/2})$ *such that*

$$
\fint_{B_R} |\nabla u - \nabla v|^{p-\delta} dx \leq \epsilon^{p-\delta}, \text{ and } \|\nabla v\|_{L^\infty(B_{R/2})} \leq C_0 = C_0(n, p, \Lambda_0, \Lambda_1).
$$

Proof. Let w and v solve (4.4) and (4.5) respectively. Since we have $v \in W^{1,p}(B_R)$, standard regularity theory gives (see, e.g., [56])

$$
\begin{aligned}
\|\nabla v\|^p_{L^\infty(B_{R/2})} & \lesssim \fint_{B_R} |\nabla v|^p dx \lesssim \fint_{B_R} |\nabla w|^p dx \\
& \lesssim \left(\fint_{B_{2R}} |\nabla w|^{p-\delta} dx \right)^{\frac{p}{p-\delta}} \lesssim \left(\fint_{B_{2R}} |\nabla u|^{p-\delta} dx \right)^{\frac{p}{p-\delta}} \leq C_0.
\end{aligned}
$$

Here we have applied Theorem 3.8. The proof of the corollary now follows from the comparison estimate in Lemma 3.13 and Lemma 4.7. $\square$

4.3 Boundary estimates

We now consider the corresponding local estimates near the boundary. Suppose that the domain Ω is (γ, R_0)-Reifenberg flat with $\gamma < 1/2$. Let $x_0 \in \partial\Omega$ and $R \in (0, R_0/20)$ and let $u \in W_0^{1,p-\delta}(\Omega)$ be a very weak solution to (3.1) for some $\delta \in (0, \min\{1, p-1\})$. On $\Omega_{20R} = \Omega_{20R}(x_0) = B_{20R}(x_0) \cap \Omega$, we let $w(x)$ be a very weak solution to the problem:

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } \Omega_{20R}, \\ w &\in u + W_0^{1,p-\delta}(\Omega_{20R}(x_0)). \end{cases} \tag{4.7}$$

We now extend u by zero to $\mathbb{R}^n \setminus \Omega$ and then extend w by u to $\mathbb{R}^n \setminus \Omega_{20R}(x_0)$.

Remark 4.9. *Analogous to Lemma 3.13, we have the boundary counterpart given in Lemma 3.24. This will imply the boundary analogue of Lemma 4.7 as given in Lemma 4.10.*

With $x_0 \in \partial\Omega$ and $0 < R < R_0/20$ as above, we now set $\rho = R(1-\gamma)$. Here γ is from the definition of (γ, R)-Reifenberg flat condition that we have assumed on Ω.

With this ρ and thanks to the existence and regularity of w in Theorem 3.21, we define another function $v \in w + W_0^{1,p}(\Omega_\rho(0))$ as the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \overline{\mathcal{A}}_{B_\rho}(\nabla v) &= 0 \quad \text{in } \Omega_\rho(0), \\ v &= w \quad \text{on } \partial\Omega_\rho(0). \end{cases} \tag{4.8}$$

We then set v to be equal to w in $\mathbb{R}^n \setminus \Omega_\rho(0)$. The following boundary difference estimate can be proved in a way just similar to the proof of Lemma 4.7.

Lemma 4.10. *Under Hypothesis 4.1, let $\delta \in (0, \delta_2)$, where δ_2 is in Theorem 3.21 and let w and v be as in (4.7) and (4.8). For $\tau = \dfrac{p}{\delta_2}\dfrac{(p+\delta_2)}{(p-1)}$, there exists a constant $C = C_{(n,p,\Lambda_0,\Lambda_1)}$ such that*

$$\fint_{B_\rho(0)} |\nabla v - \nabla w|^{p-\delta}\, dx \le C \left(\fint_{B_\rho(0)} \Upsilon(\mathcal{A}, B_\rho(0))(x)^\tau \, dx \right)^{\min\{p-\delta, \frac{p-\delta}{p-1}\}/\tau} \left(\fint_{B_{14\rho}(0)} |\nabla w|^{p-\delta}\, dx \right).$$

As the boundary of Ω can be very irregular, the L^∞-norm of ∇v up to the boundary of Ω could be unbounded. Therefore, we consider another equation:

$$\begin{cases} \operatorname{div} \overline{\mathcal{A}}_{B_\rho}(\nabla V) &= 0 \quad \text{in } B_\rho^+(0), \\ \qquad\qquad V &= 0 \quad \text{on } T_\rho, \end{cases} \tag{4.9}$$

where T_ρ is the flat portion of $\partial B_\rho^+(0)$. A function $V \in W^{1,p}(B_\rho^+(0))$ is a weak solution of (4.9) if its zero extension to $B_\rho(0)$ belongs to $W^{1,p}(B_\rho(0))$ and if

$$\int_{B_\rho^+(0)} \overline{\mathcal{A}}_{B_\rho}(\nabla V) \cdot \nabla \phi \, dx = 0$$

for all $\phi \in W_0^{1,p}(B_\rho^+(0))$.

We shall now need the following key perturbation result obtained earlier in [50, Theorem 2.12].

Theorem 4.11 ([50])**.** *Suppose that $\mathcal{A}$ satisfies Hypothesis 4.1. For any $\epsilon > 0$, there exists a small $\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,\epsilon)} > 0$ such that if $v \in W^{1,p}(\Omega_\rho(0))$ is a solutions of (4.8) under the geometric setting in Remark 2.4, then there exists a weak solution $V \in W^{1,p}(B_\rho^+(0))$ of (4.9) whose zero extension to $B_\rho(0)$ satisfies*

$$\|\nabla V\|^p_{L^\infty(B_{\rho/4}(0))} \leq C_{(n,p,\Lambda_0,\Lambda_1)} \fint_{B_\rho(0)} |\nabla v|^p \, dx,$$

and

$$\fint_{B_{\rho/8}(0)} |\nabla v - \nabla V|^p \, dx \leq \epsilon^p \fint_{B_\rho(0)} |\nabla v|^p \, dx.$$

We now have the boundary analogue of Corollary 4.8. The proof of the following corollary follows with obvious modification as in [44, Corollary 2.10].

Corollary 4.12 ([44])**.** *For any $\epsilon > 0$, there exists constants $\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,\epsilon)} > 0$ and $\tilde{\delta}_1 = \tilde{\delta}_{1(n,p,\Lambda_0,\Lambda_1,\epsilon)} \in (0, \delta_2)$, where δ_2 is as in Theorem 3.21, such that the following holds with $\tau = \dfrac{p}{\delta_2}\dfrac{(p+\delta_2)}{(p-1)}$. If Ω is $(\gamma,\, R_0)$-Reifenberg flat and for any $\delta \in (0, \tilde{\delta}_1)$, let $u \in W_0^{1,p-\delta}(\Omega)$ be a very weak solution of (4.2) with*

$$\fint_{B_{20R}(x_0)} |\nabla u|^{p-\delta} \chi_\Omega \, dx \leq 1, \qquad \fint_{B_{20R}(x_0)} |\mathbf{f}|^{p-\delta} \chi_\Omega \, dx \leq \gamma^{p-\delta} \quad \text{and } [\mathcal{A}]_\tau^{R_0} \leq \gamma,$$

where $x_0 \in \partial\Omega$ *and* $R \in (0, R_0/20)$*, then there is a function*

$$V \in W^{1,\infty}(B_{R/10}(x_0))$$

such that

$$\|\nabla V\|_{L^\infty(B_{R/10}(x_0))} \le C_0 = C_{0(n,p,\Lambda_0,\Lambda_1)},$$

and

$$\int_{B_{R/10}(x_0)} |\nabla u - \nabla V|^{p-\delta}\, dx \le \epsilon^{p-\delta}. \tag{4.10}$$

Proof. With $x_0 \in \partial\Omega$ and $R \in (0, R_0/20)$, we set $\rho = R(1-\gamma)$. Also, extend both u and $\mathbf{f}$ by zero to $\mathbb{R}^n \setminus \Omega$. By Remark (2.4) and by translating and rotating if necessary, we may assume that $0 \in \Omega$, $x_0 = (0, \dots, 0, -\rho\gamma/(1-\gamma))$ and the geometric setting

$$B^+_\rho(0) \subset \Omega_\rho(0) \subset B_\rho(0) \cap \{x_n > -4\gamma\rho\}. \tag{4.11}$$

Moreover, we shall further restrict $\gamma \in (0, 1/45)$ so that we have

$$B_{R/10}(x_0) \subset B_{\rho/8}(0).$$

We now choose w and v as in (4.7) and (4.8) corresponding to these R and ρ. Then, since $B_{14\rho}(0) \subset B_{20R}(x_0)$, there holds

$$\int_{B_\rho(0)} |\nabla v|^p dx \le C \int_{B_\rho(0)} |\nabla w|^p dx \le C \left(\int_{B_{20R}(x_0)} |\nabla u|^{p-\delta} dx \right)^{\frac{p}{p-\delta}} \le C.$$

By Theorem 4.11 for any $\eta > 0$ we can find a $\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,\eta)} \in (0, 1/45)$ such that, under (4.11), there is a function $V \in W^{1,p}(B_\rho(0)) \cap W^{1,\infty}(B_{\rho/4}(0))$ such that

$$\|\nabla V\|^p_{L^\infty(B_{R/10}(x_0))} \le C \|\nabla V\|^p_{L^\infty(B_{\rho/4}(0))} \le C \int_{B_\rho(0)} |\nabla v|^p dx \le C,$$

and

$$\int_{B_{\rho/8}(0)} |\nabla v - \nabla V|^p dx \le \eta^p \int_{B_\rho(0)} |\nabla v|^p dx \le C\eta^p.$$

By Hölder's inequality, the last bound gives

$$\int_{B_{\rho/8}(0)} |\nabla v - \nabla V|^{p-\delta} dx \le C\eta^{p-\delta}. \tag{4.12}$$

Now writing

$$\underset{B_{R/10}(x_0)}{} |\nabla u - \nabla V|^{p-\delta} dx = \underset{B_{\rho/8}(0)}{} |\nabla(u-w) + \nabla(w-v) + \nabla(v-V)|^{p-\delta} dx,$$

and using (4.12) along with Theorem 3.21 and Lemma 4.10, we obtain inequality (4.10) as desired. $\square$

4.4 Weighted estimates

We now use Corollaries 4.8 and 4.12 to obtain the following technical result.

Proposition 4.13. *Under Hypothesis 4.1, there are constants $\lambda = \lambda_{(n,p,\Lambda_0,\Lambda_1)} > 1$ and $\tau = \tau_{(n,p,\Lambda_0,\Lambda_1)} > 1$ such that the following holds. For any $\epsilon > 0$, there exist constants $\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,\epsilon)} > 0$ and $\overline{\delta} = \overline{\delta}_{(n,p,\Lambda_0,\Lambda_1,\epsilon)} > 0$ such that if $u \in W_0^{1,p-\delta}(\Omega)$ with $\delta \in (0, \overline{\delta})$, is a very weak solution to (4.2) with Ω being (γ, R_0)-Reifenberg flat, $[\mathcal{A}]_\tau^{R_0} \leq \gamma$, and if, for some ball $B_\rho(y)$ with $\rho < R_0/1200$,*

$$B_\rho(y) \cap \{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) \leq 1\} \cap \{x \in \mathbb{R}^n : \mathcal{M}(|\mathbf{f}|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}(x) \leq \gamma\} \neq \emptyset, \tag{4.13}$$

then one has

$$|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)| < \epsilon\, |B_\rho(y)|. \tag{4.14}$$

Proof. By (4.13), there exists an $x_0 \in B_\rho(y)$ such that for any $r > 0$,

$$\underset{B_r(x_0)}{} |\nabla u|^{p-\delta}\, dx \leq 1 \quad \text{and} \quad \underset{B_r(x_0)}{} \chi_\Omega |\mathbf{f}|^{p-\delta}\, dx \leq \gamma^{p-\delta}. \tag{4.15}$$

By the first inequality in (4.15), for any $x \in B_\rho(y)$, there holds

$$\mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) \leq \max\left\{\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x),\, 3^n\right\}. \tag{4.16}$$

To prove (4.14), it is enough to consider the case $B_{4\rho}(y) \subset \Omega$ and the case $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$. First we consider the latter. Let $y_0 \in B_{4\rho}(y) \cap \partial\Omega$, we then have

$$B_{2\rho}(y) \subset B_{6\rho}(y_0) \subset B_{1200\rho}(y_0) \subset B_{1205\rho}(x_0).$$

Thus by (4.15) we obtain

$$\fint_{B_{1200\rho}(y_0)} |\nabla u|^{p-\delta}\, dx \le c \quad \text{and} \quad \fint_{B_{1200\rho}(y_0)} \chi_\Omega |\mathbf{f}|^{p-\delta}\, dx \le c\,\gamma^{p-\delta},$$

where $c = (1205/1200)^n$. Since $60\rho < R_0/20$, by Corollary 4.12 (with $R = 60\rho$), there exists a $\tau = \tau_{(n,p,\Lambda_0,\Lambda_1)} > 1$ such that the following holds. For any $\eta \in (0,1)$, there are constants $\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,\eta)} > 0$ and $\overline{\delta} = \overline{\delta}_{(n,p,\Lambda_0,\Lambda_1,\eta)} > 0$ such that if Ω is a (γ, R_0)-Reifenberg flat domain and $[\mathcal{A}]_\tau^{R_0} \le \gamma$, then one can find a function $V \in W^{1,\infty}(B_{6\rho}(y_0))$ with

$$\|\nabla V\|_{L^\infty(B_{2\rho}(y))} \le \|\nabla V\|_{L^\infty(B_{6\rho}(y_0))} \le C_0, \tag{4.17}$$

and, for $\delta \in (0, \overline{\delta})$,

$$\fint_{B_{2\rho}(y)} |\nabla u - \nabla V|^{p-\delta}\, dx \le C \fint_{B_{6\rho}(y_0)} |\nabla u - \nabla V|^{p-\delta}\, dx \le C\,\eta^{p-\delta}. \tag{4.18}$$

In view of (4.16) and (4.17), we see that for $\lambda = \max\{3^n, 2C_0\}$,

$$\begin{aligned}
&\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y) \subset \\
&\qquad \subset \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y) \\
&\qquad \subset \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u - \nabla V|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda/2\} \cap B_\rho(y).
\end{aligned}$$

Thus by the weak-type $(1,1)$ Maximal function inequality and (4.18), we find

$$\begin{aligned}
|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)| &\le \frac{C}{\lambda^{p-\delta}} \int_{B_{2\rho}(y)} |\nabla u - \nabla V|^{p-\delta}\, dx \\
&\le \frac{C}{C_0^{p-\delta}} |B_{2\rho}(y)|\, \eta^{p-\delta}.
\end{aligned}$$

This gives the estimate (4.14) in the case $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$, provided η is appropriately chosen. The interior case $B_{4\rho}(y) \subset \Omega$ can be obtained in a similar was by using Corollary 4.8, instead of Corollary 4.12. $\square$

Proposition 4.13 can now be used to obtain the following result which involves A_∞ weights.

Proposition 4.14. *Under Hypothesis 4.1, there exists $\lambda = \lambda_{(n,p,\Lambda_0,\Lambda_1)} > 1$ and $\tau = \tau_{(n,p,\Lambda_0,\Lambda_1)} > 1$ such that the following holds: for any $w \in A_\infty$ and $\epsilon > 0$, there exist $\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,\epsilon,[w]_\infty)} > 0$*

and $\overline{\delta} = \overline{\delta}_{(n,p,\Lambda_0,\Lambda_1,\epsilon,[w]_\infty)} > 0$ *such that if* $u \in W_0^{1,p-\delta}(\Omega)$ *with* $\delta \in (0,\overline{\delta})$, *is a very weak solution of* (4.2) *with* Ω *being* (γ, R_0)-*Reifenberg flat,* $[\mathcal{A}]_\tau^{R_0} \leq \gamma$, *and if, for some ball* $B_\rho(y)$ *with* $\rho < R_0/1200$,

$$w(\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)) \geq \epsilon\, w(B_\rho(y)),$$

then one has

$$B_\rho(y) \subset \{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\} \cup \{x \in \mathbb{R}^n : \mathcal{M}(|\mathbf{f}|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\}. \tag{4.19}$$

Proof. Suppose that (Ξ_0, Ξ_1) is a pair of A_∞ constants of w and let λ and τ be as in Proposition 4.13. Given $\epsilon > 0$, we choose a $\gamma = \gamma_{(\Xi_0,\Xi_1,\epsilon)}$ and $\overline{\delta} = \overline{\delta}_{(\Xi_0,\Xi_1,\epsilon)}$ as in Proposition 4.13 with $[\epsilon/(2\Xi_0)]^{1/\Xi_1}$ replacing ϵ. The proof then follows by a contradiction. To that end, suppose that the inclusion in (4.19) fails for this γ, then we must have that

$$B_\rho(y) \cap \{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) \leq 1\} \cap \{x \in \mathbb{R}^n : \mathcal{M}(|\mathbf{f}|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}(x) \leq \gamma\} \neq \emptyset$$

for some $\delta \in (0, \overline{\delta})$. Hence by Proposition 4.13, if Ω is a $(\gamma,\ R_0)$-Reifenberg flat and $[\mathcal{A}]_\tau^{R_0} \leq \gamma$, there holds

$$|\{x \in \mathbb{R}^n :\ \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda \cap B_\rho(y)| \leq \left(\frac{\epsilon}{2\,\Xi_0}\right)^{1/\Xi_1} |B_\rho(y)|.$$

Thus using the A_∞ characterization of w (Lemma 2.21), we immediately get that

$$\begin{aligned} w(\{x \in \mathbb{R}^n &: \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)) \\ &\leq \Xi_0 \left[\frac{|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)|}{|B_\rho(y)|}\right]^{\Xi_1} w(B_\rho(y)) \\ &\leq \frac{\epsilon}{2} w(B_\rho(y)) < \epsilon\, w(B_\rho(y)). \end{aligned}$$

This yields a contradiction and thus the proof is complete. □

The Calderón-Zygmund decomposition type lemma 2.36 will allow us to iterate the result of Proposition 4.14 to obtain Theorem 4.15 below.

Theorem 4.15. *Under Hypothesis 4.1, let λ and τ be as in Proposition 4.14, then for any $w \in A_\infty$ and any $\epsilon > 0$, there exist $\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,\epsilon,[w]_\infty)} > 0$ and $\overline{\delta} = \overline{\delta}_{(n,p,\Lambda_0,\Lambda_1,\epsilon,[w]_\infty)} > 0$ such that the following holds: Suppose that for any solution $u \in W_0^{1,p-\delta}(\Omega)$ with $\delta \in (0, \overline{\delta})$, is a very weak solution of (3.1) in a (γ, R_0)-Reifenberg flat domain Ω, with $[\mathcal{A}]_\tau^{R_0} \leq \gamma$, suppose also that $\{B_r(y_i)\}_{i=1}^L$ is a sequence of balls with centers $y_i \in \overline{\Omega}$ and a common radius $0 < r \leq R_0/4000$ that covers Ω. If for all $i = 1, \dots, L$*

$$w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}) < \epsilon\, w(B_r(y_i)), \tag{4.20}$$

then for any $s > 0$ and any integer $k \geq 1$ there holds

$$\begin{aligned} w(\{x \in & \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^k\})^s \leq \\ & \leq \sum_{i=1}^{k} (A\epsilon)^{si}\, w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\lambda^{(k-i)}\})^s + \\ & \qquad + (A\epsilon)^{sk}\, w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^s, \end{aligned}$$

where the constant $A = A_{(n,[w]_\infty)}$.

Proof. The theorem will be proved by induction on k. Given $w \in A_\infty$ and $\epsilon > 0$, we take $\gamma = \gamma_{(\epsilon,[w]_\infty)}$ and $\overline{\delta} = \overline{\delta}_{(\epsilon,[w]_\infty)}$ as in Proposition 4.14. The case $k = 1$ follows from Proposition 4.14 and Lemma 2.36. Indeed, for $\delta \in (0, \overline{\delta})$, let

$$C = \{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}$$

$$D = \{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\} \cup \{x \in \Omega : \mathcal{M}(|\mathbf{f}|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\}.$$

Then from assumption (4.20), it follows $w(C) < \epsilon\, w(B_r(y_i))$ for all $i = 1, \dots, L$. Moreover, if $y \in \Omega$ and $\rho \in (0, 2r)$ such that $w(C \cap B_\rho(y)) \geq \epsilon\, w(B_\rho(y))$, then $0 < \rho \leq R_0/1200$ and $B_\rho(y) \cap \Omega \subset D$ by Proposition 4.14. Thus all hypotheses of Lemma 2.36 are satisfied, which yield, for a constant $B = B(n, [w]_\infty)$,

$$\begin{aligned} w(C)^s & \leq B^s\, \epsilon^s\, w(D)^s \\ & \leq B^s\, 2^s \epsilon^s\, w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^s + \\ & \qquad + B^s\, 2^s \epsilon^s\, w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\})^s \end{aligned}$$

for any given $s > 0$. This proves the case $k = 1$ with $A = 2B$. Suppose now that the conclusion of the lemma is true for some $k > 1$. Normalizing u to $u_\lambda = u/\lambda$ and $\mathbf{f}_\lambda = \mathbf{f}/\lambda$, we see that for every $i = 1, \dots, L$,

$$\begin{aligned} w(\{x \in \Omega : \mathcal{M}(|\nabla u_\lambda|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}) = \\ &= w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^2\}) \\ &\leq w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > \lambda\}) \\ &< \epsilon\, w(B_r(y_i)). \end{aligned}$$

Here we have used the fact that $\lambda > 1$ in the first inequality. Note that u_λ solves

$$\begin{cases} \operatorname{div} \tilde{\mathcal{A}}(x, \nabla u_\lambda) &= \operatorname{div} |\mathbf{f}_\lambda|^{p-2}\mathbf{f}_\lambda \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\tilde{\mathcal{A}}(x, \xi) = \mathcal{A}(x, \lambda\xi)/\lambda^{p-1}$ which obeys the same structural conditions in Hypothesis 4.1. Thus by inductive hypothesis, it follows that

$$\begin{aligned} w(\{x \in \Omega : \mathcal{M}(|\nabla u_\lambda|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^k\})^s \;\leq \\ \leq \sum_{i=1}^{k} (A\epsilon)^{si}\, w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}_\lambda|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\lambda^{(k-i)}\})^s + \\ + (A\epsilon)^{sk}\, w(\{x \in \Omega : \mathcal{M}(|\nabla u_\lambda|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^s. \end{aligned} \tag{4.21}$$

Finally, applying the case $k = 1$ to the last term in (4.21) we conclude that

$$\begin{aligned} w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^{k+1}\})^s \;\leq \\ \leq \sum_{i=1}^{k+1} (A\epsilon)^{si}\, w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\lambda^{k+1-i}\})^s \\ + \; (A\epsilon)^{s(k+1)}\, w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^s. \end{aligned}$$

This completes the proof of the theorem. □

We are now ready to obtain the main result of this section.

Theorem 4.16. *Suppose that $\mathcal{A}$ satisfies Hypothesis 4.1 and let $M > 1$ and w be an A_∞ weight. There exist constants $\tau = \tau_{(n,p,\Lambda_0,\Lambda_1)} > 1$, $\delta = \delta_{(n,p,\Lambda_0,\Lambda_1,M,[w]_\infty)} > 0$ and*

$\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,M,[w]_\infty)} > 0$ such that for any $t \in (0,\infty]$ and any $q \in (0,M]$, the following holds: If $u \in W_0^{1,p-\delta}(\Omega)$ is a very weak solution of (3.1) in a (γ, R_0)-Reifenberg flat domain Ω with $[\mathcal{A}]_\tau^{R_0} \le \gamma$, then one has the estimate

$$\|\nabla u\|_{L_w(q,t)(\Omega)} \le C\|\mathcal{M}(|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}}\|_{L_w(q,t)(\Omega)}, \tag{4.22}$$

where the constant $C = C_{(n,p,\Lambda_0,\Lambda_1,t,q,M,[w]_\infty,\mathrm{diam}(\Omega)/R_0)}$.

Remark 4.17. *The introduction of M in the above theorem is just for a technical reason. It ensures that the constant δ is independent of q as the proof of the theorem reveals.*

Remark 4.18. *It follows also from the proof of Theorem 4.16 that if (Ξ_0, Ξ_1) is pair of A_∞ constants of w such that $\max\{\Xi_0, 1/\Xi_1\} \le \overline{\varpi}$ then the constants δ, γ and C above can be chosen to depend just on the upper-bound $\overline{\varpi}$ instead of (Ξ_0, Ξ_1).*

Proof. Let $\lambda_{(n,p,\Lambda_0,\Lambda_1)}$ and $\tau_{(n,p,\Lambda_0,\Lambda_1)}$ be as in Theorem 4.15. Take $\epsilon = \lambda^{-M}A^{-1}2^{-1}$ and choose $\delta = \overline{\delta}_{(n,p,\Lambda_0,\Lambda_1,\epsilon,[w]_\infty)}/2$, where $A = A_{(n,[w]_\infty)}$ and $\overline{\delta}$ are as in Theorem 4.15; thus $\delta = \delta_{(n,p,\Lambda_0,\Lambda_1,M,[w]_\infty)}$, which is independent of q. Using Theorem 4.15 we also get a constant $\gamma = \gamma_{(n,p,\Lambda_0,\Lambda_1,M,[w]_\infty)} > 0$ for this choice of ϵ.

We shall prove (4.22) only for $t \in (0, \infty)$, as for $t = \infty$ the proof is just similar. Choose a finite number of points $\{y_i\}_{i=1}^L \subset \Omega$ and a ball B_0 of radius $2\,\mathrm{diam}(\Omega)$ such that

$$\Omega \subset \bigcup_{i=1}^{L} B_r(y_i) \subset B_0,$$

where $r = \min\{R_0/4000, \mathrm{diam}(\Omega)\}$. We claim that we can choose N large such that for $u_N = u/N$ and for all $i = 1, \dots, L$,

$$w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda) < \epsilon\, w(B_r(y_i)). \tag{4.23}$$

Indeed from the weak-type $(1,1)$ estimate for the maximal function, there exists a constant $C_{(n)} > 0$ such that

$$|\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}| < \frac{C_{(n)}}{(\lambda N)^{p-\delta}} \int_\Omega |\nabla u|^{p-\delta}\,dx.$$

If (Ξ_0, Ξ_1) is a pair of A_∞ constants of w, then using Lemma 2.21, we see that

$$w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}) < \Xi_0 \left(\frac{C_{(n)}}{(\lambda N)^{p-\delta}|B_0|} \int_\Omega |\nabla u|^{p-\delta}\, dx. \right)^{\Xi_1} w(B_0). \tag{4.24}$$

Also, there are $C_1 = C_{1(n,[w]_\infty)} \geq 1$ and $p_1 = p_{1(n,[w]_\infty)} \geq 1$ such that

$$w(B_0) \leq C_1 \left(\frac{|B_0|}{|B_r(y_i)|} \right)^{p_1} w(B_r(y_i)) \tag{4.25}$$

for every $i = 1, 2, \ldots, L$. This follows from the so-called *strong doubling property* of A_∞ weights (see, e.g., [19, Chapter 9]). In view of (4.24) and (4.25), we now choose N such that

$$\frac{C_{(n)}}{(\lambda N)^{p-\delta}|B_0|} \int_\Omega |\nabla u|^{p-\delta}\, dx = \left(\frac{|B_r(y_i)|}{|B_0|} \right)^{p_1/\Xi_1} \left(\frac{\epsilon}{\Xi_0 C_1} \right)^{1/\Xi_1}.$$

This gives the desired estimate (4.23). Note that for this N we have

$$\begin{aligned} N &\leq C|B_0|^{\frac{-1}{p-\delta}} \|\nabla u\|_{L^{p-\delta}(\Omega)} \leq C|B_0|^{\frac{-1}{p-\delta}} \|\mathbf{f}\chi_\Omega\|_{L^{p-\delta}(B_0)} \\ &\leq C\mathcal{M}(|\mathbf{f}|^{p-\delta}\chi_\Omega)(x)^{\frac{1}{p-\delta}} \end{aligned} \tag{4.26}$$

for all $x \in \Omega$. Here $C = C_{(n,p,\Lambda_0,\Lambda_1,M,[w]_\infty,\mathrm{diam}(\Omega)/R_0)}$ and the second inequality follows from Theorem 3.3.

With this N, we denote by

$$S = \sum_{k=1}^{\infty} \lambda^{tk} w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^k\})^{\frac{t}{q}}$$

and for $J \geq 1$ let

$$S_J = \sum_{k=1}^{J} \lambda^{tk} w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^k\})^{\frac{t}{q}}$$

be its partial sum. By Lemma 2.19, we see that

$$C^{-1}S \leq \|\mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}\|^t_{L_w(q,t)(\Omega)} \leq C(w(\Omega)^{\frac{t}{q}} + S). \tag{4.27}$$

By (4.23) and Theorem 4.15, we find

$$\begin{aligned} S_J \leq & \sum_{k=1}^{J} \lambda^{tk} \left[\sum_{j=1}^{k} (A\epsilon)^{\frac{t}{q}j} w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}_N|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\lambda^{(k-j)}\})^{\frac{t}{q}} \right] \\ &+ \sum_{k=1}^{J} \lambda^{tk} (A\epsilon)^{\frac{t}{q}k} w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^{\frac{t}{q}}. \end{aligned}$$

Here recall that $\epsilon = \lambda^{-M} A^{-1} 2^{-1}$ ans $A = A_{(n,[w]_\infty)}$. Now interchanging the order of summation, we get

$$
\begin{aligned}
S_J &\leq \sum_{j=1}^{J} (A\epsilon\lambda^q)^{\frac{t}{q}j} \left[\sum_{k=j}^{J} \lambda^{t(k-j)} w(\Omega \cap \{\mathcal{M}(|\mathbf{f}_N|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}} > \gamma\lambda^{(k-j)}\})^{\frac{t}{q}} \right] \\
&\quad + \sum_{k=1}^{J} (A\epsilon\lambda^q)^{\frac{t}{q}k} w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^{\frac{t}{q}} \\
&\leq C \left[\|\mathcal{M}(|\mathbf{f}_N|^{p-\delta})^{\frac{1}{p-\delta}}\|^t_{L_w(q,t)(\Omega)} + w(\Omega)^{\frac{t}{q}} \right] \sum_{j=1}^{\infty} 2^{-\frac{t}{q}j} \\
&\leq C \left[\|\mathcal{M}(|\mathbf{f}_N|^{p-\delta})^{\frac{1}{p-\delta}}\|^t_{L_w(q,t)(\Omega)} + w(\Omega)^{\frac{t}{q}} \right]
\end{aligned}
$$

for a constant $C = C_{(n,p,\Lambda_0,\Lambda_1,q,t,M,[w]_\infty)}$. Letting $J \to \infty$ and making use of (4.27), we arrive at

$$
\|\mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}\|^t_{L_w(q,t)(\Omega)} \leq C \left[\|\mathcal{M}(|\mathbf{f}_N|^{p-\delta})^{\frac{1}{p-\delta}}\|^t_{L_w(q,t)(\Omega)} + w(\Omega)^{\frac{t}{q}} \right].
$$

This gives

$$
\|\nabla u\|_{L_w(q,t)(\Omega)} \leq C \left[\|\mathcal{M}(|\mathbf{f}|^{p-\delta}\chi_\Omega)^{\frac{1}{p-\delta}}\|_{L_w(q,t)(\Omega)} + N w(\Omega)^{\frac{1}{q}} \right],
$$

which in view of (4.26) yields the desired estimate. □

4.5 Theory of Extrapolation

Theorem 4.19. *For $p > 1$, let $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n)$ be a given vector field and denote $u \in W_0^{1,p}(\Omega)$ to be the unique weak solution to (4.2). Suppose we have that*

$$
\int_\Omega |\nabla u|^p \, v(x) \, dx \leq C_{([v]_{\frac{p}{p-1}})} \int_\Omega |\mathbf{f}|^p \, v(x) \, dx \tag{4.28}
$$

holds for all weights $v \in A_{\frac{p}{p-1}}$. Then for any $p - 1 < q < \infty$, there holds

$$
\int_\Omega |\nabla u|^q w(x) \, dx \leq C_{([w]_{\frac{q}{p-1}})} \int_\Omega |\mathbf{f}(x)|^q w(x) \, dx \tag{4.29}
$$

for all weights $w \in A_{\frac{q}{p-1}}$.

Remark 4.20. *What we obtain in this thesis is the weighted bound (4.1) for all weights $w \in A_1$ which unfortunately is not enough for us to apply the above extrapolation theorem.*

However, it provides us with an alternative view on the conjecture of T. Iwaniec and gives us a different sense of how far we are from completely resolving this conjecture. Of course, one can also generalize this conjecture by proposing the bound (4.28) *for all weights* $v \in A_{\frac{p}{p-1}}$.

Proof. First we consider the sub-natural case $p-1 < q < p$. To that end, let $w \in A_{\frac{q}{p-1}}$ and suppose that $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n) \cap L^q_w(\Omega, \mathbb{R}^n)$ satisfying (4.28) for all $v \in A_{\frac{p}{p-1}}$. Extend both $\mathbf{f}$ and u by zero to $\mathbb{R}^n \setminus \Omega$ and define

$$\mathcal{R}(\mathbf{f})(x) := \sum_{k=0}^{\infty} \frac{\mathcal{M}^{(k)}(|\mathbf{f}|^{p-1})(x)}{2^k \|\mathcal{M}\|^k_{L_w^{q/(p-1)} \to L_w^{q/(p-1)}}}.$$

Here $\mathcal{M}^{(k)} = \mathcal{M} \circ \mathcal{M} \circ \cdots \circ \mathcal{M}$ (k times) and note that (see, e.g., [19, Chapter 9])

$$\|\mathcal{M}\|_{L_w^{q/(p-1)} \to L_w^{q/(p-1)}} \leq C_{(n,p,q,[w]_{\frac{q}{p-1}})}. \tag{4.30}$$

Now it is easy to observe from the definition of $\mathcal{R}(\mathbf{f})$ that

$$|\mathbf{f}(x)|^{p-1} \leq \mathcal{R}(\mathbf{f})(x), \quad \text{and} \quad \|\mathcal{R}(\mathbf{f})\|_{L_w^{q/(p-1)}} \leq 2\|\mathbf{f}\|^{p-1}_{L^q_w}. \tag{4.31}$$

An important result which we shall need is the following estimate:

$$\mathcal{R}(\mathbf{f})^{-\frac{(p-q)}{(p-1)}} w \in A_{\frac{p}{p-1}} \quad \text{with} \quad [\mathcal{R}(\mathbf{f})^{-\frac{(p-q)}{(p-1)}} w]_{\frac{p}{p-1}} \leq C_{([w]_{\frac{q}{p-1}})}. \tag{4.32}$$

The proof of (4.32) is obtained as follows: it follows from (4.30) and the definition of $\mathcal{R}(\mathbf{f})$ that

$$\mathcal{M}(\mathcal{R}(\mathbf{f})) \leq C_{([w]_{\frac{q}{p-1}})} \mathcal{R}(\mathbf{f}),$$

and thus we get that

$$\mathcal{R}(\mathbf{f})(x)^{-1} \leq C_{([w]_{\frac{q}{p-1}})} \left(\frac{1}{|B|} \int_B \mathcal{R}(\mathbf{f})\, dy \right)^{-1}$$

for any ball $B \subset \mathbb{R}^n$ containing x. Set now $s = \dfrac{(p-q)}{(p-1)} \dfrac{q}{p}$. Using the last inequality, we find for any ball $B \subset \mathbb{R}^n$,

$$\fint_B \mathcal{R}(\mathbf{f})^{-s\frac{p}{q}} w\, dx \leq C_{([w]_{\frac{q}{p-1}})} \left(\fint_B \mathcal{R}(\mathbf{f})\, dy \right)^{-s\frac{p}{q}} \left(\fint_B w(x)\, dx \right). \tag{4.33}$$

On the other hand, by Hölder's inequality there holds

$$\begin{aligned}\left({}_{B} [\mathcal{R}(\mathbf{f})^{-s\frac{p}{q}} w(x)]^{1-p}\, dx\right)^{\frac{1}{p-1}} &= \left({}_{B} \mathcal{R}(\mathbf{f})^{p-q} w(x)^{1-p}\, dx\right)^{\frac{1}{p-1}} \\ &\leq \left({}_{B} \mathcal{R}(\mathbf{f})\, dx\right)^{\frac{p-q}{p-1}} \left({}_{B} w(x)^{\frac{1-p}{1-p+q}}\, dx\right)^{\frac{1-p+q}{p-1}}.\end{aligned} \tag{4.34}$$

Multiplying (4.33) by (4.34), we obtain the conclusion stated in (4.32).

We now obtain by Hölder's inequality

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u|^q w\, dx &= \int_{\mathbb{R}^n} |\nabla u|^q\, \mathcal{R}(\mathbf{f})^{-s} \mathcal{R}(\mathbf{f})^{s} w\, dx \\ &\leq \left(\int_{\mathbb{R}^n} |\nabla u|^p \mathcal{R}(\mathbf{f})^{-s.\frac{p}{q}} w\, dx\right)^{q/p} \left(\int_{\mathbb{R}^n} \mathcal{R}(\mathbf{f})^{s.\frac{q}{p-q}} w\, dx\right)^{(p-q)/p}.\end{aligned} \tag{4.35}$$

By making use of the hypothesis of the theorem along with (4.31), we can then estimate the right hand side of (4.35) as

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u|^q w\, dx &\leq C_{\left([\mathcal{R}(\mathbf{f})^{-s.\frac{p}{q}} w]_{\frac{p}{p-1}}\right)} \left(\int_{\mathbb{R}^n} |\mathbf{f}|^p\, \mathcal{R}(\mathbf{f})^{-s.\frac{p}{q}} w\, dx\right)^{q/p} \left(\int_{\mathbb{R}^n} \mathcal{R}(\mathbf{f})^{s.\frac{q}{(p-q)}} w\, dx\right)^{(p-q)/p} \\ &\leq C_{\left([\mathcal{R}(\mathbf{f})^{-s.\frac{p}{q}} w]_{\frac{p}{p-1}}\right)} \left(\int_{\mathbb{R}^n} \mathcal{R}(\mathbf{f})^{\frac{q}{p-1}} w\, dx\right) \\ &\leq C_{\left([\mathcal{R}(\mathbf{f})^{-s.\frac{p}{q}} w]_{\frac{p}{p-1}}\right)} 2^{\frac{q}{p-1}} \|\mathbf{f}\|^q_{L^q_w}.\end{aligned}$$

Then applying (4.32), we obtain (4.29) in the case $p-1 < q < p$.

We now consider the case $p < q < \infty$ and in this regard, we fix a $w \in A_{\frac{q}{p-1}}$ and let $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n) \cap L^q_w(\Omega, \mathbb{R}^n)$ be as in the theorem. For any $h \in L^{(q/p)'}_w(\mathbb{R}^n)$, define

$$\mathcal{R}'(h)(x) := \sum_{k=0}^{\infty} \frac{(\mathcal{M}')^{(k)}(|h|^{\frac{(q/p)'}{(q/(p-1))'}})(x)}{2^k \|\mathcal{M}'\|^k_{L^{(q/(p-1))'}_w \to L^{(q/(p-1))'}_w}},$$

where $\mathcal{M}'(h) := \dfrac{\mathcal{M}(hw)}{w}$ and $(q/p)' = \dfrac{q}{q-p}$, $(q/(p-1))' = \dfrac{q}{q-p+1}$ denote the conjugate Hölder exponents. Then it is easy to observe that

$$|h|^{\frac{(q/p)'}{(q/(p-1))'}}(x) \leq \mathcal{R}'(h)(x), \text{ and } \|\mathcal{R}'(h)\|_{L^{(q/(p-1))'}_w} \leq 2\|h\|^{\frac{(q/p)'}{(q/(p-1))'}}_{L^{(q/p)'}_w}. \tag{4.36}$$

We now choose an $h \in L^{(q/p)'}_w(\mathbb{R}^n)$ with $\|h\|_{L^{(q/p)'}_w)} = 1$ such that

$$\int_{\mathbb{R}^n} |\nabla u|^q\, w(x)\, dx = \||\nabla u|^p\|^{q/p}_{L^{q/p}_w} = \left(\int_{\mathbb{R}^n} |\nabla u|^p h(x)\, w(x)\, dx\right)^{q/p}. \tag{4.37}$$

For this choice of h, define $H := [\mathcal{R}'(h)]^{\frac{(q/(p-1))'}{(q/p)'}}$. It is easy to see from (4.36) that $0 \le h \le H$. We now prove the following important estimate:

$$(Hw) \in A_{\frac{p}{p-1}} \quad \text{with} \quad [Hw]_{\frac{p}{p-1}} \le C_{([w]_{\frac{q}{p-1}})}. \tag{4.38}$$

Analogous to (4.30), we observe that $\mathcal{M}'(\mathcal{R}'(h)) \le C_{([w]_{\frac{q}{p-1}})}\mathcal{R}'(h)$. Thus for any ball B containing x,

$$(Hw)(x)^{1-p} \le C_{([w]_{\frac{q}{p-1}})} \left({}_{B} H^{\frac{(q/p)'}{(q/(p-1))'}} w(y)\, dy \right)^{\frac{(q/(p-1))'}{(q/p)'}(1-p)} w(x)^{\frac{1-p}{q-p+1}},$$

where we have used the fact that $\left(\dfrac{(q/(p-1))'}{(q/p)'} - 1\right)(p-1) = \dfrac{1-p}{q-p+1}$. With this we obtain the estimate

$$\left({}_{B} (Hw)^{1-p}\, dx \right)^{\frac{1}{p-1}} \le C_{([w]_{\frac{q}{p-1}})} \left({}_{B} H^{\frac{(q/p)'}{(q/(p-1))'}} w\, dy \right)^{-\frac{(q/(p-1))'}{(q/p)'}} \left({}_{B} w^{\frac{1-p}{q-p+1}}\, dx \right)^{\frac{1}{p-1}} \tag{4.39}$$

for all balls $B \subset \mathbb{R}^n$.

On the other hand, by Hölder's inequality, we obtain

$${}_{Q} Hw\, dx \le \left({}_{Q} H^{\frac{(q/p)'}{(q/(p-1))'}} w\, dx \right)^{\frac{(q/(p-1))'}{(q/p)'}} \left({}_{Q} w\, dx \right)^{1-\frac{(q/(p-1))'}{(q/p)'}}. \tag{4.40}$$

Multiplying (4.39) by (4.40) and observing that $1 - \dfrac{(q/(p-1))'}{(q/p)'} = \dfrac{1}{q-p+1}$, we get

$$[Hw]_{\frac{p}{p-1}} \le C_{([w]_{\frac{q}{p-1}})} \left({}_{Q} w^{\frac{1-p}{q-p+1}}\, dx \right)^{\frac{1}{p-1}} \left({}_{Q} w\, dx \right)^{\frac{1}{q-p+1}} \le C_{([w]_{\frac{q}{p-1}})},$$

which completes the proof of (4.38).

Using our hypothesis on $\mathbf{f}$ and Hölder's inequality we now obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u|^p\, h\, w\, dx &\le \int_{\mathbb{R}^n} |\nabla u|^p\, H\, w\, dx \\
&\le C_{\left([Hw]_{\frac{p}{p-1}}\right)} \int_{\mathbb{R}^n} |\mathbf{f}|^p\, H\, w\, dx \\
&\le C_{\left([Hw]_{\frac{p}{p-1}}\right)} \left(\int_{\mathbb{R}^n} |\mathbf{f}|^q\, w\, dx \right)^{p/q} \left(\int_{\mathbb{R}^n} |H|^{(q/p)'}\, w\, dx \right)^{1/(q/p)'}.
\end{aligned} \tag{4.41}$$

Concerning the last term on the right, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |H|^{(q/p)'}\, w\, dx &= \int_{\mathbb{R}^n} \mathcal{R}'(h)^{(q/(p-1))'}\, w\, dx \\ &= \|\mathcal{R}'(h)\|_{L_w^{(q/(p-1))'}}^{(q/(p-1))'} \leq 2^{(q/(p-1))'} \|h\|_{L_w^{(q/p)'}}^{(q/p)'},\end{aligned} \tag{4.42}$$

where the last inequality follows from (4.36).

Substituting (4.42) into (4.41) and recalling (4.37), we obtain the desired estimate when $p < q < \infty$. $\square$

Chapter 5

Sharp Existence Result for Riccati type Equation

In this Chapter, we will study existence of solutions to equations of the form:

$$\begin{cases} -\operatorname{div}\mathcal{A}(x,\nabla u) = |\nabla u|^p + \sigma \ \text{ in } \Omega \\ \qquad\qquad u = 0 \ \text{ on } \partial\Omega \end{cases} \tag{5.1}$$

Hypothesis 5.1. *We will assume the nonlinearity $\mathcal{A}(x,\zeta)$ satisfies* (2.1) *and* (2.2)*. Also we shall assume that $\mathcal{A}(x,\zeta)$ satisfies $(\gamma_0,\, R_0)$-BMO condition as quantified in Theorem 4.3. We further assume that the domain Ω satisfies all the conditions of Theorem 4.3.*

Remark 5.2. *The Hypothesis 5.1 ensures that we have estimates of the form:*

$$\|\nabla u\|_{L^p_w(\Omega)} \leq C_{(n,p,\Lambda_0,\Lambda_1,[w]_1,\mathrm{diam}(\Omega)/R_0)}\|\mathbf{f}\|_{L^p_w(\Omega)}$$

holds for all weights $w \in A_1$.

In what follows, we will only assume that Hypothesis 5.1 are satisfied unless explicitly stated otherwise.

We shall recall the Schauder Fixed point theorem:

Theorem 5.3 (Schauder Fixed Point Theorem)**.** *Suppose $K \subset X$ is a closed, convex set and assume also that*

$$A : K \to K$$

is precompact. Then A has a fixed point in K.

Definition 5.4. Define the set

$$\mathbb{E}_T := \{\phi \in W^{1,1}_0(\Omega) \cap W^{1,p}_0(\Omega) : \|\phi\|_{M^{1,p}} \leq T\} \tag{5.2}$$

where T is a fixed constant to be chosen later. We shall impose the subset topology from $W^{1,1}_0(\Omega)$ on $\mathbb{E}$. Note here that the norm $\|\cdot\|_{M^{1,p}}$ is defined in Definition 2.16.

Lemma 5.5. *The set $\mathbb{E}_T$ is closed and convex for a fixed T.*

Proof. Let v_n be any sequence in $\mathbb{E}_T$ with $v_n \to v$ strongly in $W_0^{1,1}(\Omega)$. Because $v_n \in W_0^{1,p}(\Omega)$ are uniformly bounded in $W_0^{1,p}(\Omega)$ norm, we have that $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$. Since $W_0^{1,p}(\Omega)$ norms are weakly lower semicontinuous, we see that

$$\|v\|_{W_0^{1,p}(K)} \leq \liminf_{n\to\infty} \|v_n\|_{W_0^{1,p}(K)} \leq T \operatorname{cap}_{1,p}(W_0^{1,p}(K))$$

which automatically implies $v \in \mathbb{E}_T$ and this shows that $\mathbb{E}_T$ is closed.

It is easy to see that $\|\cdot\|_{M^{1,p}}$ is a seminorm and seminorms are subadditive. This easily implies that $\mathbb{E}_T$ is a convex set. □

5.1 Main Theorem

Theorem 5.6. *Let Ω be a bounded domain and assume that the nonlinearity $\mathcal{A}(x, \nabla u)$ satisfies* (2.3) *for the proof of this theorem. Suppose there exists a solution $u \in W_0^{1,p}(\Omega)$ to* (5.1) *with $|\nabla u|^p \in M^{1,p}(\Omega)$, then there exists a vector field ζ such that $\sigma = -\operatorname{div}\zeta$ and satisfies the estimate*

$$\||\zeta|^{p'}\|_{M^{1,p}} \lesssim \||\nabla u|^p\|_{M^{1,p}} + \||\nabla u|^p\|_{M^{1,p}}^{p'}$$

Remark 5.7. *If we assume $\sigma \geq 0$ and compactly supported in Ω, then any weak solution $u \in W_0^{1,p}(\Omega)$ solving* (5.1) *automatically satisfies $|\nabla u|^p \in M^{1,p}(\Omega)$. To see this, consider any function $\phi \in C_c^\infty(\Omega)$ with $\phi \geq 0$ and using ϕ^p as test function in* (5.1)*, we get*

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), p\phi^{p-1}\nabla\phi\rangle \, dx = \int_\Omega |\nabla u|^p \phi^p \, dx + \int_\Omega \sigma\phi^p \, dx$$

An application of Young's inequality on the left gives

$$\int_\Omega |\nabla u|^p \phi^p \, dx + \int_\Omega \sigma\phi^p \, dx \leq \frac{1}{2}\int_\Omega |\nabla u|^p \phi^p \, dx + C\int_\Omega |\nabla\phi|^p \, dx.$$

In other words, we have that

$$\int_\Omega |\nabla u|^p \phi^p \, dx \leq C\int_\Omega |\nabla\phi|^p \, ds \qquad \textit{and} \qquad \int_\Omega \sigma\phi^p \, dx \, dx \leq C\int_\Omega |\nabla\phi|^p \, dx$$

for all $\phi \in C_c^\infty(\Omega)$ with $\phi \geq 0$. This is precisely the trace inequality and hence Theorem 2.32 implies that $\sigma \in M^{1,p}(\Omega)$ and $|\nabla u|^p \in M^{1,p}(\Omega)$.

Before we proceed with the proof of the theorem, we need the following Lemma:

Lemma 5.8. *Given any function $g \in L^s(\Omega)$, there exists a vector field $\mathbf{h} \in L^s(\Omega)$ such that $g = \operatorname{div} \mathbf{h}$ satisfying the estimate*

$$|\mathbf{h}(x)| \leq I_1(g)(x)$$

where $I_1(\cdot)(x)$ denotes the Riesz potential of order 1.

Proof. First we extend g to be zero outside Ω. Now consider the Green's function $G(x, y)$ associated to $-\Delta$ on the ball $\Omega \subset B_R$ with radius $R = \operatorname{diam}(\Omega)$, then consider at the following problem:

$$\begin{cases} -\Delta\phi = g & \text{in} \quad B_R \\ \phi = 0 & \text{on} \quad \partial B_R \end{cases}$$

We can then write $\phi = G * g := G * (-\Delta\phi)$ and from this it is easy to see that

$$\Delta\phi(x) = \int_{B_R} \Delta_x G(x, y) g(y)\, dy = \int_{B_R} \Delta_x G(x, y)(-\Delta\phi(y))\, dy.$$

Hence we can write $g(x) = (-\operatorname{div} \nabla G) * g := -\operatorname{div}(\mathbf{h})$. More specifically, we have

$$h(x) = \int_{B_R} \nabla_x G(x, y) g(y)\, dy.$$

Since the Green function $G(x, y)$ we considered was on B_R, we easily see that the Greens function $G(x, y)$ must satisfy the estimate

$$|\nabla_x G(x, y)| \leq C(R, n) \frac{1}{|x - y|^{n-1}}.$$

Hence we obtain the pointwise estimate

$$|h(x)| \leq I_1(g)(x).$$

□

Proof of Theorem 5.6. Consider a ball $B_R \supset \Omega$ and applying Lemma 5.8 with $g = |\nabla u|^p$ and from (5.1), we see that $\sigma = -\operatorname{div} \mathcal{A}x, \nabla u + \operatorname{div}(h)$ with $h(x) = \int_{B_R} \nabla_x G(x, y) g(y)\, dy$. That is, if we set $\zeta(x) = \mathcal{A}(x, \nabla u) - h(x)$, then we trivially have $\sigma = -\operatorname{div} \zeta$.

From Lemma 5.8, it is easy to see that

$$|\zeta(x)|^{p'} \leq C(p)\left[|\mathcal{A}(x,\nabla u)|^{p'} + |h(x)|^{p'}\right] \leq C_{(p,\beta)}\left[|\nabla u|^p + I_1(|\nabla u|^p)^{p'}(x)\right].$$

Integrating over a compact set $K \Subset \mathbb{R}^n$ with $\operatorname{cap}_{1,p}(K\cap\overline{\Omega}) > 0$ and dividing by $\operatorname{cap}_{1,p}(K,\overline{\Omega})$, we get

$$\begin{aligned}\frac{\int_{K\cap\Omega}|\zeta|^{p'}\,dx}{\operatorname{cap}_{1,p}(K\cap\overline{\Omega})} &\leq C_{(p,\beta)}\left(\frac{\int_{K\cap\Omega}|\nabla u|^p\,dx}{\operatorname{cap}_{1,p}(K\cap\overline{\Omega})} + \frac{\int_{K\cap\Omega}I_1(|\nabla u|^p)^{p'}}{\operatorname{cap}_{1,p}(K\cap\overline{\Omega})}\right)\\ &\leq C_{(p,\beta)}\left(\||\nabla u|^p\|_{M^{1,p}} + \|I_1(|\nabla u|^p)^{p'}\|_{M^{1,p}}\right).\end{aligned}$$

Taking supremum over all compact sets $K \Subset \mathbb{R}^n$, we get the estimate

$$\||\zeta|^{p'}\|_{M^{1,p}} \leq C_{(p,\beta)}\left(\||\nabla u|^p\|_{M^{1,p}} + \|I_1(|\nabla u|^p)^{p'}\|_{M^{1,p}}\right)$$

Note that since by assumption, we have $|\nabla u|^p \in M^{1,p}(\Omega)$, using Theorem 2.30, we see that $\||\nabla u|^p\|_{M^{1,p}} < +\infty$ if and only if $\|I_1(|\nabla u|^p)^{p'}\|_{M^{1,p}} < +\infty$. Using Proposition 2.31, we can replace $\|I_1(|\nabla u|^p)^{p'}\|_{M^{1,p}}$ by $\||\nabla u|^p\|_{M^{1,p}}^{p'}$ and this completes the proof of the theorem. □

5.2 Main Theorem - Converse

Theorem 5.9. *Consider the equation*

$$\begin{cases}\operatorname{div}\mathcal{A}(x,\nabla u) = |\nabla u|^p + \operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f}) & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega\end{cases} \tag{5.3}$$

with the nonlinearity $\mathcal{A}(x,\zeta)$ and domain Ω satisfying Hypothesis 5.1. Then there exists a constant $T_S > 0$ such that if

$$\alpha := \||\mathbf{f}|^{p'}\|_{M^{1,p}} < T_S,$$

then there exists a solution $u \in W_0^{1,p}(\Omega)$ with $|\nabla u|^p \in M^{1,p}(\Omega)$ solving (5.3).

The proof of Theorem 5.9 will be carried in several steps. First, we approximate (5.3) and then obtain existence and regularity for the approximated equation. Eventually we will use the regularity and an appropriate test function to pass through the limit.

We shall first approximate (5.3) as follows: Consider any $v \in \mathbb{E}_T$ where $\mathbb{E}_T$ is as defined in (5.2). Note that we have not yet made any choice on T which will be made later on. Here $\mathbf{f}_s$ is any approximating sequence which converges strongly in $L^{p'}(\Omega)$ to the given $\mathbf{f}$, for example, we take $\mathbf{f}_s = T_s(\mathbf{f})$ and this suffices for our purposes where $T_s(x) := \min\{|x|, s\} \operatorname{sgn}(x)$ for any $s > 0$ which is the usual truncation operator. We now extend $\mathbf{f}$ to be 0 on $\mathbb{R}^n \backslash \Omega$ and consider the equation:

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla \tilde{u}_s) = T_s(|\nabla v|^p) + \operatorname{div}(\mathbf{f}_s) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{5.4}$$

Now applying Lemma 5.8, we see that (5.4) can be written as

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla \tilde{u}_s) = \operatorname{div}(\mathbf{h}_s) + \operatorname{div}(\mathbf{f}_s) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{5.5}$$

where

$$\mathbf{h}_s(x) = -\int_{B_R} \nabla_x G(x,y) T_s(|\nabla v|^p)\, dy = -\int_{\Omega} \nabla_x G(x,y) T_s(|\nabla v|^p)\, dy$$

satisfies the estimate

$$|\mathbf{h}_s(x)| \leq I_1(|\nabla v|^p)(x). \tag{5.6}$$

We shall first show existence and regularity of a weak solution to (5.4) and then apply Schauder Fixed Point Theorem 5.3 to show the existence and regularity of solution to (5.7):

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_s) = T_s(|\nabla u_s|^p) + \operatorname{div}(\mathbf{f}_s) & \text{in} \quad \Omega \\ u_s = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{5.7}$$

Once we have existence and regularity for the solution in (5.7), we will show strong convergence in $W_0^{1,p}(\Omega)$ of u_s to some function $u \in W_0^{1,p}(\Omega)$ and this will help us pass through the limit in (5.7) and show existence of solution to (5.3). Along the way, we need to put some restrictions on the vector field $\mathbf{f}$.

The first assumption we make on $\mathbf{f}$ and hence automatically also on $\mathbf{f}_s$ is that

$$\alpha := \||\mathbf{f}|^{p'}\|_{M^{1,p}}^{1/p'} < S_1 \;\text{ and hence }\; \||\mathbf{f}_s|^{p'}\|_{M^{1,p}}^{1/p'} < S_1 \;\; .$$

Here we need to make a suitable choice of T_0 and this will be made in the following Lemma. Note that the above assumption essentially implies that

$$\int_K |\mathbf{f}|^{p'}\ dx \leq S_1 \operatorname{cap}_{1,p}(K).$$

Lemma 5.10. *There exists constants $S_1 > 0$ and $T_0 > 0$ such that for any $v \in \mathbb{E}_{T_0}$ with $\||\mathbf{f}_s|^{p'}\|_{M^{1,p}} \leq S_1$, we have a unique solution $\tilde{u}_s \in \mathbb{E}_{T_0}$ which solves* (5.4).

Proof. Since $\mathbf{h}_s+\mathbf{f}_s \in L^\infty(\Omega)$, we have the existence of a unique weak solution in $\tilde{u}_s \in W_0^{1,p}(\Omega)$ to (5.5) from the standard theory of monotone operators. All that remains now to show is $\tilde{u}_s \in \mathbb{E}_{T_0}$.

From Theorem 4.3, we have the estimate

$$\int_\Omega |\nabla \tilde{u}_s|^p w\ dx \leq \tilde{C}_1(n,p,[w]_1) \int_\Omega |\mathbf{h}_s + \mathbf{f}_s|^{p'} w\ dx.$$

By assumption, we have that $|\nabla v|^p \in M^{1,p}(\Omega)$ and $|\mathbf{f}_s|^{p'} \in M^{1,p}(\Omega)$, hence it easy to see from using (5.6) and Theorem (2.30) that

$$\||\mathbf{h}_s + \mathbf{f}_s|^{p'}\|_{M^{1,p}} \leq \tilde{S}_2.$$

Now making use of Lemma 2.34, we see that

$$\||\nabla \tilde{u}_s|^p\|_{M^{1,p}} \leq C_{(n,p,\tilde{C}_1)} \left[\||\mathbf{f}_s + \mathbf{h}_s|^{p'}\|_{M^{1,p}}^{1/p'} \right]^{p'}.$$

By using Theorem 2.30 and Proposition 2.31, we get

$$\begin{aligned} \||\nabla \tilde{u}_s|^p\|_{M^{1,p}} &\leq C_{(n,p,\tilde{C}_1)} \left[\||\mathbf{f}_s + \mathbf{h}_s|^{p'}\|_{M^{1,p}}^{1/p'} \right]^{p'} \\ &\leq C_{(n,p,\tilde{C}_1)} \left[\||\mathbf{f}_s|^{p'}\|_{M^{1,p}}^{1/p'} + \||\nabla v|^p\|_{M^{1,p}}^{1/p} \right]^{p'}. \end{aligned} \tag{5.8}$$

Define $g(t) := C_{(n,p,\tilde{C}_1)}(t+\alpha)^{p'} - t := \tilde{C}_2(t+\alpha)^{p'} - t$ where $\alpha = \||\mathbf{f}_s|^{p'}\|_{M^{1,p}}^{1/p'}$. Then $g'(t) = 0$ iff

$$t = t_0 := \left[\frac{1}{\tilde{C}_2(t + \alpha(p'-1)} \right]^{\frac{1}{p'-1}} - \alpha.$$

We can see that g is a decreasing function and

$$g(t_0) = \tilde{C}_2(t + \alpha \left(\frac{1}{\tilde{C}_2(t + \alpha(p' - 1)}\right)^{\frac{p'}{p'-1}} - \left(\frac{1}{\tilde{C}_2(t + \alpha(p' - 1)}\right)^{\frac{1}{p'-1}} + \alpha.$$

If we assume that

$$\alpha \leq \left(\frac{1}{\tilde{C}_2(p' - 1)}\right)^{\frac{1}{p'-1}} - \tilde{C}_2 \left(\frac{1}{\tilde{C}_2(p' - 1)}\right)^{\frac{p'}{p'-1}} := S_1,$$

then $g(t_0) \leq 0$ and hence there is exactly one root for g at some $T_0 \in (0, t_0]$, i.e $g(T_0) = 0$. We fix this choice of T_0 and henceforth work with $\mathbb{E}_{T_0}$.

For this $t = T_0$, we have from (5.8), that

$$\||\nabla \tilde{u}_s|^p\|_{M^{1,p}} \leq \tilde{C}_2[\alpha + T_0]^{p'} = T_0.$$

This follows by the definition of T_0 and from the assumption that $v \in \mathbb{E}_{T_0}$.

What we have just shown is that given any $v \in \mathbb{E}_{T_0}$ with $\||\mathbf{f}|^{p'}\|_{M^{1,p}} \leq S_1$, we have a unique solution $\tilde{u}_s \in \mathbb{E}_{T_0}$ solving (5.4) and this proves the lemma. □

Theorem 5.11. *Let S_1 and T_0 be defined as in Lemma 5.10. Assume the vector field $\mathbf{f}_s$ satisfies $\||\mathbf{f}_s|^{p'}\|_{M^{1,p}} \leq S_1$, then we have a solution $u_s \in \mathbb{E}_{T_0}$ solving* (5.7).

Proof. We shall apply Schauder Fixed Point Theorem 5.3 to prove existence. From Lemma 5.5, we already know that $\mathbb{E}_{T_0}$ is closed and convex. So all that remains to show before applying Theorem 5.3 is that the operator $\mathcal{B} : \mathbb{E}_{T_0} \to \mathbb{E}_{T_0}$ given by $v \mapsto \tilde{u}_s$ is precompact. Here $\tilde{u}_s$ is the unique solution solving (5.4) as obtained in Lemma 5.10. Again from Lemma 5.10 we see that the map $\mathcal{B}$ is well defined.

Consider any sequence $v^k \in \mathbb{E}_{T_0}$ with $\|v^k\|_{M^{1,p}} \leq T_0$, then the solutions $\tilde{u}_s^k$ solving (5.4) are in $\mathbb{E}_{T_0}$ and hence we have that

- $v^k \to v$ weakly in $W_0^{1,p}(\Omega)$ since $v^k \in \mathbb{E}_{T_0}$,
- $T_s(|\nabla v^k|^p)$ is uniformly bounded in $W^{-1,p'}(\Omega)$ since $v^k \in \mathbb{E}_{T_0}$,

- $\tilde{u}_s^k \to \tilde{u}_s$ weakly in $W_0^{1,p}(\Omega)$ and

Thus all the hypothesis of Theorem 2.40 are satisfied and hence we get

- $\tilde{u}_s^k \to \tilde{u}_s$ strongly in $W_0^{1,q}(\Omega)$ for any $1 \leq q < p$,

- $\tilde{u_s}$ solves

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla \tilde{u}_s) = T_s(|\nabla v|^p) + \operatorname{div}(\mathbf{f}_s) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

This shows that the operator $\mathcal{B} : \mathbb{E}_{T_0} \to \mathbb{E}_{T_0}$ is precompact under $W_0^{1,1}(\Omega)$ topology and we can now apply Schauder Fixed Point Theorem to prove the existence of a solution $u_s \in \mathbb{E}_{T_0}$ solving (5.7). This completes the proof of the Theorem. $\square$

Theorem 5.12. *Let $\delta > \max\{\frac{1}{\Lambda_0}, 1\}$ be any fixed constant where Λ_0 is defined in* (2.2) *and define $\mu := \frac{\delta}{p-1}$. Let $u_s \in \mathbb{E}_{T_0}$ be any solution of* (5.4) *and define*

$$w_s = \frac{e^{\mu|u_s|} - 1}{\mu} sgn(u_s).$$

Then there exists an $S_2 > 0$ such that if $\||\mathbf{f}_s|^{p'}\|_{M^{1,p}} \leq S_2$, then the following regularity estimate holds:

$$\|u_s\|_{W_0^{1,p}(\Omega)} + \|w_s\|_{W_0^{1,p}(\Omega)} \leq M_\delta.$$

Here the constant M_δ depends only on S_2, T_0 and is independent of the solution.

Proof. Define $v = e^{\delta|u_s|}w_s$ as the test function. This is clearly an admissible test function since $v \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, hence we see that

$$\nabla v = e^{\delta|u_s|}\nabla w_s + \delta w_s e^{\delta|u_s|}\nabla u_s \ \operatorname{sgn}(u_s).$$

Using this as a test function in (5.7), we get

$$\begin{aligned} &\int_\Omega \langle \mathcal{A}(x, \nabla u_s), \nabla w_s\rangle e^{\delta|u_s|} \ dx + \int_\Omega \delta|w_s|e^{\delta|u_s|}\langle \mathcal{A}(x, \nabla u_s), \nabla u_s\rangle \ dx \\ &\qquad = \int_\Omega T_s(|\nabla u_s|^p)e^{\delta|u_s|}w_s \ dx + \int_\Omega \langle \mathbf{f}_s, \nabla(e^{\delta|u_s|}w_s)\rangle \ dx \\ &I_1 + I_2 = I_3 + I_4 \end{aligned}$$

Estimate for I_1: Since $\nabla w_s = e^{\mu|u_s|}\nabla u_s$, using the condition (2.1), we see that

$$\begin{aligned} I_1 &= \int_\Omega \langle \mathcal{A}(x,\nabla u_s), \nabla w_s\rangle e^{\delta|u_s|}\, dx \\ &= \int_\Omega \langle \mathcal{A}(x,\nabla u_s), \nabla u_s\rangle e^{(\mu+\delta)|u_s|}\, dx \\ &\geq \Lambda_0 \int_\Omega |\nabla w_s|^p\, dx \end{aligned}$$

where we have used the fact $\mu + \delta = p - 1$.

Estimate for $I_3 - I_2$:

$$\begin{aligned} I_3 - I_2 &= \int_\Omega T_s(|\nabla u_s|^p) e^{\delta|u_s|} w_s\, dx - \int_\Omega \delta |w_s| e^{\delta|u_s|}\langle \mathcal{A}(x,\nabla u_s), \nabla u_s\rangle\, dx \\ &\leq \int_\Omega |\nabla u_s|^p e^{\delta|u_s|}|w_s|\, dx - \delta\Lambda_0 \int_\Omega |w_s| e^{\delta|u_s|}|\nabla u_s|^p\, dx \\ &= \int_\Omega (1-\delta\Lambda_0)|\nabla u_s|^p e^{\delta|u_s|}|w_s|\, dx \leq 0, \end{aligned}$$

since we assume that $\delta \geq \dfrac{1}{\Lambda_0}$. As a consequence, we can ignore these two terms.

Estimate for I_4: We have after expanding,

$$\delta \int_\Omega |\mathbf{f}_s| e^{\delta|u_s|}|\nabla u_s||w_s|\, dx + \int_\Omega |\mathbf{f}_s| e^{\delta|u_s|}|\nabla w_s|\, dx = \delta I_4^1 + I_4^2$$

Estimate for I_4^1: By Holders inequality, we have

$$\begin{aligned} \int_\Omega |\mathbf{f}_s| e^{\delta|u_s|}|\nabla u_s||w_s|\, dx &\leq \left(\int_\Omega |\mathbf{f}_s|^{p'} e^{p\mu|u_s|}\, dx\right)^{1/p'} \left(\int_\Omega |\nabla u_s|^p |w_s|^p\, dx\right)^{1/p} \\ &= \left(\int_\Omega |\mathbf{f}_s|^{p'} \left(e^{\mu|u_s|} - 1 + 1\right)^p\, dx\right)^{1/p'} \times \\ &\qquad \times \left(\int_\Omega |\nabla u_s|^p |w_s|^p\, dx\right)^{1/p} \\ &\leq C_{(p)} \left(\mu^{p-1} \left(\int_\Omega |\mathbf{f}_s|^{p'} |w_s|^p\, dx\right)^{1/p'} + \|\mathbf{f}_s\|_{L^{p'}(\Omega)}\right) \times \\ &\qquad \times \left(\int_\Omega |\nabla u_s|^p |w_s|^p\, dx\right)^{1/p}. \end{aligned}$$

Since both $|\mathbf{f}_s|^{p'} \in M^{1,p}(\Omega)$ and $|\nabla u_s|^p \in M^{1,p}(\Omega)$, we see that from applying Theorem 2.32,

$$I_4^1 \leq \mu^{p-1} C_{(p,\||\mathbf{f}_s|^{p'}\|_{M^{1,p}},T_0)} \|\nabla w_s\|^p_{L^p(\Omega)} + C_{(p,T_0)} \|\mathbf{f}_s\|_{L^{p'}(\Omega)} \|\nabla w_s\|_{L^p(\Omega)}. \tag{5.9}$$

Here we have made use of the following Trace inequalities:

$$\left(\int_\Omega |\mathbf{f}_s|^{p'}|w_s|\ dx\right)^{1/p'} \leq C_{tr}(\||\mathbf{f}_s|^{p'}\|_{M^{1,p}})^{1/p'}\left(\int_\Omega |\nabla w_s|^p\ dx\right)^{1/p'}$$

and

$$\left(\int_\Omega |\nabla u_s|^p|w_s|^p\ dx\right)^{1/p} \leq C_{tr}(T_0)^{1/p}\left(\int_\Omega |\nabla w_s|^p\ dx\right)^{1/p}.$$

Estimate for I_4^2: By Holders inequality, we get

$$\int_\Omega |\mathbf{f}|e^{\delta|u_s}|\nabla w_s|\ dx \ \leq \left(\int_\Omega |\mathbf{f}_s|^{p'}e^{p\mu|u_s|}\ dx\right)^{1/p'}\left(\int_\Omega |\nabla w_s|^p\ dx\right)^{1/p}$$

Proceding as in Estimate for I_4^1, we get

$$I_4^2 \leq \mu^{p-1}C_{(p,\||\mathbf{f}_s|^{p'}\|_{M^{1,p}})}\|\nabla w_s\|^p_{L^p(\Omega)} + C_{(p)}\|\mathbf{f}_s\|_{L^{p'}(\Omega)}\|\nabla w_s\|_{L^p(\Omega)}. \tag{5.10}$$

Hence combining estimates (5.9) and (5.10), we get

$$I_4 \leq (1+\delta)\mu^{p-1}C_{0(\||\mathbf{f}_s|^{p'}\|_{M^{1,p}},T_0)}\|\nabla w_s\|^p_{L^p(\Omega)} + C_{1(p,T_0)}\|\mathbf{f}_s\|_{L^{p'}(\Omega)}\|\nabla w_s\|_{L^p(\Omega)}.$$

Since we have assumed that $\mathbf{f}_s \in \mathbb{E}_{T_0}$, we can replace $\|\mathbf{f}_s\|_{L^{p'}(\Omega)}$ in the above equation with a constant depending on T_0.

From Remark 2.33, there is a constant which we denote by S_2 such that if $\||\mathbf{f}_s|^{p'}\|_{M^{1,p}} \leq S_2$, we then obtain

$$(1+\delta)\mu^{p-1}C_{0(\||\mathbf{f}_s|^{p'}\|_{M^{1,p}},T)} \leq \frac{\Lambda_0}{2}.$$

This completes the proof of the theorem. □

Remark 5.13. *Henceforth, we shall assume that $\||\mathbf{f}_s|^{p'}\|_{M^{1,p}} \leq \min\{S_1, S_2\} =: T_S$ with S_1 is as in Lemma 5.10 and S_2 is as given in Theorem 5.12.*

We have thus far shown existence and uniform regularity to solutions of (5.7) for a fixed $s > 0$. If we can show strong convergence of the solutions u_s in $W_0^{1,p}(\Omega)$ as $\lim_{s\to\infty}$, we can then pass through the limit in (5.7) and conclude with existence of a solution to (5.1). This next section will focus on proving this strong convergence.

5.3 Strong convergence of approximate solutions

As a consequence of Theorem 5.12, we see that the sequence of approximate solutions $\{u_n\}$ to (5.11) is uniformly bounded in $W_0^{1,p}(\Omega)$ and hence has a weakly convergent subsequence which satisfies the following:

i. $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$,

ii. $u_n \to u$ a.e in Ω and

iii. $u_n \to u$ in measure on Ω

for some function $u \in W_0^{1,p}(\Omega)$.

Henceforth, we fix this subsequence $n \to \infty$ and the weak limit u, from which we have the following theorem:

Theorem 5.14. *Let $\mathcal{A}(x, \nabla u)$ and Ω satisfy Hypothesis 5.1 and assume that $\||\mathbf{f}_n|^{p'}\|_{M^{1,p}} \leq T_S$ as given in Remark 5.13 where $\mathbf{f}_n = T_n(\mathbf{f})$. Let u_n be a solution to*

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_n) = T_n(|\nabla u_n|^p) + \operatorname{div}(\mathbf{f}_n) & \text{in} \quad \Omega \\ u_n = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.11}$$

as obtained in Theorem 5.11, then we have for any fixed $k > 0$, the strong convergence of $\nabla T_k(u_n) \xrightarrow{n} \nabla T_k(u)$ in $L^p(\Omega)$.

Proof. We shall make use of the following test function in (5.11):

$$v_n = e^{\delta |T_j(u_n)|} \psi(z_n)$$

where $z_n = T_k(u_n) - T_k(u)$ for some $j \geq k$ and ψ is a *smooth increasing* function satisfying

$$\psi(0) = 0 \quad \text{and} \quad \psi' - \left(\frac{1 + \Lambda_1 \delta}{\Lambda_0}\right)|\psi| \geq 1.$$

Clearly we see that $v_n \in L^\infty \cap W_0^{1,p}(\Omega)$ and hence this is a valid test function. Using this, we obtain

$$
\begin{aligned}
\int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla z_n\rangle e^{\delta|T_j(u_n)|}\psi'(z_n)\,dx &= \int_\Omega T_n(|\nabla u_n|^p) e^{\delta|T_j(u_n)|}\psi(z_n)\,dx \\
&- \int_\Omega \langle \mathbf{f}_n, \nabla T_j(u_n)\rangle \delta e^{\delta|T_j(u_n)|}\psi(z_n)\operatorname{sgn}(u_n)\,dx - \int_\Omega \langle \mathbf{f}_n, \nabla z_n\rangle e^{\delta|T_j(u_n)|}\psi'(z_n)\,dx \\
&- \int_\Omega \delta\langle \mathcal{A}(x,\nabla u_n), \nabla T_j(u_n)\rangle e^{\delta|T_j(u_n)|}\psi(z_n)\operatorname{sgn}(u_n)\,dx \\
I_1 &= I_2 + I_3 + I_4 + I_5
\end{aligned}
$$

We now estimate the first term I_1 as follows:

$$
\begin{aligned}
I_1 &= \int_\Omega \langle \mathcal{A}(x,\nabla u_n), \nabla T_k(u_n) - \nabla T_k(u)\rangle e^{\delta|T_j(u_n)|}\psi'(z_n)\,dx \\
&= \int_{\{|u_n|\le k\}} \langle \mathcal{A}(x,\nabla u_n) - \mathcal{A}(x,\nabla T_k(u)), \nabla T_k(u_n) - \nabla T_k(u)\rangle e^{\delta|T_j(u_n)|}\psi'(z_n)\,dx \\
&\quad + \int_{\{|u_n|\le k\}} \langle \mathcal{A}(x,\nabla T_k(u)), \nabla T_k(u_n) - \nabla T_k(u)\rangle e^{\delta|T_j(u_n)|}\psi'(z_n)\,dx \\
&\quad + \int_{\{|u_n|> k\}} \langle \mathcal{A}(x,\nabla u_n), -\nabla T_k(u)\rangle e^{\delta|T_j(u_n)|}\psi'(z_n)\,dx \\
&= I_1^1 + I_1^2 + I_1^3
\end{aligned}
$$

We now split $I_2 + I_5$ as $I_2' + X_2'$ where

$$
\begin{aligned}
I_2' &= -\delta \int_{\{|u_n|>k\}} \langle \mathcal{A}(x,\nabla u_n), \nabla T_j(u_n)\rangle e^{\delta|T_j(u_n)|}\psi(z_n)\operatorname{sgn}(u_n)\,dx \\
&\qquad + \int_{\{|u_n|>k\}} T_n(|\nabla u_n|^p) e^{\delta|T_j(u_n)|}\psi(z_n)\,dx, \\
X_2' &= -\delta \int_{\{|u_n|\le k\}} \langle \mathcal{A}(x,\nabla u_n), \nabla T_j(u_n)\rangle e^{\delta|T_j(u_n)|}\psi(z_n)\operatorname{sgn}(u_n)\,dx \\
&\qquad + \int_{\{|u_n|\le k\}} T_n(|\nabla u_n|^p) e^{\delta|T_j(u_n)|}\psi(z_n)\,dx.
\end{aligned}
$$

Since $j \geq k$, we see that $|\nabla T_j(u_n)|\chi_{\{|u_n|\leq k\}} = |\nabla u_n|\chi_{\{|u_n|\leq k\}}$ and hence we get,

$$\begin{aligned}
|X_2'| &\leq \delta\Lambda_1 \int_{\{|u_n|\leq k\}} |\nabla u_n|^p \ e^{\delta|T_j(u_n)|} \ |\psi(z_n)| \ dx + \int_{\{|u_n|\leq k\}} |\nabla u_n|^p \ e^{\delta|T_j(u_n)|} \ |\psi(z_n)| \, dx \\
&\leq \Bigg[\int_{\{|u_n|\leq k\}} \langle \mathcal{A}(x,\nabla u_n) - \mathcal{A}(x,\nabla T_k(u)), \nabla T_k(u_n) - \nabla T_k(u)\rangle e^{\delta|T_j(u_n)|} \ |\psi(z_n)| \ dx \\
&\quad + \int_{\{|u_n|\leq k\}} \langle \mathcal{A}(x,\nabla T_k(u)), \nabla T_k(u_n) - \nabla T_k(u)\rangle e^{\delta|T_j(u_n)|} \ |\psi(z_n)| \ dx \\
&\quad + \int_{\{|u_n|\leq k\}} \langle \mathcal{A}(x,\nabla T_k(u_n)), \nabla T_k(u)\rangle e^{\delta|T_j(u_n)|} \ |\psi(z_n)| \ dx \Bigg] \left(\frac{1+\Lambda_1\delta}{\Lambda_0}\right) \\
&= X_2^1 + X_2^2 + X_2^3.
\end{aligned}$$

From our choice of ψ, we see that $I_1^1 - X_2^1 \geq I_1^1$ and hence we get

$$I_1^1 \leq -I_1^2 - I_1^3 + I_3 + I_4 + X_2^2 + X_2^3 + I_2'$$

We shall now estimate each of the terms as follows:

Estimate for $-I_1^2$: We know that $u_n \xrightarrow{n} u$ a.e, from which we see that

$$\mathcal{A}(x,\nabla T_k(u))e^{\delta|T_j(u_n)|}\psi'(z_n) \xrightarrow{n} \mathcal{A}(x,\nabla T_k(u))e^{\delta|T_j(u)|}\psi'(0) \quad a.e.$$

From the pointwise estimate

$$\mathcal{A}(x,\nabla T_k(u))e^{\delta|T_j(u_n)|}\psi'(z_n) \leq \Lambda_1 e^{\delta j} \max_{s\in[-2k,2k]} |\psi'(s)||\nabla T_k(u)|^{p-1}$$

and noting that $|\nabla T_k(u)|^{p-1} \in L^{p'}(\Omega)$, we easily see from using Lemma 2.38 that

$$\mathcal{A}(x,\nabla T_k(u))e^{\delta|T_j(u_n)|}\psi'(z_n) \xrightarrow{n} \mathcal{A}(x,\nabla T_k(u))e^{\delta|T_j(u)|}\psi'(0) \ \text{ strongly in } L^{p'}(\Omega).$$

From the observation

$$\chi_{\{|u_n|\leq k\}}(\nabla T_k(u_n) - \nabla T_k(u)) = \nabla T_k(u_n) - \nabla T_k(u)\chi_{\{|u_n|\leq k\}}.$$

and using the fact that $\nabla u_n \xrightharpoonup{n} \nabla u$ weakly in L^p and $\|\nabla T_k(u_n)\|_{L^p(\Omega)}$ being uniformly bounded independent of n (since $u_n \in \mathbb{E}_{T_0}$) , we get that

$$\nabla T_k(u_n) \xrightharpoonup{n} \nabla T_k(u) \quad \text{weakly in} \quad L^p.$$

Since $\chi_{\{|u_n|\le k\}} \xrightarrow{n} \chi_{\{|u|\le k\}}$ a.e in $\Omega \setminus \{|u| = k\}$ while $\nabla T_k(u) = 0$ a.e on $\{|u| = k\}$, we must have that

$$\chi_{\{|u_n|\le k\}}(\nabla T_k(u_n) - \nabla T_k(u)) \overset{n}{\rightharpoonup} 0 \quad \text{weakly in} \quad L^p.$$

The above calculations along with using Proposition 2.39 implies that $I_1^2 \xrightarrow{n} 0$ as $n \to \infty$.

Estimate for $-I_1^3$: Similar to how we applied Theorem 2.38 to estimate I_1^2, we see that

$$\chi_{\{|u_n|>k\}} \nabla T_k(u) e^{\delta|T_j(u_n)|}\psi'(z_n) \xrightarrow{n} 0 \quad \text{strongly in} \quad L^p.$$

We also have that $\mathcal{A}(x, \nabla u_n)$ is uniformly bounded in $L^{p'}(\Omega)$ independent of n and hence has a weakly convergent limit.

Combining both the facts and using Proposition 2.39, we see that $I_1^3 \xrightarrow{n} 0$ and $n \to \infty$.

Estimate for I_3: Since $\mathbf{f}_n = T_n(\mathbf{f})$ and from the definition of z_n, we see that

$$\delta \mathbf{f}_n e^{\delta|T_j(u_n)|}\psi(z_n)\,\mathrm{sgn}(u_n) \xrightarrow{n} 0 \quad \text{a.e in} \quad \Omega.$$

Also, $\mathbf{f}_n e^{\delta|T_j(u_n)|}\psi(z_n)\,\mathrm{sgn}(u_n)$ is uniformly integrable in $L^{p'}$ independent of n and hence we can apply Vitaly Theorem 2.38 to conclude that

$$\delta \mathbf{f}_n e^{\delta|T_j(u_n)|}\psi(z_n)\,\mathrm{sgn}(u_n) \xrightarrow{n} 0 \quad \text{strongly in} \quad L^{p'}.$$

Since $\nabla T_j(u_n)$ is uniformly bounded, it has a weakly convergent limit.

Combining the above results and using Proposition 2.39, we see that $I_3 \xrightarrow{n} 0$ as $n \to \infty$.

Estimate for I_4:

$$\begin{aligned} I_4 &= \int_{\{|u_n|\le k\}} \langle \mathbf{f}_n, \nabla z_n\rangle e^{\delta|T_j(u_n)|}\psi'(z_n)\,dx + \int_{\{|u_n|>k\}} \langle \mathbf{f}_n, \nabla z_n\rangle e^{\delta|T_j(u_n)|}\psi'(z_n)\,dx \\ &= I_4^1 + I_4^2 \end{aligned}$$

Estimate for I_4^1: Similar calculations in the spirit of estimate I_1^2 shows that

$$\mathbf{f}_n e^{\delta|T_j(u_n)|}\psi'(z_n) \xrightarrow{n} \mathbf{f} e^{\delta|T_j(u)|}\psi'(0) \quad \text{strongly in} \quad L^{p'}.$$

Since

$$\chi_{\{|u_n|\le k\}}(\nabla T_k(u_n) - \nabla T_k(u)) = \nabla T_k(u_n) - \nabla T_k(u)\chi_{\{|u_n|\le k\}}$$

and $\chi_{\{|u_n|\le k\}} \xrightarrow{n} \chi_{\{|u|\le k\}}$ a.e in $\Omega \setminus \{|u| = k\}$ while $\nabla T_k(u) = 0$ a.e on $\{|u| = k\}$, we must have that

$$\chi_{\{|u_n|\le k\}}(\nabla T_k(u_n) - \nabla T_k(u)) \xrightharpoonup{n} 0 \quad \text{weakly in} \quad L^p(\Omega).$$

Thus by using Proposition 2.39, we must have $I_4^1 \xrightarrow{n} 0$ as $n \to \infty$.

Estimate I_4^2: Proceeding in similar way as we estimated I_1^3, we can easily show that $I_4^2 \xrightarrow{n} 0$ as $n \to \infty$.

Combining the estimate for I_4^1 and I_4^2, we see that $I_4 \xrightarrow{n} 0$ as $n \to \infty$.

Estimate for X_2^2: Similar calculations as in estimate I_1^1 gives

$$\chi_{\{|u_n|\le k\}}\mathcal{A}(x, \nabla T_k(u))e^{\delta|T_j(u_n)|}|\psi(z_n)| \xrightarrow{n} 0 \quad \text{strongly in} \quad L^{p'}$$

and

$$\chi_{\{|u_n|\le k\}}(\nabla T_k(u_n) - \nabla T_k(u)) \xrightharpoonup{n} 0 \quad \text{weakly in} \quad L^p.$$

Combining all this, we see that $X_2^2 \xrightarrow{n} 0$ as $n \to \infty$.

Estimate for X_2^3: Since $\mathcal{A}(x, \nabla T_k(u_n))$ is uniformly bounded in $L^{p'}$ independent of n, we see that it is weakly compact. Combining this weak compactness with the observations

$$\nabla T_k(u)e^{\delta|T_j(u_n)|}|\psi(z_n)| \xrightarrow{n} 0 \qquad \text{strongly in} \quad L^p,$$

we see that $X_2^3 \xrightarrow{n} 0$ as $n \to \infty$.

Estimate for I_2': Since $j \geq k$, we have $\chi_{\{|u_n|>k\}}\langle \mathcal{A}(x, \nabla u_n), \nabla T_j(u_n)\rangle \geq \Lambda_0 |\nabla u|^p \chi_{\{|u_n|>j\}}$.

Also, it is easy to see that $\operatorname{sgn}(u_n)\chi_{\{|u_n|>k\}}\psi(z_n) \geq 0$ a.e on Ω. From both these observations, we see that

$$\begin{aligned}
&\int_{\{|u_n|>k\}} [\operatorname{sgn}(u_n)T_n(|\nabla u_n|^p) - \delta\langle \mathcal{A}(x, \nabla u_n), \nabla T_j(u_n)\rangle] \operatorname{sgn}(u_n) e^{\delta|T_j(u_n)|}\psi(z_n)\, dx \\
&\quad \leq \int_{\{|u_n|>k\}} \left[|\nabla u_n|^p - \delta\Lambda_0 |\nabla u_n|^p \chi_{\{|u_n|\leq j\}}\right] \operatorname{sgn}(u_n) e^{\delta|T_j(u_n)|}\psi(z_n)\, dx \\
&\quad \leq \int_{\{|u_n|>j\}} |\nabla u_n|^p \operatorname{sgn}(u_n) e^{\delta|T_j(u_n)|}\psi(z_n)\, dx.
\end{aligned}$$

In the last inequality, we have used the assumption $\delta > \dfrac{1}{\Lambda_0}$ along with the fact that $|\nabla u_n|\chi_{\{|u_n|\leq j\}} = 0$ on the set $\{|u_n| > j\}$. Hence we get

$$\begin{aligned}
I_2' &\leq \int_{\{|u_n|>j\}} |\nabla u_n|^p \operatorname{sgn}(u_n) e^{\delta|T_j(u_n)|}\psi(z_n)\, dx \\
&\leq e^{\delta j}\psi(2k) \int_{\{|u_n|>j\}} |\nabla u_n|^p\, dx.
\end{aligned} \tag{5.12}$$

If we define $y_n := e^{\frac{\delta}{p-1}|u_n|} - 1$, then we get

$$\begin{aligned}
\int_{\{|u_n|>j\}} |\nabla u_n|^p\ dx &= \left(\frac{p-1}{\delta}\right)^p \int_{\{|u_n|>j\}} e^{-\frac{\delta p}{p-1}|u_n|} |\nabla y_n|^p\ dx \\
&\leq \left(\frac{p-1}{\delta}\right)^p e^{-\frac{j\delta p}{p-1}} \int_{\{|u_n|>j\}} |\nabla y_n|^p\ dx \leq \left(\frac{p-1}{\delta}\right)^p e^{-\frac{j\delta p}{p-1}} M_\delta.
\end{aligned} \tag{5.13}$$

In the last inequality, we have made use of the uniform estimate in Theorem 5.12 Combining estimates (5.12) and (5.13), we see that

$$I_2' \leq \left(\frac{p-1}{\delta}\right)^p e^{-\frac{j\delta}{p-1}} M_\delta$$

which implies $\limsup\limits_{j\to\infty} \limsup\limits_{n\to\infty} I_2' \leq 0$.

Using (2.2) and the fact that $\langle \mathcal{A}(x, \nabla T_k(u_n)) - \mathcal{A}(x, \nabla T_k(u)), \nabla T_k(u_n) - \nabla T_k(u)\rangle \geq 0$, we get

$$I_1^1 \geq \Lambda_0 \int_{\{|u_n|\leq k\}} \langle \mathcal{A}(x, \nabla T_k(u_n)) - \mathcal{A}(x, \nabla T_k(u)), \nabla T_k(u_n) - \nabla T_k(u)\rangle\ dx.$$

Note that $e^{\delta|T_j(u_n)|} \geq 1$ uniformly and $|\psi'| \geq 1$. Combining all the estimates, this shows that

$$\int_{\{|u_n|\leq k\}} \langle \mathcal{A}(x, \nabla T_k(u_n)) - \mathcal{A}(x, \nabla T_k(u)), \nabla T_k(u_n) - \nabla T_k(u)\rangle \xrightarrow{n} 0 \quad \text{as } n \to \infty. \tag{5.14}$$

By using Vitali's theorem 2.38, we see that

$$\begin{aligned}\chi_{\{|u_n|>k\}}\langle \mathcal{A}(x,\nabla T_k(u_n)) - \mathcal{A}(x,\nabla T_k(u)), \nabla T_k(u_n) - \nabla T_k(u)\rangle &= \\ = \chi_{\{|u_n|>k\}}\langle \mathcal{A}(x,\nabla T_k(u)), \nabla T_k(u)\rangle \xrightarrow{n} 0 \quad \text{strongly in } L^1(\Omega).\end{aligned} \tag{5.15}$$

Using (5.14) and (5.15) and applying (2.2), we see that

$$\int_\Omega \left(|\nabla T_k(u_n)|^2 + |\nabla T_k(u)|^2\right)^{\frac{p-2}{2}} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \, dx \xrightarrow{n} 0 \quad \text{as } n \to \infty. \tag{5.16}$$

We will split the proof into two cases:

$p \geq 2$**:** In this situation, an application of triangle inequality trivially gives:

$$\left(|\nabla T_k(u_n)|^2 + |\nabla T_k(u)|^2\right)^{\frac{p-2}{2}} \left(|\nabla T_k(u_n) - \nabla T_k(u)|\right)^2 \geq |\nabla T_k(u_n) - \nabla T_k(u)|^p \tag{5.17}$$

$1 < p < 2$**:** Applying Hölder's inequality, we get:

$$\begin{aligned}\int_\Omega |\nabla T_k(u_n) - \nabla T_k(u)|^p \, dx &\leq \left(\int_\Omega (|\nabla T_k(u_n)|^2 + |\nabla T_k(u)|^2)^{\frac{p-2}{2}} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \, dx\right)^{\frac{p}{2}} \times \\ &\quad \times \left(\int_\Omega (|\nabla T_k(u_n)|^2 + |\nabla T_k(u)|^2)^{\frac{p}{2}} \, dx\right)^{\frac{2-p}{2}} \\ &\leq C_{(M_\delta, T_0)} \left(\int_\Omega (|\nabla T_k(u_n)|^2 + |\nabla T_k(u)|^2)^{\frac{p-2}{2}} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \, dx\right)^{\frac{p}{2}}.\end{aligned} \tag{5.18}$$

In the last inequality, we used the fact that $u \in \mathbb{E}_{T_0}$ and the uniform estimate from Theorem 5.12.

Using (5.16) and (5.17) in the case $p \geq 2$ or (5.18) in the case $1 < p < 2$, we see that

$$\int_\Omega |\nabla T_k(u_n) - \nabla T_k(u)|^p \, dx \xrightarrow{n} 0 \qquad \text{as } n \to \infty.$$

This implies $\nabla T_k(u_n) \xrightarrow{n} \nabla T_k(u)$ strongly in L^p for a fixed k and this completes the proof of the Theorem. □

Theorem 5.15. *Let u_n be as in Theorem 5.14, then we have*

$$\limsup_{n\to\infty} \int_\Omega |\nabla G_k(u_n)|^p \, dx \to 0 \quad \text{as } k \to \infty$$

where $G_k(s) = s - T_k(s)$.

Proof. Define $y_n = e^{\frac{\delta}{p-1}|u_n|} - 1$ and hence we get

$$\begin{aligned} \int_{\{|u_n|>k\}} |\nabla u_n|^p \; dx &= \left(\frac{p-1}{\delta}\right)^p \int_{\{|u_n|>k\}} e^{-\frac{\delta p}{p-1}|u_n|} |\nabla y_n|^p \; dx \\ &\leq \left(\frac{p-1}{\delta}\right)^p e^{-\frac{\delta p}{p-1}k} \int_{\{|u_n|>k\}} |\nabla y_n|^p \; dx. \end{aligned}$$

Using the a priori estimate from Theorem 5.12, we get

$$\int_\Omega |\nabla G_k(u_n)|^p \; dx \leq \left(\frac{p-1}{\delta}\right)^p e^{-\frac{\delta p}{p-1}k} M_\delta^p$$

which proves the theorem. □

Combining Theorem 5.14 and Theorem 5.15, we have the following Theorem:

Theorem 5.16. *Let $\mathcal{A}$ and Ω satisfy Hypothesis 5.1 and assume that $\||\mathbf{f}_n|^{p'}\|_{M^{1,p}} \leq T_S$ as given in Remark 5.13 where $\mathbf{f}_n := T_n(\mathbf{f})$. Let u_n be a solution to*

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_n) = T_n(|\nabla u_n|^p) + \operatorname{div}(\mathbf{f}_n) & \text{in} \quad \Omega \\ u_n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

as obtained in Theorem 5.11, then we have the strong convergence of $\nabla u_n \xrightarrow{n} \nabla u$ in $L^p(\Omega)$.

We now have all the estimates needed for the proof of Theorem 5.9.

Proof of Theorem 5.9. From Theorem 5.16, we can now easily pass through the limit in (5.11) to show the existence of a solution to (5.1). This solution is in $M^{1,p}(\Omega)$ since $u_n \in \mathbb{E}_{T_0}$. Observing $\mathbb{E}_{T_0}$ is closed, we must also have that $u \in \mathbb{E}_{T_0}$. This completes the proof.

□

Chapter 6
Existence of Distributional solution

Consider the problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f}) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Our goal is to show existence of a distributional solution below the natural exponent under some mild assumptions on $\mathbf{f}$.

We assume that the domain Ω and the nonlinearity $\mathcal{A}(x, \nabla u)$ satisfy:

Hypothesis 6.1. *We will assume the nonlinearity $\mathcal{A}(x, \zeta)$ is a caratheodory function which satisfies* (2.1) *and* (2.2) *for some $\gamma \in (0, 1]$. We will also assume that the nonlinearity $\mathcal{A}(x, \nabla u)$ and the domain Ω are such that the following global a priori estimate holds:*

$$\|\nabla u\|_{L^q(\Omega)} \leq C\|\mathbf{f}\|_{L^q(\Omega)}. \tag{6.2}$$

for all $q \in (p - \delta_0, p]$ for some $\delta_0 < 1$.

Remark 6.2. *From Theorem 3.3, we see that there exists a $\delta_0 > 0$ such that if Ω^c satisfies the p-thick condition, then the estimate*

$$\|\nabla u\|_{L^q(\Omega)} \leq C\|\mathbf{f}\|_{L^q(\Omega)}.$$

holds for all $q \in (p - \delta_0, p + \delta_0)$.

Lemma 6.3. *Let $\mathbf{f} \in (L^s(\Omega))^n$ be a given vector field for some $1 < s < \infty$. Also suppose that $\operatorname{div}(\mathbf{f}) = \mu$ is a Radon measure in the sense of Definition 2.15 and $|\mu|(K) < \infty$ for all $K \Subset \Omega$. Then there exists a sequence of smooth functions $\mathbf{f}_\epsilon$ such that*

i. $\mathbf{f}_\epsilon \to \mathbf{f}$ *in* $(L^s(\Omega))^n$ *as* $\epsilon \to 0$,

ii. $\operatorname{div}(\mathbf{f}_\epsilon) = \mu_\epsilon$ *is a Radon measure in the sense of Definition 2.15 and* *iii. for every $K \Subset \Omega$, there exists an ϵ_K such that*

$$|\mu_\epsilon|(K) < \infty \quad \text{for all } \epsilon < \epsilon_K$$

with the bound being independent of ϵ.

Proof. First extend $\mathbf{f}$ to be zero outside Ω. Let ϕ_ϵ be the standard mollifier and consider the sequence

$$\mathbf{f}_\epsilon := \phi_\epsilon * \mathbf{f} = \int_{\mathbb{R}^n} \phi_\epsilon(x-y)\mathbf{f}(y)\, dy,$$

where $(f_\epsilon^1, f_\epsilon^2, \ldots, f_\epsilon^n) := (\phi_\epsilon * \ f^1, \phi_\epsilon * \ f^2, \ldots, \phi_\epsilon * \ f^n)$. By standard convolution theory, (i) and (ii) are satisfied since $\mathbf{f}_\epsilon^i$ are smooth functions.

All that remains is to show (iii). In this regard, we observe that for any $\psi \in C_c^\infty(\Omega_l)$ for a fixed $l \in \mathbb{N}$ (see Definition 2.1), the equality

$$\left|\int_\Omega \operatorname{div}(\mathbf{f}_\epsilon)\, \psi\, dx\right| = \left|\int_\Omega \phi_\epsilon * \psi\, d\mu\right|$$

holds for all $\epsilon < \epsilon_{\Omega_l}$ where ϵ_{Ω_l} is chosen such that $\operatorname{spt}(\phi_{\epsilon_{\Omega_l}} * \psi) \subseteq \Omega_l + \epsilon_{\Omega_l} \Subset \Omega$. Note that

$$\int_\Omega \phi_\epsilon * \psi\, d\mu := -\int_\Omega \langle \mathbf{f}, \nabla(\phi_\epsilon * \psi)\rangle\, dx$$

which holds in the sense of Definition 2.15. This gives, for all $\epsilon < \epsilon_K$, that

$$\begin{aligned}
|\mu_\epsilon|(K) &\le \inf_{\substack{O \\ K \Subset O}} \sup_\psi \left\{ \int_\Omega \operatorname{div}(\mathbf{f}_\epsilon)\, \psi\, dx : \ \psi \in C_c^1(O),\ |\psi| \le 1 \right\} \\
&\le \sup_\psi \left\{ \int_\Omega \operatorname{div}(\mathbf{f}_\epsilon)\, \psi\, dx : \ \psi \in C_c^1(O_K),\ |\psi| \le 1 \right\}
\end{aligned}$$

for some fixed O_K such that $K \Subset O_K \Subset \Omega$ with ϵ_K chosen such that for all $\epsilon < \epsilon_K$, we have $\operatorname{spt}(\phi_{\epsilon_K} * \psi) \Subset O_K + \epsilon_K \Subset \Omega$ holds .

Thus

$$\begin{aligned}
|\mu_\epsilon|(K) &\le \sup_\psi \left\{ \int_\Omega \operatorname{div}(\mathbf{f}_\epsilon)\, \psi\, dx : \ \psi \in C_c^1(O_K),\ |\psi| \le 1 \right\} \\
&= \sup_\psi \left\{ \int_\Omega \phi_\epsilon * \psi\, d\mu : \ \psi \in C_c^1(O_K),\ |\psi| \le 1 \right\} \\
&\le |\mu|(O_K + \epsilon_K) < \infty
\end{aligned}$$

for all $\epsilon < \epsilon_K$ with the bound being independent of ϵ. $\square$

Remark 6.4. *Since by assumption $\mathcal{A}(x,\cdot)$ is continuous and satisfies* (2.3), *we have for any sequence $\nabla u_j \to \nabla u$ a.e in Ω with $u_j, u \in W_0^{1,p-\delta}(\Omega)$, the following convergence holds:*

$$\lim_{j\to\infty} \int_\Omega \langle \mathcal{A}(x,\nabla u_j), \nabla\varphi\rangle \, dx = \int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle \, dx$$

for all $\varphi \in C_c^1(\Omega)$. This follows since by estimate (6.2), *we have that $\mathcal{A}(x,\nabla u_j)$ is uniformly bounded in $(L^{\frac{p-\delta}{p-1}}(\Omega))^n$ and this gives us the desired weak convergence of $\mathcal{A}(x,\nabla u_j)$.*

6.1 Main Theorem

We are now ready to state our main theorem:

Theorem 6.5. *Let $\mathbf{f} \in (L^{p-\delta}(\Omega))^n$ be a given vector field with $0 < \delta < \delta_0$ chosen such that Hypothesis 6.1 holds. Also assume that $\operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f}) = \mu$, a Radon measure in the sense of Definition 2.15 satisfying $|\mu|(K) < \infty$ for all $K \Subset \Omega$. Then there exists a distributional solution $u \in W_0^{1,p-\delta}(\Omega)$ to* (6.1), *i.e. the following holds for all $\varphi \in W_0^{1,\frac{p-\delta}{1-\delta}}(\Omega)$,*

$$\int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle \, dx = \int_\Omega |\mathbf{f}|^{p-2}\langle \mathbf{f}, \nabla\varphi\rangle \, dx.$$

Proof. We apply Lemma 2.37 to $\mathbf{f}$ with $s = p-\delta$ and then consider the auxiliary problem

$$\begin{cases} \operatorname{div}\mathcal{A}(x,\nabla u_\epsilon) &= \operatorname{div}(|\mathbf{f}_\epsilon|^{p-2}\mathbf{f}_\epsilon) \quad \text{in}\,\Omega \\ u_\epsilon &= 0 \quad \text{on}\,\partial\Omega \end{cases}. \tag{6.3}$$

By estimate (6.2), we have

$$\|\nabla u_\epsilon\|_{L^{p-\delta}(\Omega)} \leq C\|\mathbf{f}_\epsilon\|_{L^{p-\delta}(\Omega)} \leq C\|\mathbf{f}\|_{L^{p-\delta}(\Omega)} < \infty.$$

Hence by Rellich's theorem, we have upto a subsequence indexed by i such that

i. $u_i \to u$ strongly in $L^{p-\delta}(\Omega)$,

ii. $\nabla u_i \rightharpoonup \nabla u$ weakly in $(L^{p-\delta}(\Omega))^n$,

iii. $u_i \to u$ a.e in Ω and

iv. $u_i \to u$ in measure on Ω.

for some function $u \in W_0^{1,p-\delta}(\Omega)$.

We shall *fix this specific subsequence* henceforth and show that this subsequence $\{u_i\}$ actually gives us the convergence needed to show that u is actually the desired weak distributional solution to (6.1).

By Remark 6.4, it is enough to show that the sequence $\{\nabla u_i\}$ converges to ∇u in measure on Ω. Fix a set Ω_l and consider the test function $T_k(u_\epsilon)\phi$ for some $\phi \in C_c^\infty(\Omega_{l+2})$. with $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on Ω_l. Note that this is a valid text function that can be used in (6.3), hence we see that

$$\begin{aligned}\int_\Omega \langle \mathcal{A}(x, \nabla u_i), \nabla T_k(u_i)\rangle \, \phi \, dx &= -\int_\Omega \langle \mathcal{A}(x, \nabla u_i), \nabla \phi\rangle \, T_k(u_i) \, dx + \int_\Omega T_k(u_i) \, \phi \, d\mu_i \\ &\leq Ck \int_\Omega |\nabla u_i|^{p-1} \, dx + k \left| \int_\Omega \phi \, d\mu_i \right| \\ &\leq Ck\end{aligned}$$

which holds for all $i > i_{\Omega_{l+2}}$. By using (2.1), we get

$$\int_{\Omega_l} |\nabla T_k(u_i)|^p \, dx \leq Ck$$

for all $i > i_{\Omega_{l+2}}$ where $\nabla T_k(u_\epsilon) := \nabla u_\epsilon \cdot \chi_{\{|u_\epsilon|<k\}}$. Hence by using Lemma 2.37, we see that the whole sequence converges in the following sense:

i. $T_k(u_i) \to T_k(u)$ strongly in $L^p(\Omega_l)$ and

ii. $\nabla T_k(u_i) \rightharpoonup \nabla T_k(u)$ weakly in $(L^p(\Omega_l))^n$.

Note that the limit is the same u as from before.

We now show that $\nabla u_i \to \nabla u$ in measure. Consider the decomposition:

$$\{x \in \Omega_l : |\nabla u_i(x) - \nabla u(x)| > \delta\} \subseteq D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6,$$

where

$$D_1 = \{x \in \Omega_l : |u_i(x)| > k\},$$

$$D_2 = \{x \in \Omega_l : |u(x)| > k\},$$

$$D_3 = \{x \in \Omega_l : |\nabla u_i(x)| > k\},$$

$$D_4 = \{x \in \Omega_l : |\nabla u(x)| > k\},$$

$$D_5 = \{x \in \Omega_l : |u_i(x) - u(x)| > \eta\},$$

$$D_6 = \{x \in \Omega_l : |\nabla u_i(x) - \nabla u(x)| \geq \delta,\ |u(x)| \leq k,\ |u_i(x)| \leq k,$$
$$|\nabla u_i(x)| \leq k,\ |\nabla u(x)| \leq k,\ |u_i(x) - u(x)| \leq \eta\}.$$

Since u, u_i and $\nabla u, \nabla u_i$ are functions in $L^{p-\delta}(\Omega)$ and $(L^{p-\delta}(\Omega))^n$ respectively, we have an $k_0(\sigma) > 0$ such that

$$\operatorname{meas}\left(\bigcup_{i=1}^{4} D_i\right) < \frac{\sigma}{3}$$

for all $k \geq k_0(\sigma)$.

Also since $u_i \to u$ in measure, we have

$$\operatorname{meas}(D_5) < \frac{\sigma}{3}$$

whenever i is chosen large enough.

If we are able to show that $\operatorname{meas}(D_6) < \dfrac{\sigma}{3}$, then the proof of the theorem is complete. In order to show this, consider the following set

$$\mathbf{K} = \{(s, \zeta, \zeta') \in \mathbb{R} \times \mathbb{R}^{2N} : |s| \leq k, |\zeta| \leq k, |\zeta'| \leq k, |\zeta - \zeta'| \geq \delta\}$$

and let $\gamma(x) := \min\limits_{\mathbf{K}} \langle \mathcal{A}(x, \zeta) - \mathcal{A}(x, \zeta'), \zeta - \zeta' \rangle$. This is achieved since $\mathbf{K}$ is a compact set and $\mathcal{A}(x, \cdot)$ is a continuous function for almost every $x \in \Omega$ by assumption. Also by (2.1), we have that $\gamma(x) > 0$ for almost every $x \in \Omega$.

By the continuity of the Lebesgue integral, we have for any given $\sigma > 0$, there exists a $\sigma'_{(\sigma)} > 0$ such that; for any measurable $E \subseteq \Omega$ with $\displaystyle\int_E \gamma \, dx \leq 2\sigma'$, then we must necessarily have that $\operatorname{meas}(E) \leq \sigma/3$.

Using this fact, in order to show $\text{meas}(D_6) < \sigma/3$, it is enough to show that

$$\int_{D_6} \gamma(x)\, dx \leq 2\sigma'. \tag{6.4}$$

To show this, consider the test function $w = T_\eta(u_\epsilon - T_k(u))\phi$ with ϕ as from before. We see that

$$w = \begin{cases} \eta\phi & \text{if}\, u_\epsilon > k+\eta \\ -\eta\phi & \text{if}\, u_\epsilon < -k-\eta \end{cases}. \tag{6.5}$$

Since $u_i, T_k(u) \in W^{1,p}(\Omega_l)$, we have $u_i - T_k(u) \in W^{1,p}(\Omega_l)$ and thus we see that $T_\eta(u_i - T_k(u))\phi \in W^{1,p}(\Omega_l)$ is a valid test function. Using this, we get

$$\begin{aligned}\int_\Omega \langle \mathcal{A}(x,\nabla u_i), \nabla T_\eta(u_i - T_k(u))\rangle\, \phi\, dx = -&\int_\Omega \langle \mathcal{A}(x,\nabla u_i), \nabla\phi\rangle T_\eta(u_i - T_k(u))\, dx \\ &+ \int_\Omega T_\eta(u_i - T_k(u))\phi\, d\mu_i. \\ \leq C\eta.&\end{aligned}$$

By choosing η small, we can get

$$\int_\Omega \langle \mathcal{A}(x,\nabla u_i), \nabla T_\eta(u_i - T_k(u))\rangle\, \phi\, dx < \frac{\sigma'}{2}.$$

We also see that $\nabla T_\eta(u_i - T_k(u)) = 0$ if $|u_\epsilon| > k+\eta$ and hence we get that

$$\nabla T_\eta(u_i - T_k(u)) = \nabla T_\eta(T_{k+\eta}(u_i) - T_k(u)).$$

Since $T_{k+\eta}(u_i) - T_k(u) \rightharpoonup T_{k+\eta}(u) - T_k(u)$ weakly in $W^{1,p}(\Omega_{l+2})$, we get

$$\begin{aligned}\lim_{i\to\infty} &\int_\Omega \langle \mathcal{A}(x,\nabla T_k(u)), \nabla T_\eta(T_{k+\eta}(u_i) - T_k(u))\rangle\phi\, dx \\ &= \int_\Omega \langle \mathcal{A}(x,\nabla T_k(u)), \nabla T_\eta(T_{k+\eta}(u) - T_k(u))\rangle\, \phi\, dx.\end{aligned} \tag{6.6}$$

Finally since $T_\eta(T_{k+\eta}(u) - T_k(u))$ converges weakly to 0 in $W^{1,p}(\Omega_{l+2})$ as $\eta \to 0$, there exists an $\eta_0(\Omega_{l+2})$ such that

$$|\int_\Omega \langle \mathcal{A}(x,\nabla T_k(u)), \nabla T_\eta(T_{k+\eta}(u) - T_k(u))\rangle\phi\, dx| \leq \frac{\sigma'}{4} \tag{6.7}$$

holds for all $\eta < \eta_0(\Omega_{l+2})$.

Thus combining (6.6) and (6.7), we get for all $\eta < \eta_0(\Omega_{l+2})$ and i sufficiently large, that

$$|\int_\Omega \langle \mathcal{A}(x, \nabla T_k(u)), \nabla T_\eta(T_{k+\eta}(u_i) - T_k(u)) \rangle \, \phi \, dx| \leq \frac{\sigma'}{2}.$$

We are now in a position to prove (6.4); by the definition of $\gamma(x)$ and D_6 and (2.1) and the structure of ϕ, we have that

$$\begin{aligned}
\int_{D_6} \gamma(x) \, dx &\leq \int_{D_6} \langle \mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u), \nabla u_i - \nabla u \rangle \, \phi \, dx \\
&\leq \int_{D_6} \langle \mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u), \nabla T_\eta(T_{k+\eta}(u_i) - T_k(u)) \rangle \, \phi \, dx \\
&\leq \sigma'.
\end{aligned}$$

Here to get the second inequality, we choose an i sufficiently large.

What we have shown is; given an Ω_l, then $\nabla u_i \to \nabla u$ in measure on Ω_l. Hence by standard measure theory, there exists a subsequence i_k such that $\nabla u_{i_k} \to \nabla u$ a.e on Ω_l. But because we already have that $\nabla u_i \rightharpoonup \nabla u$ weakly on $W_0^{1,p-\delta}(\Omega)$, the stability result from Lemma 2.37 gives that the whole sequence itself converges a.e to ∇u in Ω. This completes the proof of the theorem. $\square$

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